Robotics - Homogeneous coordinates and transformations

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15 March 2012

Outline

Outline

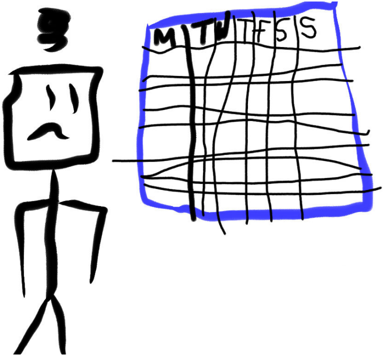
Calendar

1st part

- Wed 14/03 Homogeneous coordinate
- Thu 29/03 Computer Vision (1)
- Thu 12/04 Computer Vision (2)

2nd part

- Thu 10/05 Localization (1)
- Thu 07/06 Localization (2)
- Thu 14/06 Slam



Outline

Homogeneous coordinates

INTRODUCTION

- Introduced in 1827 (Möbius)
- Used in *projective geometry*
- Suitable for *points at the infinity*
- Easily code
  - points (2D-3D)
  - lines (2D)
  - conics (2D)
  - planes (3D)
  - quadrics (3D)
  - ...
- Transformation simpler than Cartesian
- ...

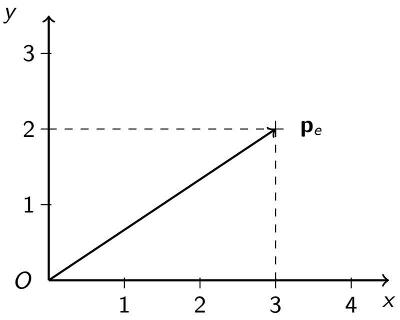
Points in Homogeneous coordinates - 2D space - Definition

Homogeneous 2D space

- Given a point $\mathbf{p}_e = \begin{bmatrix} X \\ Y \end{bmatrix} \in \mathbb{R}^2$ in Cartesian coordinates
- we can define $\mathbf{p}_h = \begin{bmatrix} x \\ y \\ w \end{bmatrix} \in \mathbb{R}^3$ in homogeneous coordinates
- under the relation $\begin{cases} X & = & x/w \\ Y & = & y/w \\ w & \neq & 0 \end{cases}$
- i.e., there is an arbitrary *scale factor* (w)

Points in Homogeneous coordinates - 2D space - Example

Example



- $\mathbf{p_e} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (euclidean)
- $\mathbf{p_{h_1}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \equiv \mathbf{p_e}$
- $\mathbf{p_{h_2}} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \equiv \mathbf{p_e}$
- $\mathbf{p_{h_3}} = \begin{bmatrix} 1.5 \\ 1 \\ 0.5 \end{bmatrix} \equiv \mathbf{p_e}$
- $\mathbf{p_{h_1}} \equiv \mathbf{p_{h_2}} \equiv \mathbf{p_{h_3}}$

Note

A Cartesian point can be represented by infinitely many homogeneous coordinates

Points in Homogeneous coordinates - 2D space - Properties

Note

A Cartesian point can be represented by infinitely many homogeneous coordinates

Property

- given $\mathbf{p_h} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}, w \neq 0$
- for $\forall \lambda \neq 0 \qquad \hat{\mathbf{p}}_\mathbf{h} = \begin{bmatrix} \lambda\, x \\ \lambda\, y \\ \lambda\, w \end{bmatrix} \equiv \mathbf{p_h}$

Proof

- $\mathbf{p_e} = \begin{bmatrix} x/w \\ y/w \end{bmatrix}$
- for $\forall \lambda \neq 0 \qquad \hat{\mathbf{p}}_\mathbf{e} = \begin{bmatrix} \frac{\lambda\, x}{\lambda\, w} \\ \frac{\lambda\, y}{\lambda\, w} \end{bmatrix} = \begin{bmatrix} x/w \\ y/w \end{bmatrix}$

Notes

- $w = 1$: *normalized* homogeneous coordinates
- *normalization* : $\left[x\, y\, w\right]^T \rightarrow \left[x/w,\ y/w,\ 1\right]^T, w \neq 0$
- *hom* $\rightarrow$ *cart* : $\left[x\, y\, w\right]^T \rightarrow \left[x/w,\ y/w\right]^T, w \neq 0$
- *cart* $\rightarrow$ *hom* : $\left[x\, y\right]^T \rightarrow \left[x,\ y,\ 1\right]^T$

Points in Homogeneous coordinates - 2D space - Improper points

What's more than Cartesian?

- All Cartesian points can be expressed in homogeneous coordinates: $\mathbf{p}_e \rightarrow \left[\mathbf{p}_e,\ 1\right]^T$
- Are homogeneous coordinates more powerful than Cartesian ones? $\rightarrow$ Yes

Improper points

- With $w = 0$ we can express *points at the infinity* $\rightarrow \left[x/0,\ y/0\right]^T$
- $\mathbf{p}_h = \left[x,\ y,\ 0\right]^T$ codes a *direction*

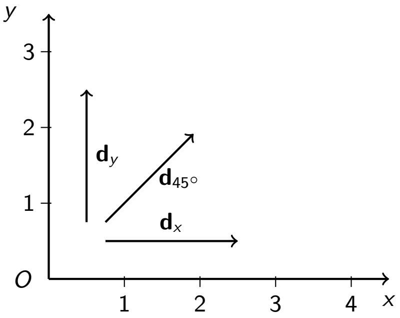
  not directly expressed in Cartesian coordinates

Property

- $\mathbf{p}_h = \left[x,\ y,\ 0\right]^T \equiv \left[\lambda x, \lambda y,\ 0\right]^T \forall \lambda \neq 0$

Points in Homogeneous coordinates - 2D space - Directions Example

EXAMPLE



- $\mathbf{d_x} = \begin{bmatrix}1\\0\\0\end{bmatrix} \equiv \begin{bmatrix}2\\0\\0\end{bmatrix}$: x-axis
- $\mathbf{d_y} = \begin{bmatrix}0\\1\\0\end{bmatrix} \equiv \begin{bmatrix}0\\-1\\0\end{bmatrix}$: y-axis
- $\mathbf{d_{45^\circ}} = \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix} \equiv \begin{bmatrix}1\\1\\0\end{bmatrix}$: 45° axis

NOTE

- A direction can be represented by infinitely many homogeneous directions
- A *unit vector* is the direction with $\|\mathbf{d}\| = 1$ (i.e., $\sqrt{x^2 + y^2} = 1$)

Points in Homogeneous coordinates - 2D space - Final Remarks

Points

$$\mathbf{p}_h = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda w \end{bmatrix} \rightarrow \mathbf{p}_e = \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

with $w \neq 0$, $\lambda \neq 0$

Origin

$$\mathbf{p}_h = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} \rightarrow \mathbf{p}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with $w \neq 0$

Improper Points - Directions

$$\mathbf{d}_h = \begin{bmatrix} \lambda x \\ \lambda y \\ 0 \end{bmatrix}$$

with $(x \neq 0 \;||\; y \neq 0)$ && $\lambda \neq 0$

Invalid homogeneous point

$$\mathbf{p}_h = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ homogeneous 2D space is defined on $\mathbb{R}^3 - [0,\, 0,\, 0]^T$

Outline

Points in Homogeneous coordinates - 3D space - Definition

Homogeneous 3D space

- Given a point $\mathbf{p}_e = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3$ in Cartesian coordinates
- we can define $\mathbf{p}_h = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$ in homogeneous coordinates
- under the relation $\begin{cases} X &= & x/w \\ Y &= & y/w \\ Z &= & z/w \\ w &\neq & 0 \end{cases}$
- i.e., there is an arbitrary *scale factor* (w)

Points in Homogeneous coordinates - 3D space - Summary

Points

$$\mathbf{p}_h = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{bmatrix} \rightarrow \mathbf{p}_e = \begin{bmatrix} x/w \\ y/w \\ z/w \end{bmatrix}$$

with $w \neq 0,\ \lambda \neq 0$

Origin

$$\mathbf{p}_h = \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \end{bmatrix} \rightarrow \mathbf{p}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with $w \neq 0$

Improper Points - Directions

$$\mathbf{d}_h = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ 0 \end{bmatrix} \text{ with}$$

$$(x \neq 0 \,||\, y \neq 0 \,||\, z \neq 0)\ \&\&\ \lambda \neq 0$$

Invalid homogeneous point

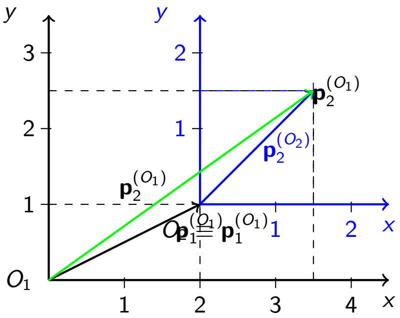
$$\mathbf{p}_h = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ homogeneous 3D space is defined on $\mathbb{R}^4 - \left[0,\ 0,\ 0,\ 0\right]^T$

Outline

Translation - Cartesian 2D

Start with an example

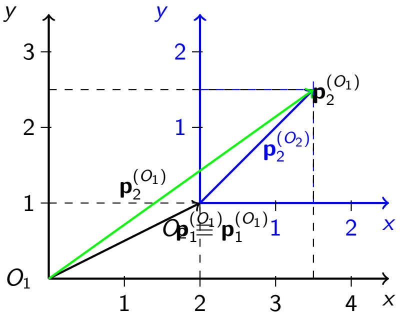


- $\mathbf{p}_1^{(O_1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $O_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \equiv \mathbf{p}_1^{(O_1)}$
- $\mathbf{p}_2^{(O_2)} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$
- $\mathbf{p}_2^{(O_1)} = \mathbf{p}_1^{(O_1)} + \mathbf{p}_2^{(O_2)}$

$$= \begin{bmatrix} 3.5 \\ 2.5 \end{bmatrix}$$

Translation - Homogeneous 2D - 1

Same example but with homogeneous coordinate



- $\mathbf{p_1^{(O_1)}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \equiv \cdots \equiv \begin{bmatrix} \lambda_1 2 \\ \lambda_1 1 \\ \lambda_1 1 \end{bmatrix}$
- $O_2 \equiv \mathbf{p_1^{(O_1)}}$
- $\mathbf{p_2^{(O_2)}} = \begin{bmatrix} 1.5 \\ 1.5 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix} \equiv \cdots \equiv \begin{bmatrix} \lambda_2 1.5 \\ \lambda_2 1.5 \\ \lambda_2 1 \end{bmatrix}$
- $\cancel{\mathbf{p_2^{(O_1)}} = \mathbf{p_1^{(O_1)}} + \mathbf{p_2^{(O_2)}}} \rightarrow$ NO
  - valid only if ${\mathbf{p_1^{(O_1)}}}_w = {\mathbf{p_2^{(O_2)}}}_w$
  - e.g., $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix} \equiv \begin{bmatrix} 1.\overline{6} \\ 1.\overline{3} \\ 1 \end{bmatrix} \not\equiv \begin{bmatrix} 3.5 \\ 2.5 \\ 1 \end{bmatrix}$
  - $\rightarrow$ we can *normalize* points ($w = 1$)
  - e.g., $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1.5 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 3.5 \\ 2.5 \\ 1 \end{bmatrix}$

Anything better?

Translation - Homogeneous 2D - 2

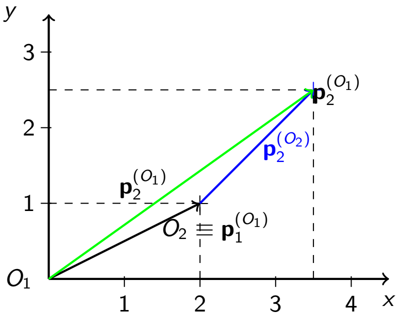
Translation: the right way with homogeneous coordinates

- $\mathbf{O} = \begin{bmatrix} x_O \\ y_O \\ 1 \end{bmatrix}$: position of the second reference frame (*normalized*)
- $\mathbf{p}^{(O)} = \begin{bmatrix} x_p \\ y_p \\ w_p \end{bmatrix}$: point w.r.t. the second reference frame (*homogeneous*)

$$\mathbf{p} = \begin{bmatrix} 1 & 0 & x_O \\ 0 & 1 & y_O \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ w_p \end{bmatrix} = \begin{bmatrix} x_p + x_O w_p \\ y_p + y_O w_p \\ w_p \end{bmatrix}$$

Translation - Homogeneous 2D - 3

Let's try on the example

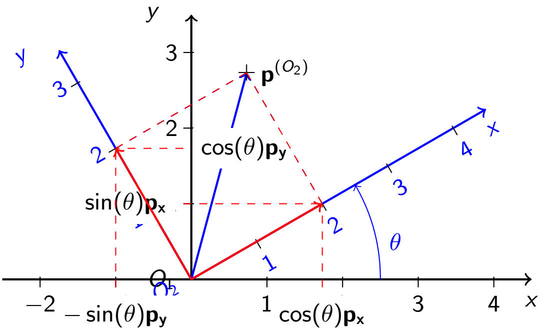


- $\mathbf{p_1^{(O_1)}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \equiv \cdots \equiv \begin{bmatrix} \lambda_1 2 \\ \lambda_1 1 \\ \lambda_1 1 \end{bmatrix}$
- $O_2 \equiv \mathbf{p_1^{(O_1)}}$ *normalized*
- $\mathbf{p_2^{(O_2)}} = \begin{bmatrix} 1.5 \\ 1.5 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix} \equiv \cdots \equiv \begin{bmatrix} \lambda_2 1.5 \\ \lambda_2 1.5 \\ \lambda_2 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_2 1.5 \\ \lambda_2 1.5 \\ \lambda_2 1 \end{bmatrix} = \begin{bmatrix} \lambda_2 1.5 + \lambda_2 2 \\ \lambda_2 1.5 + \lambda_2 1 \\ \lambda_2 \end{bmatrix} \equiv \begin{bmatrix} 3.5 \\ 2.5 \\ 1 \end{bmatrix}$$

Rotation - Cartesian 2D

Derivation from example



- $O_1 \equiv O_2$
- rotated of $\theta = 30^\circ$
- $\mathbf{p}^{(O_2)} = \begin{bmatrix} \mathbf{p}_x \\ \mathbf{p}_y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
- $\cos(\theta + 90^\circ) = -\sin(\theta)$
- $\sin(\theta + 90^\circ) = \cos(\theta)$

$$\mathbf{p}^{(O_1)} = \begin{bmatrix} \cos(\theta)\mathbf{p}_x - \sin(\theta)\mathbf{p}_y \\ \sin(\theta)\mathbf{p}_x + \cos(\theta)\mathbf{p}_y \end{bmatrix} = \begin{bmatrix} \cos(30^\circ)2 - \sin(30^\circ)2 \\ \sin(20^\circ)2 + \cos(30^\circ)2 \end{bmatrix} = \begin{bmatrix} 0.73 \\ 2.73 \end{bmatrix}$$

Rotation - Homogeneous 2D

- From $\mathbf{p}^{(O_1)} = \begin{bmatrix} \cos(\theta)\mathbf{p}_x - \sin(\theta)\mathbf{p}_y \\ \sin(\theta)\mathbf{p}_x + \cos(\theta)\mathbf{p}_y \end{bmatrix}$

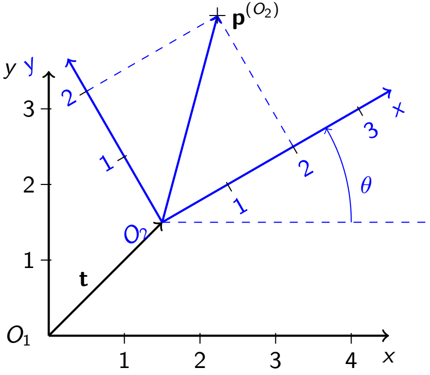
- Rewrite with matrices $\mathbf{p}^{(O_1)} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{p}^{(O_2)}$

- Pass to homogeneous $\mathbf{p}_h^{(O_1)} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_h^{(O_2)}$

  where $\mathbf{p}_h^{(O_2)} = \begin{bmatrix} \lambda\mathbf{p}_x \\ \lambda\mathbf{p}_y \\ \lambda \end{bmatrix}$

Rototranslation - Homogeneous 2D

PUTTING THINGS TOGETHER



- O_2 translated of $\mathbf{t}$ w.r.t. O_1
- O_2 rotated of θ w.r.t. O_1

TWO STEPS:

$$\mathbf{p}_h^{(O_2')} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_h^{(O_2)}$$

$$\mathbf{p}_h^{(O_1)} = \begin{bmatrix} 1 & 0 & \mathbf{t}_x \\ 0 & 1 & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_h^{(O_2')}$$

ONE STEP:

$$\mathbf{p}_h^{(O^1)} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & \mathbf{t}_x \\ \sin(\theta) & \cos(\theta) & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_h^{(O_2)}$$

Rototranslation - Homogeneous 2D - Some notes

Consider $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & \mathbf{t}_x \\ \sin(\theta) & \cos(\theta) & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n}_x & \mathbf{m}_x & \mathbf{t}_x \\ \mathbf{n}_y & \mathbf{m}_y & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \mathbf{T}$

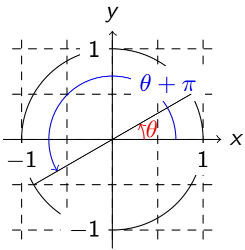
- $\mathbf{n} = \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ 0 \end{bmatrix}$ is the *direction vector* (improper point) of the x axis
- $\mathbf{m} = \begin{bmatrix} \mathbf{m}_x \\ \mathbf{m}_y \\ 0 \end{bmatrix}$ is the *direction vector* (improper point) of the x axis
- $\|\mathbf{n}\| = \|\mathbf{m}\| = 1$ they are unit vectors
- $\|[\mathbf{n}_x, \mathbf{m}_x]\| = \|[\mathbf{n}_y, \mathbf{m}_y]\| = 1$ are unit vectors too
- $\mathbf{R}$ is an *orthogonal matrix* $\rightarrow$ $\mathbf{R}^{-1} = \mathbf{R}^T$
- $\mathbf{T}$ is homogeneous too! i.e., $\mathbf{T} \equiv \lambda\mathbf{T}$ $\quad \mathbf{T}\,\mathbf{p}_2 \equiv \lambda\mathbf{T}\,\mathbf{p}_2$

Rototranslation - Homogeneous 2D - Get parameters

Consider $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ 0 & 0 & 1 \end{bmatrix}$

- remember $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & \mathbf{t}_x \\ \sin(\theta) & \cos(\theta) & \mathbf{t}_y \\ 0 & 0 & 1 \end{bmatrix}$
- $\mathbf{t} = \begin{bmatrix} T_{13} \\ T_{23} \end{bmatrix}$
- $\theta = \text{atan2}(T_{21}, T_{11})$

ATAN2



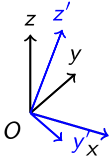
- $\arctan(y/x) =\rightarrow [0, \pi]$
- $\text{atan2}(y, x) \rightarrow [-\pi, \pi]$

$$\text{atan2}(y, x) = \begin{cases} \arctan(y/x) & x > 0 \\ \arctan(y/x) + \pi & y \geq 0, x < 0 \\ \arctan(y/x) - \pi & y < 0, x < 0 \\ +\frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & x = 0, y = 0 \end{cases}$$

Outline

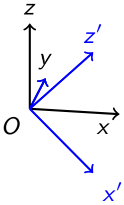
Rotations in 3D - 1

Rotate around x



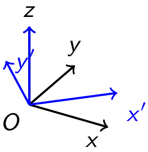
ρ, *roll* ("rollio")
around x-axis

Rotate around y



θ, *pitch* ("beccheggio")
around y-axis

Rotate around z



ϕ, *yaw* ("imbardata")
around z-axis

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\rho) & -\sin(\rho) \\ 0 & \sin(\rho) & \cos(\rho) \end{bmatrix} \quad \mathbf{R}_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \quad \mathbf{R}_z = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations in 3D - 2

To create a 3D rotation

- Compose 3 planar rotation
- 24 possible conventions
  non-commutative matrix products

Common convention

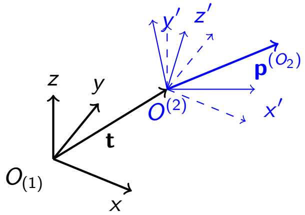
- Rotate around x (roll)
- Rotate around y (pitch)
- Rotate around z (yaw)

Derive a complete rotation matrix

- Let $\mathbf{p}^{O^{\mathbf{R}_{xyz}}}$ in the rotated system
- $\mathbf{p}^{O^{\mathbf{R}_{yz}}} = \mathbf{R}_x \mathbf{p}^{O^{\mathbf{R}_{xyz}}}$
- $\mathbf{p}^{O^{\mathbf{R}_{z}}} = \mathbf{R}_y \mathbf{p}^{O^{\mathbf{R}_{yz}}}$
- $\mathbf{p}^{O} = \mathbf{R}_z \mathbf{p}^{O^{\mathbf{R}_{z}}}$ in the original system
- $\mathbf{R}_{xyz} = \mathbf{R}_z\, \mathbf{R}_y\, \mathbf{R}_x$

$$\mathbf{R}_{xyz} = \begin{bmatrix} \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta)\sin(\rho)-\sin(\phi)\cos(\rho) & \cos(\phi)\sin(\theta)\cos(\rho)+\sin(\phi)\sin(\rho) \\ \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta)\sin(\rho)+\cos(\phi)\cos(\rho) & \sin(\phi)\sin(\theta)\cos(\rho)-\cos(\phi)\sin(\rho) \\ -\sin(\theta) & \cos(\theta)\sin(\rho) & \cos(\theta)\cos(\rho) \end{bmatrix}$$

Rototranslations - Homogeneous 3D



Complete Transformation

- $\mathbf{p}^{(O_2)}$ w.r.t. O_2 reference
- Let consider O_2' rotated as O_1, but translated by $\mathbf{t}$
- $\mathbf{R}$ rotation of O_2 wrt O_2'
- $\mathbf{p}^{(O_2')} = \mathbf{R}\,\mathbf{p}^{(O_2)}$
- $\mathbf{p}^{(O_1)} = \mathbf{t} + \mathbf{p}^{(O_2')}$

In Homogeneous Coordinates

$$\mathbf{p}_h^{(O_1)} = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{p}_h^{(O_2)} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{p}_h^{(O_2)}$$

where $\mathbf{p}_h^{(O_2)}$ is $\mathbf{p}^{(O_2)}$ in homogeneous coordinates

and $\mathbf{p}_h^{(O_1)}$ is $\mathbf{p}^{(O_1)}$ in homogeneous coordinates

Rototranslation - Homogeneous 3D - Some notes

Consider $\begin{bmatrix} \mathbf{n}_x & \mathbf{m}_x & \mathbf{a}_x & \mathbf{t}_x \\ \mathbf{n}_y & \mathbf{m}_y & \mathbf{a}_y & \mathbf{t}_y \\ \mathbf{n}_z & \mathbf{m}_z & \mathbf{a}_z & \mathbf{t}_z \\ 0 & 0 & 1 & \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \mathbf{T}$

- $\mathbf{n} = \begin{bmatrix} \mathbf{n}_x \\ \mathbf{n}_y \\ \mathbf{n}_z \\ 0 \end{bmatrix}$, $\mathbf{m} = \begin{bmatrix} \mathbf{m}_x \\ \mathbf{m}_y \\ \mathbf{m}_z \\ 0 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \\ 0 \end{bmatrix}$ are the *direction vectors* of the x,y,z, axes
- $\|\mathbf{n}\| = \|\mathbf{m}\| = \|\mathbf{a}\| = 1$ unit vectors
- $\left\|[\mathbf{n}_x, \mathbf{m}_x, \mathbf{a}_x]\right\| = \left\|[\mathbf{n}_y, \mathbf{m}_y, \mathbf{a}_y]\right\| = \left\|[\mathbf{n}_z, \mathbf{m}_z, \mathbf{a}_z]\right\| = 1$ are unit vectors too
- $\mathbf{R}$ is an *orthogonal matrix* $\rightarrow$ $\mathbf{R}^{-1} = \mathbf{R}^T$
- $\mathbf{T}$ is homogeneous too! i.e., $\mathbf{T} \equiv \lambda\mathbf{T}$ $\quad \mathbf{T}\,\mathbf{p}_2 \equiv \lambda\mathbf{T}\,\mathbf{p}_2$

Rototranslation - Homogeneous 3D - Get parameters

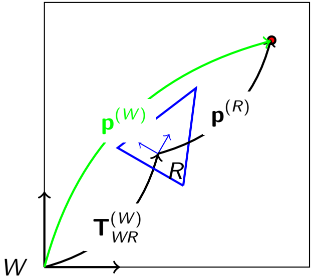
Consider $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- remember $\begin{bmatrix} \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta)\sin(\rho)-\sin(\phi)\cos(\rho) & \cos(\phi)\sin(\theta)\cos(\rho)+\sin(\phi)\sin(\rho) \\ \sin(\phi)\cos(\theta) & \sin(\phi)\sin(\theta)\sin(\rho)+\cos(\phi)\cos(\rho) & \sin(\phi)\sin(\theta)\cos(\rho)-\cos(\phi)\sin(\rho) \\ -\sin(\theta) & \cos(\theta)\sin(\rho) & \cos(\theta)\cos(\rho) \end{bmatrix}$
- $\mathbf{t} = \begin{bmatrix} T_{14} \\ T_{24} \\ T_{34} \end{bmatrix}$
- $\phi = \text{atan2}\left(T_{21}, T_{11}\right)$
- $\theta = \text{atan2}\left(-T_{31}, \sqrt{T_{32}^2 + T_{33}^2}\right)$
- $\rho = \text{atan2}\left(T_{32}, T_{33}\right)$

Outline

Transformations - Why?

Think about...



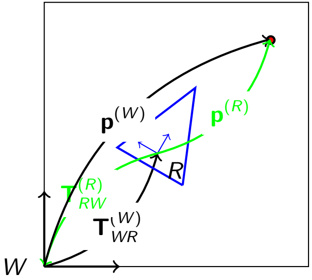
- W is the *world* reference frame.
- R is the *robot* reference frame.
- $\mathbf{T}_{WR}^{(W)}$ is the transformation that codes position and orientation of the robot w.r.t. W.
- The robot perceives the *red point*, it knows the point $\mathbf{p}^{(R)}$ in robot reference frame.
- $\mathbf{p}^{(W)} = \mathbf{T}_{WR}^{(W)} \mathbf{p}^{(R)}$ is the point in world coordinates.

$\mathbf{T}_{WR}^{(W)} = \begin{bmatrix} \mathbf{R}_{WR}^{(W)} & \mathbf{t}_{WR}^{(W)} \\ \mathbf{0} & 1 \end{bmatrix}$ is the transformation matrix

- map points in R reference frame in W frame
- $\mathbf{t}_{WR}^{(W)}$ is the position of R w.r.t. W
- $\mathbf{R}_{WR}^{(W)}$ is the rotation applied to a reference frame rotated as W with origin on O

Transformations - Inversion

Inverse transformation



- W is the *world* reference frame.
- R is the *robot* reference frame.
- $\mathbf{T}_{WR}^{(W)}$ is the transformation that codes position and orientation of the robot w.r.t. W.
- You know the *red point* $\left(\mathbf{p}^{(W)}\right)$ in world coordinates,

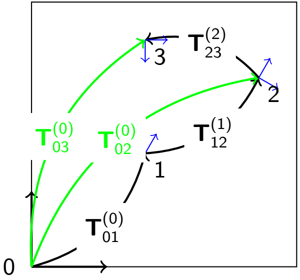
Position of the world w.r.t. the robot

$$\mathbf{T}_{RW}^{(R)} = \left(\mathbf{T}_{WR}^{(W)}\right)^{-1} = \begin{bmatrix} \mathbf{R}_{WR}^{(W)\,T} & -\mathbf{R}_{WR}^{(W)\,T}\,\mathbf{t}_{WR}^{(W)} \\ \mathbf{0} & 1 \end{bmatrix}$$ is the transformation matrix

$\mathbf{p}^{(R)} = \mathbf{T}_{RW}^{(R)}\,\mathbf{p}^{(W)}$ is the point in robot coordinates.

Transformations - Composition

Composition of transformations



- $\mathbf{T}_{01}^{(0)}$: pose of 1 w.r.t. 0
- $\mathbf{T}_{12}^{(1)}$: pose of 2 w.r.t. 1
- $\mathbf{T}_{02}^{(0)} = \mathbf{T}_{01}^{(0)}\mathbf{T}_{12}^{(1)}$: pose of 2 w.r.t. 0
- $\mathbf{T}_{23}^{(2)}$: pose of 3 w.r.t. 2
- $\mathbf{T}_{03}^{(0)} = \mathbf{T}_{01}^{(0)}\mathbf{T}_{12}^{(1)}\mathbf{T}_{23}^{(2)} = \mathbf{T}_{02}^{(0)}\mathbf{T}_{23}^{(2)}$: pose of 3 w.r.t. 0

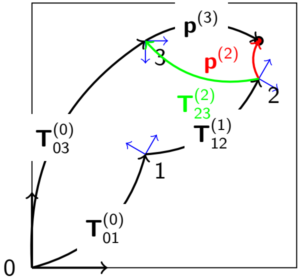
Graphical method

- Post-multiplication following arrows verse
- Pre-multiplication coming back in arrows verse

Note: Not unique convention about arrow direction!!

Transformations - Composition Example

Example



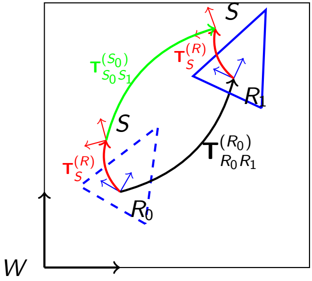
- $\mathbf{T}^{(0)}_{01}$: pose of 1 w.r.t. 0
- $\mathbf{T}^{(1)}_{12}$: pose of 2 w.r.t. 1
- $\mathbf{T}^{(0)}_{03}$: pose of 3 w.r.t. 0
- $\mathbf{p}^{(3)}$: position of $\mathbf{p}$ w.r.t 3
- $\mathbf{p}^{(2)}$: position of $\mathbf{p}$ w.r.t 2?

Solution

- $\mathbf{T}^{(2)}_{23} = \left(\mathbf{T}^{(1)}_{12}\right)^{-1} \left(\mathbf{T}^{(0)}_{01}\right)^{-1} \mathbf{T}^{(0)}_{03} = \mathbf{T}^{(2)}_{21}\, \mathbf{T}^{(1)}_{10}\, \mathbf{T}^{(0)}_{03}$
- Note: $\left(\mathbf{T}^{(1)}_{12}\right)^{-1} \left(\mathbf{T}^{(0)}_{01}\right)^{-1} = \left(\mathbf{T}^{(0)}_{01}\, \mathbf{T}^{(1)}_{12}\right)^{-1} = \left(\mathbf{T}^{(0)}_{02}\right)^{-1}$
- $\mathbf{p}^{(2)} = \mathbf{T}^{(2)}_{23}\mathbf{p}^{(3)}$

Transformations - Composition - Practical Case

Change reference system of motion



- $\mathbf{T}_{R_0R_1}^{(R0)}$: pose of robot R at time $t = 1$
  w.r.t. robot at time $t = 0$
  i.e., the relative motion of the robot
- $\mathbf{T}_{S}^{(R)}$: pose of a sensor S w.r.t. the robot R
  Note: fixed in time
- $\mathbf{T}_{S_0S_1}^{(S_0)}$: pose of sensor S at time $t = 1$
  w.r.t. sensor at time $t = 0$?
  i.e., the relative motion of the sensor

Solution

- $\mathbf{T}_{S_0S_1}^{(S_0)} = \left(\mathbf{T}_{RS}^{(R)}\right)^{-1} \mathbf{T}_{R_0R_1}^{(R_0)} \mathbf{T}_{RS}^{(R)}$

Outline

Lines in homogeneous coordinates

Lines definition

- Slope-intercept form: $y = mx + q$

  vertical lines $m = \infty$

- Linear equation: $a\,x + b\,y + c = 0$,

  $(a, b) \in \mathbb{R}^2 - \{0, 0\}$

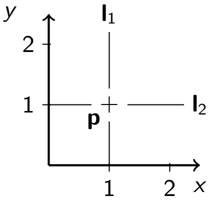
Homogeneous coordinates

- Line: $\mathbf{l} = \begin{bmatrix} a, & b, & c \end{bmatrix}^T$
- Point: $\mathbf{p} = \begin{bmatrix} x, & y, & 1 \end{bmatrix}^T$
- $\mathbf{p}$ lies on $\mathbf{l} \Leftrightarrow \mathbf{l}^T\mathbf{p} = \mathbf{p}^T\mathbf{l} = 0$
- Homogeneous property:
  - $\mathbf{l} \equiv \lambda_1 \mathbf{l}$
  - $\mathbf{p} \equiv \lambda_2 \mathbf{p}$

Point from Lines & Lines from Points

Intersection of lines

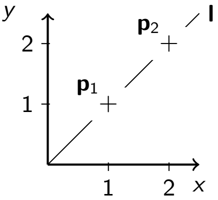
- $\mathbf{p} = \mathbf{l}_1 \cap \mathbf{l}_2 = \mathbf{l}_1 \times \mathbf{l}_2$



- $\mathbf{l}_1 = \begin{bmatrix}1, & 0, & -1\end{bmatrix}^T \rightarrow \mathbf{x} = 1$
- $\mathbf{l}_2 = \begin{bmatrix}0, & 1, & -1\end{bmatrix}^T \rightarrow \mathbf{y} = 1$
- $\mathbf{p} = \begin{bmatrix}1, & 1, & 1\end{bmatrix}^T$

Line joining two points

- $\mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$



- $\mathbf{p}_1 = \begin{bmatrix}1, & 1, & 1\end{bmatrix}^T$
- $\mathbf{p}_2 = \begin{bmatrix}2, & 2, & 1\end{bmatrix}^T$
- $\mathbf{l} = \begin{bmatrix}-1, & 1, & 0\end{bmatrix}^T \rightarrow y = x$

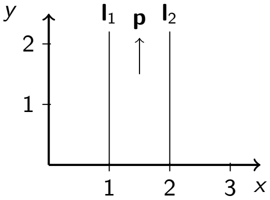
Cross product reminder

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \quad \mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \vec{\mathbf{u}}_x & \vec{\mathbf{u}}_y & \vec{\mathbf{u}}_z \\ \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \mathbf{b}_x & \mathbf{b}_y & \mathbf{b}_z \end{bmatrix} = \begin{bmatrix} \mathbf{a}_y\mathbf{b}_z - \mathbf{a}_z\mathbf{b}_y \\ \mathbf{a}_z\mathbf{b}_x - \mathbf{a}_x\mathbf{b}_z \\ \mathbf{a}_x\mathbf{b}_y - \mathbf{a}_y\mathbf{b}_x \end{bmatrix}$$

Ideal points and $\mathbf{l}_\infty$

Intersection of parallel lines

- $\mathbf{p} = \mathbf{l}_1 \cap \mathbf{l}_2 = \mathbf{l}_1 \times \mathbf{l}_2$



- $\mathbf{l}_1 = \left[1,\ 0,\ -1\right]^T \rightarrow \mathbf{x} = 1$
- $\mathbf{l}_2 = \left[1,\ 0,\ -2\right]^T \rightarrow \mathbf{x} = 2$
- $\mathbf{p} = \left[0,\ 1,\ 0\right]^T \rightarrow$ *improper point*

  direction of y axis

Line that join improper points ($\mathbf{l}_\infty$)

- $\mathbf{p}_1 = \left[x_1,\ y_1,\ 0\right]^T$
- $\mathbf{p}_2 = \left[x_2,\ y_2,\ 0\right]^T$
- $\mathbf{p}_1 \times \mathbf{p}_1 \equiv \left[0,\ 0,\ 1\right]^T$
- $\mathbf{l}_\infty = \left[0,\ 0,\ 1\right]^T$:

  join $\forall$ pair of improper points,

  i.e., $\mathbf{l}_\infty^T \left[x,\ y,\ 0\right]^T = 0$

Duality principle

Duality

$$\mathbf{p} \quad \longleftrightarrow \quad \mathbf{l}$$

$$\mathbf{p}^T\mathbf{l} = 0 \quad \longleftrightarrow \quad \mathbf{l}^T\mathbf{p} = 0$$

$$\mathbf{p} = \mathbf{l}_1 \times \mathbf{l}_2 \quad \longleftrightarrow \quad \mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$$

To any theorem in 2D projective geometry there correspond a *dual theorem*,

derived by interchanging the role of *points* and *lines*

Conics

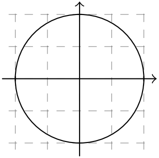
Definition

- $2^n d$ degree equations
- planar curve
- Equation: $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- Homogeneous: $ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$
- Matrix form:
  - $\mathbf{x} = \begin{bmatrix} x, & y, & w \end{bmatrix}^T$
  - $\mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$
  - $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$

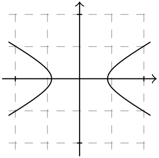
$\rightarrow$ $\mathbf{C}$ is homogeneous too, i.e., 6 parameters, 5 D.O.F.

Conics - Summary

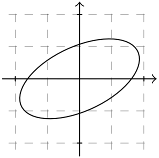
PROPER CONICS: rank $(\mathbf{C}) = 3$



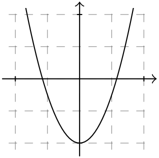
Circle



Hyperbola

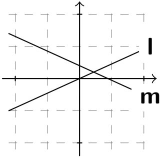


Ellipse

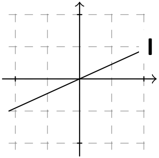


Parabola

DEGENERATE CONICS: rank $(\mathbf{C}) < 3$



$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$
2-lines,
rank $(\mathbf{C}) = 2$



$\mathbf{C} = \mathbf{l}\mathbf{l}^T$
repeated line,
rank $(\mathbf{C}) = 1$

Conics Parameters Estimation

Parameters estimation

- Given a point x_i, y_i, it satisfies
  $ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$
- Rewrite: $\begin{bmatrix} x_i^2, & x_iy_i, & y_i^2, & x_i, & y_i, & 1 \end{bmatrix} \begin{bmatrix} a, & b, & c, & d, & e, & f \end{bmatrix}^T = 0$
- Stacking constraints on ≥ 5 points:

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solve the linear system

Details

"Multiple View Geometry in computer vision" - Hartley, Zisserman, Chapter 2.

Outline

Projective Transformations - Definition

Definition

A *projectivity* is an invertible mapping $h(\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

x_1, x_2, x_3 lie on the same line $\iff$ $h(x_1), h(x_2), h(x_3)$ do

i.e., a projectivity maintains collinearity

Theorem

A mapping $h(\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a projectivity

$\iff$

$\exists$ a non-singular 3×3 matrix $\mathbf{H}$ such that

$\forall \mathbf{p} \in \mathbb{R}^2$ expressed with its homogeneous vector $\mathbf{p}_h$

$h(\mathbf{p}_h) = \mathbf{H}\,\mathbf{p}_h$

Projective Transformations - Practice

Projective Transformation

$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Note

- $\mathbf{H}$ has 9 elements
- $\mathbf{H}$ is homogeneous too: $\lambda\mathbf{H} \equiv \mathbf{H}$

  normalized if $h_{33} = 1$
- $\rightarrow$ only 8 D.O.F.

synonymous

- Projectivity
- *Projective transformation*
- Collineation
- *Homography*

Projective Transformations - Mapping between planes

ORIGINAL IMAGE

RECTIFIED IMAGE

ESTIMATION

- Take four point on first image $\mathbf{x}_i$
- Map on four known destination points $\mathbf{x}'_i$
- Solve $\begin{cases} \mathbf{x}'_1 & = & \mathbf{H}\,\mathbf{x}_1 \\ \mathbf{x}'_2 & = & \mathbf{H}\,\mathbf{x}_2 \\ \mathbf{x}'_3 & = & \mathbf{H}\,\mathbf{x}_3 \\ \mathbf{x}'_4 & = & \mathbf{H}\,\mathbf{x}_4 \end{cases}$ on h_{ij}.

NOTE

- $\mathbf{H}$ has 8 D.O.F. ($\lambda\mathbf{H} = \mathbf{H}$)
- each point impose 2 constraint

Details

"Multiple View Geometry in computer vision"
Hartley Zisserman
Chapter 4.