Certifiable Robustness

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October 30, 2024

Today's Agenda

Randomized Smoothing

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Certified Adversarial Robustness via Randomized Smoothing

Jeremy Cohen [1] **Elan Rosenfeld** [1] **J. Zico Kolter** [1 2]

Certifiable Robustness

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- Classifier $f : \mathbb{R}^d \to [0,1]^K$ is **ϵ-robust** at x, if

$$\forall \, \|\delta\|_p \leq \epsilon, \quad \underset{i \in [K]}{argmax} \; f_i(x+\delta) = \underset{i \in [K]}{argmax} \; f_i(x)$$

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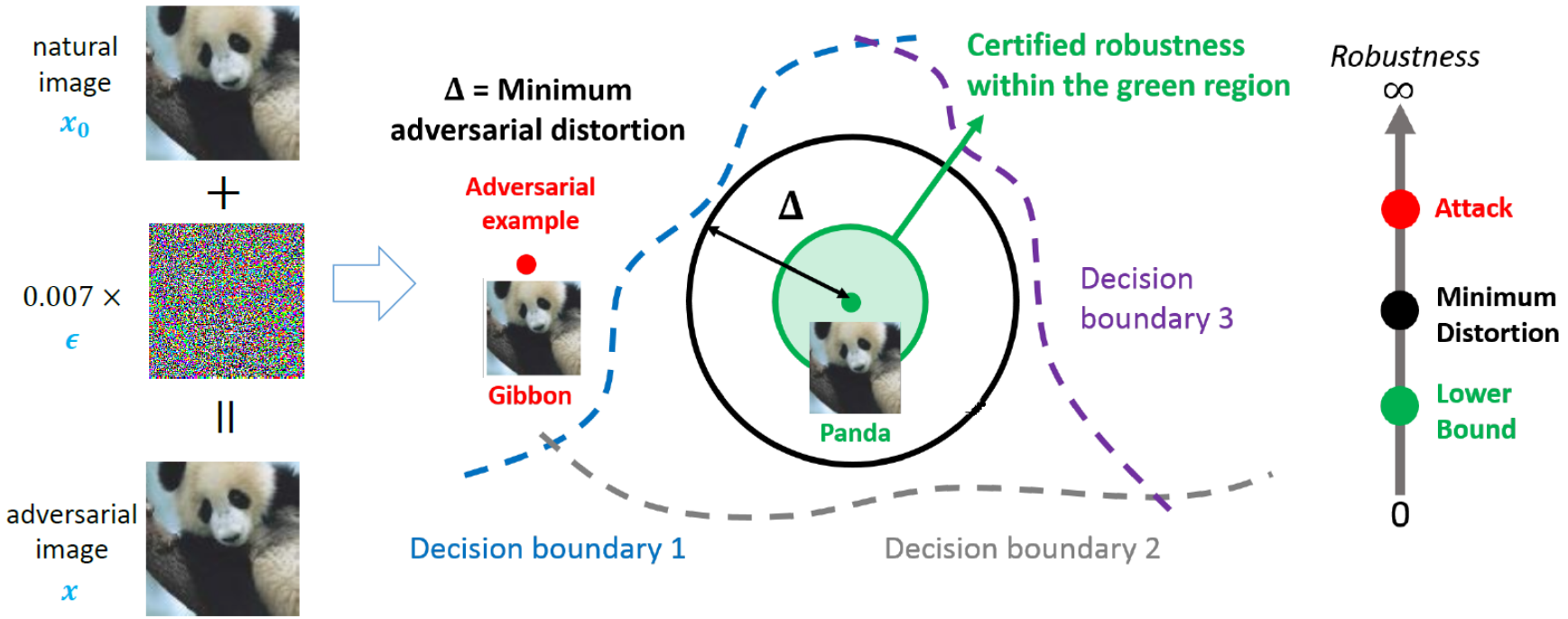

Lipschitz continuity

A function $f : I \to R$ over some set $I \subseteq \mathbb{R}^d$ is called Lipschitz continuous if there exists a positive real constant L such that, for all $x, y \in I$,

$$|f(y) - f(x)| \leq L\|y - x\|_2$$

or

$$f(x) - L\|y - x\|_2 \leq f(y) \leq f(x) + L\|y - x\|_2$$

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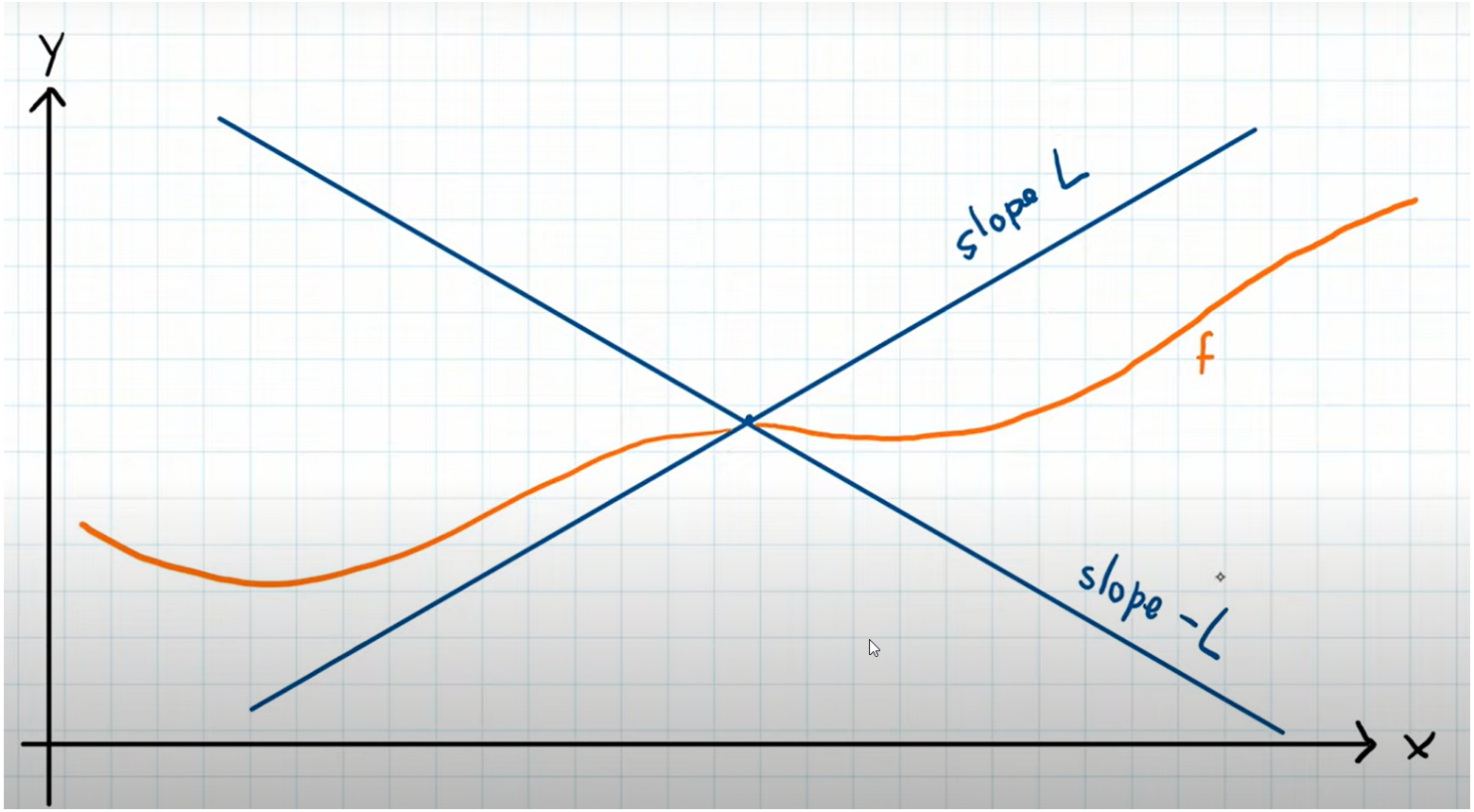
Generally, Let $f = f_1 \circ f_2 \circ .... \circ f_K$ and the Lipschitz constant of f_i be L_i for all $i \in \{1, 2, ..., K\}$, then the Lipschitz constant of f is $L \leq \prod_{k=1}^{K} L_k$.

Lipschitz continuity



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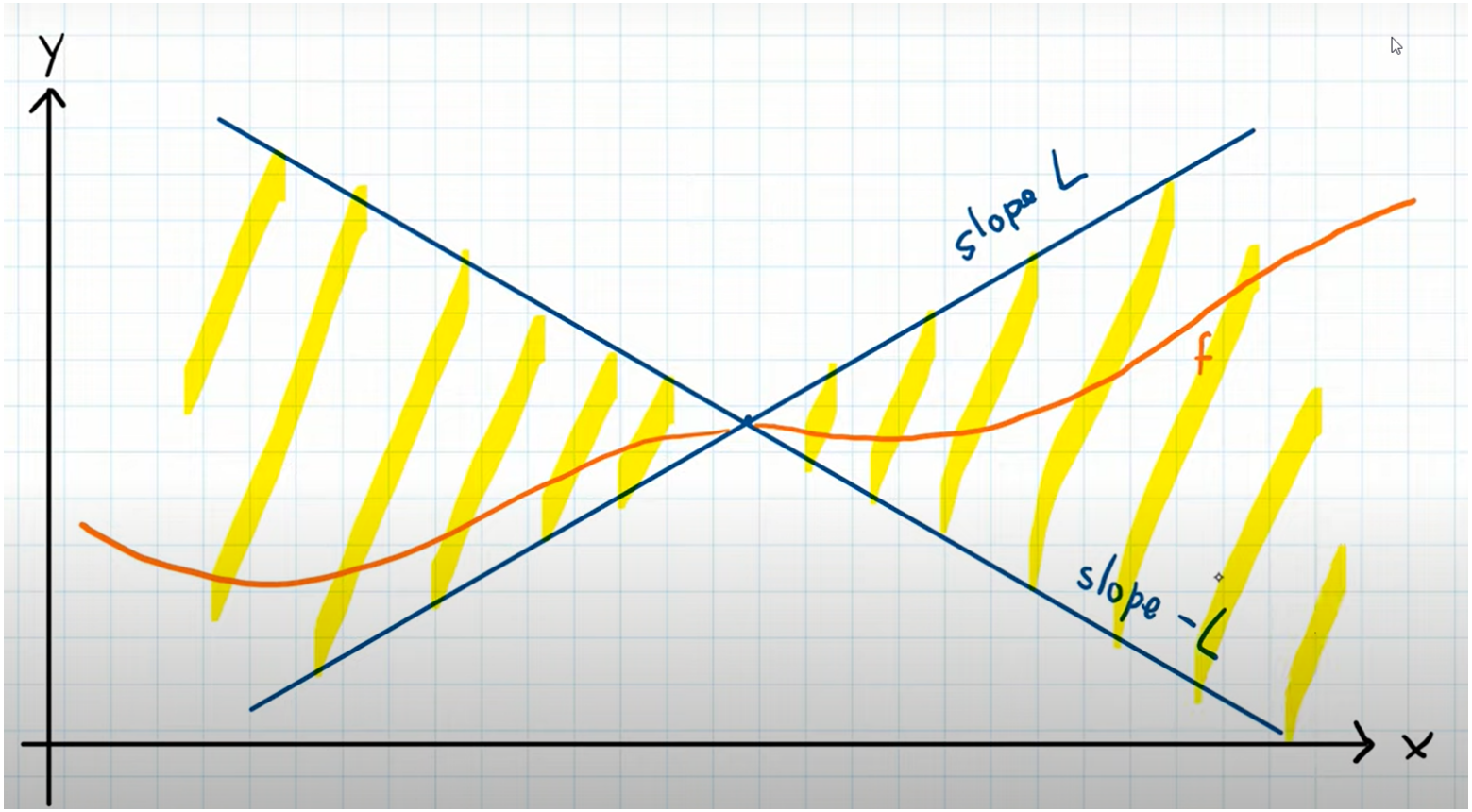

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Let $f : I \rightarrow R$ be a continuous and differntiable function over some set $I \subseteq \mathbb{R}^d$, if we have $\|f'(x)\|_2 \leq m$ for all $x \in I$, then m is the upper Lipschitz constant of f ($L \leq m$).

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Proof sketch:
Mean value theorem: Let $f : I \to R$ be a continuous and differntiable function over some set $I \subseteq \mathbb{R}^d$, For all $a, b \in I$ $(b > a)$, there exists some $c \in (a, b)$ such that:

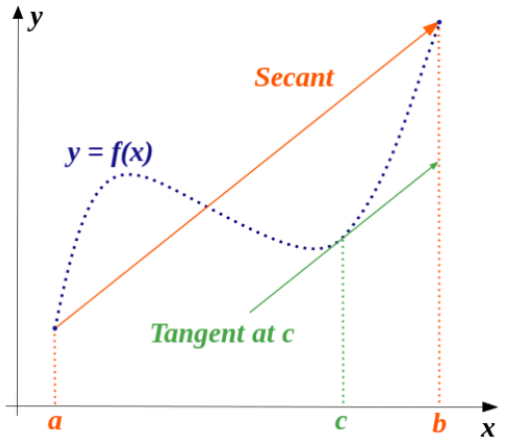
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

For all $a, b \in I$, there exist $c \in (a, b)$, such that:

$$|f(b) - f(a)| = \|f'(c).b - a\|_2 \leq \|f'(c)\|_2 \|b - a\|_2.$$

Since we know that $\|f'(c)\|_2 \leq m$, we have

$$|f(b) - f(a)| \leq m\|b - a\|_2.$$



Certifiable Robustness for L-Lipschitz classifier

Theorem 0: If $f : \mathbb{R}^d \to [0,1]^K$ is L-lipschitz, then f is ϵ-robust at x with $\epsilon = \frac{1}{2L}(P_A - P_B)$, where $P_A = \max_i f_i(x)$, $P_B = \max_{j \neq i} f_j(x)$, and $f_k(x)$ is the k-th element of the probability vector $f(x)$.

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Proof.

Since f is L-lipschitz, we have

$$\forall x, y \in \mathbb{R}^d, \|f(y) - f(x)\|_2 \leq L\|y - x\|_2$$

Denote $x' = x + \delta$ and assume that $\|\delta\| \leq \epsilon$. we get

$$\|f(x') - f(x)\|_2 \leq L\|x' - x\|_2 \to \|f(x') - f(x)\|_2 \leq L\|\delta\|_2 \leq L\epsilon$$

Hence, P_A can be reduced at most by $L\epsilon$ and P_B can be increased at most by $L\epsilon$. We have ($P'_i = f_i(x')$)

$$P'_A \geq P_A - L\epsilon \quad \text{and} \quad P'_B \leq P_B + L\epsilon$$

Since we want that the label of x' be the same as x, P'_A must be greater than P'_B ($P'_A \geq P'_B$). We have

$$P_A - L\epsilon \geq P_B + L\epsilon \;\to\; 2L\epsilon \leq P_A - P_B \;\to\; \epsilon \leq \frac{1}{2L}(P_A - P_B)$$

□

Smoothed Classifier

If we compute the (upper bound of) Lipschitz constant of the classifier, we can determine the radius (ϵ) of the robustness for each sample.

- However, **the lipschitz constant of deep neural networks** is very large and it **grows exponentially** with the number of layers. Hence, the certification is useless ($\epsilon \to 0$).

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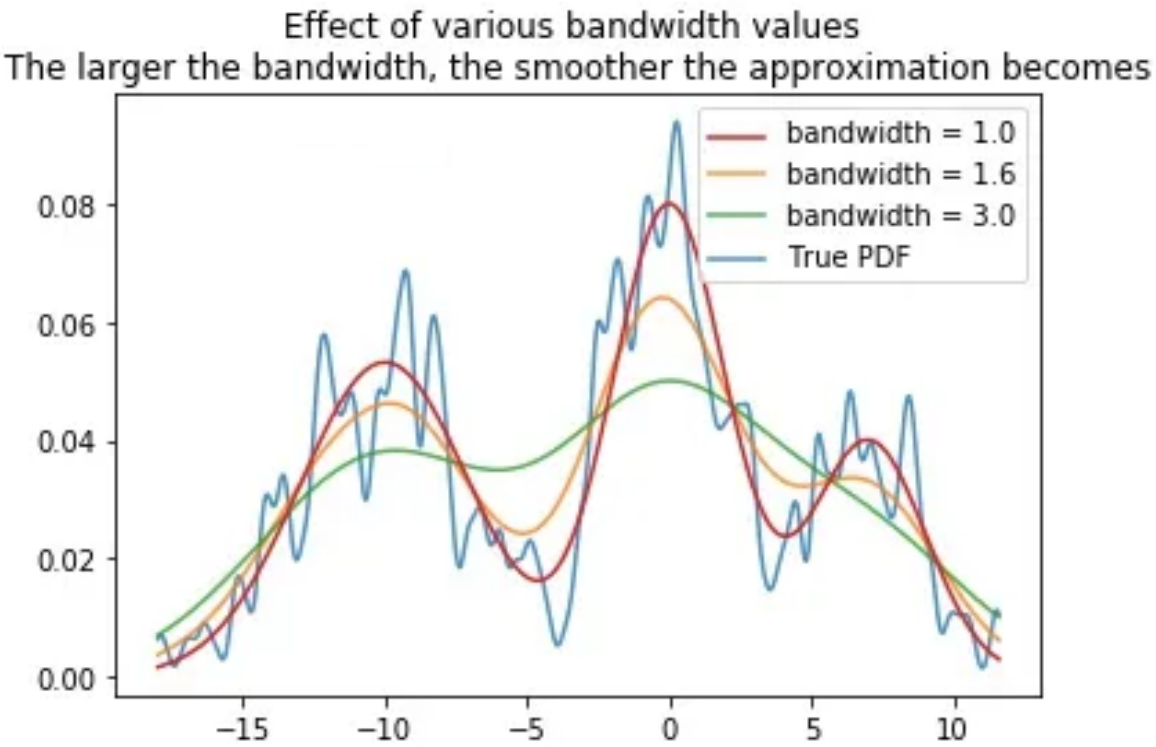

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Randomized Smoothing

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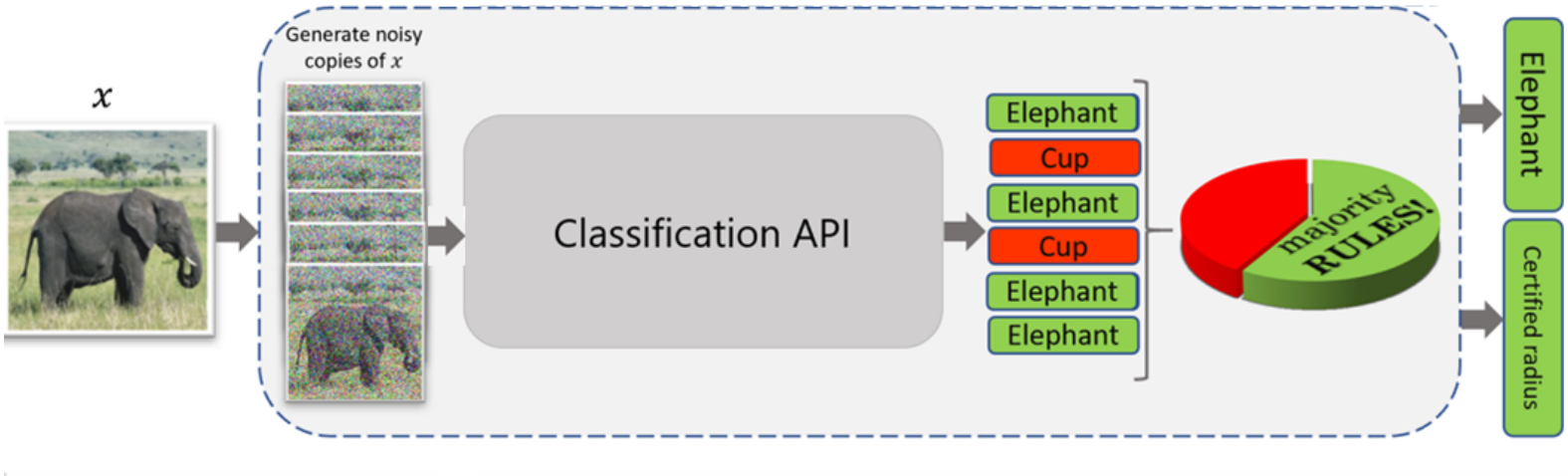
Consider a classification problem from $\mathbb{R}^d$ to classes $\mathcal{Y}$. Randomized smoothing is a method for constructing a new, **smoothed classifier** $\hat{f}$ from an arbitrary base classifier f.

- When queried at x, the smoothed classifier $\hat{f}$ returns whichever class the base classifier f is most likely to return when x is perturbed by isotropic Gaussian noise:

$$\hat{f}(x) = \underset{c \in \mathcal{Y}}{argmax}\ \mathbb{P}(f(x+\epsilon) = c) \quad \text{where} \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I) \tag{1}$$

  The noise level σ is a hyperparameter of the smoothed classifier $\hat{f}$ which controls a robustness/accuracy tradeoff.



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Notation

Suppose that when the base classifier f classifies $\mathcal{N}(x, \sigma^2 I)$, the most probable class c_A is returned with probability P_A, and the "runner-up" class is returned with probability P_B.

- $\underline{P_A}$ is a lower bound for P_A and $\overline{P_B}$ is a lower bound for P_B.

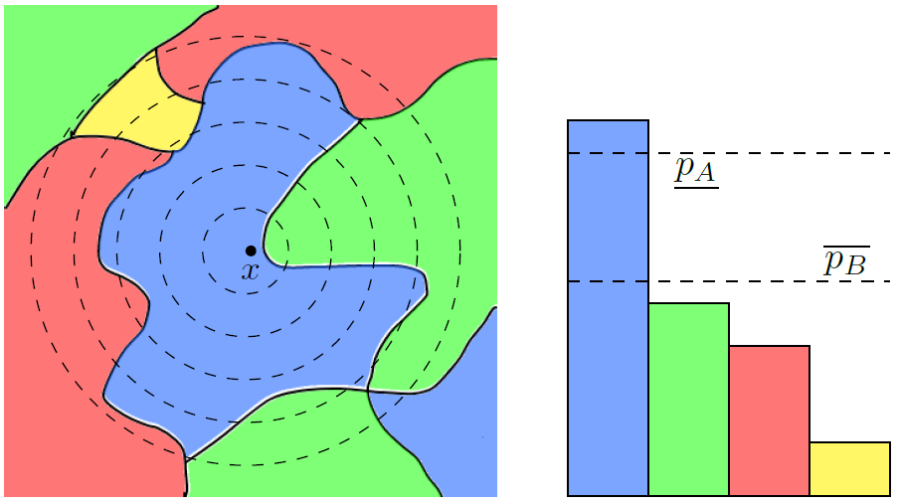

Figure 1. Evaluating the smoothed classifier at an input x. **Left**: the decision regions of the base classifier f are drawn in different colors. The dotted lines are the level sets of the distribution $\mathcal{N}(x, \sigma^2 I)$. **Right**: the distribution $f(\mathcal{N}(x, \sigma^2 I))$. As discussed below, $\underline{p_A}$ is a lower bound on the probability of the top class and $\overline{p_B}$ is an upper bound on the probability of each other class. Here, $g(x)$ is "blue."

Robustness guarantee

Theorem 1. Let $f : \mathbb{R}^d \to \mathcal{Y}$ be any deterministic or random function, and let $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Let $\hat{f}$ be defined as in (1). Suppose $C_A \in \mathcal{Y}$ and $\underline{P_A}, \overline{P_B} \in [0, 1]$ satisfy:

$$\mathbb{P}(f(x + \epsilon) = C_A) \geq \underline{P_A} \geq \overline{P_B} \geq \max_{C \neq C_A} \mathbb{P}(f(x + \epsilon) = C)$$

Then $\hat{f}(x + \delta) = C_A$ for all $\|\delta\|_2 \leq R$, where

$$R = \frac{\sigma}{2}(\Phi^{-1}(\underline{P_A}) - \Phi^{-1}(\overline{P_B}))$$

where Φ^{-1} is the inverse of the standard Gaussian CDF.

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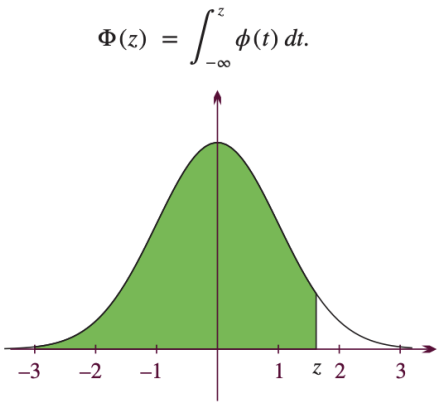
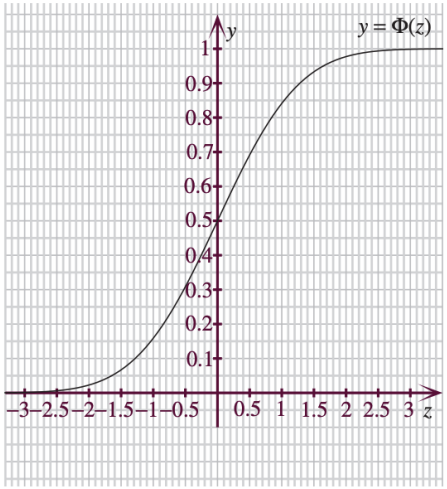
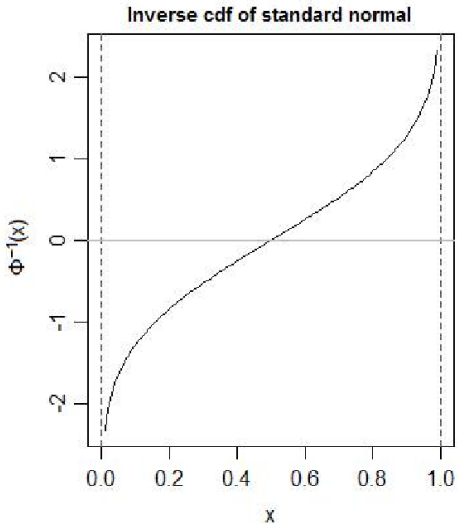


This is the graph of the standard normal probability density function $\phi(z)$.



This is the graph of the standard normal cumulative distribution function $\Phi(z)$.



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- **Theorem 1 assumes nothing about f**. This is crucial since it is unclear which well-behavedness assumptions, if any, are satisfied by modern deep architectures.
- The certified radius R is large when: (1) the noise level σ is high, (2) the probability of the top class C_A is high, and (3) the probability of each other class is low.
- **The certified radius R goes to ∞ as $\underline{P_A} \to 1$ and $\overline{P_B} \to 0$.** This should sound reasonable: the Gaussian distributi is supported on all of $\mathbb{R}^d$, so the only way that $f(x + \epsilon) = C_A$ with probability 1 is if $f = C_A$ almost everywhere.

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Let

$$f : \mathbb{R}^d \to [0, 1]$$
$$\hat{f}(x) = \mathbb{E}_{z \sim \mathcal{N}(0, I_d)}[f(x + \sigma z)]$$

It is well-known that $\hat{f}$ is Lipschitz (it has uniform bound on the Lipschitz constant). In practice, we can approximate $\hat{f}$ by empirical average

$$y^{(k)} = \sum_{i=1}^{k} f(x + \sigma z), \quad \text{where } z \sim \mathcal{N}(0, I_d)$$

It can be shown that if $k \to \infty$, $y^{(k)}$ almost surely converges to $\hat{f}$.

Lipschitz Constant of Randomized Smoothed Classifier

Recall: Expected Value

The expectation, or expected value, of some function $f(x)$ with respect to a probability distribution $P_X(x)$ is the average, or mean value, that f takes on when x is drawn from P.

For discrete random variable X, $P_X(x)$ is **Probability Mass Function (PMF)** and expected value can be computed with a summation:

$$\mathbb{E}_{X\sim P}[f(x)] = \sum_x f(x)P_X(x)$$

For continuous random variable X, $P_X(x)$ is **Probability Density Function (PDF)** and expected value is computed with an integral:

$$\mathbb{E}_{X\sim P}[f(x)] = \int f(x)P_X(x)dx$$

Lipschitz Constant of Randomized Smoothed Classifier

Recall

- If $z \sim \mathcal{N}(0, I_d)$, then $\mu + \sigma z \sim \mathcal{N}(\mu, \sigma^2 I_d)$.
- The identity matrix is often denoted by I_n, where n is the dimension. The determinant of the identity matrix is 1.
- The Probability Density Function (PDF) of multivariate normal distribution

$$f_X(x) = \frac{1}{(2\pi)^{d/2} det(\Sigma)^{1/2}} exp\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}$$

where X is normally distributed random variable with mean $\mu \in \mathbb{R}^d$ and covariance $\Sigma \in \mathbb{R}^{d \times d}$, and d is the dimension of x.

The Cumulative Distribution Function (CDF) of multivariate normal distribution

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

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We have

$$\hat{f}(x) = \mathbb{E}_{z \sim \mathcal{N}(0, I_d)}[f(x + \sigma z)] = \frac{1}{(2\pi)^{d/2}} \int f(x + \sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz$$

$\hat{f}(x)$ is the weighted average of $f(x)$ in the vicinity of x.

Lipschitz Constant of Randomized Smoothed Classifier

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Change of variable: $w = -\sigma z$.

Lipschitz Constant of Randomized Smoothed Classifier

Recall: change of variable

Double Integral.
Suppose that we want to integrate $f(x, y)$ over the region R. Under the transformation $x = g(u, v)$, $y = h(u, v)$ the region becomes S and the integral becomes

$$\iint_R f(x, y)\, dxdy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

where $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the absolute value of inverse Jacobian determinant of the transformation.

Vector Integral.
Suppose $x, y \in \mathbb{R}^d$ that we want to integrate $f(x)$ over the region R. Under the transformation $y = g(x)$ the region becomes S and the integral becomes

$$\int_R f(x)\, dx = \int_S f(g(x)) \left| \frac{\partial x}{\partial y} \right| dy$$

where $\left| \frac{\partial x}{\partial y} \right|$ is the absolute value of inverse Jacobian determinant of the transformation.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = \frac{1}{(2\pi)^d} \int f(x + \sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz$$

Change of variable: $w = -\sigma z$. Hence, $dz = \frac{1}{\sigma^d} dw$. We get

$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x+\sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz = \int f(x-w) \frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|w\|_2^2}{2\sigma^2}\} dw$$

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Recall: Convolution

Convolution of two functions f and g over a infinite range $(-\infty, +\infty)$ is given by

$$f * g = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau = \int_{-\infty}^{+\infty} f(t - \tau) g(\tau) d\tau = g * f$$

where the symbol $[f * g](t)$ denotes convolution of f and g.

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$$\hat{f}(x) = \int f(x-w) g_\sigma(w) dw = \int f(w) g_\sigma(x-w) dw = f * g_\sigma$$

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$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x+\sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz = \int f(x-w) \underbrace{\frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|w\|_2^2}{2\sigma^2}\}}_{g_\sigma(w)} dw$$

$$\hat{f}(x) = \int f(x-w) g_\sigma(w) dw = \int f(w) g_\sigma(x-w) dw = f * g_\sigma$$

Recall

Let $f : I \to R$ be a continuous and differentiable function over some set $I \subseteq \mathbb{R}^d$, if we have $\|f'(x)\|_2 \leq m$ for all $x \in I$, then m is the upper Lipschitz constant of f ($L \leq m$).

Hence, We calculate $\|\nabla_x \hat{f}(x)\|_2$ in order to find the **upper bound on Lipschitz constant** of $\hat{f}$.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = f * g_\sigma$$

$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma)$$

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$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma) = f * \nabla_x g_\sigma$$

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Thus, we should compute $\nabla_x g_\sigma$

$$\nabla_x g_\sigma = \nabla_x \left(\frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|x-w\|_2^2}{2\sigma^2}\} \right) = \frac{1}{(2\pi)^{d/2}\sigma^d} \frac{-2(x-w)}{2\sigma^2} \exp\{-\frac{\|x-w\|_2^2}{2\sigma^2}\}$$
$$= \frac{-(x-w)}{\sigma^2} g_\sigma$$

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We have

$$\nabla_x \hat{f}(x) = \int f(w) \frac{-(x-w)}{\sigma^2} g_\sigma(x-w) dw = \int f(x-w) \frac{-w}{\sigma^2} g_\sigma(w) dw$$

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$$= \int f(x+\sigma z)\frac{z}{\sigma}\underbrace{\frac{1}{(2\pi)^{d/2}}\exp\{-\frac{\|z\|_2^2}{2}\}}_{\mathcal{N}(0,I_d)} dz = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)\frac{z}{\sigma}]$$

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Therefore, for Lipschitz constant of $\hat{f}$, we have

$$L_{\hat{f}} \leq \|\nabla_x \hat{f}(x)\|_2 \Rightarrow L_{\hat{f}} \leq \|\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)_{\in[0,1]}\frac{z}{\sigma}]\|_2$$

Lipschitz Constant of Randomized Smoothed Classifier

Recal: Triangel inequality

Triangle Inequality. Let $a_k \in \mathbb{R}^d$,

$$|\sum_{k=1}^{N} a_k| \leq \sum_{k=1}^{N} |a_k|.$$

Corollary: For random variable X, if X has a finite expectation, then

$$|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$$

Proof sketch:

$$|\mathbb{E}[X]| = |\sum_{x} x P_X(x)| \underset{\text{Triangle Inequality}}{\leq} \sum_{x} |x| P_X(x) = \mathbb{E}[|X|]$$

Lipschitz Constant of Randomized Smoothed Classifier

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$$\leq \frac{1}{\sigma}\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2]$$

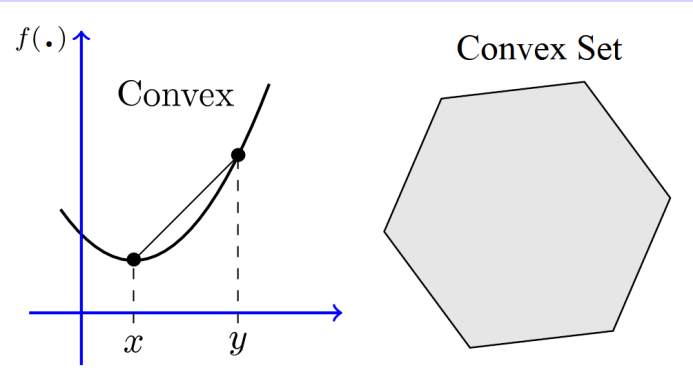
Lipschitz Constant of Randomized Smoothed Classifier

Recall: Convex Function and Jensen's Inequality

Convex set. A set C is convex if the line segment between any two points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and any $\lambda \in [0, 1]$, we have: $\lambda x_1 + (1 - \lambda)x_2 \in C$

Convex Function. Consider a function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is a convex set. We say that f is a convex function if, for any two points x and y in I and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$



Jensen's Inequality. If $f(x)$ is a convex function on I , and $\mathbb{E}[f(X)]$ and $f(\mathbb{E}[X])$ are finite, then

$$f(E[X]) \leq E[f(X)].$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\nabla_x \hat{f}(x) = \int f(x-w)\frac{-w}{\sigma^2} g_\sigma(w) dw$$

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$$\leq \frac{1}{\sigma}\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2] \leq \frac{1}{\sigma}\sqrt{\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2]}$$

Note: Since $f(x) = x^2$ is a convex function, for random variable X, we have

$$f(\mathbb{E}[g(X)]) \underset{\text{Jensen's Inequality}}{\leq} \mathbb{E}[f(g(X))] \Rightarrow \mathbb{E}^2[g(X)] \leq \mathbb{E}[g^2(X)] \Rightarrow \mathbb{E}[g(X)] \leq \sqrt{\mathbb{E}[g^2(X)]}$$

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$$L_{\hat{f}} \leq \frac{1}{\sigma}\sqrt{\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2]}$$

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$$\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2] = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2 + z_2^2 + ... + z_d^2]$$
$$= \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2] + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_2^2] + ... + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_d^2] = 1 + 1 + ... + 1 = d$$

Recall: For random variable X, $var(X) = E(X^2) - E[X]^2$

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Recall: For random variable X, $var(X) = E(X^2) - E[X]^2$

Finally, we have

$$L_{\hat{f}} \leq \frac{\sqrt{d}}{\sigma}$$

However, since the upper bound depends on the dimension of x, it increases as d grows. As $L_{\hat{f}}$ increases, the radios of our certify bound decreases.

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The main issue is that we calculated the Lipschitz constant working for all x in the input space. We know that the robustness of various inputs is different. Hence, we should compute the **local Lipschitz constant** that depends on specific input x.

Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y: \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

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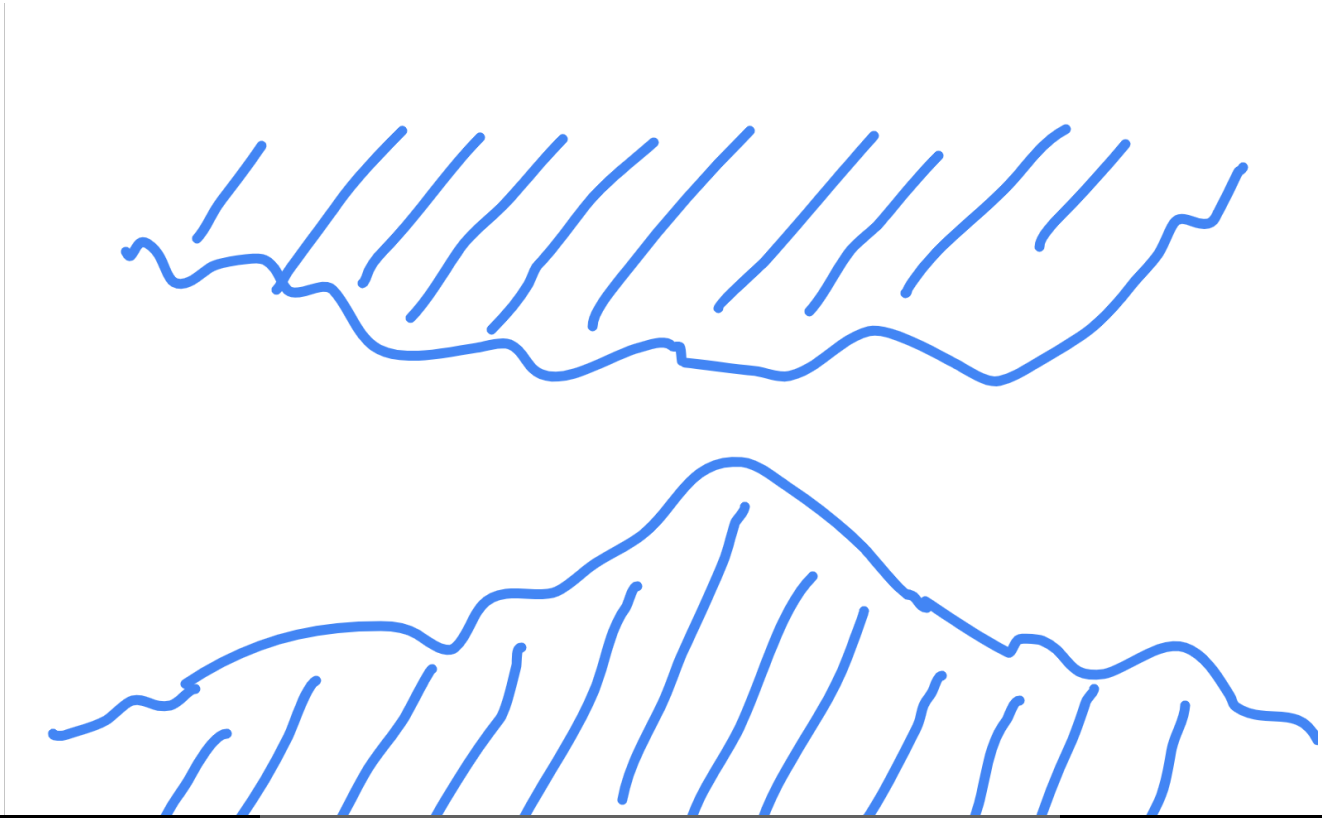

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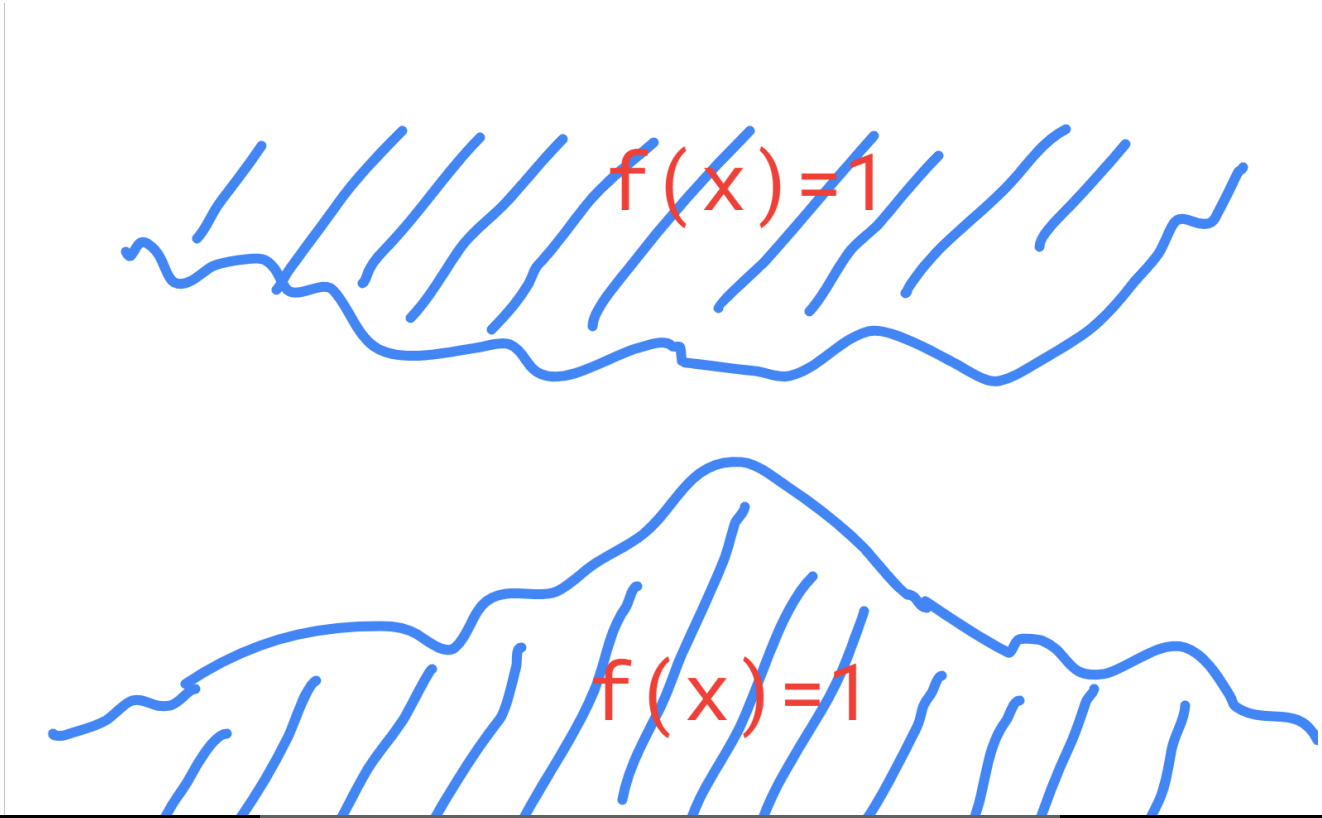

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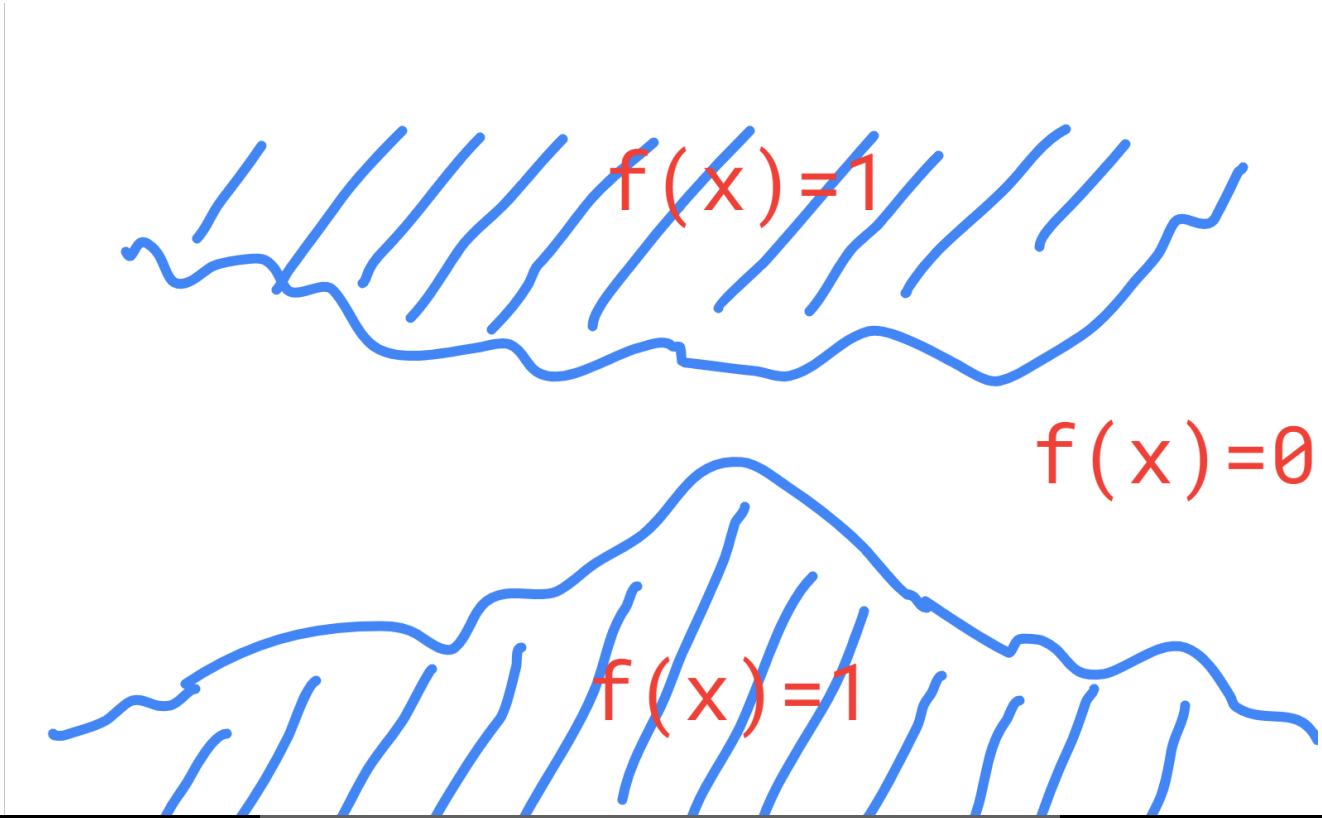

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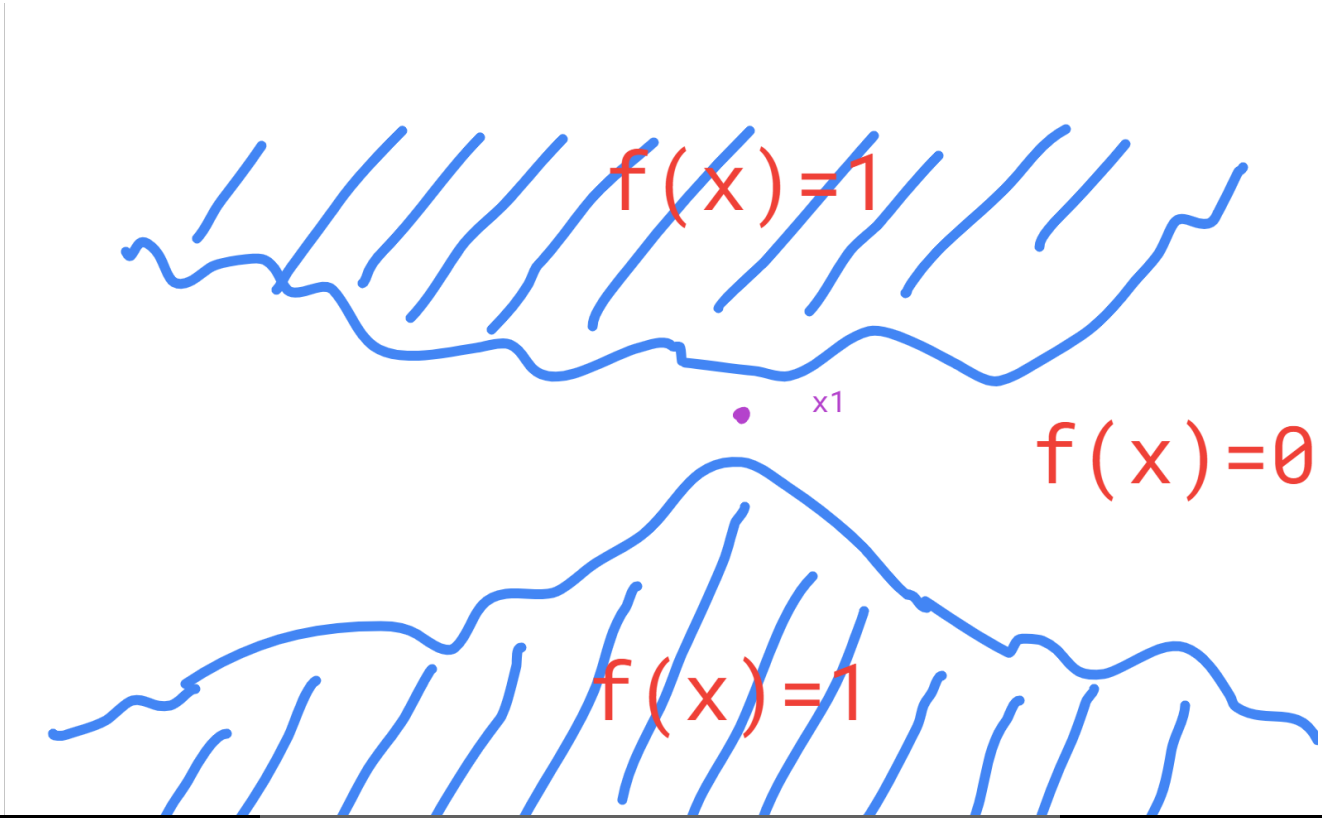

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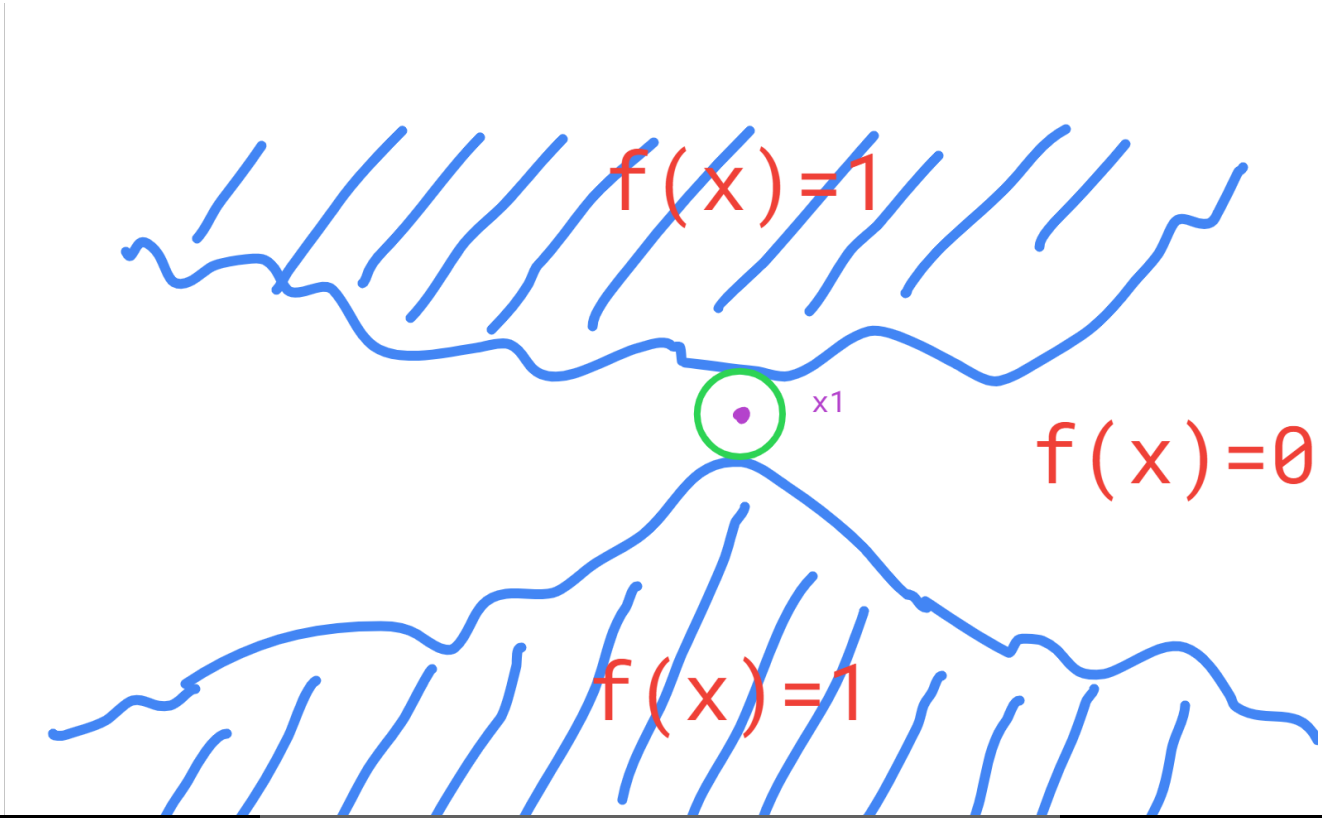

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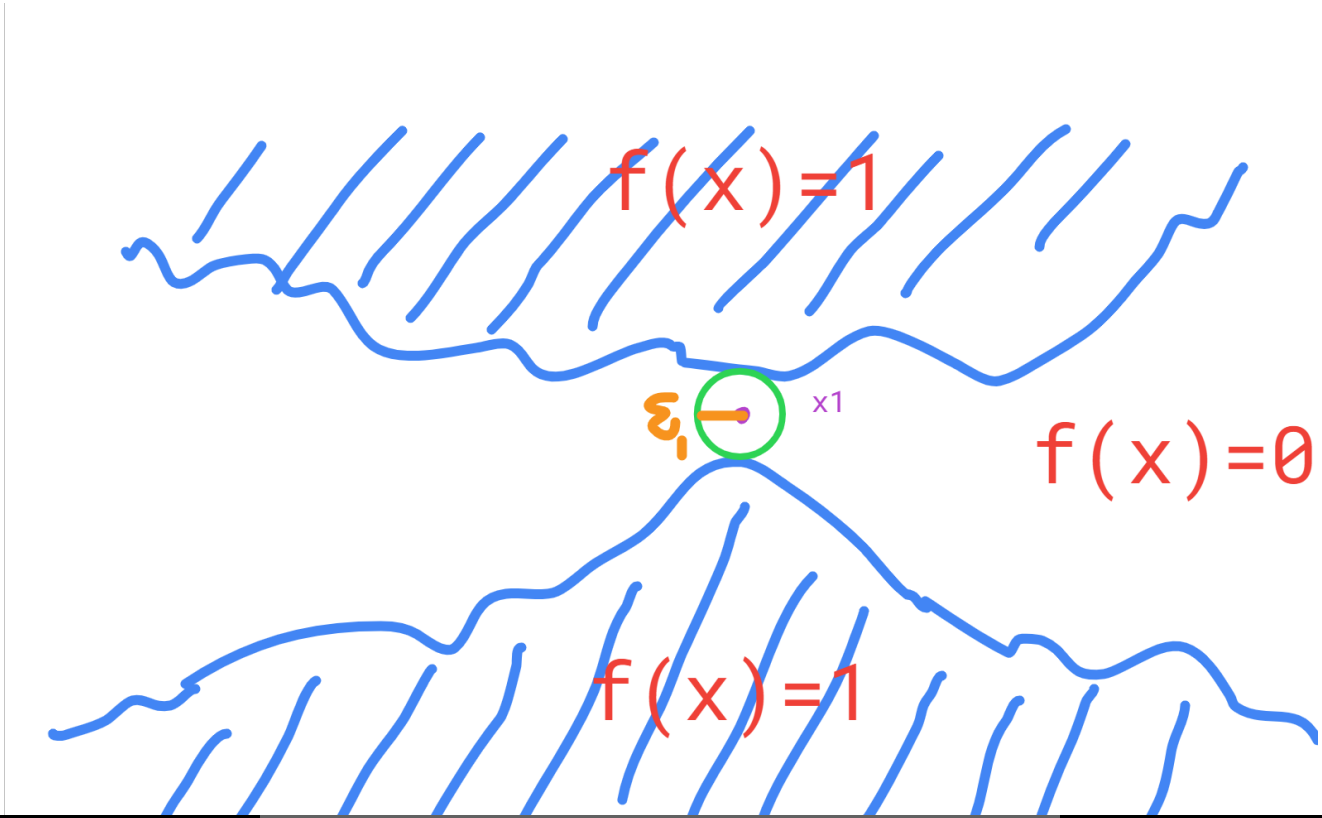

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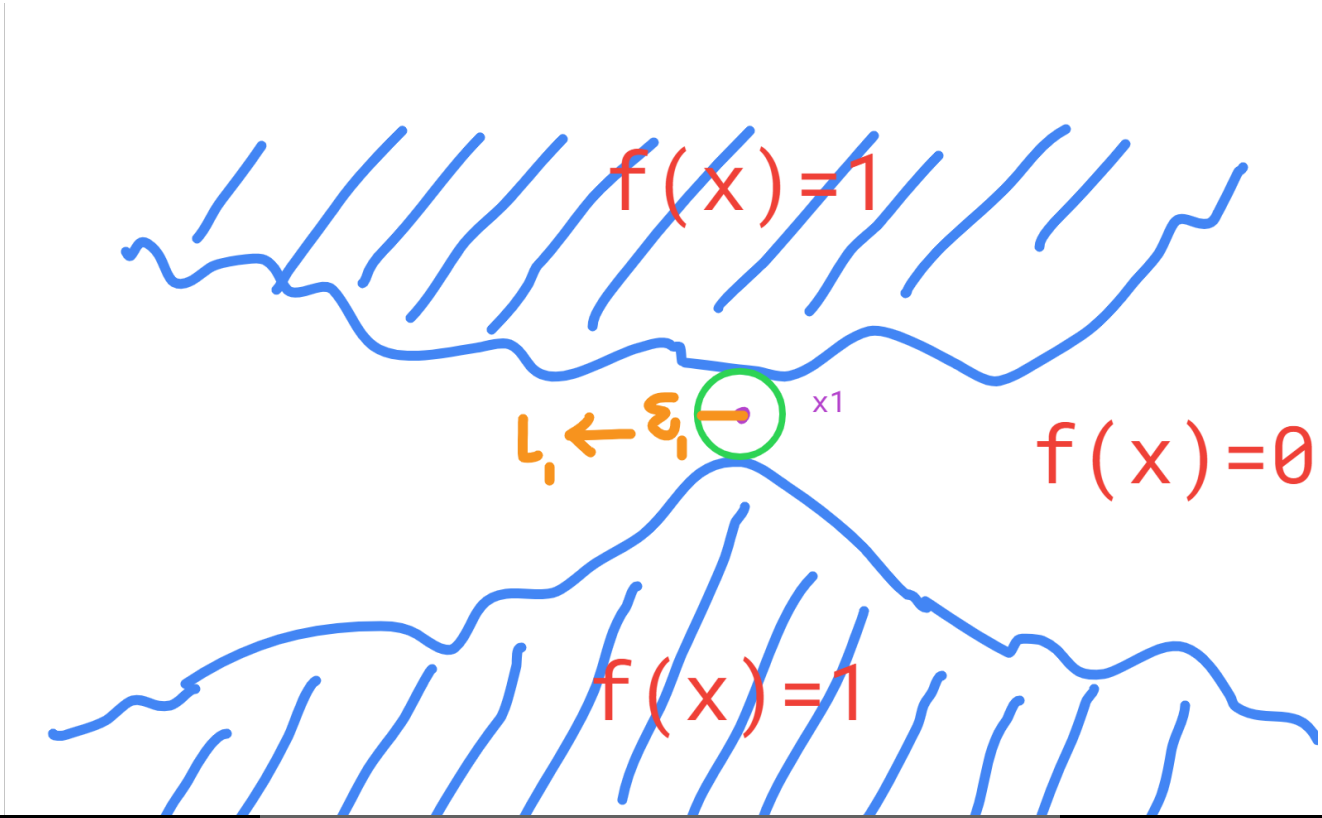

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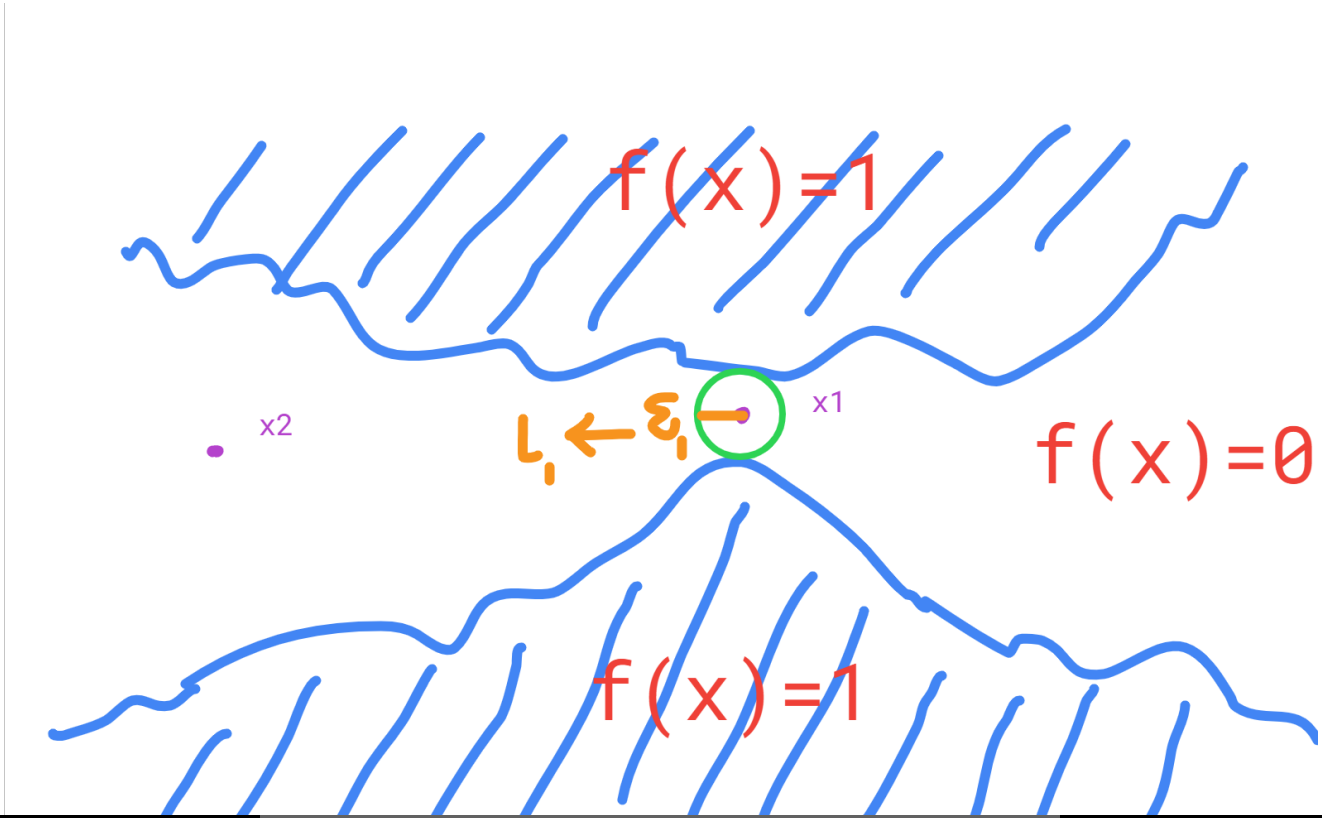

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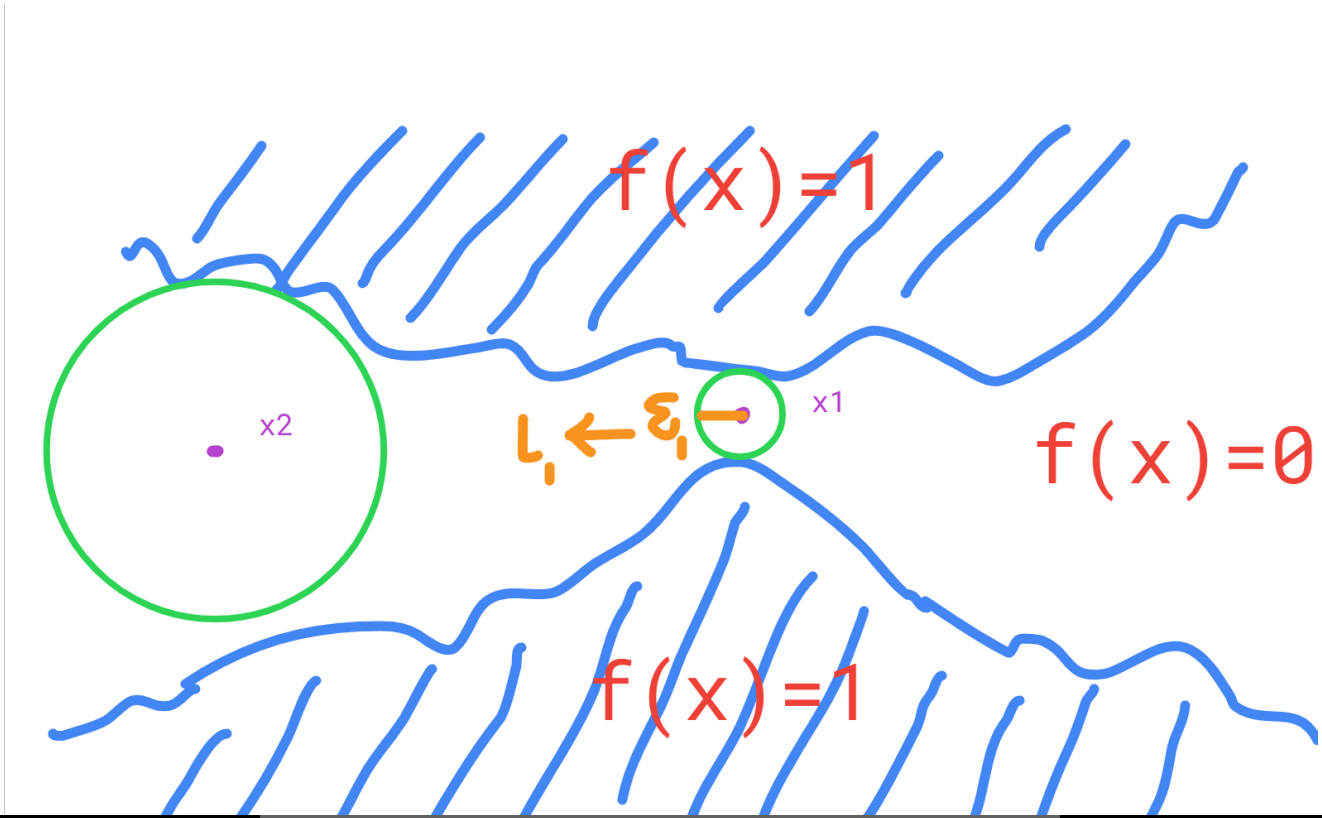

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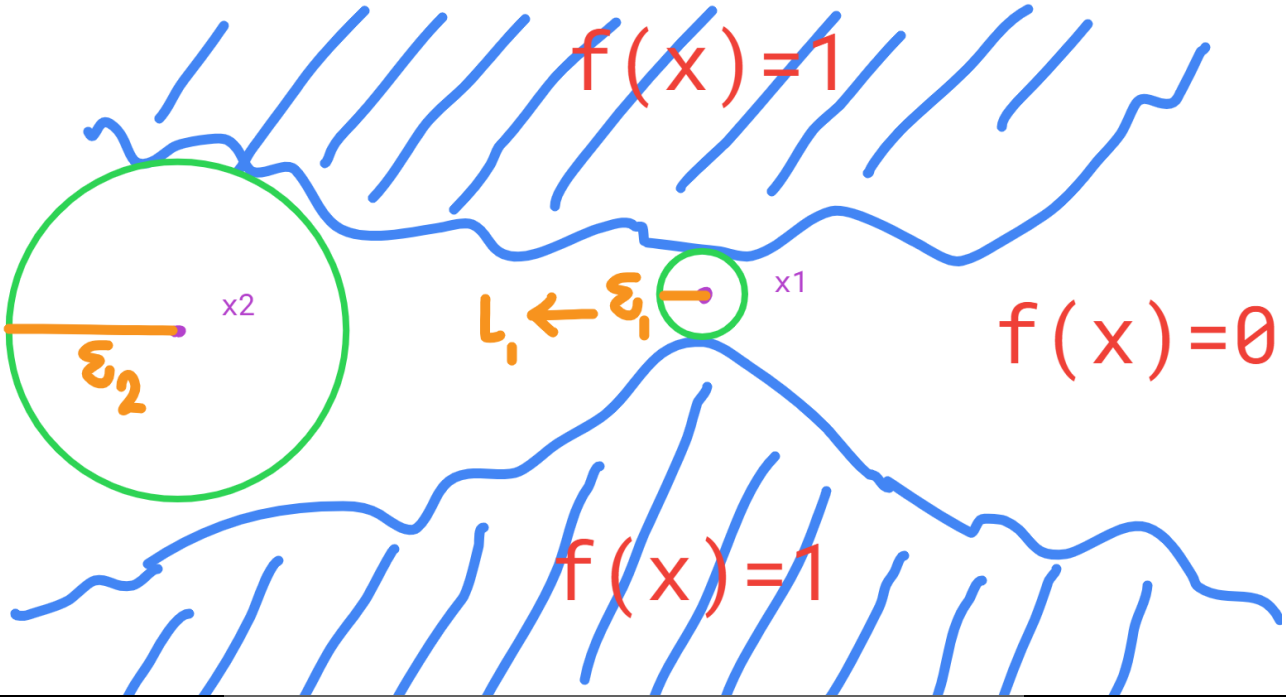

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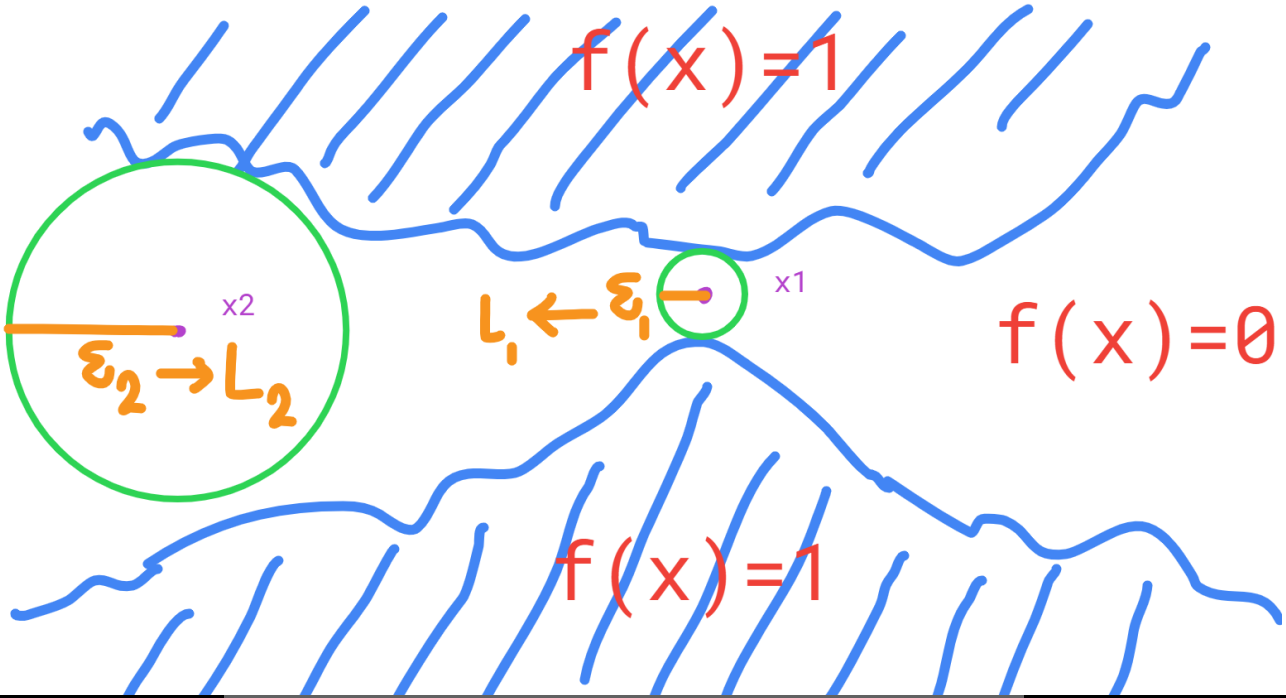

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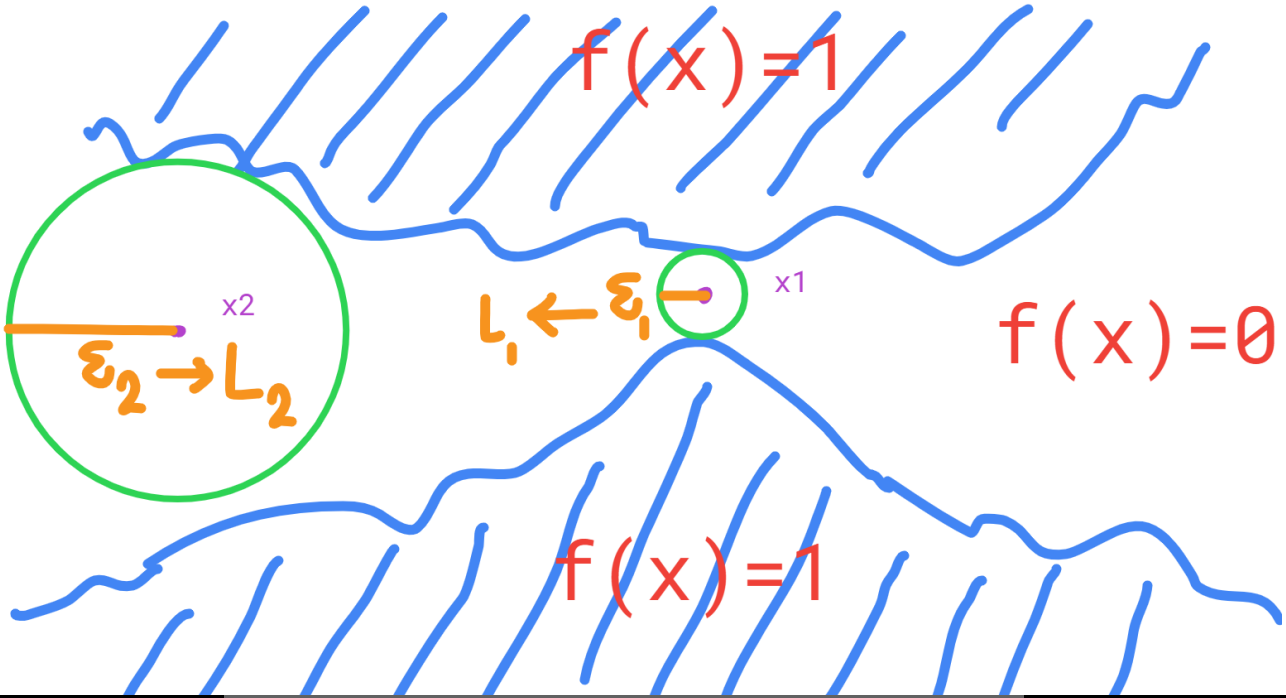

Local Lipschitz Constant

We can write L_2 as the supremum of an inner product

$$\|\nabla_x \hat{f}\|_2 = \underset{\|u\|_2=1}{sup} \; u.\nabla_x \hat{f} \quad (\text{where } u^* = \frac{\nabla_x \hat{f}}{\|\nabla_x \hat{f}\|_2})$$

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We know $\nabla_x \hat{f}(x) = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma} f(x+\sigma z)]$. Therefor, we have

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Since expectation and inner product are linear operators, we have

$$\|\nabla_x \hat{f}\|_2 = \underset{\|u\|_2=1}{sup}\ \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z.u}{\sigma} f(x+\sigma z)]$$

Local Lipschitz Constant

We can write L_2 as the supremum of an inner product

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} u.\nabla_x \hat{f} \quad (\text{where } u^* = \frac{\nabla_x \hat{f}}{\|\nabla_x \hat{f}\|_2})$$

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We want to $\|\nabla_x \hat{f}\|_2$ becomes input-dependent. We assume $p_x = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)]$. Thus, we have

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z.u}{\sigma} f(x+\sigma z)]$$
$$\text{such that} \quad \mathbb{E}_z[f(x+\sigma z)] = p_x$$

(See **this video** for the rest of proof.)

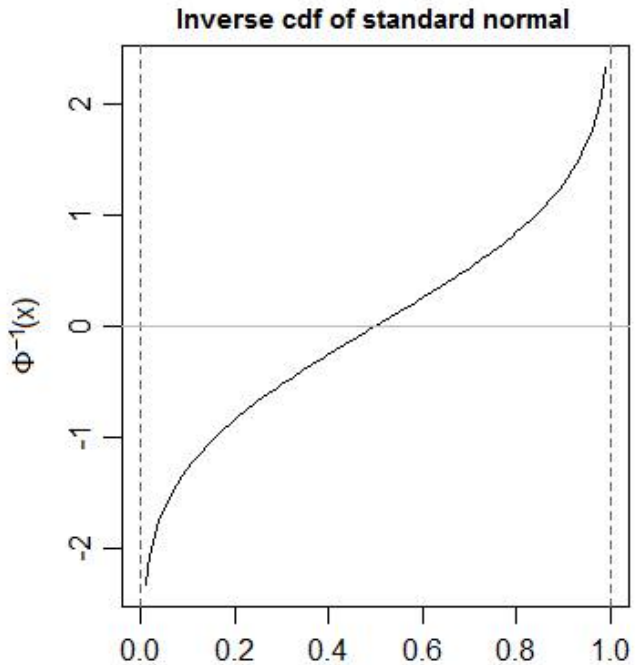
Local Lipschitz Constant

By taking supremum over u and f, the local Lipschitz constant L_x for input x is as follows

$$L_x \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$$

Where Φ is the Cumulative Density Function (CDF) of standard Gaussian distribution $\mathcal{N}(0, I_d)$.

- The bound is input-dependent.
- The bound does not depend on the input dimension d.
- L_x decreases as p_x rises.
- L_x decreases as σ rises.



Local Lipschitz Constant

We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x).\nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$
$$\|\nabla_x \psi(\hat{f}(x))\|_2 = \|\nabla_x \hat{f}(x)\|_2 |\nabla_{p_x} \psi(p_x)|$$

where $\nabla_{p_x} \psi(p_x)$ is a scalar. We know $\|\nabla_x \hat{f}(x)\|_2 \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$. In order to the Lipschitz constant of $\psi(\hat{f}(x))$ be $\frac{1}{\sigma}$, $\nabla_{p_x} \psi(p_x)$ should be

$$\frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Local Lipschitz Constant

Recall: The Derivative of the CDF of Standard Normal Distribution

The derivative of the CDF of standard normal distribution $\mathcal{N}(0, I)$

$$\Phi'(t) = \frac{d}{dt} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Recall: Fundamental Theorem of Calculus

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_{a}^{x} f(t)dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

Another way of stating the conclusion of the fundamental theorem of calculus is:

$$\frac{d}{dx} \int_{a}^{x} f(t)dt = f(x)$$

The conclusion of the fundamental theorem of calculus can be loosely expressed in words as: ”the derivative of an integral of a function is that original function”, or ”differentiation undoes the result of integration”.

Local Lipschitz Constant

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$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x).\nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$
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Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}} = \left[\Phi'(\Phi^{-1}(p_x))\right]^{-1}$$

Local Lipschitz Constant

Recall:The Derivative of Inverse Function

Given an invertible function $y = f(x)$, the derivative of its inverse function $f^{-1}(y)$ is

$$\left[f^{-1}\right]'(y) = \frac{1}{f'(f^{-1}(y))} = \left[f'(f^{-1}(y))\right]^{-1}$$

Proof

To see why this is true, start with the function $x = f^{-1}(y)$. Write this as $y = f(x)$ and differentiate both sides implicitly with respect to y using the Chain Rule

$$1 = f'(x).\frac{dx}{dy}$$

Thus

$$\frac{dx}{dy} = \frac{1}{f'(x)}$$

but $x = f^{-1}(y)$. Thus,

$$\left[f^{-1}\right]'(y) = \frac{1}{f'\left[f^{-1}(y)\right]}$$

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We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

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Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}} = \left[\Phi'(\Phi^{-1}(p_x))\right]^{-1} = \left[\Phi^{-1}\right]'(p_x)$$

Thus

$$\psi(p_x) = \Phi^{-1}(p_x) + c$$

where c is a constant (we set $c = 0$).

Certifiable Robustness for $1/\sigma$-Lipschitz smoothed classifier

Since, the Lipschitz constant of $\Phi^{-1}(\hat{f}(x))$ is $\frac{1}{\sigma}$, similar to **Theorem 0**, we can show that

$$\underset{i}{\text{argmax}}\ \hat{f}_i(x+\delta) = \underset{i}{\text{argmax}}\ \hat{f}_i(x)$$

for all $\|\delta\|_2 \leq R$, where

$$R = \frac{1}{2L}(\Phi^{-1}(P_A) - \Phi^{-1}(P_B)) = \frac{\sigma}{2}(\Phi^{-1}(P_A) - \Phi^{-1}(P_B)),$$

$P_A = \underset{i}{max}\ \hat{f}_i(x)$, $P_B = \underset{j\neq i}{max}\ \hat{f}_j(x)$, and $\hat{f}_k(x)$ is the k-th element of the probability vector $\hat{f}(x)$.

Training the Base Classifier

Theorem 1 holds regardless of how the base classifier f is trained.

- If the base classifier f is trained via standard supervised learning on the data distribution, it will **see no noisy images during training**, and hence **will not necessarily learn to classify** $\mathcal{N}(x, \sigma^2 I)$ with x's true label.
- Therefore, we **train the base classifier with Gaussian data augmentation** at variance σ^2.

Experiments

Evaluation metric is **certified test set accuracy** at radius r defined as

- The fraction of the test set which $\hat{f}$ classifies correctly with a prediction that is certifiably robust within an ℓ_2 ball of radius r.
- Thus, to compute **certified accuracy**, we pick a target radius and count the number of points in the test set whose certified radius $r \geq T$ and where the predicted c_A matches the test set label. Standard accuracy is instantiated with $T = 0$.

CIFAR10 model

- ResNet-110
- $n = 100000$ (use n samples from $f(x + \epsilon)$ to obtain some $\underline{P_A}$ and $\overline{P_B}$.)
- Certifying each example took 15 seconds on an NVIDIA RTX 2080 Ti.

ImageNet model

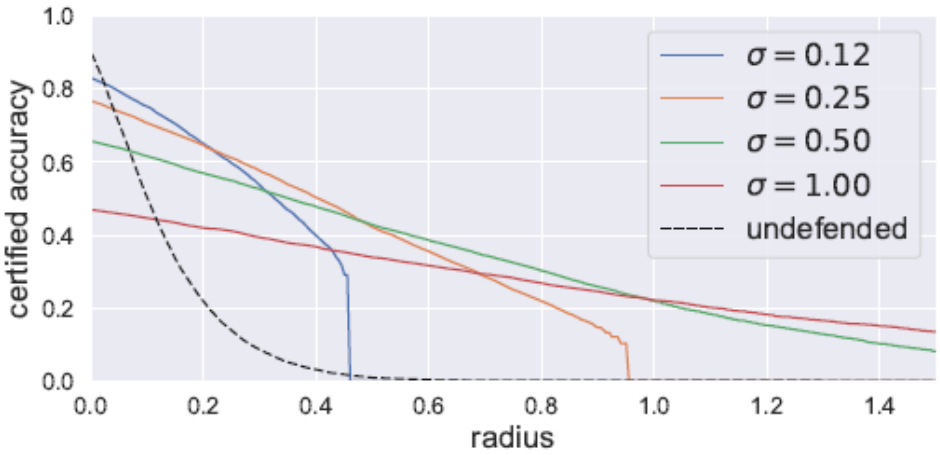
- ResNet-50
- $n = 100000$
- certifying each example took 110 seconds.

Experiments

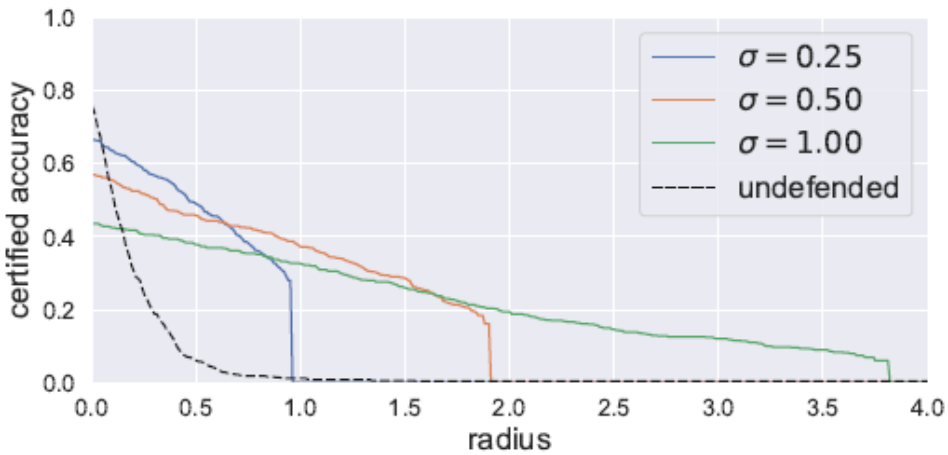
	$r = 0.0$	$r = 0.5$	$r = 1.0$	$r = 1.5$	$r = 2.0$	$r = 2.5$	$r = 3.0$
$\sigma = 0.25$	**0.67**	**0.49**	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.50$	0.57	0.46	**0.37**	**0.29**	0.00	0.00	0.00
$\sigma = 1.00$	0.44	0.38	0.33	0.26	**0.19**	**0.15**	**0.12**

Table 2. Approximate certified test accuracy on ImageNet. Each row is a setting of the hyperparameter σ, each column is an ℓ_2 radius. The entry of the best σ for each radius is bolded. For comparison, random guessing would attain 0.001 accuracy.

Experiments



CIFAR10



ImageNet

Figure 6. Approximate certified accuracy attained by randomized smoothing on CIFAR-10 (left) and ImageNet (right). The hyper-parameter σ controls a robustness/accuracy tradeoff. The dashed black line is an upper bound on the empirical robust accuracy of an undefended classifier with the base classifier's architecture.

Gaussian Smoothing for L_P Attacks

If we use Gaussian smoothing against L_P attacks, for $p \geq 2$ we get

$$r_p = \frac{\sigma}{2d^{\frac{1}{2}-\frac{1}{p}}}(\Phi^{-1}(p_1(x)) - \Phi^{-1}(p_2(x)))$$

Curse of dimensionality: For L_P attacks where $p > 2$, the smoothing-based certificate upper bound decreases as d increases.

(See: Curse of Dimensionality on Randomized Smoothing for Certifiable Robustness, Kumar et al., 2020)

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- Create a surrogate model with high **fidelity** to the taget model.
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Query-based

- Based on the target model responses for consecutive queries
  - Gradient estimation
    - Based on zero-order (ZO) optimization algorithms
  - Search-based
    - Based on choosing a search strategy using a search distribution.

Jacobian-based Dataset Augmentation

Jacobian-based Dataset Augmentation

Practical Black-Box Attacks against Machine Learning

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Abstract

The first black-box attack against DNN classifiers for real-world adversaries with no knowledge about the model.

- The only capability of the black-box adversary is to observe **labels** given by the DNN to chosen inputs.

The attack strategy

- **Training a local model** to substitute for the target DNN.
  - Using **inputs synthetically generated** by an adversary and labeled by the target DNN.
- The local substitute is used to craft adversarial examples, and find that they are misclassified by the targeted DNN.

To perform a real-world and properly-blinded evaluation, we attack a DNN hosted by **MetaMind**, an online deep learning API.

Threat Model

In the black-box setting, adversaries do not know internal details of a system to compromise it.

- The adversary has **no information about the structure or parameters** of the target DNN.
- The adversary has **no knowledge of the training data** used to learn the DNN's parameters.
- The adversary does **not have access to any large training dataset**.
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A data-limited adversary

- Many modern machine learning systems require large and expensive training sets for training.
  - This makes attacks based on training substitute model **unfeasible for adversaries without large labeled datasets**.
- To enable the adversary to train a substitute model without a real labeled dataset
  - The adversary **uses the target DNN as an oracle to construct a synthetic dataset**.

Adversarial Capabilities

The oracle O is the targeted DNN.

- Its name refers to the only capability of the adversary: accessing the label $\tilde{O}(x)$ for any input x by querying oracle O.

The output label $\tilde{O}(x)$ is the index of the class assigned the largest probability by the DNN

$$\tilde{O}(x) = \underset{j \in 0,\ldots,N-1}{argmax} \; O_j(x)$$

where $O_j(x)$ is the j-th component of the probability vector $O(x)$ output by DNN O, and N is the number of classes.

Accessing labels $\tilde{O}$ produced by the DNN O is the only capability assumed in our threat model.

Adversarial Goal

The adversary wants to produce a minimally altered version of any input x, named **adversarial example**, and denoted x^*, misclassified by oracle O

$$x^* = x + argmin\{z : \tilde{O}(x+z) \neq \tilde{O}(x)\} = x + \delta_x$$
$$\text{such that : } \tilde{O}(x^*) \neq \tilde{O}(x)$$

Black-box Attack

Black-box Attack Strategy

1. **Substitute Model Training**: the attacker queries the oracle with **synthetic inputs** selected by a Jacobian based heuristic to build a **model F approximating the oracle model O's decision boundaries**.
2. **Adversarial Sample Crafting**: the attacker uses substitute network F to craft adversarial samples, which are then misclassified by oracle O due to the transferability of adversarial samples.

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Challenges

- **Select an architecture** for our substitute without knowledge of the targeted oracle's architecture
- **Limit the number of queries** made to the oracle in order to ensure that the approach is tractable.

Generating a Synthetic Dataset

The heuristic used to generate **synthetic training inputs** is based on identifying **directions in which the model's output is varying**, around an initial set of training points.

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- These directions are identified with the substitute DNN's **Jacobian matrix** J_F , which is evaluated at several input points x.
- Precisely, the adversary evaluates the sign of the Jacobian matrix dimension corresponding to the **label assigned to input** x **by the oracle**

$$sign(J_F(x)[\tilde{O}(x)]) = sign(\nabla_x F(x)_{\tilde{O}(x)})$$

where $F(x)_i$ is the i-th element of the probability vector $F(x)$ output by substitute model F.

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where $F(x)_i$ is the i-th element of the probability vector $F(x)$ output by substitute model F.

The new synthetic training point x' is created as follows

$$x' = x + \lambda.sign(J_F(x)[\tilde{O}(x)]) = x + \lambda.sign(\nabla_x F(x)_{\tilde{O}(x)})$$

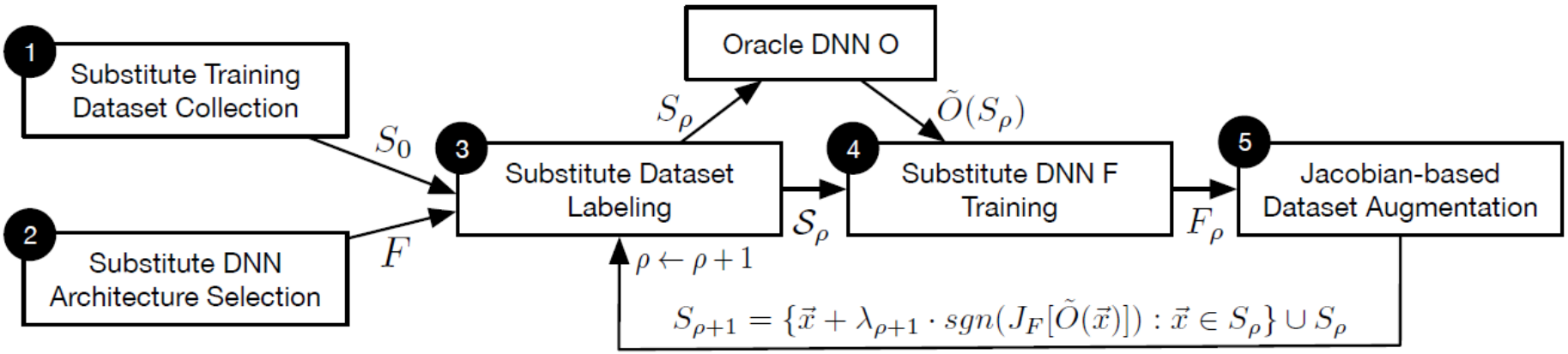
where x is already in the training set, and λ is a parameter of the augmentation.

This technique is called **Jacobian-based Dataset Augmentation**.

Jacobian-based Dataset Augmentation

- **Initial Collection**: The adversary collects a very small set S_0 of inputs representative of the input domain.
- **Architecture Selection**: The adversary selects an architecture to be trained as the substitute model F.
- **Substitute Training**: the adversary iteratively trains more accurate substitute model F_ρ by repeating the following for max_ρ rounds:
  1. **Labeling** S_ρ by oracle O
  2. **Training** F_ρ on S_ρ
  3. Create $S_{\rho+1}$ by **augmenting** S_ρ with more synthetic training points.

$$S_{\rho+1} = \{x + \lambda_{\rho+1}.sign(J_F(x)[\tilde{O}(x)]) : \ x \in S_\rho\} \cup S_\rho$$



Algorithm

Algorithm 1 - Substitute DNN Training: for oracle $\tilde{O}$, a maximum number max_ρ of substitute training epochs, a substitute architecture F, and an initial training set S_0.

Input: $\tilde{O}$, max_ρ, S_0, λ

Define architecture F
for $\rho \in 0 \,..\, max_\rho - 1$ **do**
    // Label the substitute training set
    $D \leftarrow \left\{(\vec{x}, \tilde{O}(\vec{x})) : \vec{x} \in S_\rho\right\}$
    // Train F on D to evaluate parameters θ_F
    $\theta_F \leftarrow \text{train}(F, D)$
    // Perform Jacobian-based dataset augmentation
    $S_{\rho+1} \leftarrow \{\vec{x} + \lambda \cdot \text{sgn}(J_F[\tilde{O}(\vec{x})]) : \vec{x} \in S_\rho\} \cup S_\rho$
end for
return θ_F

Next Papers

They introduce the *Jacobian-based Dataset Augmentation* (JbDA) technique for generating synthetic samples (row 13). It relies on computing the Jacobian matrices with the current F' evaluated on the already labeled samples in L. Each element $x \in L$ is modified by adding the sign of the Jacobian matrix $\nabla_x \mathcal{L}(F'(x, c_i))$ dimension corresponding to the label assigned to x by F, evaluated with regards to the classification loss $\mathcal{L}$. Thus, the set U is extended with $\{x + \lambda \cdot sign(\nabla_x \mathcal{L}(F'(x, c_i)))\}$, $\forall x \in L$. U has the same

PRADA, IEEE European S&P, 2018

```
# Get augmented inputs
X, Y = X.to(self.device), Y.to(self.device)
delta_i = self.fgsm_untargeted(model_adv, X, Y.argmax(dim=1), device=self.device, epsilon=step_size)
# Get correnponding outputs from blackbox
if self.aug_strategy == 'jbda':
    Y_i = self.blackbox(X + delta_i)
```

Prediction Poisoning, ICLR, 2020

2. *Jacobian Based Dataset Augmentation (JBDA)* (Papernot et al., 2017) uses synthetic data to query the target model. JBDA starts with a small set of in-distribution "seed" examples $\{x_i\}$. The attack iteratively trains the clone model by performing the following steps: (i) Obtain a labeled dataset $\{x_i, y_i\}$ by querying the target model (ii) Train the clone model on the labeled dataset (iii) Augment the dataset with synthetic examples x_i' by perturbing the original input x_i to change the prediction of the clone model by using the Jacobian of the loss function: $x_i' = x_i + \beta sign\left(\nabla_x \mathcal{L}\left(C\left(x_i; \theta_c\right), y_i\right)\right)$.

EDM, ICLR, 2021

generates a synthetic example x', by perturbing it using the jacobian of the clone model's loss function: $x' = x + \lambda sign\left(\nabla_x \mathcal{L}\left(f'\left(x; \theta'\right)\right)\right)$. These synthetic examples are labeled using the predictions of the defender's model $y' = f(x')$ and the labeled synthetic examples thus generated: $\mathcal{D}_{syn} = \{x', y'\}$, are used to augment the adversary's dataset: $\mathcal{D}_{seed} = \mathcal{D}_{seed} \cup \mathcal{D}_{syn}$ and retrain f'.

Adaptive Misinformation, CVPR, 2019

Attack on MetaMind MNIST Model

Initial Substitute Training Sets

- **MNIST subset**: This initial substitute training set is made of 150 samples from the MNIST test set.
- **Handcrafted set**: 100 samples by handwriting 10 digits for each class between 0 and 9 with a laptop trackpad.

Implementation details

- $max_\rho = 6$.
- During each of these 6 rounds, the model is trained for 10 epochs from scratch.
- $\lambda = 0.1$

The Accuracy of the Two Substitute Models

Substitute Epoch	Initial Substitute Training Set from MNIST test set	Handcrafted digits
0	24.86%	18.70%
1	41.37%	19.89%
2	65.38%	29.79%
3	74.86%	36.87%
4	80.36%	40.64%
5	79.18%	56.95%
6	81.20%	67.00%

Figure 4: **Substitute DNN Accuracies:** each column corresponds to an initial substitute training set: 150 MNIST test samples, and handcrafted digits. Accuracy is reported on the unused 9,850 MNIST test samples.

Evaluation

- MNIST test samples are used to generate adversarial examples using FGSM.
- The success rate is the proportion of adversarial samples misclassified by the substitute model.
- The transferability of adversarial samples refers to the oracle misclassification rate of adversarial samples crafted using the substitute DNN.

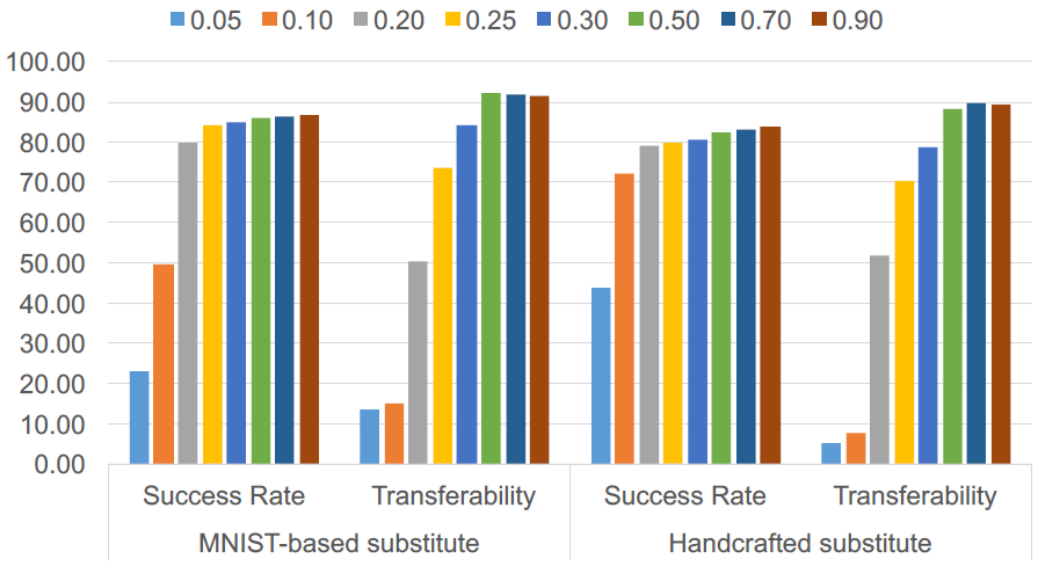


Figure 5: **Success Rate and Transferability of Adversarial Samples for the MetaMind attacks:** performed using MNIST-based and handcrafted substitutes: each bar corresponds to a different perturbation input variation.

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- Having the **step size periodically alternating between positive and negative values improves the quality** of the oracle approximation made by the substitute.
- **Reducing Oracle Querying**: randomly select κ samples from a list of samples. The adversary after σ iterations selects κ new inputs for Jacobian-based dataset augmentation.