



Certifiable Robustness

A. M. Sadeghzadeh, Ph.D.

Sharif University of Technology
Computer Engineering Department (CE)
Data and Network Security Lab (DNSL)



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Today's Agenda

1 Randomized Smoothing



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Certified Adversarial Robustness via Randomized Smoothing

Jeremy Cohen¹ Elan Rosenfeld¹ J. Zico Kolter^{1,2}

Certifiable Robustness

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$$\forall \|\delta\|_p \leq \epsilon, \quad \underset{i \in [K]}{\operatorname{argmax}} f_i(x + \delta) = \underset{i \in [K]}{\operatorname{argmax}} f_i(x)$$

where K is the number of classes.

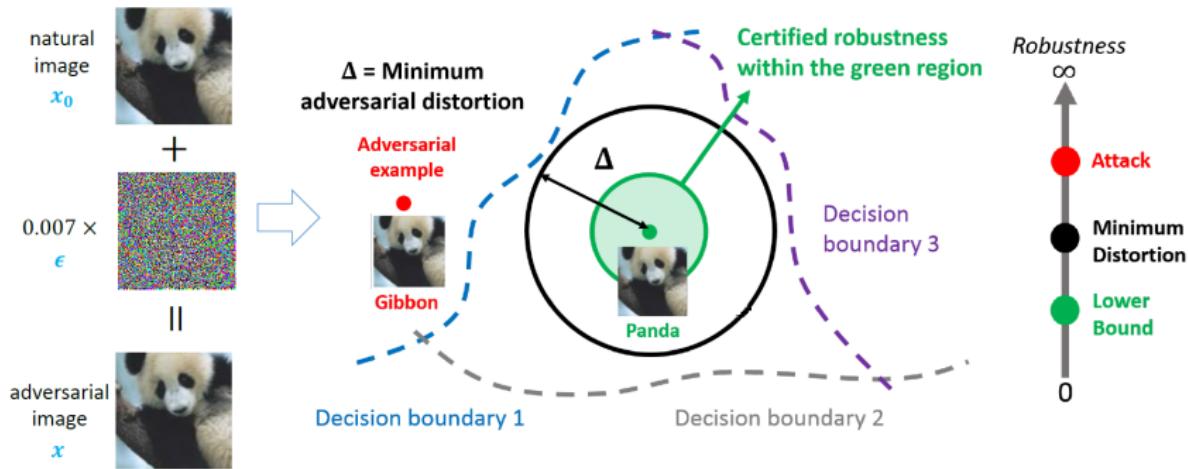
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Lipschitz continuity

A function $f : I \rightarrow R$ over some set $I \subseteq \mathbb{R}^d$ is called Lipschitz continuous if there exists a positive real constant L such that, for all $x, y \in I$,

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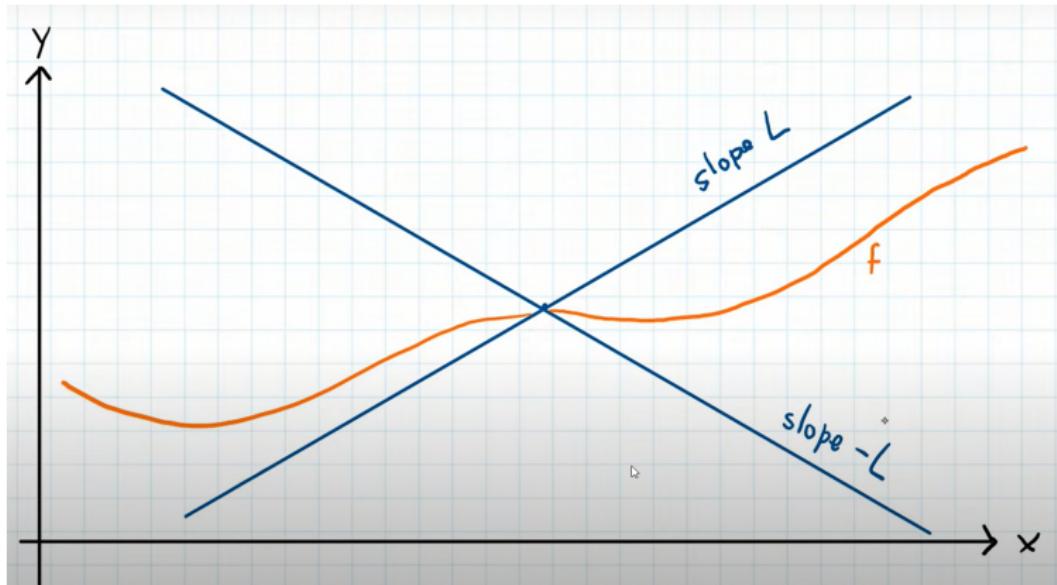
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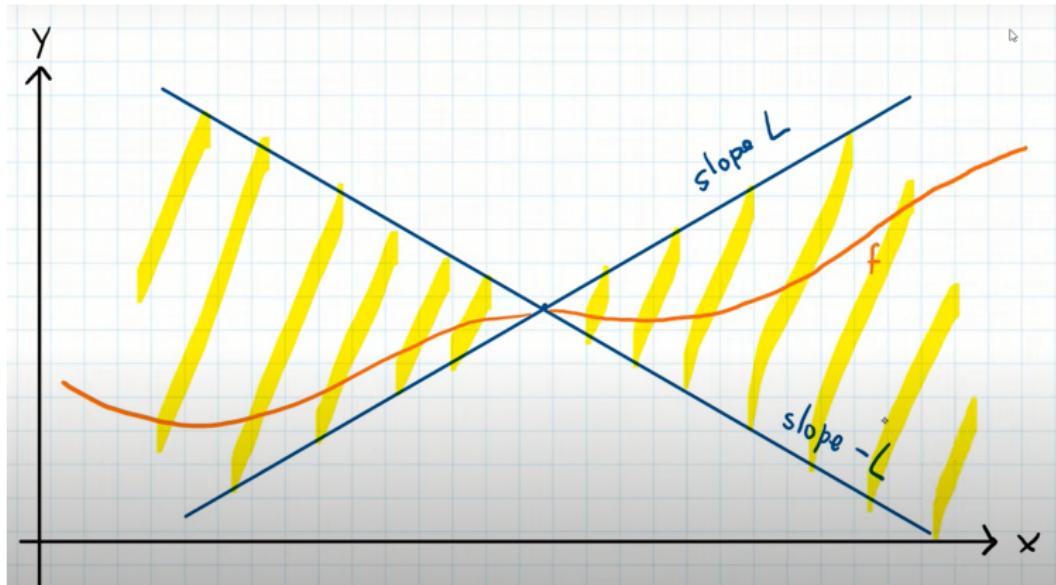
$$|f_1(f_2(y)) - f_1(f_2(x))| \leq L_1 |f_2(y) - f_2(x)| \leq L_1 L_2 \|y - x\|_2$$

Generally, Let $f = f_1 \circ f_2 \circ \dots \circ f_K$ and the Lipschitz constant of f_i be L_i for all $i \in \{1, 2, \dots, K\}$, then the Lipschitz constant of f is $L \leq \prod_{k=1}^K L_k$.

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Let $f : I \rightarrow R$ be a continuous and differentiable function over some set $I \subseteq \mathbb{R}^d$, if we have $\|f'(x)\|_2 \leq m$ for all $x \in I$, then m is the upper Lipschitz constant of f ($L \leq m$).

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Proof sketch:

Mean value theorem: Let $f : I \rightarrow R$ be a continuous and differentiable function over some set $I \subseteq \mathbb{R}^d$, For all $a, b \in I$ ($b > a$), there exists some $c \in (a, b)$ such that:

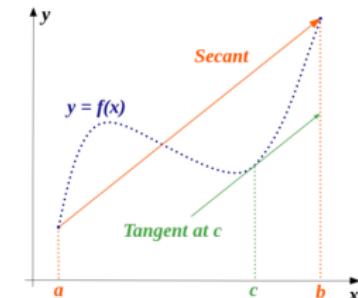
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

For all $a, b \in I$, there exist $c \in (a, b)$, such that:

$$|f(b) - f(a)| = \|f'(c).b - a\|_2 \leq \|f'(c)\|_2 \|b - a\|_2.$$

Since we know that $\|f'(c)\|_2 \leq m$, we have

$$|f(b) - f(a)| \leq m \|b - a\|_2.$$



Certifiable Robustness for L-Lipschitz classifier

Theorem 0: If $f : \mathbb{R}^d \rightarrow [0, 1]^K$ is L -lipschitz, then f is ϵ -robust at x with $\epsilon = \frac{1}{2L}(P_A - P_B)$, where $P_A = \max_i f_i(x)$, $P_B = \max_{j \neq i} f_j(x)$, and $f_k(x)$ is the k -th element of the probability vector $f(x)$.

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Proof.

Since f is L -lipschitz, we have

$$\forall x, y \in \mathbb{R}^d, \|f(y) - f(x)\|_2 \leq L\|y - x\|_2$$

Denote $x' = x + \delta$ and assume that $\|\delta\| \leq \epsilon$. we get

$$\|f(x') - f(x)\|_2 \leq L\|x' - x\|_2 \rightarrow \|f(x') - f(x)\|_2 \leq L\|\delta\|_2 \leq L\epsilon$$

Hence, P_A can be reduced at most by $L\epsilon$ and P_B can be increased at most by $L\epsilon$. We have ($P'_i = f_i(x')$)

$$P'_A \geq P_A - L\epsilon \quad \text{and} \quad P'_B \leq P_B + L\epsilon$$

Since we want that the label of x' be the same as x , P'_A must be greater than P'_B ($P'_A \geq P'_B$). We have

$$P_A - L\epsilon \geq P_B + L\epsilon \rightarrow 2L\epsilon \leq P_A - P_B \rightarrow \epsilon \leq \frac{1}{2L}(P_A - P_B)$$

□

Smoothed Classifier

If we compute the (upper bound of) Lipschitz constant of the classifier, we can determine the radius (ϵ) of the robustness for each sample.

- However, **the lipschitz constant of deep neural networks** is very large and it **grows exponentially** with the number of layers. Hence, the certification is useless ($\epsilon \rightarrow 0$).

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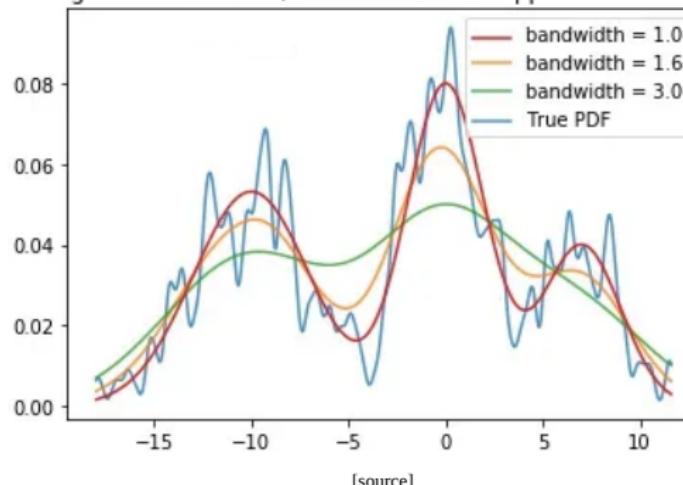
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Effect of various bandwidth values
The larger the bandwidth, the smoother the approximation becomes



Randomized Smoothing

Consider a classification problem from \mathbb{R}^d to classes \mathcal{Y} . Randomized smoothing is a method for constructing a new, **smoothed classifier** \hat{f} from an arbitrary base classifier f .

Randomized Smoothing

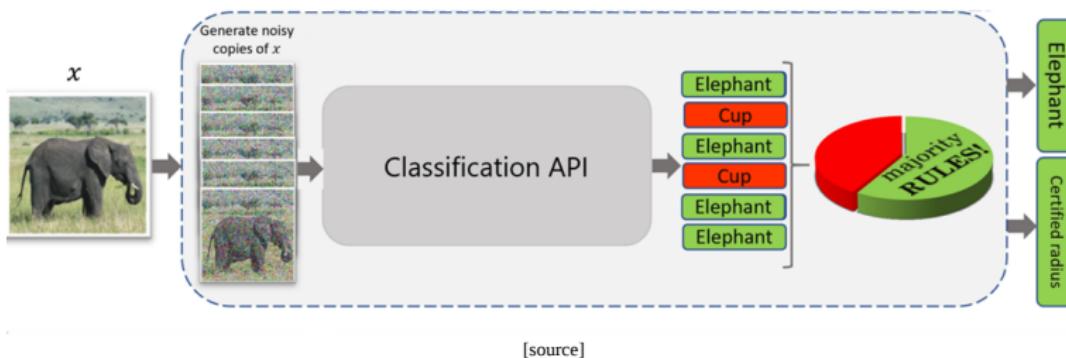
Consider a classification problem from \mathbb{R}^d to classes \mathcal{Y} . Randomized smoothing is a method for constructing a new, **smoothed classifier** \hat{f} from an arbitrary base classifier f .

- When queried at x , the smoothed classifier \hat{f} returns whichever class the base classifier f is most likely to return when x is perturbed by isotropic Gaussian noise:

$$\hat{f}(x) = \underset{c \in \mathcal{Y}}{\operatorname{argmax}} \mathbb{P}(f(x + \epsilon) = c) \quad (1)$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$

The noise level σ is a hyperparameter of the smoothed classifier \hat{f} which controls a robustness/accuracy tradeoff.



Notation

Suppose that when the base classifier f classifies $\mathcal{N}(x, \sigma^2 I)$, the most probable class c_A is returned with probability P_A , and the “runner-up” class is returned with probability P_B .

- \underline{P}_A is a lower bound for P_A and \overline{P}_B is a lower bound for P_B .

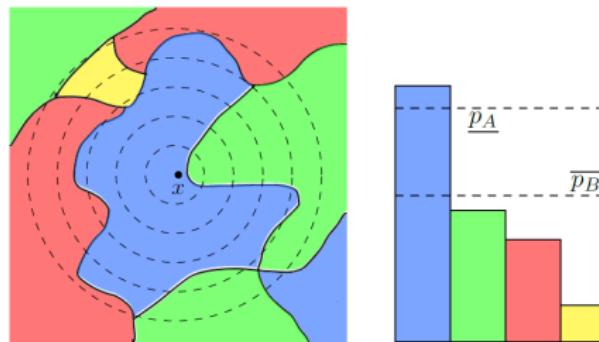


Figure 1. Evaluating the smoothed classifier at an input x . **Left:** the decision regions of the base classifier f are drawn in different colors. The dotted lines are the level sets of the distribution $\mathcal{N}(x, \sigma^2 I)$. **Right:** the distribution $f(\mathcal{N}(x, \sigma^2 I))$. As discussed below, \underline{p}_A is a lower bound on the probability of the top class and \overline{p}_B is an upper bound on the probability of each other class. Here, $g(x)$ is “blue.”

Robustness guarantee

Theorem 1. Let $f : \mathbb{R}^d \rightarrow \mathcal{Y}$ be any deterministic or random function, and let $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Let \hat{f} be defined as in (1). Suppose $C_A \in \mathcal{Y}$ and $\underline{P}_A, \overline{P}_B \in [0, 1]$ satisfy:

$$\mathbb{P}(f(x + \epsilon) = C_A) \geq \underline{P}_A \geq \overline{P}_B \geq \max_{C \neq C_A} \mathbb{P}(f(x + \epsilon) = C)$$

Then $\hat{f}(x + \delta) = C_A$ for all $\|\delta\|_2 \leq R$, where

$$R = \frac{\sigma}{2} (\Phi^{-1}(\underline{P}_A) - \Phi^{-1}(\overline{P}_B))$$

where Φ^{-1} is the inverse of the standard Gaussian CDF.

Robustness guarantee

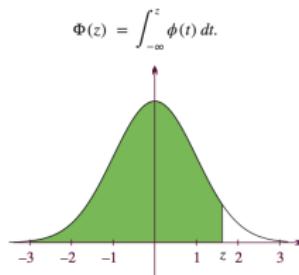
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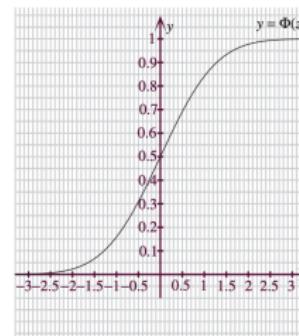
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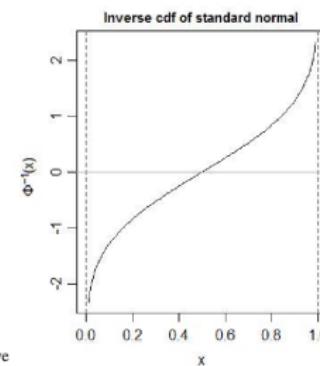
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This is the graph of the standard normal probability density function $\phi(z)$.



This is the graph of the standard normal cumulative distribution function $\Phi(z)$.



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- **Theorem 1 assumes nothing about f .** This is crucial since it is unclear which well-behavedness assumptions, if any, are satisfied by modern deep architectures.
- The certified radius R is large when: (1) the noise level σ is high, (2) the probability of the top class C_A is high, and (3) the probability of each other class is low.
- **The certified radius R goes to ∞ as $\underline{P}_A \rightarrow 1$ and $\overline{P}_B \rightarrow 0$.** This should sound reasonable: the Gaussian distribution is supported on all of \mathbb{R}^d , so the only way that $f(x + \epsilon) = C_A$ with probability 1 is if $f = C_A$ almost everywhere.

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Let

$$\begin{aligned} f : \mathbb{R}^d &\rightarrow [0, 1] \\ \hat{f}(x) &= \mathbb{E}_{z \sim \mathcal{N}(0, I_d)}[f(x + \sigma z)] \end{aligned}$$

It is well-known that \hat{f} is Lipschitz (it has uniform bound on the Lipschitz constant). In practice, we can approximate \hat{f} by empirical average

$$y^{(k)} = \sum_{i=1}^k f(x + \sigma z), \quad \text{where } z \sim \mathcal{N}(0, I_d)$$

It can be shown that if $k \rightarrow \infty$, $y^{(k)}$ almost surely converges to \hat{f} .

Lipschitz Constant of Randomized Smoothed Classifier

Recall: Expected Value

The expectation, or expected value, of some function $f(x)$ with respect to a probability distribution $P_X(x)$ is the average, or mean value, that f takes on when x is drawn from P .

For discrete random variable X , $P_X(x)$ is **Probability Mass Function (PMF)** and expected value can be computed with a summation:

$$\mathbb{E}_{X \sim P}[f(x)] = \sum_x f(x)P_X(x)$$

For continuous random variable X , $P_X(x)$ is **Probability Density Function (PDF)** and expected value is computed with an integral:

$$\mathbb{E}_{X \sim P}[f(x)] = \int f(x)P_X(x)dx$$

Lipschitz Constant of Randomized Smoothed Classifier

Recall

- If $z \sim \mathcal{N}(0, I_d)$, then $\mu + \sigma z \sim \mathcal{N}(\mu, \sigma^2 I_d)$.
- The identity matrix is often denoted by I_n , where n is the dimension. The determinant of the identity matrix is 1.
- The Probability Density Function (PDF) of multivariate normal distribution

$$f_X(x) = \frac{1}{(2\pi)^{d/2} \det(\Sigma)^{1/2}} \exp\left\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right\}$$

where X is normally distributed random variable with mean $\mu \in \mathbb{R}^d$ and covariance $\Sigma \in \mathbb{R}^{d \times d}$, and d is the dimension of x .

The Cumulative Distribution Function (CDF) of multivariate normal distribution

$$P(X \leq x) = F_X(x) = \int_{-\infty}^x f_X(t) dt$$

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It can be shown that if $k \rightarrow \infty$, $y^{(k)}$ almost surely converges to \hat{f} .

We have

$$\hat{f}(x) = \mathbb{E}_{z \sim \mathcal{N}(0, I_d)}[f(x + \sigma z)] = \frac{1}{(2\pi)^{d/2}} \int f(x + \sigma z) \exp\left\{-\frac{\|z\|_2^2}{2}\right\} dz$$

$\hat{f}(x)$ is the weighted average of $f(x)$ in the vicinity of x .

Lipschitz Constant of Randomized Smoothed Classifier

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Change of variable: $w = -\sigma z$.

Lipschitz Constant of Randomized Smoothed Classifier

Recall: change of variable

Double Integral.

Suppose that we want to integrate $f(x, y)$ over the region R . Under the transformation $x = g(u, v), y = h(u, v)$ the region becomes S and the integral becomes

$$\iint_R f(x, y) \, dx dy = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du dv$$

where $\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the absolute value of inverse Jacobian determinant of the transformation.

Vector Integral.

Suppose $x, y \in \mathbb{R}^d$ that we want to integrate $f(x)$ over the region R . Under the transformation $y = g(x)$ the region becomes S and the integral becomes

$$\int_R f(x) \, dx = \int_S f(g(x)) \left| \frac{\partial x}{\partial y} \right| \, dy$$

where $\left| \frac{\partial x}{\partial y} \right|$ is the absolute value of inverse Jacobian determinant of the transformation.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = \frac{1}{(2\pi)^d} \int f(x + \sigma z) \exp\left\{-\frac{\|z\|_2^2}{2}\right\} dz$$

Change of variable: $w = -\sigma z$. Hence, $dz = \frac{1}{\sigma^d} dw$. We get

$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x + \sigma z) \exp\left\{-\frac{\|z\|_2^2}{2}\right\} dz = \int f(x-w) \frac{1}{(2\pi)^{d/2} \sigma^d} \exp\left\{-\frac{\|w\|_2^2}{2\sigma^2}\right\} dw$$

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Recall: Convolution

Convolution of two functions f and g over a infinite range $(-\infty, +\infty)$ is given by

$$f * g = \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)d\tau = \int_{-\infty}^{+\infty} f(t - \tau)g(\tau)d\tau = g * f$$

where the symbol $[f * g](t)$ denotes convolution of f and g .

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$$\hat{f}(x) = \int f(x-w) g_\sigma(w) dw = \int f(w) g_\sigma(x-w) dw = f * g_\sigma$$

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Recall

Let $f : I \rightarrow R$ be a continuous and differentiable function over some set $I \subseteq \mathbb{R}^d$, if we have $\|f'(x)\|_2 \leq m$ for all $x \in I$, then m is the upper Lipschitz constant of f ($L \leq m$).

Hence, We calculate $\|\nabla_x \hat{f}(x)\|_2$ in order to find the **upper bound on Lipschitz constant** of \hat{f} .

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = f * g_\sigma$$

$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma)$$

Lipschitz Constant of Randomized Smoothed Classifier

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Thus, we should compute $\nabla_x g_\sigma$

$$\begin{aligned}\nabla_x g_\sigma &= \nabla_x\left(\frac{1}{(2\pi)^{d/2}\sigma^d} \exp\left\{-\frac{\|x-w\|_2^2}{2\sigma^2}\right\}\right) = \frac{1}{(2\pi)^{d/2}\sigma^d} \frac{-2(x-w)}{2\sigma^2} \exp\left\{-\frac{\|x-w\|_2^2}{2\sigma^2}\right\} \\ &= \frac{-(x-w)}{\sigma^2} g_\sigma\end{aligned}$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\begin{aligned}\hat{f}(x) &= f * g_\sigma \\ \nabla_x \hat{f}(x) &= \nabla_x(f * g_\sigma)\end{aligned}$$

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We have

$$\nabla_x \hat{f}(x) = \int f(w) \frac{-(x-w)}{\sigma^2} g_\sigma(x-w) dw = \int f(x-w) \frac{-w}{\sigma^2} g_\sigma(w) dw$$

Lipschitz Constant of Randomized Smoothed Classifier

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Change of variable: $w = -\sigma z$. We get

$$\begin{aligned}\nabla_x \hat{f}(x) &= \int f(x + \sigma z) \frac{\sigma z}{\sigma^2} g_\sigma(-\sigma z) \sigma^d dz = \int f(x + \sigma z) \frac{z}{\sigma} \frac{\sigma^d}{(2\pi)^{d/2} \sigma^d} \exp\left\{-\frac{\|-\sigma z\|_2^2}{2\sigma^2}\right\} dz \\ &= \int f(x + \sigma z) \underbrace{\frac{z}{\sigma} \frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{\|z\|_2^2}{2}\right\}}_{\mathcal{N}(0, I_d)} dz = \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [f(x + \sigma z) \frac{z}{\sigma}]\end{aligned}$$

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Lipschitz Constant of Randomized Smoothed Classifier

Recal: Triangel inequality

Triangle Inequality. Let $a_k \in \mathbb{R}^d$,

$$\left| \sum_{k=1}^N a_k \right| \leq \sum_{k=1}^N |a_k|.$$

Corollary: For random variable X , if X has a finite expectation, then

$$|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$$

Proof sketch:

$$|\mathbb{E}[X]| = \left| \sum_x x P_X(x) \right| \stackrel{\text{Triangle Inequality}}{\leq} \sum_x |x| P_X(x) = \mathbb{E}[|X|]$$

Lipschitz Constant of Randomized Smoothed Classifier

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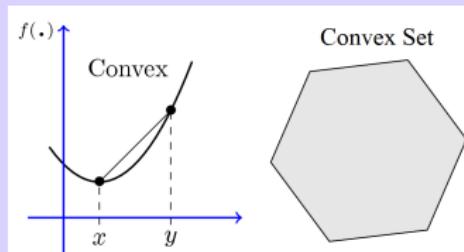
Lipschitz Constant of Randomized Smoothed Classifier

Recall: Convex Function and Jensen's Inequality

Convex set. A set C is convex if the line segment between any two points in C lies in C , i.e., if for any $x_1, x_2 \in C$ and any $\lambda \in [0, 1]$, we have: $\lambda x_1 + (1 - \lambda)x_2 \in C$

Convex Function. Consider a function $f : I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is a convex set. We say that f is a convex function if, for any two points x and y in I and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$



Jensen's Inequality. If $f(x)$ is a convex function on I , and $\mathbb{E}[f(X)]$ and $f(\mathbb{E}[X])$ are finite, then

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Lipschitz Constant of Randomized Smoothed Classifier

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Note: Since $f(x) = x^2$ is a convex function, for random variable X , we have

$$f(\mathbb{E}[g(X)]) \underset{\text{Jensen's Inequality}}{\leq} \mathbb{E}[f(g(X))] \Rightarrow \mathbb{E}^2[g(X)] \leq \mathbb{E}[g^2(X)] \Rightarrow \mathbb{E}[g(X)] \leq \sqrt{\mathbb{E}[g^2(X)]}$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$L_{\hat{f}} \leq \frac{1}{\sigma} \sqrt{\mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [\|z\|_2^2]}$$

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We know

$$\begin{aligned} \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [\|z\|_2^2] &= \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [z_1^2 + z_2^2 + \dots + z_d^2] \\ &= \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [z_1^2] + \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [z_2^2] + \dots + \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [z_d^2] = 1 + 1 + \dots + 1 = d \end{aligned}$$

Recall: For random variable X, $\text{var}(X) = E(X^2) - E[X]^2$

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Finally, we have

$$L_{\hat{f}} \leq \frac{\sqrt{d}}{\sigma}$$

However, since the upper bound depends on the dimension of x , it increases as d grows. As $L_{\hat{f}}$ increases, the radius of our certify bound decreases.

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The main issue is that we calculated the Lipschitz constant working for all x in the input space. We know that the robustness of various inputs is different. Hence, we should compute the **local Lipschitz constant** that depends on specific input x .

Local Lipschitz Constant

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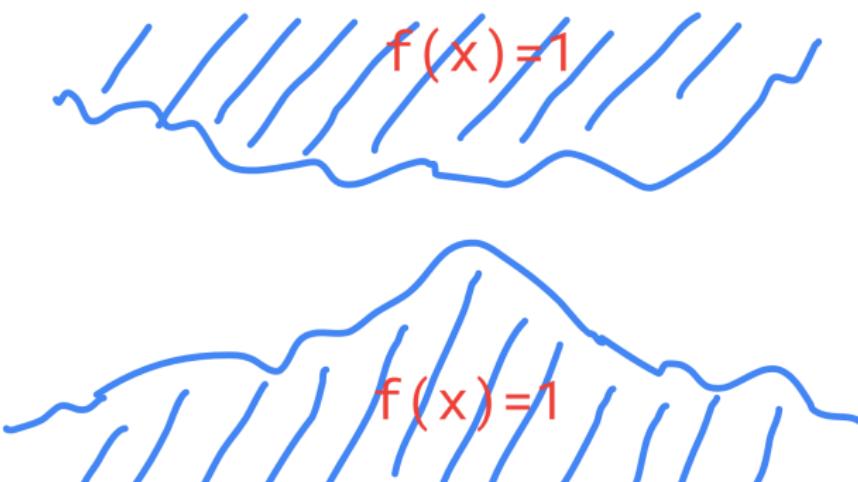
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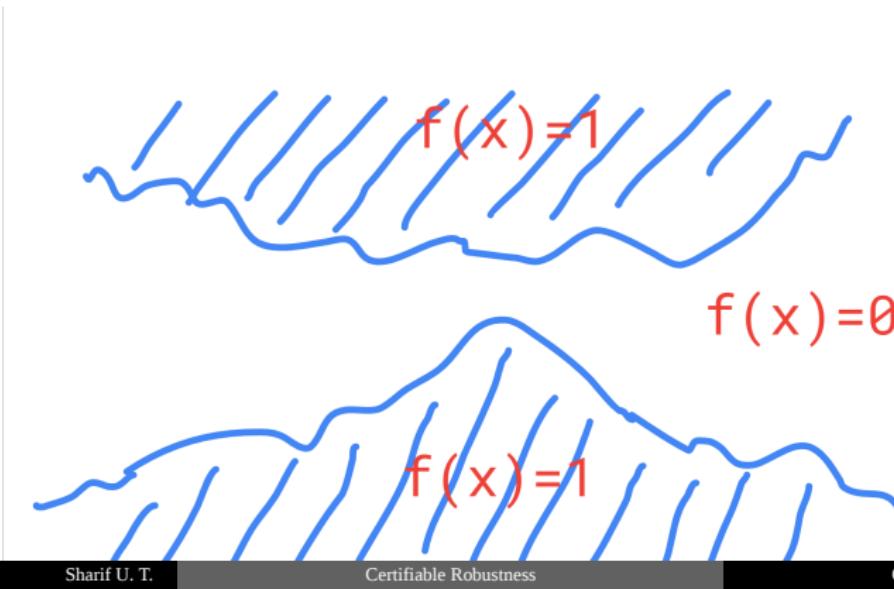
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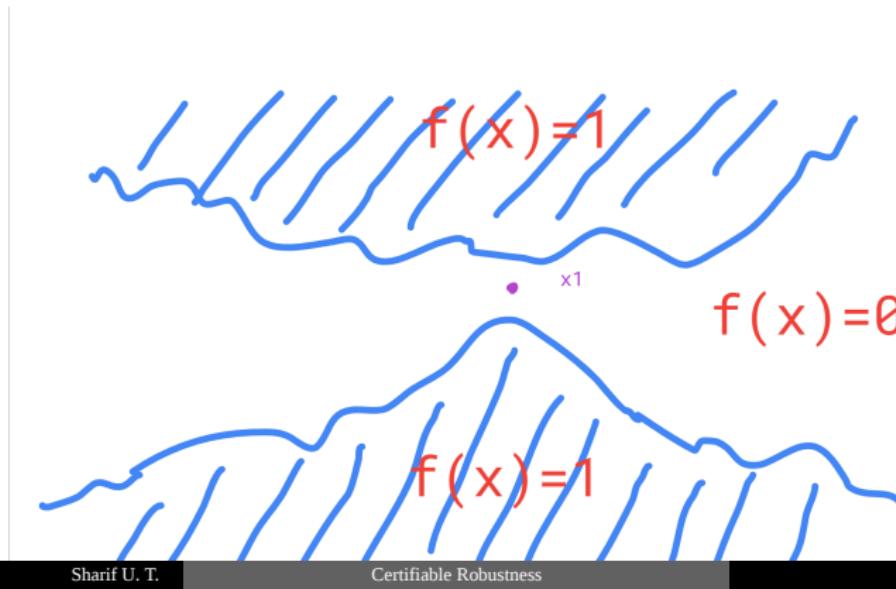
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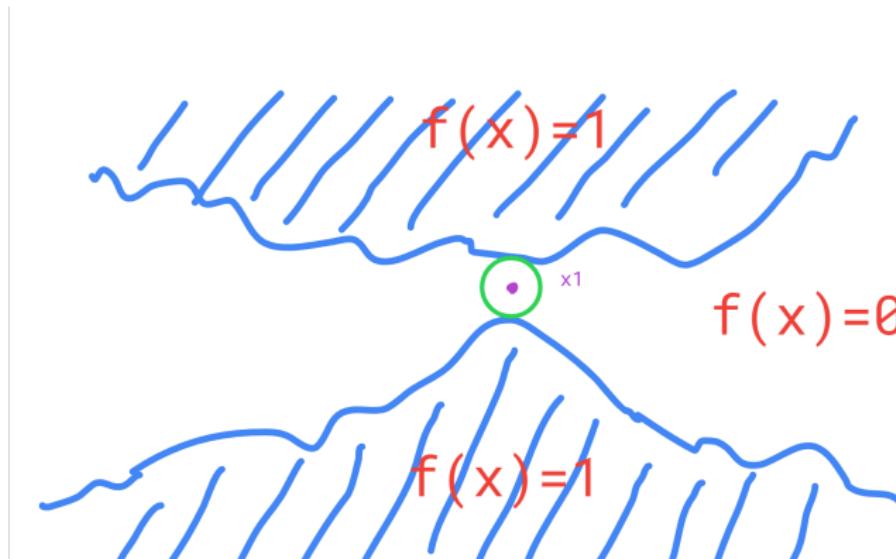
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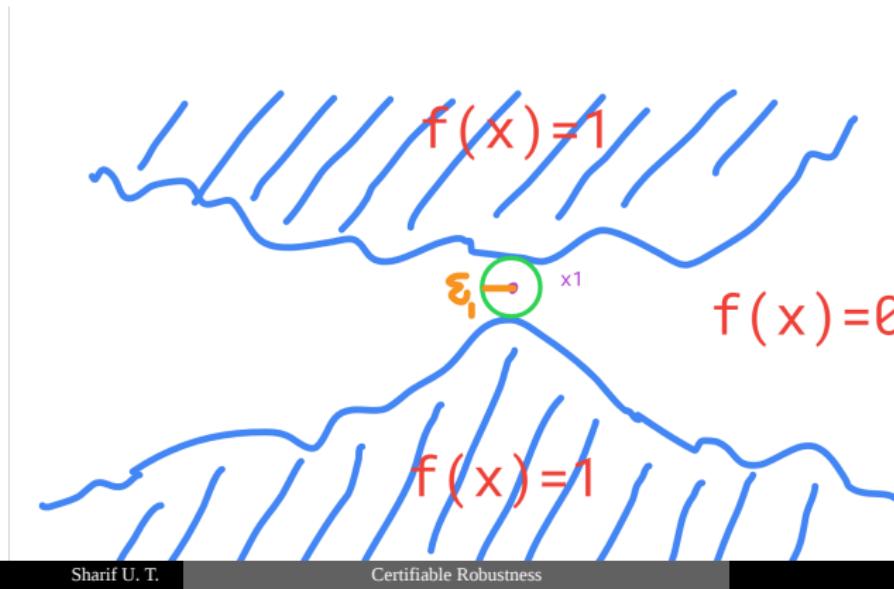
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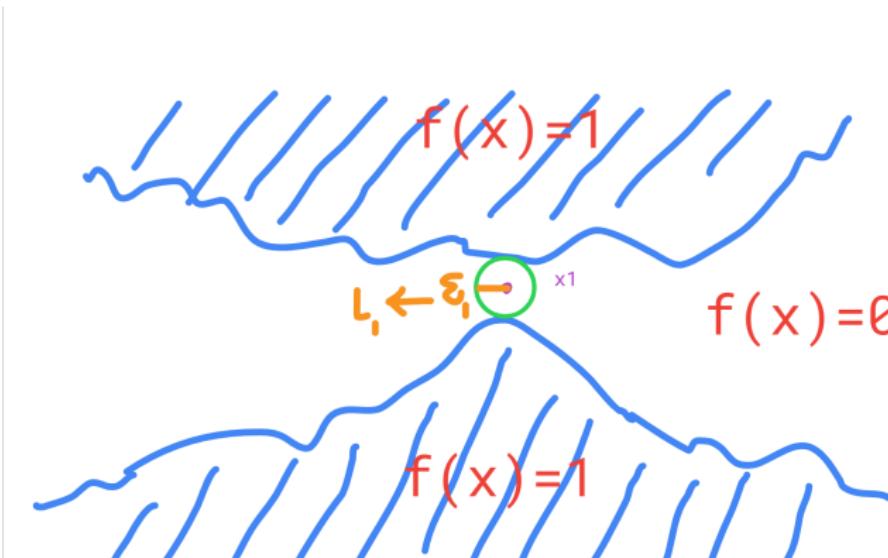
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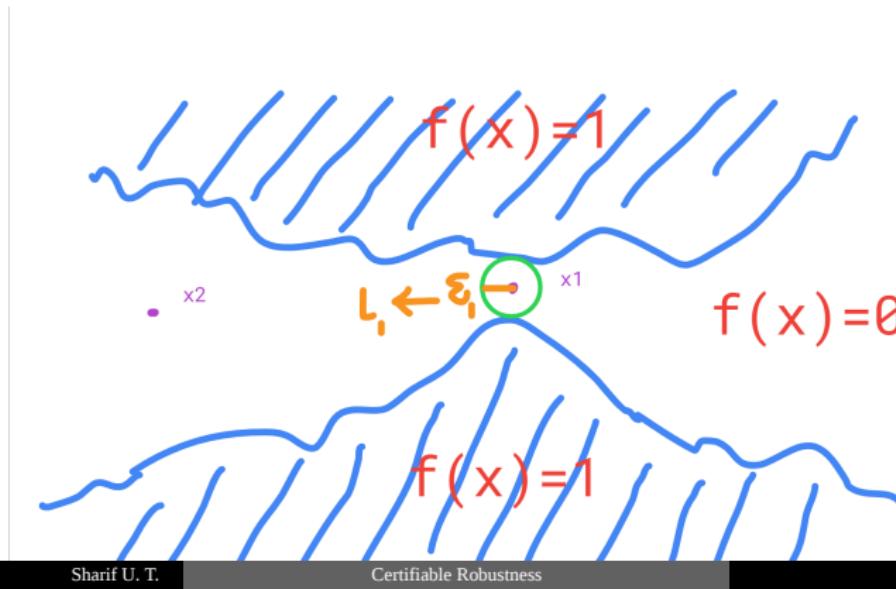
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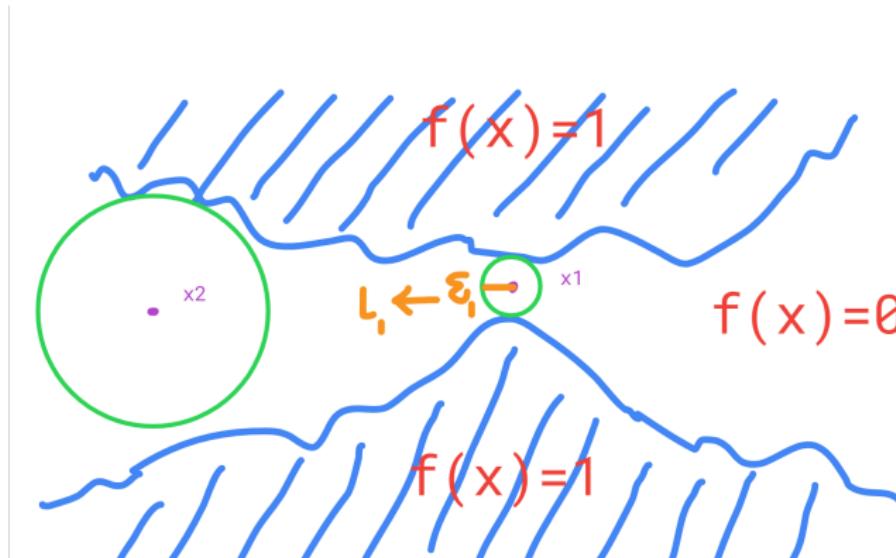
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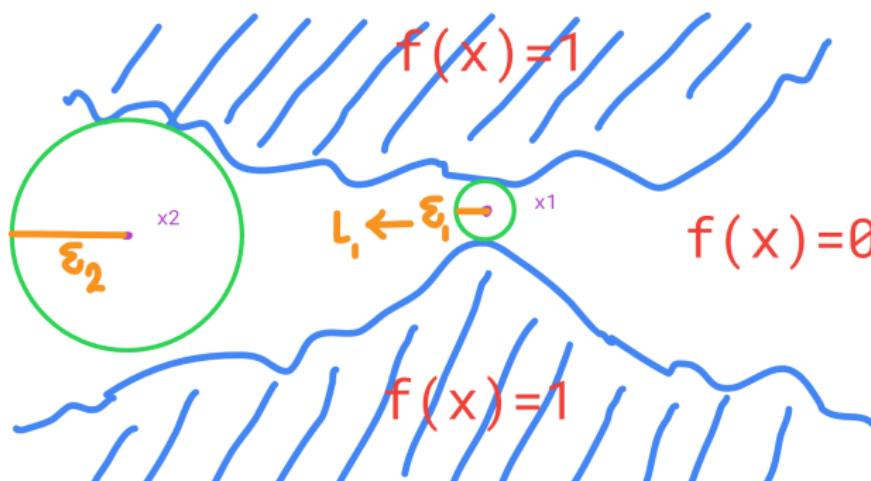
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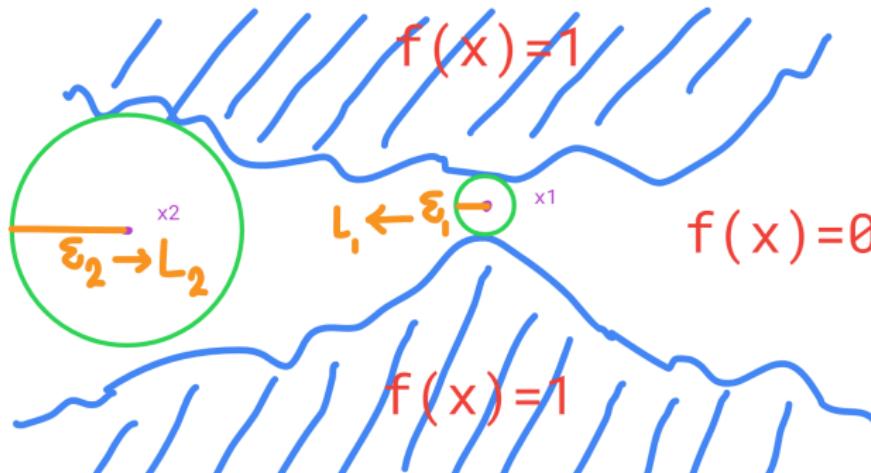
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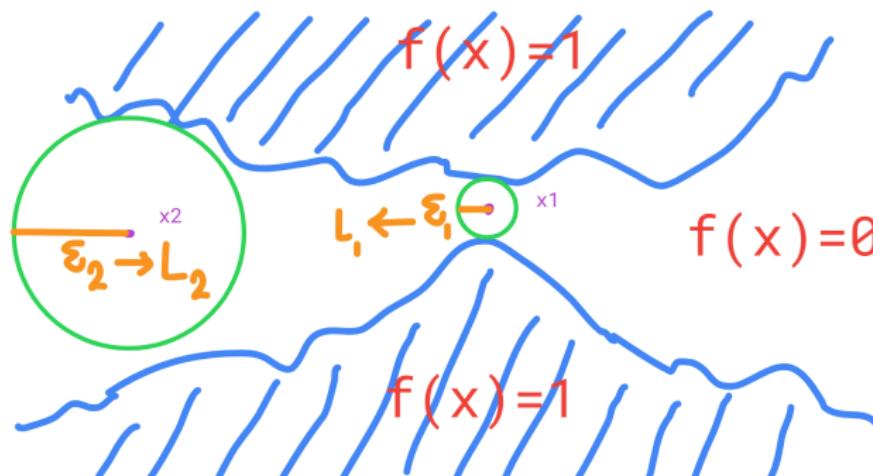
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Local Lipschitz Constant

We can write L_2 as the supremum of an inner product

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} u \cdot \nabla_x \hat{f} \quad (\text{where } u^* = \frac{\nabla_x \hat{f}}{\|\nabla_x \hat{f}\|_2})$$

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We know $\nabla_x \hat{f}(x) = \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [\frac{z}{\sigma} f(x + \sigma z)]$. Therefor, we have

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Since expectation and inner product are linear operators, we have

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$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [\frac{z \cdot u}{\sigma} f(x + \sigma z)]$$

We want to $\|\nabla_x \hat{f}\|_2$ becomes input-dependent. We assume $p_x = \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [f(x + \sigma z)]$. Thus, we have

$$\begin{aligned} \|\nabla_x \hat{f}\|_2 &= \sup_{\|u\|_2=1} \mathbb{E}_{z \sim \mathcal{N}(0, I_d)} [\frac{z \cdot u}{\sigma} f(x + \sigma z)] \\ &\text{such that } \mathbb{E}_z [f(x + \sigma z)] = p_x \end{aligned}$$

(See [this video](#) for the rest of proof.)

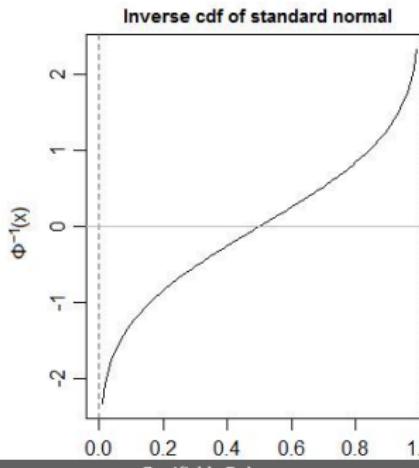
Local Lipschitz Constant

By taking supremum over u and f , the local Lipschitz constant L_x for input x is as follows

$$L_x \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$$

Where Φ is the Cumulative Density Function (CDF) of standard Gaussian distribution $\mathcal{N}(0, I_d)$.

- The bound is input-dependent.
- The bound does not depend on the input dimension d .
- L_x decreases as p_x rises.
- L_x decreases as σ rises.



Local Lipschitz Constant

We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x) \cdot \nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$

$$\|\nabla_x \psi(\hat{f}(x))\|_2 = \|\nabla_x \hat{f}(x)\|_2 |\nabla_{p_x} \psi(p_x)|$$

where $\nabla_{p_x} \psi(p_x)$ is a scalar. We know $\|\nabla_x \hat{f}(x)\|_2 \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$. In order to the Lipschitz constant of $\psi(\hat{f}(x))$ be $\frac{1}{\sigma}$, $\nabla_{p_x} \psi(p_x)$ should be

$$\frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Local Lipschitz Constant

Recall: The Derivative of the CDF of Standard Normal Distribution

The derivative of the CDF of standard normal distribution $\mathcal{N}(0, I)$

$$\Phi'(t) = \frac{d}{dt} \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Recall: Fundamental Theorem of Calculus

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t) dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

Another way of stating the conclusion of the fundamental theorem of calculus is:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

The conclusion of the fundamental theorem of calculus can be loosely expressed in words as: "the derivative of an integral of a function is that original function", or "differentiation undoes the result of integration".

Local Lipschitz Constant

We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x) \cdot \nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$

$$\|\nabla_x \psi(\hat{f}(x))\|_2 = \|\nabla_x \hat{f}(x)\|_2 |\nabla_{p_x} \psi(p_x)|$$

where $\nabla_{p_x} \psi(p_x)$ is a scalar. We know $\|\nabla_x \hat{f}(x)\|_2 \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$. In order to the Lipschitz constant of $\psi(\hat{f}(x))$ be $\frac{1}{\sigma}$, $\nabla_{p_x} \psi(p_x)$ should be

$$\frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}} = [\Phi'(\Phi^{-1}(p_x))]^{-1}$$

Local Lipschitz Constant

Recall: The Derivative of Inverse Function

Given an invertible function $y = f(x)$, the derivative of its inverse function $f^{-1}(y)$ is

$$[f^{-1}]'(y) = \frac{1}{f'(f^{-1}(y))} = [f'(f^{-1}(y))]^{-1}$$

Proof

To see why this is true, start with the function $x = f^{-1}(y)$. Write this as $y = f(x)$ and differentiate both sides implicitly with respect to y using the Chain Rule

$$1 = f'(x) \cdot \frac{dx}{dy}$$

Thus

$$\frac{dx}{dy} = \frac{1}{f'(x)}$$

but $x = f^{-1}(y)$. Thus,

$$[f^{-1}]'(y) = \frac{1}{f' [f^{-1}(y)]}$$

Local Lipschitz Constant

We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x) \cdot \nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$

$$\|\nabla_x \psi(\hat{f}(x))\|_2 = \|\nabla_x \hat{f}(x)\|_2 |\nabla_{p_x} \psi(p_x)|$$

where $\nabla_{p_x} \psi(p_x)$ is a scalar. We know $\|\nabla_x \hat{f}(x)\|_2 \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$. In order to the Lipschitz constant of $\psi(\hat{f}(x))$ be $\frac{1}{\sigma}$, $\nabla_{p_x} \psi(p_x)$ should be

$$\frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}} = [\Phi'(\Phi^{-1}(p_x))]^{-1} = [\Phi^{-1}]'(p_x)$$

Thus

$$\psi(p_x) = \Phi^{-1}(p_x) + c$$

where c is a constant (we set $c = 0$).

Certifiable Robustness for $1/\sigma$ -Lipschitz smoothed classifier

Since, the Lipschitz constant of $\Phi^{-1}(\hat{f}(x))$ is $\frac{1}{\sigma}$, similar to **Theorem 0**, we can show that

$$\operatorname{argmax}_i \hat{f}_i(x + \delta) = \operatorname{argmax}_i \hat{f}_i(x)$$

for all $\|\delta\|_2 \leq R$, where

$$R = \frac{1}{2L}(\Phi^{-1}(P_A) - \Phi^{-1}(P_B)) = \frac{\sigma}{2}(\Phi^{-1}(P_A) - \Phi^{-1}(P_B)),$$

$P_A = \max_i \hat{f}_i(x)$, $P_B = \max_{j \neq i} \hat{f}_j(x)$, and $\hat{f}_k(x)$ is the k -th element of the probability vector $\hat{f}(x)$.

Training the Base Classifier

Theorem 1 holds regardless of how the base classifier f is trained.

- If the base classifier f is trained via standard supervised learning on the data distribution, it will see **no noisy images during training**, and hence **will not necessarily learn to classify $\mathcal{N}(x, \sigma^2 I)$ with x 's true label**.
- Therefore, we **train the base classifier with Gaussian data augmentation** at variance σ^2 .

Experiments

Evaluation metric is **certified test set accuracy** at radius r defined as

- The fraction of the test set which \hat{f} classifies correctly with a prediction that is certifiably robust within an ℓ_2 ball of radius r .
- Thus, to compute **certified accuracy**, we pick a target radius T and count the number of points in the test set whose certified radius $r \geq T$ and where the predicted c_A matches the test set label. Standard accuracy is instantiated with $T = 0$.

CIFAR10 model

- ResNet-110
- $n = 100000$ (use n samples from $f(x + \epsilon)$ to obtain some \underline{P}_A and \overline{P}_B .)
- Certifying each example took 15 seconds on an NVIDIA RTX 2080 Ti.

ImageNet model

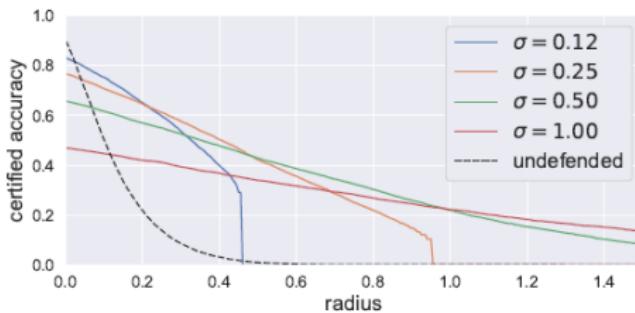
- ResNet-50
- $n = 100000$
- certifying each example took 110 seconds.

Experiments

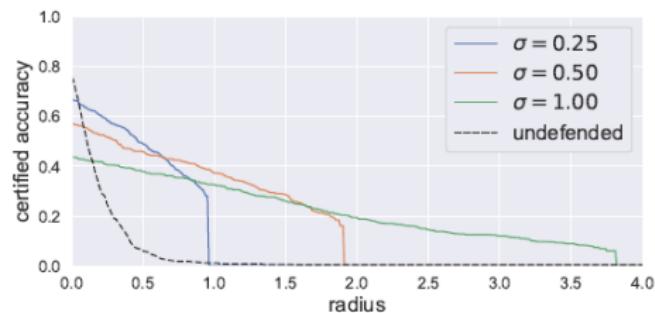
	$r = 0.0$	$r = 0.5$	$r = 1.0$	$r = 1.5$	$r = 2.0$	$r = 2.5$	$r = 3.0$
$\sigma = 0.25$	0.67	0.49	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.50$	0.57	0.46	0.37	0.29	0.00	0.00	0.00
$\sigma = 1.00$	0.44	0.38	0.33	0.26	0.19	0.15	0.12

Table 2. Approximate certified test accuracy on ImageNet. Each row is a setting of the hyperparameter σ , each column is an ℓ_2 radius. The entry of the best σ for each radius is bolded. For comparison, random guessing would attain 0.001 accuracy.

Experiments



CIFAR10



ImageNet

Figure 6. Approximate certified accuracy attained by randomized smoothing on CIFAR-10 (left) and ImageNet (right). The hyper-parameter σ controls a robustness/accuracy tradeoff. The dashed black line is an upper bound on the empirical robust accuracy of an undefended classifier with the base classifier's architecture.

Gaussian Smoothing for L_P Attacks

If we use Gaussian smoothing against L_P attacks, for $p \geq 2$ we get

$$r_p = \frac{\sigma}{2d^{\frac{1}{2} - \frac{1}{p}}} (\Phi^{-1}(p_1(x)) - \Phi^{-1}(p_2(x)))$$

Curse of dimensionality: For L_P attacks where $p > 2$, the smoothing-based certificate upper bound decreases as d increases.

(See: Curse of Dimensionality on Randomized Smoothing for Certifiable Robustness, Kumar et al., 2020)