Faculty of Sciences
Faculty of Mathematics, Statistics, and Computer Science
Stochastic Process Assignment 1
Due Date: Wednesday, 12th March

Problem 1. Suppose we have two boxes and $2d$ balls, of which d are black and d are red. Initially, d of the balls are placed in box 1, and the remainder of the balls are placed in box 2. At each trial, a ball is chosen at random from each of the boxes, and the two balls are put back in the opposite boxes. Let X_0 denote the number of black balls initially in box 1 and, for $n \geq 1$, let X_n denote the number of black balls in box 1 after the n-th trial. Find the transition function of the Markov chain X_n, $n \geq 0$.

Problem 2. Let $\{X_n\}$ be a homogeneous Markov chain on a space of states $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.8 & 0.2 & 0 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

Compute the following probabilities:

1. $P(X_1 = 2, X_2 = 3, X_3 = 1, X_4 = 3, X_5 = 1, X_6 = 3, X_7 = 2 \mid X_0 = 3)$

2. $P(X_{n+2} = 3 \mid X_n = 2),$ for $n \geq 0$

3. $P(X_{n-1} = 1, X_{n+1} = 1 \mid X_{n-2} = 1),$ for $n \geq 2$

Problem 3. Consider the Ehrenfest chain, which is a simple model of the exchange of heat or gas molecules between two isolated bodies. Suppose we have two boxes, labeled 1 and 2, and d balls labeled $1, 2, \ldots, d$. Initially, some of these balls are in box 1 and the remainder are in box 2. An integer is selected at random from $1, 2, \ldots, d$, and the ball labeled by that integer is removed from its box and placed in the opposite box. This procedure is repeated indefinitely with the selections being independent from trial to trial. Let X_n denote the number of balls in box 1 after the n-th trial. Then $X_n, n \geq 0$, is a Markov chain on $\mathcal{S} = \{0, 1, 2, \ldots, d\}$.

1. Write the transition probability function for the Ehrenfest chain.

2. Let $X_n, n \geq 0$, be the Ehrenfest chain and suppose that X_0 has a binomial distribution with parameter d and $\frac{1}{2}$, i.e.

$$P(X_0 = x) = \binom{d}{x} \left(\frac{1}{2}\right)^d, \quad x = 0, 1, \ldots, d.$$

   Find the distribution of X_1.

Problem 4. Let $X_n, n \geq 0$ be a Markov chain. Show that

$$P(X_0 = x_0 \mid X_1 = x_1, \ldots, X_n = x_n) = P(X_0 = x_0 \mid X_1 = x_1).$$

Problem 5. Consider an experiment of mating rabbits. We watch the evolution of a particular gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid–the order is irrelevant, so gG is the same as Gg) or gg (recessive). In mating two rabbits, the offspring inherits a gene from each of its parents with equal probability. Thus, if we mate a dominant (GG) with a hybrid (Gg), the offspring is dominant with probability 1/2 or hybrid with probability 1/2.

Start with a rabbit of a given character (GG, Gg, or gg) and mate it with a hybrid. The offspring produced is again mated with a hybrid, and the process is repeated through a number of generations, always mating with a hybrid.

1. Find P^n by first calculating $P^2, P^3, \ldots$ and identifying the pattern.

2. **Bonus:** Now use Eigenvalue decomposition or Singular Value Decomposition (SVD) to find P^n.