**Faculty of Sciences**

**Faculty of Mathematics, Statistics, and Computer Science**

# Stochastic Process Assignment 11

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**Problem 1.** Consider a birth and death process $X(t)$, $t \geq 0$, such as the branching process, that has state space $\{0, 1, 2, \ldots\}$ and birth and death rates of the form

$$\lambda_x = x\lambda \quad \text{and} \quad \mu_x = x\mu, \quad x \geq 0,$$

where $\lambda$ and $\mu$ are nonnegative constants. Set

$$m_x(t) = \mathbb{E}_x(X(t)) = \sum_{y=0}^{\infty} y P_{xy}(t).$$

(a) Write the forward equation for the process.

(b) Use the forward equation to show that $m_x'(t) = (\lambda - \mu) m_x(t)$.

(c) Conclude that

$$m_x(t) = x e^{(\lambda - \mu)t}.$$

**Problem 2.** Let $X(t)$, $t > 0$, be as in previous question. Set

$$s_x(t) = \mathbb{E}_x(X^2(t)) = \sum_{y=0}^{\infty} y^2 P_{xy}(t).$$

(a) Use the forward equation to show that

$$s_x'(t) = 2(\lambda - \mu) s_x(t) + (\lambda + \mu) m_x(t).$$

(b) Find $s_x(t)$.

(c) Find $\text{Var}(X(t))$ under the condition that $X(0) = x$.

**Problem 3.** A factory has three machines in use and one repairman. Suppose each machine works for an exponential amount of time with mean 60 days between breakdowns, but each breakdown requires an exponential repair time with mean 4 days. Let $\{X_t\}_{t\geq 0}$ be a Markov jump process to describe the number of working machines. Find the rate matrix $D$ and Markov matrix $Q$.

**Problem 4.** Let $\xi_n$, $n = 0, 1, \ldots$, be a two-state Markov chain with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1-\alpha & \alpha \end{pmatrix}.$$

Let $\{N(t); t \geq 0\}$ be a Poisson process with parameter $\lambda$. Show that

$$X(t) = \xi_{N(t)}, \quad t \geq 0,$$

is a two-state birth and death process and determine the parameters $\lambda_0$ and $\mu_1$ in terms of $\alpha$ and $\lambda$.

**Problem 5.** Consider a birth and death process having three states 0, 1, and 2, and birth and death rates such that $\lambda_0 = \mu_2$. Use the forward equation to find $P_{0y}(t)$, $y = 0, 1, 2$.