Faculty of Sciences

Faculty of Mathematics, Statistics, and Computer Science

Stochastic Process Tutorial

Problem 1. Arrivals of passengers at a bus stop form a Poisson process $X(t)$ with rate $\lambda = 2$ per unit time. Assume that a bus departed at time $t = 0$, leaving no customers behind. Let T denote the arrival time of the next bus. Then the number of passengers present when it arrives is $X(T)$. Suppose that the bus arrival time T is independent of the Poisson process and that T has the uniform probability density function

$$f_T(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

- Determine the conditional moments $\mathbb{E}[X(T) \mid T = t]$ and $\mathbb{E}[X(T)^2 \mid T = t]$.
- Determine the mean $\mathbb{E}[X(T)]$ and variance $\mathrm{Var}[X(T)]$.

Problem 2. Shocks occur to a system according to a Poisson process of rate λ. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time t?

Problem 3. Let $X(t)$ and $Y(t)$ be two independent homogeneous Poisson processes with intensities $\lambda_X > 0$ and $\lambda_Y > 0$.

- Verify that $m_{X(t)}(s) = \mathbb{E}[e^{sX(t)}] = e^{t\lambda_X(e^s - 1)}$ for all $t \ge 0$, $s \in \mathbb{R}$.
- For $0 < s < t$, determine the conditional distribution of $X(s)$ given $X(t)$ in terms of a common distribution.
  (Find $\mathbb{P}\{X(s) = k \mid X(t) = n\}$.)

- Let T be a random variable having an exponential distribution with parameter ν. Suppose that for all times $t \in [0, 1)$, T and $X(t)$ are independent. Find the distribution of the random variable $X(T)$, i.e., find $\mathbb{P}\{X(T) = n\}$.

- Find the probability that $X(t)$ jumps to 1 before $Y(t)$ does.

Problem 4. Suppose $N(t)$ is a Poisson process with rate parameter λ.

(a) Let $S_1 < S_2 < \ldots$ be the waiting times (arrival epochs), independent of $N(t)$. Find $\mathbb{P}(N(3) = 5 \mid N(7) = 2)$ and $\mathbb{E}[S_k]$ for each $k \in \mathbb{N}$.

(b) Suppose that passengers arrive at a train station according to a Poisson process with rate λ. The only train departs after a deterministic time T. Let W be the combined waiting time for all passengers. Compute $\mathbb{E}[W]$.

Problem 5. You are given the following information:

- Taxis arrive according to a Poisson process at a rate of 12 per hour.

- Buses arrive according to a Poisson process at a rate of 6 per hour.

- The arrival of buses and taxis are independent.

- You get a ride to university from either a bus or a taxi, whichever arrives first.

(a) Find the distribution of your waiting time.

(b) Find the probability you will have to wait more than 10 minutes for a ride to university.