1. Basic Master Theorem (First Form)

The basic Master Theorem solves recurrences of the form: $T(n) = aT(n/b) + f(n)$, where $a \geq 1$, $b > 1$. Let $p = \log\_b(a)$. Then: 1. If $f(n) = O(n^{p - \varepsilon})$ for some $\varepsilon > 0 \Rightarrow T(n) = \Theta(n^p)$. 2. If $f(n) = \Theta(n^p) \Rightarrow T(n) = \Theta(n^p \log n)$. 3. If $f(n) = \Omega(n^{p + \varepsilon})$ and the regularity condition $a f(n/b) \leq c f(n)$ (for $c < 1$, n large) holds $\Rightarrow T(n) = \Theta(f(n))$.

Examples

1. $T(n) = 2T(n/2) + n \Rightarrow p = 1 \Rightarrow$ Case 2 $\Rightarrow T(n) = \Theta(n \log n)$. 2. $T(n) = 3T(n/2) + n \Rightarrow p = \log\_2(3) \approx 1.585 \Rightarrow$ Case 1 $\Rightarrow T(n) = \Theta(n^{1.585})$. 3. $T(n) = 2T(n/2) + 2^n \Rightarrow$ f dominates, regular $\Rightarrow T(n) = \Theta(2^n)$.

2. Extended Master Theorem (Polynomial × Logarithmic Form)

The extended Master Theorem applies when $f(n) = \Theta(n^{\log_b a} \log^k n)$: 1. If $k < 0 \Rightarrow T(n) = \Theta(n^{\log_b a})$. 2. If $k = 0 \Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$. 3. If $k > 0 \Rightarrow T(n) = \Theta(n^{\log_b a} \log^{k} n)$.

Examples

1. $T(n) = 2T(n/2) + n/\log n \Rightarrow k = -1 \Rightarrow T(n) = \Theta(n)$. 2. $T(n) = 2T(n/2) + n \Rightarrow k = 0 \Rightarrow T(n) = \Theta(n \log n)$. 3. $T(n) = 2T(n/2) + n \log^2 n \Rightarrow k = 2 \Rightarrow T(n) = \Theta(n \log^3 n)$.

4. Akra–Bazzi Theorem

The Akra–Bazzi Theorem generalizes the Master Theorem for recurrences of the form: $T(x) = \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) + g(x)$, where: • $a_i > 0$, $0 < b_i < 1$, and $h_i(x) = O(x / \log^2 x)$. • $g(x)$ is positive and continuous. Then: If p satisfies $\Sigma a_i b_i^p = 1$, we have: $T(x) = \Theta(x^p (1 + \int_1^x g(u)/u^{p+1} du))$.

Examples

1. $T(n) = T(n/2) + T(n/4) + n \Rightarrow p$ solves $(1/2)^p + (1/4)^p = 1 \Rightarrow p \approx 0.7 \Rightarrow T(n) = \Theta(n)$. 2. $T(n) = 2T(n/2) + n/\log n \Rightarrow p = 1 \Rightarrow T(n) = \Theta(n \log \log n)$. 3. $T(n) = 3T(n/2) + n \Rightarrow p = \log_2(3) \approx 1.585 \Rightarrow T(n) = \Theta(n^{1.585})$.