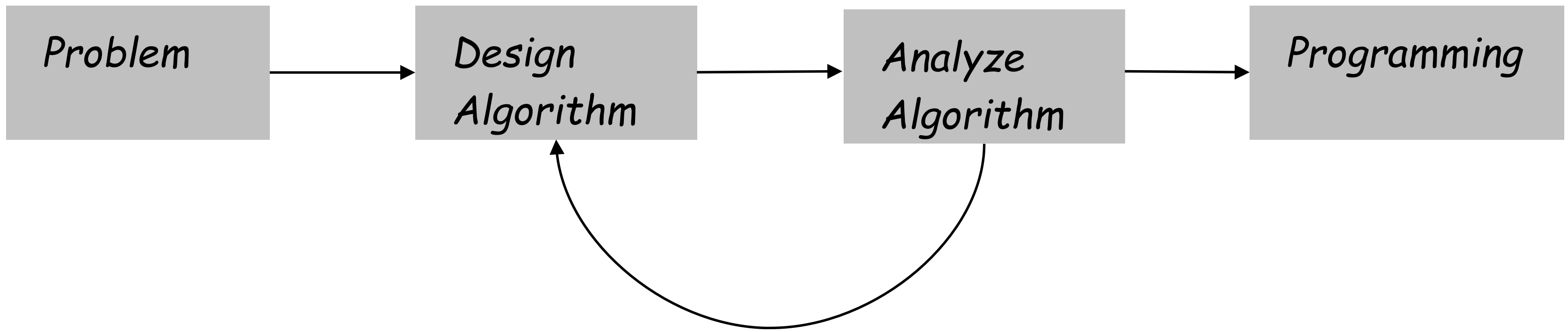
Data Structures & Algorithms

Algorithms Design and Algorithm Analysis

- Introduction
- Fibonacci Number
- Maximum Sum Subarray
- Asymptotic Order of Growth
- A Survey of Common Running Times

Problem Solving



Algorithmic Paradigms

Design and analysis of computer algorithms.

- Greedy.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomized algorithms.

Applications

Wide range of applications.

- Caching.
- Compilers.
- Databases.
- Scheduling.
- Networking.
- Data analysis.
- Signal processing.
- Computer graphics.
- Scientific computing.
- Operations research.
- Artificial intelligence.
- Computational biology.
- . . .

Algorithm Analysis

Analysis refers to mathematical techniques for establishing both the correctness and efficiency of algorithms.

Efficiency: Given an algorithm *A*, we want to know how efficient it is. This includes several possible criteria:

- What is the running time of algorithm *A*?
- Is *A* the most efficient algorithm to solve the given problem? (For example, can we find a lower bound on the running time of any algorithm to solve the given problem?)

Running Time Analysis

The followings affect the running time:

- Hardware
- Compiler
- Input size
- Input arrangement
- …

T(n): the running time when the input size is n

The running time may depend on several variables

Example: T(n, m) for A graph with m edges and n vertices

How Analyze Algorithms?

First option is to compute the execution time for different input sizes

- ***Not good***: times are specific to a particular computer or compiler and not applicable to any input size

Second option is to count the number of primitive operations

- ***good***: but there are several primitive operations and each has its own running time: Add (A=B+C), Multiply (A=B*C), Increment (A=A+1), Assignment (A=B), Comparision (A<B), Logic (A or B), …

We select the second option with the following assumption to make life easier

Assumption: all primitive operations cost 1 unit.

- Eliminates dependence on the speed of our computer, otherwise impossible to verify and to compare

The problem of sorting

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

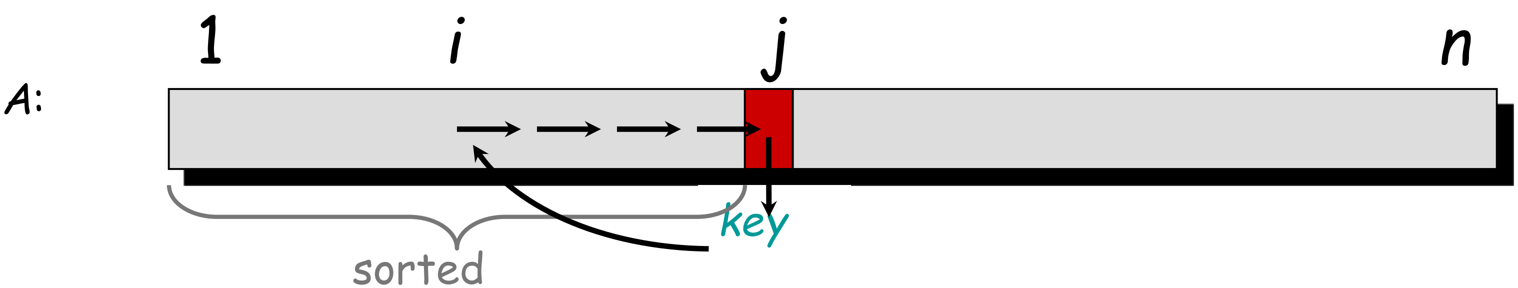
Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)        ⊳ A[1 . . n]
    for j ← 2 to n
         do key ← A[ j]
            i ← j – 1
            while i > 0 and A[i] > key
                 do A[i+1] ← A[i]
                    i ← i – 1
            A[i+1] = key
```



Example of insertion sort

8	2	4	9	3	6

Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

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Example of insertion sort

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Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*

Kinds of analyses

Worst-case: (usually)

- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (NEVER)

- Cheat with a slow algorithm that works fast on some input.

Insertion sort analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} j = n^2$$

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} j/2 = n^2$$

Best case: Input sorted.

$$T(n) = \sum_{j=2}^{n} 1 = n$$

Comparing Functions Using Rate of Growth ≡ Asymptotic Analysis

- Using *rate of growth* as a measure to compare different functions implies comparing them **asymptotically**.

- If $f(x)$ is *faster growing* than $g(x)$, then $f(x)$ always eventually becomes larger than $g(x)$ **in the limit** (for large enough values of x).

Example

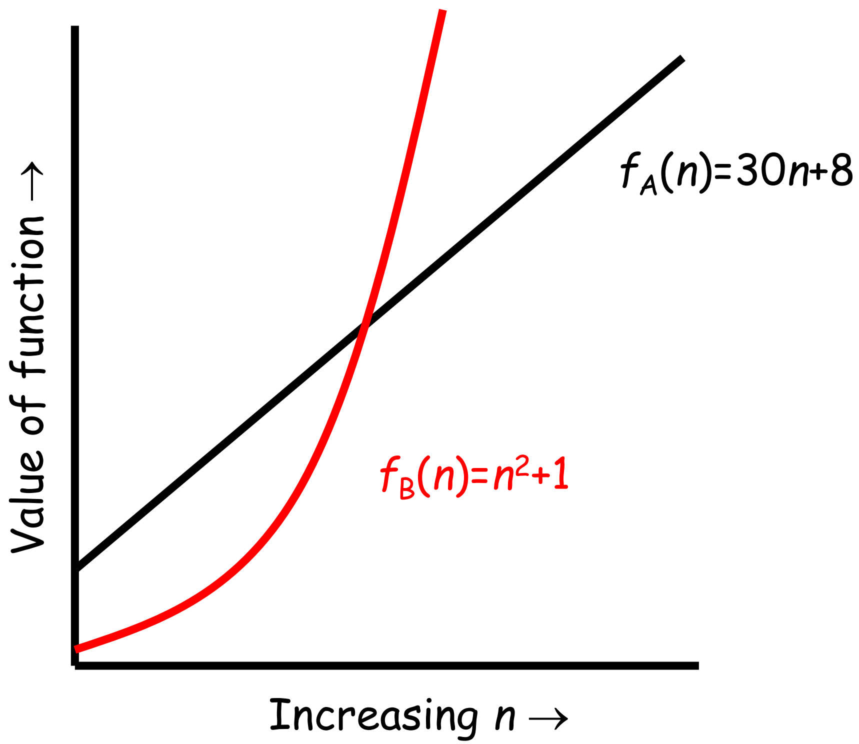
- Suppose you are designing a web site to process user data (*e.g.*, financial records).

- Suppose program A takes $\boldsymbol{f_A(n)=30n+8}$ microseconds to process any n records, while program B takes $\boldsymbol{f_B(n)=n^2+1}$ microseconds to process the n records.

- Which program would you choose, knowing you'll want to support millions of users?

Visualizing Orders of Growth

- On a graph a faster growing function eventually becomes larger...



Big-O Notation

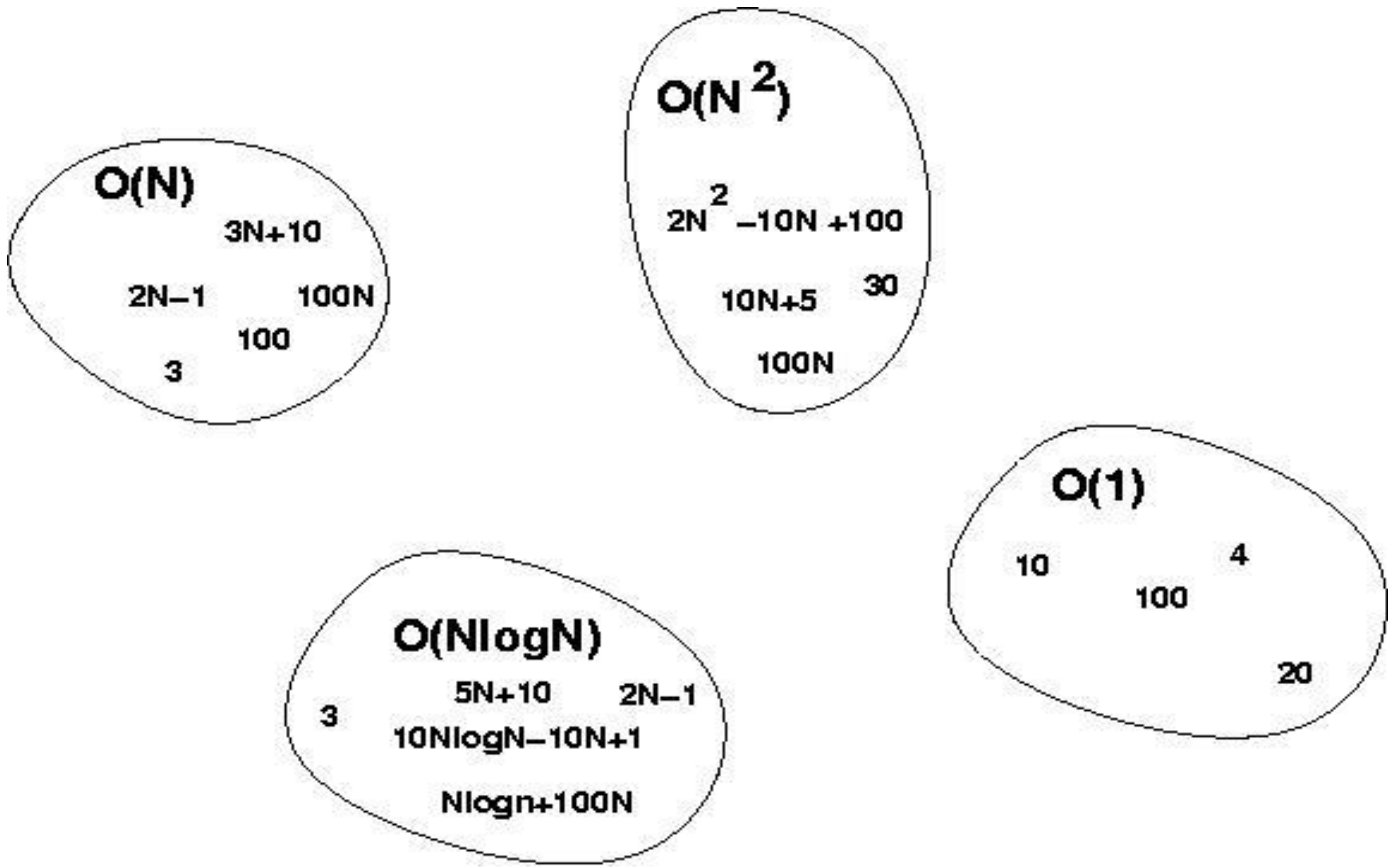
- We say $f_A(n)$=30n+8 is *order n*, or $O(n)$.
-

  It is, **at most**, roughly *proportional* to n.
- $f_B(n)=n^2+1$ is *order* n^2, or $O(n^2)$. It is, **at most**, roughly proportional to n^2.

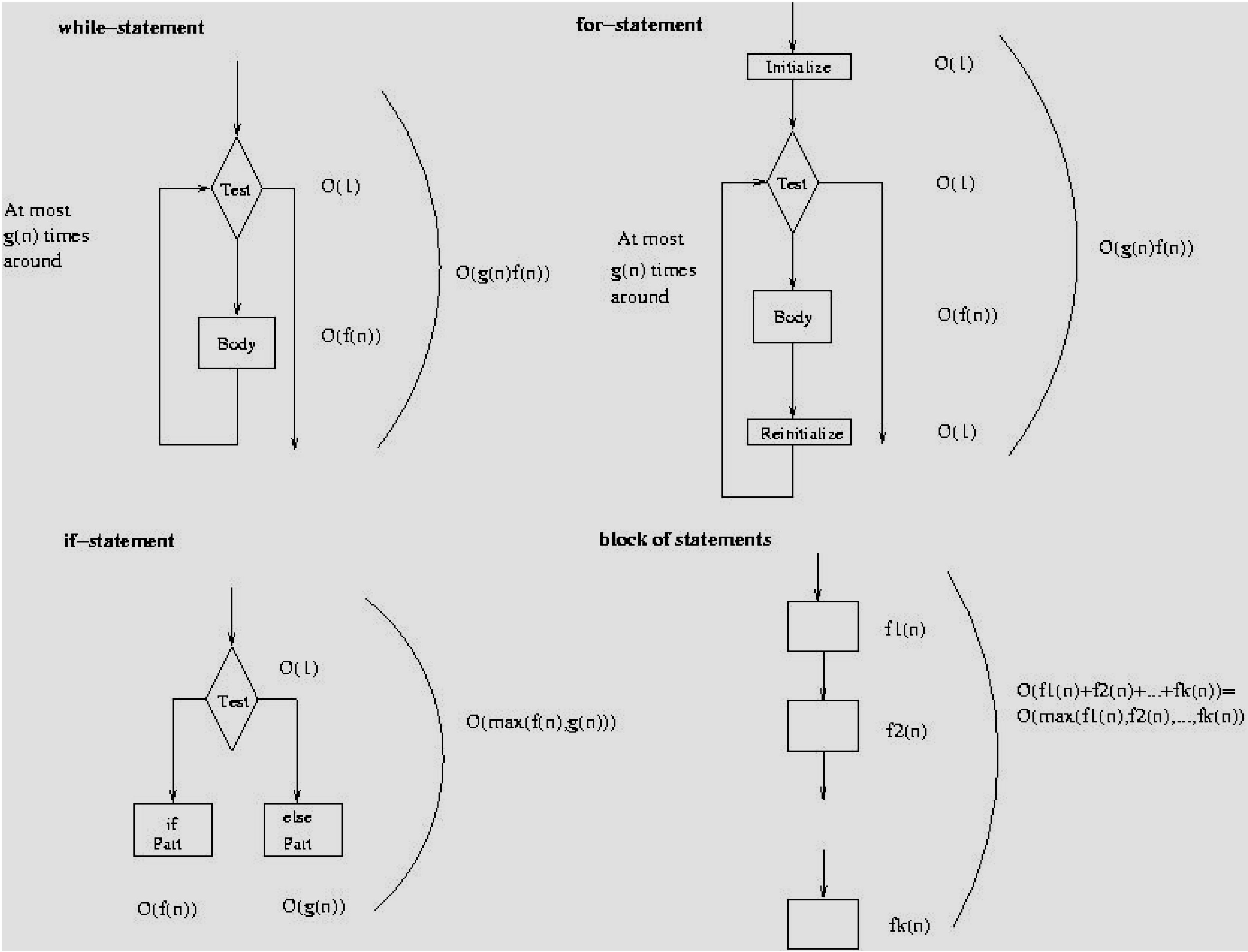
- In general, an $O(n^2)$ algorithm will be <u>slower</u> than $O(n)$ algorithm.

- Note: an $O(n^2)$ function will grow <u>faster</u> than an $O(n)$ function.

Big-O Visualization



Running time of various statements



Polynomial-Time Algorithm

Brute force. For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.

- Typically takes 2^N time or worse for inputs of size N.
- Unacceptable in practice.

Def. An algorithm is *poly-time* if the below property holds.

> There exists constants $c > 0$ and $d > 0$ such that on every input of size N, its running time is bounded by $c N^d$ steps or simply the running time is $O(N^d)$

Worst-Case Polynomial-Time

Def. An algorithm is practical if its running time is polynomial.

Justification: It really works in practice!

- Although $6.02 \times 10^{23} \times N^{20}$ is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.

Exceptions.

- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Fibonacci Number

Definition and Problem

$F_{\mathrm{n}} = F_{n-1} + F_{n-2}, F_1 = F_2 = 1$

Problem: Compute $\mathrm{F_n}$ for given n.

Example: $n = 10$

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$

Algorithm 1

```
F(n):
if n ≤ 2
   return 1
else
  return F(n-1)+F(n-2)
```

- $T(n) = T(n-1) + T(n-2)$
- $T(n) = O(\varphi^n), \varphi = (1+\sqrt{5})/2$

Algorithm 2

```
F(n):
a=b=1
for i = 3 to n {
    b=a+b
    a=b-a
}
return b
```

- $T(n) = O(n)$

Algorithm 3

Use the following observations:

- $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$. So, $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$
- To compute a^n, we can compute only $a, a^2, a^4, a^8, \ldots$

Running time: $\mathrm{T(n)} = \mathrm{O}(\log n)$

Running Time

n	10	20	30	40	50	100
Alg1	44μs	5300μs	0.5s	64s	2.2h	7myrs
Alg2	4μs	7μs	10μs	14μs	18μs	45μs
Alg3	11μs	12μs	13μs	15μs	17μs	21μs

Remark: Assume a computer does 10^8 operations per second. Then, alg1 takes $\frac{\varphi^{100}}{10^8} s = 7myrs$. Note that $\varphi = (1+\sqrt{5})/2$.

Maximum Sum Subarray

Maximum Sum Subarray

Problem: Given a one dimensional array A[1..n] of numbers.
Find a contiguous subarray with largest sum within A.

Assume an empty subarray has sum 0.

Example:

4	-7	12	5	-2	3	-5	1	5	-8	2	5

4	-7	12	5	-2	3	-5	1	5	-8	2	5

Algorithm 1 (brute-force)

```
sol = 0
for i = 1 to n do
    for j = i to n do
        sum = 0
        for k = i to j do
            sum = sum + a[k]
        if sum > sol then
           sol = sum
return sol
```

Running time: $T(n) = O(n^3)$

Algorithm 2 (brute-force)

Observation: Let S[i] = A[1]+…+A[i]. We have A[i]+…+A[j]=S[j]-S[i-1]

```
Pre-Processing
S[0] = 0
for i = 1 to n do
    S[i] = S[i-1]+A[i]
```

Running time of pre-processing: $T(n) = O(n)$

```
sol = 0
for i = 1 to n do
   for j = i to n do
        if S[j]-S[i-1] > sol then
            sol = S[j]-S[i-1]
return sol
```

Running time: $T(n) = O(n^2)$

Asymptotic Order of Growth

Asymptotic Order of Growth

Upper bounds. $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $f(n) \leq c \cdot g(n)$.

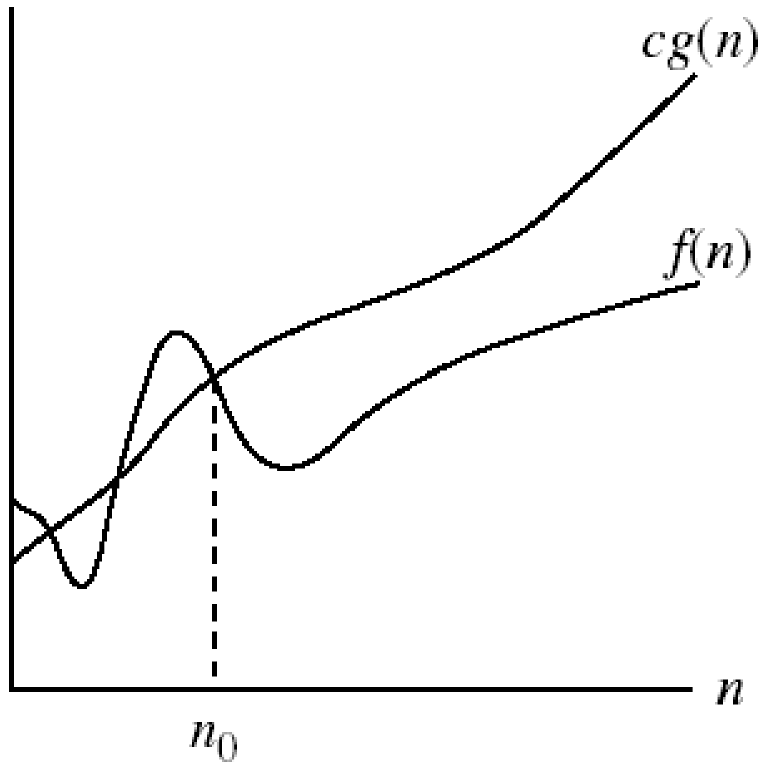
Lower bounds. $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $f(n) \geq c \cdot g(n)$.

Tight bounds. $f(n)$ is $\Theta(g(n))$ if $f(n)$ is both $O(g(n))$ and $\Omega(g(n))$.

Ex: $f(n) = 32n^2 + 17n + 32$.

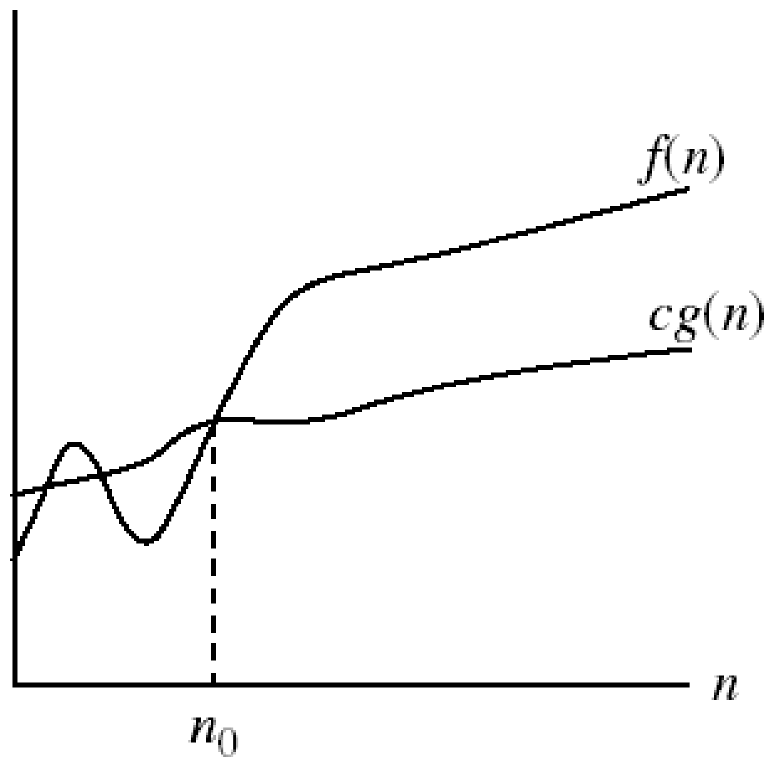
- $f(n)$ is $O(n^2)$, $O(n^3)$, $\Omega(n^2)$, $\Omega(n)$, and $\Theta(n^2)$.
- $f(n)$ is not $O(n)$, $\Omega(n^3)$, $\Theta(n)$, or $\Theta(n^3)$.

Big-O Visualization



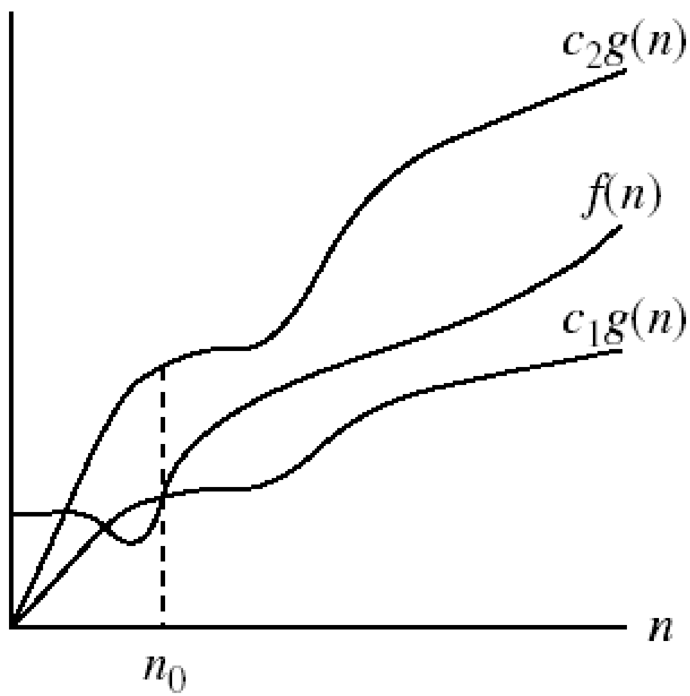
$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Big- Ω Visualization



$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Big- Θ Visualization



$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Notation

Slight abuse of notation. $f(n) = O(g(n))$.

- Not transitive:
  - $f(n) = 5n^3$; $g(n) = 3n^2$
  - $f(n) = O(n^3) = g(n)$
  - but $f(n) \neq g(n)$.
- Better notation: $f(n) \in O(g(n))$.

Meaningless statement. Any comparison-based sorting algorithm requires at least $O(n \log n)$ comparisons.

- Use Ω for lower bounds.

Properties

Transitivity.

- If f = $O(g)$ and $g = O(h)$ then $f = O(h)$.
- If f = $\Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
- If f = $\Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

- If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
- If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
- If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Asymptotic Bounds for Some Common Functions

Polynomials. $a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$.

Polynomial time. Running time is $O(n^d)$ for some constant d independent of the input size n.

Logarithms. $O(\log_a n) = O(\log_b n)$ for any constants $a, b > 0$.

can avoid specifying the base

Logarithms. For every $x > 0$, $\log n = O(n^x)$.

log grows slower than every polynomial

Exponentials. For every $r > 1$ and every $d > 0$, $n^d = O(r^n)$.

every exponential grows faster than every polynomial

A Survey of Common Running Times

Linear Time: O(n)

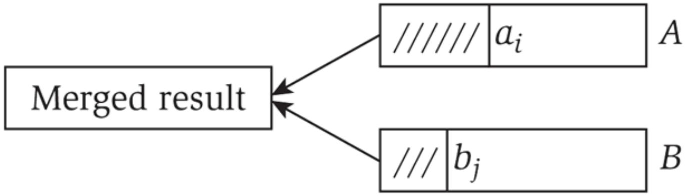
Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of n numbers $a_1, \ldots, a_n$.

```
max ← a_1
for i = 2 to n {
   if (a_i > max)
      max ← a_i
}
```

Linear Time: O(n)

Merge. Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole.




```
i = 1, j = 1
while (both lists are nonempty) {
   if (a_i ≤ b_j) append a_i to output list and increment i
   else           append b_j to output list and increment j
}
append remainder of nonempty list to output list
```

Claim. Merging two lists of size n takes O(n) time.

Pf. After each comparison, the length of output list increases by 1.

O(n log n) Time

O(n log n) time. Arises in divide-and-conquer algorithms.

also referred to as linearithmic time

Sorting. Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons.

Largest empty interval. Given n time-stamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

O(n log n) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

```
min ← (x1 - x2)^2 + (y1 - y2)^2
for i = 1 to n {
   for j = i+1 to n {
      d ← (xi - xj)^2 + (yi - yj)^2
      if (d < min)
         min ← d
   }
}
```

don't need to take square roots

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion.

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, …, n, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```
foreach set Si {
   foreach other set Sj {
      foreach element p of Si {
         determine whether p also belongs to Sj
      }
      if (no element of Si belongs to Sj)
         report that Si and Sj are disjoint
   }
}
```

Polynomial Time: $O(n^k)$ Time

Independent set of size k. Given a graph, are there k nodes such that no two are joined by an edge?

k is a constant

$O(n^k)$ solution. Enumerate all subsets of k nodes.

```
foreach subset S of k nodes {
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
   }
}
```

- Check whether S is an independent set = $O(k^2)$.
- Number of k element subsets = $\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k\,(k-1)\,(k-2)\cdots(2)\,(1)} \le \frac{n^k}{k!}$
- $O(k^2 n^k / k!) = O(n^k)$.

poly-time for k=17,
but not practical

Exponential Time

Independent set. Given a graph, what is maximum size of an independent set?

$O(n^2\, 2^n)$ solution. Enumerate all subsets.

```
S* ← ϕ
foreach subset S of nodes {
   check whether S in an independent set
   if (S is largest independent set seen so far)
      update S* ← S
   }
}
```

References

- Section 5.8 of the text book “introduction to algorithms: a creative approach” by Udi Manber, 1989.
- Section 4.1 of the text book “introduction to algorithms” by CLRS, 3rd edition.
- The original slides were prepared by Kevin Wayne. The slides are distributed by Pearson Addison-Wesley.