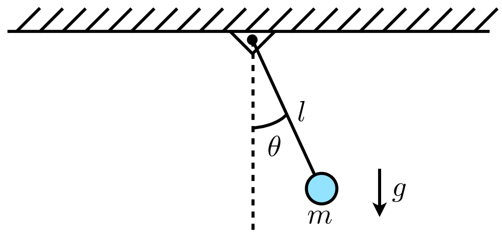
Exercise 3–1: Consider the schematic of the single pendulum.



The kinetic energy T and potential energy V may be written as:

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$
$$V = -gml\cos(\theta)$$

The Lagrangian $\mathcal{L}$ is given by $\mathcal{L} = T - V$, and the Euler-Lagrange equations for the motion of the pendulum are given by the following second order differential equation in θ:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

Write down the second order ODE using the specific T and V defined above. Please write this ODE in the form $\ddot{\theta} = f(\theta, \dot{\theta})$. Notice that this ODE is not linear!!

Now you may assume that $l = m = g = 1$ for the remainder of the problem.

You may still suspend variables to get a system of two first order (nonlinear) ODEs by writing the ODE as:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = f(\theta, \omega)$$

(a) What are the fixed points of this system where all derivatives are zero?

(b) Write down the linearized equations in a neighborhood of each fixed point and determine the linear stability. You may formally linearize the nonlinear ODE or you may use a small angle approximation for $\sin(\theta)$; the two approaches are equivalent.

(c) Sketch the linearized phase portrait in a small neighborhood around each of these fixed points.

(d) How do the above answers match your physical intuition about the fixed points of this system?

Exercise 3–2: Consider the ODE for the pendulum:

$$\ddot{\theta} = -\sin(\theta)$$

Draw the full nonlinear phase portrait for this ODE (i.e., plot trajectories on the θ vs $\dot{\theta}$ axes).

Next, add damping:

$$\ddot{\theta} = -\sin(\theta) - \dot{\theta}$$

Draw the phase portrait for the damped ODE. For both cases, feel free to use `pplane` for help with the phase portraits.

Exercise 3–3: Consider the ODE for the pendulum:

$$\ddot{\theta} = -\sin(\theta)$$

Now, we are going to add stabilizing feedback control, assuming that we can measure θ and $\dot{\theta}$:

$$\ddot{\theta} = -\sin(\theta) + \tau$$

where $\tau = -2\dot{\theta} - 2(\theta - \pi)$ is an applied torque at the base of the pendulum.

What is the new stability of the inverted position, when $\theta = \pi$?

Exercise 3–4: Please solve the following differential equation (with initial conditions) for the three cases below (by hand!). You may use whatever method you find simplest. You may check your work in MATLAB.

$$\begin{aligned}
&\ddot{x} + 3\dot{x} + 2x = f(t), \\
&x(0) = 2, \\
&\dot{x}(0) = -3.
\end{aligned}$$

(a) For $f(t) = 0$. Note that this is just the unforced ODE $\ddot{x} + 3\dot{x} + 2x = 0$.

(b) For $f(t) = 2e^{-3t}$.

(c) For $f(t) = 20\sin(t)$. Hint: try a particular solution $x_P = A\cos(t) + B\sin(t)$ and solve for A and B.

In all cases, be sure to make sure that your initial conditions are still satisfied!

ME564 - Autumn 2016 **Due: 4:00:00pm Friday November 4^{th}, 2016**

Homework 3

Exercise 3–5: Consider the physical system of a bead constrained to move in the potential field $\mathbb{V}(x)$. Note: there are no equations given, and so you must use physical intuition.

- Please sketch the phase portrait (position x vs. velocity v). Try to make your sketch as accurate as possible using as much information about the potential as you can. In both cases, you will draw the phase portrait with and without damping.
- How many fixed points does the system have and what is their stability?
- What can we say about the eigenvalues of the linearization around each point?
- Finally, please pick a trajectory on the phase portrait and explain in one sentence what this trajectory means physically.