Exercise 3-1

$T = \frac{1}{2} m L^2 \dot{\theta}^2 \qquad V = -gmL\cos\theta$

$\mathcal{L} = T - V = \frac{1}{2} m L^2 \dot{\theta}^2 + gmL\cos\theta$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \longrightarrow mL^2\ddot{\theta} + gmL\sin\theta = 0$

$\frac{\partial \mathcal{L}}{\partial \theta} \to -gmL\sin\theta$

$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} \longrightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta}$

Thus, $\ddot{\theta} = -\frac{g}{L}\sin\theta$

$g = L = 1 \longrightarrow \ddot{\theta} = -\sin\theta$

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\sin\theta \end{cases}$$

Fixed points $\dot{\theta} = 0$ and $\dot{\omega} = 0 \to \sin\theta = 0 \longrightarrow \theta = n\pi \to \theta = 0, \pi$

fixed points $\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$$\frac{Df}{D\underset{\sim}{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix} \qquad \underset{\sim}{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \text{ or } \underset{\sim}{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

a) $\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \cos\theta = \cos(0) = 1 \qquad \frac{Df}{D\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \frac{Df}{D\underset{\sim}{x}}(\bar{x}) \cdot (x - \bar{x})$

thus $\dot{\theta} = \omega$
and
$\dot{\omega} = \ddot{\theta} = -\theta$

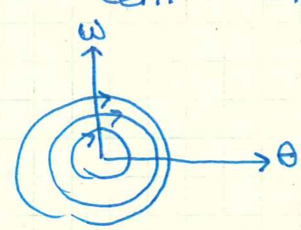
b) Similarly for $\begin{bmatrix} 0 \\ \pi \end{bmatrix}$ $\qquad \dot{\theta} = \omega$

$\ddot{\theta} = (\theta - \pi)$

---

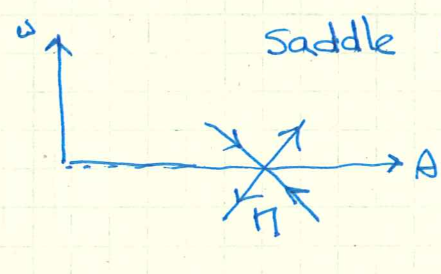
for $\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \frac{Df}{D\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \lambda^2 + 1 = 0 \qquad \lambda = \pm i$

Center stable



for $\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \end{bmatrix}$ $\qquad \frac{Df}{D\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda^2 - 1 = 0 \qquad \lambda = \pm 1$

Saddle unstable

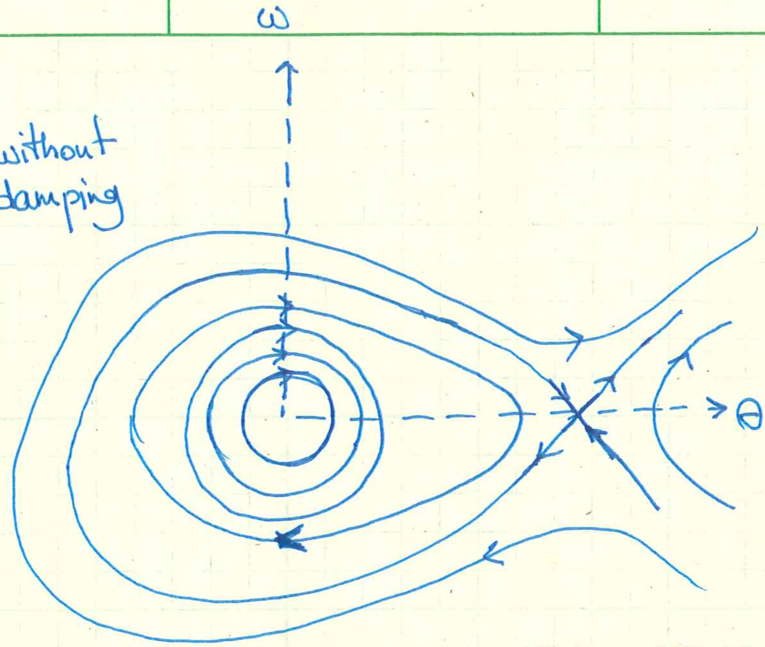


note that our intution tells us that the fixed point at top is unstable while the one at the bottom is stable

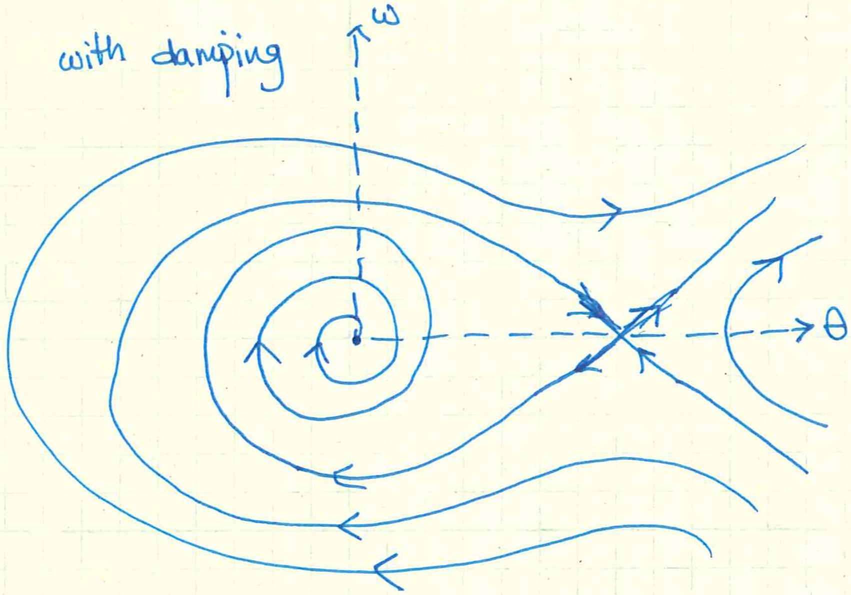
$\theta = \pi$ $\qquad \theta = 0$

Ex 3-2

without damping



with damping



Ex 3-3

$$\ddot{\theta} = -\sin\theta - 2\dot{\theta} - 2(\theta - \pi)$$

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\sin\theta - 2\dot{\theta} - 2(\theta - \pi)$$

$$\frac{D\underset{\sim}{f}}{D\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -\cos\theta - 2 & -2 \end{bmatrix} \quad \text{at} \begin{bmatrix} 0 \\ \pi \end{bmatrix} \longrightarrow \frac{D\underset{\sim}{f}}{D\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\cdots \begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} \qquad \lambda(\lambda+2)+1 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0 \qquad \lambda = -1$$

double root

stable now

Exercise 3-4

$\ddot{x} + 3\dot{x} + 2x = f(t)$

$x(0) = 2$

$\dot{x}(0) = -3$

$x(t) = x_h + x_p \quad x_h \to$ Ch. Eqn $\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda = -1, -2$

thus $x_h = c_1 e^{-t} + c_2 e^{-2t}$

a) $f(t) = 0$

$x(t) = x_h \longrightarrow x(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad \dot{x}(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

$x(0) \Rightarrow c_1 + c_2 = 2$

$\dot{x}(0) \Rightarrow -c_1 - 2c_2 = -3$

$c_1 = c_2 = 1$

$x(t) = e^{-t} + e^{-2t}$

---

b) $f(t) = 2e^{-3t}$

$x_p = K e^{\alpha t}$ where $\alpha = -3 \to x_p = K e^{-3t}$

plug x_p into the eqn $\to 9Ke^{-3t} + 3 * (-3Ke^{-3t}) + 2Ke^{-3t} = 2e^{-3t}$

$\boxed{K=1}$

thus $x(t) = x_h + x_p = c_1 e^{-t} + c_2 e^{-2t} + e^{-3t}$

$x(0) = 2 \qquad c_1 + c_2 + 1 = 2$

$\dot{x}(0) = -3 \qquad -c_1 - 2c_2 - 3 = -3$

$\left\{ \begin{array}{l} c_1 + c_2 = 1 \\ c_1 + 2c_2 = 0 \end{array} \right.$

$c_2 = -1$ and $c_1 = 2$

thus $x(t) = 2e^{-t} - e^{-2t} + e^{-3t}$

c) $f(t) = 20\sin(t)$

$$x_p = A\cos t + B\sin t$$

$$\dot{x}_p = -A\sin t + B\cos t$$

$$\ddot{x}_p = -A\cos t - B\sin t$$

plug into the equation

$$(-A+3B+2A)\cos t + (-B-3A+2B)\sin t = 20\sin t$$

$$A + 3B = 0$$

$$B - 3A = 20$$

$$B = -\frac{5}{2} = -2.5$$

$$A = 7.5$$

$$x = c_1 e^{-t} + c_2 e^{-2t} + 7.5\cos t - 2.5\sin t$$

$$x(0) = 2 \longrightarrow c_1 + c_2 + 7.5 = 2 \qquad c_1 + c_2 = -5.5$$

$$\dot{x}(0) = -3 \qquad -c_1 - 2c_2 - 2.5 = -3 \qquad c_1 + 2c_2 = 0.5$$

$$c_2 = 6 \quad c_1 = -11.5$$

$$x(t) = -11.5e^{-t} + 6e^{-2t} + 7.5\cos t - 2.5\sin t$$

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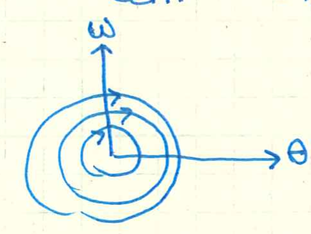
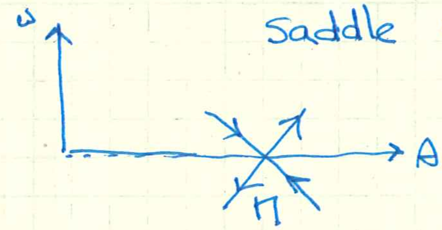
Center stable



For $\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \end{bmatrix}$ $\quad \frac{D\underset{\sim}{f}}{D\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \lambda^2 - 1 = 0 \quad \lambda = \pm 1$

Saddle unstable



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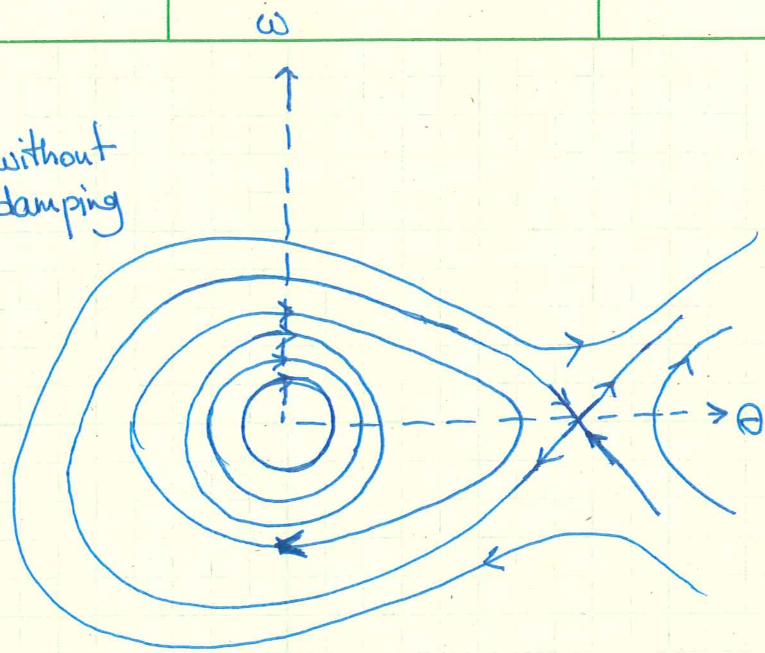
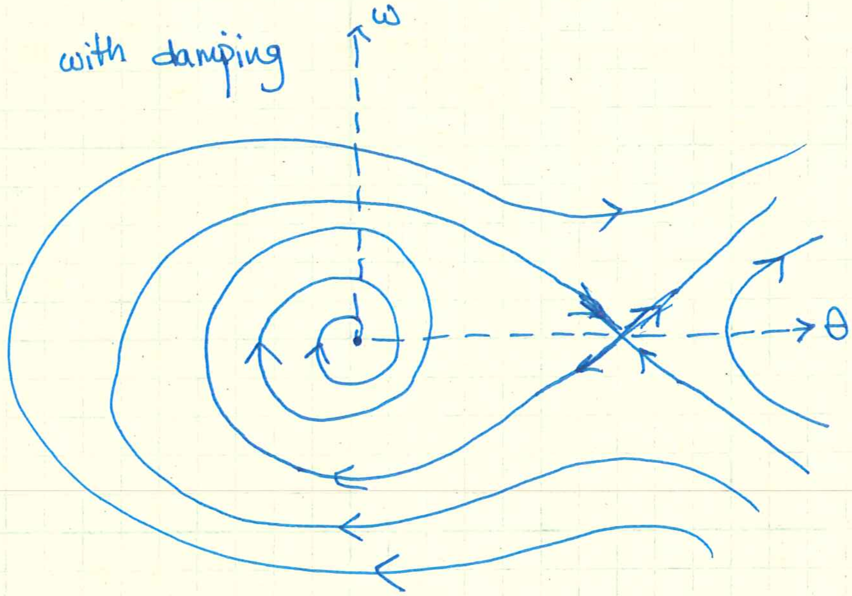
top → $\theta = \pi$ $\qquad$ bottom → $\theta = 0$

Ex 3-2

without damping



with damping



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$X(t) = X_h \longrightarrow X(t) = C_1 e^{-t} + C_2 e^{-2t}$ $\quad \dot{x}(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$

$X(0) \Rightarrow C_1 + C_2 = 2$

$C_1 = C_2 = 1$

$\dot{x}(0) \Rightarrow -C_1 - 2C_2 = -3$

$x(t) = e^{-t} + e^{-2t}$

---

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$X_p = Ke^{\alpha t}$ where $\alpha = -3 \rightarrow X_p = Ke^{-3t}$

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$\boxed{K=1}$

thus $X(t) = X_h + X_p = C_1 e^{-t} + C_2 e^{-2t} + e^{-3t}$

$X(0) = 2 \qquad C_1 + C_2 + 1 = 2$

$\dot{X}(0) = -3 \qquad -C_1 - 2C_2 - 3 = -3$

$\left.\begin{matrix} C_1 + C_2 = 1 \\ C_1 + 2C_2 = 0 \end{matrix}\right\}$

$C_2 = -1$ and $C_1 = 2$

thus $X(t) = 2e^{-t} - e^{-2t} + e^{-3t}$

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plug into the equation

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$$A+3B=0$$
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$$B = -\frac{5}{2} = -2.5$$
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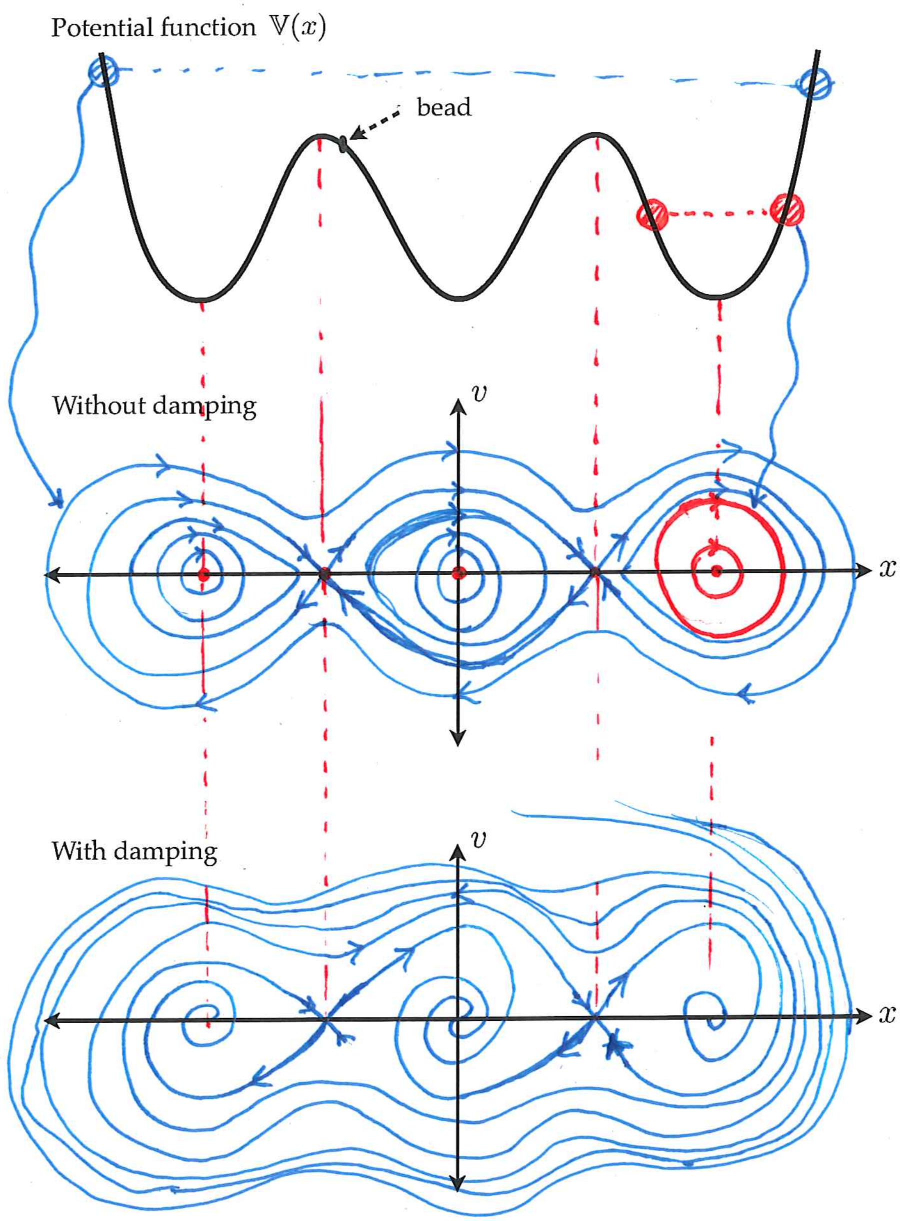
$c_2 = 6 \quad c_1 = -11.5$

$$x(t) = -11.5e^{-t} + 6e^{-2t} + 7.5\cos t - 2.5\sin t$$

Exercise 3–5: Consider the physical system of a bead constrained to move in the potential field $\mathbb{V}(x)$. Note: there are no equations given, and so you must use physical intuition.

- Please sketch the phase portrait (position x vs. velocity v). Try to make your sketch as accurate as possible using as much information about the potential as you can. In both cases, you will draw the phase portrait with and without damping.
- How many fixed points does the system have and what is their stability?
- What can we say about the eigenvalues of the linearization around each point?
- Finally, please pick a trajectory on the phase portrait and explain in one sentence what this trajectory means physically.



Ex 3-5:

b) There are 5 fixed points which are corresponding to points at which $\frac{\partial V}{\partial x} = 0$

3 stable centers associated with valleys

and 2 unstable saddles associated with peaks

c) for the centers i.e. the valleys the eigenvalues are pure imaginary and for saddle i.e. the peaks the eigenvalues are real with opposite signs