Exercise 2-1 $\quad \ddot{x}+5\dot{x}+6x=0$

a) $x(t)=e^{\lambda t} \rightarrow$ ch. Eqn $\quad \lambda^2+5\lambda+6=0 \quad (\lambda+3)(\lambda+2)=0$

$\lambda_1=-3$ and $\lambda_2=-2$

$x(t)=C_1 e^{-3t}+C_2 e^{-2t}$

b) $v=\dot{x}$

$\dot{v}=-5v+6x$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{A=\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}\begin{bmatrix} x \\ v \end{bmatrix}$$

Eigen values $\quad \det(A-\lambda I)=0 \quad \begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix}=0$

$$\lambda(5+\lambda)+6=0 \longrightarrow \lambda_1=-3 \quad \lambda_2=-2$$

$A\underset{\sim}{v}_1=\lambda_1\underset{\sim}{v}_1 \qquad \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}=\lambda_1\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$

$\cancel{0a_1}+b_1=-3a_1 \qquad b_1=-3a_1$

$-6a_1-5b_1=-3b_1 \qquad b_1=-3a_1$

$$\underset{\sim}{v}_1=\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$A\underset{\sim}{v}_2=\lambda_2 v_2 \quad \rightarrow b_1=-2a_1 \rightarrow \quad v_2=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

following the notes $\underset{\sim}{x}(t) = T e^{Dt} T^{-1} \underset{\sim}{x}(0)$ is the solution to $\dot{\underset{\sim}{x}} = A \underset{\sim}{x}$

$$\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}}_{T} e^{\wedge}\underbrace{\begin{bmatrix} -3t & 0 \\ 0 & -2t \end{bmatrix}}_{Dt} \underbrace{\begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}}_{T^{-1}} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

note $\rightarrow$ $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -2e^{-3t} & -e^{-3t} \\ 3e^{-2t} & +e^{-2t} \end{bmatrix} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -2e^{-3t} + 3e^{-2t} & -e^{-3t} + e^{-2t} \\ 6e^{-3t} - 6e^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} x(0)\left[-2e^{-3t} + 3e^{-2t}\right] - v(0)\left[e^{-3t} - e^{-2t}\right] \\ 6x(0)\left[e^{-3t} - e^{-2t}\right] + v(0)\left[3e^{-3t} - 2e^{-2t}\right] \end{bmatrix}$$

thus $X(t) = \left(-2e^{-3t} + 3e^{-2t}\right)x(0) - \left(e^{-3t} - e^{-2t}\right)v(0)$

c) a) $x(t) = c_1 e^{-3t} + c_2 e^{-2t}$ $\quad x(0) = -1 \quad \dot{x}(0) = 4$

$c_1 + c_2 = -1 \qquad c_1 = -2$

$-3c_1 - 2c_2 = 4 \qquad c_2 = 1$

thus $x(t) = -2e^{-3t} + e^{-2t}$

c) b) $x(t) = -\left(-2e^{-3t} + 3e^{-2t}\right) - 4\left(e^{-3t} - e^{-2t}\right)$

$x(t) = -2e^{-3t} + e^{-2t}$ ✓

Ex 2-2

$\dddot{x} + \ddot{x} - 4\dot{x} - 4x = 0$

$u = \dot{x}$

$v = \ddot{x} \longrightarrow v = \dot{u}$

$-v + 4u + 4x = \dddot{x} \longrightarrow \dot{v} = -v + 4u + 4x$

$$\frac{d}{dt}\begin{bmatrix} x \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ u \\ v \end{bmatrix}$$

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix}$$

$\lambda_1 = -2, \quad \lambda_2 = -1, \quad \lambda_3 = 2$

$\downarrow \quad \downarrow \quad \downarrow$ using Matlab or python

$$\underset{\sim}{v_1} = \begin{bmatrix} -0.22 \\ 0.44 \\ -0.87 \end{bmatrix} \quad \underset{\sim}{v_2} = \begin{bmatrix} -0.58 \\ 0.58 \\ -0.58 \end{bmatrix} \quad \underset{\sim}{v_3} = \begin{bmatrix} -0.22 \\ -0.44 \\ -0.87 \end{bmatrix}$$

more beautiful eigenvectors

$\lambda_1 = -2 \qquad \lambda_2 = -1 \qquad \lambda_3 = 2$

$$\underset{\sim}{v_1} = \begin{bmatrix} +1 \\ -2 \\ 4 \end{bmatrix} \quad \underset{\sim}{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \underset{\sim}{v_3} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

the eigenvector $\longrightarrow$ associated with the unstable eigenvalue $\lambda = 2$

$$\underset{\sim}{v_3} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

the eigenvectors associated with Stable eigenvalues

$$\begin{array}{cc} \lambda_1 = -2 & \lambda_2 = -1 \\ \underset{\sim}{v_1} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} & \underset{\sim}{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{array}$$

for $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ the solution will be stable since the initial condition is in the direction of the stable eigenvetor

for $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ the solution will be unstable since the initial condition is in the direction of the unstable eigenvector

for $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

Since the system is linear we can superpose the solutions for $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ to get the solution for $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

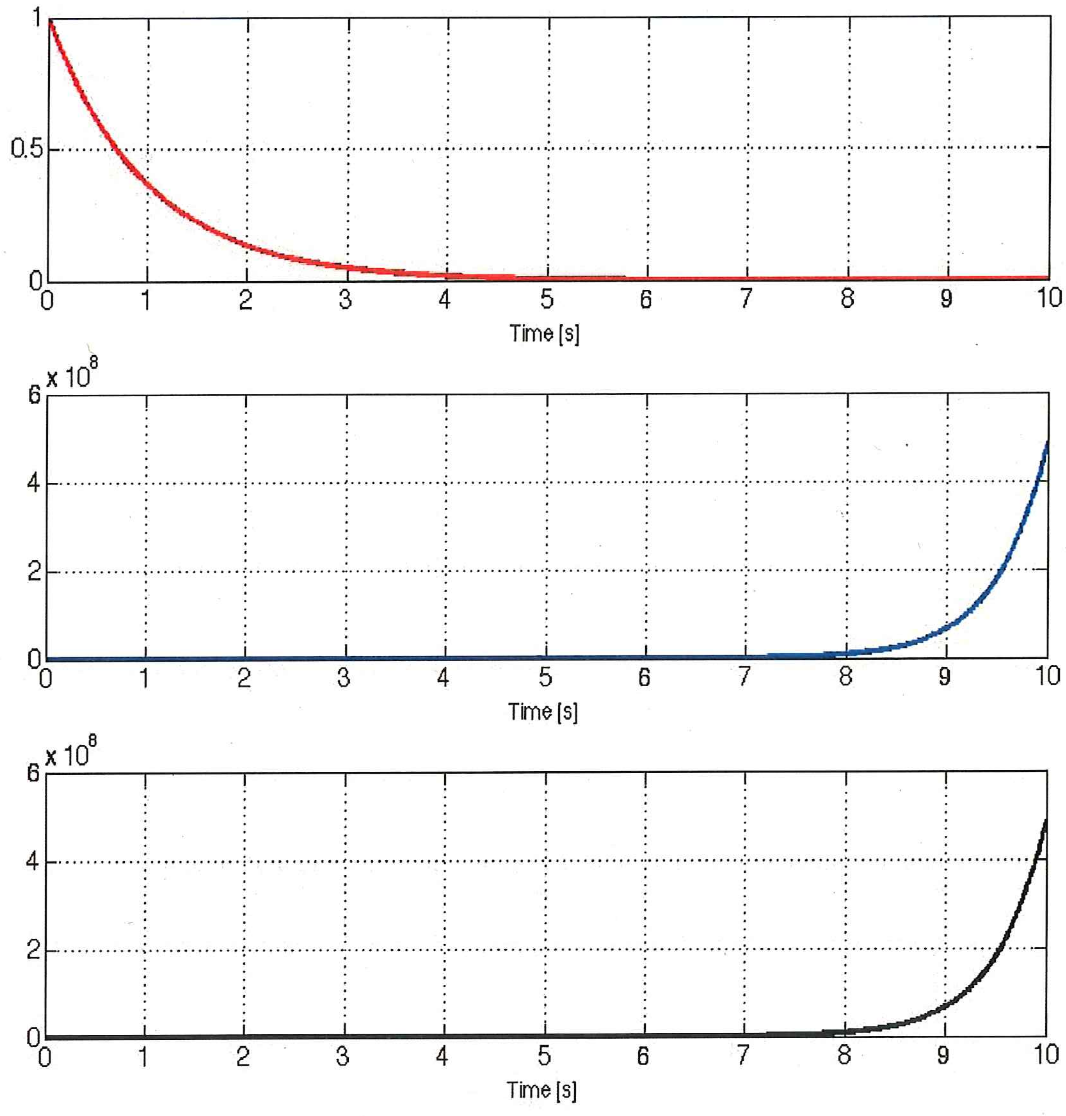
and therefore the solution for $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ is unstable

Since $\left\{ \begin{matrix} \text{Stable} \\ \text{Solution} \end{matrix} \right\} + \left\{ \begin{matrix} \text{unstable} \\ \text{Solution} \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \text{unstable} \\ \text{Solution} \end{matrix} \right\}$

These are validated by plots provided in the next page:

Solutions for Ex 2-2:

First, second, and third initial conditions from top to bottom, respectively:



I) $m_1\ddot{x}_1 + K_1 x_1 + K_2(x_1 - x_2) = 0$

II) $m_2\ddot{x}_2 + K_2(x_2 - x_1) = 0$

a) Single 4th order ode in x_1

$$\text{I)} \rightarrow \quad x_2 = \frac{m_1\ddot{x}_1 + K_1 x_1 + K_2 x_1}{K_2} = \frac{m_1\ddot{x}_1 + (K_1+K_2)x_1}{K_2}$$

$$\text{thus} \quad \ddot{x}_2 = \frac{m_1\ddddot{x}_1 + (K_1+K_2)\ddot{x}_1}{K_2}$$

$$\text{Plug into II)} \quad m_2\left\{\frac{m_1\ddddot{x}_1 + (K_1+K_2)\ddot{x}_1}{K_2}\right\} + K_2\left(\frac{m_1\ddot{x}_1 + (K_1+K_2)x_1}{K_2} - x_1\right) = 0$$

$$\frac{m_2 m_1}{K_2}\ddddot{x}_1 + \frac{m_2}{K_2}(K_1+K_2)\ddot{x}_1 + m_1\ddot{x}_1 + (K_1+K_2)x_1 - K_2 x_1 = 0$$

$$\text{thus} \quad \frac{m_2 m_1}{K_2}\ddddot{x}_1 + \left(\frac{m_2}{K_2}K_1 + (m_1+m_2)\right)\ddot{x}_1 + K_1 x_1 = 0$$

$$\boxed{\frac{m_1 m_2}{K_2}\ddddot{X}_1 + \left(\frac{m_2}{K_2}K_1 + m_1 + m_2\right)\ddot{X}_1 + K_1 X_1 = 0}$$

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b) single 4th order in x_2

$$\text{II)} \rightarrow \quad \frac{m_2\ddot{X}_2 + K_2 X_2}{K_2} = X_1 \quad \rightarrow \quad \frac{m_2}{K_2}\ddot{X}_2 + X_2 = X_1$$

thus

$$\ddot{X}_1 = \frac{m_2}{K_2}\ddddot{X}_2 + \ddot{X}_2$$

I) $$m_1\left(\frac{m_2}{k_2}\ddddot{x}_2+\ddot{x}_2\right)+(k_1+k_2)\left(\frac{m_2}{k_2}\ddot{x}_2+x_2\right)-k_2x_2=0$$

$$\boxed{\frac{m_1m_2}{k_2}\ddddot{x}_2+\left(\frac{m_2k_1}{k_2}+m_1+m_2\right)\ddot{x}_2+k_1x_2=0}$$

$u_1=x_1$

$u_2=\dot{x}_1 \rightarrow \dot{u}_2=\ddot{x}_1$

$$\rightarrow \begin{cases} u_2=\dot{u}_1 \\ \dot{u}_2=\frac{1}{m_1}\left(-k_1u_1-k_2u_1+k_2w_1\right)\end{cases}$$

$w_1=x_2$

$w_2=\dot{x}_2 \rightarrow \dot{w}_2=\ddot{x}_2$

$$\begin{cases} \dot{w}_1=w_2 \\ \dot{w}_2=\frac{1}{m_2}\left(-k_2w_1+k_2u_1\right)\end{cases}$$

$$\frac{d}{dt}\begin{bmatrix}u_1\\u_2\\w_1\\w_2\end{bmatrix}=\underbrace{\begin{bmatrix}0&1&0&0\\ \frac{-(k_1+k_2)}{m_1}&0&k_2&0\\ 0&0&0&1\\ \frac{k_2}{m_2}&0&\frac{-k_2}{m_2}&0\end{bmatrix}}_{A}\begin{bmatrix}u_1\\u_2\\w_1\\w_2\end{bmatrix}$$

Take $k_1=k_2=m_1=m_2=1$

$$A=\begin{bmatrix}0&1&0&0\\-2&0&1&0\\0&0&0&1\\1&0&-1&0\end{bmatrix}$$

$\lambda_1=1.618j$

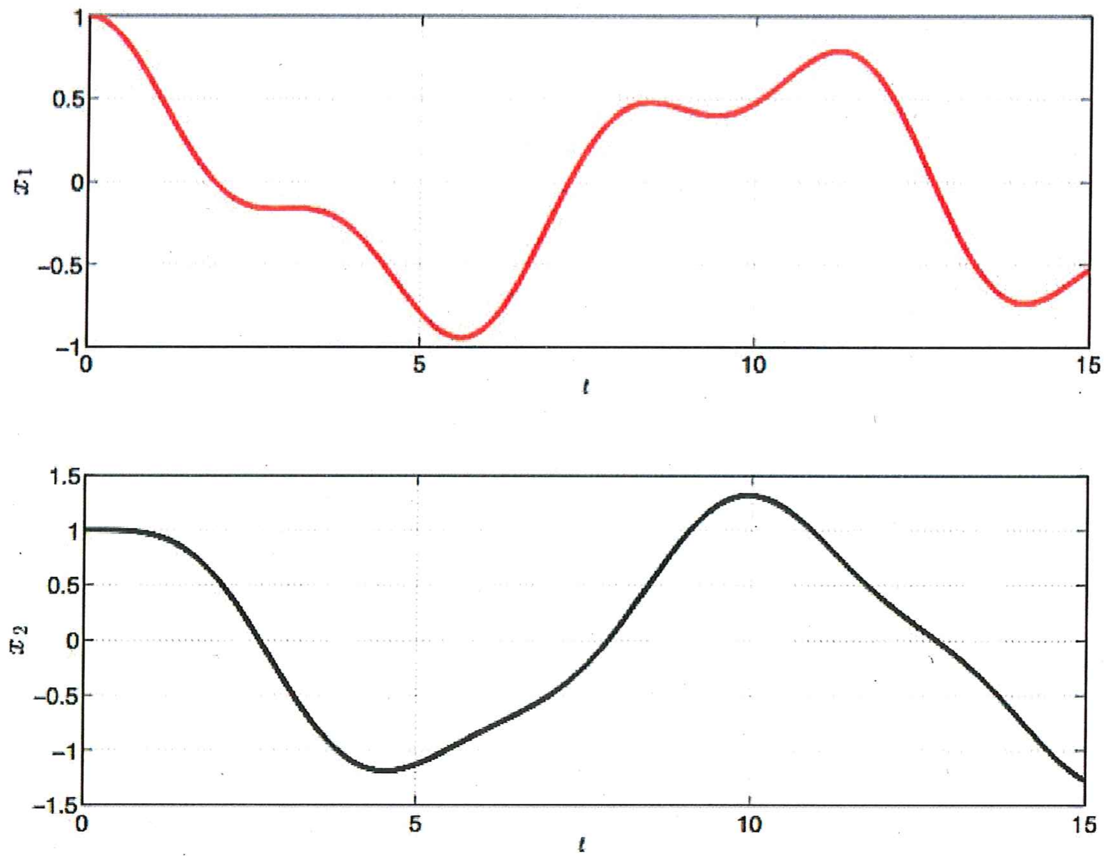
$\lambda_2=-1.618j$

$\lambda_3=0.618j$

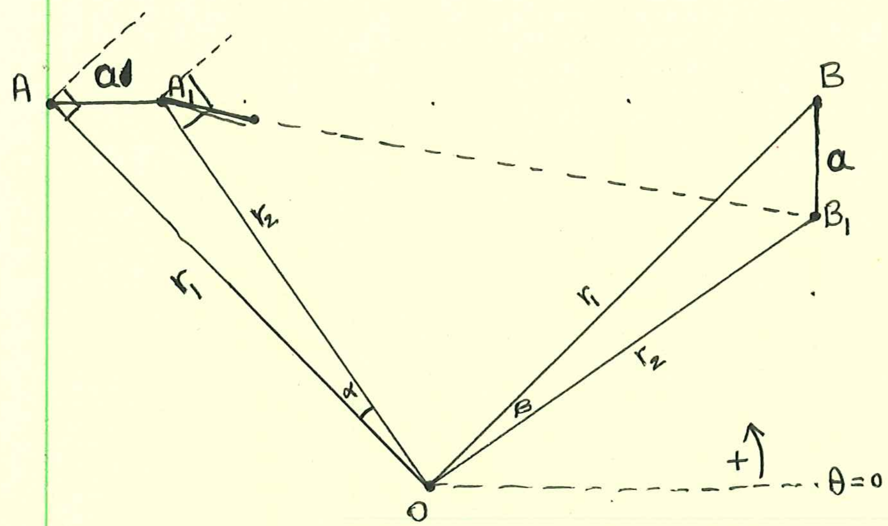
$\lambda_4=-0.618j$

all eigenvalues are imaginery
thus we will have oscillatory solutions
i.e. sines and cosines

Solutions for Ex 2-3:



Ex 2-4



due to the symmetry of the problem

$\overset{\triangle}{AOA_1} = \overset{\triangle}{BOB_1}$

thus $\alpha=\beta$

and as result of that and considering that

$\widehat{AOB} = 90^\circ$

we have that

A_1OB_1 is also 90°,

and since $OA_1 = OB_1$

we have $\widehat{OA_1B_1} = \frac{\pi}{4}$

Therefore, in a polar coordinate with O as the center,

The velocity of the boat is

$$\underset{\sim}{V} = -c\sin\frac{\pi}{4}\,\underset{\sim}{e_r} - c\cos\frac{\pi}{4}\,\underset{\sim}{e_\theta}$$

where c is the magnitude of velocity

Since $\underset{\sim}{V} = \dot{r}\underset{\sim}{e_r} + r\dot{\theta}\,\underset{\sim}{e_\theta}$

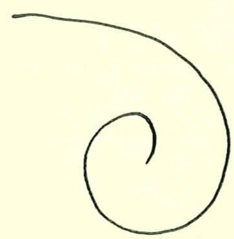
$$\dot{r} = -c\sin\frac{\pi}{4} \quad \text{and} \quad r\dot{\theta} = -c\cos\frac{\pi}{4}$$

Since $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} \longrightarrow \boxed{\dot{r} = r\dot{\theta}}$

$$\frac{dr}{dt} = r\frac{d\theta}{dt} \quad \text{or} \quad \frac{dr}{d\theta} = r \rightarrow \frac{dr}{r} = d\theta$$

$$\int \frac{dr}{r} = \int d\theta \qquad \ln r = \theta + \ln r_0$$

$$\frac{r}{r_0} = e^{\theta} \rightarrow \text{spiral}$$



We can also solve it WRT t

$$\dot{r} = -C \sin\frac{\pi}{4} \rightarrow r - r_0 = -C\sin\frac{\pi}{4}\, t$$

at the middle $r\big|_{\text{middle}} = 0 \longrightarrow$ thus $t = \dfrac{r_0}{C\sin\frac{\pi}{4}}$

for our problem we have $r_0 = \frac{\sqrt{2}}{2}$ mile and $C = 1$ mph

thus $t = 1$ hour

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one could also solve this problem by "considering a coordinate system which always follows the vector directly connecting two adjacent boats. Since the first boat is always pointed at the second one and the second one is moving perpendicular to this direction the motion gives $V=1$ in the direction that brings the boats together. So this is another way to get 1hr."