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Group 1: Tuesday Tutorial

1. CONVEX SETS AND CONVEX FUNCTIONS THEORY

(a) A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if its domain is a convex set and for all x_1, x_2 in its domain, and all $\theta \in [0,1]$, we have

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

~~In othe~~ This means that if one takes any two points x_1, x_2, then f, which is calculated on a convex combination of these 2 points, ~~the these~~ should not be greater that the same convex combination of $f(x_1)$ and $f(x_2)$.

~~Optimization~~ :

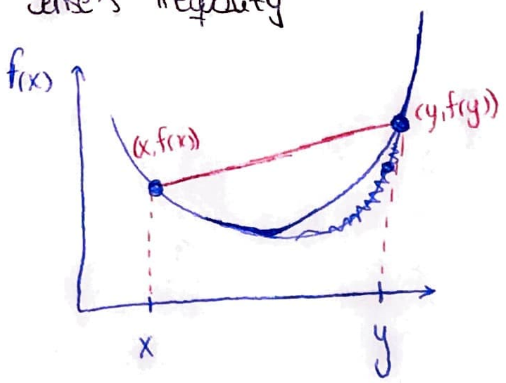
Convexity is important in optimization because many fields face with optimization problems where one wants to minimize a function with n variables. Most of the numerical methods for minimazing a function can only find a local optimal solution, but our goal ~~is~~ is to find a point x^* such that $f(x^*) \leq f(x)$, holds for all other points x. This solution would be a globally optimal solution.

If the function f is convex, then we can make sure that a locally optimal solution is globally optimal.

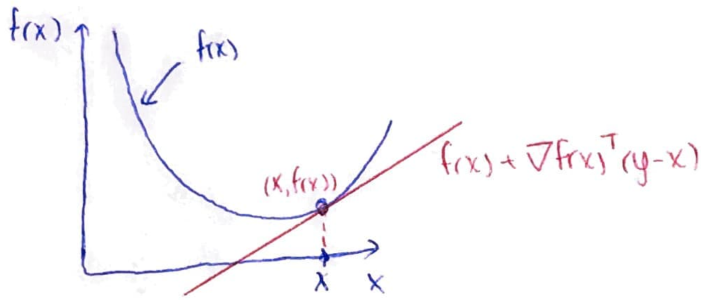
(b) A convex set is a set that contains the line segment between any two points in the set. and a line segment between any 2 points x_1, x_2 is the set of all points.

~~As mentioned before, a convex function~~
A convex function is a real-valued function if the line segment between any 2 points on the graph of the function does not lie below the graph ~~of~~ between the 2 points.

(d) (i) Jense's inequality

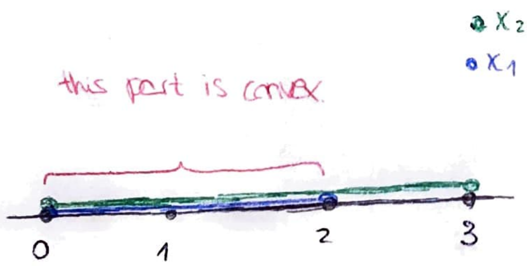


(ii) First-order condition

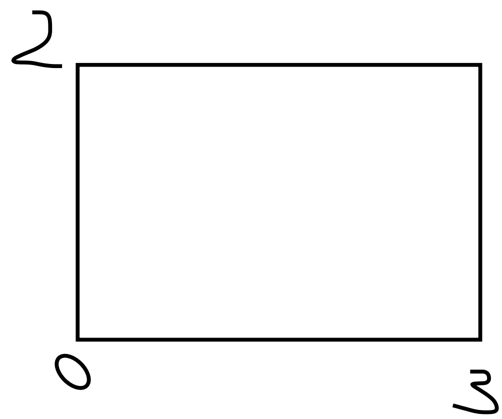


(iii) A twice differentiable function f is convex iff the domain of f is a convex set, for all $x \in \mathrm{dom} f$ and if $\nabla^2 f(x) \geq 0$

(c)



2.1



2. SECOND ORDER CONVEXITY CONDITION

(a) (i) $f(x,y,z) = ax^2 + y^2 + 2z^2 - 4axy + 2yz$

$f'_x(x,y,z) = 2ax - 4ay$

$f'_y(x,y,z) = 2y - 4ax + 2z$

$f'_z(x,y,z) = 4z + 2y$

$f''_{xx}(x,y,z) = 2a$

$f''_{xy}(x,y,z) = -4a$

$f''_{xz}(x,y,z) = 0$

$f''_{yx}(x,y,z) = -4a$

$f''_{yy}(x,y,z) = 2$

$f''_{yz}(x,y,z) = 2$

$f''_{zx}(x,y,z) = 0$

$f''_{zy}(x,y,z) = 2$

$f''_{zz}(x,y,z) = 4$

$$H = \begin{vmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yx} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{vmatrix} = \begin{vmatrix} 2a & -4a & 0 \\ -4a & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} =$$

$$= 2a \cdot 2 \cdot 2 + 0 + 0 - 0 - (-4a)\cdot(-4a)\cdot 4 - 2 \cdot 2 \cdot 2a = 8a - 64a^2$$

3.1

$8a - 64a^2 = 0$

$\cancel{8(a - 8a^2) = 0}$

$8a(1 - 8a) = 0 \rightarrow 8a = 0 \rightarrow a = 0$

$\rightarrow 1 - 8a = 0$

$a = \frac{1}{8}$

$f(x,y,z)$ will be convex when a is $[0, \frac{1}{8}]$

(ii) $f(x,y) = 4ax^2 + 8xy + by^2$

$f'_x(x,y) = 8ax + 8y$

$f'_y(x,y) = 8x + 2by$

$$H = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix} = \begin{vmatrix} 8a & 8 \\ 8 & 2b \end{vmatrix} = 16ab - 64$$

$16ab - 64 > 0$

$16ab = 64 \rightarrow a = \frac{64}{16b} = \frac{4}{b}$

For any value of a and $b \geq \frac{4}{a}$, the $f(x,y)$ will be strictly convex. or any value b and $a \geq \frac{4}{b}$.

3.2

(b) $f_1(x,y) = \ln(\sqrt[2]{xy})$

$\sqrt{xy} > 0$ The domain is $\{(x,y) \mid xy \geq 0\}$ and the range $[0, \infty)$

$xy \geq 0$

$f_x(x,y) = \cancel{\frac{1}{2}\ln(x)} \; \frac{1}{2}\ln(xy)$

$f'_x(x,y) = \frac{1}{2} \cdot \frac{1}{x\cancel{y}} \cdot \cancel{y} = \frac{1}{2x}$

$f'_y(x,y) = \frac{1}{2} \cdot \frac{1}{\cancel{x}y} \cdot \cancel{x} = \frac{1}{2y}$

$f''_{xx} = -2x^{-2} = \frac{-2}{x^2}$

$f''_{xy} = 0$

$f_{yx} = 0$

$f_{yy} = -\frac{2}{y^2}$

$$H = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix} = \begin{vmatrix} -2/x^2 & 0 \\ 0 & -2/y^2 \end{vmatrix} = (-2/x^2) \times (-2/y^2) = \frac{4}{x^2y^2}$$

~~x and~~ x and y need to be greater or smaller than 0 and $f_1(x,y)$ will be strictly convex or strictly concav and therefore the domain is $\{x,y \mid x^2y^2 \neq 0\}$.

4.1

$f_2(x,y) = (x-1)^2 + xy^2 = x^2 - 2x + 1 + xy^2$.

$f'_x(x,y) = 2x - 2 + y^2$

$f'_y(x,y) = 2xy$

$$H = \begin{vmatrix} 2 & 2y \\ 2x & 2y \end{vmatrix} = 4y - 4xy \geq 0$$

$$4y(1-4x) \geq 0$$

There is no restriction for the values, therefore the domain is all real numbers $\mathbb{R}$.

4.2

Index der Kommentare

2.1 the set as a whole is convex and it could be drawn as a rectangle

3.1 the final answer is correct but you should have proved all upper left determinant positive

3.2 a > 0 due to the upper left determinant being positive 8a > 0

4.1 here the function is always concave due to the odd determinant being always negative (look up negative definite matrix)

4.2 the inequality here is a restriction if you solve it you will get x > y^2