Modern Optimization Techniques

1. Theory

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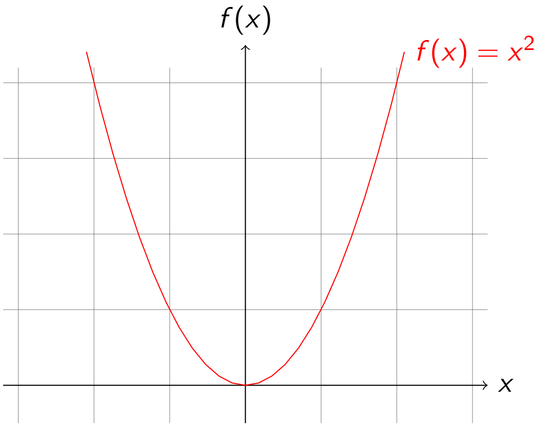
Syllabus

Mon. 07.11.	(1)	0. Overview
		1. Theory
Mon. 14.11.	(2)	1. Convex Sets and Functions
		2. Unconstrained Optimization
Mon. 21.11.	(3)	2.1 Gradient Descent
Mon. 28.11.	(4)	2.2 Stochastic Gradient Descent
Mon. 05.12.	(5)	2.3 Newton's Method
Mon. 12.12.	(6)	2.4 Quasi-Newton Methods
Mon. 19.12.	(7)	*canceled*
	—	— *Christmas Break* —
Mon. 09.01.	(8)	2.5 Subgradient Methods
Fri.. 13.01.	(9)	2.6 Coordinate Descent
		3. Equality Constrained Optimization
Mon. 23.01.	(10)	3.1 Duality
Mon. 30.01.	(11)	3.2 Methods
		4. Inequality Constrained Optimization
Mon. 06.01.	(12)	4.1 Primal Methods
Mon. 13.02.	(13)	4.2 Barrier and Penalty Methods

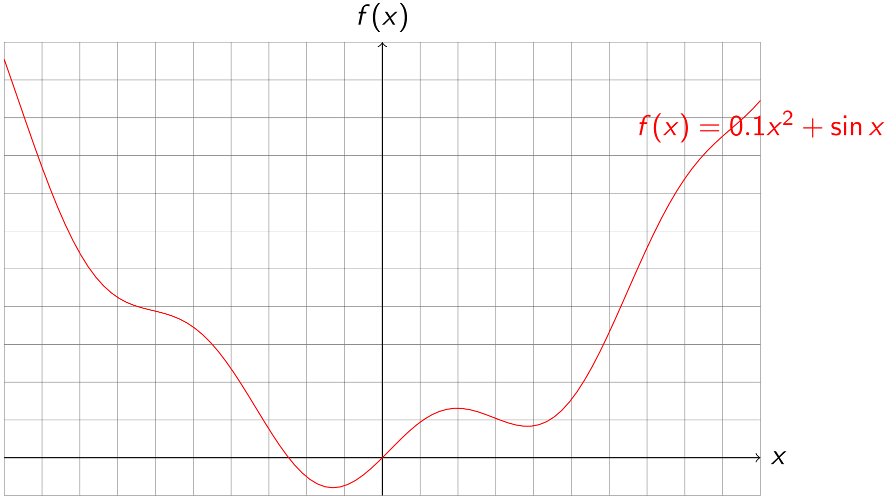
Outline

Outline

A convex function



A non-convex function



Convex Optimization Problem

An **optimization problem**

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\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & h_q(x) \leq 0, \quad q = 1, \ldots, Q \\
& Ax = b
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is said to be convex if $f, h_1 \ldots h_Q$ are convex.

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How do we know if a function is convex or not?

Note: The equality constraints also are convex, even linear.

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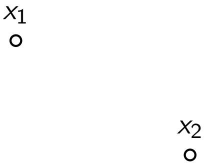

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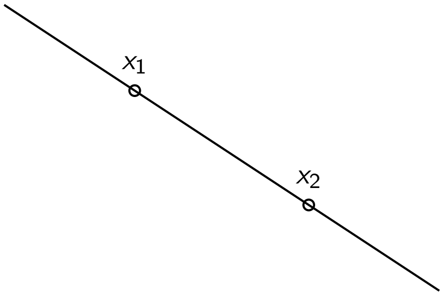

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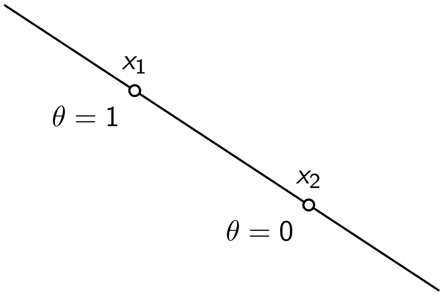

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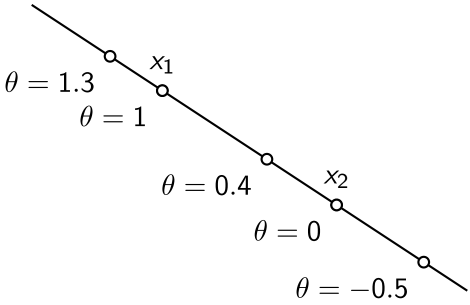

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Examples:

- $\mathbb{R}^N$ for $N \in \mathbb{N}^+$
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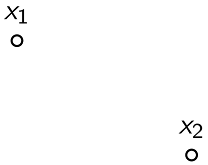

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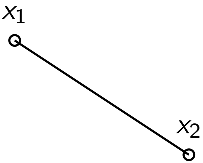

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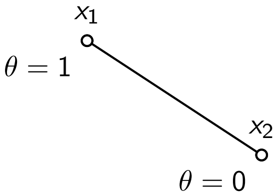

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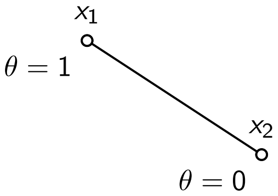
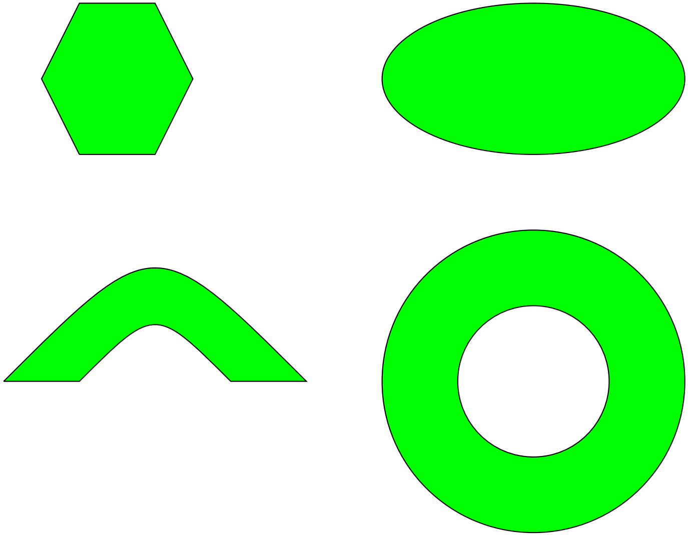


A **convex set** contains the line segment between any two points in the set.

Convex Sets - Examples: Which ones are Convex?



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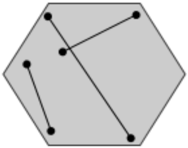
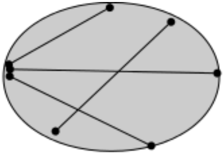
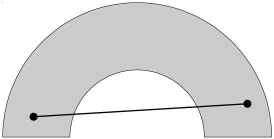
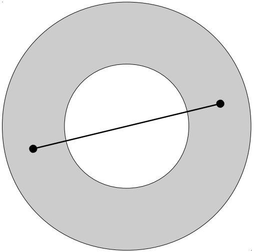




Non-convex Sets:





Convex Sets - Examples

All affine sets are also convex:

- $\mathbb{R}^N$ for $N \in \mathbb{N}^+$

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Convex sets (but in general not affine sets):

- solution set of linear inequalities $X := \{x \in \mathbb{R}^N \mid Ax \leq b\}$
  - half spaces, e.g. $X := \{x \in \mathbb{R}^N \mid a^T x \leq b\}$
    e.g., $X := \{x \in \mathbb{R}^N \mid x_1 \geq 0\}$

  - convex polygons (2d) / polyhedrons (3d) / polytopes (nd)

Convex Combination and Convex Hull

(standard) simplex:

$$\begin{aligned}\Delta^N :=& \{\theta \in \mathbb{R}^N \mid \theta_n \geq 0, n = 1, \ldots, N; \sum_{n=1}^{N} \theta_n = 1\} \\ =& \{\theta \in [0,1]^N \mid \mathbb{1}^T \theta = 1\}\end{aligned}$$

convex combination of some points $x_1, \ldots x_N \in \mathbb{R}^M$: any point x with

$$x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_N x_N, \quad \theta \in \Delta^N$$

convex hull of a set $X \subseteq \mathbb{R}^M$ of points:

$$\text{conv}(X) := \{\theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_N x_N \mid N \in \mathbb{N}, x_1, \ldots, x_N \in X, \theta \in \Delta^N\}$$

i.e., the set of all convex combinations of points in X.

Note: $\mathbb{1} := (1, 1, \ldots, 1)^T$ vector of all ones.

Outline

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- for all $x_1, x_2 \in \operatorname{dom} f$ and $0 \leq \theta \leq 1$ it satistfies

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

(the function is below of its secant segments/chords.)

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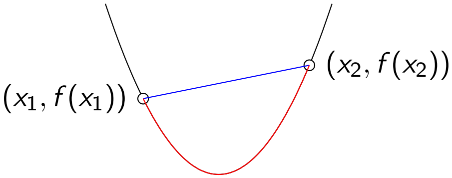

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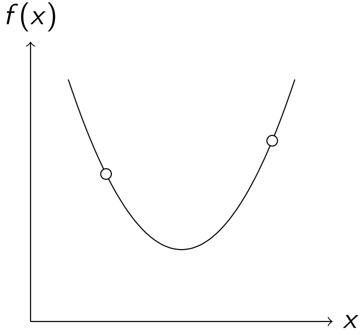

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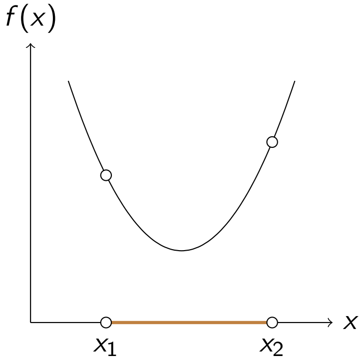

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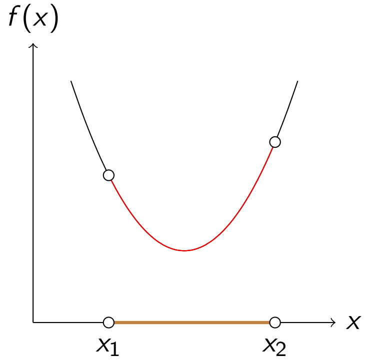

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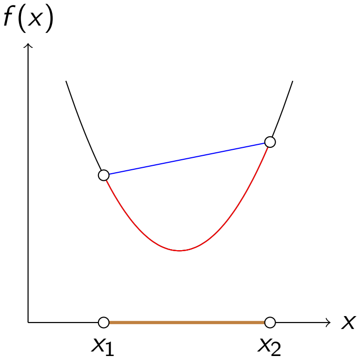

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How are Convex Functions Related to Convex Sets?

epigraph of a function $f : X \to \mathbb{R}, X \subseteq \mathbb{R}^N$:

$$\text{epi}(f) := \{(x, y) \in X \times \mathbb{R} \mid y \geq f(x)\}$$

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f is convex (as function) $\iff$ epi(f) is convex (as set).

proof is straight-forward (try it!)

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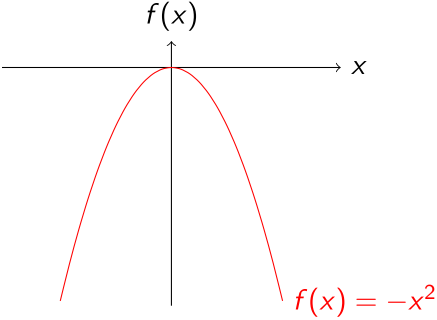
A Concave Function



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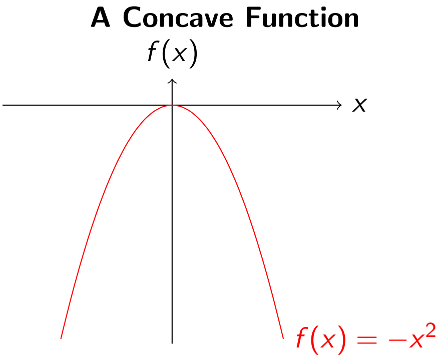

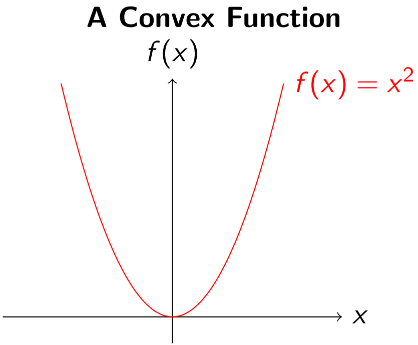

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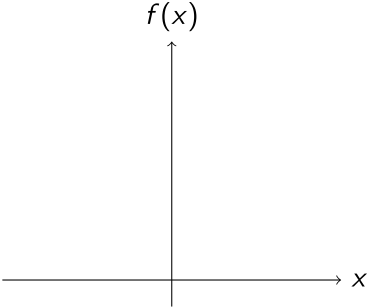

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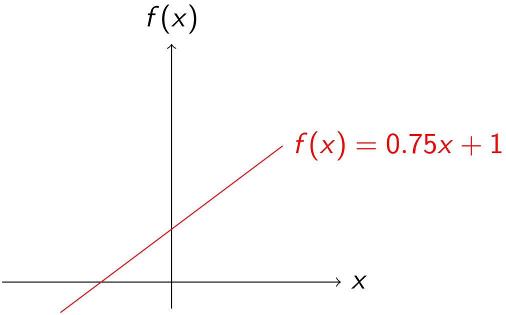

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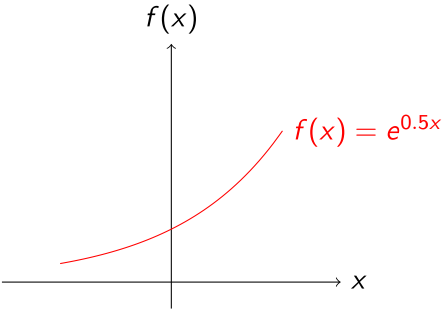

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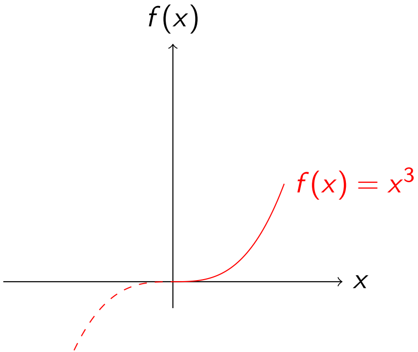

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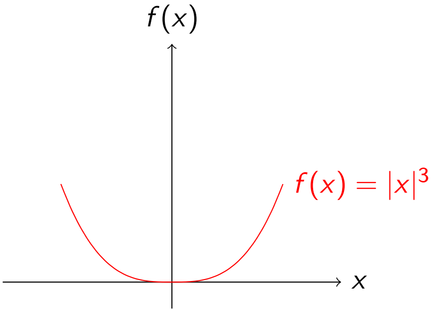

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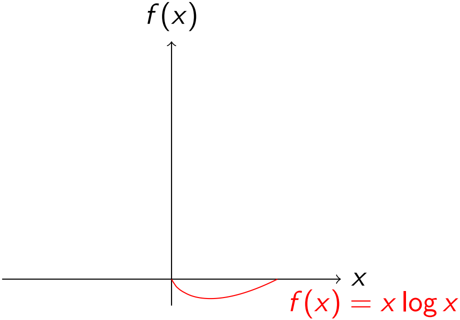

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- logarithm: $f(x) := \log x$, with $\operatorname{dom} f := \mathbb{R}^+$

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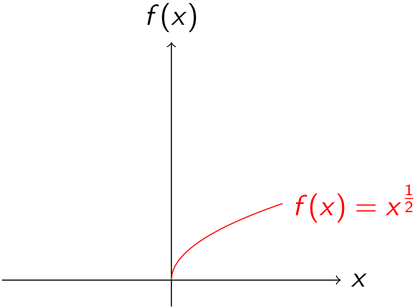

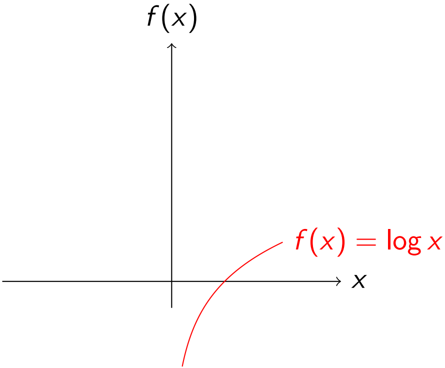

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Examples of Convex functions:

All norms are convex!

- Immediate consequence of the triangle inequality and absolute homogeneity.

$$||\theta x + (1-\theta)y|| \leq ||\theta x|| + ||(1-\theta)y|| = \theta||x|| + (1-\theta)||y||$$

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All norms are convex!

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Affine functions on vectors are also convex: $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$

Outline

1st-Order Condition

f is **differentiable** if $\mathrm{dom}\, f$ is open and the gradient

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_n} \right)^T$$

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(the function is above (or on) any of its tangents.)

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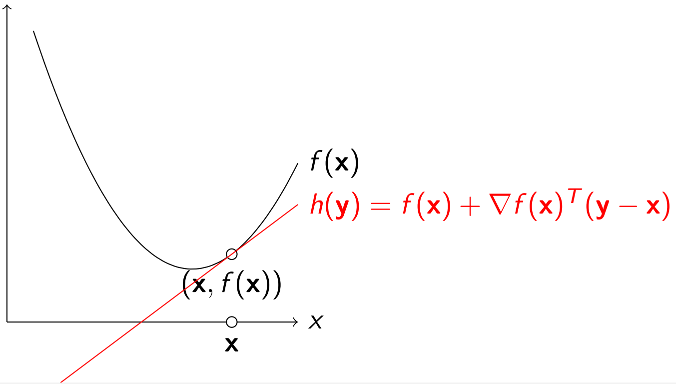

1st-Order Condition / Proof

Let $\operatorname{dom} f = X$ be convex.

$$f : X \to \mathbb{R} \text{ convex} \Leftrightarrow f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y}$$

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$$\begin{aligned} \rightsquigarrow \theta f(x) + (1 - \theta)f(y) \quad &\geq f(z) + \nabla f(z)^T(\theta x + (1 - \theta)y) - \nabla f(z)^T z \\ &= f(z) + \nabla f(z)^T z - \nabla f(z)^T z \\ &= f(z) = f(\theta x + (1 - \theta)y) \end{aligned}$$

1st-Order Condition / Strict Variant

strict 1st-order condition: a differentiable function f is strictly convex iff

- $\mathrm{dom}\, f$ is a convex set
- for all $\mathbf{x}, \mathbf{y} \in \mathrm{dom}\, f, \mathbf{x} \neq \mathbf{y}$

$$f(\mathbf{y}) > f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x})$$

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Q: What does this imply for points $\mathbf{x}$ with $\nabla f(\mathbf{x}) = 0$?

Global Minima

Let $\operatorname{dom} f = X$ be convex.

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Consequence: Points x with $\nabla f(x) = 0$ are (equivalent) global minima.

- minima form a convex set
- if f is strictly convex: there is exactly one global minimum x^*.

2nd-Order Condition

f is **twice differentiable** if $\operatorname{dom} f$ is open and the Hessian $\nabla^2 f(x)$

$$\nabla^2 f(\mathbf{x})_{n,m} = \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_m}$$

exists everywhere.

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Furthermore:

- for functions f on $\mathrm{dom}\, f \subseteq \mathbb{R}$ simply $f''(x) \geq 0$ for all $x \in \mathrm{dom}\, f$
- if $\nabla^2 f(\mathbf{x}) \succ 0$ for all $\mathbf{x} \in \mathrm{dom}\, f$, then f is strictly convex
  - the converse is not true,
    e.g., $f(x) = x^4$ is strictly convex, but has 0 derivative at 0.

Positive Semidefinite Matrices (Review)

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is **positive semidefinite ($A \succeq 0$)**:

$$x^T A x \geq 0, \quad \forall x \in \mathbb{R}^N$$

Equivalent:

(i) all eigenvalues of A are ≥ 0.

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A symmetric matrix $A \in \mathbb{R}^{N \times N}$ is **positive definite ($A \succ 0$)**:

$$x^T A x > 0, \quad \forall x \in \mathbb{R}^N \setminus \{0\}$$

Equivalent:

(i) all eigenvalues of A are > 0.

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Example: Quadratic Functions

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Q: Are quadratic functions convex?

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A quadratic function f is convex $\iff$ A is pos. semidef.

Recognizing Convex Functions

- There are a number of operations that preserve the convexity of a function.

- If f can be obtained by applying those operations to a convex function, f is also convex.

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Q: Is the sum of two conex functions g and h convex again?

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Sum:

- if g and h are convex functions, then $g + h$ is convex.

- Example: $f(x) = e^x + x \log x$ with $\operatorname{dom} f = \mathbb{R}^+$ is convex since e^x and $x \log x$ are convex

Recognizing Convex Functions / Composition

Composition of two convex functions:

- let $g : \mathbb{R}^N \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ be both convex and

$$f(x) := h(g(x))$$

- in general f is **not** convex
- counter example $N = 1, \quad g(x) = h(x) = e^{-x}$:

$$\begin{aligned}
\left(e^{-e^{-x}}\right)'' &= \left(e^{-e^{-x}}(-e^{-x})(-1)\right)' \\
&= \left(e^{-e^{-x}} e^{-x}\right)' \\
&= e^{-e^{-x}} e^{-x} e^{-x} + e^{-e^{-x}} e^{-x}(-1) \\
&= e^{-e^{-x}} e^{-x}(e^{-x} - 1) < 0 \quad \text{for } x > 0
\end{aligned}$$

Recognizing Convex Functions / Composition

Composition with affine functions:

- if f is convex then $f(A\mathbf{x} + \mathbf{b})$ is convex.

Recognizing Convex Functions / Composition

Composition with affine functions:

- if f is convex then $f(A\mathbf{x} + \mathbf{b})$ is convex.
- Example: norm of an affine function $||A\mathbf{x} + \mathbf{b}||$

Recognizing Convex Functions / Composition

Composition with nondecreasing functions:

- if $g : \mathbb{R}^N \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ and

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- f is convex if:
  - g is convex, h is convex and nondecreasing *or*
  - g is concave, h is convex and nonincreasing

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- f is convex if:
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  - g is concave, h is convex and nonincreasing
- proof:

$$\begin{aligned}\nabla^2 h(g(\mathbf{x})) &= \nabla \left(h'(g(\mathbf{x}))\nabla g(\mathbf{x})\right) \\ &= h''(g(\mathbf{x}))\nabla g(\mathbf{x})\nabla g(\mathbf{x})^T + h'(g(\mathbf{x}))\nabla^2 g(\mathbf{x})\end{aligned}$$

- Examples:
  - $e^{g(\mathbf{x})}$ is convex if g is convex
  - $\frac{1}{g(\mathbf{x})}$ is convex if g is concave and positive

Recognizing Convex Functions

Pointwise Maximum:

- if $f_1, \ldots, f_M$ are convex functions then
  $f(\mathbf{x}) := \max\{f_1(\mathbf{x}), \ldots, f_M(\mathbf{x})\}$ is convex.

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- Example: $f(\mathbf{x}) := \max_{m=1,\ldots,M}(a_m^T \mathbf{x} + b_m)$ is convex

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There are many different ways to establish the convexity of a function:

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2. Show that $\nabla^2 f(\mathbf{x}) \succeq 0$ for twice differentiable functions.

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There are many different ways to establish the convexity of a function:

1. Check the definition.
2. Show that $\nabla^2 f(\mathbf{x}) \succeq 0$ for twice differentiable functions.
3. Show that f can be obtained from other convex functions by operations that preserve convexity.

Summary

- **Convex sets** are closed under line segments (convex combinations).
- **Convex functions** are defined on a convex domain and
  - are below any of their secant segments / chords (definition),
  - are globally above (or on) their tangents (1st-order condition),
  - have a positive semidefinite Hessian (2nd-order condition).
- For convex functions, points with vanishing gradients are (equivalent) **global minima**.
- Operations that preserve convexity:
  - scaling with a nonnegative constant
  - sums
  - pointwise maximum
  - composition with an affine function
  - composition with a nondecreasing convex scalar function
  - composition of a noninc. convex scalar function with a concave funct.
    - esp. $-g$ for a concave g

Further Readings

- Convex sets:
  - Boyd and Vandenberghe, 2004, chapter 2, esp. 2.1
  - see also ch. 2.2 and 2.3
- Convex functions:
  - Boyd and Vandenberghe, 2004, chapter 3, esp. 3.1.1–7, 3.2.1–5
- Convex optimization:
  - Boyd and Vandenberghe, 2004, chapter 4, esp. 4.1–3
  - see also ch. 4.4

References

Boyd, Stephen and Lieven Vandenberghe (2004). *Convex Optimization*. Cambridge University Press.