Modern Optimization Techniques – Group 01

Exercise Sheet 05

Submitted by: Muhammad Inaam Ashraf (Matrikel-Nr: 307524)

Semester 2 MSc. Data Analytics

Question 1: Exact Newton Method

1. Let us (theoretically) optimize the following function using a Newton Descent approach:

$$\boldsymbol{f}: \boldsymbol{R}^2 \to \boldsymbol{R}, \qquad \boldsymbol{f(x,y) = exp(x^2 + y^2) = e^{x^2+y^2}}$$

(a) Compute the gradient $\nabla \boldsymbol{f(x,y)}$ and the Hessian $\nabla^2 \boldsymbol{f(x,y)}$!

$$\nabla f(x,y) = \begin{bmatrix} \nabla_x f(x,y) \\ \nabla_y f(x,y) \end{bmatrix} = \begin{bmatrix} \boldsymbol{e^{x^2+y^2} 2x} \\ \boldsymbol{e^{x^2+y^2} 2y} \end{bmatrix}$$

$$\nabla^2 f(x,y) = \begin{bmatrix} \nabla_x(\nabla_x f(x,y)) & \nabla_y(\nabla_x f(x,y)) \\ \nabla_x(\nabla_y f(x,y)) & \nabla_y(\nabla_y f(x,y)) \end{bmatrix} = \begin{bmatrix} e^{x^2+y^2} 2(2x^2+1) & e^{x^2+y^2} 4xy \\ e^{x^2+y^2} 4xy & e^{x^2+y^2} 2(2y^2+1) \end{bmatrix}$$

(b) Compute $\nabla^2 \boldsymbol{f(x,y)^{-1}}$ using Cramer's Rule:

$$\nabla^2 f(x,y)^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -\mathrm{b} \\ -\mathrm{c} & a \end{bmatrix}$$

$$ad - bc = \left(e^{x^2+y^2} 4x^2 + e^{x^2+y^2} 2\right)\left(e^{x^2+y^2} 4y^2 + e^{x^2+y^2} 2\right) - (e^{x^2+y^2} 4xy)(e^{x^2+y^2} 4xy)$$

$$ad - bc = \left((e^{x^2+y^2})^2 4x^2 4y^2 + (e^{x^2+y^2})^2 8x^2 + (e^{x^2+y^2})^2 8y^2 + (e^{x^2+y^2})^2 4\right) - ((e^{x^2+y^2})^2 4x^2 4y^2)$$

$$ad - bc = (e^{x^2+y^2})^2 8x^2 + (e^{x^2+y^2})^2 8y^2 + (e^{x^2+y^2})^2 4$$

$$\nabla^2 f(x,y)^{-1} = \frac{1}{(e^{x^2+y^2})^2 8x^2 + (e^{x^2+y^2})^2 8y^2 + (e^{x^2+y^2})^2 4} \begin{bmatrix} e^{x^2+y^2} 4y^2 + e^{x^2+y^2} 2 & -e^{x^2+y^2} 4xy \\ -e^{x^2+y^2} 4xy & e^{x^2+y^2} 4x^2 + e^{x^2+y^2} 2 \end{bmatrix}$$

$$\nabla^2 f(x,y)^{-1} = \frac{1}{e^{x^2+y^2} 8x^2 + e^{x^2+y^2} 8y^2 + e^{x^2+y^2} 4} \begin{bmatrix} 4y^2 + 2 & -4xy \\ -4xy & 4x^2 + 2 \end{bmatrix}$$

(c) Compute the update step of the Newton Algorithm:

$$\Delta_{x.y} = -\nabla^2 f(x,y)^{-1}\nabla f(x,y) = -\frac{1}{e^{x^2+y^2}8x^2 + e^{x^2+y^2}8y^2 + e^{x^2+y^2}4}\begin{bmatrix} 4y^2+2 & -4xy \\ -4xy & 4x^2+2 \end{bmatrix}\begin{bmatrix} e^{x^2+y^2}2x \\ e^{x^2+y^2}2y \end{bmatrix}$$

$$\Delta_{x.y} = -\frac{1}{e^{x^2+y^2}8x^2 + e^{x^2+y^2}8y^2 + e^{x^2+y^2}4}\begin{bmatrix} e^{x^2+y^2}8xy^2 + e^{x^2+y^2}4x - e^{x^2+y^2}8xy^2 \\ -e^{x^2+y^2}8x^2y + e^{x^2+y^2}8x^2y + e^{x^2+y^2}4y \end{bmatrix}$$

$$\Delta_{x.y} = -\frac{1}{8x^2+8y^2+4}\begin{bmatrix} 4x \\ 4y \end{bmatrix} = \begin{bmatrix} -\dfrac{\boldsymbol{x}}{\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{2}\boldsymbol{y}^{\mathbf{2}} + \mathbf{1}} \\ -\dfrac{\boldsymbol{y}}{\mathbf{2}\boldsymbol{x}^{\mathbf{2}} + \mathbf{2}\boldsymbol{y}^{\mathbf{2}} + \mathbf{1}} \end{bmatrix}$$

Question 2: Quasi-Newton Method: BFGS

Learn a logistic regression using the BFGS Quasi-Newton Method from the lecture, for the following data:

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{5} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{-2} & \mathbf{6} \\ \mathbf{1} & \mathbf{1} & \mathbf{4} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{3} \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

Start with a random initialization of $\boldsymbol{\beta}_0$ and show the convergence of the algorithm! Also start with a diagonal Hessian in the first iteration and $\boldsymbol{\mu}_0 = \mathbf{0.001}$
Note: Show the working for two iterations without coding the method.

We have

$$\mathcal{L}(X,\beta,Y) = -\sum_{i=1}^{m} y_i log\big(\sigma(x_i\beta)\big) + (1-y_i)log(1-\sigma(x_i\beta))$$

$$\nabla_\beta \mathcal{L}(X,\beta,Y) = -X^T\big(Y-\hat{Y}\big)$$

Initial Hessian inverse and $\beta^{(0)}$ are given by

$$A^{(0)} = I, \qquad \beta^{(0)} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We have

$$\Delta\beta^{(0)} = -A^{(0)}\nabla_{\beta^{(0)}}\mathcal{L}\big(X,\beta^{(0)},Y\big)$$

Computing $\hat{Y}$

$$X\beta^{(0)} = \begin{bmatrix} 1 & 1 & 5 & 2 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 3 & 9 & -2 & 6 \\ 1 & 1 & 4 & 0 & -1 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ -4 \\ 2 \end{bmatrix}, and\ \hat{Y} = \frac{1}{1+e^{-X\beta^{(0)}}} = \begin{bmatrix} 0.269 \\ 0.881 \\ 0.119 \\ 0.018 \\ 0.881 \end{bmatrix}$$

Now

$$\nabla_{\beta^{(0)}}\mathcal{L}\left(X,\beta^{(0)},Y\right) = -X^T\left(Y-\hat{Y}\right) = -\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 5 & 1 & 9 & 4 & 2 \\ 2 & 1 & -2 & 0 & 1 \\ 3 & 2 & 6 & -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.269 \\ 0.881 \\ 0.119 \\ 0.018 \\ 0.881 \end{bmatrix}\right) = \begin{bmatrix} -0.832 \\ -0.832 \\ -2.868 \\ -1.939 \\ -2.092 \end{bmatrix}$$

$$\Delta\beta^{(0)} = -A^{(0)}\nabla_{\beta^{(0)}}\mathcal{L}\left(X,\beta^{(0)},Y\right) = -\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -0.832 \\ -0.832 \\ -2.868 \\ -1.939 \\ -2.092 \end{bmatrix} = \begin{bmatrix} 0.832 \\ 0.832 \\ 2.868 \\ 1.939 \\ 2.092 \end{bmatrix}$$

So

$$\beta^{(1)} = \beta^{(0)} + \mu_0\Delta\beta^{(0)} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + 0.001\begin{bmatrix} 0.832 \\ 0.832 \\ 2.868 \\ 1.939 \\ 2.092 \end{bmatrix} = \begin{bmatrix} \mathbf{1.0008} \\ \mathbf{.0008} \\ \mathbf{-.9971} \\ \mathbf{.0019} \\ \mathbf{1.0021} \end{bmatrix}$$

$$s^{(1)} = \beta^{(1)} - \beta^{(0)} = \begin{bmatrix} 1.0008 \\ .0008 \\ -.9971 \\ .0019 \\ 1.0021 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{.0008} \\ \mathbf{.0008} \\ \mathbf{.0029} \\ \mathbf{.0019} \\ \mathbf{.0021} \end{bmatrix}$$

$$g^{(1)} = \nabla_{\beta^{(1)}}\mathcal{L}\left(X,\beta^{(1)},Y\right) - \nabla_{\beta^{(0)}}\mathcal{L}\left(X,\beta^{(0)},Y\right)$$

$$\nabla_{\beta^{(1)}}\mathcal{L}\left(X,\beta^{(1)},Y\right) = -X^T\left(Y-\hat{Y}\right)$$

Computing $\hat{Y}$

$$X\beta^{(1)} = \begin{bmatrix} 1 & 1 & 5 & 2 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 3 & 9 & -2 & 6 \\ 1 & 1 & 4 & 0 & -1 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1.0008 \\ .0008 \\ -.9971 \\ .0019 \\ 1.0021 \end{bmatrix} = \begin{bmatrix} -.974 \\ 2.011 \\ -1.962 \\ -3.989 \\ 2.016 \end{bmatrix}, and\ \hat{Y} = \frac{1}{1+e^{-X\beta^{(1)}}} = \begin{bmatrix} 0.274 \\ 0.882 \\ 0.123 \\ 0.018 \\ 0.883 \end{bmatrix}$$

$$\nabla_{\beta^{(1)}}\mathcal{L}\left(X,\beta^{(1)},Y\right) = -X^T\left(Y-\hat{Y}\right) = -\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 5 & 1 & 9 & 4 & 2 \\ 2 & 1 & -2 & 0 & 1 \\ 3 & 2 & 6 & -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.274 \\ 0.882 \\ 0.123 \\ 0.018 \\ 0.883 \end{bmatrix}\right) = \begin{bmatrix} -0.82 \\ -0.81 \\ -2.8 \\ -1.93 \\ -2.04 \end{bmatrix}$$

$$g^{(1)} = \nabla_{\beta^{(1)}}\mathcal{L}\left(X,\beta^{(1)},Y\right) - \nabla_{\beta^{(0)}}\mathcal{L}\left(X,\beta^{(0)},Y\right) = \begin{bmatrix} -0.82 \\ -0.81 \\ -2.8 \\ -1.93 \\ -2.04 \end{bmatrix} - \begin{bmatrix} -0.832 \\ -0.832 \\ -2.868 \\ -1.939 \\ -2.092 \end{bmatrix} = \begin{bmatrix} \mathbf{0.012} \\ \mathbf{0.023} \\ \mathbf{0.067} \\ \mathbf{0.005} \\ \mathbf{0.047} \end{bmatrix}$$

Finally,

$$A^{(1)} = A^{(0)} + \frac{\left(s^{(1)} - A^{(0)}g^{(1)}\right)s^{(1)^T} + s^{(1)}(s^{(1)} - A^{(0)}g^{(1)})^T}{s^{(1)^T}g^{(1)}} - \frac{(s^{(1)} - A^{(0)}g^{(1)})^T g^{(1)}}{(s^{(1)^T}g^{(1)})^2} s^{(1)}s^{(1)^T}$$

Now,

$$s^{(1)} - A^{(0)}g^{(1)} = \begin{bmatrix} .0008 \\ .0008 \\ .0029 \\ .0019 \\ .0021 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.012 \\ 0.023 \\ 0.067 \\ 0.005 \\ 0.047 \end{bmatrix} = \begin{bmatrix} -.011 \\ -.022 \\ -.065 \\ -.003 \\ -.045 \end{bmatrix}$$

$$A^{(1)} = A^{(0)} + \frac{\begin{bmatrix} -.011 \\ -.022 \\ -.065 \\ -.003 \\ -.045 \end{bmatrix} [.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021] + \begin{bmatrix} .0008 \\ .0008 \\ .0029 \\ .0019 \\ .0021 \end{bmatrix} [-.011 \quad -.022 \quad -.065 \quad -.003 \quad -.045]}{[.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021] \begin{bmatrix} -.011 \\ -.022 \\ -.065 \\ -.003 \\ -.045 \end{bmatrix}}$$

$$- \frac{[-.011 \quad -.022 \quad -.065 \quad -.003 \quad -.045] \begin{bmatrix} -.011 \\ -.022 \\ -.065 \\ -.003 \\ -.045 \end{bmatrix}}{\left([.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021] \begin{bmatrix} -.011 \\ -.022 \\ -.065 \\ -.003 \\ -.045 \end{bmatrix}\right)^2} \begin{bmatrix} .0008 \\ .0008 \\ .0029 \\ .0019 \\ .0021 \end{bmatrix} [.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021]$$

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -.057 & -.085 & -.261 & -.075 & -.185 \\ -.085 & -.113 & -.356 & -.139 & -.254 \\ -.261 & -.356 & -1.118 & -.406 & -.796 \\ -.075 & -.139 & -.406 & -.038 & -.283 \\ -.185 & -.254 & -.796 & -.283 & -.567 \end{bmatrix} - \begin{bmatrix} -.045 & -.045 & -.155 & -.105 & -.113 \\ -.045 & -.045 & -.155 & -.105 & -.113 \\ -.155 & -.155 & -.535 & -.362 & -.390 \\ -.105 & -.105 & -.362 & -.245 & -.264 \\ -.113 & -.113 & -.390 & -.264 & -.285 \end{bmatrix}$$

$$\boldsymbol{A^{(1)}} = \begin{bmatrix} \mathbf{.99} & \mathbf{-.04} & \mathbf{-.11} & \mathbf{.03} & \mathbf{-.07} \\ \mathbf{-.04} & \mathbf{.93} & \mathbf{-.2} & \mathbf{-.03} & \mathbf{-.14} \\ \mathbf{-.11} & \mathbf{-.2} & \mathbf{.42} & \mathbf{-.04} & \mathbf{-.41} \\ \mathbf{.03} & \mathbf{-.03} & \mathbf{-.04} & \mathbf{1.21} & \mathbf{-.02} \\ \mathbf{-.07} & \mathbf{-.14} & \mathbf{-.41} & \mathbf{-.02} & \mathbf{.72} \end{bmatrix}$$

Second iteration:

We have

$$\Delta\beta^{(1)} = -A^{(1)} \nabla_{\beta^{(1)}} \mathcal{L}\left(X, \beta^{(1)}, Y\right)$$

$$\Delta\beta^{(1)} = -\begin{bmatrix} .99 & -.04 & -.11 & .03 & -.07 \\ -.04 & .93 & -.2 & -.03 & -.14 \\ -.11 & -.2 & .42 & -.04 & -.41 \\ .03 & -.03 & -.04 & 1.21 & -.02 \\ -.07 & -.14 & -.41 & -.02 & .72 \end{bmatrix} \begin{bmatrix} -0.82 \\ -0.81 \\ -2.8 \\ -1.93 \\ -2.04 \end{bmatrix} = \begin{bmatrix} 0.39 \\ -0.20 \\ 0 \\ 2.17 \\ 0.12 \end{bmatrix}$$

So

$$\beta^{(2)} = \beta^{(1)} + \mu_0 \Delta\beta^{(1)} = \begin{bmatrix} 1.0008 \\ .0008 \\ -.9971 \\ .0019 \\ 1.0021 \end{bmatrix} + 0.001 \begin{bmatrix} 0.39 \\ -0.20 \\ 0 \\ 2.17 \\ 0.12 \end{bmatrix} = \begin{bmatrix} \mathbf{1.0012} \\ \mathbf{.0006} \\ \mathbf{-.9971} \\ \mathbf{4.1042} \\ \mathbf{1.0022} \end{bmatrix}$$

$$s^{(2)} = \beta^{(2)} - \beta^{(1)} = \begin{bmatrix} 1.0012 \\ .0006 \\ -.9971 \\ 4.1042 \\ 1.0022 \end{bmatrix} - \begin{bmatrix} 1.0008 \\ .0008 \\ -.9971 \\ .0019 \\ 1.0021 \end{bmatrix} = \begin{bmatrix} \mathbf{.0008} \\ \mathbf{.0008} \\ \mathbf{.0029} \\ \mathbf{.0019} \\ \mathbf{.0021} \end{bmatrix}$$

$$g^{(2)} = \nabla_{\beta^{(2)}} \mathcal{L}\left(X, \beta^{(2)}, Y\right) - \nabla_{\beta^{(1)}} \mathcal{L}\left(X, \beta^{(1)}, Y\right)$$

$$\nabla_{\beta^{(2)}} \mathcal{L}\left(X, \beta^{(2)}, Y\right) = -X^T\left(Y - \hat{Y}\right)$$

Computing $\hat{Y}$

$$X\beta^{(2)} = \begin{bmatrix} 1 & 1 & 5 & 2 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 3 & 9 & -2 & 6 \\ 1 & 1 & 4 & 0 & -1 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1.0012 \\ .0006 \\ -.9971 \\ 4.1042 \\ 1.0022 \end{bmatrix} = \begin{bmatrix} -.969 \\ 2.014 \\ -1.966 \\ -3.989 \\ 2.019 \end{bmatrix}, and\ \hat{Y} = \frac{1}{1 + e^{-X\beta^{(2)}}} = \begin{bmatrix} 0.275 \\ 0.882 \\ 0.123 \\ 0.018 \\ 0.883 \end{bmatrix}$$

$$\nabla_{\beta^{(2)}} \mathcal{L}\left(X, \beta^{(2)}, Y\right) = -X^T\left(Y - \hat{Y}\right) = -\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 5 & 1 & 9 & 4 & 2 \\ 2 & 1 & -2 & 0 & 1 \\ 3 & 2 & 6 & -1 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.275 \\ 0.882 \\ 0.123 \\ 0.018 \\ 0.883 \end{bmatrix} \right) = \begin{bmatrix} -0.8189 \\ -0.808 \\ -2.798 \\ -1.93 \\ -2.043 \end{bmatrix}$$

$$g^{(2)} = \nabla_{\beta^{(2)}} \mathcal{L}\left(X, \beta^{(2)}, Y\right) - \nabla_{\beta^{(1)}} \mathcal{L}\left(X, \beta^{(1)}, Y\right) = \begin{bmatrix} -0.8189 \\ -0.808 \\ -2.798 \\ -1.93 \\ -2.043 \end{bmatrix} - \begin{bmatrix} -0.82 \\ -0.81 \\ -2.8 \\ -1.93 \\ -2.04 \end{bmatrix} = \begin{bmatrix} \mathbf{0.001} \\ \mathbf{0.0008} \\ \mathbf{0.002} \\ \mathbf{0.0033} \\ \mathbf{0.0018} \end{bmatrix}$$

Finally,

$$A^{(2)} = A^{(1)} + \frac{\left(s^{(2)} - A^{(1)} g^{(2)}\right) {s^{(2)}}^T + s^{(2)} (s^{(2)} - A^{(1)} g^{(2)})^T}{{s^{(2)}}^T g^{(2)}} - \frac{(s^{(2)} - A^{(1)} g^{(2)})^T g^{(2)}}{({s^{(2)}}^T g^{(2)})^2} s^{(2)} {s^{(2)}}^T$$

Now,

$$s^{(2)} - A^{(1)} g^{(2)} = \begin{bmatrix} .0008 \\ .0008 \\ .0029 \\ .0019 \\ .0021 \end{bmatrix} - \begin{bmatrix} .99 & -.04 & -.11 & .03 & -.07 \\ -.04 & .93 & -.2 & -.03 & -.14 \\ -.11 & -.2 & .42 & -.04 & -.41 \\ .03 & -.03 & -.04 & 1.21 & -.02 \\ -.07 & -.14 & -.41 & -.02 & .72 \end{bmatrix} \begin{bmatrix} 0.001 \\ 0.0008 \\ 0.002 \\ 0.0033 \\ 0.0018 \end{bmatrix} = \begin{bmatrix} -.0004 \\ -.0001 \\ .0003 \\ -.0017 \\ -.00009 \end{bmatrix}$$

$$A^{(1)} = A^{(0)} + \frac{\begin{bmatrix} -.0004 \\ -.0001 \\ .0003 \\ -.0017 \\ -.00009 \end{bmatrix} [.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021] + \begin{bmatrix} .0008 \\ .0008 \\ .0029 \\ .0019 \\ .0021 \end{bmatrix} [-.0004 \quad -.0001 \quad .0003 \quad -.0017 \quad -.00009]}{[.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021] \begin{bmatrix} 0.001 \\ 0.0008 \\ 0.002 \\ 0.0033 \\ 0.0018 \end{bmatrix}}$$

$$- \frac{[-.0004 \quad -.0001 \quad .0003 \quad -.0017 \quad -.00009] \begin{bmatrix} -.011 \\ -.022 \\ -.065 \\ -.003 \\ -.045 \end{bmatrix}}{\left([.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021] \begin{bmatrix} 0.001 \\ 0.0008 \\ 0.002 \\ 0.0033 \\ 0.0018 \end{bmatrix}\right)^2} \begin{bmatrix} .0008 \\ .0008 \\ .0029 \\ .0019 \\ .0021 \end{bmatrix} [.0008 \quad .0008 \quad .0029 \quad .0019 \quad .0021]$$

$$\boldsymbol{A^{(2)}} = \begin{bmatrix} \mathbf{.96} & \mathbf{-.04} & \mathbf{-.09} & \mathbf{.09} & \mathbf{-.07} \\ \mathbf{-.04} & \mathbf{.94} & \mathbf{-.21} & \mathbf{-.066} & \mathbf{-.142} \\ \mathbf{-.09} & \mathbf{-.21} & \mathbf{.418} & \mathbf{.043} & \mathbf{-.4} \\ \mathbf{-.09} & \mathbf{-.066} & \mathbf{.043} & \mathbf{.704} & \mathbf{-.046} \\ \mathbf{-.07} & \mathbf{-.14} & \mathbf{-.40} & \mathbf{-.046} & \mathbf{.716} \end{bmatrix}$$