Modern Optimization Techniques

3. Equality Constrained Optimization / 3.2. Methods

Lars Schmidt-Thieme

Information Systems and Machine Learning Lab (ISMLL)
Institute for Computer Science
University of Hildesheim, Germany

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Outline

Equality Constrained Optimization Problems

A **constrained optimization problem** has the form:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_p(\mathbf{x}) = 0, \quad p = 1, \ldots, P \end{aligned}$$

Where:

- $f : \mathbb{R}^N \rightarrow \mathbb{R}$ objective function
- $g_1, \ldots, g_P : \mathbb{R}^N \rightarrow \mathbb{R}$ equality constraints
- a feasible, optimal $\mathbf{x}^*$ exists

Convex Equality Constrained Optimization Problems

An equality constrained optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_p(\mathbf{x}) = 0, \quad p = 1, \ldots, P \end{aligned}$$

is **convex** iff:

- f is convex
- $g_1, \ldots, g_P$ are affine

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{a}, \quad A \in \mathbb{R}^{P \times N}, a \in \mathbb{R}^P \end{aligned}$$

Affine Equality Constraints $Ax = a$

- Always can assume: A has rank $P \leq N$.
  - otherwise delete extra rows in A (by Gauss elimination).

- each row in A is a normal vector for $\mathcal{X}$.

- the feasible set $\mathcal{X}$ is simple, just an affine set.

$P = \text{rank}(A)$	feasible set $\mathcal{X}$	$\dim(\mathcal{X})$
N	point	0
N-1	line	1
N-2	plane	2
N-3	3d volume	3
$\vdots$	$\vdots$	$\vdots$
1	hyperplane	$N - 1$
0	unconstrained	N

Optimality criterion

Given a convex equality constrained optimization problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{a}, \quad A \in \mathbb{R}^{P \times N}, a \in \mathbb{R}^P \end{aligned}$$

Its Lagrangian is given by:

$$L(\mathbf{x}, \nu) = f(\mathbf{x}) + \nu^T (A\mathbf{x} - \mathbf{a})$$

with derivative:

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \nu) = \nabla_{\mathbf{x}} f(\mathbf{x}) + A^T \nu$$

Optimality criterion

Given a convex equality constrained optimization problem

$$\begin{array}{rl} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & A\mathbf{x} = \mathbf{a}, \quad A \in \mathbb{R}^{P \times N}, a \in \mathbb{R}^{P} \end{array}$$

The optimal solution $\mathbf{x}^*$ must fulfill the KKT conditions:

Optimality criterion

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The optimal solution $\mathbf{x}^*$ must fulfill the KKT conditions:

1. **primal feasibility:** $g_p(\mathbf{x}) = 0$ and $h_q(\mathbf{x}) \leq 0, \quad \forall p, q$
2. **dual feasibility:** $\lambda \geq 0$
3. **complementary slackness:** $\lambda_q \, h_q(\mathbf{x}) = 0, \quad \forall q$
4. **stationarity:** $\nabla f(\mathbf{x}) \quad + \sum_{p=1}^{p} \nu_p \nabla g_p(\mathbf{x}) + \sum_{q=1}^{Q} \lambda_q \nabla h_q(\mathbf{x}) = 0$

Optimality criterion

Given a convex equality constrained optimization problem

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The optimal solution $\mathbf{x}^*$ must fulfill the KKT conditions:

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2. **dual feasibility:** $\cancel{\lambda \geq 0}$
3. **complementary slackness:** $\cancel{\lambda_q h_q(\mathbf{x}) = 0, \quad \forall q}$
4. **stationarity:** $\nabla f(\mathbf{x}) + \sum_{p=1}^{P} \nu_p \nabla g_p(\mathbf{x}) \cancel{+ \sum_{q=1}^{Q} \lambda_q \nabla h_q(\mathbf{x})} = 0$

- Since there are no inequality constraints, stroke-through conditions are irrelevant.

Optimality criterion

Given a convex equality constrained optimization problem

$$\begin{array}{rl} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & A\mathbf{x} = \mathbf{a}, \quad A \in \mathbb{R}^{P \times N}, \mathbf{a} \in \mathbb{R}^P \end{array}$$

The optimal solution $\mathbf{x}^*$ must fulfill the KKT conditions:

1. **primal feasibility:** $A\mathbf{x} = \mathbf{a}$
2. **stationarity:** $\nabla f(\mathbf{x}) + A^T \nu^* = 0$

- i.e., a feasible x^* is optimal,
  if there exists a ν^* with $\nabla f(\mathbf{x}^*) + A^T \nu^* = 0$

Example

Given the following problem:

$$\begin{aligned} \text{minimize} \quad & (x_1 - 2)^2 + 2(x_2 - 1)^2 - 5 \\ \text{subject to} \quad & x_1 + 4x_2 = 3 \end{aligned}$$

Q: Can you sketch the problem?

Example

Given the following problem:

$$\begin{array}{rl} \text{minimize} & (x_1 - 2)^2 + 2(x_2 - 1)^2 - 5 \\ \text{subject to} & x_1 + 4x_2 = 3 \end{array}$$

optimality condition:

1. **primal feasibility:** $$A\mathbf{x} = \mathbf{a}$$
2. **stationarity:** $$\nabla f(\mathbf{x}) + A^T \nu^* = 0$$

instantiated for the example problem:

1. **primal feasibility:** $$x_1 + 4x_2 = 3$$
2. **stationarity:** $$\begin{pmatrix} 2x_1 - 4 \\ 4x_2 - 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}^T v = 0$$

Example

Given the following problem:

$$\begin{array}{rl} \text{minimize} & (x_1 - 2)^2 + 2(x_2 - 1)^2 - 5 \\ \text{subject to} & x_1 + 4x_2 = 3 \end{array}$$

instantiated for the example problem:

1. **primal feasibility:** $$x_1 + 4x_2 = 3$$
2. **stationarity:** $$\begin{pmatrix} 2x_1 - 4 \\ 4x_2 - 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}^T v = 0$$

can be simplified to:

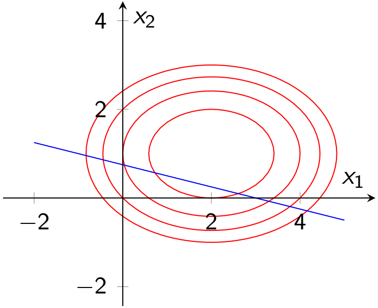
$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 4 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \nu \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

with solution $x_1 = \frac{5}{3}, x_2 = \frac{1}{3}, \nu = \frac{2}{3}$

Example

Given the following problem:

$$\begin{array}{rl} \text{minimize} & (x_1 - 2)^2 + 2(x_2 - 1)^2 - 5 \\ \text{subject to} & x_1 + 4x_2 = 3 \end{array}$$



Note: red: contour lines of objective function, blue: feasible set $\mathcal{X}$ defined by the equality constraint

Generic Handling of Equality Constraints

Two generic ways to handle equality constraints:

1. **Eliminate affine equality constraints**
   - and then use any unconstrained optimization method.
   - limited to **affine** equality constraints
2. **Represent equality constraints as inequality constraints**
   - and then use any optimization method for inequality constraints.

1. Eliminating Affine Equality Constraints

Reparametrize feasible points:

$$\{x \mid Ax = a\} = x_0 + \{x \mid Ax = 0\} = x_0 + \{Fz \mid z \in \mathbb{R}^{N-P}\}$$

with

- $x_0 \in \mathbb{R}^N$: any feasible point: $Ax_0 = a$
- $F \in \mathbb{R}^{N\times(N-P)}$ composed of $N - P$ basis vectors of the nullspace of A.
  - $AF = 0$ (e.g., compute F by Gauss elimination)

equality constrained problem: $\underset{x^* = x_0 + Fz^*}{\Longleftrightarrow}$ reduced unconstrained problem:

$$\min_x f(x) \qquad\qquad \min_z \tilde{f}(z) := f(x_0 + Fz)$$

subject to $Ax = a$

1. Eliminating Affine Eq. Constr. / KKT Conditions

Be z^* the solution of the reduced unconstrained problem, i.e., $\nabla \tilde{f}(z^*) = 0$.
Then $x^* := x_0 + Fz^*$ fulfills the KKT conditions with

$$\nu^* := -(AA^T)^{-1} A \nabla f(x^*)$$

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$$\nu^* := -(AA^T)^{-1} A \nabla f(x^*)$$

Proof:

i. primal feasibility: $Ax^* = Ax_0 + AFz^* = a + 0 = a$

ii. stationarity: $\nabla f(x^*) + A^T \nu^* \stackrel{?}{=} 0$

$$\begin{pmatrix} F^T \\ A \end{pmatrix} (\nabla f(x^*) + A^T \nu^*) = \begin{pmatrix} F^T \nabla f(x^*) - F^T A^T (AA^T)^{-1} A \nabla f(x^*) \\ A \nabla f(x^*) - AA^T (AA^T)^{-1} A \nabla f(x^*) \end{pmatrix}$$

$$= \begin{pmatrix} \nabla \tilde{f}(z^*) - (AF)^T (\ldots) \\ A \nabla f(x^*) - A \nabla f(x^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and as $\begin{pmatrix} F^T \\ A \end{pmatrix}$ has full rank / is invertible

$$\nabla f(x^*) + A^T \nu^* = 0$$

2. Reducing to Inequality Constraints

- Q: How can we reduce equality constraints to inequality constraints?
- Q: If the equality constraints are affine,
  will the inequality constraints also be affine?
- Q: If the equality constraints are convex,
  will the inequality constraints also be convex?

2. Reducing to Inequality Constraints

- P equality constraints obviously can be represented as $2P$ inequality constraints:

$$g_p(x) = 0, \quad p = 1, \ldots, P \quad \iff \quad \begin{aligned} -g_p(x) &\leq 0, \quad p = 1, \ldots, P \\ g_p(x) &\leq 0, \quad p = 1, \ldots, P \end{aligned}$$

- Then any method for inequality constraints can be used (see next chapter).
- Q: If the equality constraints are affine, will the inequality constraints also be affine?
- Q: If the equality constraints are convex, will the inequality constraints also be convex?

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- For non-linear equality constraints, the problem is not convex.
  - remember: the equality constrained problem also was not convex in this case.

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  - remember: the equality constrained problem also was not convex in this case.

- The inequality constrained problem cannot be strictly feasible.

Equality Constraints / Algorithms

1. Reparametrize:

```
min-eq-reparam(f, A, a, ...) :
  x_0 := solve(Ax = a)
  F := solve-all(Ax = 0)
  z* := min-unconstrained(f~(z) := f(x_0 + Fz), ...)
  return x_0 + Fz*
```

min-eq-reparam$(f, A, a, \ldots)$:
  $x_0 :=$ solve$(Ax = a)$
  $F :=$ solve-all$(Ax = 0)$
  $z^* :=$ min-unconstrained$(\tilde{f}(z) := f(x_0 + Fz), \ldots)$
  return $x_0 + Fz^*$

2. Represent as inequalities:

min-eq-represent-ineq$(f, g_{1:P}, \ldots)$:
  $h_{1:P} := g_{1:P}$
  $h_{P+1:2P} := -g_{1:P}$
  $x^* :=$ min-ineq$(f, h_{1:2P}, \ldots)$
  return x^*

Outline

Quadratic Programming

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\mathbf{x}^T P \mathbf{x} + \mathbf{q}^T \mathbf{x} + r \\ \text{subject to} & A\mathbf{x} = \mathbf{a} \end{array}$$

with given $P \in \mathbb{R}^{N \times N}$ pos. semidef., $\mathbf{q} \in \mathbb{R}^N$, $r \in \mathbb{R}$.

Optimality Condition:

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}^* \\ \nu^* \end{pmatrix} = \begin{pmatrix} -\mathbf{q} \\ \mathbf{a} \end{pmatrix}$$

- KKT Matrix
- solve the linear system of equations to compute a solution/minimum.
  - unique if the *KKT* matrix is invertible/non-singular:

$$\begin{pmatrix} \mathbf{x}^* \\ \nu^* \end{pmatrix} = \begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\mathbf{q} \\ \mathbf{a} \end{pmatrix}$$

Quadratic Programming / Unique Solutions

Unconstrained quadratic programs have a unique solution,
iff P is pos.def.: $\mathbf{x} \neq 0 \quad \Rightarrow \quad \mathbf{x}^T P \mathbf{x} > 0$

Linearly constrained quadratic programs have a unique solution,
iff P is pos.def. on the nullspace of A:

$$A\mathbf{x} = 0, \quad \mathbf{x} \neq 0 \quad \Rightarrow \quad \mathbf{x}^T P \mathbf{x} > 0$$

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Proof: show that the KKT matrix is invertible:

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \nu \end{pmatrix} = 0 \quad \rightsquigarrow \text{(i) } Px + A^T\nu = 0, \quad \text{(ii) } Ax = 0$$

$$\underset{(i)}{\rightsquigarrow} \quad 0 = x^T(Px + A^T\nu) = x^T Px + (Ax)^T\nu \underset{(ii)}{=} x^T Px \quad \underset{ass.}{\rightsquigarrow} x = 0$$

$$\underset{(i)}{\rightsquigarrow} \quad A^T\nu = 0 \quad \rightsquigarrow \quad \nu = 0 \text{ as } A \text{ has full rank}$$

Example

$$
\begin{array}{rl}
\text{minimize} & (x_1 - 2)^2 + 2(x_2 - 1)^2 - 5 \\
\text{subject to} & x_1 + 4x_2 = 3
\end{array}
$$

is an example for a quadratic programming problem:

$$
\begin{aligned}
f(x) &= (x_1 - 2)^2 + 2(x_2 - 1)^2 - 5 \\
&= x_1^2 - 4x_1 + 4 + 2x_2^2 - 2x_2 + 1 - 5 \\
&= x_1^2 + 2x_2^2 - 4x_1 - 2x_2 \\
P &:= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad \mathbf{q} := \begin{pmatrix} -4 \\ -2 \end{pmatrix}, \quad r := 0 \\
A &:= \begin{pmatrix} 1 & 4 \end{pmatrix}, \quad \mathbf{a} := \begin{pmatrix} 3 \end{pmatrix}
\end{aligned}
$$

Outline

Descent step for equality constrained problems

Given the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{a} \end{aligned}$$

- start with a feasible solution $\mathbf{x}$
- compute a step $\Delta\mathbf{x}$ such that
  - f decreases: $f(\mathbf{x} + \Delta\mathbf{x}) \leq f(\mathbf{x})$
  - yields feasible point: $A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{a}$
- which means solving the following problem for $\Delta\mathbf{x}$:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x} + \Delta\mathbf{x}) \\ \text{subject to} \quad & A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{a} \end{aligned}$$

Newton Step

The Newton Step is the solution for the minimization of the second order approximation of f:

$$
\begin{aligned}
\text{minimize} \quad & \hat{f}(\mathbf{x} + \Delta\mathbf{x}) := f(\mathbf{x}) + \nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 f(\mathbf{x}) \Delta\mathbf{x} \\
\text{subject to} \quad & A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{a}
\end{aligned}
$$

which can be simplified to

$$A\Delta\mathbf{x} = 0$$

if the last iterate is feasible already

$$A\mathbf{x} = \mathbf{a}$$

Newton Step

The Newton Step is the solution for the minimization of the second order approximation of f:

$$\begin{array}{ll} \text{minimize} & \hat{f}(\mathbf{x} + \Delta\mathbf{x}) := f(\mathbf{x}) + \nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 f(\mathbf{x}) \Delta\mathbf{x} \\ \text{subject to} & A\Delta\mathbf{x} = \mathbf{0} \end{array}$$

This is a quadratic programming problem with:

- $P := \nabla^2 f(\mathbf{x})$
- $\mathbf{q} := \nabla f(\mathbf{x})$
- $r := f(\mathbf{x})$

and thus optimality conditions:

- $A\Delta\mathbf{x} = \mathbf{0}$
- $\nabla_{\Delta\mathbf{x}} \hat{f}(\mathbf{x} + \Delta\mathbf{x}) + A^T \nu = \nabla f(\mathbf{x}) + \nabla^2 f(\mathbf{x}) \Delta\mathbf{x} + A^T \nu = 0$

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$$\begin{aligned} \text{minimize} \quad & \hat{f}(\mathbf{x} + \Delta\mathbf{x}) := f(\mathbf{x}) + \nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 f(\mathbf{x}) \Delta\mathbf{x} \\ \text{subject to} \quad & A\Delta\mathbf{x} = \mathbf{0} \end{aligned}$$

Is computed by solving the following system:

$$\begin{pmatrix} \nabla^2 f(\mathbf{x}) & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \nu \end{pmatrix} = \begin{pmatrix} -\nabla f(\mathbf{x}) \\ \mathbf{0} \end{pmatrix}$$

Newton's Method for Unconstrained Problems (Review)

min-newton$(f, \nabla f, \nabla^2 f, x^{(0)}, \mu, \epsilon, K)$:
  for $k := 1, \ldots, K$:
    $\Delta x^{(k-1)} := -\nabla^2 f(x^{(k-1)})^{-1} \nabla f(x^{(k-1)})$
    if $-\nabla f(x^{(k-1)})^T \Delta x^{(k-1)} < \epsilon$:
      return $x^{(k-1)}$
    $\mu^{(k-1)} := \mu(f, x^{(k-1)}, \Delta x^{(k-1)})$
    $x^{(k)} := x^{(k-1)} + \mu^{(k-1)} \Delta x^{(k-1)}$
  return "not converged"

where

- f objective function
- ∇f, $\nabla^2 f$ gradient and Hessian of objective function f
- $x^{(0)}$ starting value
- μ step length controller
- ϵ convergence threshold for Newton's decrement
- K maximal number of iterations

Newton's Method for Affine Equality Constraints

min-newton-eq$(f, \nabla f, \nabla^2 f, A, x^{(0)}, \mu, \epsilon, K)$:
  for $k := 1, \ldots, K$:
    $$\begin{pmatrix} \Delta x^{(k-1)} \\ \nu^{(k-1)} \end{pmatrix} := - \begin{pmatrix} \nabla^2 f(x^{(k-1)}) & A^T \\ A & 0 \end{pmatrix}^{-1} \begin{pmatrix} \nabla f(x^{(k-1)}) \\ 0 \end{pmatrix}$$
    if $-\nabla f(x^{(k-1)})^T \Delta x^{(k-1)} < \epsilon$:
      return $x^{(k-1)}$
    $\mu^{(k-1)} := \mu(f, x^{(k-1)}, \Delta x^{(k-1)})$
    $x^{(k)} := x^{(k-1)} + \mu^{(k-1)} \Delta x^{(k-1)}$
  return "not converged"

where

- A affine equality constraints
- $x^{(0)}$ feasible starting value (i.e., $Ax^{(0)} = a$)

Newton's Method for Aff. Eq. Cstrs. / Reduction

min-newton-eq-red$(f, \nabla f, \nabla^2 f, A, a, \mu, \epsilon, K)$:
  $x_0 := \text{solve}(Ax = a)$
  $F := \text{solve-all}(Ax = 0)$
  $z^{(0)} := 0$
  for $k := 1, \ldots, K$:
    $\Delta z^{(k-1)} := \text{solve}((F^T \nabla^2 f(x_0 + F z^{(k-1)}) F) \Delta z = -F^T \nabla f(x_0 + F z^{(k-1)}))$
    if $-F^T \nabla f(x_0 + F z^{(k-1)})^T \Delta z^{(k-1)} < \epsilon$:
      return $x_0 + F z^{(k-1)}$
    $\mu^{(k-1)} := \mu(z \mapsto f(x_0 + Fz), z^{(k-1)}, \Delta z^{(k-1)})$
    $z^{(k)} := z^{(k-1)} + \mu^{(k-1)} \Delta z^{(k-1)}$
  return "not converged"

where

- A, a affine equality constraints

Convergence

- The iterates $x^{(k)}$ are the same as those of the Newton algorithm for the eliminated unconstrained problem

$$\tilde{f}(z) := f(x_0 + Fz), \quad x^{(k)} = x_0 + Fz^{(k)}$$

  - as the Newton steps $\Delta x = F\Delta z$ coincide
    as they fulfil the KKT conditions of the quadratic approximation

- Thus convergence is the same as in the unconstrained case.

Outline

Newton Step at Infeasible Points

If $\mathbf{x}$ is infeasible, i.e. $A\mathbf{x} \neq \mathbf{a}$, we have the following problem:

$$\begin{aligned} \text{minimize} \quad & \hat{f}(\mathbf{x} + \Delta\mathbf{x}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T \Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T \nabla^2 f(\mathbf{x}) \Delta\mathbf{x} \\ \text{subject to} \quad & A\Delta\mathbf{x} = \mathbf{a} - A\mathbf{x} \end{aligned}$$

which can be solved for $\Delta\mathbf{x}$ by solving the following system of equations:

$$\begin{pmatrix} \nabla^2 f(\mathbf{x}) & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \nu \end{pmatrix} = - \begin{pmatrix} \nabla f(\mathbf{x}) \\ A\mathbf{x} - \mathbf{a} \end{pmatrix}$$

- An undamped iteration of this algorithm yields a feasible point.
- With step length control: points will stay infeasible in general.

Step Length Control

- Δx is not necessarily a descent direction for f
- but $(\Delta x \ \nu)$ is a descent direction for the norm of the **primal-dual residuum**:

$$r(x,\nu) := || \begin{pmatrix} \nabla f(x) + A^T \nu \\ Ax - a \end{pmatrix} ||$$

- The Infeasible Start Newton algorithm requires a proper convergence analysis (see Boyd and Vandenberghe, 2004, ch. 10.3.3)

Newton's Method for Lin. Eq. Cstr. / Infeasible Start

min-newton-eq-inf$(f, \nabla f, \nabla^2 f, A, a, x^{(0)}, \nu^{(0)}, \mu, \epsilon, K)$:

  for $k := 1, \ldots, K$:

    if $r(x^{(k-1)}, \nu^{(k-1)}) < \epsilon$:

      return $x^{(k-1)}$

    $\begin{pmatrix} \Delta x^{(k-1)} \\ \Delta \nu^{(k-1)} \end{pmatrix} := -\begin{pmatrix} \nabla^2 f(x^{(k-1)}) & A^T \\ A & 0 \end{pmatrix}^{-1} \begin{pmatrix} \nabla f(x^{(k-1)}) + A^T \nu^{(k-1)} \\ A x^{(k-1)} - a \end{pmatrix}$

    $\mu^{(k-1)} := \mu\left(r, \begin{pmatrix} x^{(k-1)} \\ \nu^{(k-1)} \end{pmatrix}, \begin{pmatrix} \Delta x^{(k-1)} \\ \Delta \nu^{(k-1)} \end{pmatrix}\right)$

    $x^{(k)} := x^{(k-1)} + \mu^{(k-1)} \Delta x^{(k-1)}$

    $\nu^{(k)} := \nu^{(k-1)} + \mu^{(k-1)} \Delta \nu^{(k-1)}$

  return "not converged"

where

- A, a affine equality constraints
- $x^{(0)}$ possibly infeasible starting value (i.e., $Ax^{(0)} \neq a$)
- $\nu^{(0)}$ starting multiplier (e.g., random)
- r is the norm of the primal-dual residuum (see previous slide)

Summary

- Optimal solutions for equality constrained optimization problems
  - have to fulfill KKT conditions:
    1. primal feasibility: $g_p(x) = 0, \quad p = 1, \ldots, P$
    2. stationarity: $\nabla f(x) + \sum_{p=1}^{P} \nu_p \nabla g_p(x) = 0$
  - for convex equality contrained problems,
    1. primal feasibility: $Ax = a$
    2. stationarity: $\nabla f(x) + A^T \nu = 0$
- Equality problems can be handled two ways:
  1. if they are affine, eliminate them.
     - **reparametrize** feasible values
       $$\{x \mid Ax = a\} = x_0 + \{x \mid Ax = 0\} = x_0 + \{Fz \mid z \in \mathbb{R}^{N-P}\}$$
     - then solve **reduced unconstrained problem** in z
  2. represent them as two inequality constraints each.

Summary (2/2)

- **quadratic programming:** affine constrained quadratic objectives can be optimized by solving a linear system of equations.

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}^* \\ \nu^* \end{pmatrix} = \begin{pmatrix} -\mathbf{q} \\ \mathbf{a} \end{pmatrix}$$

- Equality constraints can be **integrated into Newton's method** by extending the linear system for the descent direction:

$$\begin{pmatrix} \nabla^2 f(\mathbf{x}) & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \nu \end{pmatrix} = \begin{pmatrix} -\nabla f(\mathbf{x}) \\ \mathbf{0} \end{pmatrix}$$

  - if the last iterate was already feasible

- Alternatively, for **infeasible starting points**,

$$\begin{pmatrix} \nabla^2 f(\mathbf{x}) & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \nu \end{pmatrix} = -\begin{pmatrix} \nabla f(\mathbf{x}) \\ A\mathbf{x} - \mathbf{a} \end{pmatrix}$$

  - either an undamped step to become feasible or
  - damped steps to reduce the primal-dual residuum

Further Readings

- equality constrained problems, quadratic programming, Newton's method for affine/linear equality constrained problems:
  - Boyd and Vandenberghe, 2004, ch. 10
- further methods for non-linear equality constrained optimization:
  - Murray, 2008

References

- Boyd, Stephen and Lieven Vandenberghe (2004). *Convex Optimization*. Cambridge University Press.
- Murray, Walter (2008). *Lecture Notes on Nonlinear Constraints / Chapter 3: Nonlinear Constraints*.