MODERN OPTIMIZATION TECHNIQUES

THIRD TAKE HOME EXAM : GROUP 01

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Q2A. The original backtracking line search condition is

$$f(x+\mu\Delta x) > f(x) + \mu \Delta f$$

where,
$$\Delta f = \alpha \nabla f(x)^T \Delta x$$

For inequality constraints, we can modify it as

$$f(x+\mu\Delta x) > f(x) + \mu \Delta f \quad \text{or} \quad \text{not } h(x+\mu\Delta x) \leq 0$$

For affine inequality constraints, feasibility of an update can be guaranteed by a maximal stepsize

$$h(x+\mu\Delta x) = B(x+\mu\Delta x) - b \leq 0$$

which gives,

$$\mu \leq \min \left\{ \frac{-(Bx-b)_q}{(B\Delta x)_q} \,\middle|\, q \in \{1,\dots,Q\} : (B\Delta x)_q > 0 \right\}$$

So, we start with the value of μ above and then perform the backtracking line search as before.

02

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Q2B:

$$\min \quad f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 5x_2$$

s.t.

$$-x_1 + 2x_2 - 2 \le 0 \quad \ldots \quad (1)$$

$$x_1 + 2x_2 - 6 \le 0 \quad \ldots \quad (2)$$

$$x_1 - 2x_2 - 2 \le 0 \quad \ldots \quad (3)$$

$$-x_1 \le 0 \quad \ldots \quad (4)$$

$$-x_2 \le 0 \quad \ldots \quad (5)$$

There is a problem in Question, So I am using this

$$\mu = 1, \quad x^{(0)} = [2 \quad 0]^T$$

Checking which inequalities are active

$-2 + 0 - 2 \le 0$

$-4 < 0$

inactive

$2 + 0 - 6 \le 0$

$-4 \le 0$

inactive

$2 - 0 - 2 \le 0$

$0 = 0$

active

$-2 < 0$ inactive

$0 = 0$ active.

So,

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \quad a = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 5 \end{bmatrix}$$

Newton Method is

$$\begin{bmatrix} \Delta x^{(k-1)} \\ v^{(k-1)} \end{bmatrix} = -\begin{bmatrix} \nabla^2 f(x^{k-1}) & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla f(x^{k-1}) \\ 0 \end{bmatrix}$$

So,

$$\begin{bmatrix} \Delta x^{(0)} \\ v^{(0)} \end{bmatrix} = -\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 \\ 1 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \end{bmatrix}$$

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Q2B:

Now $v^{(0)} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \begin{matrix} \rightarrow \text{Constraint } 3 \\ \rightarrow \text{Contrant } 5 \end{matrix}$

~~Dropping Constraint 3~~ $\Delta x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

~~$A = \{0 \quad -1\}$~~ ~~$a = 0$~~

$$x^{(1)} = x^{(0)} + \mu \Delta x^{(0)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Now, dropping constraint 3 with ~~lowest~~ most -ve value

$$A = \{0 \quad -1\}, \quad a = 0 \qquad \begin{bmatrix} 2(2)-2 \\ 2(0)-5 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x^{(1)} \\ v^{(1)} \end{bmatrix} = - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3.1

Dropping constraint 5 as well, $A = \{\}$, $a = \phi$

$$\begin{bmatrix} \Delta x^{(2)} \\ v^{(2)} \end{bmatrix} = - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2(1)-2 \\ 2(0)-5 \end{bmatrix} = - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$$

Now, the point $x^{(3)}$ is infeasible. So we have to make it feasible as follows.

04

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Q2B

Let $p_1 = x^{(3)} - x^{(2)} = \begin{bmatrix}1\\2.5\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\2.5\end{bmatrix}$

& $\hat{x}^{(3)} = x^{(2)} + t\,p_1 = \begin{bmatrix}1\\0\end{bmatrix} + t\begin{bmatrix}0\\2.5\end{bmatrix} = \begin{bmatrix}1\\2.5t\end{bmatrix}$

Now, for all inequality constraints, we check t.

$$A\left(x^{(2)} + t\left(x^{(3)} - x^{(2)}\right)\right) \leq a$$

$$\begin{bmatrix}-1 & 2\\1 & 2\\1 & -2\\-1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\2.5t\end{bmatrix} \leq \begin{bmatrix}2\\6\\2\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}-1+5t \leq 2 & \Rightarrow & t \leq 3/5\\ 1+5t \leq 6 & \Rightarrow & t \leq 1\\ 1-5t \leq 2 & \Rightarrow & t \geq -1/5\\ -1 \leq 0 & & \\ -2.5t \leq 0 & \Rightarrow & t \geq 0\end{matrix}$$

Since constraint 2 is active and constraint 1 is violated, we use $t = 3/5$. Hence,

$$\hat{x}^{(3)} = \begin{bmatrix}1\\0\end{bmatrix} + \frac{3}{5}\begin{bmatrix}0\\2.5\end{bmatrix} = \begin{bmatrix}1\\1.5\end{bmatrix}$$

Now, adding constraint 1 to active set, because it is active now

$$A = \begin{bmatrix}-1 & 2\end{bmatrix}, \quad a = 6$$

$$\begin{bmatrix}\Delta x^{(3)}\\ \gamma^{(3)}\end{bmatrix} = -\begin{bmatrix}2 & 0 & -1\\0 & 2 & 2\\-1 & 2 & 0\end{bmatrix}^{-1}\begin{bmatrix}2(1)-2\\2(1.5)-5\\0\end{bmatrix} = -\begin{bmatrix}2 & 0 & -1\\0 & 2 & 2\\-1 & 2 & 0\end{bmatrix}^{-1}\begin{bmatrix}0\\-2\\0\end{bmatrix}$$

$$\begin{bmatrix}\Delta x^{(3)}\\ \gamma^{(3)}\end{bmatrix} = \begin{bmatrix}0.4\\0.2\\0.8\end{bmatrix} \rightarrow \text{+ve, so we stop.}$$

Hence,

$$x^{*} = x^{(4)} = \begin{bmatrix}1\\1.5\end{bmatrix} + (1)\begin{bmatrix}0.4\\0.2\end{bmatrix} = \begin{bmatrix}1.4\\1.7\end{bmatrix} \quad \text{Answer.}$$

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Q 2C The given objective function $4x_1 + x_2 + 3$ is linear. However, ~~on~~ the equality constraint is quadratic. Primal problem is given by.

$$\mathcal{L}(x, \nu, \lambda) = 4x_1 + x_2 + 3 + \nu(2x_1^2 + 2x_2^2 - 6) + \lambda_1(3x_1 - 2x_2) + \lambda_2(x_1 + x_2 + 1)$$

Dual is

?

$$g(\nu, \lambda) = \text{infimum} \left[\mathcal{L}(x, \nu, \lambda) \right]$$

Taking derivative of $\mathcal{L}$ and putting $= 0$

$$\nabla \mathcal{L} = \begin{bmatrix} 4 + 4\nu x_1 + 3\lambda_1 + \lambda_2 \\ 1 + 4\nu x_2 - 2\lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we have $4\nu x_1 = -3\lambda_1 - \lambda_2 - 4 \Rightarrow x_1 = \dfrac{-3\lambda_1 - \lambda_2 - 4}{4\nu}$

$4\nu x_2 = 2\lambda_1 - \lambda_2 - 1 \Rightarrow x_2 = \dfrac{2\lambda_1 - \lambda_2 - 1}{4\nu}$

Putting x_1 & x_2 in $g(\nu, \lambda)$.

$$g(\nu, \lambda) = \frac{-3\lambda_1 - \lambda_2 - 4}{\nu} + \frac{2\lambda_1 - \lambda_2 - 1}{4\nu} + 3$$

$$+ \nu \left(2\left(\frac{-3\lambda_1 - \lambda_2 - 4}{4\nu}\right)^2 + 2\left(\frac{2\lambda_1 - \lambda_2 - 1}{4\nu}\right)^2 - 6 \right)$$

$$+ \lambda_1 \left(3\left(\frac{-3\lambda_1 - \lambda_2 - 4}{4\nu}\right) - 2\left(\frac{2\lambda_1 - \lambda_2 - 1}{4\nu}\right) \right) + \lambda_2 \left(\frac{-3\lambda_1 - \lambda_2 - 4}{4\nu} + \frac{2\lambda_1 - \lambda_2 - 1}{4\nu} + 1 \right)$$

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Q2C

$g(v,\lambda)$ can be simplified further but I think it is not required here.

Advantage of dual $g(v,\lambda)$ here is that it is not linear anymore. Moreover it is concave and can be solved as a maximization problem. Our primal problem had the objective function linear so it was difficult to solve.

Q3A:

$$f_0(x_1, x_2) = t(x_1 + 2x_2) - \log(x_1) - \log(x_2 - 3)$$

s.t. $\quad x_1 + x_2 = 4.$

Barrier function is given by.

$$t \cdot f(x_1, x_2) + B(x_1, x_2)$$

where $\quad B(x_1, x_2) = -\sum_{q=1}^{Q} \log(-h_q(x))$ we have $Q=2$

$$= -\log(-h_1(x)) - \log(-h_2(x)).$$

Comparing, we get original problem as

$$f(x_1, x_2) = x_1 + 2x_2$$

s.t. $\quad h_1(x) \Rightarrow -x_1 \leq 0,$

$h_2(x) \Rightarrow -x_2 + 3 \leq 0$

$g(x) \Rightarrow x_1 + x_2 = 4$

Q3A. Primal problem will be.

$$\mathcal{L}(x_1, x_2, v, \lambda) = x_1 + 2x_2 + v(x_1 + x_2 - 4) + \lambda_1(-x_1) + \lambda_2(-x_2 + 3)$$

Dual is

$$g(v, \lambda) = \text{infinimum}\left[\mathcal{L}(x, v, \lambda)\right]$$

Taking derivative of $\mathcal{L}$ and putting equal to 0.

$$\nabla \mathcal{L} = \begin{bmatrix} 1 + v - \lambda_1 \\ 2 + v - \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As can be seen, we cannot write x_1 and x_2 in terms of v, λ_1, λ_2 here and thus cannot substitute in dual langrangian. Hence, we cannot ~~compt~~ solve the dual problem here. The reason is that it is a constrained linear program for which there is no analytical solution as discussed by Prof. Lars. For such a problem, we have certain specialized algorithms like the Simplex Tableau.

Q3B. Quadratic penalty function is given by

$$P(x) = \sum_{p=1}^{P} (g_p(x))^2 \quad , \text{ we have } P = 1$$

So,

$$f(x_1, x_2) + c\,P(x) = x_1^2 + x_2^2 + c\left(-0.5x_1 + x_2 - 1\right)^2$$

08

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Q3B

For $x = [0 \quad 0]^T$, the penalty function is not 0.

So, we taking derivative and put it equal to 0.

$$\nabla \left(f(x_1,x_2) + c\,P(x_1,x_2) \right) = \begin{bmatrix} 2x_1 + 2c(-0.5x_1+x_2-1)(-0.5) \\ 2x_2 + 2c(-0.5x_1+x_2-1)(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we have

$$2x_1 - c(-0.5x_1 + x_2 - 1) \Rightarrow x_1 = 0.5c(-0.5x_1 + x_2 - 1)$$

$$2x_2 + 2c(-0.5x_1 + x_2 - 1) \Rightarrow x_2 = -c(-0.5x_1 + x_2 - 1)$$

$$\text{or} \quad -x_2 = c(-0.5x_1 + x_2 - 1)$$

comparing, we get

$$x_1 = 0.5(-x_2) = -0.5x_2$$

Putting in x_2 derivative term

$$-x_2 = c(-0.5(-0.5x_2) + x_2 - 1)$$

$$\text{or} \quad x_2 + c(0.25x_2 + x_2 - 1) = 0$$

$$x_2 + 1.25cx_2 = c \Rightarrow x_2 = \frac{c}{1+1.25c}$$

&

$$x_1 = -0.5x_2 = \frac{-0.5c}{1+1.25c}$$

Answer.

8.1

−1

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Q3C: Advantages of Penalty over Barrier

- we can start from any point as the problem is solved in an unconstrained manner.
- Unlike ~~Barr~~ Barrier, Penalty method works for both equality and inequality constraints. In cases where we have inequality constraints these are converted to equality constraints and the resulting functions is not differentiable at the border, so we use subgradients.

Yes, both methods can be combined. For instance, given the problem

$$\min f(x)$$
$$\text{s.t. } g_i(x) = 0 \quad i = 1, \dots p$$
$$h_i(x) \le 0, \quad i = 1, \dots, q.$$

We can form a logarithmic - quadratic function using log Barrier and quadratic penalty functions as follows.

$$f(x) + c\left(-\sum_{i=1}^{q} \log(-h_i(x))\right) + \frac{1}{c}\sum_{i=1}^{p}(g_i(x))^2$$

This works similar to the pure penalty and Barrier functions.. We can also combine other

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Q3c Barrier and Penalty function, similarly to form different combinations of mixed Barrier and Penalty objective functions.

Q1B

$$\min \quad f(x_1, x_2) = 2x_1^2 + x_2^2 + x_1 x_2$$

$$\text{s.t.} \quad x_1 - x_2 = 2$$

$$x^{(0)} = [4/3, -2/3]^T, \quad \mu = 0.05$$

Checking if $x^{(0)}$ is feasible.

$$\frac{4}{3} - \left(-\frac{2}{3}\right) = 2 \Rightarrow \frac{6}{3} = 2 \Rightarrow 2 = 2 \checkmark$$

The equations: $A = [1 \quad -1]$, $a = 2$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 4x_1 + 1 \ x_2 \\ 2x_2 + 1 \ x_1 \end{bmatrix} \quad \nabla^2 f(x_1, x_2) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Newton Method.

$$\begin{bmatrix} \Delta x^{(k-1)} \\ v^{(k-1)} \end{bmatrix} = -\begin{bmatrix} \nabla^2 f(x_1^{(k-1)}, x_2^{(k-1)}) & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} \nabla f(x_1^{k-1}, x_2^{k-1}) \\ 0 \end{bmatrix}$$

So

$$\begin{bmatrix} \Delta x^{(0)} \\ v^{(0)} \end{bmatrix} = -\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4(4/3) + 1 \\ 2(-2/3) + 1 \\ 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6.33 \\ -0.33 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2.33 \end{bmatrix}$$

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Q1B:

$$x^{(1)} = x^{(0)} + \mu \Delta x^{(0)} = \begin{bmatrix} 4/3 \\ -2/3 \end{bmatrix} + 0.05 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 4/3 \\ -2/3 \end{bmatrix} + \begin{bmatrix} -0.05 \\ -0.05 \end{bmatrix} = \begin{bmatrix} 1.283 \\ -0.717 \end{bmatrix}$$ Answer

Next iteration

$$\begin{bmatrix} \Delta x^{(1)} \\ v^{(1)} \end{bmatrix} = - \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4(1.283) + 1 \\ 2(-0.717) + 1 \\ 0 \end{bmatrix}$$

$$= - \begin{bmatrix} 0.167 & 0.167 & 0.33 \\ 0.167 & 0.167 & -0.67 \\ 0.33 & -0.667 & -1.33 \end{bmatrix} \begin{bmatrix} 6.132 \\ -0.434 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.9496 \\ -0.9496 \\ -2.33 \end{bmatrix}$$

11.1

So,

$$x^{(2)} = x^{(1)} + \mu \Delta x^{(1)} = \begin{bmatrix} 1.283 \\ -0.717 \end{bmatrix} + (0.05) \begin{bmatrix} -0.9496 \\ -0.9496 \end{bmatrix}$$

-1

$$x^{(2)} = \begin{bmatrix} 1.2355 \\ -0.7645 \end{bmatrix}$$ Answer.

Q1C

$$f(x) = x^2 + 1 \quad \text{s.t.} \quad (x-2)(x-4) \le 0$$

The given objective function is clearly quadratic since the highest-degree term is of second degree. For feasible set, we write the constraint in standard form ~~Ax = b~~ ~~Based~~

so, $h(x) = x^2 - 6x + 8 \le 0$

12)

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Q1C

Taking derivative.

$\nabla h(x) = 2x - 6 \le 0$ or $x \le 3$

This give the feasible set.

Now,

KKT conditions

1. Primal feasibility $h(x \le 0 \Rightarrow x^2 - 6x + 8 \le 0$

2. dual feasibility $\lambda \geq 0$

3. complementary slackness. $\lambda h(x) = 0$, $x^2\lambda - 6\lambda x + 8\lambda = 0$

4. Stationarity. $\nabla f(x) + \lambda \nabla h(x) = 0$

$$2x + \lambda(2x-6) = 0$$

$$2x + 2\lambda x - 6\lambda = 0$$

So, $x^2 - 6x + 8 \le 0 \Rightarrow (x-2)(x-4) \le 0$, $x \le 2, 4$

Putting in (4) $x = 2$

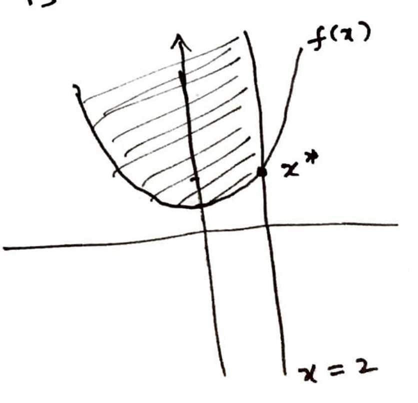
$4 + 4\lambda - 6\lambda = 0 \Rightarrow \lambda = 2$

$x = 4$, $8 + 8\lambda - 6\lambda = 0 \Rightarrow \lambda = -4$

Since, $\lambda \geq 0$, the correct λ is 2

and our feasible solution is

$$x^* = 2$$



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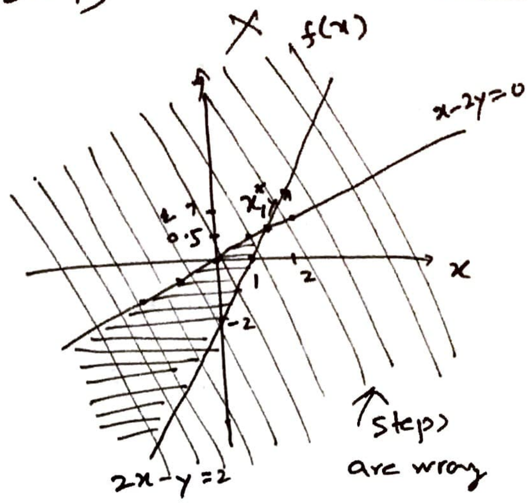
13

Q1A: (a) $\max \quad 4x - y$

$$\text{s.t.} \quad 2x - y \le 2$$
$$x - 2y \le 0$$



It can be shown graphically that it is possible to solve this problem. Our unconstrained objective function does not have a maximum as shown by the steps in graph. Now, both inequalities are plotted and these bound the unconstrained function to a feasible region marked with horizontal lines in the plot. Clearly, this region has a maximum which can also be computed as follows:

$$x - 2y \le 0 \quad \Rightarrow \quad x = 2y$$

$$2(2y) - y \le 2 \quad \Rightarrow \quad y = 2/3, \quad x = 4/3$$

13.1

-0,75

So, optimal maximum is

$$(x^*, y^*) = \left(\frac{4}{3}, \frac{2}{3}\right)$$

This can also be seen in the plot.

Please see next page for correct plot

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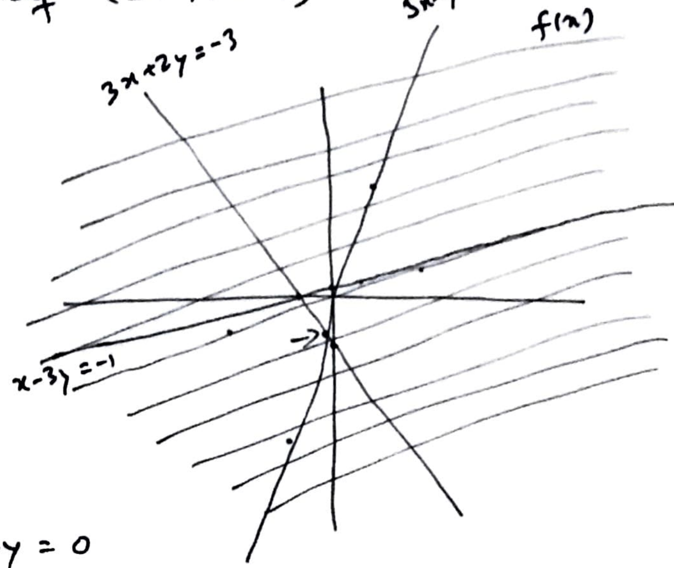
Q1A: (b) max $x-3y$

s.t. $x-3y \leq -1$

$3x+2y \leq -3$

$5x-y = 0$



As shown graphically, a point on the line $5x-y=0$ will be a maximum. There will not be a region because we have an equality here.

Solving. $y = 5x \Rightarrow 3x + 10x = -3$

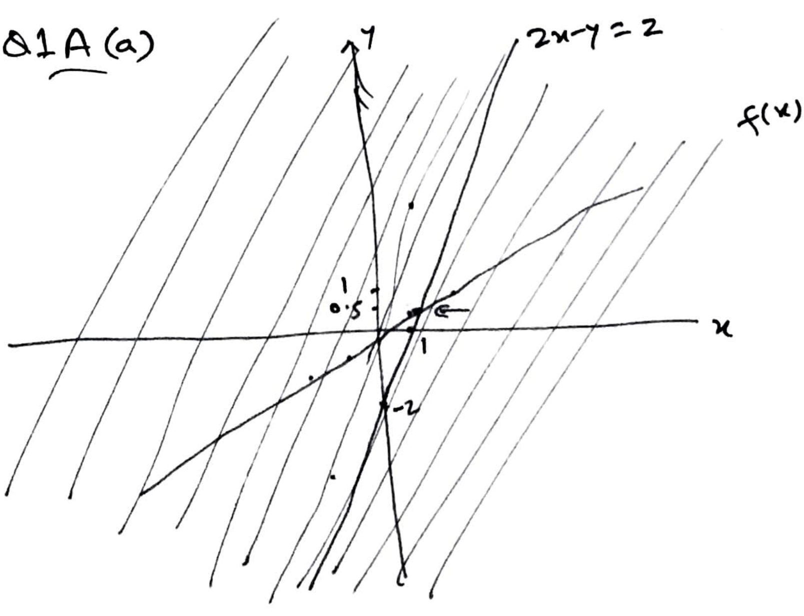
$\Rightarrow x = -3/13$

& $y = -15/13$

The other point.

$x - 15x = -1$, $x = 1/14$
$y = 5/14$ } is not feasible as it violates $3x+2y \leq -3$.

Correct plot of Q1A (a)



Index der Kommentare

3.1 you can stop here as the working set is empty as mentioned in the question

8.1 you have to compute optimal points by taking the limit when c--> infinity

11.1 computation errors

13.1 reasoning is not correct. this problem has unbounded solution