Amir Hossein Eyvazkhani

1747696

Group 1

1

a) Convexity is generally defined as:

$$f(\alpha x+\beta y) \leqslant \alpha f(x)+\beta f(y)$$

for all $x, y \in R^n$ and $\alpha, \beta \in R$ with $\alpha+\beta=1$ and $\alpha, \beta \geqslant 0$.

One of the benefits of convexity is that it guarantees the existence of local minima. Also, Convex Optimization Problems can be Solved efficiently using algorithmic approaches.

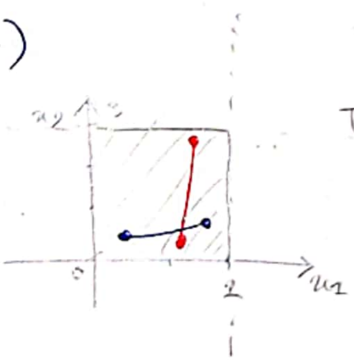
b) A set C is convex if the line segment between any two points in C lies in C. e.g.: for any $x, y \in C$ and any $\theta \in [0,1]$ we have

$$\theta y+(1-\theta) x \in C$$

A function $f: R^n \to R$ is convex if $\operatorname{dom} f$ is convex set and if for all $x, y \in \operatorname{dom} f$ and θ with $0 \leqslant \theta \leqslant 1$ we have

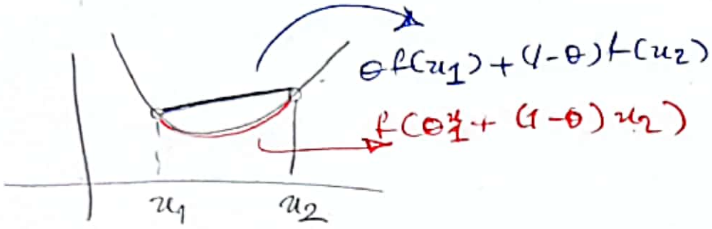
$$f(\theta x+(1-\theta) y) \leqslant \theta f(x)+(1-\theta) f(y)$$

c)

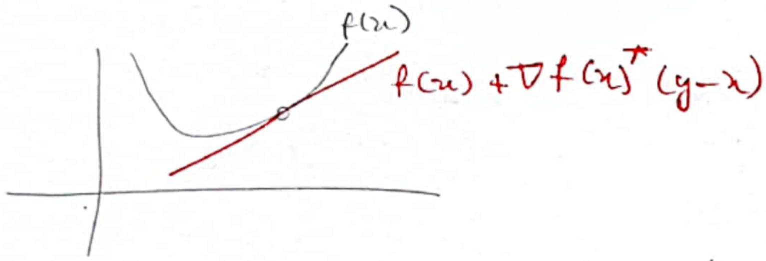


The colored area is the set. As we can see, a line between any two points of D, is also inside D.

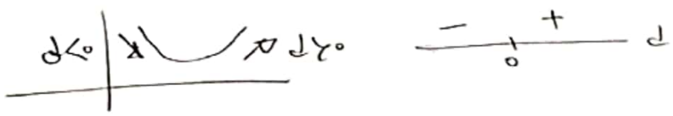
d) i) Jensen's inequality shows that the function is always lower/less than the interpolation (straight line) between two points in the domain of that function.



ii) This shows that any first order Taylor approximation of f near x, namely tangent lines in x, is lesser than the actual function. e.g.: It is a global underestimator.



iii) This shows that the derivative is non-decreasing. This means that the derivative cure is a non-decreasing curve and the function itself, is a curve with this around the minima:



2

i) Theory 1) a matrix is possitive definite if and only if the determinent of each square matrix in it's upper left are all possitive.

$$\frac{df}{dx} = 2ax - 4ay \qquad \frac{d^2f}{dxdx} = 2a, \ \frac{d^2f}{dxdy} = -4a, \ \frac{d^2f}{dxdz} = 0$$

$$\frac{df}{dy} = 2y - 4ax + 2z \qquad \frac{d^2f}{dydx} = -4a, \ \frac{d^2f}{d^2y} = 2, \ \frac{d^2f}{dydz} = 2$$

$$\frac{df}{dz} = 4z + 2y \qquad \frac{d^2f}{dzdx} = 0, \ \frac{df}{dzdy} = 2, \ \frac{df}{d^2z} = 4$$

$$H_f = \begin{vmatrix} 2a & -4a & 0 \\ -4a & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix}$$

i) it is symetric ✓

ii) by theoriem 1 :

$2a \geq 0 \Rightarrow a \geq 0$ I

$\det \begin{vmatrix} 2a & -4a \\ -4a & 2 \end{vmatrix} \geq 0 \Rightarrow 4a - 16a^2 \geq 0 \Rightarrow a(4-16a) \geq 0 \Rightarrow 0 \leq a \leq 1/4$ II

$\det(H_f) \geq 0 \Rightarrow 16a + 0 + 0 - 0 - 8a - 64a^2 \geq 0 \Rightarrow 8a(1-8a) \geq 0$

$\Rightarrow 0 \leq a \leq \frac{1}{8}$ III

I, II, III $\Rightarrow 0 \leq a \leq 1/8$

ii)
$$\frac{df}{dx} = 8ax + 8y \qquad \frac{df}{d^2x} = 8a, \ \frac{df}{dxdy} = 8$$

$$\frac{df}{dy} = 8x + 2by \qquad \frac{df}{dydx} = 8, \ \frac{df}{d^2y} = 2b$$

$$H_f = \begin{pmatrix} 8a & 8 \\ 8 & 2b \end{pmatrix}$$

i) it is symetric ✓

ii) (theoriem 1)

$8a \geq 0 \Rightarrow a \geq 0$ I

$16ab - 64 \geq 0 \Rightarrow ab \geq 4$ II

$\Bigg\} \to$ I, II $\Rightarrow a \geq 0, \ b \geq \frac{4}{a}$

(b)

i) $\frac{df}{dx} = \frac{1}{\sqrt{xy}} \times \frac{y}{2\sqrt{xy}} = \frac{1}{2x}$ $\quad \frac{df}{d^2x} = \frac{-1}{2x^2}$, $\frac{df}{dxdy} = 0$

$\frac{df}{dy} = \frac{1}{\sqrt{xy}} \times \frac{x}{2\sqrt{xy}} = \frac{1}{2y}$ $\quad \frac{df}{dydx} = 0$, $\frac{df}{d^2y} = \frac{-1}{2y^2}$

$$H_f = \begin{pmatrix} \frac{-1}{2x^2} & 0 \\ 0 & \frac{-1}{2y^2} \end{pmatrix}$$

i) symetric ✓

ii) theoriem 2: if the det of square matrixes from left hand side alternate signs, then the matrix is negative definite. (II)

$\det\left(\frac{-1}{2x^2}\right) < 0$

$\det(H_f) = \frac{1}{4y^2x^2} \geq 0$ } (I)

(I), (II) $\longrightarrow$ f is concave.

ii) $\frac{df}{dx} = 2(x-1) + y^2$ $\quad \frac{df}{d^2x} = 2$, $\frac{df}{dxdy} = 2y$

$\frac{df}{dy} = 2xy$ $\quad \frac{df}{dydx} = 2y$, $\frac{df}{d^2y} = 2x$

$$H_f = \begin{pmatrix} 2 & 2y \\ 2y & 2x \end{pmatrix}$$

i) it is symetric ✓

ii) theoriem 1):

$2 > 0$ ✓

$\det(H_f) \geq 0 \Rightarrow 4x - 4y^2 \geq 0 \Rightarrow x \geq y^2 \Rightarrow \text{dom} f = \{(x,y) \in R \mid x \geq y^2\}$