# Modern Optimization Techniques

## 2. Unconstrained Optimization / 2.6. Coordinate Descent

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# Syllabus

| | | |
|---|---|---|
| Mon. 30.10. | (1) | 0. Overview |
| | | **1. Theory** |
| Mon. 6.11. | (2) | 1. Convex Sets and Functions |
| | | **2. Unconstrained Optimization** |
| Mon. 13.11. | (3) | 2.1 Gradient Descent |
| Mon. 20.11. | (4) | 2.2 Stochastic Gradient Descent |
| Mon. 27.11. | (5) | 2.3 Newton's Method |
| Mon. 4.12. | (6) | 2.4 Quasi-Newton Methods |
| Mon. 11.12. | (7) | 2.5 Subgradient Methods |
| Mon. 18.12. | — | — *canceled* — |
| | — | — *Christmas Break* — |
| Mon. 8.01. | (8) | 2.6 Coordinate Descent |
| | | **3. Equality Constrained Optimization** |
| Mon. 15.01. | (9) | 3.1 Duality |
| Mon. 22.01. | (10) | 3.2 Methods |
| | | **4. Inequality Constrained Optimization** |
| Mon. 29.01. | (11) | 4.1 Primal Methods |
| Mon. 5.02. | (12) | 4.2 Barrier and Penalty Methods |
| Mon. 12.02. | (13) | 4.3 Cutting Plane Methods |
| | | Q & A |

# Outline

1. Idea and Optimality

2. Coordinate Descent Algorithm

3. Convergence

# Outline

# Coordinate Descent

- Gradient Descent and Stochastic Gradient Descent:
  - update **all** variables simultaneously.
  - use the gradient to do so.
    (first order methods)
- **Coordinate Descent**:
  - update **one** variable at a time.
  - use an **analytic solver** to do so
    (**derivative-free method**)
    - if not possible: use one-dimensional gradient steps
      (first order method; often slow)

# Coordinate Descent

Start with an initial guess $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}, \ldots, \mathbf{x}_N^{(0)}) \in \mathbb{R}^N$

For $k = 1, 2, 3, \ldots$:

# Coordinate Descent

Start with an initial guess $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}, \ldots, \mathbf{x}_N^{(0)}) \in \mathbb{R}^N$

For $k = 1, 2, 3, \ldots$:

$$\mathbf{x}_1^{(k)} \leftarrow \arg\min_{\mathbf{x}_1} f(\mathbf{x}_1, \mathbf{x}_2^{(k-1)}, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

# Coordinate Descent

Start with an initial guess $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}, \ldots, \mathbf{x}_N^{(0)}) \in \mathbb{R}^N$

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$$\mathbf{x}_2^{(k)} \leftarrow \arg\min_{\mathbf{x}_2} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

# Coordinate Descent

Start with an initial guess $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}, \ldots, \mathbf{x}_N^{(0)}) \in \mathbb{R}^N$

For $k = 1, 2, 3, \ldots$:

$$\mathbf{x}_1^{(k)} \leftarrow \arg\min_{\mathbf{x}_1} f(\mathbf{x}_1, \mathbf{x}_2^{(k-1)}, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

$$\mathbf{x}_2^{(k)} \leftarrow \arg\min_{\mathbf{x}_2} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

$$\mathbf{x}_3^{(k)} \leftarrow \arg\min_{\mathbf{x}_3} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \mathbf{x}_3, \ldots, \mathbf{x}_N^{(k-1)})$$

# Coordinate Descent

Start with an initial guess $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}, \ldots, \mathbf{x}_N^{(0)}) \in \mathbb{R}^N$

For $k = 1, 2, 3, \ldots$:

$$\mathbf{x}_1^{(k)} \leftarrow \arg\min_{\mathbf{x}_1} f(\mathbf{x}_1, \mathbf{x}_2^{(k-1)}, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

$$\mathbf{x}_2^{(k)} \leftarrow \arg\min_{\mathbf{x}_2} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

$$\mathbf{x}_3^{(k)} \leftarrow \arg\min_{\mathbf{x}_3} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \mathbf{x}_3, \ldots, \mathbf{x}_N^{(k-1)})$$

$$\vdots$$

$$\mathbf{x}_n^{(k)} \leftarrow \arg\min_{\mathbf{x}_n} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \ldots, \mathbf{x}_{n-1}^{(k)}, \mathbf{x}_n, \mathbf{x}_{n+1}^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

# Coordinate Descent

Start with an initial guess $\mathbf{x}^{(0)} = (\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}, \ldots, \mathbf{x}_N^{(0)}) \in \mathbb{R}^N$

For $k = 1, 2, 3, \ldots$:

$$\mathbf{x}_1^{(k)} \leftarrow \arg\min_{\mathbf{x}_1} f(\mathbf{x}_1, \mathbf{x}_2^{(k-1)}, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

$$\mathbf{x}_2^{(k)} \leftarrow \arg\min_{\mathbf{x}_2} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2, \mathbf{x}_3^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

$$\mathbf{x}_3^{(k)} \leftarrow \arg\min_{\mathbf{x}_3} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \mathbf{x}_3, \ldots, \mathbf{x}_N^{(k-1)})$$

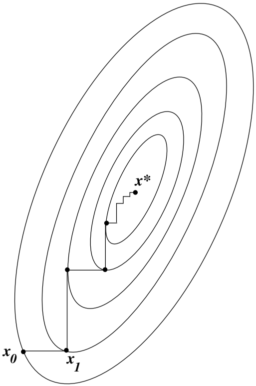
$$\vdots$$

$$\mathbf{x}_n^{(k)} \leftarrow \arg\min_{\mathbf{x}_n} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \ldots, \mathbf{x}_{n-1}^{(k)}, \mathbf{x}_n, \mathbf{x}_{n+1}^{(k-1)}, \ldots, \mathbf{x}_N^{(k-1)})$$

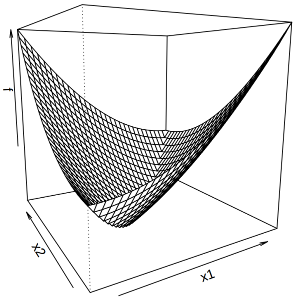
$$\vdots$$

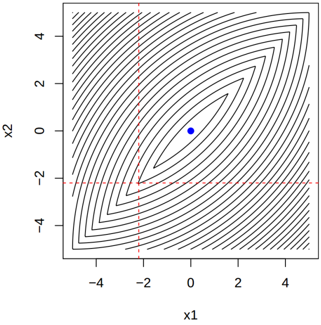
$$\mathbf{x}_N^{(k)} \leftarrow \arg\min_{\mathbf{x}_N} f(\mathbf{x}_1^{(k)}, \mathbf{x}_2^{(k)}, \ldots, \mathbf{x}_{N-1}^{(k)}, \mathbf{x}_N)$$

# Coordinate Descent Algorithm




Nocedal and Wright, 2006, p.249




[https://www.cs.cmu.edu/~ggordon/10725-F12/slides/25-coord-desc.pdf]

- assume the current iterate $x^{(k)}$ is the intersection of the two red lines.
- Q: Where will a coordinate descent step along axis $x_1$ go to?
- Q: Where will a coordinate descent step along axis $x_2$ go to?
- Q: Where will the next coordinate descent step go to?

# Does Coordinate-wise Minimization Lead to Global Minima?

**Question:** If a point $\mathbf{x}$ is minimal along each axis, is it a global minimum?

$$f(\mathbf{x} + t\,\mathbf{e^{(n)}}) \geq f(\mathbf{x}) \quad \forall t \in \mathbb{R}, \forall n \in \{1, \ldots, N\}$$
$$\stackrel{?}{\Longrightarrow} f(\mathbf{x}) = \min_{\mathbf{y}} f(\mathbf{y})$$

where:

- $\mathbf{e^{(n)}} \in \mathbb{R}^N$ is the $n$-th unit vector with $\mathbf{e}^{(n)}_n := 1$ and $\mathbf{e}^{(n)}_m := 0$ for $m \neq n$.

# Does Coordinate-wise Minimization Lead to Global Minima?

If $f$ is **convex** and **differentiable**: yes.

Proof: $g^{(i)}(\mu) := f(\mathbf{x} + \mu\, \mathbf{e^{(i)}})$ are convex and differentiable.

$\mu = 0$ is their minimum, thus

$$0 = \frac{\partial g^{(i)}}{\partial \mu}(\mu) = \frac{\partial f}{\partial \mathbf{x}_i}(\mathbf{x})$$

And then

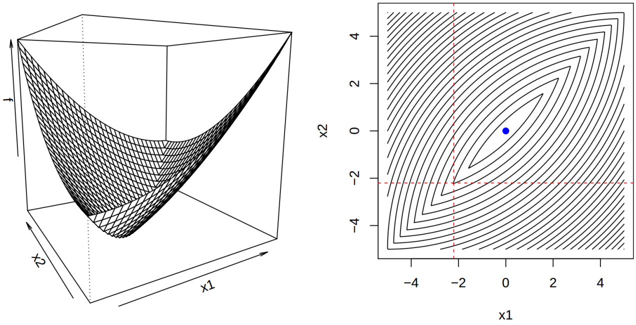
$$\nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_1}, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_2}, \ldots, \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_n}\right)^T = \mathbf{0}$$

and thus $x$ is the global optimum.

# Does Coordinate-wise Minimization Lead to Global Minima?

If $f$ is **convex**, but **not differentiable**: in general, no.

Counter example:



# Does Coordinate-wise Minimization Lead to Global Minima?

If $f$ is a sum of

- a differentiable convex and
- a separable convex function,

then: yes.

$$f(\mathbf{x}) = g(\mathbf{x}) + \sum_{n=1}^{N} h_n(\mathbf{x}_n)$$

with

i. $g$ is differentiable and convex

ii. all $h_n$ are convex
(but each $h_n$ depends only on a single $\mathbf{x}_n$)

## Does Coordinate-wise Minimization Lead to Global Minima? / Proof

Proof: For any $\mathbf{y} \in \operatorname{dom} f$:

$$
\begin{aligned}
f(\mathbf{y}) - f(\mathbf{x}) &= g(\mathbf{y}) - g(\mathbf{x}) + \sum_{n=1}^{N} h_n(\mathbf{y}_n) - h_n(\mathbf{x}_n) \\
&\underset{\text{convex}}{\geq} \nabla g(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \sum_{n=1}^{N} s_n(\mathbf{y}_n - \mathbf{x}_n), \quad \forall s_n \in \partial h_n(x_n) \\
&= \sum_{n=1}^{N} \underbrace{(\nabla_n g(\mathbf{x}) + s_n)}_{=0 \text{ for } s_n = -\nabla_n g(\mathbf{x})}(\mathbf{y}_n - \mathbf{x}_n) = 0
\end{aligned}
$$

as $s_n := -\nabla_n g(\mathbf{x})$ is a subgradient of $h_n$:

$$
\begin{aligned}
& x \text{ is minimal along axis } n \\
\rightsquigarrow \quad & f(X_n; X_{-n}) \text{ is convex and has minimum at } X_n = x_n \\
\rightsquigarrow \quad & 0 \text{ is a subgradient of } f(X_n; X_{-n}) \text{ at } x_n: \\
& \quad 0 \in \partial f(X_n; X_{-n}) = \nabla_n g(x) + \partial h_n(x_n) \\
\rightsquigarrow \quad & \exists s_n \in \partial h_n(x_n) : \nabla_n g(x) + s_n = 0
\end{aligned}
$$

# Outline

# One-dimensional Subproblems

Let $f : X \to \mathbb{R}, X \subseteq \mathbb{R}^N$ be a function, $x \in \operatorname{dom} f$ a point.

***n*-th one-dimensional subproblem at *x***:

$$
\begin{aligned}
g_n^{(x)} &: T_n^{(x)} \to \mathbb{R} \\
g_n^{(x)}(t) &:= f(x + t\, e^{(n)}) \\
T_n^{(x)} &:= \{t \in \mathbb{R} \mid x + t\, e^{(n)} \in \operatorname{dom} f\}
\end{aligned}
$$

***n*-th one-dimensional subproblem solver at *x***:

$$
h_n(x) := \arg\min_{t \in T_n^{(x)}} g_n^{(x)}(t)
$$

# One-dimensional Subproblems / Example 1

$$f(x_1, x_2) := ax_1^2 + bx_2^2 + cx_1 x_2$$

Q: What are the one-dimensional problems $g_1, g_2$ for $x_1$ and $x_2$?
What are their solvers $h_1, h_2$?

remember:
***n*-th one-dimensional subproblem at *x***:

$$\begin{aligned} g_n^{(x)} &: T_n^{(x)} \to \mathbb{R} \\ g_n^{(x)}(t) &:= f(x + t\, e^{(n)}) \\ T_n^{(x)} &:= \{t \in \mathbb{R} \mid x + t\, e^{(n)} \in \operatorname{dom} f\} \end{aligned}$$

***n*-th one-dimensional subproblem solver at *x***:

$$h_n(x) := \arg\min_{t \in T_n^{(x)}} g_n^{(x)}(t)$$

# One-dimensional Subproblems / Example 2

Solving a linear system of equations / least squares / linear regression:

$$\begin{aligned} f(x) &:= x^T A x - b^T x, \quad A \in \mathbb{R}^{N\times N} \text{ sym. pos.def.}, b \in \mathbb{R}^N \\ g_n^{(x)}(t) &= f(X_n = x_n + t; x_{-n}), \quad T_n^{(x)} = \mathbb{R} \\ f(X_n; x_{-n}) &= A_{n,n}X_n^2 + (2(x_{-n}^T A_{-n,.})_n - b_n)X_n + x_{-n}^T A_{-n,-n}x_{-n} - b_{-n}^T x_{-n} \\ &= A_{n,n}X_n^2 + (2A_{n,-n}x_{-n} - b_n)X_n + x_{-n}^T A_{-n,-n}x_{-n} - b_{-n}^T x_{-n} \end{aligned}$$

analytic minimum:

$$\begin{aligned} f'(X_n; x_{-n}) &= 2A_{n,n}X_n + 2A_{n,-n}x_{-n} - b_n \overset{!}{=} 0 \\ \rightsquigarrow \quad X_n &= \frac{b_n - 2A_{n,-n}x_{-n}}{2A_{n,n}} \\ \text{i.e.,} \quad h_n(x) &:= \frac{b_n - 2A_{n,-n}x_{-n}}{2A_{n,n}} \end{aligned}$$

# Coordinate Descent / Random Coordinate

**min-cd**$(f, h, x^{(0)}, K, \epsilon)$:
  for $k := 1, \ldots, K$:
    draw $n^{(k)} \sim \text{unif}(\{1, \ldots, N\})$
    $x_m^{(k)} := x_m^{(k-1)}$ for $m \in \{1, \ldots, N\}, m \neq n^{(k)}$
    $x_{n^{(k)}}^{(k)} := h_{n^{(k)}}(x^{(k-1)})$
    if $k \geq N$ and $||x^{(k)} - x^{(k-N)}|| \leq \epsilon$
      return $x^{(k)}$
  return "not converged"

where

- $h$ solvers for one-dimensional subproblems $g_n^{(x)}$.
- $\epsilon$ convergence threshold step in an epoche

# Coordinate Descent / Cyclic Epochs

**min-cd**$(f, h, x^{(0)}, K, \epsilon)$:
  for $k := 1, \ldots, K$:
    $x_1^{(k)} := h_1(x_1^{(k-1)}, \ldots, x_N^{(k-1)})$
    $x_2^{(k)} := h_2(x_1^{(k)}, x_2^{(k-1)}, \ldots, x_N^{(k-1)})$
    $x_3^{(k)} := h_3(x_1^{(k)}, x_2^{(k)}, x_3^{(k-1)}, \ldots, x_N^{(k-1)})$
    $\vdots$
    $x_N^{(k)} := h_N(x_1^{(k)}, \ldots, x_{N-1}^{(k)}, x_N^{(k-1)})$
    if $||x^{(k)} - x^{(k-1)}|| \leq \epsilon$
      return $x^{(k)}$
  return "not converged"

where

- $h$ solvers for one-dimensional subproblems $g_n^{(x)}$.
- $\epsilon$ convergence threshold step in an epoche

# Coordinate Descent / Cyclic Epochs

$$
\begin{array}{l}
\textbf{min-cd}(f, h, x^{(0)}, K, \epsilon)\text{:} \\
\quad x^{(0,N)} := x^{(0)} \\
\quad \text{for } \ k := 1, \ldots, K\text{:} \\
\quad\quad x^{(k,0)} := x^{(k-1,N)} \\
\quad\quad \text{for } \ n := 1, \ldots, N\text{:} \\
\quad\quad\quad x_m^{(k,n)} := x_m^{(k,n-1)} \text{ for } m \in \{1, \ldots, N\}, m \neq n \\
\quad\quad\quad x_n^{(k,n)} := h_n(x^{(k,n-1)}) \\
\quad\quad \text{if } \ ||x^{(k,N)} - x^{(k-1,N)}|| \leq \epsilon \\
\quad\quad\quad \text{return } \ x^{(k,N)} \\
\quad \text{return } \ \text{"not converged"}
\end{array}
$$

where

- $h$ solvers for one-dimensional subproblems $g_n^{(x)}$.
- $\epsilon$ convergence threshold step in an epoche

# Coordinate Descent / 1-dim Gradient Steps

**min-cd**$(f, \nabla f, x^{(0)}, K, \epsilon)$:
  for $k := 1, \ldots, K$:
    draw $n^{(k)} \sim \text{unif}(\{1, \ldots, N\})$
    $x^{(k)} := x^{(k-1)} - \mu^{(k)} (\nabla f(x^{(k-1)}))_{n^{(k)}} e^{(n^{(k)})}$
    if $k \geq N$ and $||x^{(k)} - x^{(k-N)}|| \leq \epsilon$
      return $x^{(k)}$
  return "not converged"

where

- $\nabla f$ gradients of objective function.
- $\mu \in (\mathbb{R}^+)^*$ step length schedule (or controller)
- $\epsilon$ convergence threshold step in an epoche

# Coordinate Descent / Considerations

- The order in which we cycle through the coordinates is arbitrary.
  - e.g., cyclic
  - better should be randomized.
- We may update **blocks of coordinates** at a time instead of only one (**block coordinate descent**).
- No need to adjust a step-size!
  - if we have exact solvers $h$ for the 1-dim. subproblems.
- Does not work in general with non-differentiable functions.
  - but with sums of differentiable and separable ones.
- Depends strongly on the choice of coordinates !
  (contrary to affine invariant methods like Newton)

# Outline

# Coordinate Lipschitz Constant

- standard **Lipschitz constant** $L$:

$$||f(x) - f(y)|| \leq L||x - y|| \qquad \forall x,y \in \text{dom } f$$

  or equivalently

$$||f(x + d) - f(x)|| \leq L||d|| \qquad \forall x \in \text{dom } f, d \in \mathbb{R}^N : x + d \in \text{dom } f$$

- **component Lipschitz constant** $L_n$: $(n \in \{1, \ldots, N\})$

$$||f(x + t\, e^{(n)}) - f(x)|| \leq |t|\, L_n \qquad \forall x \in \text{dom } f, t \in \mathbb{R} : x + t\, e^{(n)} \in \text{dom } f$$

- **coordinate Lipschitz constant** $L_{\max}$:

$$L_{\max} := \max_{n \in \{1,\ldots,N\}} L_n$$

  - always $L_{\max} \leq L$.

Note: In the following $|| \cdot ||$ denotes the L2-norm.

# Lipschitz Continuous Functions / Bounded Derivative

Lemma

*A differentiable function $f : X \to \mathbb{R}, X \subseteq \mathbb{R}$ is Lipschitz-continuous iff its derivative is bounded:*

$$\begin{array}{ccc} |f(x) - f(y)| \leq L|x-y| & \iff & |f'(x)| \leq L \\ \forall x, y \in \operatorname{dom} f & & \forall x \in \operatorname{dom} f \end{array}$$

*Proof: "$\Rightarrow$":*

$$\left|\frac{f(x) - f(x+t)}{|t|}\right| = \frac{|f(x) - f(x+t)|}{|t|} \leq \frac{|t|L}{|t|} \leq L \quad | t \to 0$$

$$|f'(x)| \leq L$$

*"$\Leftarrow$":*

$$f(x) \underset{Taylor}{=} f(y) + f'(\xi)(x-y) \quad \textit{for a } \xi \in [x, y]$$

$$\begin{aligned} |f(x) - f(y)| &= |f'(\xi)(x-y)| = |f'(\xi)||x-y| \\ &\leq L|x-y| \end{aligned}$$

## Theorem (convergence of coordinate descent)

*If*

i. $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^N$ *is convex and differentiable,*

ii. $\nabla f$ *is uniformly Lipschitz-continuous:*
$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||, \quad L \in \mathbb{R}_0^+$

iii. *the sublevel set of* $x^{(0)}$ *is bounded:*
$$\max_{x^*: f(x^*) = \min_x f(x)} \ \max_{x \in S_f(x^{(0)})} ||x - x^*|| \leq R, \quad R \in \mathbb{R}_0^+$$

iv. *constant steplength* $\mu^{(k)} := 1/L_{\max}$ *is used*
*(with coordinate Lipschitz constant* $L_{\max} \leq L$*),*

*then coordinate descent converges and*

$$\mathbb{E}(f(x^{(k)})) - f(x^*) \leq \frac{2NL_{\max}R^2}{k}$$

# Convergence / Proof (1/3)

$$
\begin{aligned}
f(x^{\text{next}}) &= f(x - \mu(\nabla f(x))_n e^{(n)}) \\
&\underset{\text{Taylor}}{=} f(x) - \nabla f(x)^T \mu(\nabla f(x))_n e^{(n)} \\
&\quad\quad + \frac{1}{2}(\mu(\nabla f(x))_n e^{(n)})^T \underbrace{\nabla^2 f(x - \xi\mu(\nabla f(x))_n e^{(n)})}_{\underset{\text{ii. bound.deriv.}}{\leq} L_n} \mu(\nabla f(x))_n e^{(n)} \\
&\leq f(x) - \nabla f(x))_n^2 \,\mu + \frac{1}{2}\mu^2(\nabla f(x))_n^2 L_n \\
&= f(x) - (\nabla f(x))_n^2 \,\mu(1 - \frac{\mu L_n}{2}) \\
&\leq f(x) - (\nabla f(x))_n^2 \,\mu(1 - \frac{\mu L_{\max}}{2}) \\
&\underset{iv.}{=} f(x) - \frac{(\nabla f(x))_n^2}{2L_{\max}}
\end{aligned}
$$

# Convergence / Proof (2/3)

$$
\begin{aligned}
f(x^{\text{next}}) &\leq f(x) - \frac{(\nabla f(x))_n^2}{2L_{\max}} \quad |\mathbb{E}_n(\ldots) \\
\mathbb{E}_n(f(x^{\text{next}})) &\leq \mathbb{E}_n(f(x) - \frac{(\nabla f(x))_n^2}{2L_{\max}}) \\
&= f(x) - \frac{1}{N}\sum_{n=1}^{N} \frac{(\nabla f(x))_n^2}{2L_{\max}}) = f(x) - \frac{1}{2NL_{\max}}||\nabla f(x)||^2 \quad (1)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_{n^{(1:k)}} &:= \mathbb{E}_{n^{(1)},n^{(2)},\ldots,n^{(k)}} \quad \text{expectation over } \textbf{all } n^{(1)},\ldots,n^{(k)} \\
\phi^{(k)} &:= \mathbb{E}_{n^{(1:k)}}(f(x^{(k)})) - f(x^*) \\
f(x) - f(x^*) &\underset{i.}{\leq} \nabla f(x)^T(x - x^*) \leq ||\nabla f(x)||\,||x - x^*|| \underset{iii.}{\leq} R||\nabla f(x)|| \\
\mathbb{E}_{n^{(1:k)}}(||\nabla f(x^{(k)})||) &\geq \frac{1}{R}\phi^{(k)} \quad (2)
\end{aligned}
$$

# Convergence / Proof (3/3)

$$\mathbb{E}_{n^{(k)}}(f(x^{(k+1)})) \underset{(1)}{\le} f(x^{(k)}) - \frac{1}{2NL_{\max}}||\nabla f(x^{(k)})||^2 \quad |\mathbb{E}_{n^{(1:k+1)}}$$

$$\phi^{(k+1)} \le \phi^{(k)} - \frac{1}{2NL_{\max}}\mathbb{E}_{n^{(1:k+1)}}(||\nabla f(x^{(k)})||^2)$$

$$\underset{\text{Jensen's Ineq.}}{\le} \phi^{(k)} - \frac{1}{2NL_{\max}}(\mathbb{E}_{n^{(1:k+1)}}(||\nabla f(x^{(k)})||))^2$$

$$\underset{(2)}{\le} \phi^{(k)} - \frac{1}{2NL_{\max}R^2}(\phi^{(k)})^2 \tag{3}$$

$$\frac{1}{\phi^{(k+1)}} - \frac{1}{\phi^{(k)}} = \frac{\phi^{(k)} - \phi^{(k+1)}}{\phi^{(k)}\phi^{(k+1)}} \ge \frac{\phi^{(k)} - \phi^{(k+1)}}{(\phi^{(k)})^2} \underset{(3)}{\ge} \frac{1}{2NL_{\max}R^2}$$

$$\frac{1}{\phi^{(k+1)}} \underset{\text{rec}}{\ge} \frac{1}{\phi^{(0)}} + \frac{k+1}{2NL_{\max}R^2} \ge \frac{k+1}{2NL_{\max}R^2}$$

# Summary

- **Coordinate descent** minimizes one-dimensional subproblems for **a single variable $x_n$** at a time.
  - in cyclic or random order
- Coordinate descent can be fast if the **one-variable subproblems can be solved analytically**.
- If an $x^*$ **minimizes all single-variable problems**, then it also is a minimizer for the all-variable problem
  - if $f$ is convex and differentiable, or
  - if $f = g + h$ is the sum of
    - a convex and differentiable $g$ and
    - a convex, possibly **non-differentiable**, but **separable** $h(x_1, \ldots, x_N) = h_1(x_1) + h_2(x_2) + \cdots + h_N(x_N)$
- **Convergence** of coordinate descent can be proven for mild conditions
  - e.g., $f$ differentiable, $\nabla f$ Lipschitz, sublevel set of $x^{(0)}$ bounded, and constant steplength $1/L_{\max}$.

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# Further Readings

- The coordinate descent method is not covered by Boyd and Vandenberghe, 2004

- Coordinate descent:
  - very briefly Nocedal and Wright, 2006, ch. 9.3

  - A brief, but dense survey: Wright, 2015

- Convergence proof: Wright, 2015

# References


Boyd, Stephen and Lieven Vandenberghe (2004). *Convex Optimization*. Cambridge University Press.

Nocedal, Jorge and Stephen J. Wright (2006). *Numerical Optimization*. Springer Science+ Business Media.

Wright, Stephen J. (2015). *Coordinate Descent Algorithms*. In: *Mathematical Programming* 151.1, pp. 3–34.

# Jensen's Inequality (review)

For a convex function f:

$$f(\mathbb{E}(x)) \leq \mathbb{E}(f(x))$$

# Outline

# Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$

$$\min_{\mathbf{x}} \quad (a_1 x_1 - a_2 x_2 + a_3)^2$$

**Algorithm:**

- Initialize $\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}$
- Repeat until convergence:
  - $\mathbf{x}_1^{(k)} \leftarrow \arg\min_{x_1} (a_1 x_1 - a_2 \mathbf{x}_2^{(k-1)} + a_3)^2$
  - $\mathbf{x}_2^{(k)} \leftarrow \arg\min_{x_2} (a_1 \mathbf{x}_1^{(k)} - a_2 x_2 + a_3)^2$

# Coordinate Descent - Example

$$\arg\min_{x_1} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_1}(a_1 x_1 - a_2 x_2 + a_3)^2$$

# Coordinate Descent - Example

$$\arg\min_{x_1} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_1}(a_1 x_1 - a_2 x_2 + a_3)^2 = 2(a_1 x_1 - a_2 x_2 + a_3)a_1$$

# Coordinate Descent - Example

$$\arg\min_{x_1} (a_1 x_1 - a_2 x_2 + a_3)^2$$

Find a closed form solution:

$$0 \overset{!}{=} \frac{d}{dx_1}(a_1 x_1 - a_2 x_2 + a_3)^2 = 2(a_1 x_1 - a_2 x_2 + a_3) a_1$$

$$x_1 = \frac{a_2 x_2 - a_3}{a_1}$$

# Coordinate Descent - Example

$$\arg\min_{x_2}(a_1x_1 - a_2x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_2}(a_1x_1 - a_2x_2 + a_3)^2$$

# Coordinate Descent - Example

$$\arg\min_{\textcolor{red}{x_2}} (a_1 x_1 - a_2 \textcolor{red}{x_2} + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_2}(a_1 x_1 - a_2 x_2 + a_3)^2 = -2(a_1 x_1 - a_2 x_2 + a_3) a_2$$

# Coordinate Descent - Example

$$\arg\min_{x_2}(a_1x_1 - a_2x_2 + a_3)^2$$

Find a closed form solution:

$$0 \stackrel{!}{=} \frac{d}{dx_2}(a_1x_1 - a_2x_2 + a_3)^2 = -2(a_1x_1 - a_2x_2 + a_3)a_2$$

$$x_2 = \frac{a_1x_1 + a_3}{a_2}$$

# Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$

$$\min_{\mathbf{x}} \quad (a_1 x_1 - a_2 x_2 + a_3)^2$$

**Algorithm:**

- Initialize $\mathbf{x}_1^{(0)}, \mathbf{x}_2^{(0)}$
- Repeat until convergence:
  - $\mathbf{x}_1^{(k)} \leftarrow \frac{a_2 \mathbf{x}_2^{(k-1)} - a_3}{a_1}$
  - $\mathbf{x}_2^{(k)} \leftarrow \frac{a_1 \mathbf{x}_1^{(k)} + a_3}{a_2}$

# Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$, $a_1 = 0.1$, $a_2 = 2$, $a_3 = 1$

$$\min_{\mathbf{x}} \quad (0.1x_1 - 2x_2 + 1)^2$$

$$\mathbf{x}_1^{(k)} \leftarrow \frac{2\mathbf{x}_2^{(k-1)} - 1}{0.1} \quad \mathbf{x}_2^{(k)} \leftarrow \frac{0.1\mathbf{x}_1^{(k)} + 1}{2}$$

# Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$, $a_1 = 0.1$, $a_2 = 2$, $a_3 = 1$

$$\min_{\mathbf{x}} \quad (0.1x_1 - 2x_2 + 1)^2$$

$$\mathbf{x}_1^{(k)} \leftarrow \frac{2\mathbf{x}_2^{(k-1)} - 1}{0.1} \quad \mathbf{x}_2^{(k)} \leftarrow \frac{0.1\mathbf{x}_1^{(k)} + 1}{2}$$

Start with $\mathbf{x}_1^{(0)} = 1, \mathbf{x}_2^{(0)} = 2$

- $\mathbf{x}_1^{(1)} \leftarrow \frac{2 \cdot 2 - 1}{0.1} = 30$
- $\mathbf{x}_2^{(1)} \leftarrow \frac{0.1 \cdot 30 + 1}{2} = 2$

# Coordinate Descent - Example

For $\mathbf{x} \in \mathbb{R}^2$, $a_1 = 0.1$, $a_2 = 2$, $a_3 = 1$

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- $\mathbf{x}_1^{(1)} \leftarrow \frac{2 \cdot 2 - 1}{0.1} = 30$
- $\mathbf{x}_2^{(1)} \leftarrow \frac{0.1 \cdot 30 + 1}{2} = 2$

$\mathbf{x}_1^{(1)} = 30, \mathbf{x}_2^{(1)} = 2$

- $\mathbf{x}_1^{(2)} \leftarrow \frac{2 \cdot 2 - 1}{0.1} = 30$
- $\mathbf{x}_2^{(2)} \leftarrow \frac{0.1 \cdot 30 + 1}{2} = 2$

# Coordinate Descent for Linear Regression

For the problem

$$\text{minimize} \quad \sum_{i=1}^{m}(y_i - \mathbf{x}^T\mathbf{a_i})^2 = \sum_{i=1}^{m}\left(y_i - \sum_{j=1}^{n} x_j a_{ij}\right)^2$$

We can compute the update rule for a specific $x_k$:

$$\frac{\partial f(\mathbf{x})}{\partial x_k} \stackrel{!}{=} 0$$

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = 2 \cdot \sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij}\right)$$

# Coordinate Descent for Linear Regression

$$2 \cdot \sum_{i=1}^{m} a_{ik} \cdot \left( y_i - \sum_{j=1}^{n} x_j a_{ij} \right) = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1}^{n} x_j a_{ij} = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot \left( x_k a_{ik} + \sum_{j=1,\, j \neq k}^{n} x_j a_{ij} \right) = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot x_k a_{ik} - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\, j \neq k}^{n} x_j a_{ij} = 0$$

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - x_k \sum_{i=1}^{m} a_{ik}^2 - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\, j \neq k}^{n} x_j a_{ij} = 0$$

# Coordinate Descent for Linear Regression

$$\sum_{i=1}^{m} a_{ik} \cdot y_i - x_k \sum_{i=1}^{m} a_{ik}^2 - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\ j\neq k}^{n} x_j a_{ij} = 0$$

$$\begin{aligned} x_k \cdot \sum_{i=1}^{m} a_{ik}^2 &= \sum_{i=1}^{m} a_{ik} \cdot y_i - \sum_{i=1}^{m} a_{ik} \cdot \sum_{j=1,\ j\neq k}^{n} x_j a_{ij} \\ x_k &= \frac{\sum_{i=1}^{m} a_{ik} \cdot \left( y_i - \sum_{j=1,\ j\neq k}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2} \end{aligned}$$

# Linear Regression Coordinate Descent - Simple Algorithm

**procedure** LINEAR REGRESSION-CD
**input:** $f$
  Get initial point $\mathbf{x}^{(0)}$
  **repeat**
    **for** $k \in 1, \ldots n$ **do**
      $x_k \leftarrow \frac{\sum_{i=1}^m a_{ik} \cdot \left(y_i - \sum_{j=1,\ j \neq k}^n x_j a_{ij}\right)}{\sum_{i=1}^m a_{ik}^2}$
    **end for**
  **until** convergence
  **return** $\mathbf{x}$, $f(\mathbf{x})$
**end procedure**

# Coordinate Descent for Linear Regression

For each parameter we have the following update rule:

$$x_k = \frac{\sum_{i=1}^{m} a_{ik} \cdot \left( y_i - \sum_{j=1,\, j\neq k}^{n} x_j a_{ij} \right)}{\sum_{i=1}^{m} a_{ik}^2}$$

One Coordinate descent epoch has a cost of $O(m \cdot n^2)$!

**Can we do it faster?**

# CD for Linear Regression - Smart Update

For each parameter we have the following update rule:

$$x_k = \frac{\sum_{i=1}^m a_{ik} \cdot \left( y_i - \sum_{j=1,\ j\neq k}^n x_j a_{ij} \right)}{\sum_{i=1}^m a_{ik}^2}$$

We can rewrite:

$$\begin{aligned}
\sum_{i=1}^m \left( y_i - \sum_{j=1,\ j\neq k}^n x_j a_{ij} \right) &= \sum_{i=1}^m \left( y_i - \sum_{j=1}^n x_j a_{ij} + x_k a_{ik} \right) \\
&= \sum_{i=1}^m \left( y_i - \sum_{j=1}^n x_j a_{ij} \right) + \sum_{i=1}^m x_k a_{ik} \\
&= \sum_{i=1}^m \left( y_i - \sum_{j=1}^n x_j a_{ij} \right) + x_k \sum_{i=1}^m a_{ik}
\end{aligned}$$

# CD for Linear Regression - Smart Update

From which we have:

$$
\begin{aligned}
x_k &= \frac{\sum_{i=1}^m a_{ik} \cdot \left( y_i - \sum_{j=1,\, j\neq k}^n x_j a_{ij} \right)}{\sum_{i=1}^m a_{ik}^2} \\
&= \frac{\sum_{i=1}^m a_{ik} \cdot \left( y_i - \sum_{j=1}^n x_j a_{ij} + \mathbf{x}_k^{old} a_{ik} \right)}{\sum_{i=1}^m a_{ik}^2} \\
&= \frac{\sum_{i=1}^m a_{ik} \cdot \left( y_i - \sum_{j=1}^n x_j a_{ij} \right)}{\sum_{i=1}^m a_{ik}^2} + \frac{\sum_{i=1}^m a_{ik} \cdot \left( \mathbf{x}_k^{old} a_{ik} \right)}{\sum_{i=1}^m a_{ik}^2} \\
&= \frac{\sum_{i=1}^m a_{ik} \cdot \left( y_i - \sum_{j=1}^n x_j a_{ij} \right)}{\sum_{i=1}^m a_{ik}^2} + \frac{\mathbf{x}_k^{old} \cdot \sum_{i=1}^m a_{ik}^2}{\sum_{i=1}^m a_{ik}^2} \\
&= \frac{\sum_{i=1}^m a_{ik} \cdot \left( y_i - \sum_{j=1}^n x_j a_{ij} \right)}{\sum_{i=1}^m a_{ik}^2} + \mathbf{x}_k^{old}
\end{aligned}
$$

# CD for Linear Regression - Smart Update

Now we have:

$$x_k = \frac{\sum_{i=1}^{m} a_{ik} \cdot \left(y_i - \sum_{j=1}^{n} x_j a_{ij}\right)}{\sum_{i=1}^{m} a_{ik}^2} + \mathbf{x}_k^{old}$$

So we can define our residual vector $\mathbf{r} \in \mathbb{R}^m$ such that

$$r_i = y_i - \sum_{j=1}^{n} x_j a_{ij}$$

# CD for Linear Regression - Smart Update

After each update of $x_k$, we can maintain $r_i$ instead of recomputing it given the old value $\mathbf{x}_k^{old}$:

$$r_i^{new} = y_i - \left( \sum_{j=1}^{n} x_j a_{ij} - \mathbf{x}_k^{old} a_{ik} + x_k a_{ik} \right)$$
$$= r_i^{old} + (\mathbf{x}_k^{old} - x_k) a_{ik}$$

# CD for Linear Regression - Smart Update

Now our algorithm looks like:

1. Initialize $\mathbf{x}$
2. Compute $r_i = y_i - \sum_{j=1}^{n} x_j a_{ij}$
3. While Not Converged
   3.1 For each $k = 1, \dots, n$
      3.1.1 $\mathbf{x}_k^{old} \leftarrow x_k$
      3.1.2 $x_k \leftarrow \frac{\sum_{i=1}^{m} a_{ik} \cdot (r_i)}{\sum_{i=1}^{m} a_{ik}^2} + \mathbf{x}_k^{old}$
      3.1.3 For all i $r_i \leftarrow r_i + (\mathbf{x}_k^{old} - x_k) a_{ik}$

**This algorithm is now $O(m \cdot n)$!**

# Linear Regression Coordinate Descent Algorithm

**procedure** LINEAR REGRESSION-CD
**input:** $f$
  Get initial point $\mathbf{x}^{(0)}$
  $\mathbf{r} \leftarrow \mathbf{y} - A\mathbf{x}^{(0)}$
  **repeat**
    **for** $k \in 1, \ldots n$ **do**
      $\mathbf{x}_k^{old} \leftarrow x_k$
      $x_k \leftarrow \frac{\sum_{i=1}^m a_{ik} \cdot r_i}{\sum_{i=1}^m a_{ik}^2} + \mathbf{x}_k^{old}$
      **for** $i \in 1, \ldots m$ **do**
        $r_i \leftarrow r_i + (\mathbf{x}_k^{old} - x_k) a_{ik}$
      **end for**
    **end for**
  **until** convergence
  **return** $\mathbf{x}$, $f(\mathbf{x})$
**end procedure**

# Real World Dataset: Body Fat prediction

We want to estimate the percentage of body fat based on various attributes:

- Age (years)
- Weight (lbs)
- Height (inches)
- Neck circumference (cm)
- Chest circumference (cm)
- Abdomen 2 circumference (cm)
- Hip circumference (cm)
- Thigh circumference (cm)
- Knee circumference (cm)
- ...

`http://lib.stat.cmu.edu/datasets/bodyfat`

# Real World Dataset: Body Fat prediction

The data is represented it as:

$$A_{m,n} = \begin{pmatrix} 1 & a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 1 & a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix} \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

with $m = 252$, $n = 14$

We can model the percentage of body fat $y$ is a linear combination of the body measurements with parameters $\mathbf{x}$:

$$\hat{y}_i = \mathbf{x}^T \mathbf{a_i} = x_0 1 + x_1 a_{i,1} + x_2 a_{i,2} + \ldots + x_n a_{i,n}$$

# Coordinate Descent - Body fat dataset

# Year Prediction Data Set

- Least Squares Problem
- Prediction of the release year of a song from audio features
- 90 features
- Experiments done on a subset of 1000 instances of the data

# Coordinate Descent - Year Prediction