Quiz: SVMs

Lecture series "Machine Learning"

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Quiz: Primal Hard-Margin SVM

- Assume a two-class classification problem with one-dimensional input space $\mathcal{X} = \mathbb{R}$ and classes $\{-1,1\}$.
- Let $\mathcal{D} = \{(x_1, y_1), (x_2, y_2)\}$ with $(x_1, y_1) = (0, -1)$ and $(x_2, y_2) = (1, 1)$ denote the training data set
- We train a primal hard-margin SVM of the form $f_{\theta}(x) = \theta x + b$
- Question: What is the resulting solution for θ and b?

$$-\theta = 0.5, b = 1$$

$$-\theta = 4, b = -2$$

$$-\theta = 1, b = -1$$

$$-\theta = 2, b = -1$$

$$-\theta = -1, b = 0$$

Solution: Primal Hard-Margin SVM

• **Solution**: the optimization problem is

Find
$$\boldsymbol{\theta}, b$$

such that $y_n(\mathbf{x}_n^T \boldsymbol{\theta} + b) \ge 1$ for $n \in \{1, ..., N\}$
and $\frac{1}{2} ||\boldsymbol{\theta}||^2$ minimal

Write down the constraints and plug in the values for the data points:

$$y_1(x_1\theta+b) \ge 1 \quad \Rightarrow \quad (-1) \cdot (0 \cdot \theta+b) \ge 1 \quad \Rightarrow \quad -b \ge 1$$

$$y_2(x_2\theta+b) \ge 1 \implies 1 \cdot (1 \cdot \theta+b) \ge 1 \implies \theta+b \ge 1$$

- Adding up the two final inqualities implies $\theta \ge 2$
- We see that for $\theta=2,\ b=-1$ the constraints are satisfied. This is also the solution with minimum $\|\theta\|$ (because of the condition $\theta\geq 2$)

Quiz: Dual Hard-Margin SVM

- Assume still the same problem setting and the same data set
- We now train a dual hard-margin SVM
- Question: what is the resulting dual parameter vector $\alpha = (\alpha_1, \alpha_2)$?

$$-\alpha_1 = 1, \alpha_2 = 0$$

$$-\alpha_1 = 2, \ \alpha_2 = -2$$

$$-\alpha_1 = 2, \ \alpha_2 = 2$$

$$-\alpha_1 = -1, \ \alpha_2 = 1$$

$$-\alpha_1 = 1, \alpha_2 = 1$$

Solution: Dual Hard-Margin SVM

• Solution: the dual optimization problem is

$$\alpha^* = \arg \max_{\alpha} \tilde{L}(\alpha)$$

$$\Rightarrow \qquad \alpha_1 y_1 + \alpha_2 y_2 = 0$$

$$\Rightarrow \qquad -\alpha_1 + \alpha_2 = 0$$

$$\Rightarrow \qquad \alpha_1 = \alpha_2$$
subject to
$$\sum_{n=1}^{N} \alpha_n y_n = 0 \text{ and } \alpha_n \ge 0 \text{ for all } n \in \{1, ..., N\}$$

where
$$\tilde{L}(\boldsymbol{\alpha}) = -\frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m \mathbf{x}_n^{\mathsf{T}} \mathbf{x}_m + \sum_{n=1}^{N} \alpha_n$$

• We also know from the lecture the primal solution can be recovered by $\mathbf{\theta}^* = \sum_{n=1}^N \alpha_n^* y_n \mathbf{x}_n$

$$\Rightarrow \theta = \alpha_1 y_1 x_1 + \alpha_2 y_2 x_2$$

$$\Rightarrow$$
 2 = $\alpha_1 \cdot (-1) \cdot 0 + \alpha_2 \cdot 1 \cdot 1$

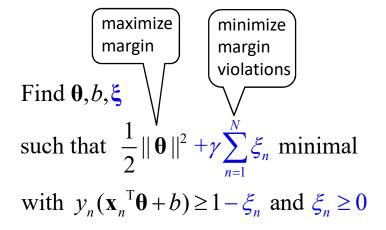
$$\Rightarrow \alpha_2 = 2$$

• So the correct solution is $\alpha_1 = 2$, $\alpha_2 = 2$



Quiz: Primal Soft-Margin SVM

- Now assume any binary classification problem with instance space $\mathcal{X} = \mathbb{R}^M$
- When training a primal soft-margin SVM, we solve the optimization problem



- Question: Given a data set \mathcal{D} , if we train a model with small γ and a model with larger γ , which of the two models will be more susceptible to overfitting?
 - The model with larger γ
 - The model with smaller γ
 - They are equally susceptible to overfitting
 - Cannot say anything based on the information provided



Solution: Primal Soft-Margin SVM

- Solution: the model with larger γ will be more susceptible to overfitting
- The easiest way to see this is by reformulating the problem as regularized loss minimization (Slide 21 in lecture):

$$\arg\min_{\boldsymbol{\theta},b} \frac{1}{2} \|\boldsymbol{\theta}\|^2 + \gamma \sum_{n=1}^{N} \max(0,1 - y_n f_{\boldsymbol{\theta}}(\mathbf{x}_n))$$

$$= \arg\min_{\boldsymbol{\theta},b} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}(\mathbf{x}_n), y_n) + \frac{1}{2N\gamma} \|\boldsymbol{\theta}\|^2$$

$$\ell(f_{\boldsymbol{\theta}}(\mathbf{x}_n), y_n) = \max(0,1 - y_n f_{\boldsymbol{\theta}}(\mathbf{x}_n))$$
regularization weight

• The larger the γ , the smaller the regularization weight