Modern Optimization Techniques

Third Take-home Exam

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Note:

- Time: 240 minutes
- Add your name and matriculation number on top of every page.
- Please provide clear and detailed answers to get full points.
- Please make sure the provided solution is clearly written and scanned.
- Solutions need to be submitted before the deadline. Please consider submitting their solutions 5 minutes before the deadline in order to ensure that the solution is uploaded and no internet problem interrupts it.
- Plagiarism, cheating and group submissions are not allowed. Any suspicious solution will be further investigated and the student will fail the course for this semester in case we prove it.

1. Feasibility and Solutions of Constrained Problems

1A. Feasibility of Optimization Problems (3 points)

Discuss whether it is possible to solve the following optimization problems (without solving it!):

a.

$$\begin{aligned} &\max\ 4x - y \\ &\text{s.t. } 2x - y \leq 2 \\ &x - 2y \leq 0 \end{aligned}$$

b.

$$\begin{aligned} &\max\ x - 3y \\ &\text{s.t. } x - 3y \leq -1 \\ &3x + 2y \leq -3 \\ &5x - y = 0 \end{aligned}$$

1B. Newton's Method for Equality Constraints (5 points)

Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2) = 2x_1^2 + x_2^2 + x_1 x_2 \\ \text{subject to} \quad & x_1 - x_2 = 2 \end{aligned}$$

Use the equality constrained Newton Algorithm with a starting point $x^{(0)} = (4/3, -2/3)^\top$ and learning rate $\mu = 0.05$. Write down the system of equations that you would have to solve in the first step of the algorithm. **Solve two steps by computing** $x^{(1)}$ and $x^{(2)}$.

1C. Analytical solution of a Constrained Problem (2 points)

Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x) = x^2 + 1 \\ \text{subject to} \quad & (x-2)(x-4) \leq 0 \end{aligned}$$

Is the given problem quadratic? Find the feasible set and compute the optimum x^* using the KKT conditions

2. Inequality Constrained: Duality and Active Set methods

2A. Backtracking Line Search for Active Set methods (2 points)

The Backtracking Line Search condition is modified when used with the Active Set method for affine inequality constraints. What is the condition added and how would this condition impact the starting step size μ?

2B. Active Set with Newton Method (5 points)

Solve the following optimization problem using the Active Set method with the equality constraint Newton method to find the search direction and the starting point $x^{(0)} = (2, 0)^T$. Stop when the working set is empty.

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 5x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 - 2 \leq 0 \\ & x_1 + 2x_2 - 6 \leq 0 \\ & x_1 - 2x_2 - 2 \leq 0 \\ & x1, x2 \leq 0 \end{aligned}$$

Hint: For simplicity, use the most negative value of the Lagrangian multiplier λ to drop the constraint and set the step size $\mu = 1$. Stop when the working set is empty.

2C. Duality of Constrained Problem (3 points)

Is the following constrained problem linear? Compute the dual Lagrangian (no need to solve it, just write the dual problem). What is the advantage of using the dual problem over the primal in this case?

$$\begin{aligned} \text{minimize} \quad & 4x_1 + x_2 + 3 \\ \text{subject to} \quad & 2x_1^2 + 2x_2^2 = 6 \\ & 3x_1 - 2x_2 \leq 0 \\ & x_1 + x_2 \leq -1 \end{aligned}$$

3. Barrier and Penalty Methods

3A. Barrier Method (2 points)

Consider the following minimization problem:

$$\begin{array}{rl} \text{minimize} & f_0(x_1, x_2) = t(x_1 + 2x_2) - \log(x_1) - \log(x_2 - 3) \\ \text{subject to} & x_1 + x_2 = 4 \end{array}$$

This is a derived optimization problem using the logarithmic barrier function. What is the original problem? What is the dual problem and is it solvable ?

3B. Penalty Method (5 points)

Consider the following optimization problem:

$$\begin{array}{rl} \text{minimize} & f(x_1, x_2) = x_1^2 + x_2^2 \\ \text{subject to} & x_2 - \dfrac{1}{2}x_1 = 1 \end{array}$$

Write down the derived optimization problem for the penalty method using the quadratic penalty function. Solve it analytically with a starting point $x^{(0)} = (0, 0)^\top$, i.e., compute the limit of the minimum as function of the penalty weight c.

3C. Barrier vs Penalty (3 points)

What are, the advantages of the penalty methods over the barrier methods? Is it possible to combine them in one method? Propose one mixed Barrier-Penalty function and discuss your choice.