Statistical Modeling Dashboard for Sensor Network Load Analysis

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Abstract

Problem

Let $x_1, \ldots, x_n$ be i.i.d. continuous random variables with common cumulative distribution function (CDF) $F(x)$ and density $f(x) = F'(x)$. Let y be an independent continuous random variable with CDF $G(y)$ and density $g(y) = G'(y)$. All variables are mutually independent.

Define

$$T = \max\{x_1, \ldots, x_n, y\}, \qquad U = \min\{x_1, \ldots, x_n, y\}.$$

We want to derive:

1. The CDF and PDF of T.
2. The CDF and PDF of U.
3. The joint density $f_{U,T}(u, t)$.

1 solution

1. Distribution of the Maximum T

CDF of T:

$$P(T \le t) = P(\max\{x_1, \ldots, x_n, y\} \le t) = F(t)^n\, G(t).$$

PDF of T:

$$\boxed{f_T(t) \;=\; nf(t)F(t)^{n-1}G(t) + F(t)^n g(t).}$$

2. Distribution of the Minimum U

Survival function:

$$P(U > u) = P(\min\{x_1, \ldots, x_n, y\} > u) = (1 - F(u))^n (1 - G(u)).$$

Thus CDF of U:

$$P(U \le u) = 1 - (1 - F(u))^n (1 - G(u)).$$

PDF of U:

$$\boxed{f_U(u) \;=\; nf(u)\,[1 - F(u)]^{n-1}[1 - G(u)] + [1 - F(u)]^n g(u).}$$

3. Joint Density of (U, T)

For $u < t$,

$$\begin{aligned} f_{U,T}(u,t) = n(n-1)\, f(u)\, f(t)\, &[F(t) - F(u)]^{n-2}[G(t) - G(u)] \\ &+ n\, g(u)\, f(t)\, [F(t) - F(u)]^{n-1} \\ &+ n\, f(u)\, g(t)\, [F(t) - F(u)]^{n-1}. \end{aligned}$$

For $u \ge t$ we have $f_{U,T}(u,t) = 0$.

$$\boxed{f_{U,T}(u,t) = \begin{cases} n(n-1)\, f(u)\, f(t)\, [F(t) - F(u)]^{n-2}[G(t) - G(u)] & \\ \quad +n\, g(u)\, f(t)\, [F(t) - F(u)]^{n-1} & \\ \quad +n\, f(u)\, g(t)\, [F(t) - F(u)]^{n-1}, & u < t, \\ & \\ 0, & u \ge t. \end{cases}}$$

4. Example: Exponential Case

If $x_i \sim \mathrm{Exp}(\lambda)$ and $y \sim \mathrm{Exp}(\mu)$, then

$$F(x) = 1 - e^{-\lambda x}, \quad f(x) = \lambda e^{-\lambda x}, \qquad G(y) = 1 - e^{-\mu y}, \quad g(y) = \mu e^{-\mu y}, \quad x, y \geq 0.$$

PDF of T:

$$f_T(t) = n\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}(1 - e^{-\mu t}) + (1 - e^{-\lambda t})^n \mu e^{-\mu t}, \qquad t \geq 0.$$

PDF of U:

$$f_U(u) = (n\lambda + \mu)e^{-(n\lambda+\mu)u}, \qquad u \geq 0.$$

So $U \sim \mathrm{Exp}(n\lambda + \mu)$.

Joint density:

$$\begin{aligned} f_{U,T}(u,t) = {} & n(n-1)\,\lambda e^{-\lambda u}\,\lambda e^{-\lambda t}\,(e^{-\lambda u} - e^{-\lambda t})^{n-2}(e^{-\mu u} - e^{-\mu t}) \\ & + n\,\mu e^{-\mu u}\,\lambda e^{-\lambda t}(e^{-\lambda u} - e^{-\lambda t})^{n-1} \\ & + n\,\lambda e^{-\lambda u}\,\mu e^{-\mu t}(e^{-\lambda u} - e^{-\lambda t})^{n-1}, \end{aligned}$$

for $0 \leq u < t$.

References

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