CSE 547: Homework One

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Question 1

1). 1.10

First we are going to prove that Q_n is equal to $2R_{n-1}+1$. Let's assume that we want to go from A to B by using reverse algorithm. In this case our available moves are:

$$B \to A$$
$$A \to C$$
$$C \to B$$

Our approach is going to be like the Tower of Hanoi's solution:

$$\text{move}(n-1) : A \to C \Rightarrow R_{n-1}$$
$$\text{move the largest disk} : A \to C \to B \Rightarrow 1$$
$$\text{move}(n-1) : C \to A \Rightarrow R_{n-1}$$
$$\text{overall cost is} : R_{n-1} + 1 + R_{n-1} = 2R_{n-1} + 1$$

Now that we proved the first relation, we are going to use the same approach to prove that R_n is equal to $Q_n + Q_{n-1} + 1$. Assume we want to go back to B from A:

$$\text{move}(n-1) : B \to A \Rightarrow R_{n-1}$$
$$\text{move}(n-1) : A \to C \Rightarrow R_{n-1}$$
$$\text{move the largest disk} : B \to A \Rightarrow 1$$
$$\text{move n-1 disks (it's like two reverse move in order)} : C \to B \to A \Rightarrow 2R_{n-2} + 2$$
$$\text{overall cost is} : R_{n-1} + R_{n-1} + 1 + 2R_{n-2} + 2$$
$$= (2R_{n-1} + 1) + (2R_{n-2} + 1) + 1 = Q_n + Q_{n-1} + 1$$

Not to mention that in all cases when $n = 0$, then we don't have any disks to move. Therefore, the equations for $n = 0$ remains as $R_0 = Q_0 = 0$.

Question 2

2). 1.14

As this is like the *Line in the plane* problem, we are going to go with the same approach.

$$P(n) = P(n-1) + \text{new spaces created by adding the } n\text{th plane}$$

Now, for counting the new spaces created by adding the new plane, the nth plane can intersect each of the previous $n-1$ planes in a line. These intersections divide nth plane into new spaces. So, it is a combination of the new and old planes. Therefore:

$$P(n) = P(n-1) + \binom{n}{2} + n + 1$$
$$P(n) = P(n-1) + \frac{n \times (n-1)}{2} + n + 1$$

For solving $P(n)$, we are going to use the *Line in the plane* solution:

$$P(n) = P(n-1) + \frac{n \times (n-1)}{2} + n + 1$$
$$= P(n-2) + \frac{(n-1) \times (n-2)}{2} + (n-1) + 1 + \frac{n \times (n-1)}{2} + n + 1$$
$$= P(n-3) + \frac{(n-2) \times (n-3)}{2} + (n-2) + 1 + \frac{(n-1) \times (n-2)}{2} + (n-1) + 1 + \frac{n \times (n-1)}{2} + n + 1$$
$$= 1 + \sum_{j=0}^{n} 1 + \sum_{j=0}^{n} \frac{j}{2} + \sum_{j=0}^{n} \binom{n-j}{2}$$
$$= 1 + n + \frac{n \times (n-1)}{2} + \frac{n \times (n-1) \times (n-2)}{2 \times 3}$$

To test our answer:

$$P(1) = 1$$
$$P(2) = 1 + 2 + 1 = 4$$
$$P(3) = 1 + 3 + \frac{3 \times (3-1)}{2} + \frac{3 \times (3-1) \times (3-2)}{6} = 8$$

Question 3

3). 1.16

First we are going to simplify our recurrence function to use the repertoire method.

$$g(1) = \alpha \ (n = 0)$$
$$g(2n + j) = 3g(n) + \gamma n + \beta_j \ (n \geq 1, j = 0, 1)$$
$$\rightarrow g(2n) = 3g(n) + \gamma n + \beta_0$$
$$\rightarrow g(2n+1) = 3g(n) + \gamma n + \beta_1$$
$$\Rightarrow g(n) = A(n)\alpha + B(n)\gamma + C(n)\beta_0 + D(n)\beta_1$$

Now first we are going to assign $\gamma = 0$, then we have:

$$g(1) = \alpha$$
$$g(2n) = 3g(n) + \beta_0$$
$$g(2n+1) = 3g(n) + \beta_1$$
$$\Rightarrow g(n) = A(n)\alpha + C(n)\beta_0 + D(n)\beta_1$$

After removing gamma, now we are going to solve this three-parameters equation by using repertoire method. First we are finding $g(n)$ by using small values of n. As in the table bellow, we are going to start from $n = 1$ to $n = 11$.

n	$g(n)$
1	α
2	$3\alpha + \beta_0$
3	$3\alpha + \beta_1$
4	$9\alpha + 4\beta_0$
5	$9\alpha + 3\beta_0 + \beta_1$
6	$9\alpha + \beta_0 + 3\beta_1$
7	$9\alpha + 4\beta_1$
8	$27\alpha + 13\beta_0$
9	$27\alpha + 12\beta_0 + \beta_1$
10	$27\alpha + 10\beta_0 + 3\beta_1$
11	$27\alpha + 9\beta_0 + 4\beta_1$

Table 1: general table for small values of n

As it looks, we are going to have:

$$A(n) = 3^m, \text{where } n = 2^m + l \text{ and } 0 \le l < 2^m$$
$$A(n) - C(n) - D(n) = 1$$
$$A(n) + C(n) = n$$

Therefore, we have $A(n), C(n), andD(n)$. Now for finding $B(n)$, we are going to use $g(n) = n$.

$$n = A(n)\alpha + B(n)\gamma + C(n)\beta_0 + D(n)\beta_1$$
$$\text{a satisfying value set is } (\alpha, \gamma, \beta_0, \beta_1) = (1, -1, 0, 1)$$
$$\to n = A(n) - B(n) + D(n) \to B(n) = A(n) + D(n) - n$$

Now that we have every parameter written based on others, we have:

$$A(n) = 3^m, \text{where } n = 2^m + l \text{ and } 0 \le l < 2^m$$
$$C(n) = n - 3^m = 2^m - 3^m + l$$
$$D(n) = A(n) - 1 - C(n) = 3^m - 1 - 2^m + 3^m - l = 2^m - 2^m - l$$
$$B(n) = A(n) + D(n) - n = 3^m + 2^m - 2^m - l - n = 3^{m+1} - 2^{m+1} - 2l$$

Question 4

4). 1.21

In order to see if an answer exists for this problem, first we are going to check what is our position after kth step. We start from number 1 and each time we move m places to kill the person on that position. To meet the condition of our problem, we need to make sure that our position is always in $(n, n+1, \ldots, 2n)$th positions. Now let's see what are the available positions in each step:

step	available positions
1	$P(1) : m \mod 2n \in (n, \dots, 2n)$
2	$P(2) : P(1) + m \mod 2n - 1 \in (n, \dots, 2n - 1)$
3	$P(3) : P(2) + m \mod 2n - 2 \in (n, \dots, 2n - 2)$
...	...
k	$P(k) : P(k-1) + m \mod 2n - k + 1 \in (n, \dots, 2n - k + 1)$

Table 2: available positions after each step

In general:

$$P(k) = P(k-1) + m \mod (2n - k + 1)$$
$$= P(k-2) + m \mod (2n - k) + m \mod (2n - k + 1)$$
$$= m \mod 2n + m \mod 2n + m \mod (2n - 1) + \dots$$

Based on the available positions, m should be a number that $P(k)$is always in$[n+1, 2n-k]$. To meet that, it needs a common multiple with each number in the equation. Therefore, m can be any number that is in the common multiple of number $n+1, n+2, \dots, 2n$.

Question 5

5). 2.14

First we try to convert our summation:

$$\sum_{k=0}^{n} k2^k \to \sum_{1 \le j \le k \le n} 2^k$$

$$(I) \text{ we know that: } \sum_{j=0}^{n} 1 = n$$

$$\text{therefore, we can replace the innser } k \text{ with } (I)\text{: } k = \sum_{j=0}^{k} 1$$

$$S_n = \sum_{1 \le j \le k \le n} 2^k$$

Now to solve the problem, according to the textbook, we have:

$$(I)\ 2S_{UT} = \sum_{1 \le j,k \le n} a_j a_k - \sum_{1 \le j = k \le n} a_j a_k$$

$$(II)\ \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

As for this problem:

$$2S_{UT} = \sum 1 \le j, k \le n2^k - \sum 1 \le j = k \le n2^k$$
$$(I), (II)\ 2S_{UT} = n2^{n+1} - 1 - (2^{n+1} + 1)$$
$$2S_{UT} = n2^n \times 2 - 2^n \times 2 - 2$$
$$S_{UT} = S_n = n2^n - 2^n - 1$$

Question 6

6). 2.21

To solve these summations using perturbation method, we need to take the first and last element out of the summation. Let's start from S_n because the two other summations are an extended version of it.

$$S_n = \sum_{k=0}^{n} -1^{n-k}$$

$$S_{n+1} = 1 - S_n = -1^{n+1} + \sum_{k=1}^{n+1} -1^{n+1-k} = -1^{n+1} + \sum_{k=0}^{n} -1^{n-k}$$

$$= -1^{n+1} + S_n \Rightarrow 2S_n = 1 - -1^{n+1} \rightarrow S_n = \frac{1}{2}(1 + -1^n)$$

$$S_n = \begin{cases} 0 & \text{if n is odd} \\ 1 & \text{if n is even} \end{cases}$$

Now we are going to solve T_n and U_n with the same approach:

$$T_n = \sum_{k=0}^{n} -1^{n-k}k$$

$$T_{n+1} = n + 1 - T_n = \sum_{k=0}^{n} -1^{n-k}(k+1) = \sum_{k=0}^{n} -1^{n-k} + \sum_{k=0}^{n} -1^{n-k}k$$

$$= S_n + T_n \Rightarrow 2T_n = n + 1 - S_n \rightarrow T_n = \frac{1}{2}(n + 1 - S_n)$$

$$T_n = \begin{cases} \frac{n+1}{2} & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$$

$$U_n = \sum_{k=0}^{n} -1^{n-k}k^2$$

$$U_{n+1} = (n+1)^2 - U_n = \sum_{k=0}^{n} -1^{n-k+1}(k+1)^2 = \sum_{k=0}^{n} -1^{n-k} + \sum_{k=0}^{n} -1^{n-k}(k+1)^2$$

$$= S_n + 2T_n + U_n \Rightarrow 2U_n = n^2 + 2n + 1 - 2T_n - S_n$$

$$T_n = \begin{cases} n^2 + n & \text{if n is odd} \\ n^2 + n & \text{if n is even} \end{cases}$$

Question 7

9).

To solve this problem, we are going to expand our recurrence relation:

$$T(n) = T(\frac{N}{2}) + n \to n = 2^k$$
$$T(2^k) = T(2^{k-1}) + 2^k = T(2^{k-2}) + 2^{k-1} + 2^k + \dots$$
$$T(2^k) = 1 + 2 + 4 + \dots + 2^k = \sum_{j=0}^{k} 2^j$$

For a geometric sequence we have:

$$S_n = \frac{a_0(r^n - 1)}{r - 1}$$

In this case, for $a_0 = 1$ and $r = 2$ we have:

$$S_n = 2^{n+1} - 1$$

Therefore:

$$T(2^k) = 2^{k+1} - 1 = 2 \times 2^k - 1$$
$$T(n) = 2n - 1$$

Question 8

10).

In order to prove this equation, we are going to use a *mathematical induction.* In the Fibonacci sequence we have:

$$F_n = F_{n-1} + F_{n-2}$$
$$F_0 = 0, F_1 = 1, F_2 = 1$$

For $n = 1$ we have:

$$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^1$$

Now let's assume for $n - 1$ we have:

$$\begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^{n-1}$$

Now we are going to multiply both sides by $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$:

$$\begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_n \\ F_{n-1} + F_{n-2} & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^n$$

And now that we proved the nth step, we can say by the power of *mathematical induction*:

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^n$$