A **linear equation** in the variables x1, …, xn is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Or we can say: $\sum_{i=0}^{n} ai\ xi = b$

System of linear equations (or a **linear system**):

A collection of one or more linear equations involving the same variables—say, x1,…,xn

Ex:

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - \quad 4x_3 &= -7 \end{aligned}$$

A system can have 3 type of answers:

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

Also we say => consistent for at least one answer

And => inconsistent for no answers

Essential information of a linear system can be recorded compactly in a rectangular array called a **matrix.**

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

coefficient matrix **augmented matrix**

What is matrix:

Traditionally, a matrix over F is a two-dimensional array whose entries are elements of F. Here is a matrix over R:

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$$

Transpose: Transposing a matrix means swapping its rows and columns.

Definition: The transpose of an $P \times Q$ matrix, written MT, is a $Q \times P$ matrix such that $(MT)j,i = Mi,$j for every $i \in P,\ j \in Q$.

Definition (Linear-combinations definition of vector-matrix multiplication): Let M be an R × C matrix. Let w be an R-vector. Then w ∗ M is the linear combination:

$$\sum_{r \in R} \boldsymbol{w}[r]\ (\text{row } r \text{ of } M)$$

Another operation is multiplying a number into a row and add it to another row which will not change the basics of our matrix.

A solution for solving the linear systems:

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

- ✓ Row operations can be applied to any matrix.
- ✓ Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

Echelon matrix:

Echelon form is a generalization of triangular matrices. Here is an example of a matrix in echelon form:

$$\begin{bmatrix} 0 & 2 & 3 & 0 & 5 & 6 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

Note that

- The first nonzero entry in row 0 is in column 1,
- The first nonzero entry in row 1 is in column 2,
- The first nonzero entry in row 2 is in column 4, and
- The first nonzero entry in row 4 is in column 5.

Definition:

An m × n matrix A is in echelon form if it satisfies the following condition:

'For any row, if that row's first nonzero entry is in position k then every previous row's first nonzero entry is in some position less than k.'

This definition implies that, as you iterate through the rows of A, the first nonzero entries per row move strictly right, forming a sort of staircase that descends to the right:

2	1	0	4	1	3	9	7
0	6	0	1	3	0	4	1
0	0	0	0	2	1	3	2
0	0	0	0	0	0	0	1

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

echelon form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Reduced echelon form

Row reduction:

Any nonzero matrix may transformed by elementary row operations into matrix in echelon form (more than one matrix in echelon form):

- reduced echelon form one obtains from a matrix is unique
- but echelon can be multiply

Pivot:

DEFINITION A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \quad \begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Row Reduction Algorithm:

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

3. Use row replacement operations to create zeros in all positions below the pivot.

4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the sub-matrix that remains. Repeat the process until there are no more nonzero rows to modify.

5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Solutions of Linear Systems:

Variables x1 and x2 corresponding to pivot columns in the matrix are called basic variables.2 the other variable, x3, is called a free variable.

$$\begin{aligned} x_1 \quad\quad - 5x_3 &= 1 \\ x_2 + \ \ x_3 &= 4 \\ 0 \quad &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced echelon form places each basic variable in one and only one equation.

Existence and Uniqueness:

THEOREM 2

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.