# CS364A: Algorithmic Game Theory
# Lecture #3: Myerson's Lemma*

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## 1 The Story So Far

Last time, we introduced the Vickrey auction and proved that it enjoys three desirable and different guarantees:

(1) [**strong incentive guarantees**] DSIC. That is, truthful bidding should be a dominant strategy (and never leads to negative utility).

Don't forget the two reasons we're after the DSIC guarantee. First, such an auction is easy to play for bidders — just play the obvious dominant strategy. Second, assuming only that bidders will play a dominant strategy when they have one, we can confidently predict the outcome of the auction.

(2) [**strong performance guarantees**] Social surplus maximization. That is, assuming truthful bids (which is justified by (1)), the allocation of goods to bidders should maximize $\sum_{i=1}^{n} v_i x_i$, where $x_i$ the amount of stuff allocated to bidder $i$.

(3) [**computational efficiency**] The auction can be implemented in polynomial (indeed, linear) time.

To extend these guarantees beyond single-item auctions to more complex problems, like the sponsored search auctions introduced last lecture, we advocated a two-step design approach.

**Step 1:** Assume, without justification, that bidders bid truthfully. Then, how should we assign bidders to slots so that the above properties (2) and (3) hold?

**Step 2:** Given our answer to Step 1, how should we set selling prices so that the above property (1) holds?



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For instance, in sponsored search auctions, the first step can be implemented using a simple greedy algorithm (assign the $j$th highest bidder the $j$th best slot). But what about the second step?

This lecture states and proves *Myerson's Lemma*, a powerful and general tool for implementing Step 2. This lemma applies to sponsored search auctions as a special case, and we'll also see further applications later.

## 2 Single-Parameter Environments

A good level of abstraction at which to state Myerson's Lemma is *single-parameter environments*. Such an environment has some number $n$ of bidders. Each bidder $i$ has a private valuation $v_i$, its value "per unit of stuff" that it gets. Finally, there is a *feasible set* $X$. Each element of $X$ is an $n$-vector $(x_1, x_2, \ldots, x_n)$, where $x_i$ denotes the "amount of stuff" given to bidder $i$. For example:

- In a single-item auction, $X$ is the set of 0-1 vectors that have at most one 1 (i.e., $\sum_{i=1}^n x_i \leq 1$).
- With $k$ identical goods and the constraint the each customer gets at most one, the feasible set is the 0-1 vectors satisfying $\sum_{i=1}^n x_i \leq k$.
- In sponsored search, $X$ is the set of $n$-vectors corresponding to assignments of bidders to slots, where each slot is assigned at most one bidder and each bidder is assigned at most one slot. If bidder $i$ is assigned to slot $j$, then the component $x_i$ equals the CTR $\alpha_j$ of its slot.

## 3 Allocation and Payment Rules

Recall that a sealed-bid auction has to make two choices: who wins and who pays what. These two decisions are formalized via an *allocation rule* and a *payment rule*, respectively. That is, a sealed-bid auction has three steps:

(1) Collect bids $\mathbf{b} = (b_1, \ldots, b_n)$

(2) **[allocation rule]** Choose a feasible allocation $\mathbf{x}(\mathbf{b}) \in X \subseteq \mathcal{R}^n$ as a function of the bids.

(3) **[payment rule]** Choose payments $\mathbf{p}(\mathbf{b}) \in \mathcal{R}^n$ as a function of the bids.

We continue to use a quasilinear utility model, so, in an auction with allocation and payment rules $\mathbf{x}$ and $\mathbf{p}$, respectively, bidder $i$ has utility

$$u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$$

on the bid profile (i.e., bid vector) $\mathbf{b}$.

In lecture, we will focus on payment rules that satisfy

$$p_i(\mathbf{b}) \in [0, b_i \cdot x_i(\mathbf{b})] \tag{1}$$

for every $i$ and $\mathbf{b}$. The constraint that $p_i(\mathbf{b}) \geq 0$ is equivalent to prohibiting the seller from paying the bidders. The constraint that $p_i(\mathbf{b}) \leq b_i \cdot x_i(\mathbf{b})$ ensures that a truthtelling bidder receives nonnegative utility (do you see why?).

There are applications where it makes sense to relax one or both of these restrictions on payments, but we won't cover any in these lectures.

# 4 Statement of Myerson's Lemma

We now come to two important definitions. Both articulate a property of allocation rules.

**Definition 4.1 (Implementable Allocation Rule)** An allocation rule $\mathbf{x}$ for a single-parameter environment is *implementable* if there is a payment rule $\mathbf{p}$ such the sealed-bid auction $(\mathbf{x}, \mathbf{p})$ is DSIC.

That is, the implementable allocation rules are those that extend to DSIC mechanisms. Equivalently, the projection of DSIC mechanisms onto their allocation rules is the set of implementable rules. If our aim is to design a DSIC mechanism, we must confine ourselves to implementable allocation rules — they form our "design space." In this terminology, we can rephrase the cliffhanger from the end of last lecture as: is the surplus-maximizing allocation rule for sponsored search, which assigns the $j$th highest bidder to the $j$th best slot, implementable?

For instance, consider a single-item auction. Is the allocation rule that awards the good to the highest bidder implementable? Sure — we've already constructed a payment rule, the second-price rule, that renders it DSIC. What about the allocation rule that awards the good to the *second-highest* bidder? Here, the answer is not clear: we haven't seen a payment rule that extends it to a DSIC mechanism, but it also seems tricky to argue that no payment rule could conceivably work.

**Definition 4.2 (Monotone Allocation Rule)** An allocation rule $\mathbf{x}$ for a single-parameter environment is *monotone* if for every bidder $i$ and bids $\mathbf{b}_{-i}$ by the other bidders, the allocation $x_i(z, \mathbf{b}_{-i})$ to $i$ is nondecreasing in its bid $z$.

That is, in a monotone allocation rule, bidding higher can only get you more stuff.

For example, the single-item auction allocation rule that awards the good to the highest bidder is monotone: if you're the winner and you raise your bid (keeping other bids constant), you continue to win. By contrast, awarding the good to the second-highest bidder is a non-monotone allocation rule: if you're the winner and raise your bid high enough, you lose.

The surplus-maximizing allocation rule for sponsored search, with the $j$th highest bidder awarded the $j$th slot, is monotone. The reason is that raising your bid can only increase your position in the sorted order of bids, which can only net you a higher slot, which can only increase the click-through-rate of your slot.

We state Myerson's Lemma in three parts; each is conceptually interesting and will be useful in later applications.

**Theorem 4.3 (Myerson's Lemma [2])** *Fix a single-parameter environment.*

*(a) An allocation rule* $\mathbf{x}$ *is implementable if and only if it is monotone.*

*(b) If* $\mathbf{x}$ *is monotone, then there is a unique payment rule such that the sealed-bid mechanism* $(\mathbf{x}, \mathbf{p})$ *is DSIC [assuming the normalization that* $b_i = 0$ *implies* $p_i(\mathbf{b}) = 0$*].*

*(c) The payment rule in (b) is given by an explicit formula (see* (6)*, below).*

Myerson's Lemma is the foundation on which we'll build most of our mechanism design theory. Part (a) states that Definitions 4.1 and 4.2 define exactly the same class of allocation rules. This equivalence is incredibly powerful: Definition 4.1 describes our design goal but is unwieldy to work with and verify, while Definition 4.2 is far more "operational." Usually, it's not difficult to check whether or not an allocation rule is monotone. Part (b) states that, when an allocation rule is implementable, there is no ambiguity in how to assign payments to achieve the DSIC property — there is only one way to do it. (Assuming bidding zero guarantees zero payment; note this follows from our standing assumption (1).) Moreover, there is a relatively simple and explicit formula for this payment rule (part (c)), a property we apply to sponsored search auctions below and to revenue-maximization auction design in future lectures.

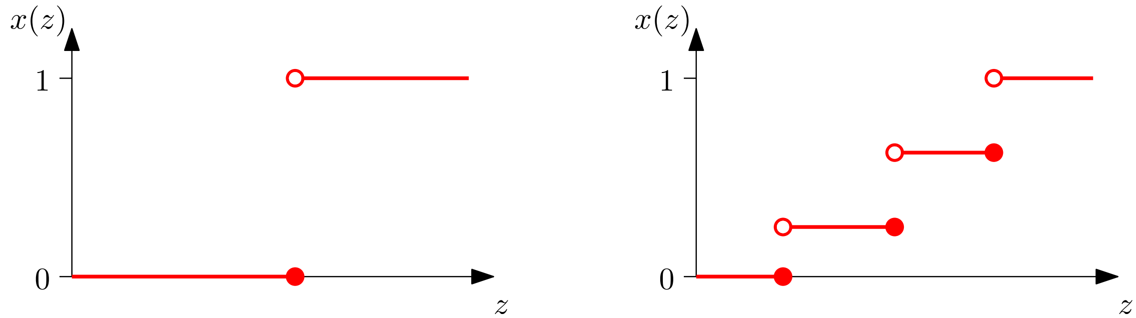
# 5 Proof of Myerson's Lemma (Theorem 4.3)

Consider an allocation rule $\mathbf{x}$, which may or may not be monotone. Suppose there is a payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is a DSIC mechanism — what could $\mathbf{p}$ look like? The plan of this proof is to cleverly invoke the stringent DSIC constraint to whittle the possibilities for $\mathbf{p}$ down to a single candidate. We will establish all three parts of the theorem in one fell swoop.

Recall the DSIC condition: for every bidder $i$, every possible private valuation $b_i$, every set of bids $\mathbf{b}_{-i}$ by the other players, it must be that $i$'s utility is maximized by bidding truthfully. For now, fix $i$ and $\mathbf{b}_{-i}$ arbitrarily. As shorthand, write $x(z)$ and $p(z)$ for the allocation $x_i(z, \mathbf{b}_{-i})$ and payment $p_i(z, \mathbf{b}_{-i})$ of $i$ when it bids $z$, respectively. Figure 1 gives two examples of what the function $x$ might look like.

We invoke the DSIC constraint via a simple but clever swapping trick. Suppose $(\mathbf{x}, \mathbf{p})$ is DSIC, and consider any $0 \leq y < z$. Because bidder $i$ might well have private valuation $z$ and can submit the false bid $y$ if it wants, DSIC demands that

$$\underbrace{z \cdot x(z) - p(z)}_{\text{utility of bidding } z} \geq \underbrace{z \cdot x(y) - p(y)}_{\text{utility of bidding } y} \tag{2}$$



(a) 0-1 Monotone Curve (b) Piecewise Constant Monotone Curve

Figure 1: Examples of allocation curves $x(\cdot)$.

Similarly, since bidder $i$ might well have the private valuation $y$ and could submit the false bid $z$, $(\mathbf{x}, \mathbf{p})$ must satisfy

$$\underbrace{y \cdot x(y) - p(y)}_{\text{utility of bidding } y} \geq \underbrace{y \cdot x(z) - p(z)}_{\text{utility of bidding } z} \tag{3}$$

Myerson's Lemma is, in effect, trying to solve for the payment rule $\mathbf{p}$ given the allocation rule $\mathbf{x}$. Rearranging inequalities (2) and (3) yields the following "payment difference sandwich," bounding $p(y) - p(z)$ from below and above:

$$z \cdot [x(y) - x(z)] \leq p(y) - p(z) \leq y \cdot [x(y) - x(z)] \tag{4}$$

The payment difference sandwich already implies one major component of Myerson's Lemma — do you see why?

Thus, we can assume for the rest of the proof that $\mathbf{x}$ is monotone. We will be slightly informal in the following argument, but will cover all of the major ideas of the proof.

In (4), fix $z$ and let $y$ tends to $z$ from above. We focus primarily on the case where $x$ is piecewise constant, as in Figure 1. In this case, $x$ is flat except for a finite number of "jumps". Taking the limit $y \downarrow z$ in (4), the left- and right-hand sides become 0 if there is no jump in $x$ at $z$. If there is a jump of magnitude $h$ at $z$, then the left- and right-hand sides both tend to $z \cdot h$. This implies the following constraint on $p$, for every $z$:

$$\text{jump in } p \text{ at } z = z \cdot \text{jump in } x \text{ at } z \tag{5}$$

Thus, assuming the normalization $p(0) = 0$, we've derived the following *payment formula*, for every bidder $i$, bids $\mathbf{b}_{-i}$ by other bidders, and bid $b_i$ by $i$:

$$p_i(b_i, \mathbf{b}_{-i}) = \sum_{j=1}^{\ell} z_j \cdot \text{jump in } x_i(\cdot, \mathbf{b}_{-i}) \text{ at } z_j, \tag{6}$$

where $z_1, \ldots, z_\ell$ are the breakpoints of the allocation function $x_i(\cdot, \mathbf{b}_{-i})$ in the range $[0, b_i]$

A similar argument applies when $x$ is a monotone function that is not necessarily piecewise constant. For instance, suppose that $x$ is differentiable. Dividing the payment difference sandwich (4) by $y - z$ and taking the limit as $y \downarrow z$ yields the constraint

$$p'(z) = z \cdot x'(z)$$

and, assuming $p(0) = 0$, the payment formula

$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \cdot \tfrac{d}{dz} x_i(z, \mathbf{b}_{-i}) dz \tag{7}$$

for every bidder $i$, bid $b_i$, and bids $\mathbf{b}_{-i}$ by the others.

We reiterate that the payment formula in (6) is the *only* payment rule with a chance of extending the given piecewise constant allocation rule $\mathbf{x}$ into a DSIC mechanism. Thus, for every allocation rule $\mathbf{x}$, there is at most one payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is DSIC (cf., part (b) of Theorem 4.3). But the proof is not complete — we still have to check that this payment rule works provided $\mathbf{x}$ is monotone! Indeed, we already know that even this payment rule fails when $\mathbf{x}$ is not monotone.

We give a proof by picture that, when $\mathbf{x}$ is monotone and piecewise constant and $\mathbf{p}$ is defined by (6), then $(\mathbf{x}, \mathbf{p})$ is a DSIC mechanism. The same argument works more generally for monotone allocation rules that are not piecewise constant, with payments defined as in (7). This will complete the proof of all three parts of Myerson's Lemma.

Figures 2(a)–(i) depict the utility of a bidder when it bids truthfully, overbids, and underbids, respectively. The allocation curve $x(z)$ and the private valuation $v$ of the bidder is the same in all three cases. Recall that the bidder's utility when it bids $b$ is $v \cdot x(b) - p(b)$. We depict the first term $v \cdot x(b)$ as a shaded rectangle of width $v$ and height $x(b)$ (Figures 2(a)–(c)). Using the formula (6), we see that the payment $p(b)$ can be represented as the shaded area to the left of the allocation curve in the range $[0, b]$ (Figures 2(d)-(f)). The bidder's utility is the difference between these two terms (Figures 2(g)-(i)). When the bidder bids truthfully, its utility is precisely the area under the allocation curve in the range $[0, v]$ (Figure 2(g)).[1] When the bidder overbids, its utility is this same area, minus the area above the allocation curve in the range $[v, b]$ (Figure 2(h)). When the bidder underbids, its utility is a subset of the area under the allocation curve in the range $[0, v]$ (Figure 2(i)). Since the bidder's utility is the largest in the first case, the proof is complete.

# 6 Applying the Payment Formula: Sponsored Search Solved

Myerson's payment formula (6) is easy to understand and apply in many applications. For starters, consider a single-item auction with the allocation rule that allocates the good to

[1] In this case, the social surplus contributed by this bidder ($v \cdot x(v)$) naturally splits into its utility (or "consumer surplus"), the area under the curve, and the seller revenue, the area above the curve (in the range $[0, v]$).

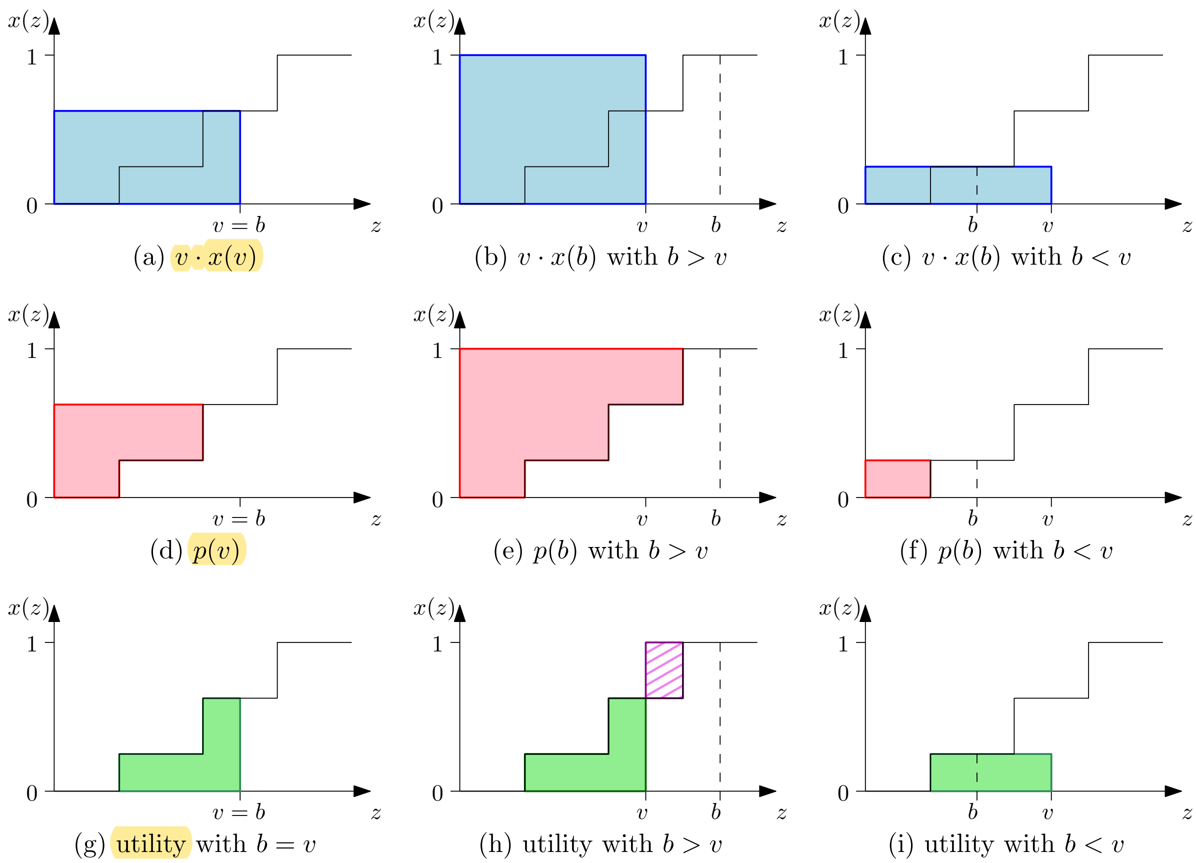



Figure 2: Proof by picture that the payment rule in (6), coupled with the given monotone and piecewise constant allocation rule, yields a DSIC mechanism. The three columns consider the cases of truthful bidding, overbidding, and underbidding, respectively. The three rows show the surplus $v \cdot x(b)$, the payment $p(b)$, and the utility $v \cdot x(b) - p(b)$, respectively. In (h), the solid region represents positive utility and the lined region represents negative utility.

the highest bidder. Fixing $i$ and $\mathbf{b}_{-i}$, the function $x_i(\cdot, \mathbf{b}_{-i})$ is 0 up to $B = \max_{j \neq i} b_j$ and 1 thereafter. The formula (6) is either 0 (if $b_i < B$) or, if $b_i > B$, there is a single breakpoint (a jump of 1 at $B$) and the payment is $p_i(b_i, \mathbf{b}_{-i}) = B$. Thus, Myerson's Lemma regenerates the Vickrey auction as a special case.

Now let's return to sponsored search auctions. Recall from last lecture that we have $k$ slots with click-through-rates (CTRs) $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Let $\mathbf{x}(\mathbf{b})$ be the allocation rule that assigns the $j$th highest bidder to the $j$th highest slot, for $j = 1, 2, \ldots, k$. We've noted previously that this rule is surplus-maximizing (assuming truthful bids) and monotone. Applying Myerson's Lemma, we can derive a unique payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is DSIC. To describe it, consider a bid profile $\mathbf{b}$; we can re-index the bidders so that $b_1 \geq b_2 \geq \cdots \geq b_n$. For intuition, focus on the first bidder and imagine increasing its bid from 0 to $b_1$, holding the other bids fixed. The allocation $x_i(z, \mathbf{b}_{-i})$ ranges from 0 to $\alpha_1$ as $z$ ranges from 0 to $b_1$, with a jump of $\alpha_j - \alpha_{j+1}$ at the point where $z$ becomes the $j$th highest bid in the profile $(z, \mathbf{b}_{-i})$, namely $b_{j+1}$. Thus, in general, Myerson's payment formula specializes to

$$p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1}) \tag{8}$$

for the $i$th highest bidder (where $\alpha_{k+1} = 0$).

Recall our assumption that bidders don't care about impressions (i.e., having their link shown), except inasmuch as it leads to a click. This motivates charging bidders per click, rather than per impression. The per-click payment for bidder/slot $i$ is simply that in (8), scaled by $\frac{1}{\alpha_i}$:

$$p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1} \frac{\alpha_j - \alpha_{j+1}}{\alpha_i}. \tag{9}$$

Observe that the formula in (9) has the pleasing interpretation that, when its link its clicked, an advertiser pays a suitable convex combination of the lower bids.

By historical accident, the sponsored search auctions used in real life are based on the "Generalized Second Price (GSP)" auction [1, 3], which is a simpler (and perhaps incorrectly implemented) version of the DSIC auction above. The per-click payment in GSP is simply the next lowest bid. Since Myerson's Lemma implies that the payment rule in (9) is the unique one that yields the DSIC property, we can immediately conclude that the GSP auction is *not* DSIC. It still has a number of nice properties, however, and is "partially equivalent" to the DSIC auction in a precise sense. The Problems ask you to explore this equivalence in detail.

# References

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