

# Training Neural Networks Optimization II

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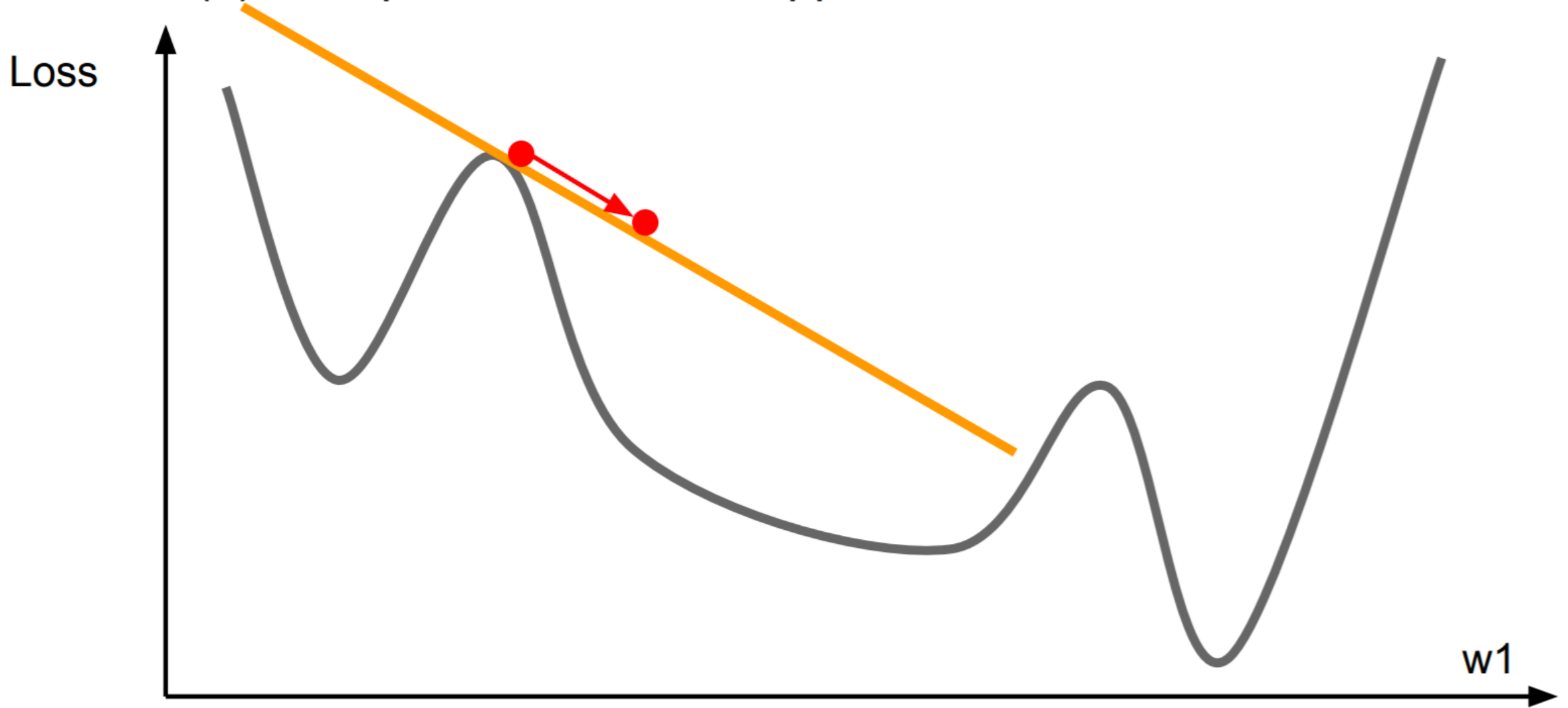
Sharif University of Technology

Spring 2024

Most slides have been adapted from Bhiksha Raj, 11-785, CMU  
and some from Fei Fei Li et. al, cs231n, Stanford

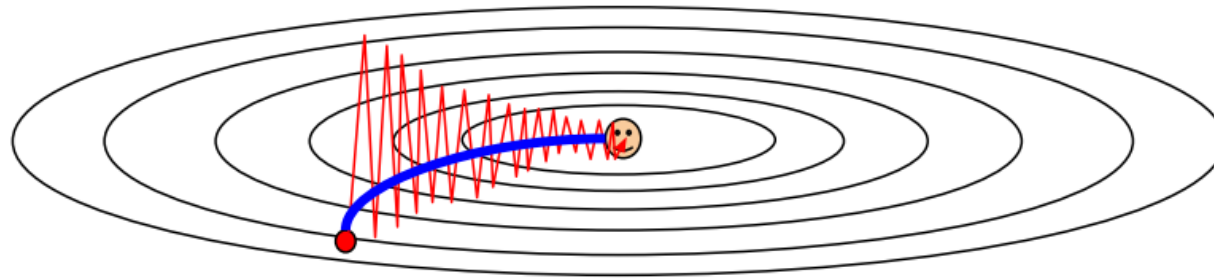
# Recap: First-order optimization

- (1) Use gradient form linear approximation
- (2) Step to minimize the approximation



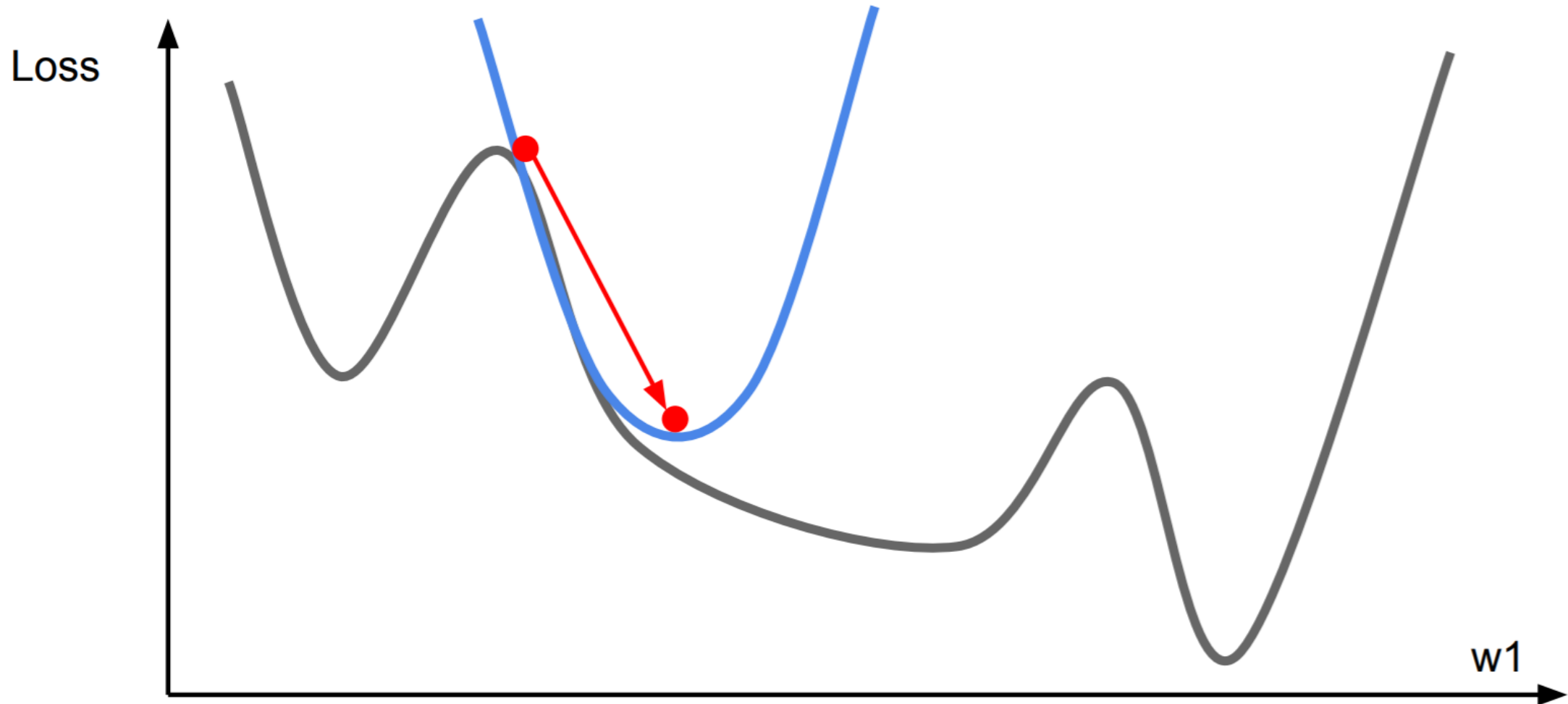
# Recap: Poor conditioning problem

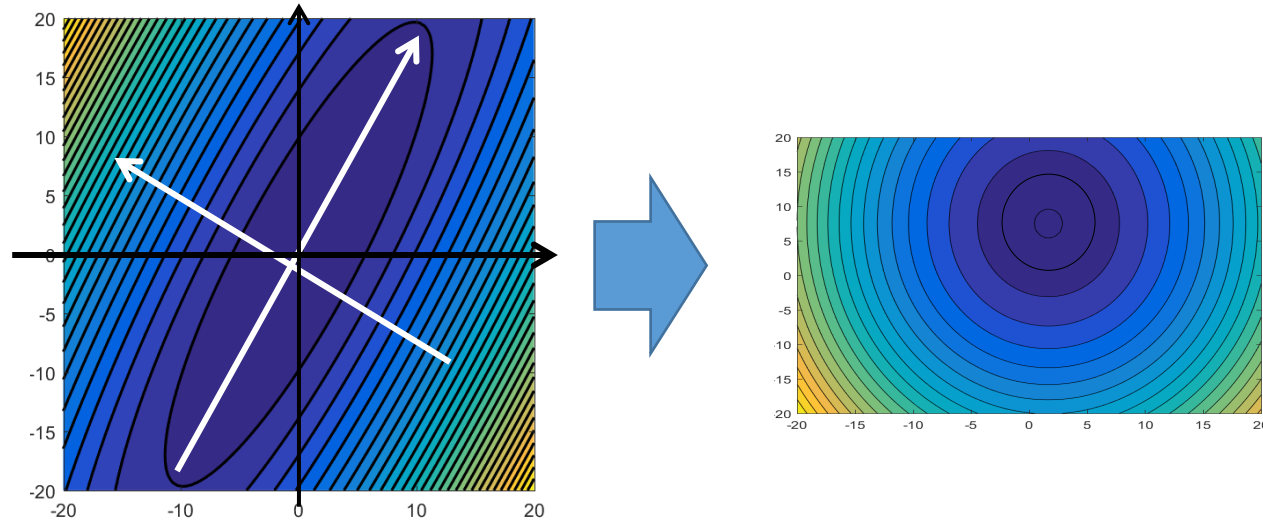
Poor Conditioning



# Recap: Second-order optimization

- (1) Use gradient **and Hessian** to form **quadratic** approximation
- (2) Step to the **minima** of the approximation





$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta \mathbf{H}^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})$$

$$\eta = 1$$

# Recap: Second-order optimization

- Taylor expansion (second-order):

$$E(\mathbf{w})$$

$$\approx E(\mathbf{w}^{(k)}) + \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T (\mathbf{w} - \mathbf{w}^{(k)}) + \frac{1}{2} (\mathbf{w} - \mathbf{w}^{(k)})^T H_E(\mathbf{w}^{(k)}) (\mathbf{w} - \mathbf{w}^{(k)}) + \dots$$

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})$$

No hyperparameters!

No learning rate!

Why is this bad for deep learning?

× Hessian has  $O(n^2)$  elements

× Inverting takes  $O(n^3)$

$n$  = (Tens or Hundreds or ... of) Millions

# Recap: Second-order optimization: L-BFGS

- Quasi-Newton methods (BFGS most popular):
  - instead of inverting the Hessian (requiring  $O(n^3)$ ), approximate inverse Hessian with rank 1 updates over time ( $O(n^2)$  each).
- L-BFGS (Limited memory BFGS):
  - Does not form/store the full inverse Hessian.
  - usually works very well in full batch, deterministic mode
    - i.e. work very well when you have a single, deterministic cost function
  - But does not transfer very well to mini-batch setting.
    - Gives bad results
    - Adapting L-BFGS to large-scale, stochastic setting is an active area of research.

# Recap: Adaptive learning rates

- ***Advanced methods:*** Adaptive updates, where the learning rate of each parameter is itself adjusted as part of the estimation
  - RMS Prop, ADAM, ...

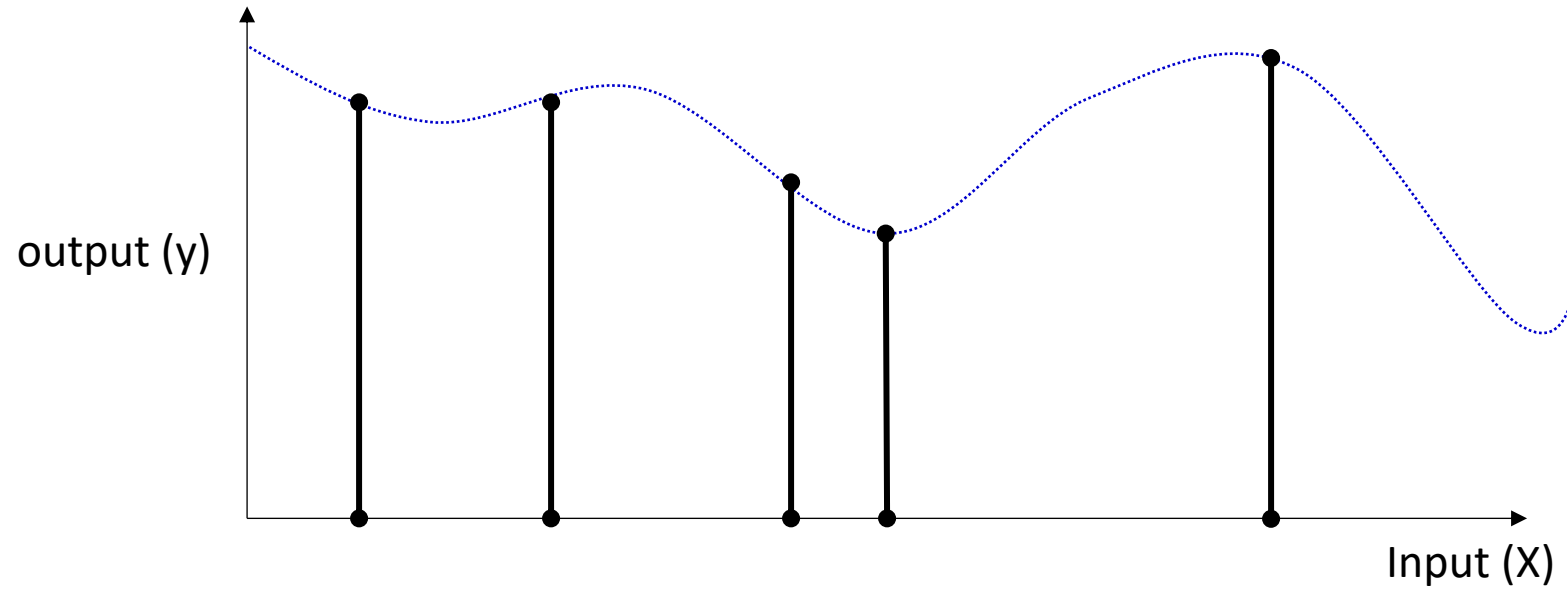


# In practice

- **Adam** is a good default choice in most cases
- **SGD** or **SGD+Momentum** can outperform Adam but may require more tuning of learning rate and decay schedule
- For some applications, the solutions found by adaptive methods **generalize worse** (often significantly worse) than SGD

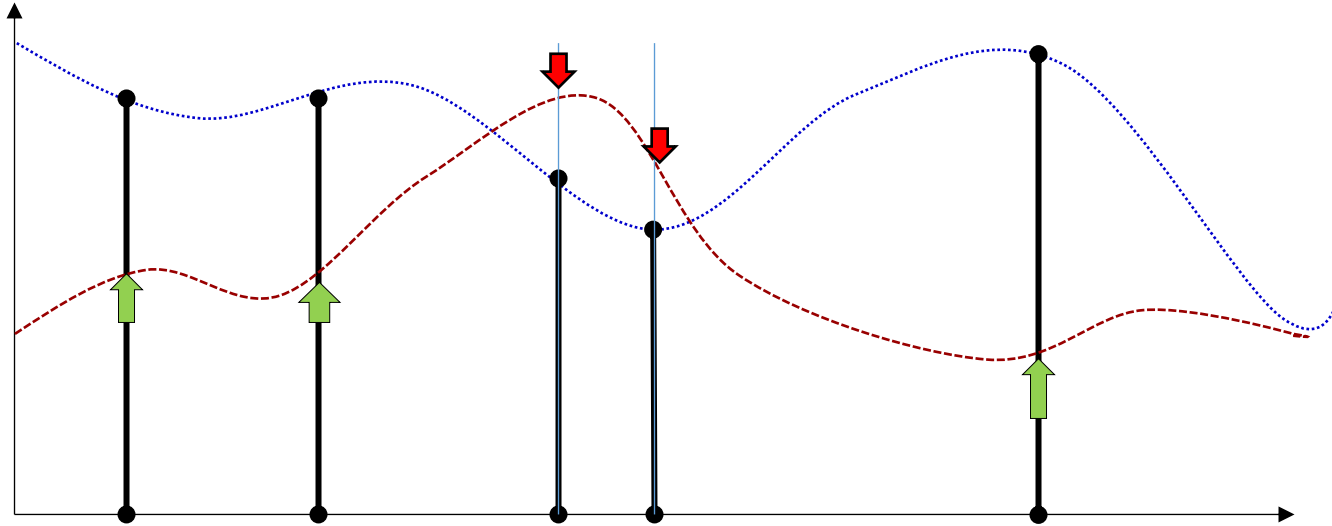
Winson et al., NeurIPS 2017

# The training formulation



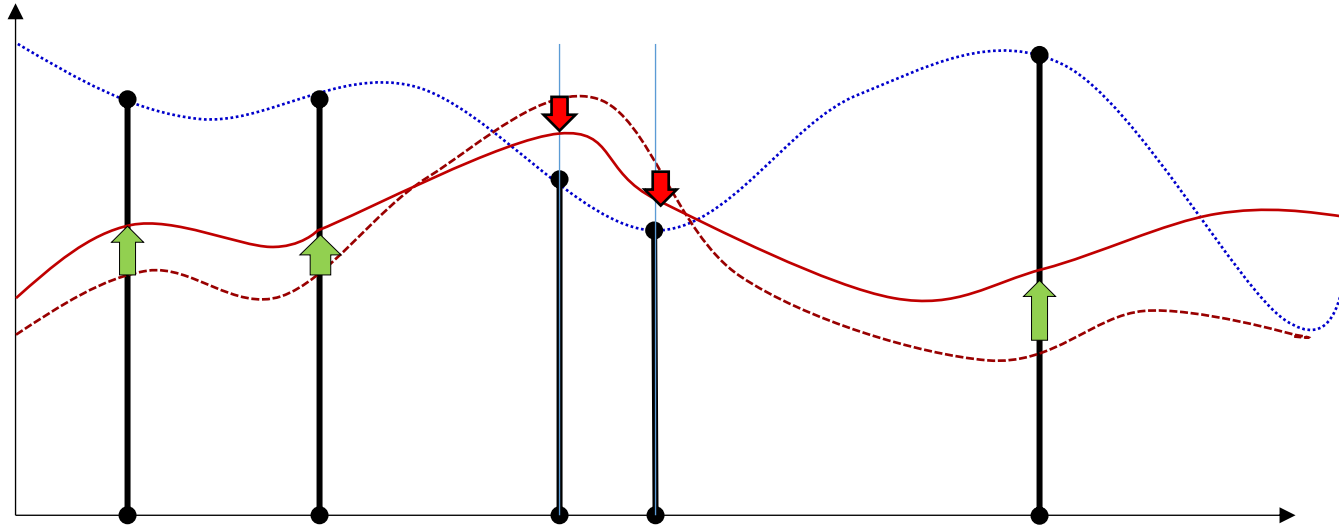
- Given input output pairs at a number of locations, estimate the entire function

# Gradient descent



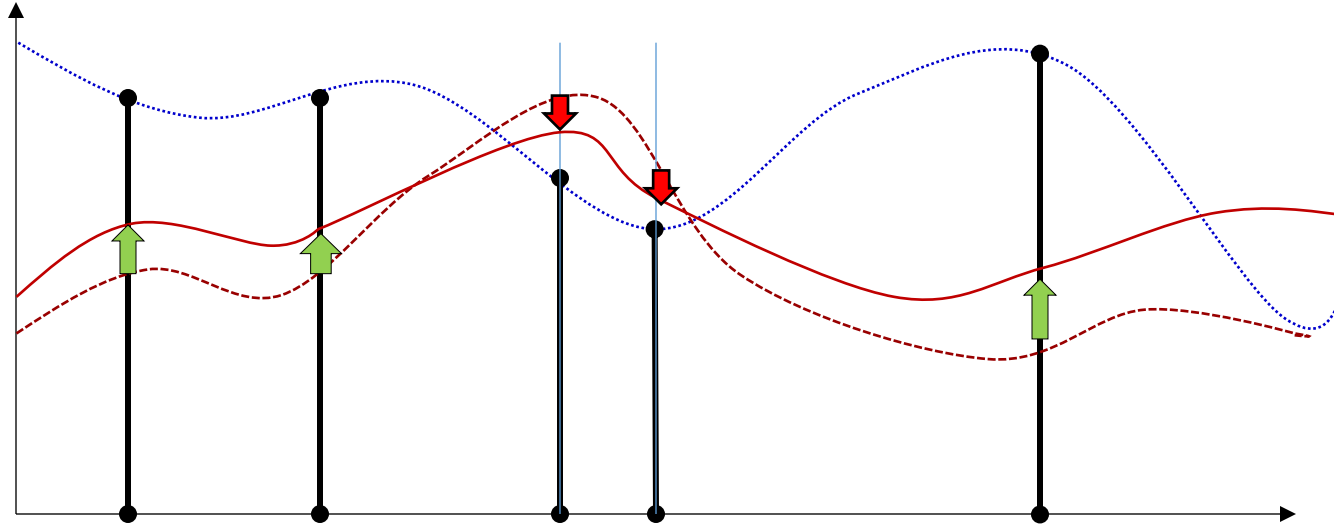
- Start with an **initial function**
- Adjust its value at *all* points to make the outputs closer to the required value
  - Gradient descent adjusts parameters to adjust the function value at *all* points
  - Repeat this iteratively until we get arbitrarily close to the target function at the training points

# Gradient descent



- Start with an initial function
- Adjust its value at *all* points to make the outputs closer to the required value
  - Gradient descent adjusts parameters to adjust the function value at *all* points
  - Repeat this iteratively until we get arbitrarily close to the target function at the training points

# Effect of number of samples

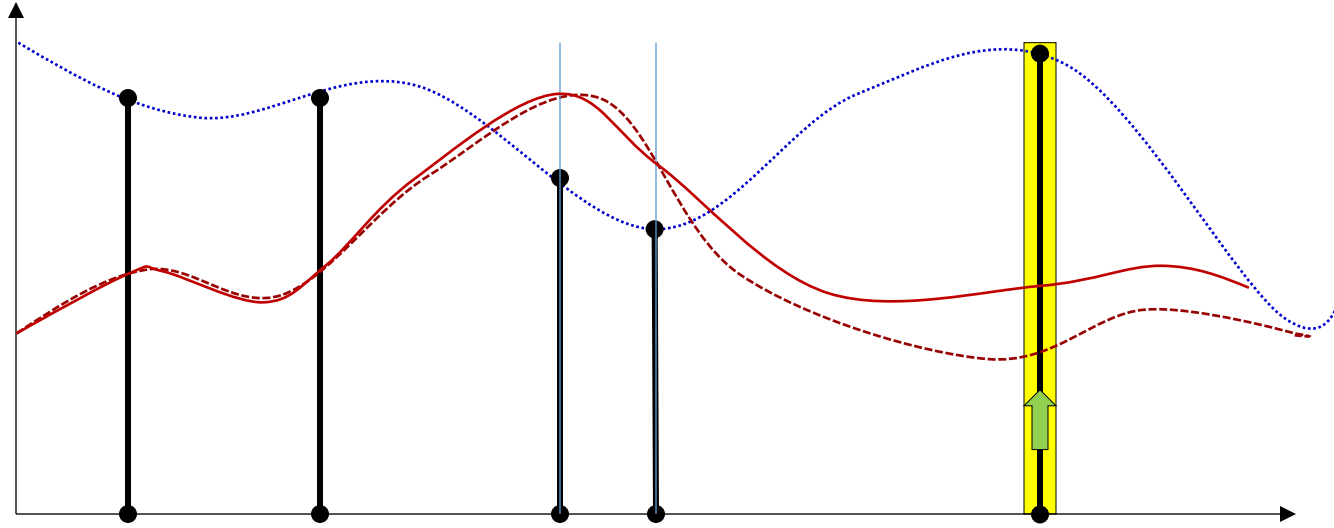


- Problem with conventional gradient descent: we try to simultaneously adjust the function at *all* training points
  - We must process *all* training points before making a single adjustment
  - “Batch” update

# Incremental Update: Stochastic Gradient Descent

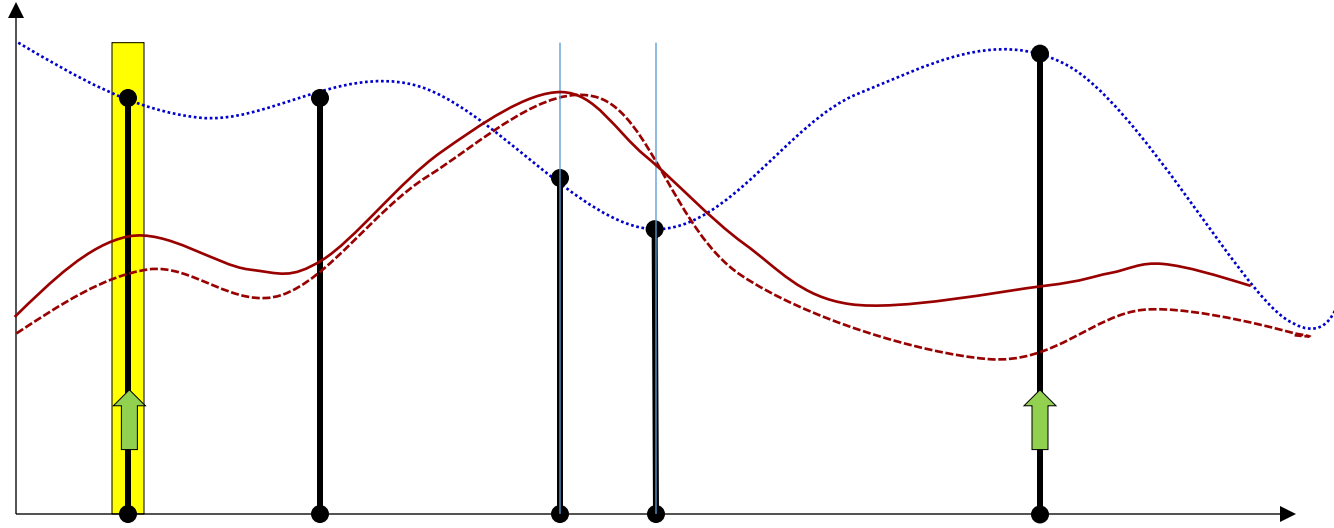
- Given  $(\mathbf{x}^{(1)}, \mathbf{y}^{(1)}), (\mathbf{x}^{(2)}, \mathbf{y}^{(2)}), \dots, (\mathbf{x}^{(N)}, \mathbf{y}^{(N)})$
- Initialize all weights
- Do:
  - Randomly permute data
  - For all  $n = 1:N$ 
    - Update.  $\mathbf{W} = \mathbf{W} - \eta \nabla_{\mathbf{W}} \text{loss}(\mathbf{o}^{(n)}, \mathbf{y}^{(n)})$
- Until convergence

# Alternative: Incremental update



- Alternative: adjust the function at one training point at a time
  - Keep adjustments small

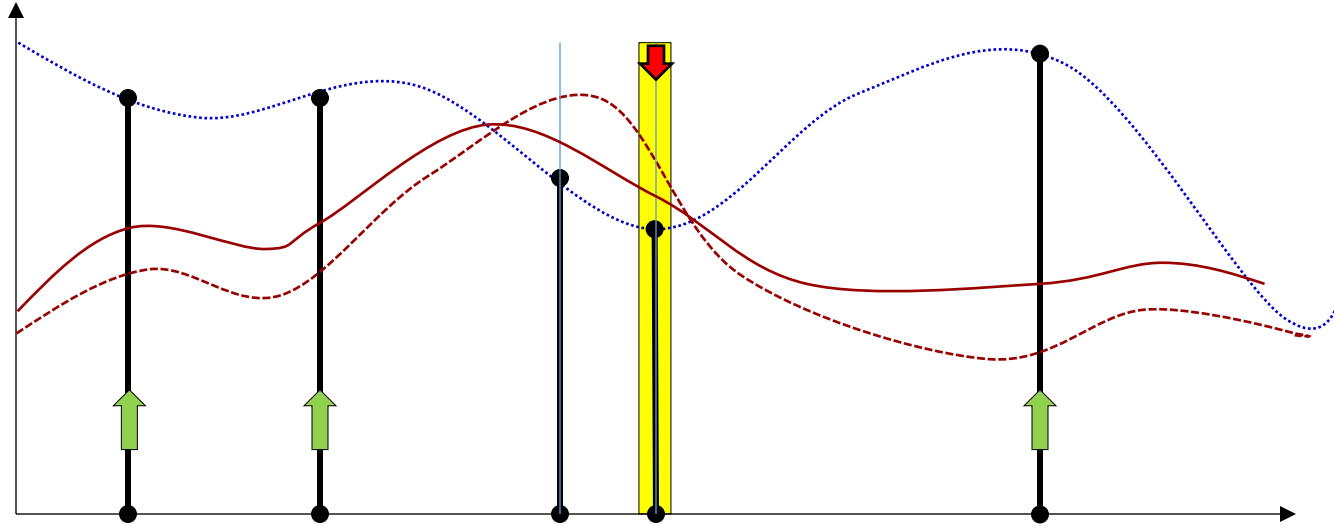
# Alternative: Incremental update



- Alternative: adjust the function at one training point at a time
  - Keep adjustments small



# Alternative: Incremental update



- Alternative: adjust the function at one training point at a time
  - Keep adjustments small
  - Eventually, when we have processed all the training points, we will have adjusted the entire function
    - With *greater* overall adjustment than we would if we made a single “Batch” update

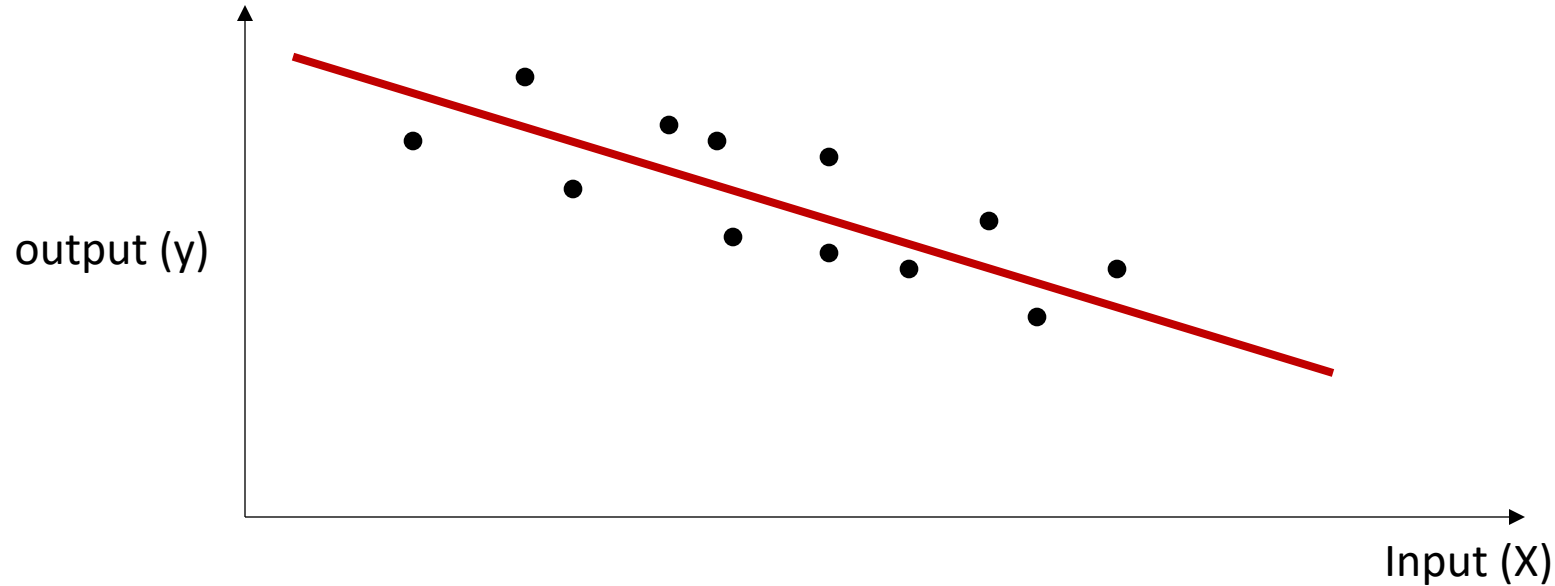
# Story so far

- In any gradient descent optimization, presenting training instances incrementally can be more effective than presenting them all at once
  - Provided training instances must be presented in random order
  - **“Stochastic Gradient Descent”**
- This also holds for training neural networks

# Explanations and restrictions

- So why does this process of incremental updates work?
- Under what conditions?

# Caveats: learning rate



- Except in the case of a perfect fit, even an optimal overall fit will look incorrect to *individual* instances
  - Correcting the function for individual instances will lead to never-ending, non-convergent updates
  - We must *shrink* the learning rate with iterations to prevent this

# Incremental Update: Stochastic Gradient Descent

- Given  $(\mathbf{x}^{(1)}, \mathbf{y}^{(1)}), (\mathbf{x}^{(2)}, \mathbf{y}^{(2)}), \dots, (\mathbf{x}^{(N)}, \mathbf{y}^{(N)})$

- Initialize all weights

- $j = 0$

- Do:

- Randomly permute data

Randomize input order

- For all  $n = 1:N$

- $j = j + 1$

- Update.  $W = W - \eta_j \nabla_W \text{loss}(\mathbf{o}^{(n)}, \mathbf{y}^{(n)})$

Learning rate reduces with j

- Until *Err* has converged

An epoch

# Stochastic Gradient Descent

- The iterations can make multiple passes over the data
- A single pass through the entire training data is called an “epoch”
  - An epoch over a training set with  $N$  samples results in  $N$  updates of parameters

# SGD convergence

- SGD converges “almost surely” to a global or local minimum for most functions
  - Sufficient condition: step sizes follow the following conditions

$$\sum_k \eta_k = \infty$$

- Eventually the entire parameter space can be searched

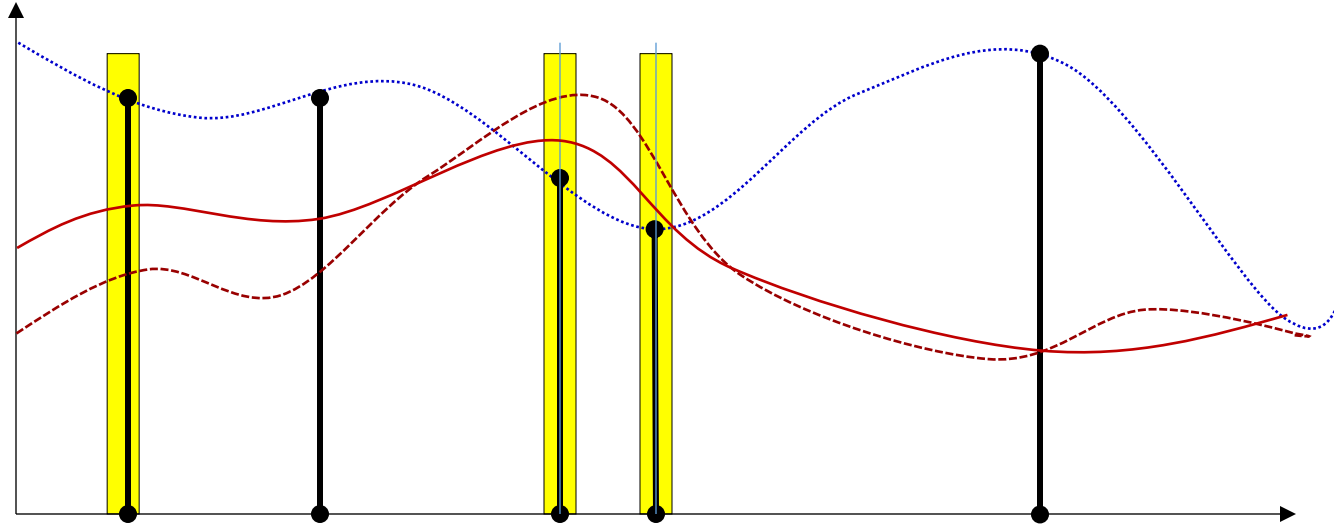
$$\sum_k \eta_k^2 < \infty$$

- The steps shrink
  - The fastest converging series that satisfies both above requirements is

$$\eta_k \propto \frac{1}{k}$$

- This is the optimal rate of shrinking the step size for strongly convex functions
  - More generally, the learning rates are heuristically determined
- If the loss is convex, SGD converges to the optimal solution
- For non-convex losses SGD converges to a stationary point

# Alternative: Mini-batch update



- Alternative: adjust the function at a small, randomly chosen subset of points
  - Keep adjustments small
  - If the subsets cover the training set, we will have adjusted the entire function
- As before, vary the subsets randomly in different passes through the training data
  - Shuffle first and reshuffle between epochs



# Mini-batch Gradient Descent

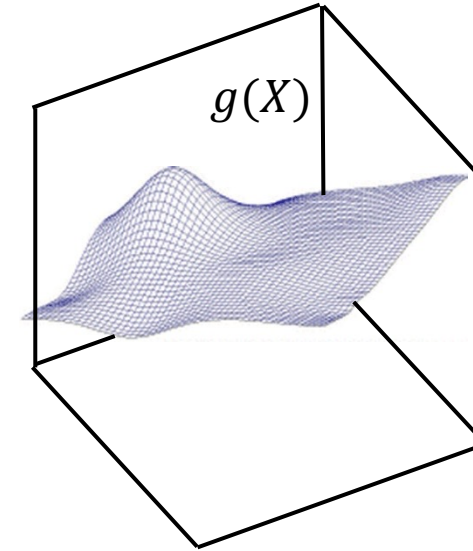
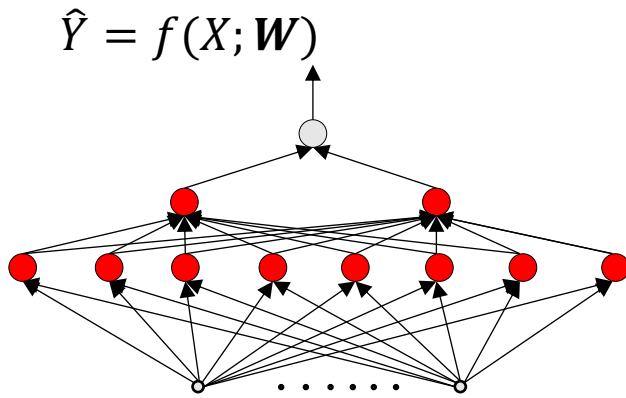
- Given  $(\mathbf{x}^{(1)}, \mathbf{y}^{(1)}), (\mathbf{x}^{(2)}, \mathbf{y}^{(2)}), \dots, (\mathbf{x}^{(N)}, \mathbf{y}^{(N)})$
  - Initialize all weights
  - $j = 0$
  - Do:
    - Randomly permute data
    - while  $n < N$
- ← Randomize input order

An epoch

- $j = j + 1$
- Update.  $W = W - \eta_j \nabla_W \sum_{i=n}^{n+B-1} \text{loss}(\mathbf{o}^{(i)}, \mathbf{y}^{(i)})$
- $n = n + B$

- Until *Err* has converged

# Recall: Modelling a function



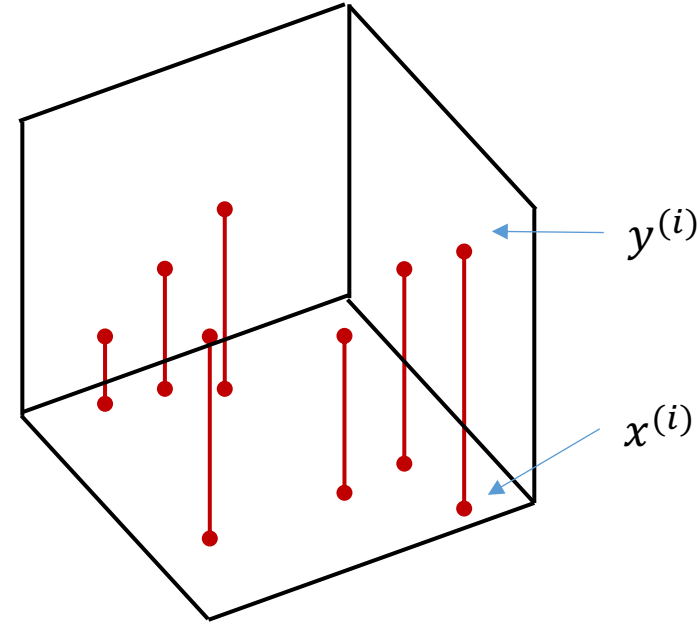
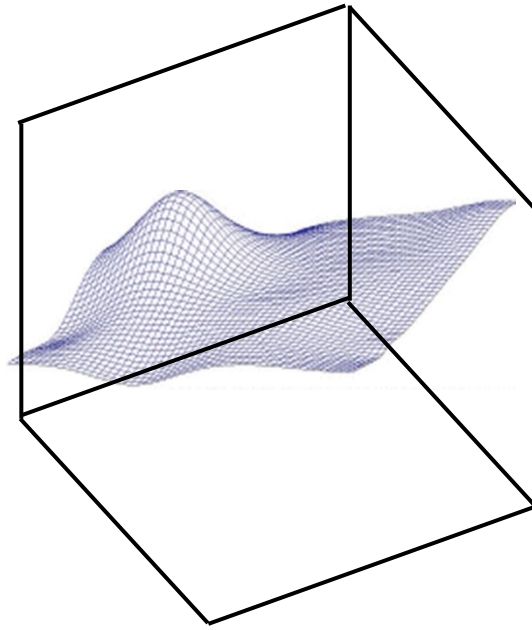
- To learn a network  $f(X; \mathbf{W})$  to model a function  $g(X)$ , it is ideal to minimize the *expected loss*

$$\begin{aligned} & E[\text{loss}(f(X; W), g(X))] \\ &= \int_X \text{loss}(f(X; W), g(X)) P(X) dX \end{aligned}$$

$$\mathbf{W}^* = \underset{W}{\operatorname{argmin}} E[\text{loss}(f(X; W), g(X))]$$

minimizes the expected error

# Recall: The *Empirical* risk



- In practice, we minimize the *empirical error*

$$Err(W) = \frac{1}{N} \sum_{i=1}^N \text{loss}(f(x^{(i)}; W), y^{(i)})$$

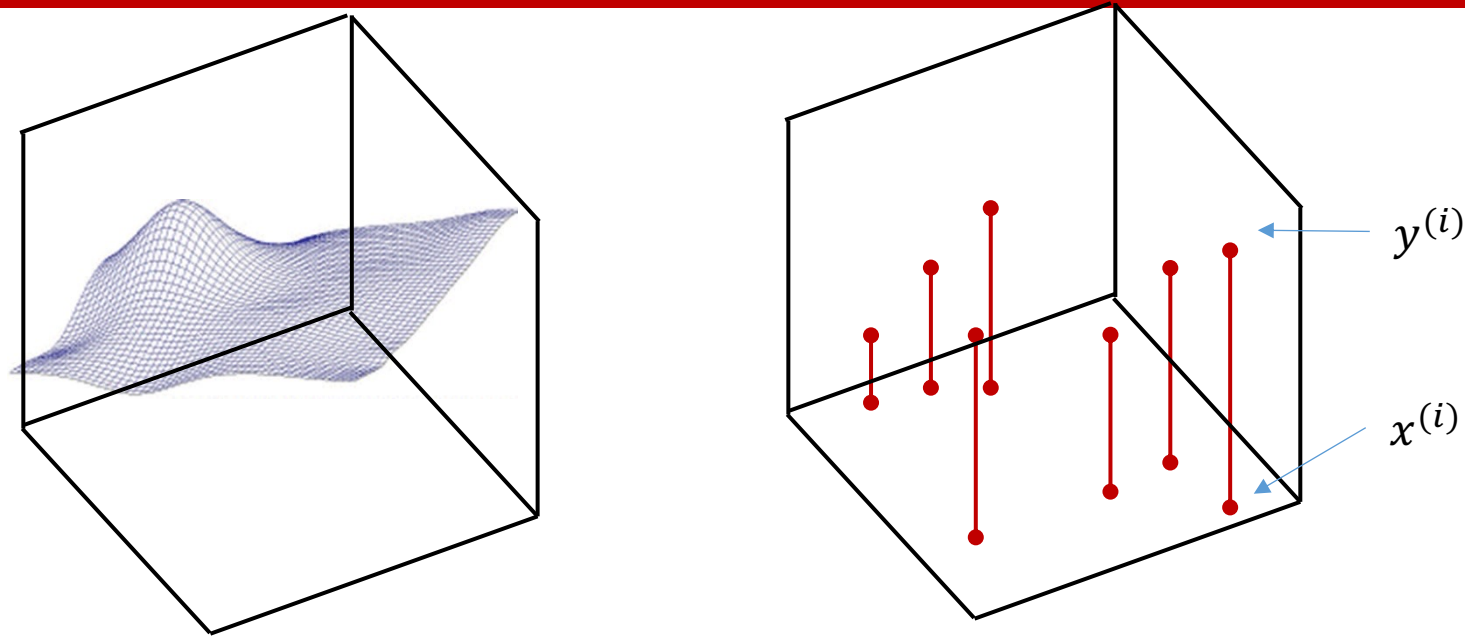
$$\widehat{W} = \underset{W}{\operatorname{argmin}} Err(f(X; W), g(X))$$

minimizes the empirical error

- The *expected value* of the *empirical error* is actually the *expected loss*

$$E[Err(W)] = E[\text{loss}(f(X; W), g(X))]$$

# The *Empirical* risk



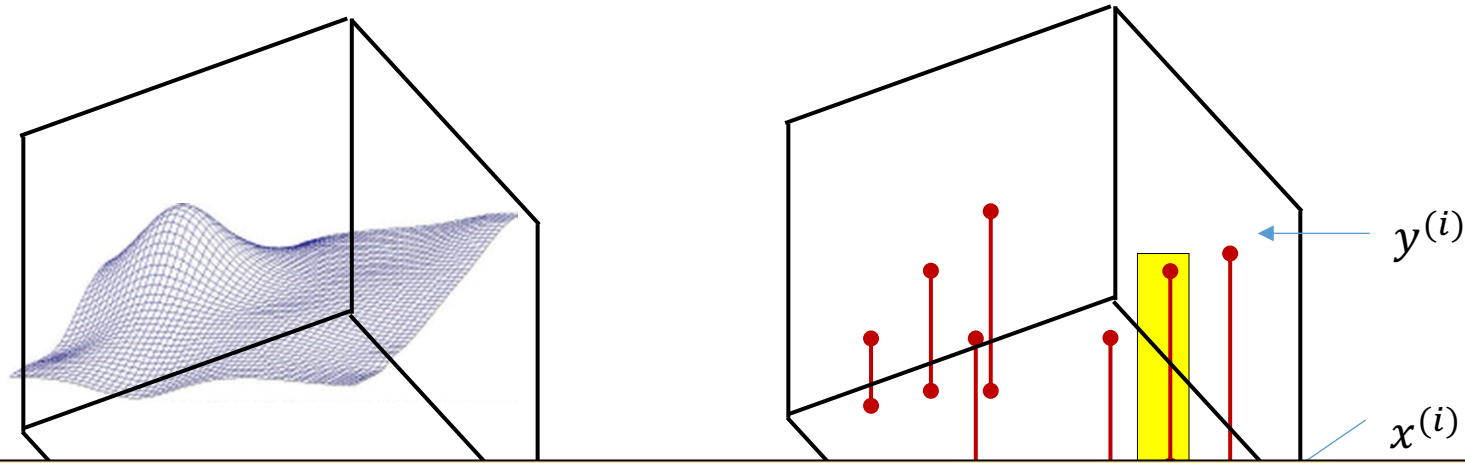
The empirical error is an *unbiased* estimate of the expected error

$$Err(f(X; W), g(X)) = \frac{1}{N} \sum_{i=1}^N loss(f(x^{(i)}; W), y^{(i)})$$
$$\widehat{W} = \underset{W}{\operatorname{argmin}} Err(f(X; W), g(X))$$

- The *expected value* of the *empirical error* is actually the *expected loss*

$$E[Err(W)] = E[loss(f(X; W), g(X))]$$

# SGD



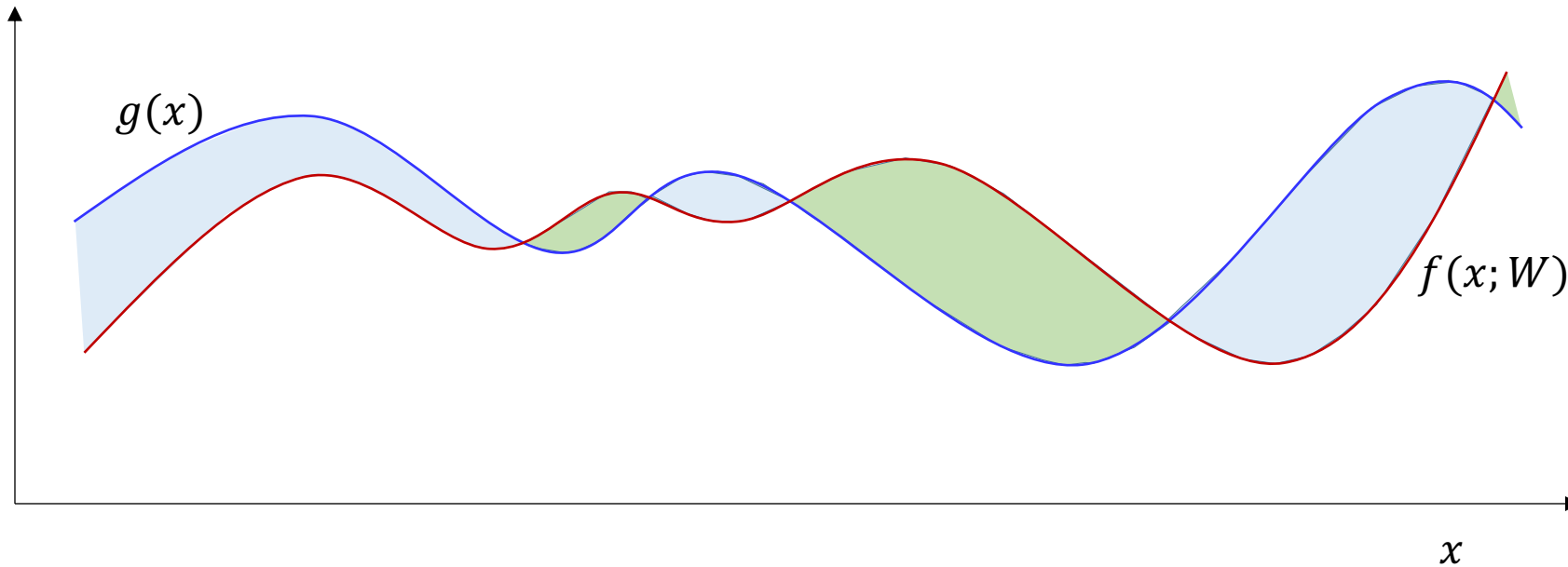
The variance of the sample error is the variance of the loss itself:  $\text{var}(\text{Err}) = \text{var}(\text{loss})$

This is **N times the variance of** the empirical average minimized by **batch update**

The sample error is also an **unbiased estimate** of the expected error

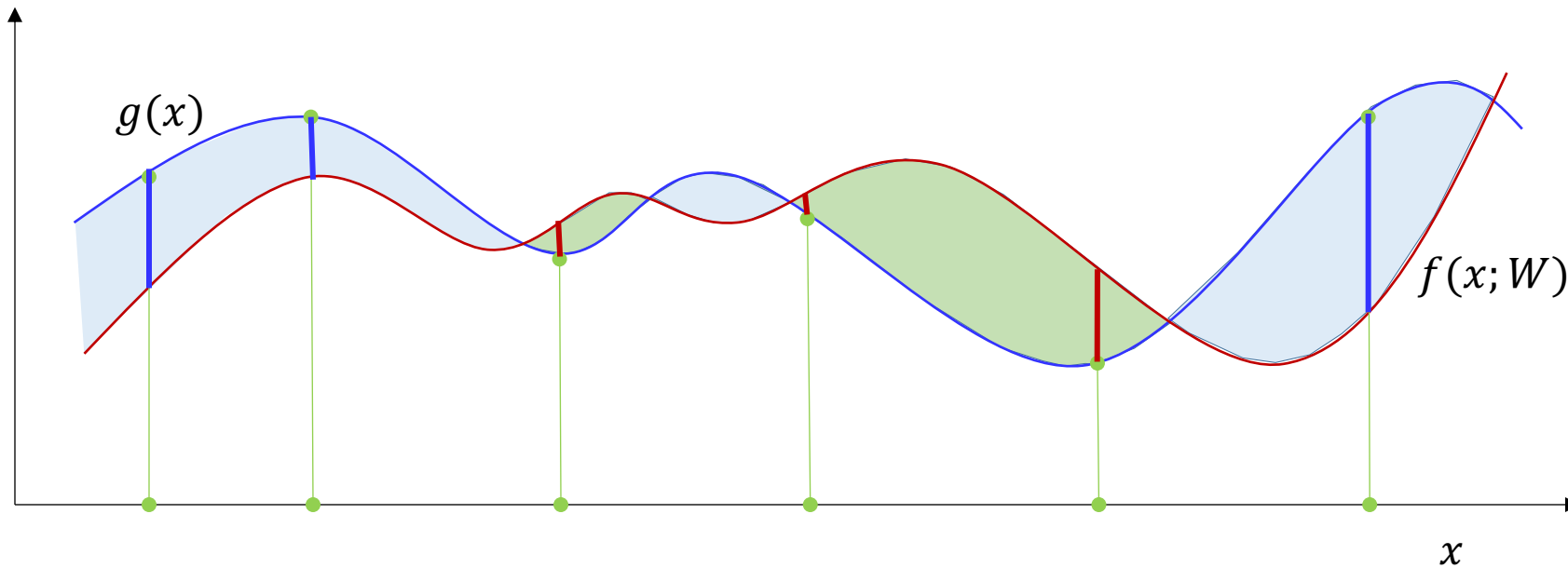
- At each iteration, **SGD** focuses on the loss of a **single** sample  $\text{loss}(f(x^{(i)}; W), y^{(i)})$
- The *expected value* of the *sample error* is **still** the *expected loss*  $E[\text{loss}(f(x; W), y)]$

# Explaining the variance



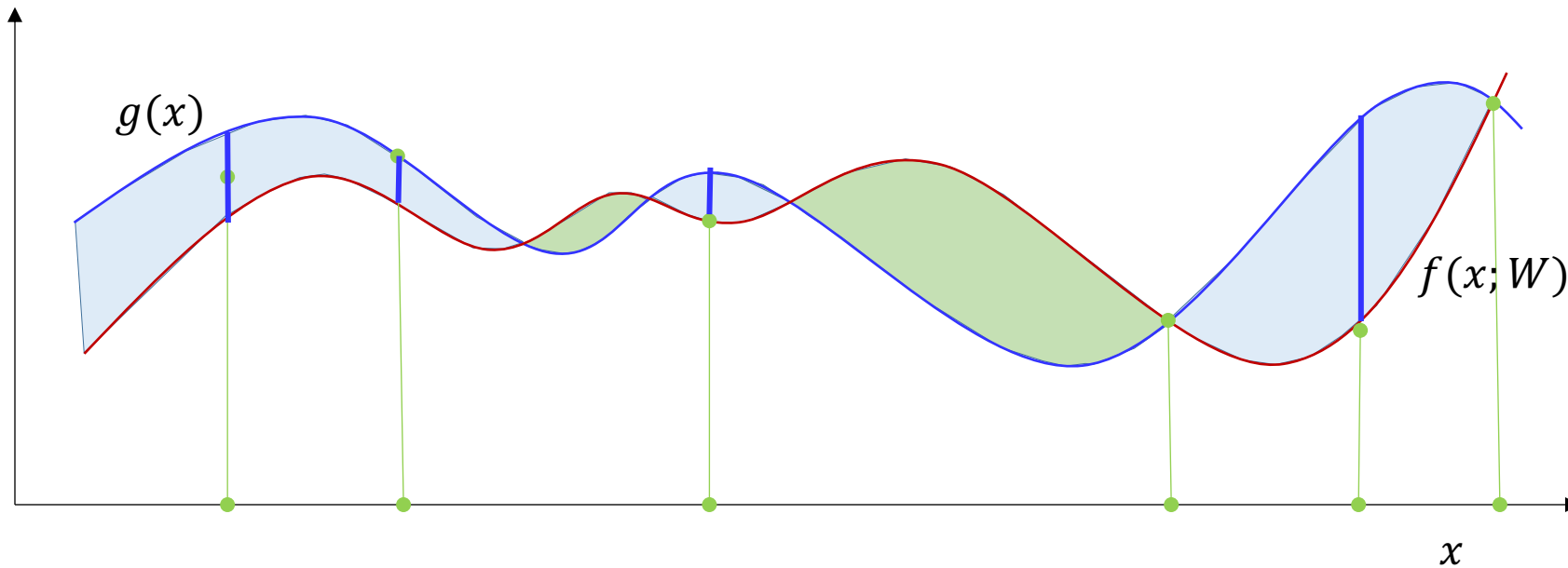
- The blue curve is the function being approximated
- The red curve is the approximation by the model at a given  $W$
- The heights of the shaded regions represent the point-by-point error
  - We want to find the  $W$  that minimizes the average loss

# Explaining the variance



- Sample estimate approximates the shaded area with the average length of the lines

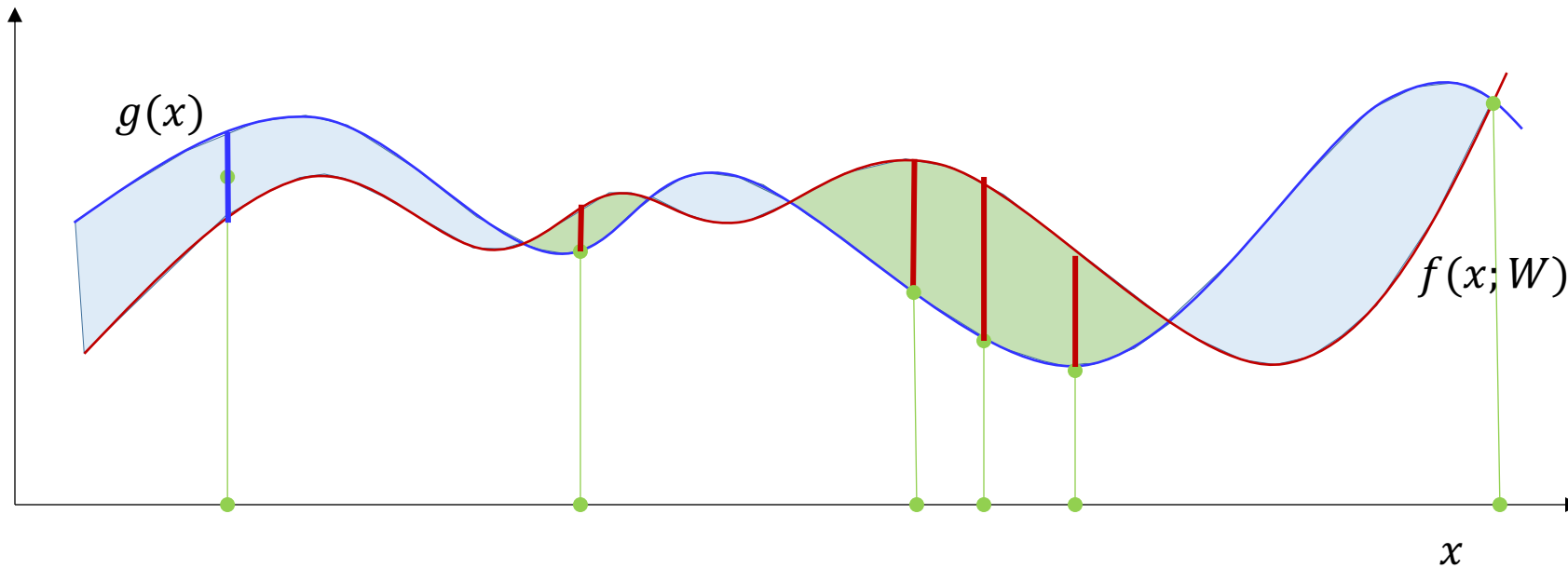
# Explaining the variance



- Sample estimate approximates the shaded area with the average length of the lines
- This average length will change with position of the samples

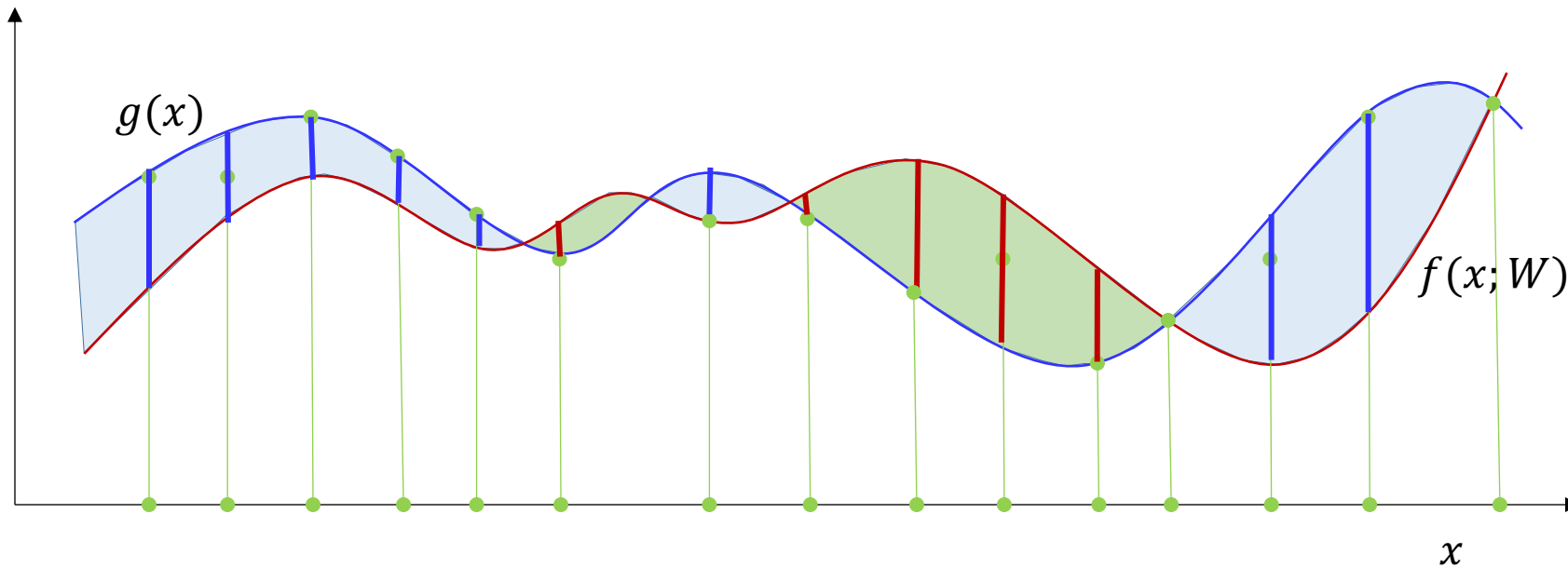


# Explaining the variance



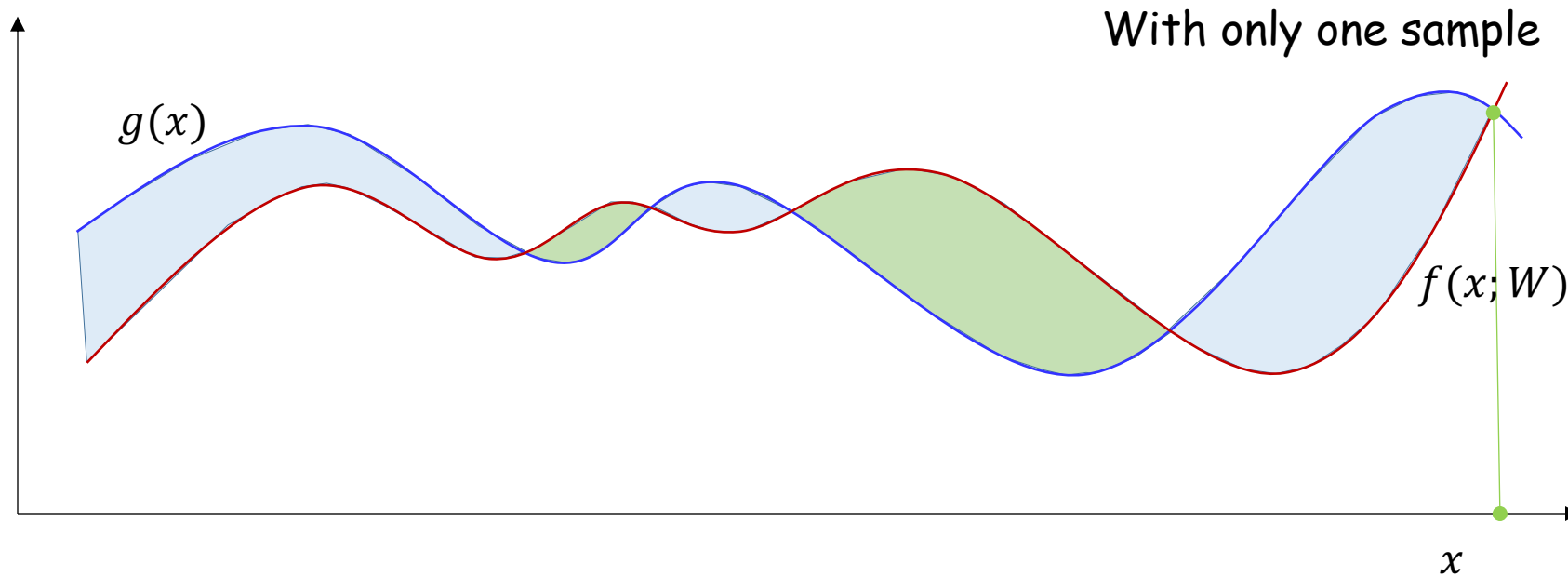
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- This average length will change with position of the samples

# Explaining the variance



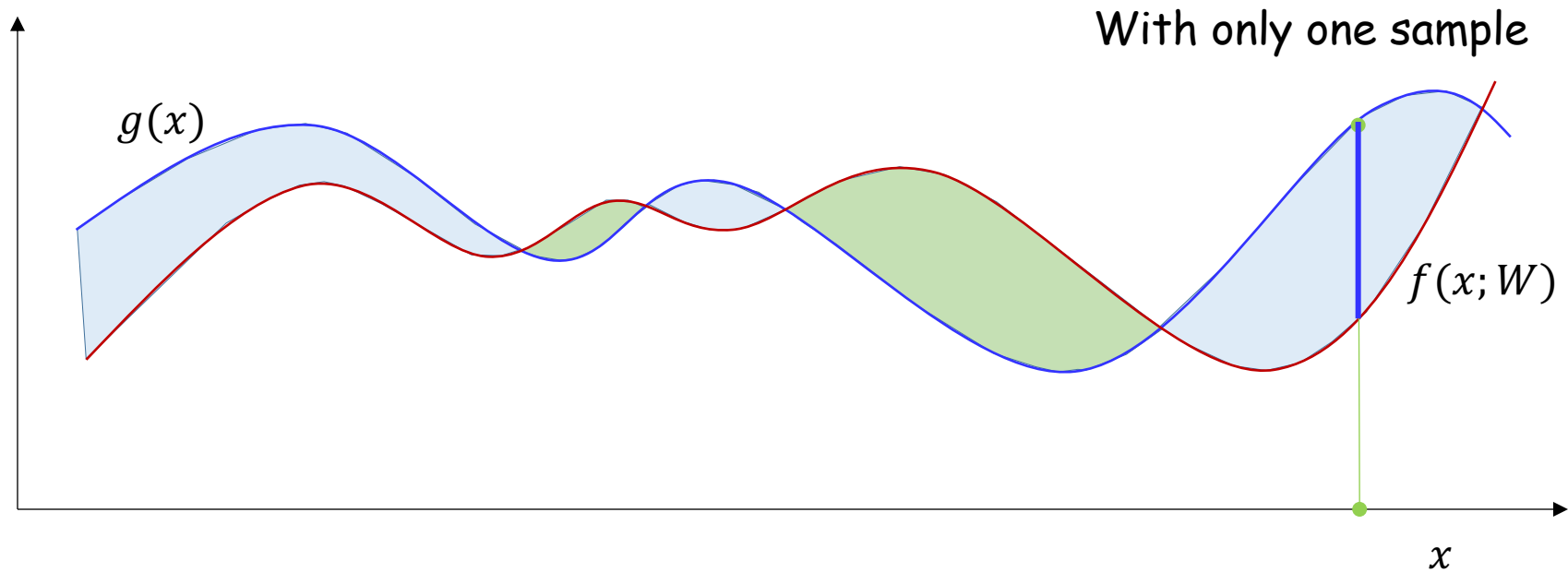
- Having more samples makes the estimate more robust to changes in the position of samples
  - The variance of the estimate is smaller

# Explaining the variance



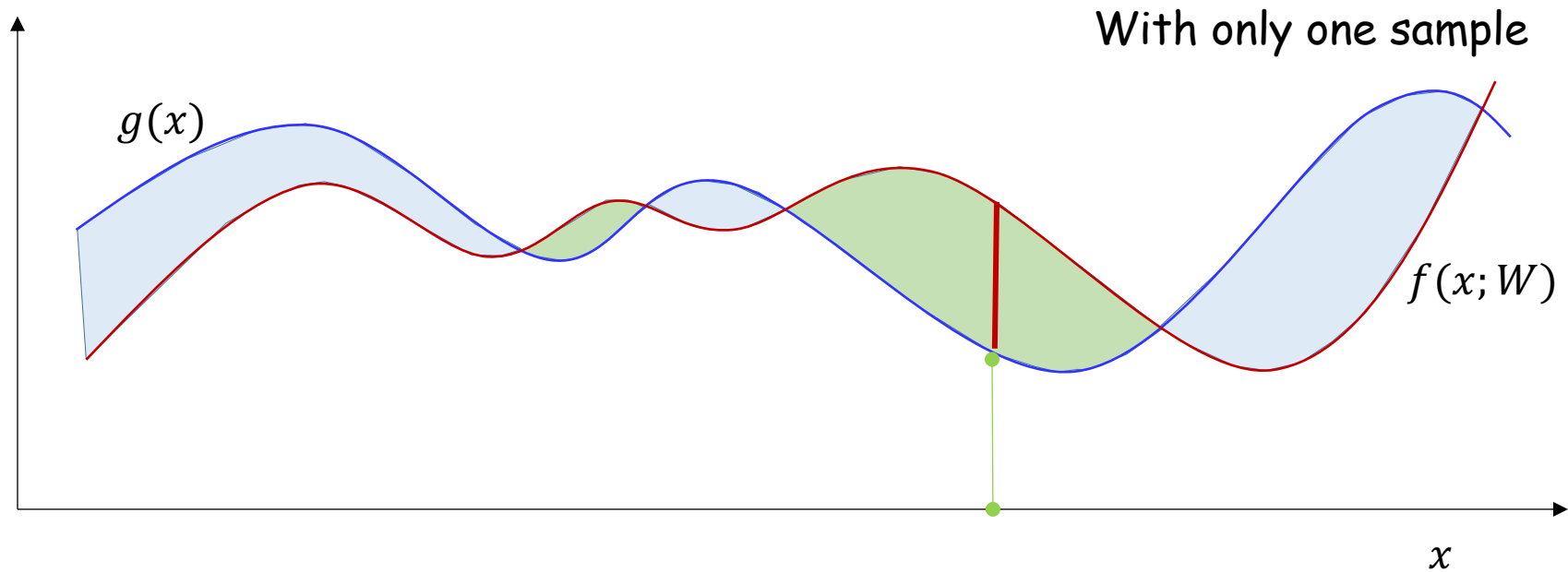
- Having very few samples makes the estimate swing wildly with the sample position
  - Since our estimator learns the  $W$  to minimize this estimate, the learned  $W$  too can swing wildly

# Explaining the variance



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# Explaining the variance

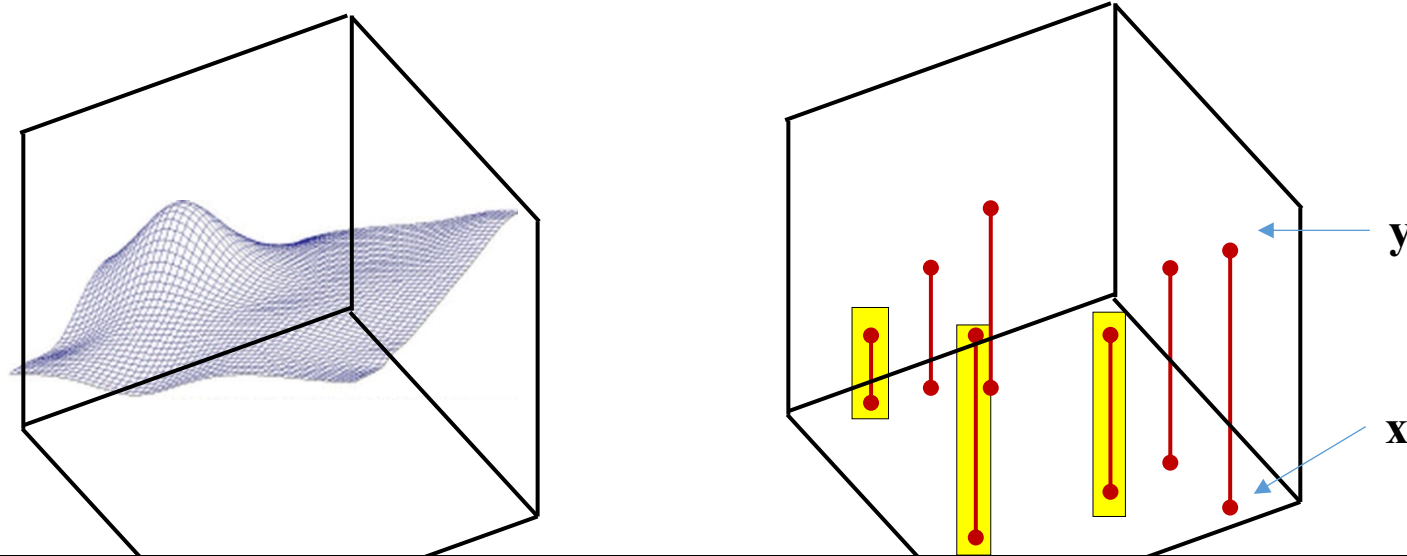


- Having very few samples makes the estimate swing wildly with the sample position
  - Since our estimator learns the  $W$  to minimize this estimate, the learned  $W$  too can swing wildly

# SGD vs batch

- SGD uses the gradient from only one sample at a time, and is consequently high variance
- But also provides significantly quicker updates than batch
- Is there a good medium?

# Mini Batches



The batch error is also an unbiased estimate of the expected error

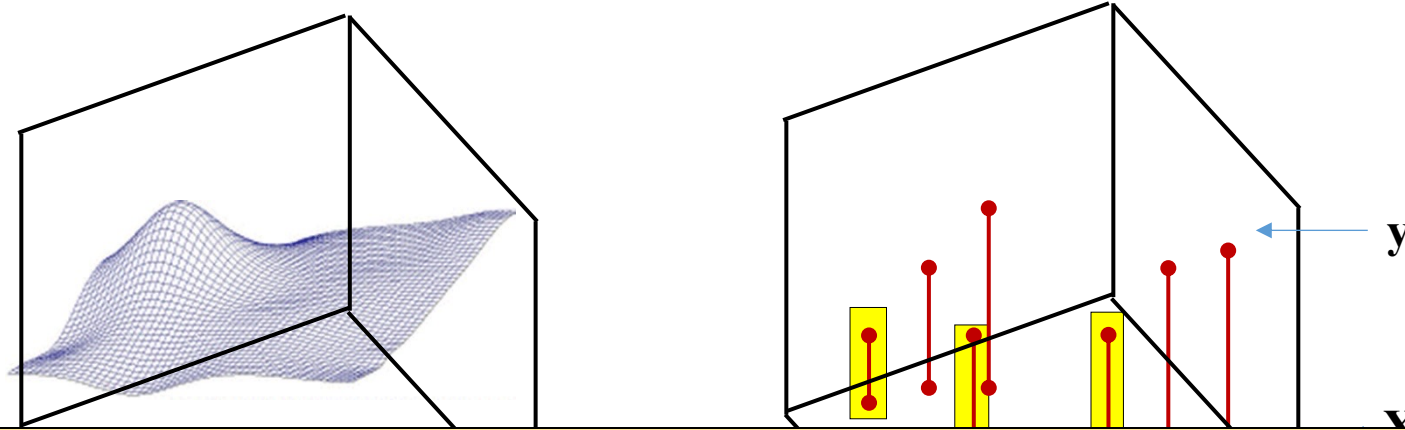
- Mini-batch updates compute and minimize a *batch error*

$$\text{BatchErr}(f(X; W), g(X)) = \frac{1}{b} \sum_{i=1}^b \text{loss}(f(x^{(i)}; W), y^{(i)})$$

- The *expected value* of the *batch error* is also the *expected divergence*

$$E[\text{BatchErr}(f(X; W), g(X))] = E[\text{loss}(f(X; W), g(X))]$$

# Mini Batches



The variance of the batch error:  $\text{var}(E) = 1/b \text{ var}(\text{loss})$

This will be much smaller than the variance of the sample error in SGD

The batch error is also an unbiased estimate of the expected error

- Mini-batch updates compute and minimize a *batch error*

$$\text{BatchErr}(f(X; W), g(X)) = \frac{1}{b} \sum_{i=1}^b \text{loss}(f(x^{(i)}; W), y^{(i)})$$

- The *expected value* of the *batch error* is also the *expected loss*

$$E[\text{BatchErr}(f(X; W), g(X))] = E[\text{loss}(f(X; W), g(X))]$$



# Mini-batch gradient descent

- Large datasets
  - Divide dataset into smaller batches containing one subset of the main training set
  - Weights are updated after seeing training data in each of these batches
- Vectorization provides efficiency

# Gradient descent methods

Stochastic gradient



Batch size=1

Stochastic mini-batch gradient

e.g., Batch size= 32, 64, 128, 256

Batch gradient



Batch size=n  
(the size of training set)

n: whole no of training data

bs: the size of batches

$m = \left\lceil \frac{n}{b} \right\rceil$ : the number of batches

Batch 1 $X^{\{1\}}, Y^{\{1\}}$	Batch 2 $X^{\{2\}}, Y^{\{2\}}$								Batch m $X^{\{m\}}, Y^{\{m\}}$
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# Mini-batch gradient descent

For epoch=1,...,k

shuffle training data

For t=1,...,m

Forward propagation on  $X^{\{t\}}$

$$J^{\{t\}} = \frac{1}{m} \sum_{n \in \text{Batch}_t} L(\hat{Y}_n^{\{t\}}, Y_n^{\{t\}}) + \lambda R(W)$$

Backpropagation on  $J^{\{t\}}$  to compute gradients  $dW$

For  $l = 1, \dots, L$

$$W^{[l]} = W^{[l]} - \alpha dW^{[l]}$$

$$A^{[0]} = X^{\{t\}}$$

For  $l = 1, \dots, L$

$$Z^{[l]} = W^{[l]} A^{[l-1]}$$

$$A^{[l]} = f^{[l]}(Z^{[l]})$$

$$\hat{Y}_n^{\{t\}} = A_n^{[L]}$$

Vectorized computation

1 **epoch**:  
Single pass  
over all  
training  
samples

Batch 1 $X^{\{1\}}, Y^{\{1\}}$	Batch 2 $X^{\{2\}}, Y^{\{2\}}$								Batch m $X^{\{m\}}, Y^{\{m\}}$
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# Gradient descent methods

Stochastic gradient descent

Batch size=1

- Does not use vectorized form and thus not computationally efficient

Stochastic mini-batch gradient

e.g., Batch size= 32, 64, 128, 256

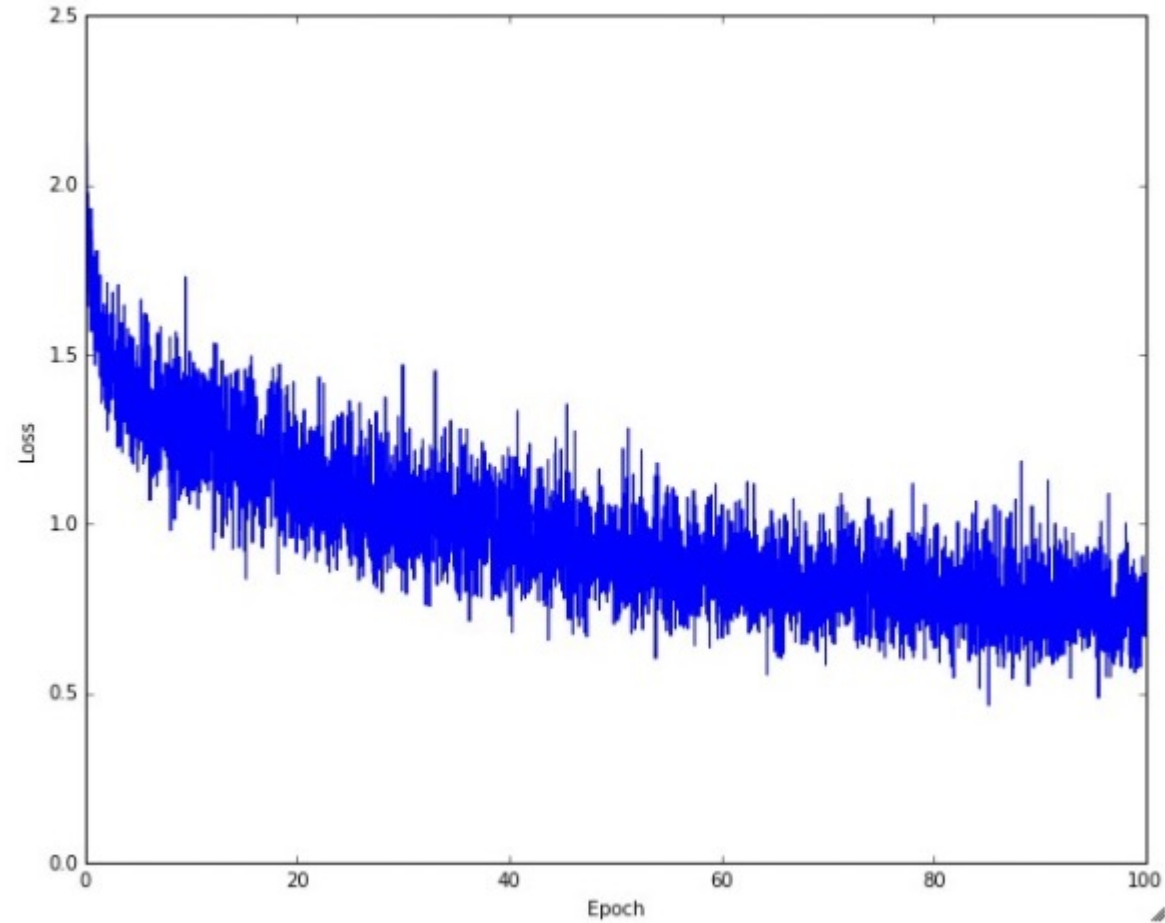
- Vectorization
- Fastest learning (for proper batch size)

Batch gradient descent

Batch size=n  
(the size of training set)

- Need to process whole training set for weight update

# Mini-batch gradient descent: loss-#epoch curve

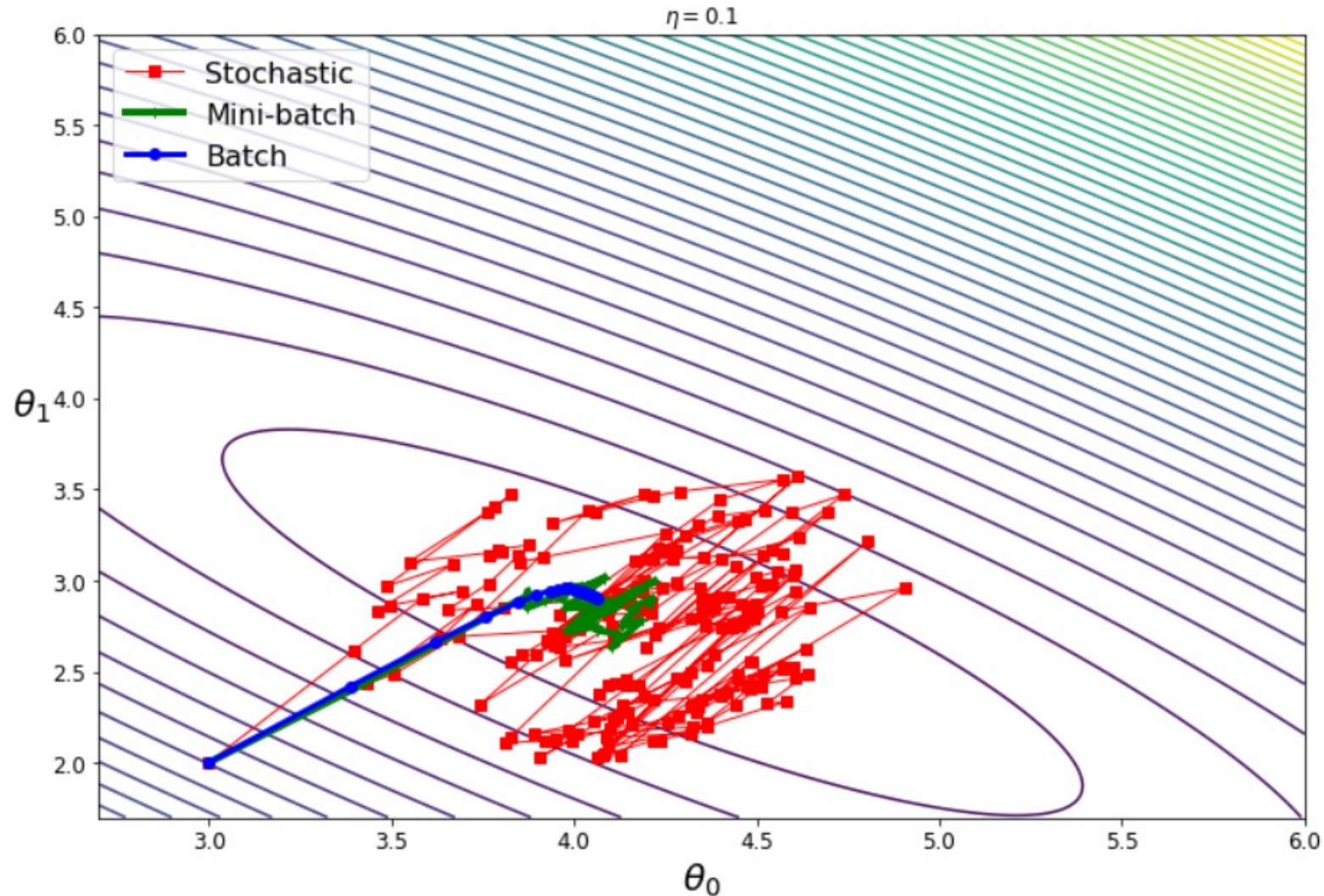


# Measuring error

- Convergence is generally defined in terms of the *overall training* error
  - Not sample or batch error
- Infeasible to actually measure the overall training error after each iteration
- More typically, we estimate is as
  - Average sample/batch error over the past  $N$  samples/batches

# Batch size

- Full batch (batch size =  $N$ )
- SGB (batch size = 1)
- SGD (batch size = 10)



# Choosing mini-batch size

- For small training sets (e.g.,  $n < 2000$ ) you can use full-batch gradient descent
- Typical mini-batch sizes for larger training sets:
  - 64, 128, 256, 512, 1024
- Make sure one batch of training data and the corresponding forward, backward required to be cached can fit in GPU memory



# Story so far

- Gradient descent can be sped up by incremental updates
  - Convergence is guaranteed under most conditions
    - Learning rate must shrink with time for convergence
  - Stochastic gradient descent: update after each observation. Can be much faster than batch learning
  - Mini-batch updates: update after batches. Can be more efficient than SGD
- Convergence can be improved using smoothed updates
  - RMSprop and Adam

# Story so far

- SGD: Presenting training instances one-at-a-time can be more effective than full-batch training
  - Provided they are provided in random order
- For SGD to converge, the learning rate must shrink sufficiently rapidly with iterations
  - Otherwise the learning will continuously “chase” the latest sample
- SGD estimates have higher variance than batch estimates
- Minibatch updates operate on *batches* of instances at a time
  - Estimates have lower variance than SGD
  - Convergence rate is theoretically worse than SGD
  - But we compensate by being able to perform batch processing

# Story so far: Training and minibatches

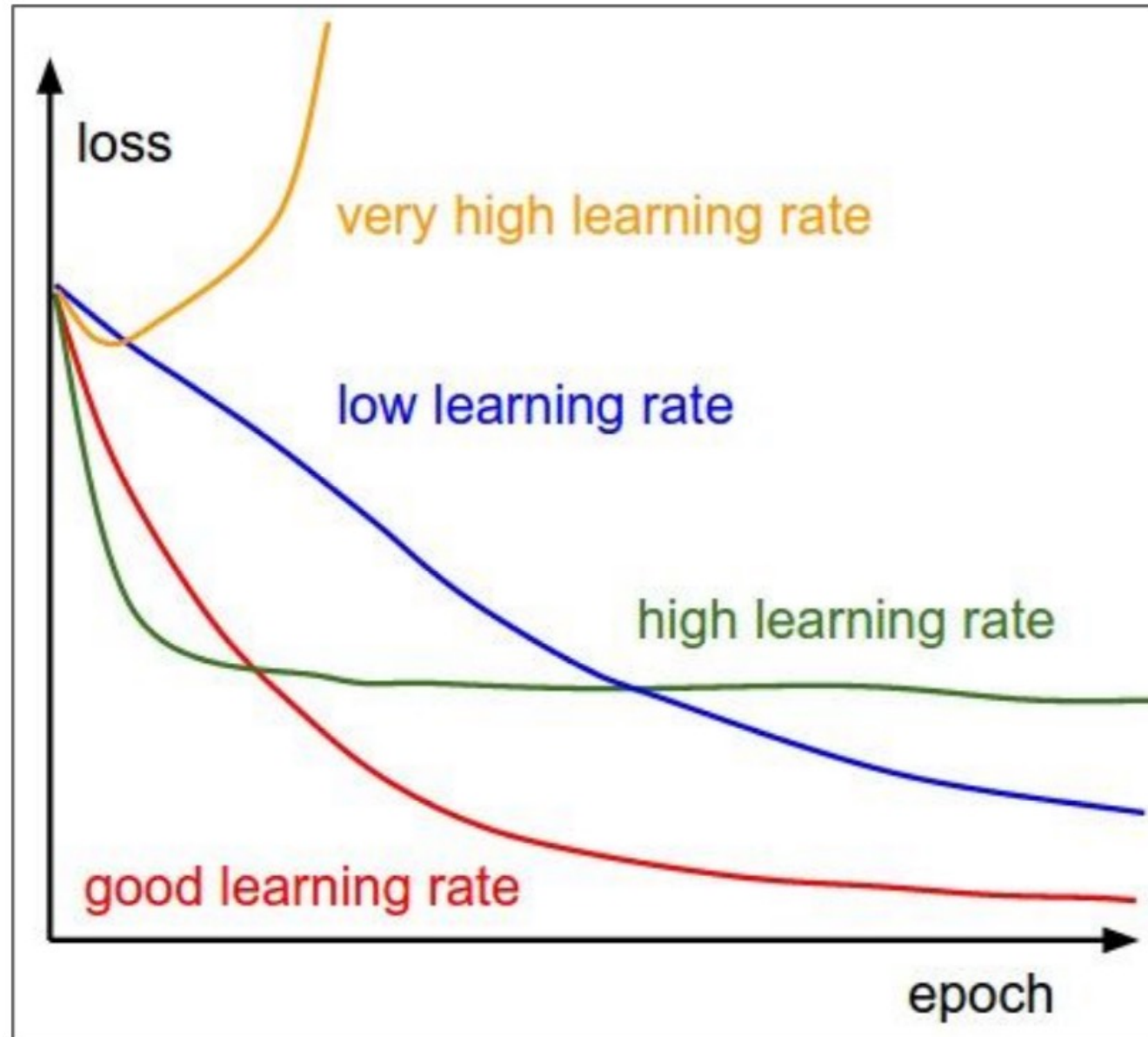
- Convergence depends on learning rate
  - Simple technique: fix learning rate until the error plateaus, then reduce learning rate by a fixed factor (e.g. 10)

Some practical issues about  
learning rate

# Learning rate

- SGD, SGD+Momentum, Adagrad, RMSProp, Adam all have learning rate as a hyperparameter.
- Learning rate is an important hyperparameter that usually adjust first

Which one of these learning rates is best to use?



# Learning rate

- Start with small regularization and find learning rate that makes the loss go down.
- loss not going down: learning rate too low

```
model = init_two_layer_model(32*32*3, 50, 10) # input size, hidden size, number of classes
trainer = ClassifierTrainer()
best_model, stats = trainer.train(X_train, y_train, X_val, y_val,
                                  model, two_layer_net,
                                  num_epochs=10, reg=0.000001,
                                  update='sgd', learning_rate_decay=1,
                                  sample_batches = True,
                                  learning_rate=1e-6, verbose=True)
```

```
Finished epoch 1 / 10: cost 2.302576, train: 0.080000, val 0.103000, lr 1.000000e-06
Finished epoch 2 / 10: cost 2.302582, train: 0.121000, val 0.124000, lr 1.000000e-06
Finished epoch 3 / 10: cost 2.302558, train: 0.119000, val 0.138000, lr 1.000000e-06
Finished epoch 4 / 10: cost 2.302519, train: 0.127000, val 0.151000, lr 1.000000e-06
Finished epoch 5 / 10: cost 2.302517, train: 0.158000, val 0.171000, lr 1.000000e-06
Finished epoch 6 / 10: cost 2.302518, train: 0.179000, val 0.172000, lr 1.000000e-06
Finished epoch 7 / 10: cost 2.302466, train: 0.180000, val 0.176000, lr 1.000000e-06
Finished epoch 8 / 10: cost 2.302452, train: 0.175000, val 0.185000, lr 1.000000e-06
Finished epoch 9 / 10: cost 2.302459, train: 0.206000, val 0.192000, lr 1.000000e-06
Finished epoch 10 / 10: cost 2.302420, train: 0.190000, val 0.192000, lr 1.000000e-06
finished optimization. best validation accuracy: 0.192000
```

Loss barely changing: Learning rate is probably too low

Notice train/val accuracy goes to 20% though, what's up with that? (remember this is softmax)

# Choosing learning rate parameter

- loss not going down: learning rate too low

```
Finished epoch 1 / 10: cost 2.302576, train: 0.080000, val 0.103000,  
Finished epoch 2 / 10: cost 2.302582, train: 0.121000, val 0.124000,  
Finished epoch 3 / 10: cost 2.302558, train: 0.119000, val 0.138000,  
Finished epoch 4 / 10: cost 2.302519, train: 0.127000, val 0.151000,  
Finished epoch 5 / 10: cost 2.302517, train: 0.158000, val 0.171000,  
Finished epoch 6 / 10: cost 2.302518, train: 0.179000, val 0.172000,  
Finished epoch 7 / 10: cost 2.302466, train: 0.180000, val 0.176000,  
Finished epoch 8 / 10: cost 2.302452, train: 0.175000, val 0.185000,  
Finished epoch 9 / 10: cost 2.302459, train: 0.206000, val 0.192000,  
Finished epoch 10 / 10: cost 2.302420, train: 0.190000, val 0.192000,
```

- loss exploding: learning rate too high

```
Finished epoch 1 / 10: cost nan, train: 0.091000, val 0.087000, lr 1.000000e+06  
Finished epoch 2 / 10: cost nan, train: 0.095000, val 0.087000, lr 1.000000e+06  
Finished epoch 3 / 10: cost nan, train: 0.100000, val 0.087000, lr 1.000000e+06
```

cost: NaN almost always means high learning rate...

- Rough range for learning rate we should be cross-validating is somewhere  $[1e-3 \dots 1e-5]$



# Learning rate

- Start with small regularization and find learning rate that makes the loss go down.
- **loss not going down: learning rate too low**
- **loss exploding: learning rate too high**

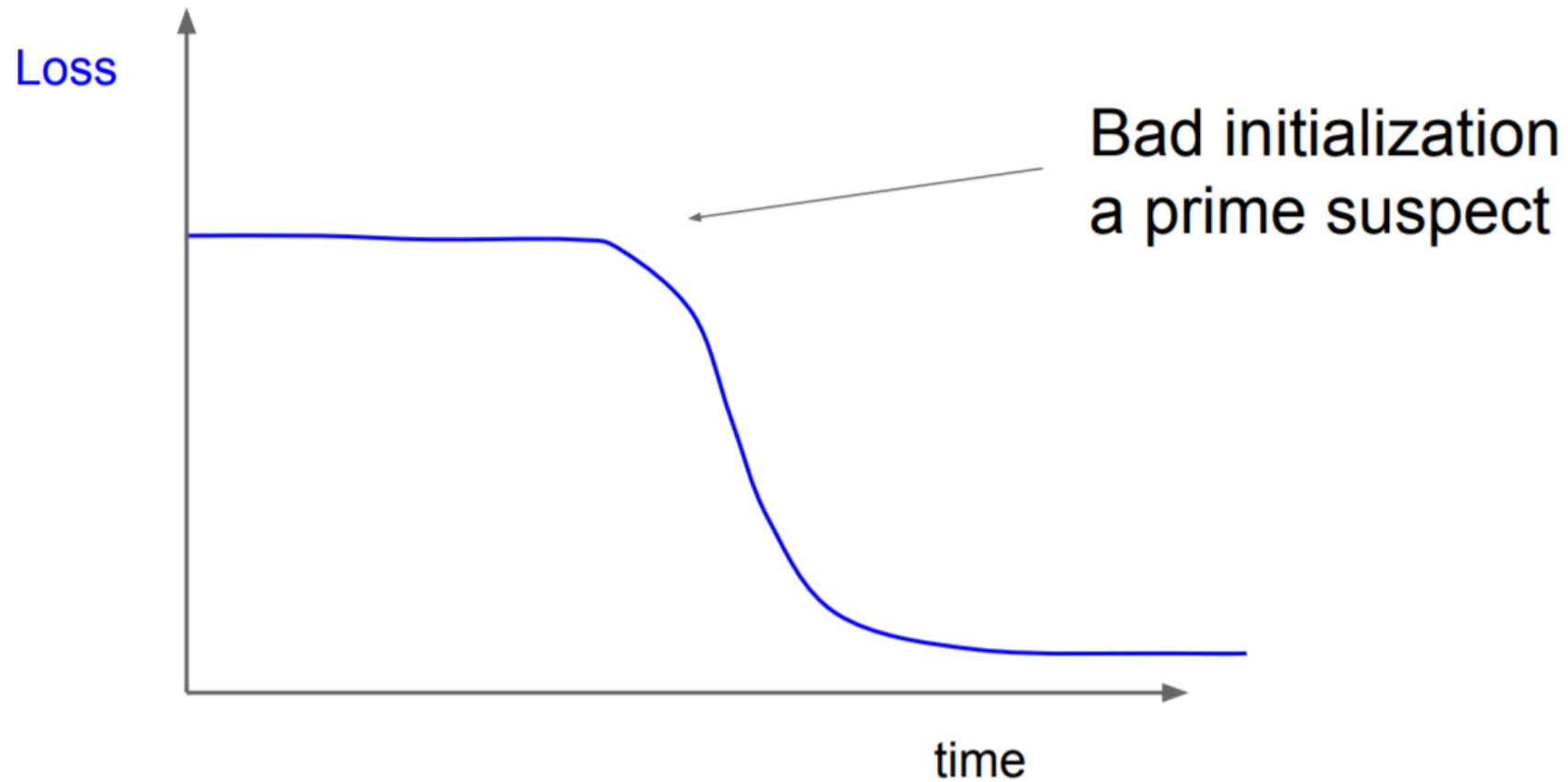
```
model = init_two_layer_model(32*32*3, 50, 10) # input size, hidden size, number of classes
trainer = ClassifierTrainer()
best_model, stats = trainer.train(X_train, y_train, X_val, y_val,
                                  model, two_layer_net,
                                  num_epochs=10, reg=0.000001,
                                  update='sgd', learning_rate_decay=1,
                                  sample_batches = True,
                                  learning_rate=3e-3, verbose=True)
```

```
Finished epoch 1 / 10: cost 2.186654, train: 0.308000, val 0.306000, lr 3.000000e-03
Finished epoch 2 / 10: cost 2.176230, train: 0.330000, val 0.350000, lr 3.000000e-03
Finished epoch 3 / 10: cost 1.942257, train: 0.376000, val 0.352000, lr 3.000000e-03
Finished epoch 4 / 10: cost 1.827868, train: 0.329000, val 0.310000, lr 3.000000e-03
Finished epoch 5 / 10: cost inf, train: 0.128000, val 0.128000, lr 3.000000e-03
Finished epoch 6 / 10: cost inf, train: 0.144000, val 0.147000, lr 3.000000e-03
```

3e-3 is still too high.

Cost explodes....=> Rough range for learning rate we should be cross-validating is somewhere [1e-3 ... 1e-5]

# Monitoring loss function during iterations



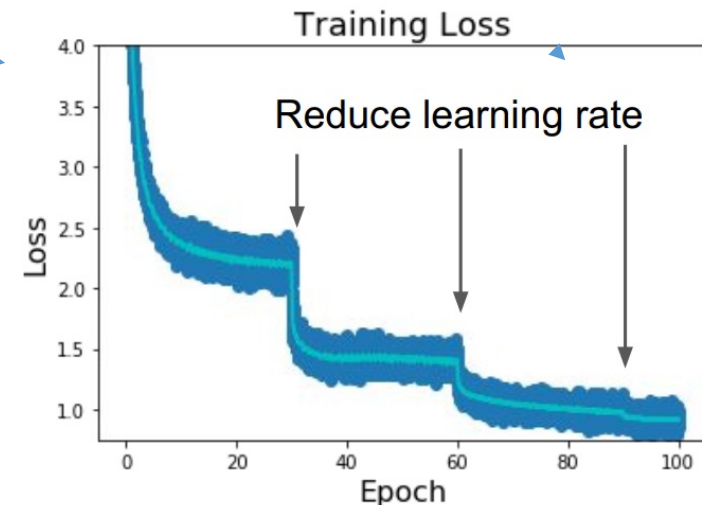
# Learning rate decay

- Maybe during the initial steps of learning, you could afford to take much bigger steps
- But then as learning approaches convergence, then having a slower learning rate allows you to take smaller steps

# Learning rate scheduling

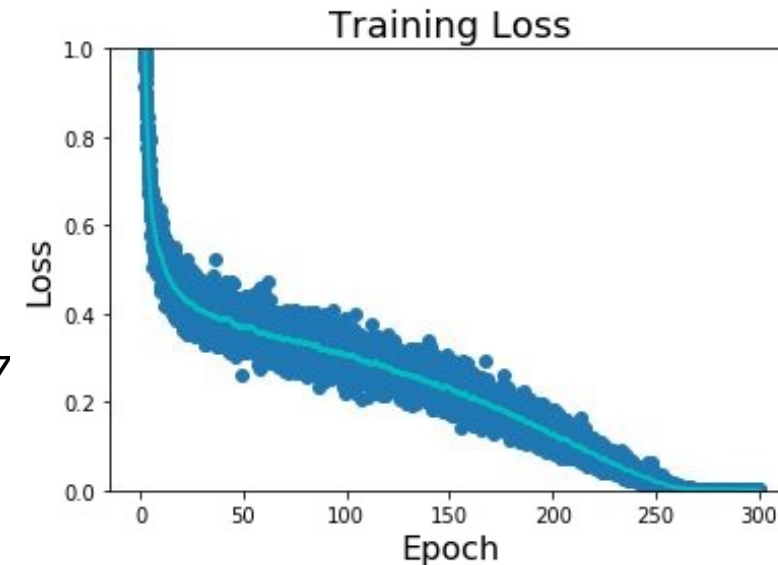
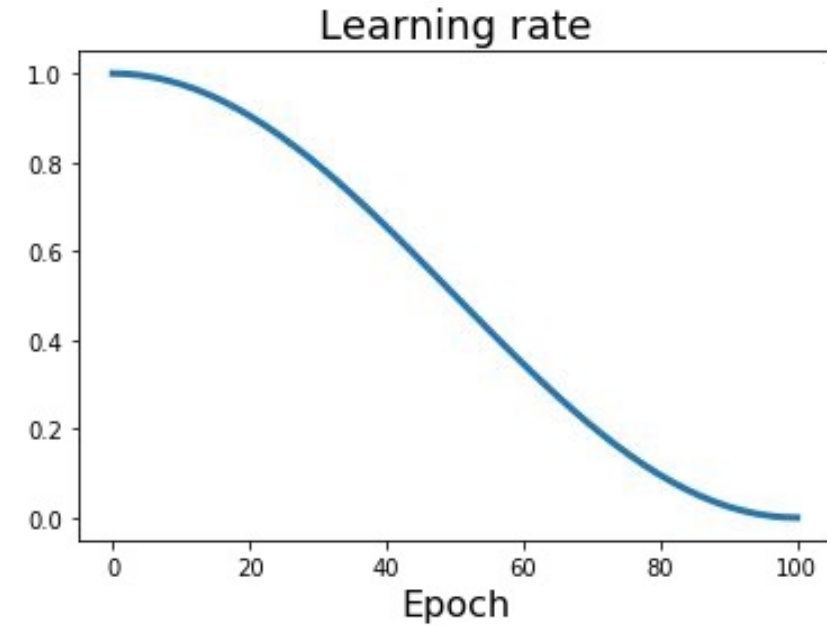
- Need for learning rate schedules
  - Benefits
    - Converge Faster
    - Higher accuracy
- Top Basic Learning Rate Schedules
  - Step-wise decay
  - Reduce on loss plateau decay
  - Cosine decay (Loshchilov & Hutter, 2017)
  - trapezoidal schedule (Xing et al., 2018)
  - ...

**Step-wise:** Reduce learning rate at a few fixed points. e.g. for ResNets, multiply LR by 0.1 after epochs 30, 60, and 90



# Learning rate decay

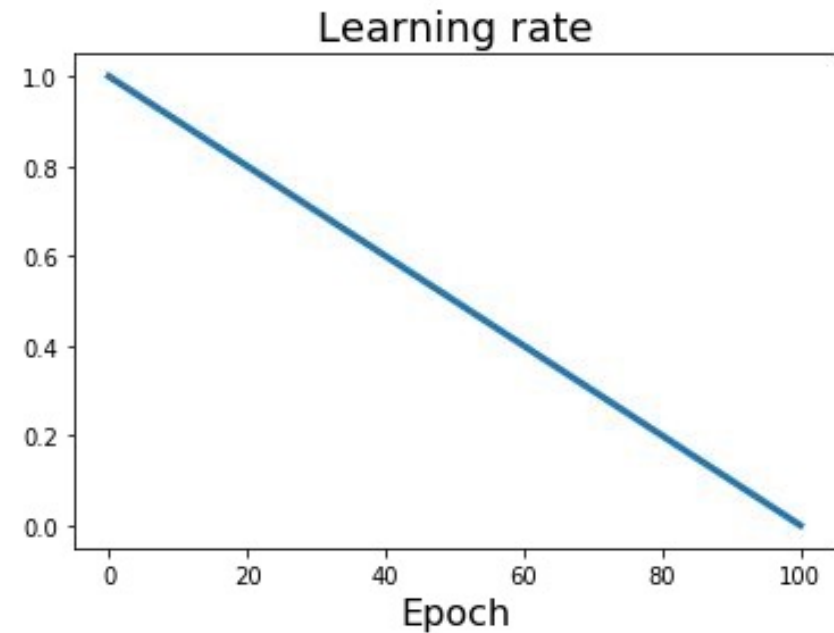
- **Cosine:**  $\alpha_t = \frac{1}{2} \alpha_0 \left( 1 + \cos \left( \frac{t\pi}{T} \right) \right)$ 
  - $\alpha_0$ : Initial learning rate
  - $\alpha_t$ : Learning rate at epoch  $t$
  - $T$ : Total number of epochs



Loshchilov and Hutter, "SGDR: Stochastic Gradient Descent with Warm Restarts", ICLR 2017  
Radford et al, "Improving Language Understanding by Generative Pre-Training", 2018  
Feichtenhofer et al, "SlowFast Networks for Video Recognition", arXiv 2018  
Child et al, "Generating Long Sequences with Sparse Transformers", arXiv 2019

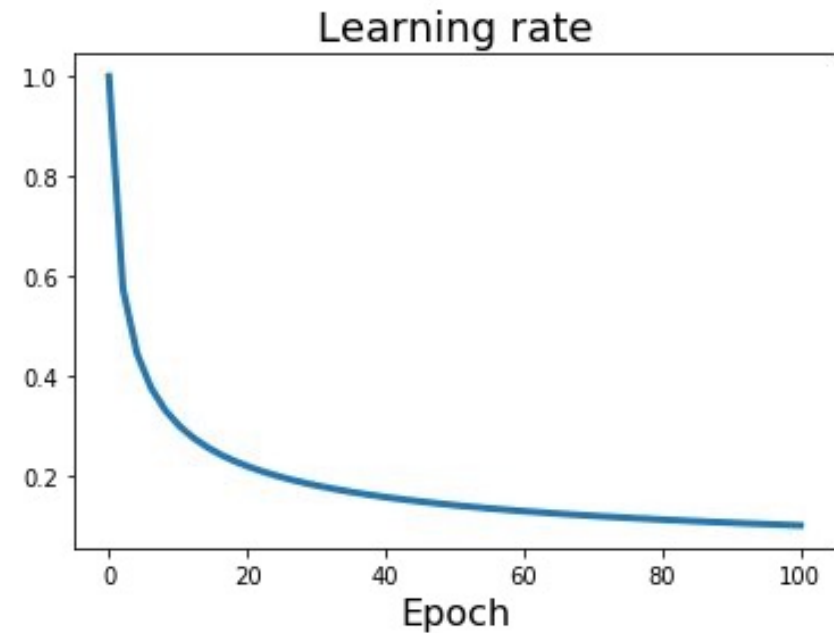
# Learning rate decay

- **Linear:**  $\alpha_t = \alpha_0 \left(1 - \frac{t}{T}\right)$ 
  - $\alpha_0$ : Initial learning rate
  - $\alpha_t$ : Learning rate at epoch  $t$
  - $T$ : Total number of epochs



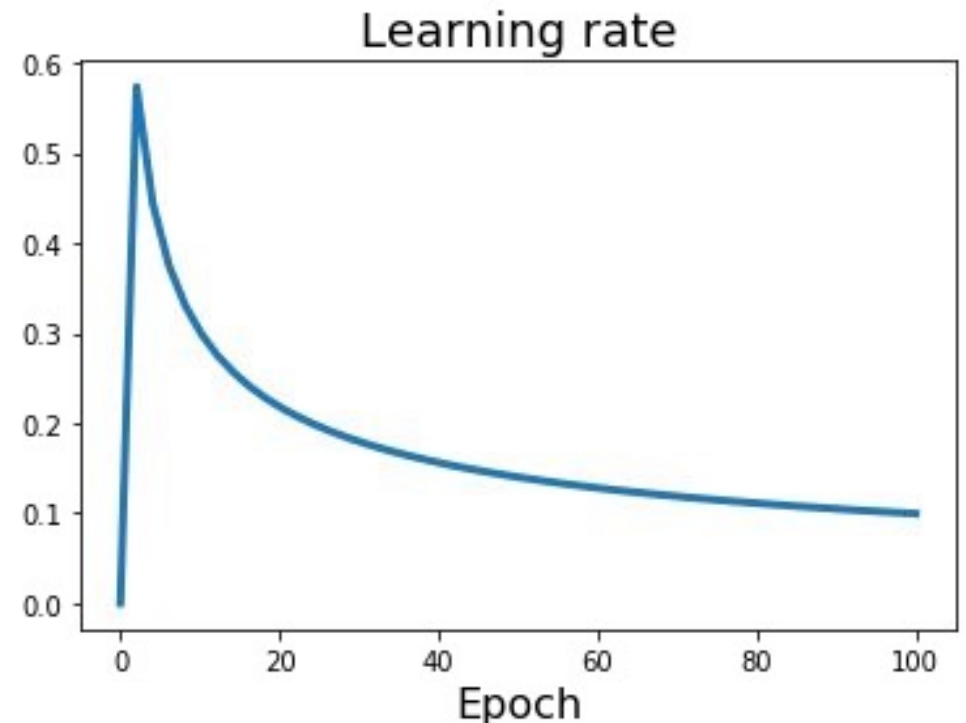
# Learning Rate Decay

- **Inverse sqrt:**  $\alpha_t = \frac{\alpha_0}{\sqrt{t}}$ 
  - $\alpha_0$ : Initial learning rate
  - $\alpha_t$ : Learning rate at epoch  $t$
  - $T$ : Total number of epochs



# Learning rate decay: Linear warmup

- High initial learning rates can make loss explode; linearly increasing learning rate from 0 over the first  $\sim 5,000$  iterations can prevent this.
- Empirical rule of thumb: If you increase the batch size by  $N$ , also scale the initial learning rate by  $N$





# Summary

- Stochastic Gradient Descent (SGD)
- Mini-batch update
- Adjusting learning rate