Cache-Oblivious B-Trees

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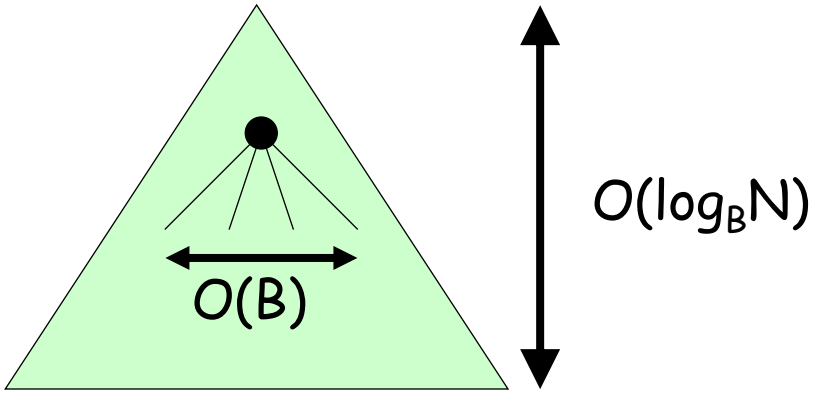
Presented by: Itai Lahan

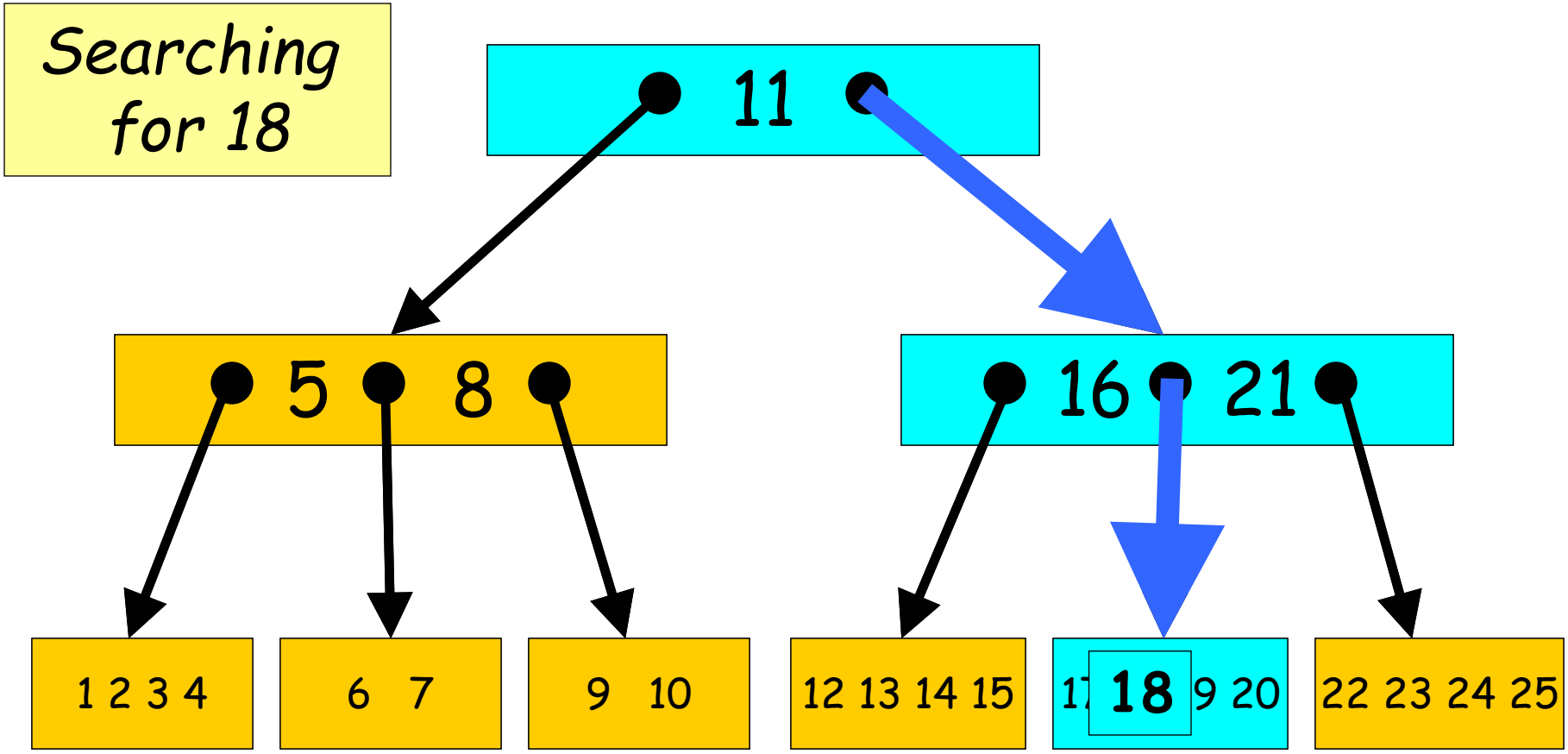
B-Trees – A reminder

[R. Bayer and E. M. McCreight. 1972]

- Balanced search tree
- All leaves have the same depth
- All nodes have degree at most B
  - The root has degree at least two
  - All internal nodes have degree at least B/2



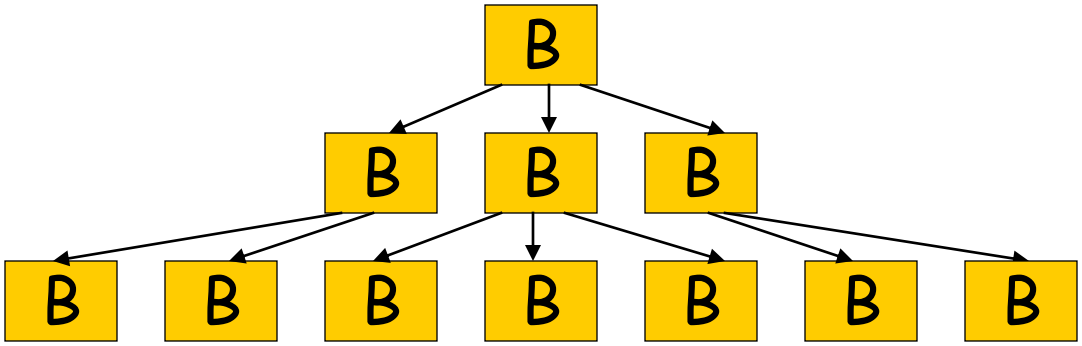
B-Trees - example



B-Trees

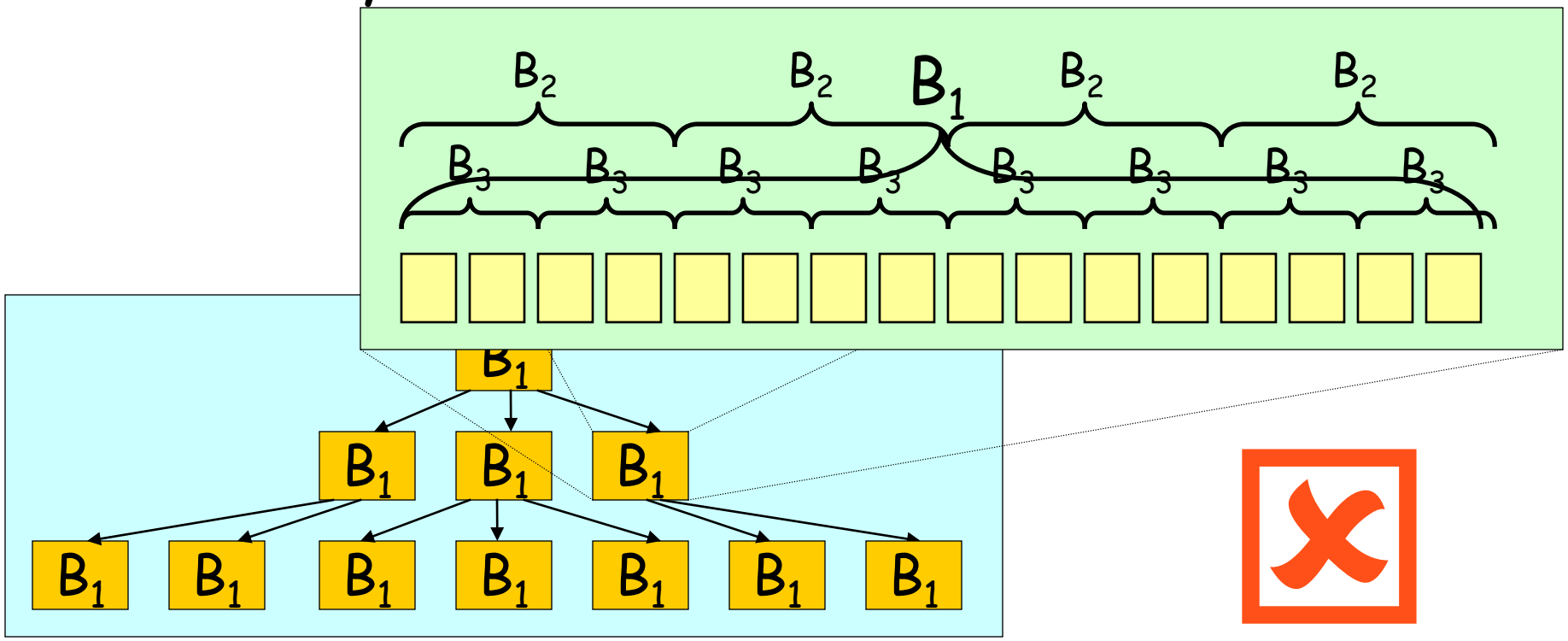
[R. Bayer and E. M. McCreight. 1972]

- Balanced search tree
- Fan-out is proportional to memory block size
- A single block read determines the next node
- Classical two-level memory model solution



B-Trees in multilevel memory hierarchy ?

- $B_1 > B_2 > B_3 > \ldots > B_k$ block sizes between memory levels



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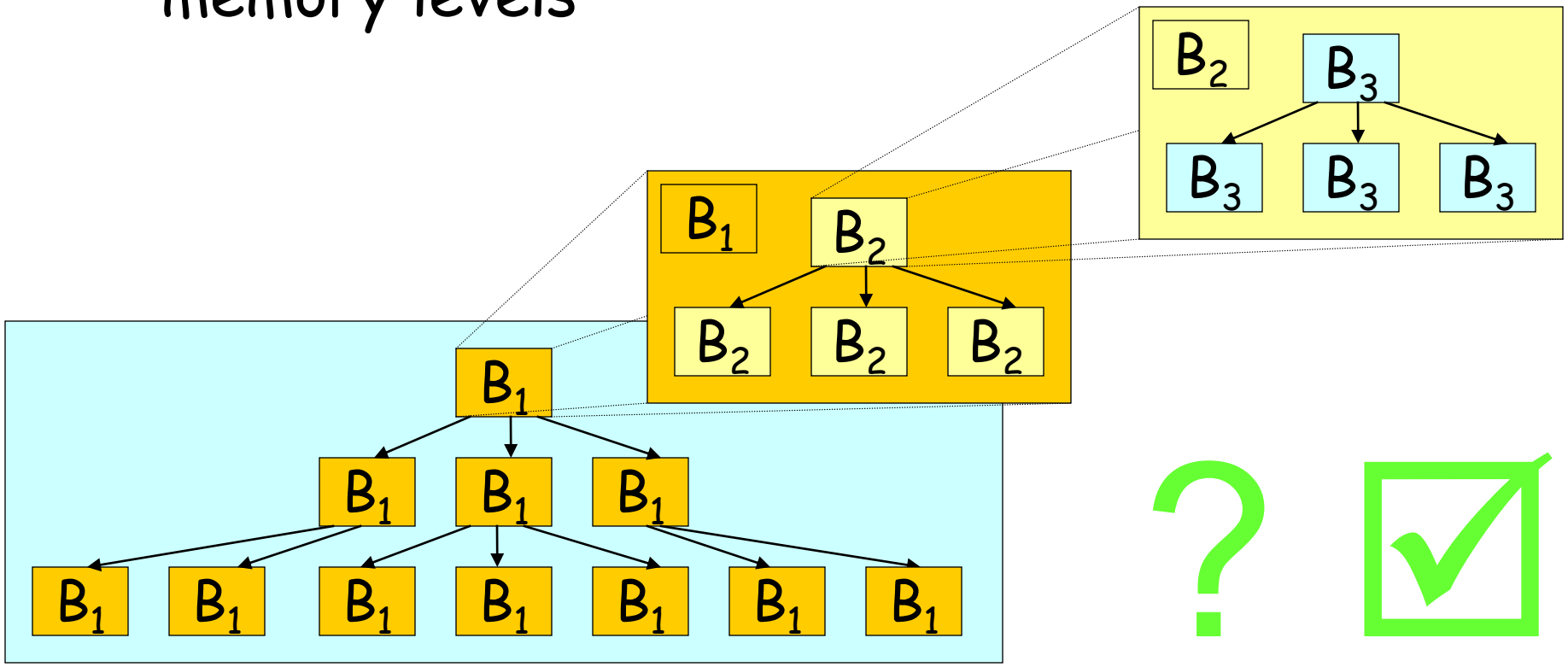
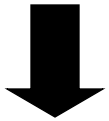
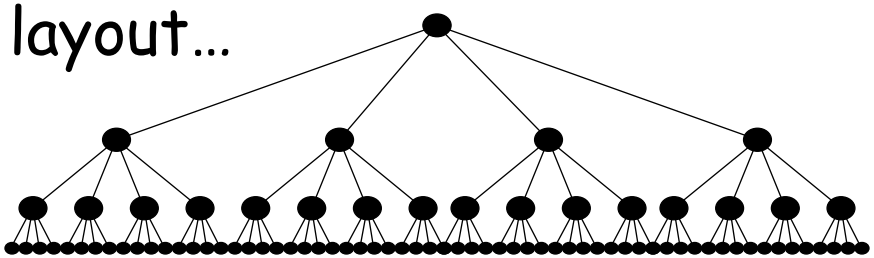


Cache Oblivious Algorithms

- No variable depends on hardware parameters
- Reason about a simple two level memory

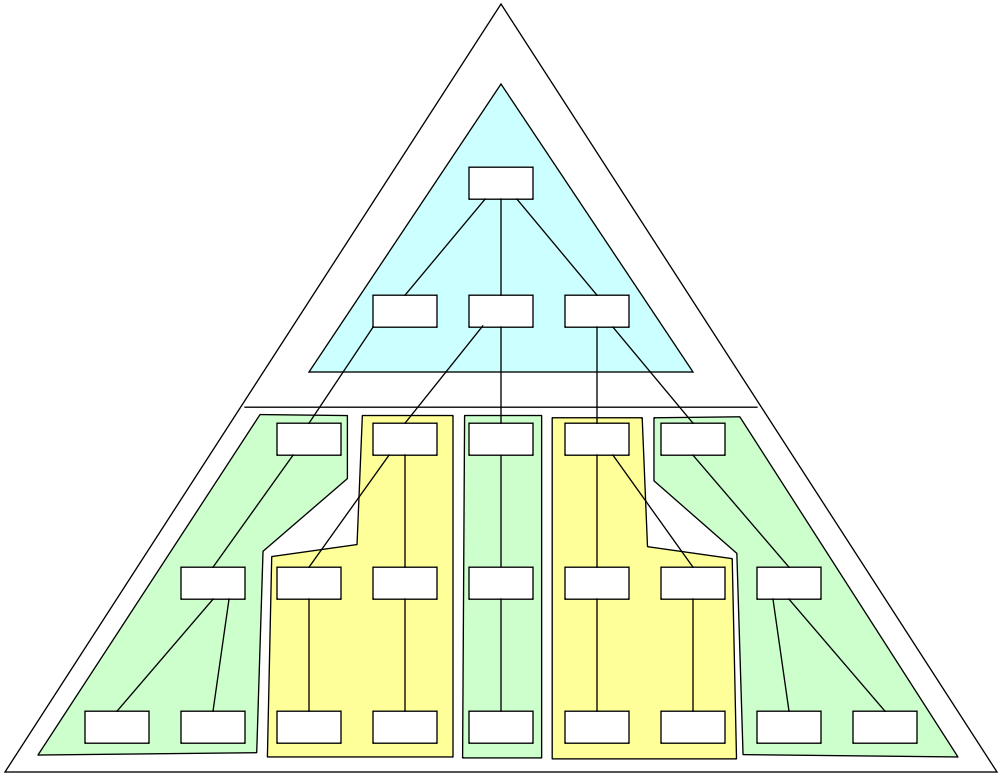


- Prove results about an unknown multilevel memory
- Cache oblivious B-Tree ?
  - How do we achieve $O(\log_B N)$ without knowing B ??
  - It's all about the memory layout…



Van Emde Boas Layout

[P. van Emde Boas, R. Kaas, and E. Zijlstra. 1977]



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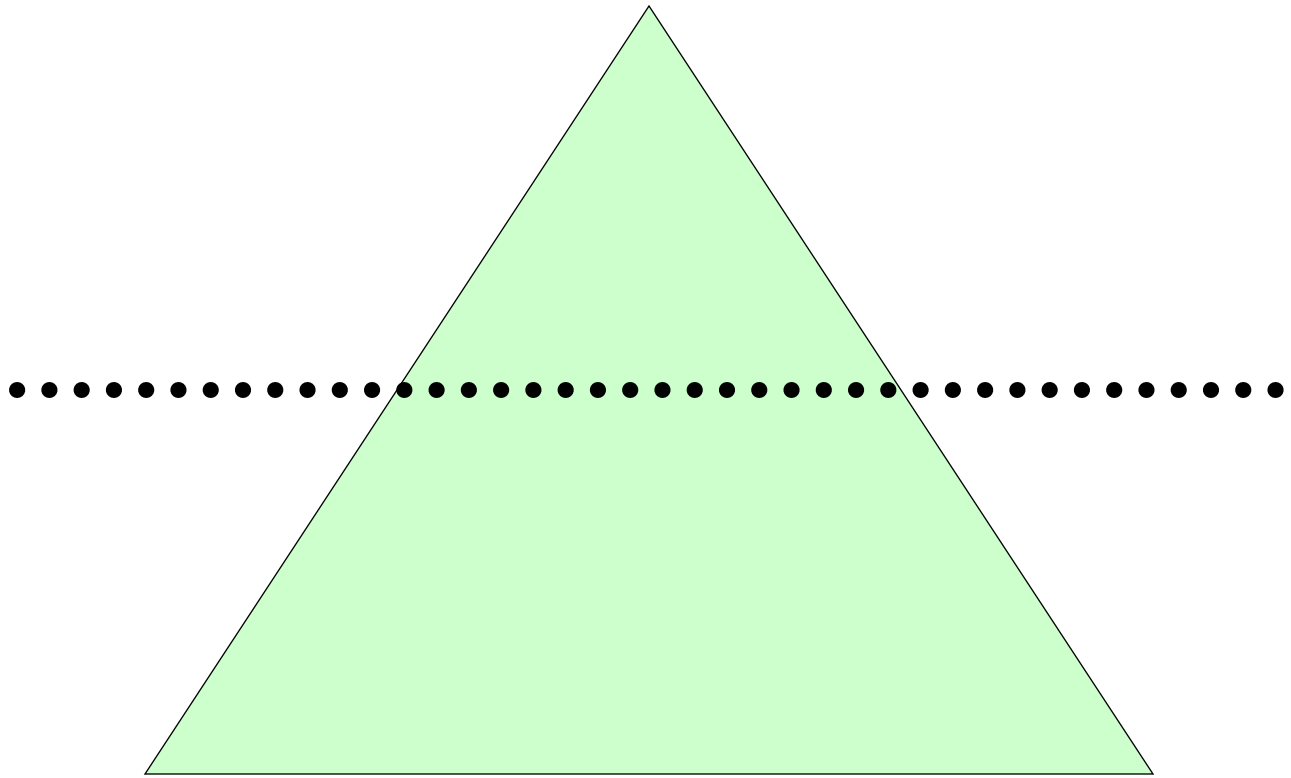
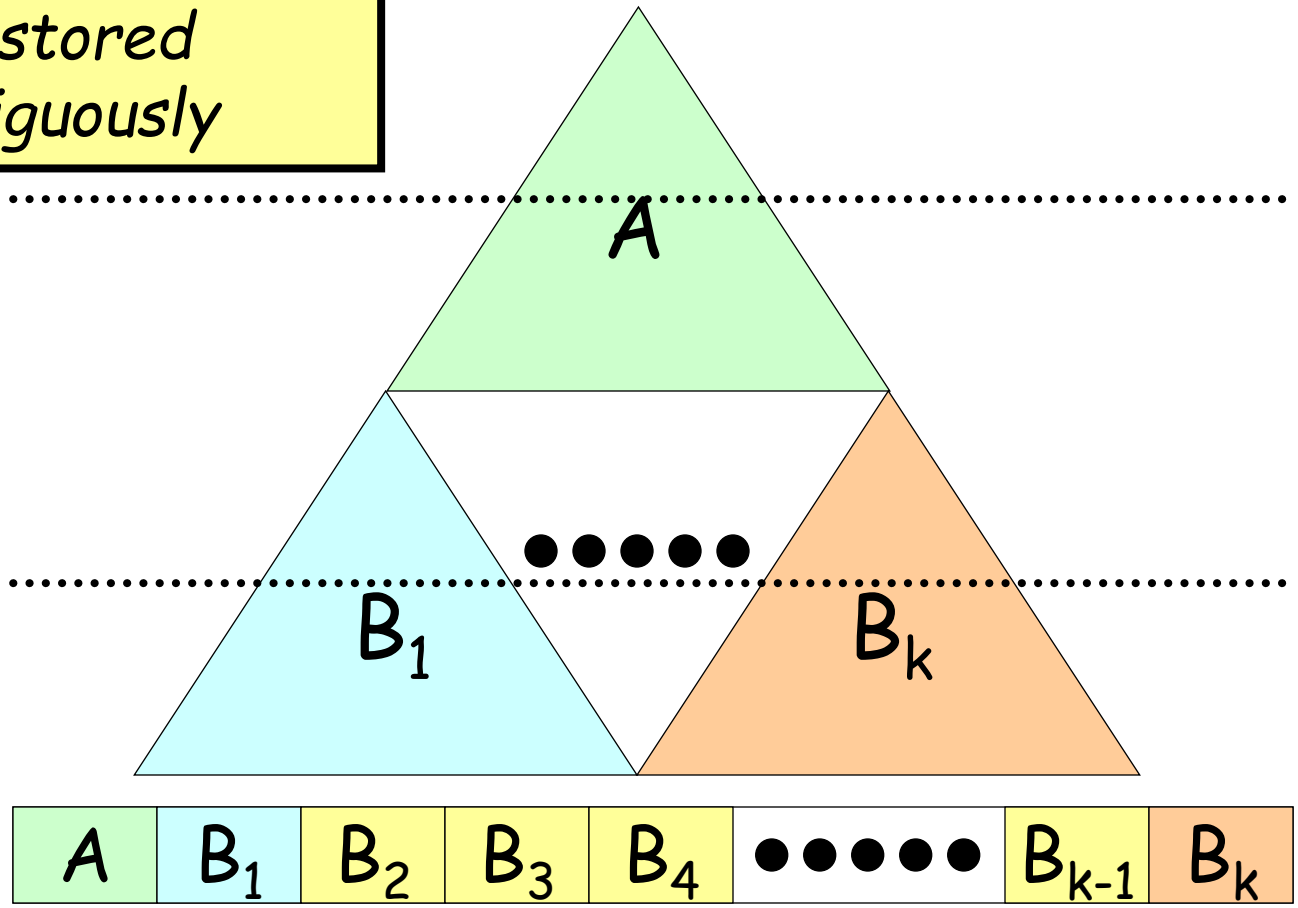


Van Emde Boas Layout

Recursive structures are stored contiguously

[P. van Emde Boas, R. Kaas, and E. Zijlstra. 1977]



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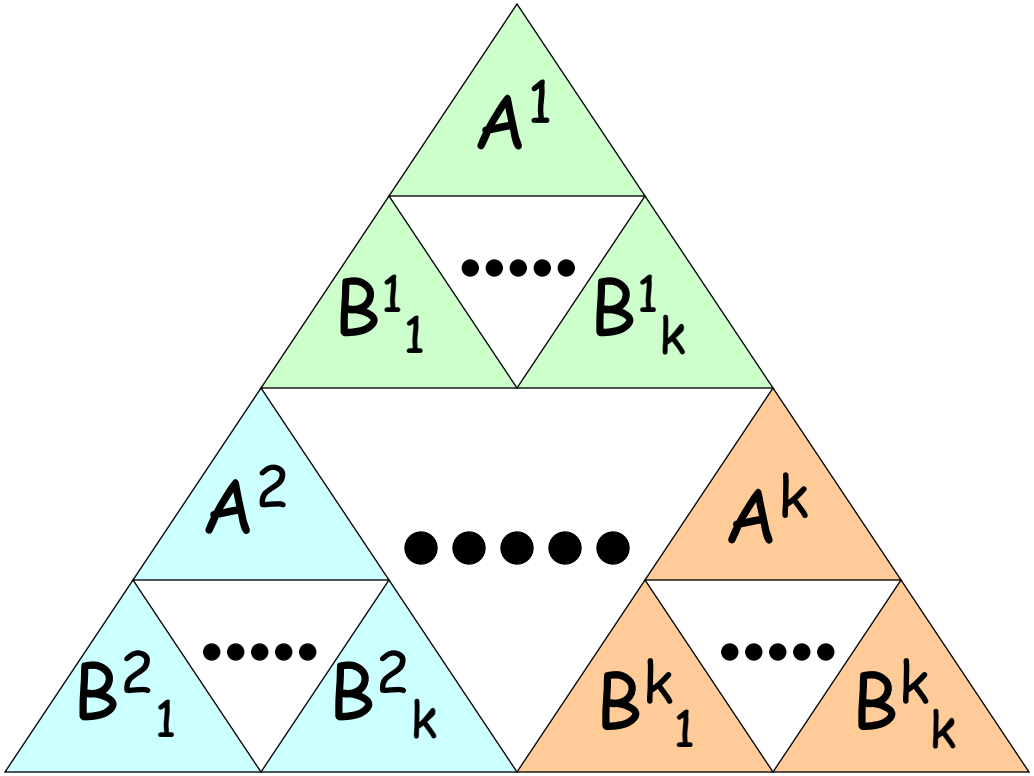

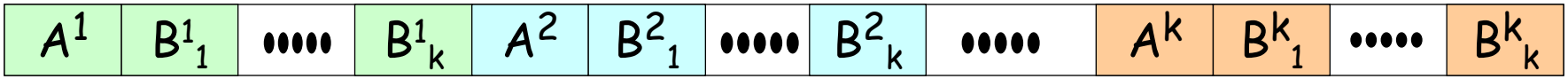

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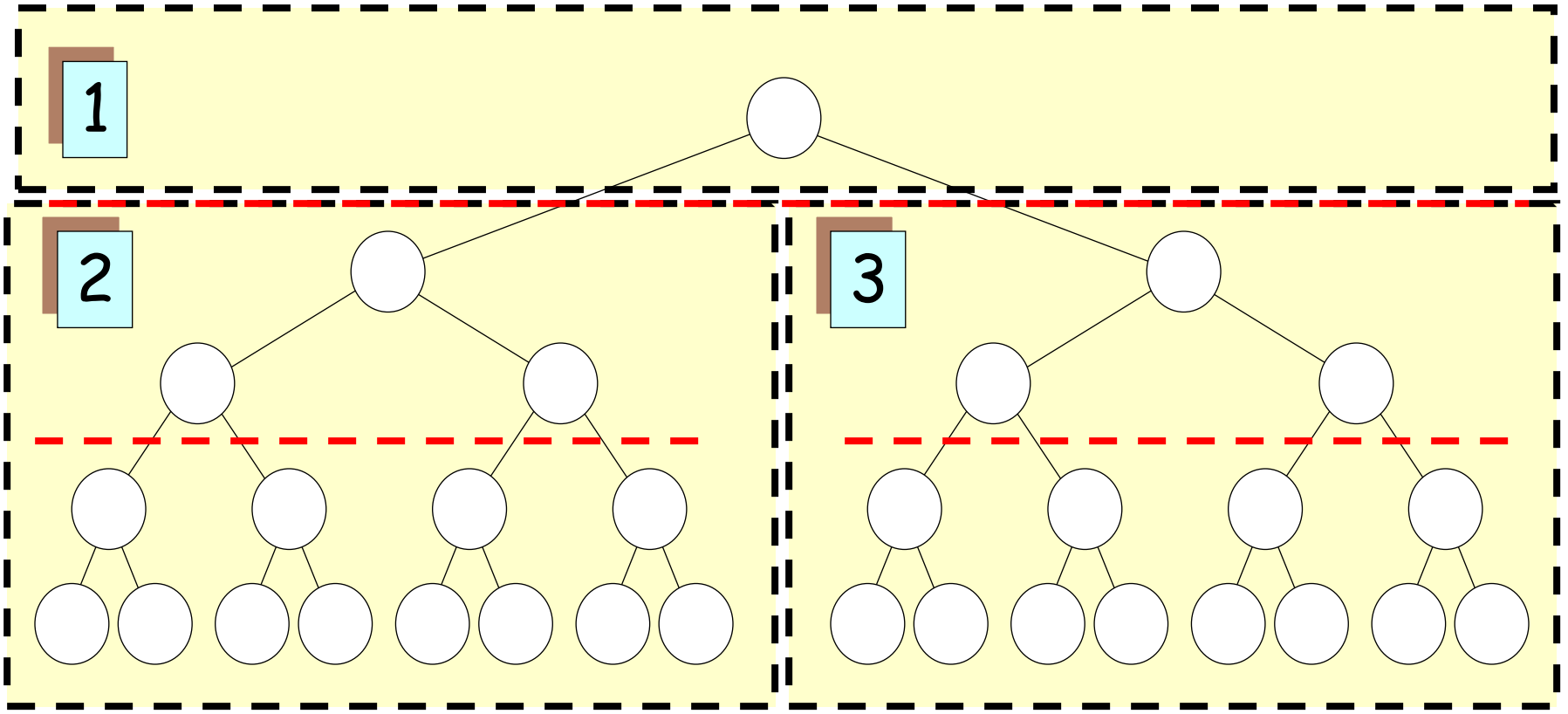

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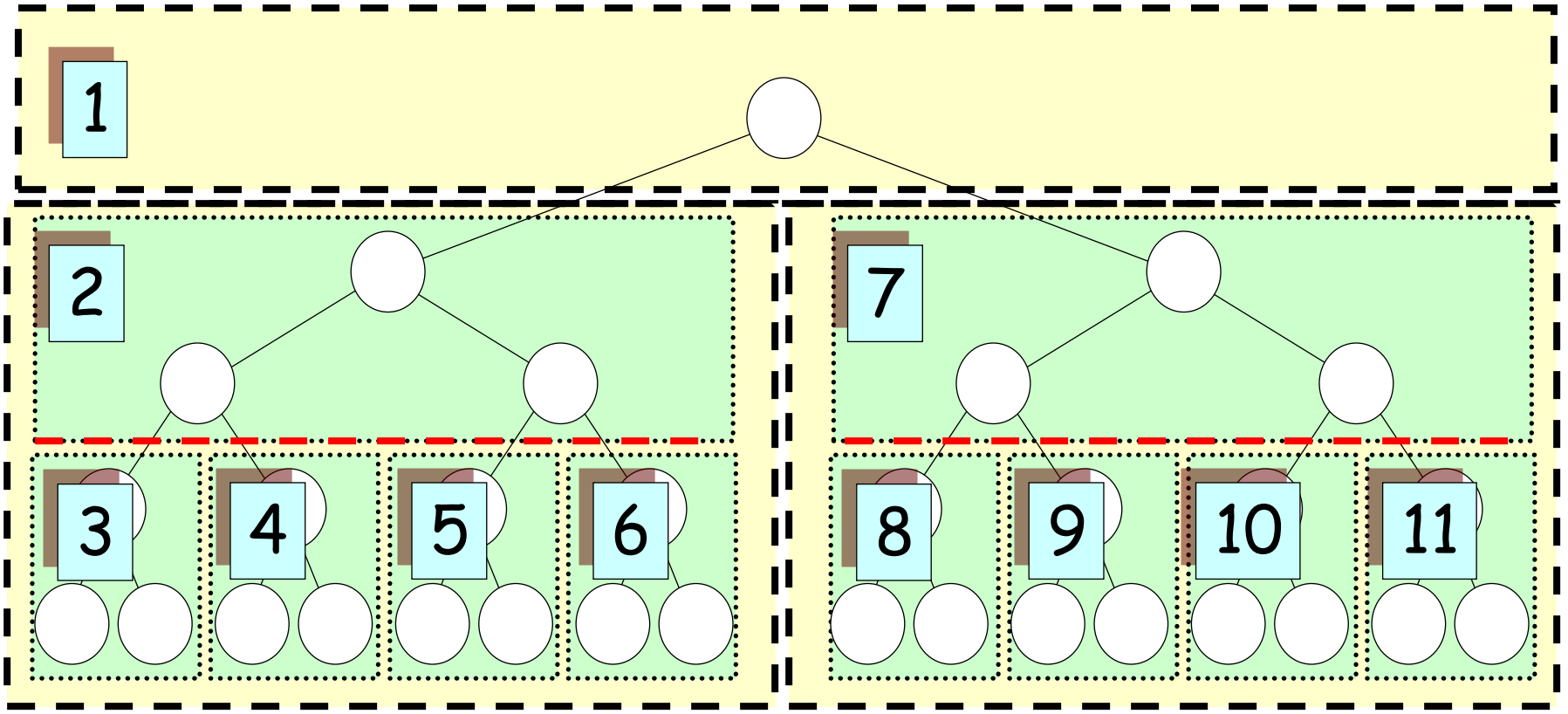

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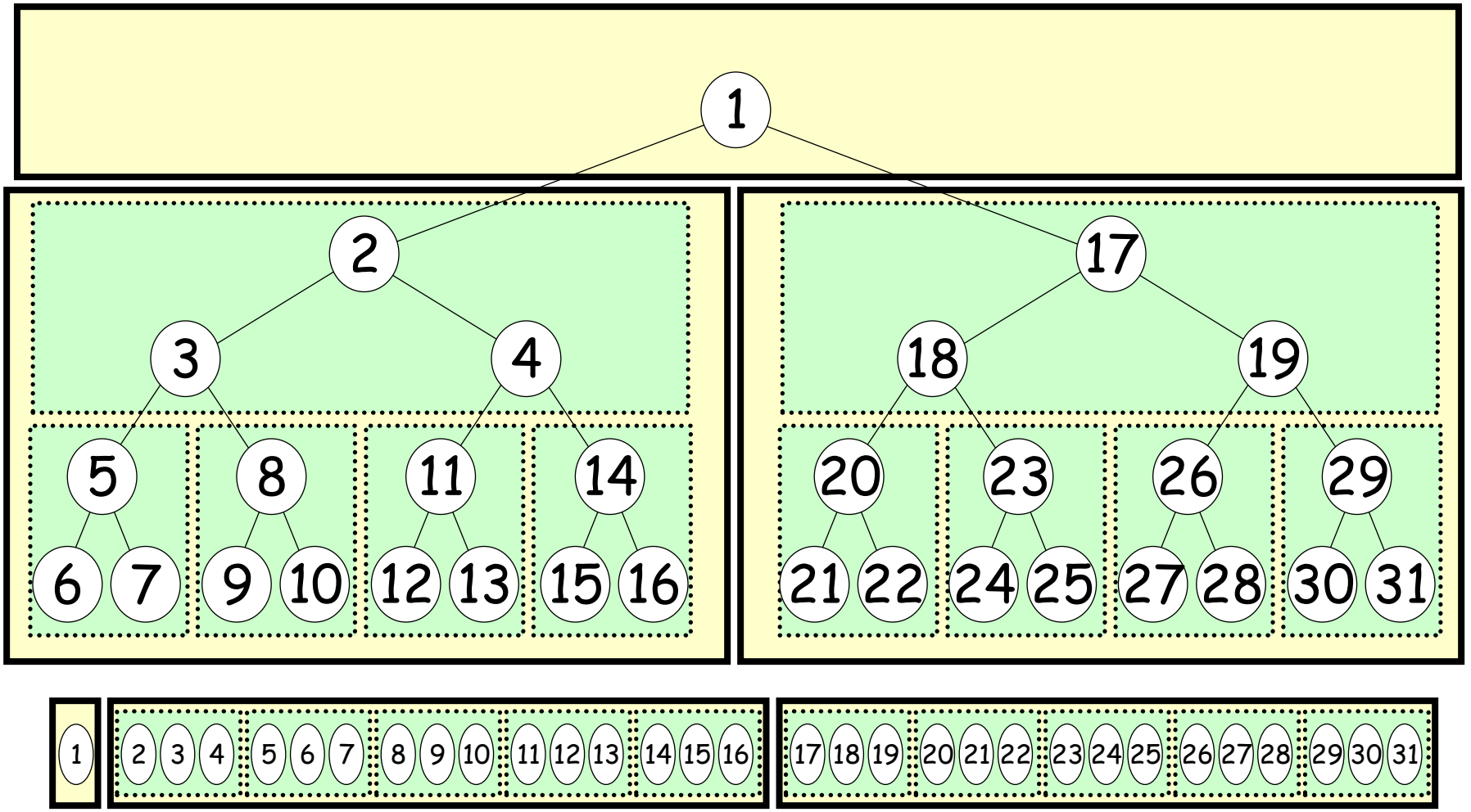

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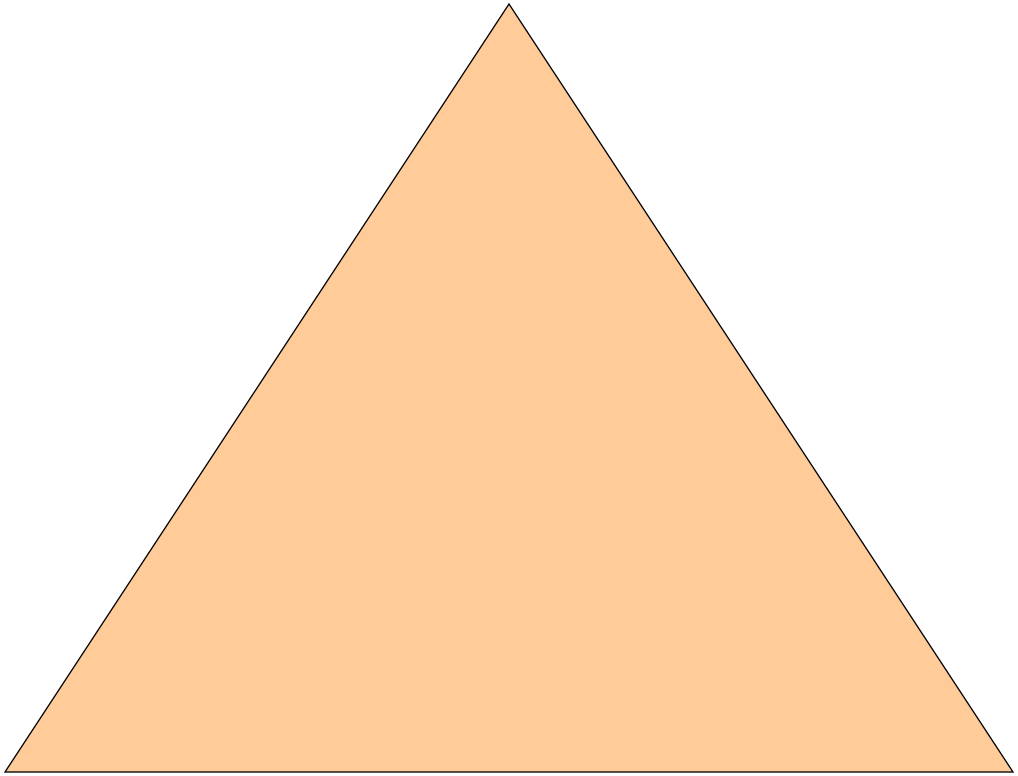

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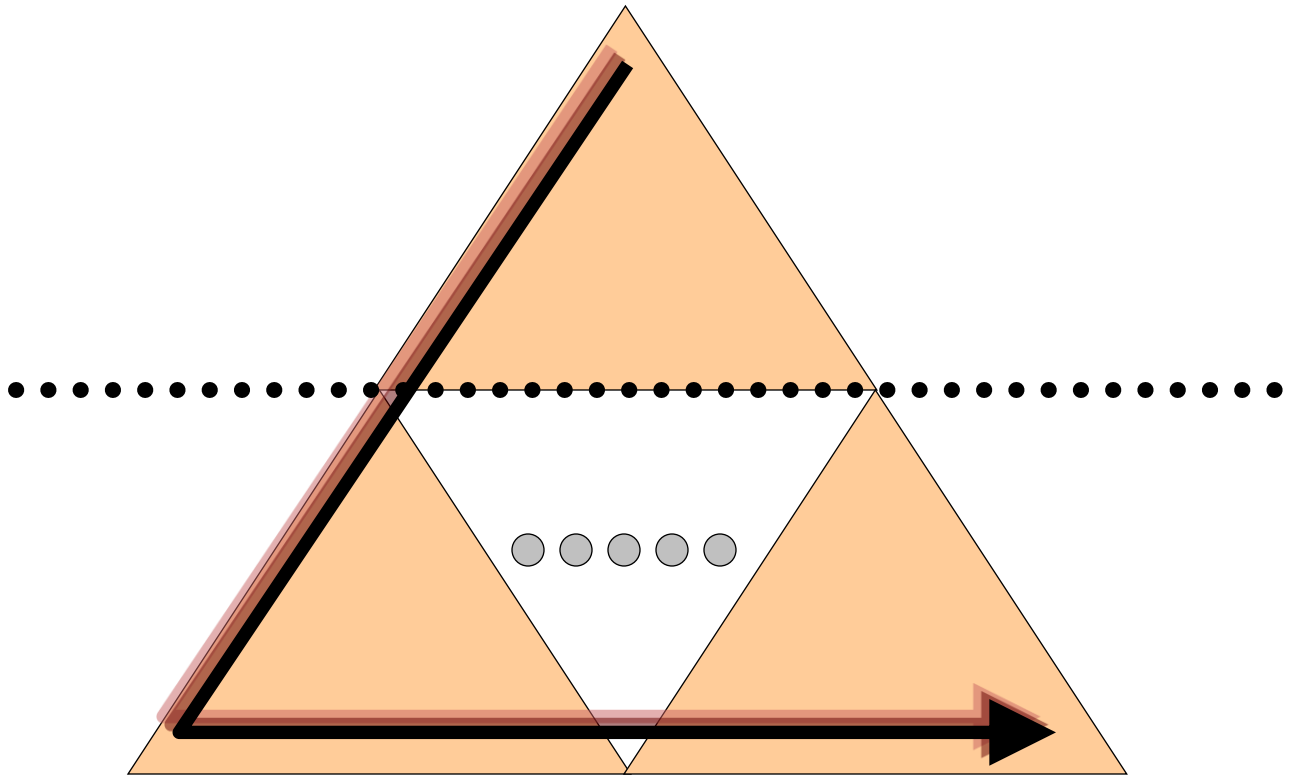

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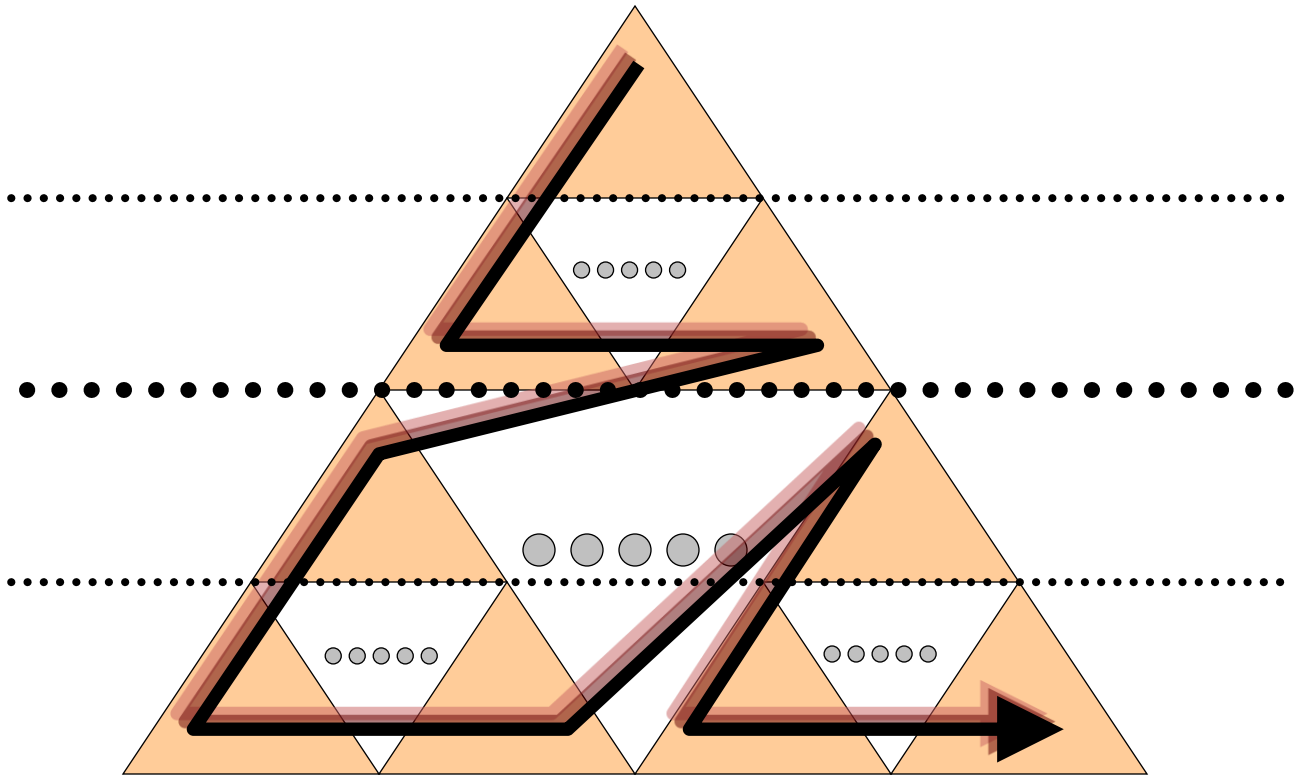

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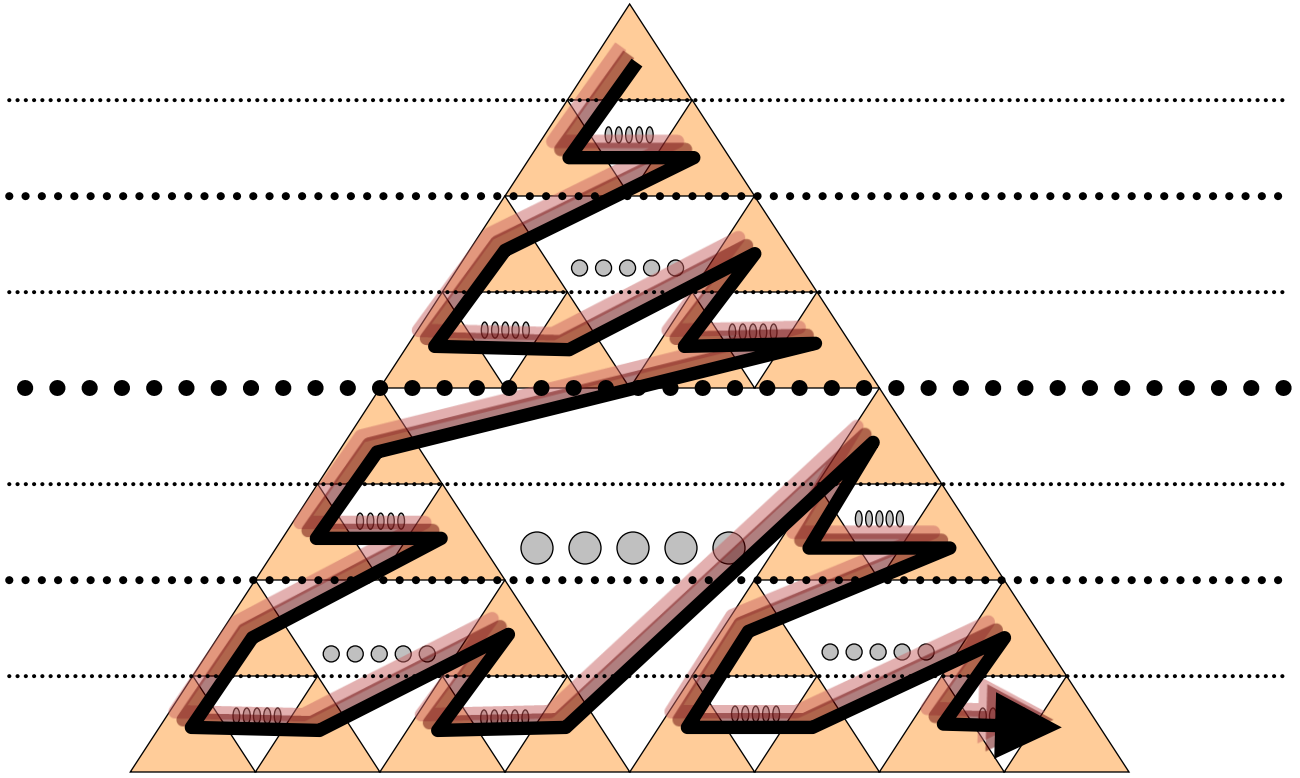

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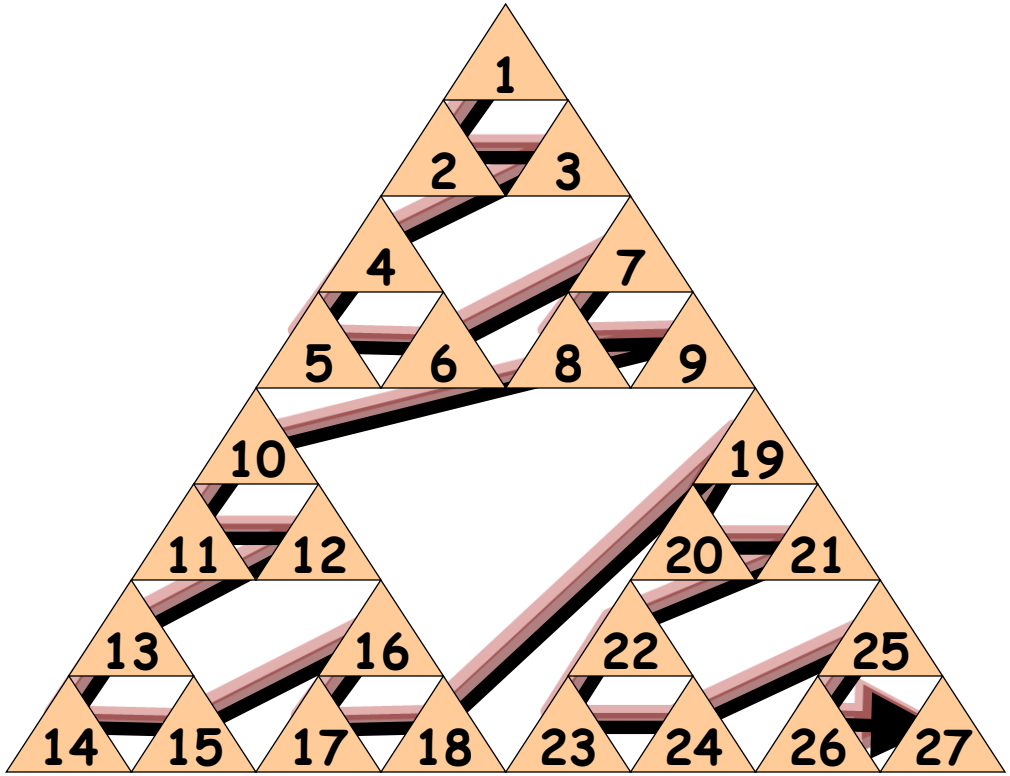
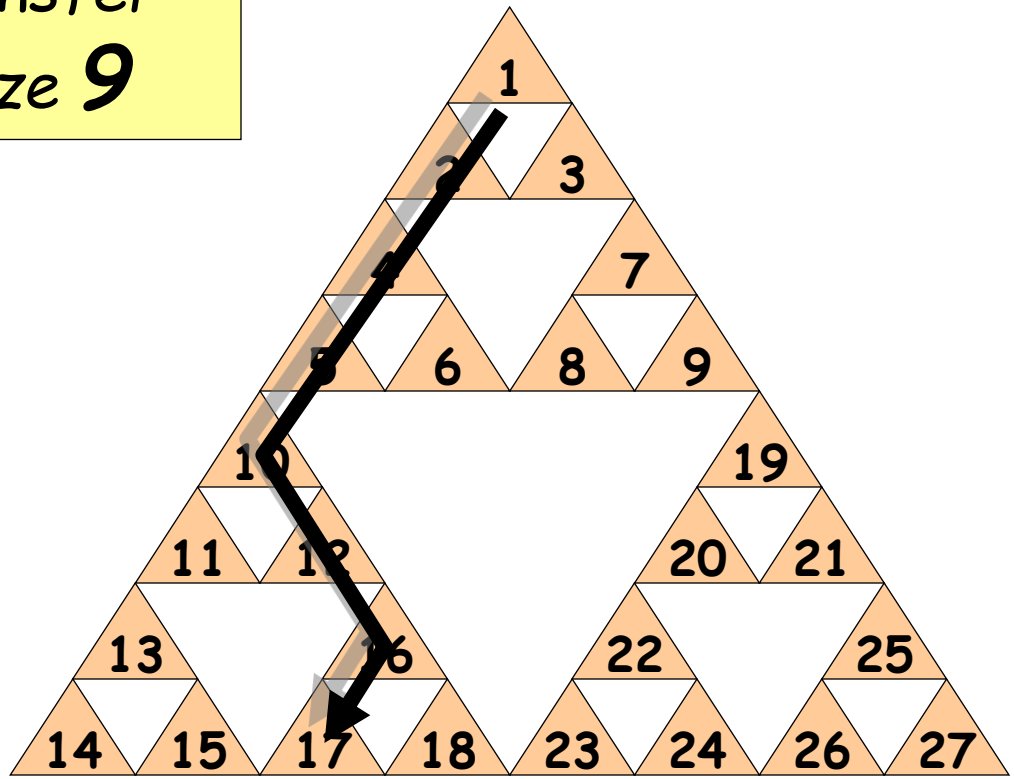


1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Van Emde Boas Layout

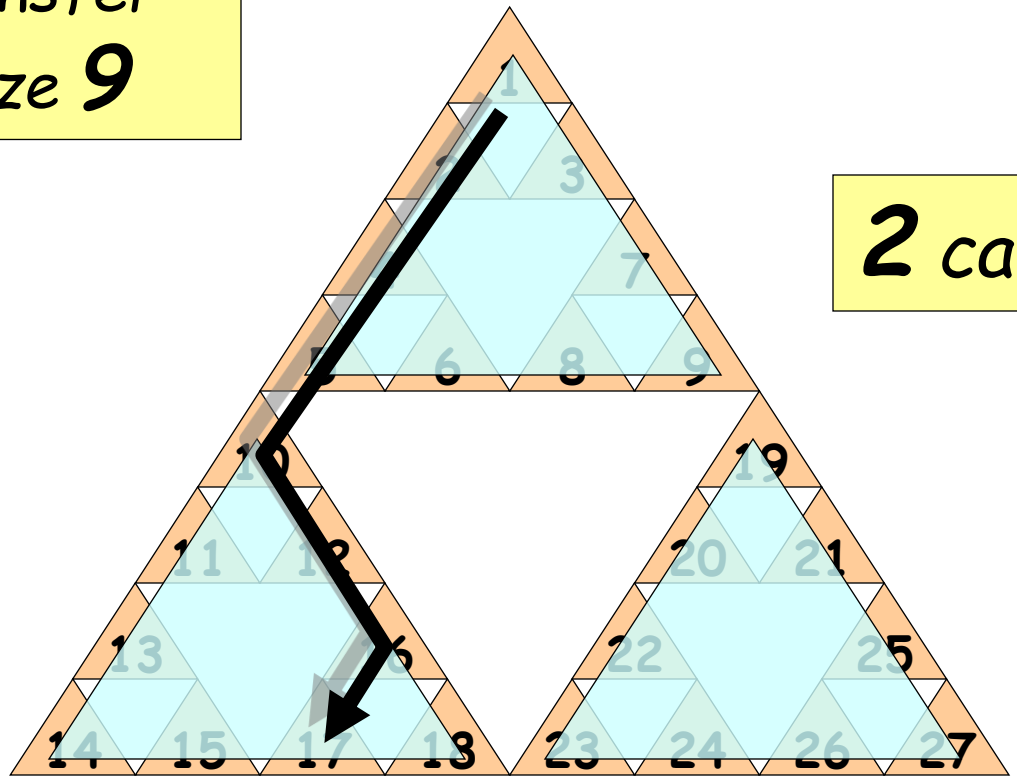
Searching with memory transfer block of size **9**



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Van Emde Boas Layout

Searching with memory transfer block of size **9**

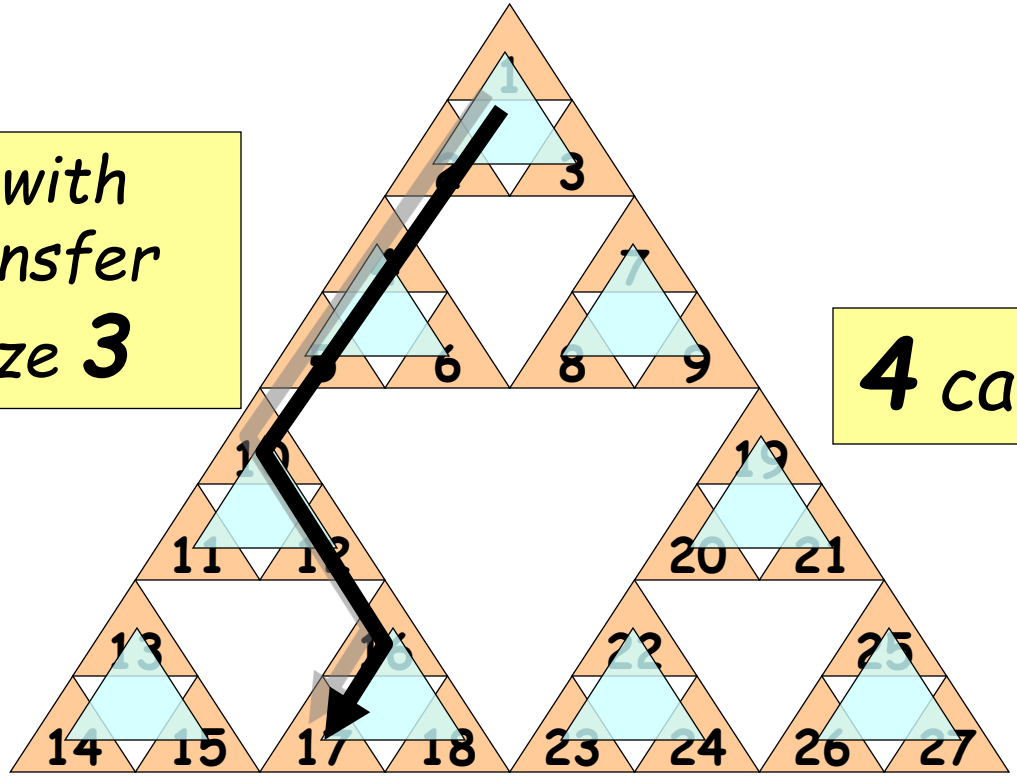


2 *cache misses*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Van Emde Boas Layout

Searching with memory transfer block of size ***3***



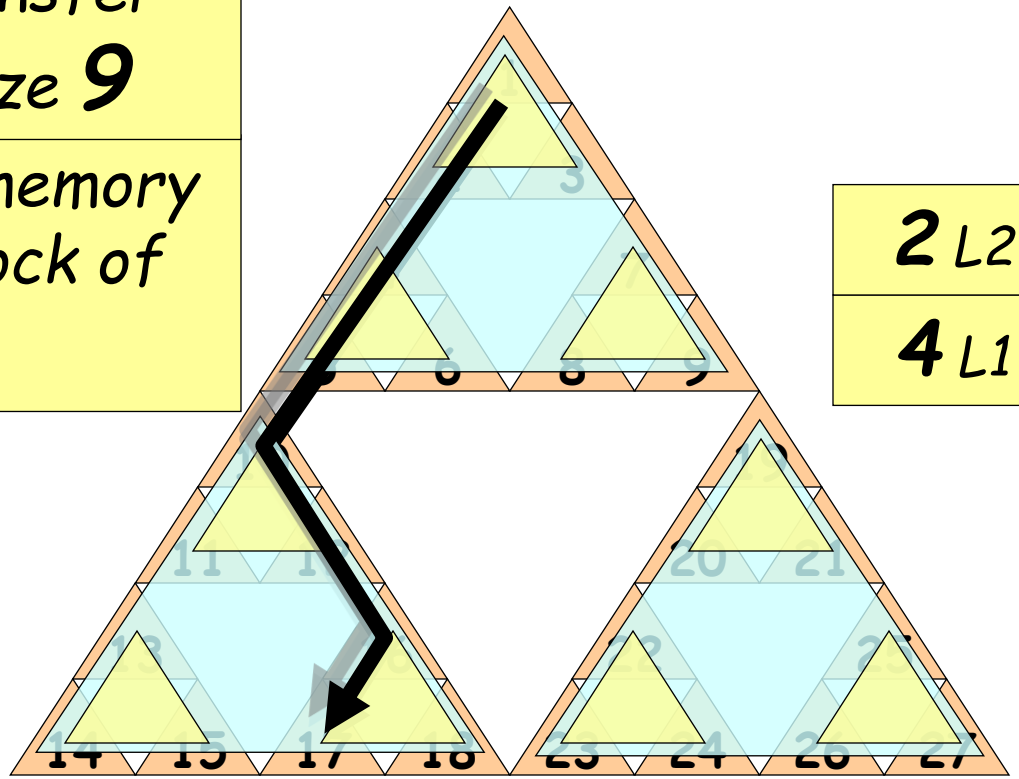
4 *cache misses*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

Van Emde Boas Layout

Searching with L2 memory transfer block of size **9**

and with L1 memory transfer block of size **3**



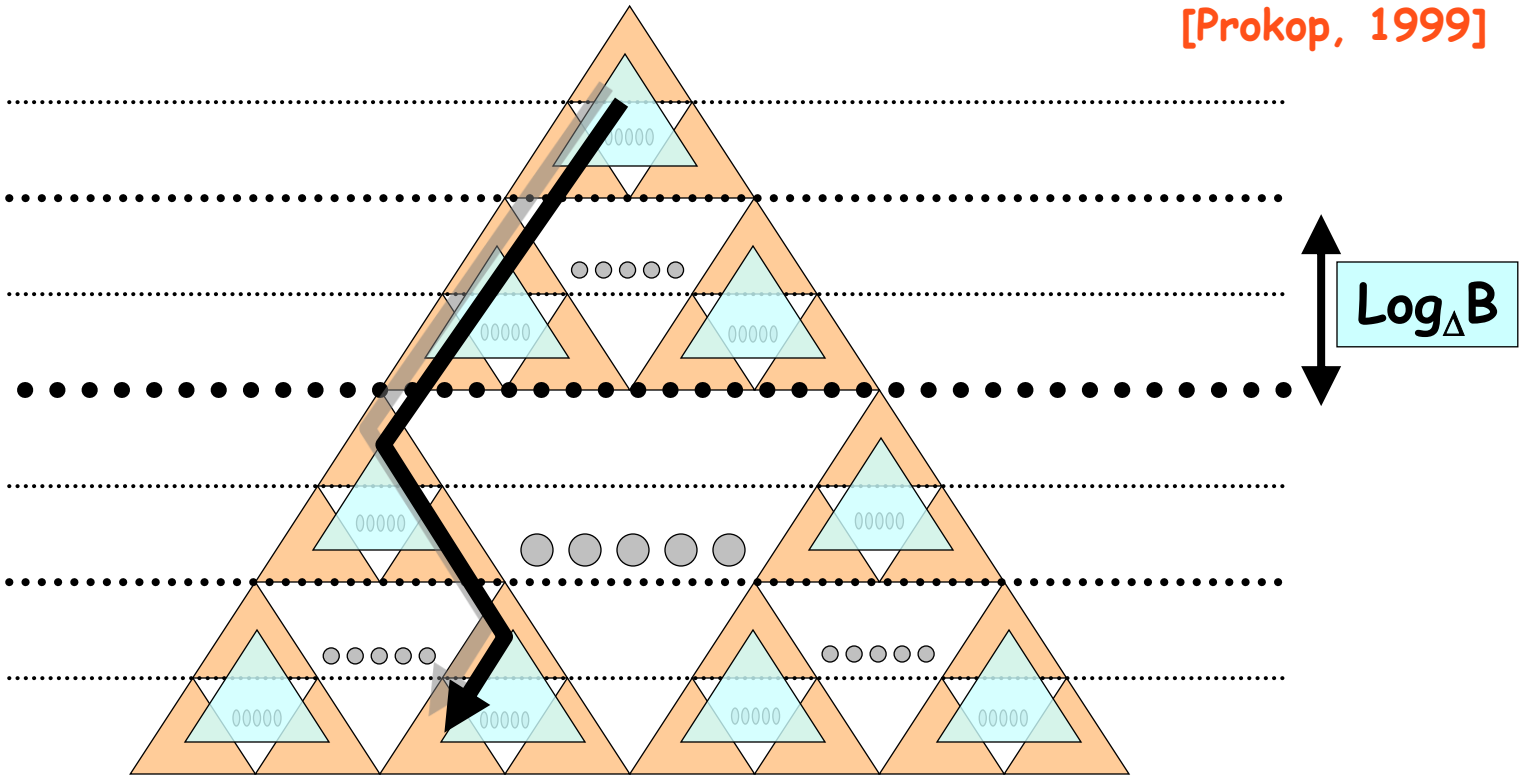
2 *L2 cache misses*

4 *L1 cache misses*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

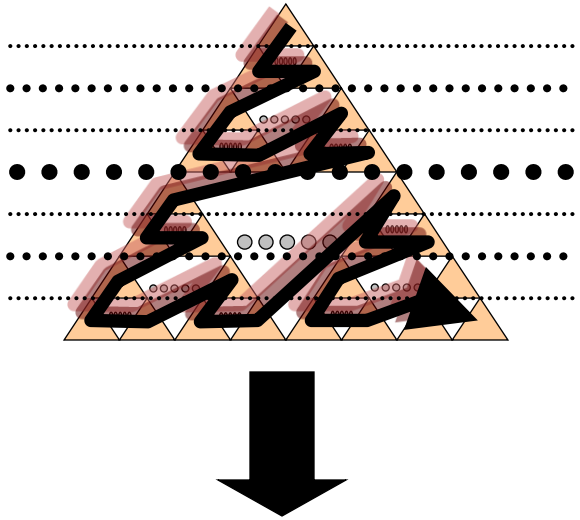
Static Cache Oblivious Tree

[Prokop, 1999]



#IOs per search = $O\left(\frac{\log_\Delta N}{\log_\Delta B}\right) = O(\log_B N)$

From Static to Dynamic

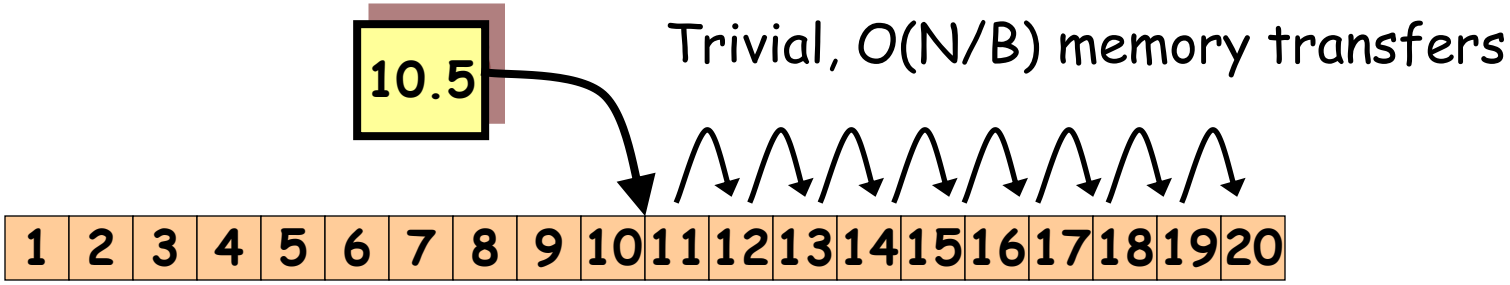


1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

- Static is fine, but… A Dynamic B-Tree should allow the addition and removal of nodes
- How do we add an element efficiently to an ordered array ?

Dynamic B-Tree – restating the problem

- How do we add an element efficiently to an ordered array ?



Can we do better ?

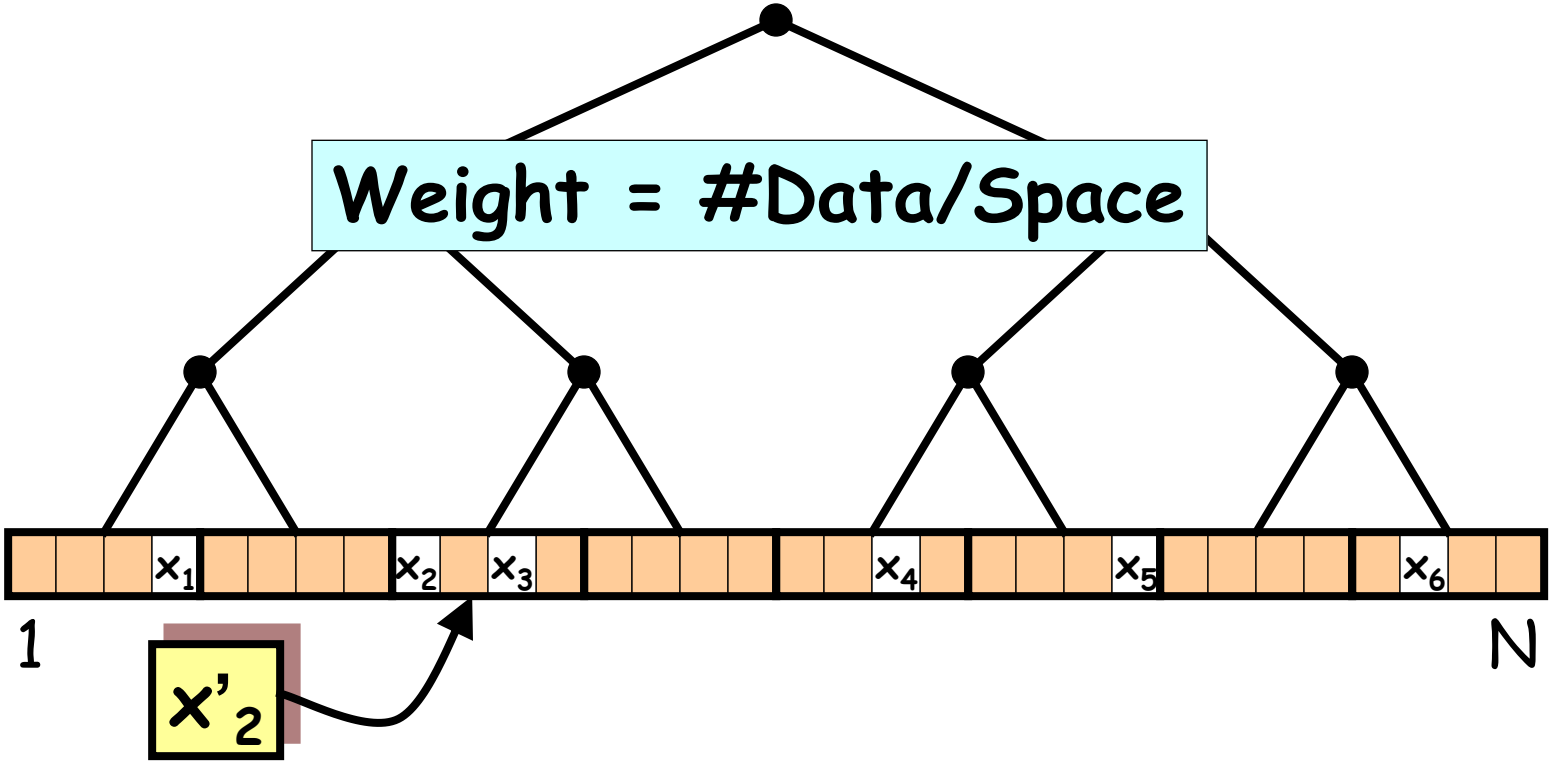
Dynamic B-Tree – restating the problem

- General idea – Array with enough extra space between nodes for future insertions

- Contradictory goals – nodes should be packed densely for locality of reference
- Guideline - Distribute elements evenly in the array
  - Store any set of k contiguous elements xi1,…, xik in a contiguous subarray of size O(k)

Packed Memory Management

[A. Itai, A. G. Konheim, and M. Rodeh 1981]



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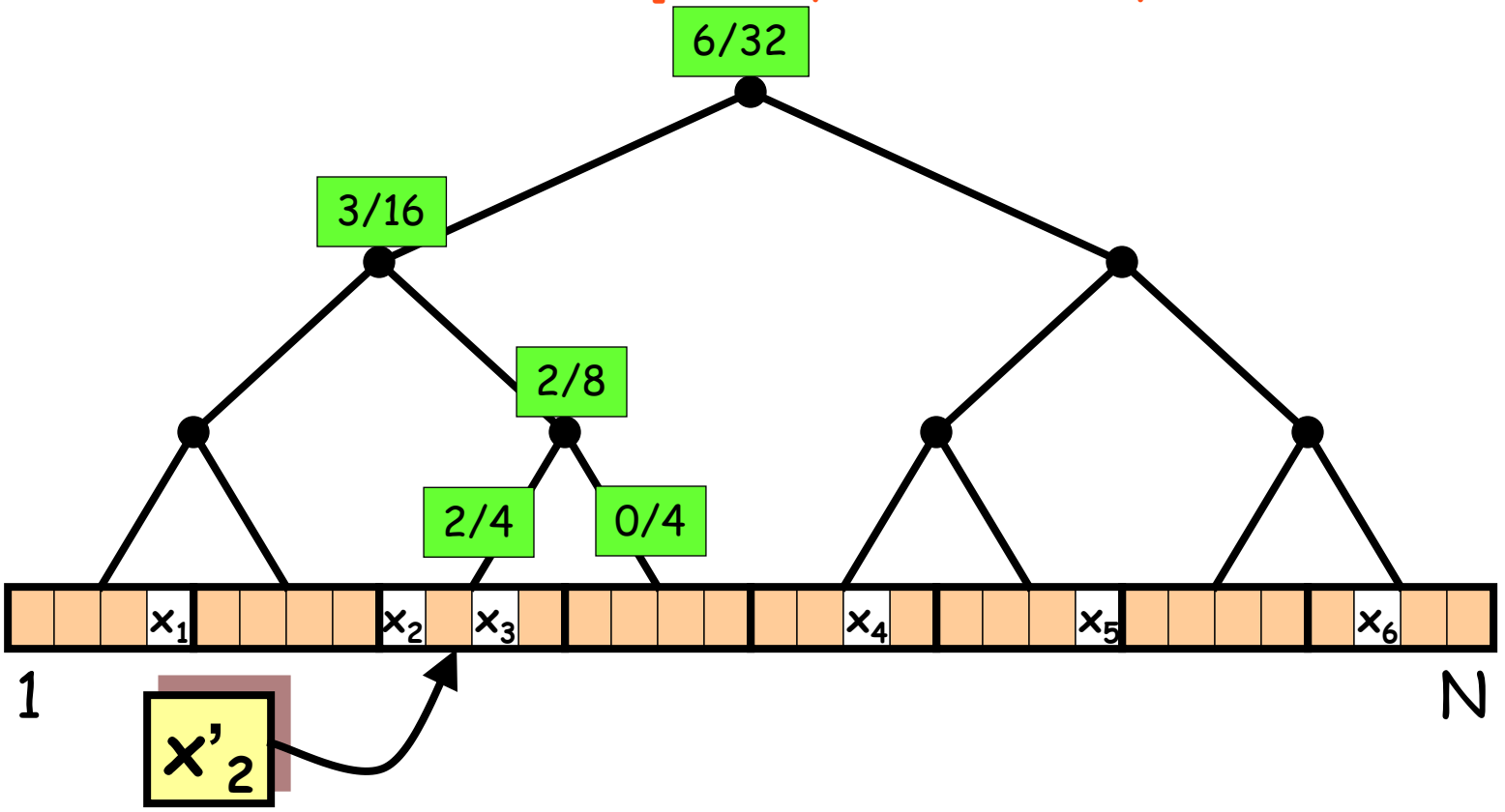

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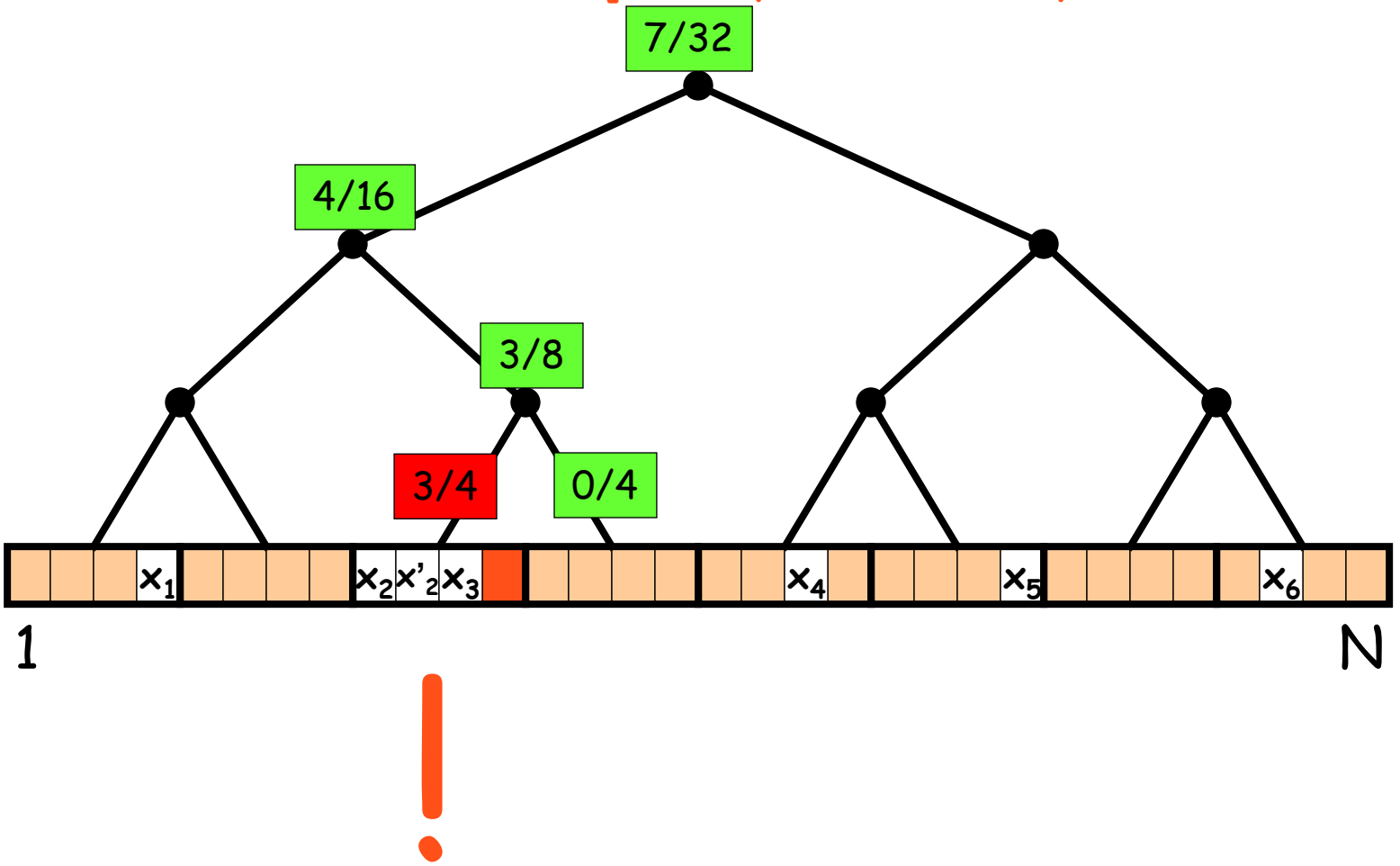

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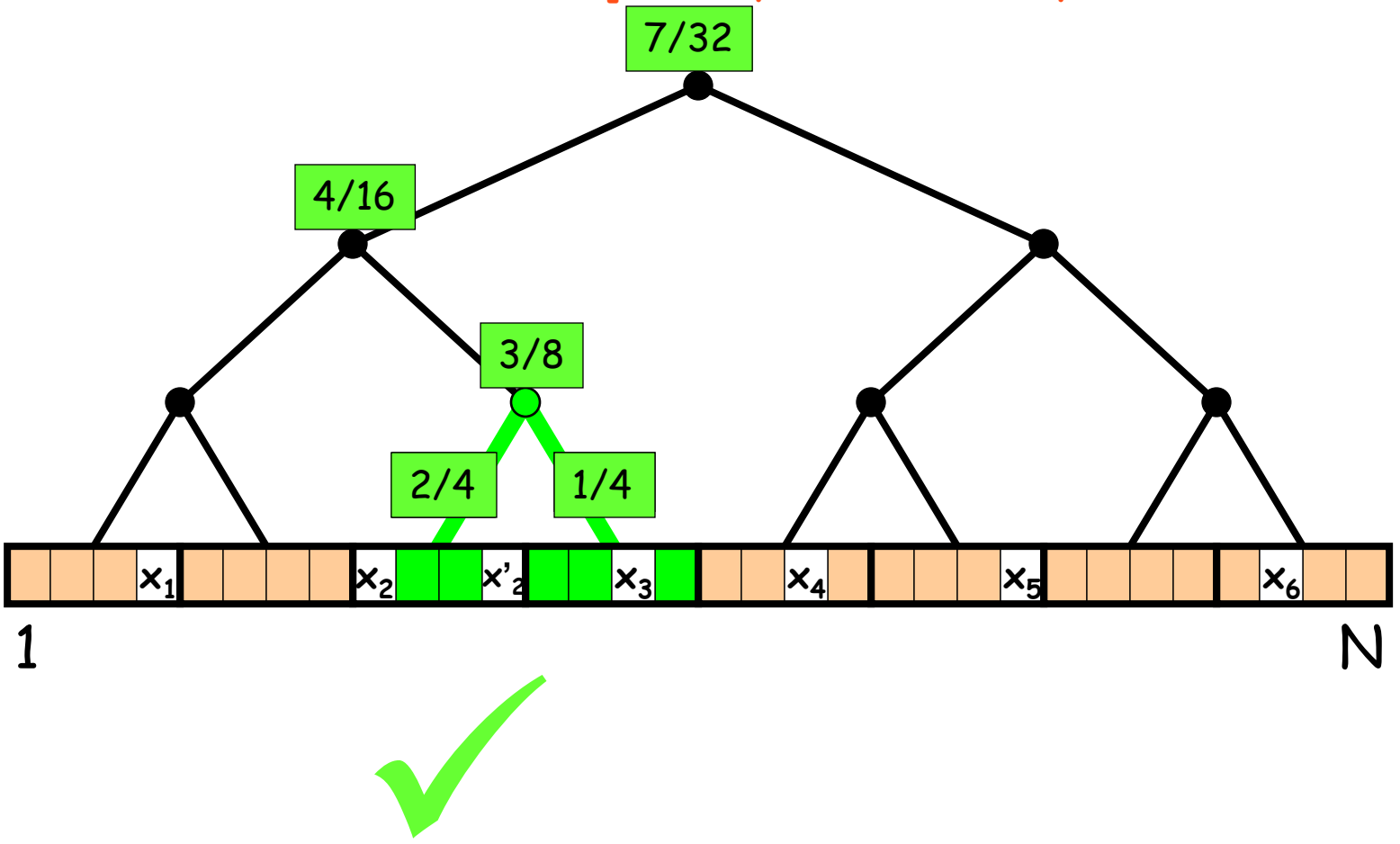
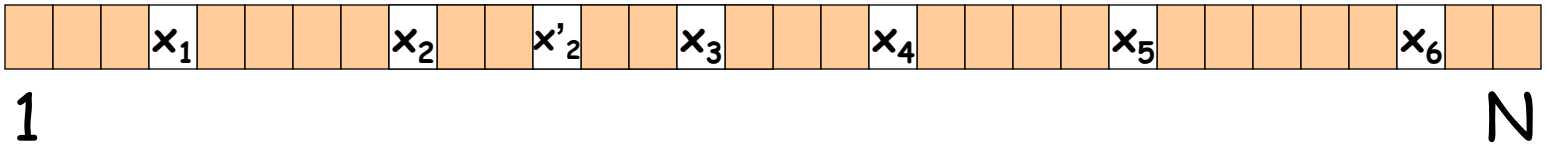


Packed Memory Management

[A. Itai, A. G. Konheim, and M. Rodeh 1981]

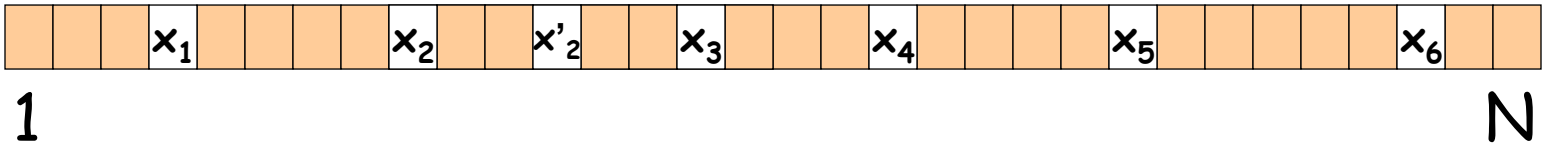
- Size of the interval to rebalance ?
- Thresholds for interval over/underflows ?



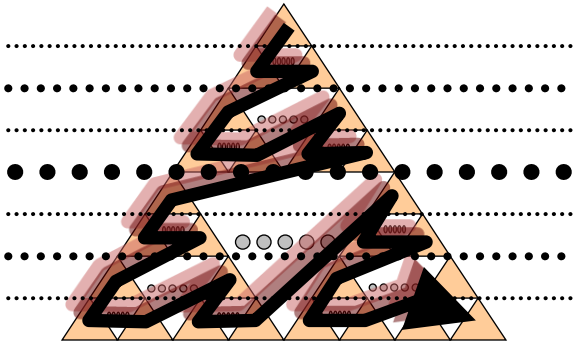
Packed Memory Management

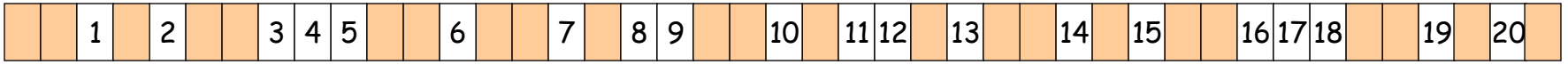
[A. Itai, A. G. Konheim, and M. Rodeh 1981]

- Scanning a k consecutive elements
  - k Elements are stored in O(k) memory cells
  - O(1+k/B) memory transfers
- Inserting/deleting an element
  - Moves $O(\log^2 N)$ amortized consecutive cells
  - $O(1 + \log^2 N / B)$ amortized memory transfers



Putting it all together… ?





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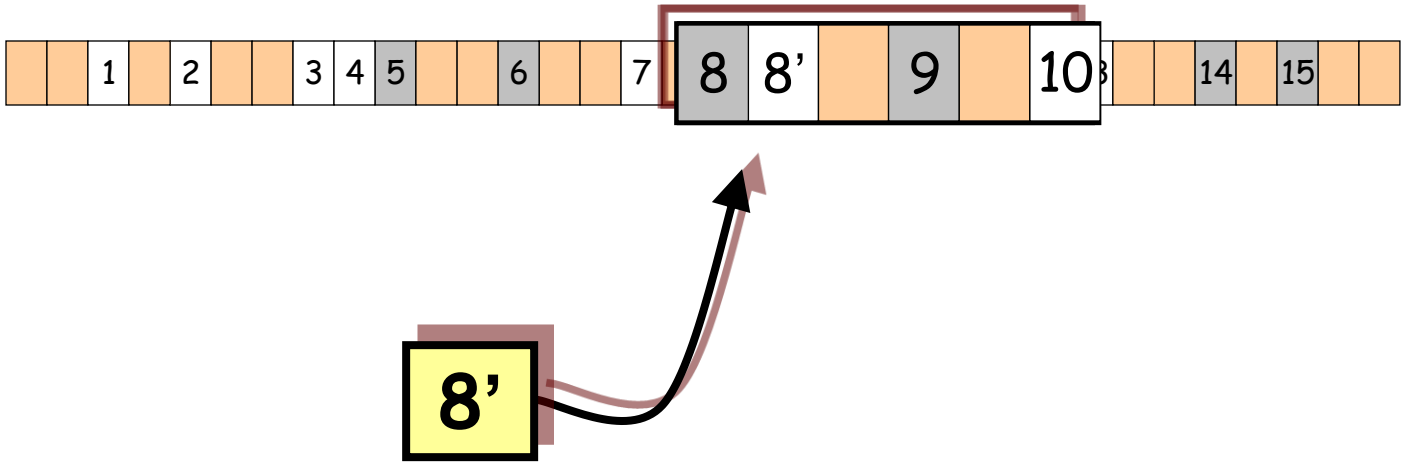
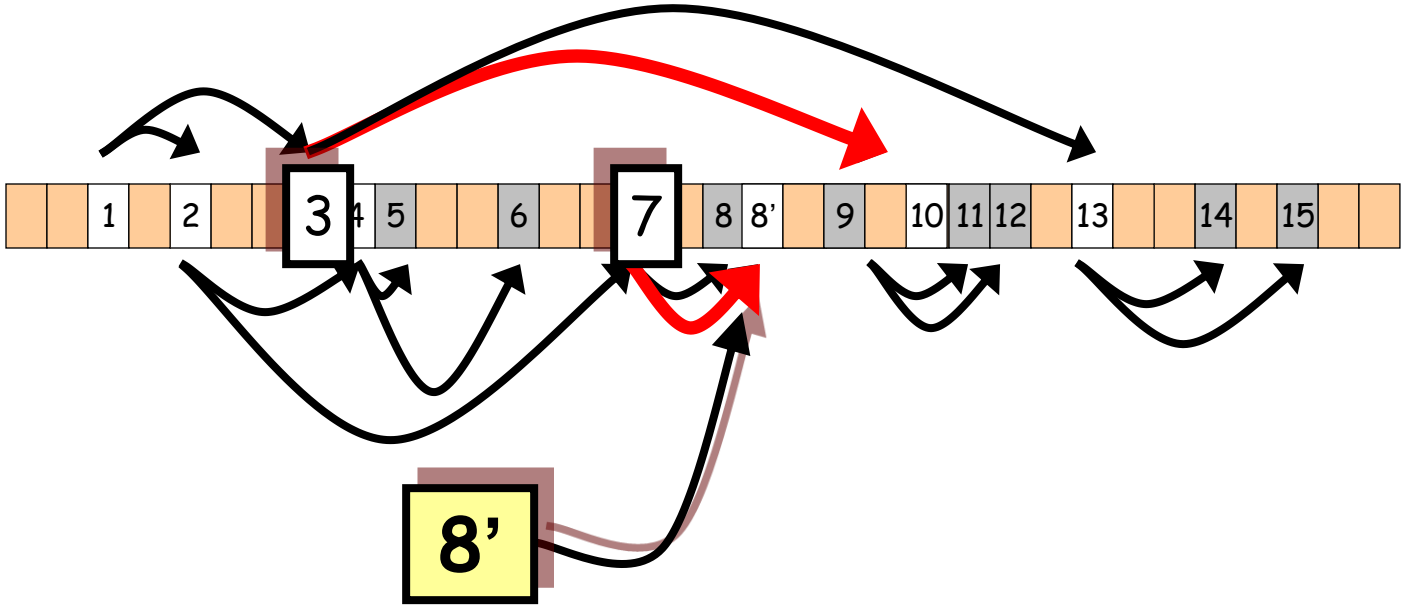
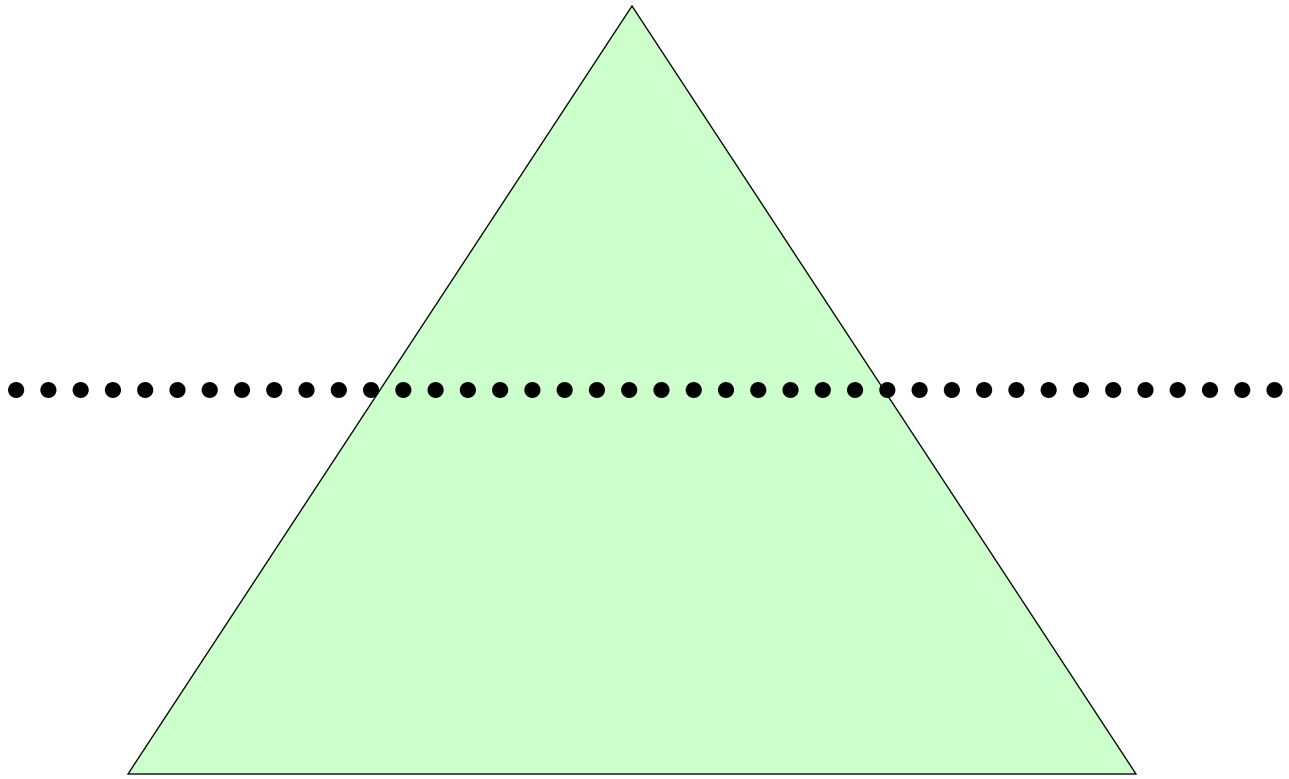


It's still a tree !



- Insertion moves $O(\log^2 N)$ amortized nodes
- Requires $O(\log^2 N)$ back pointers updates
- And what about the van Emde layout ?

Splits and Merges



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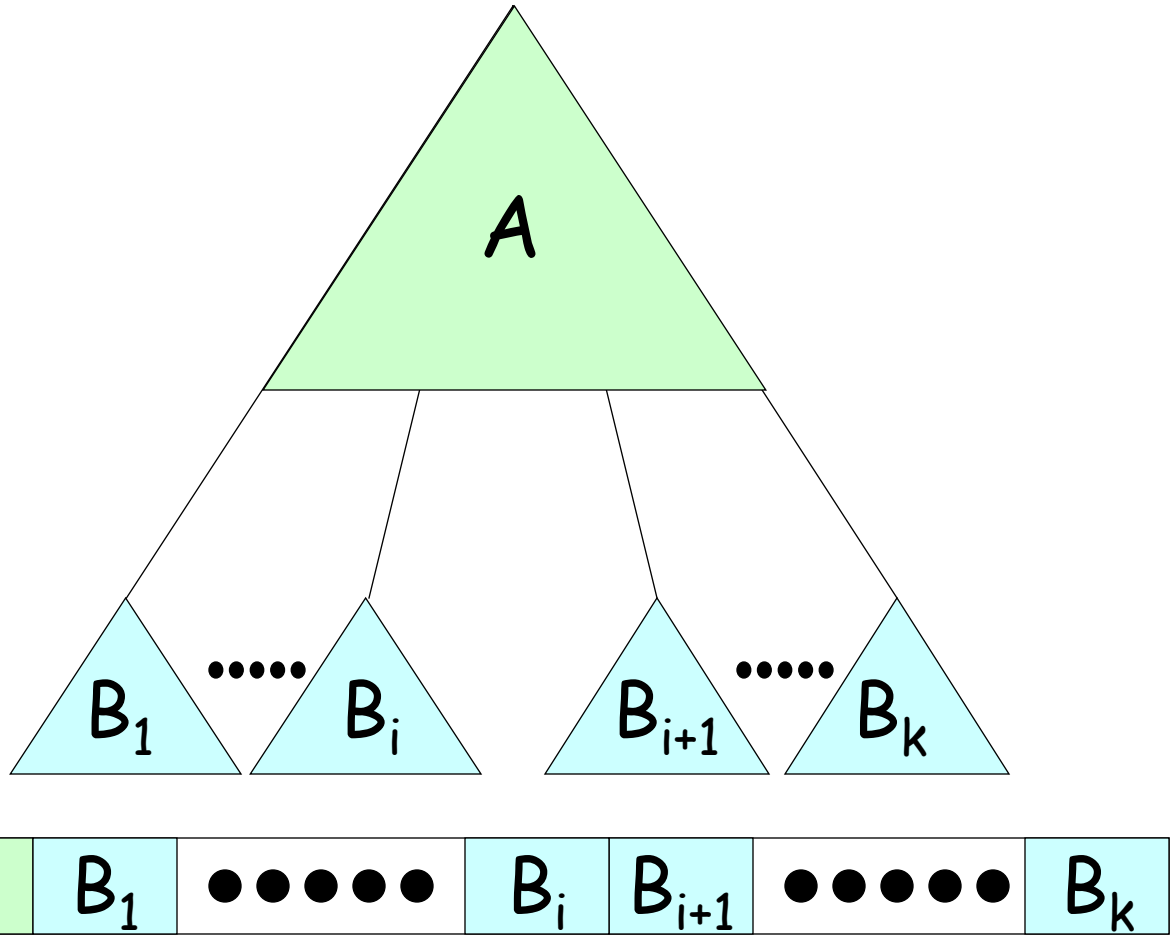

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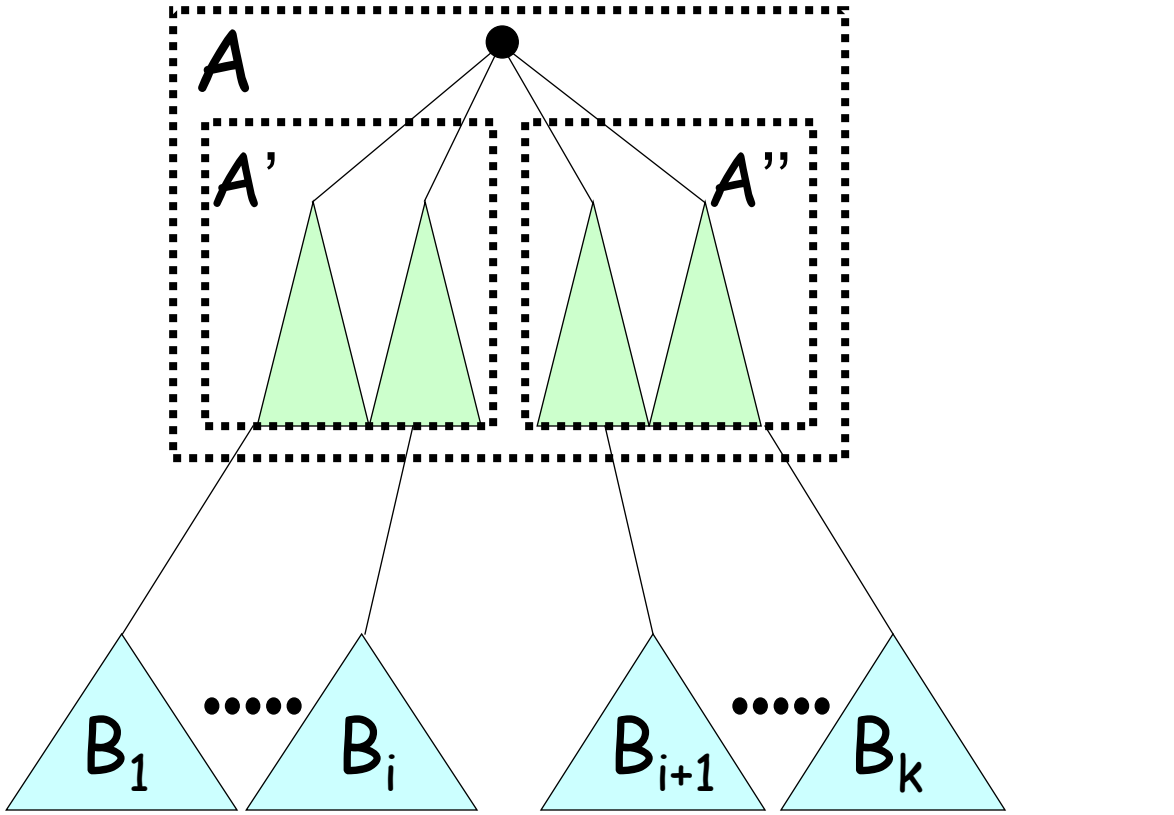

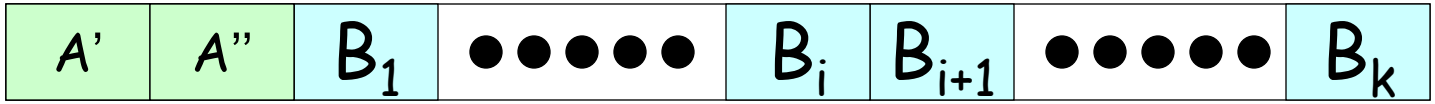

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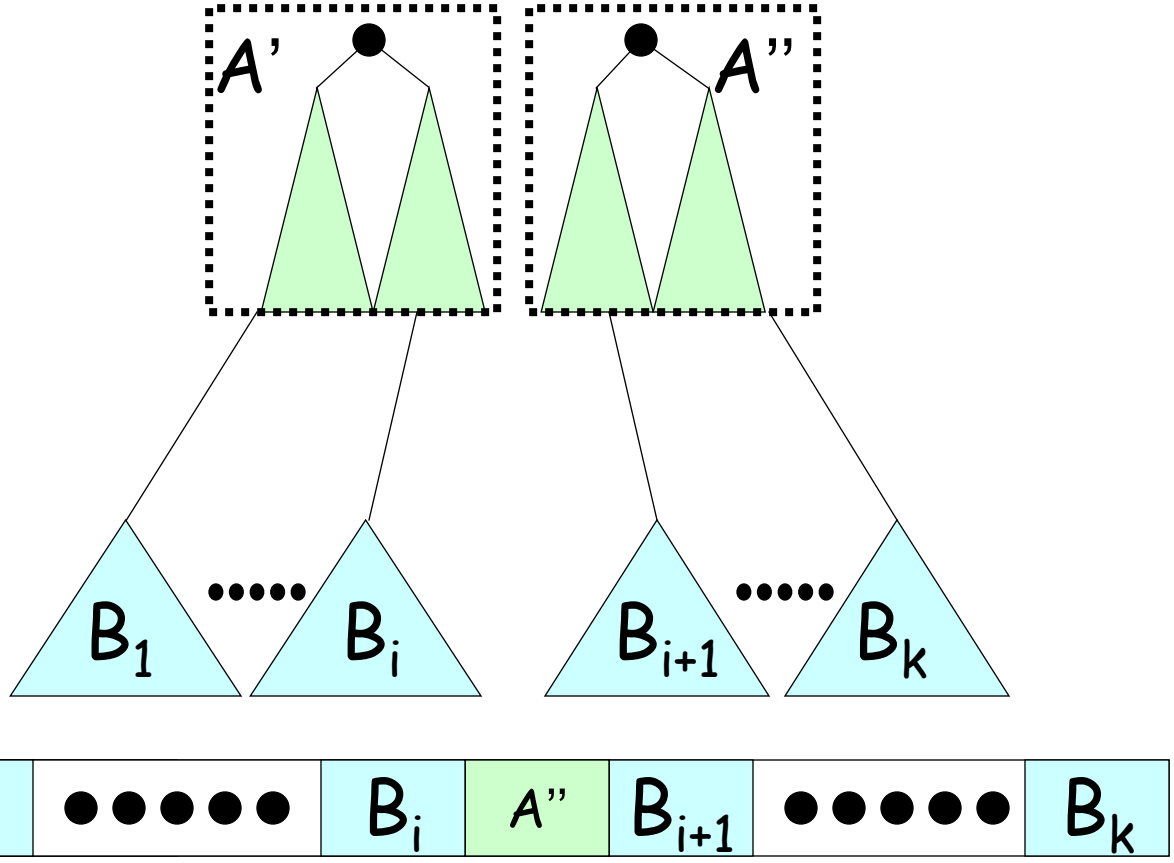

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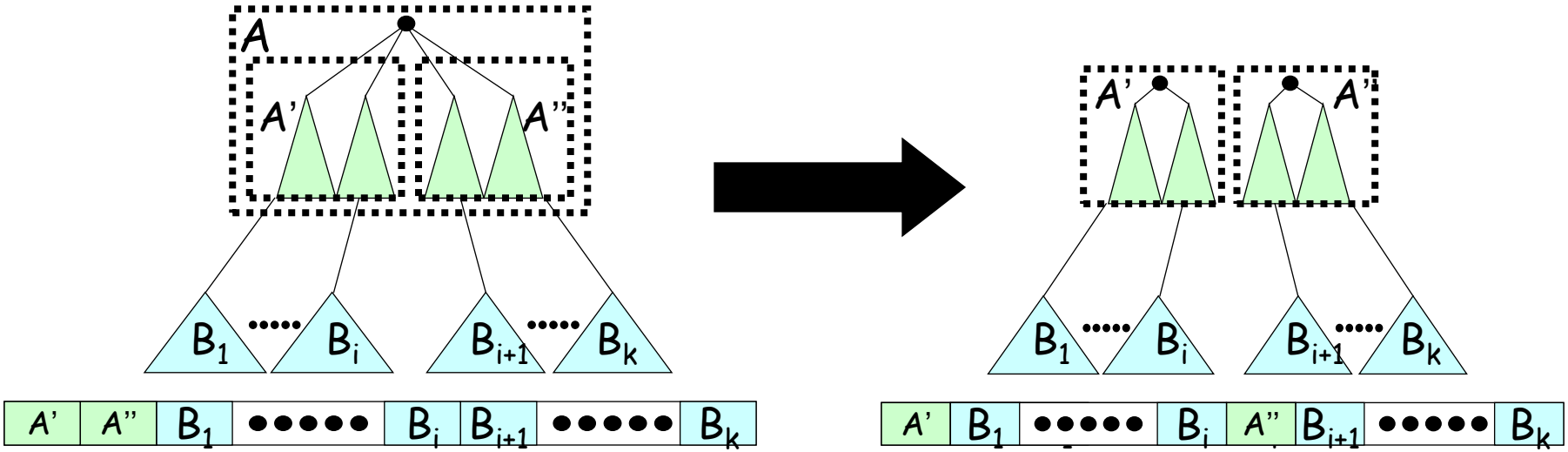
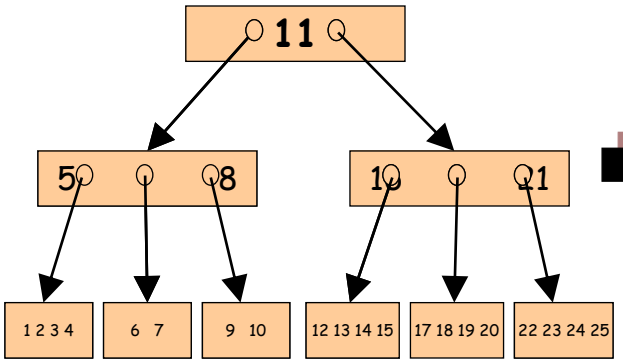


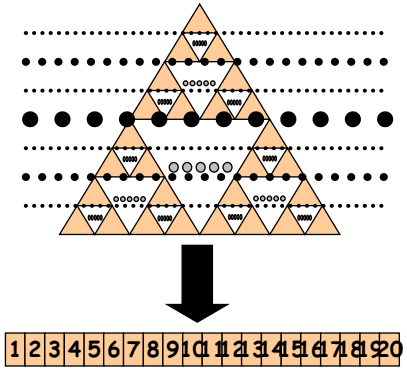
- Each update requires O(1 + logN/B) amortized memory transfers

Putting it all together… again

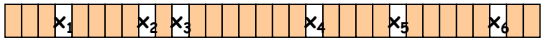
Weight-balanced B-tree



van Emde Boas layout



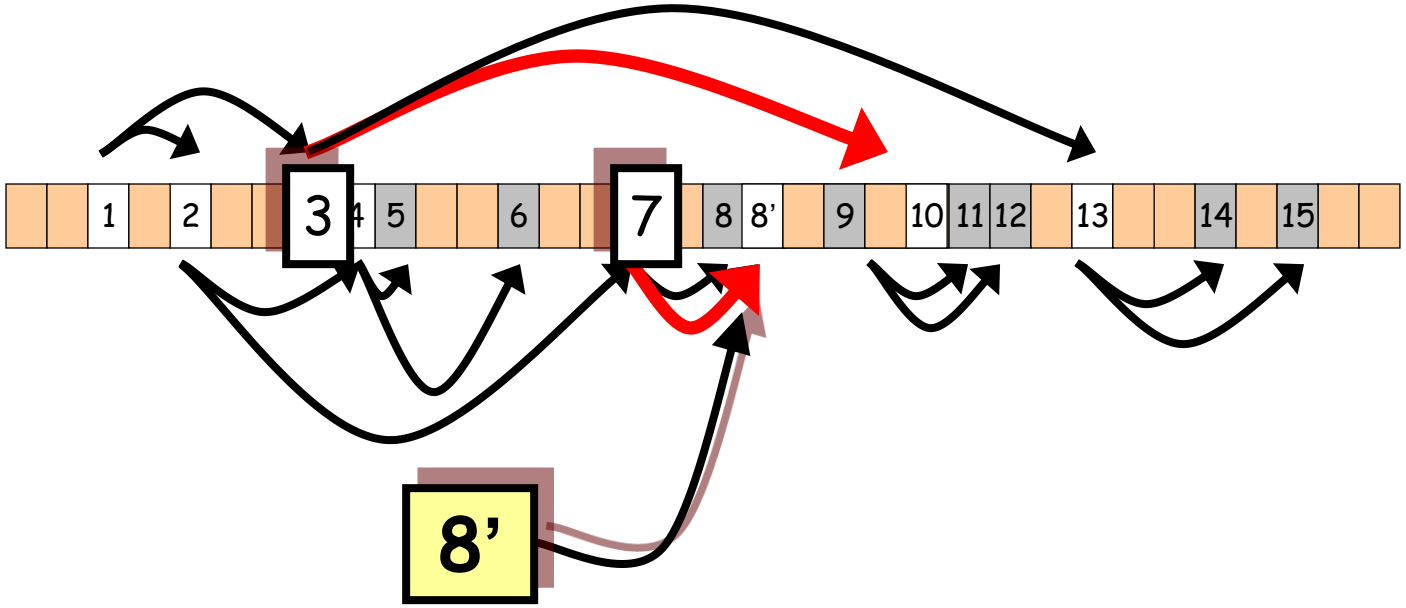
Packed Memory Maintenance



- Search $O(\log_B N)$
- Scan $O(1 + k/B)$

- Update $O(\log_B N + \log^2 N)$
  - Search for key $O(\log_B N)$
  - Create/delete leaf $O(1)$
  - Rearange packed array $O(1+\log^2 N/B)$
  - Update back pointers $O(1+\log^2 N)$
  - Split/Merge nodes $O(1+\log N/B)$

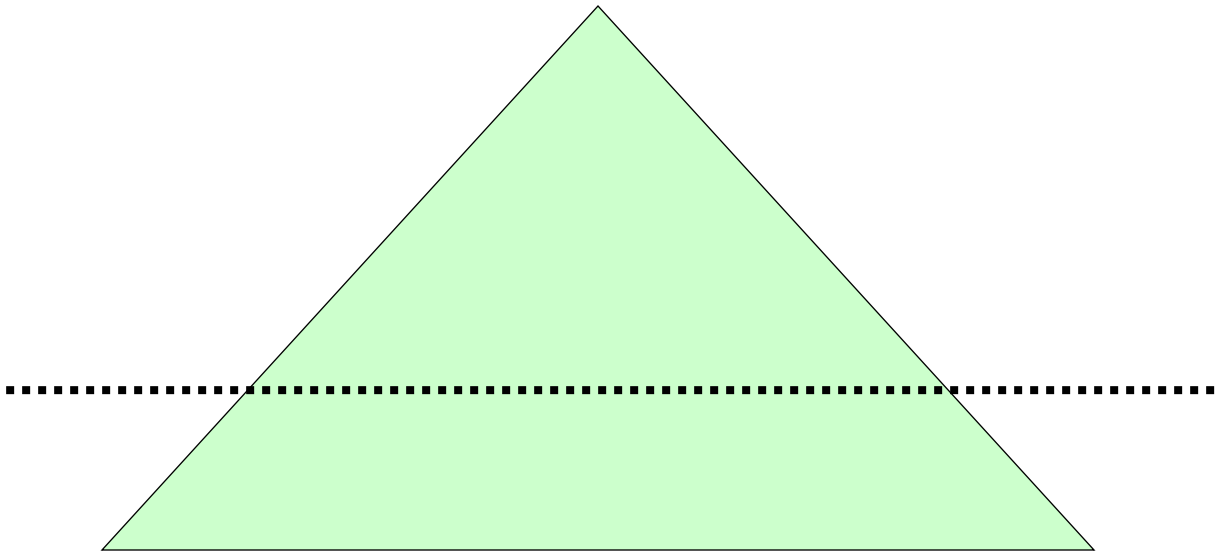
Saving on back pointer updates ?



- How can we reduce back pointer updates ?

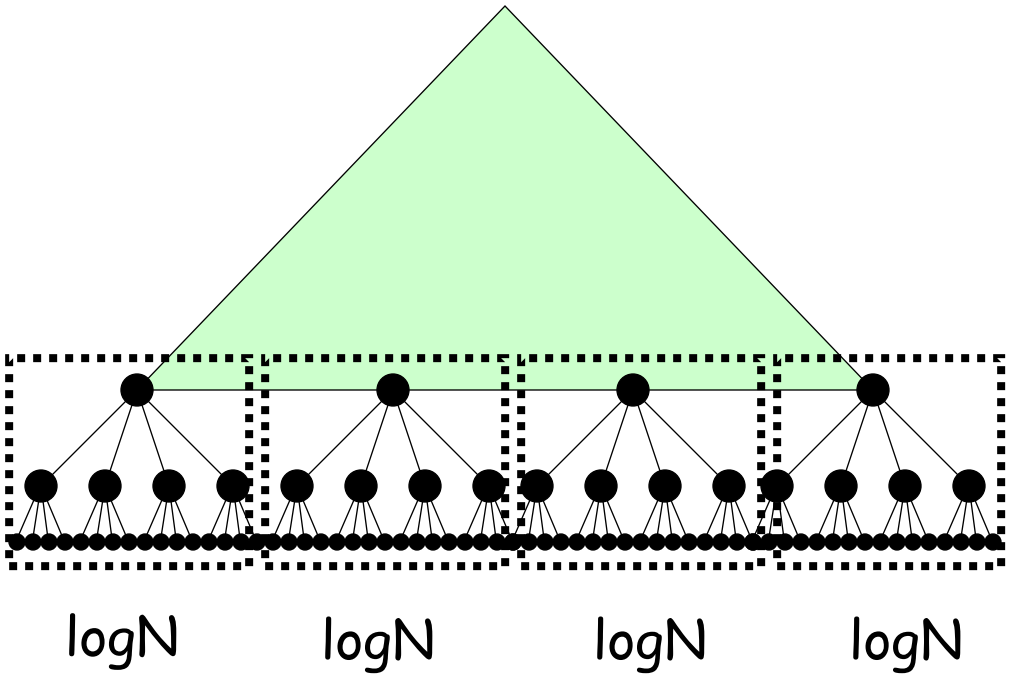
Saving on back pointer updates ?

- Idea – pack groups of leaves together beneath a single parent



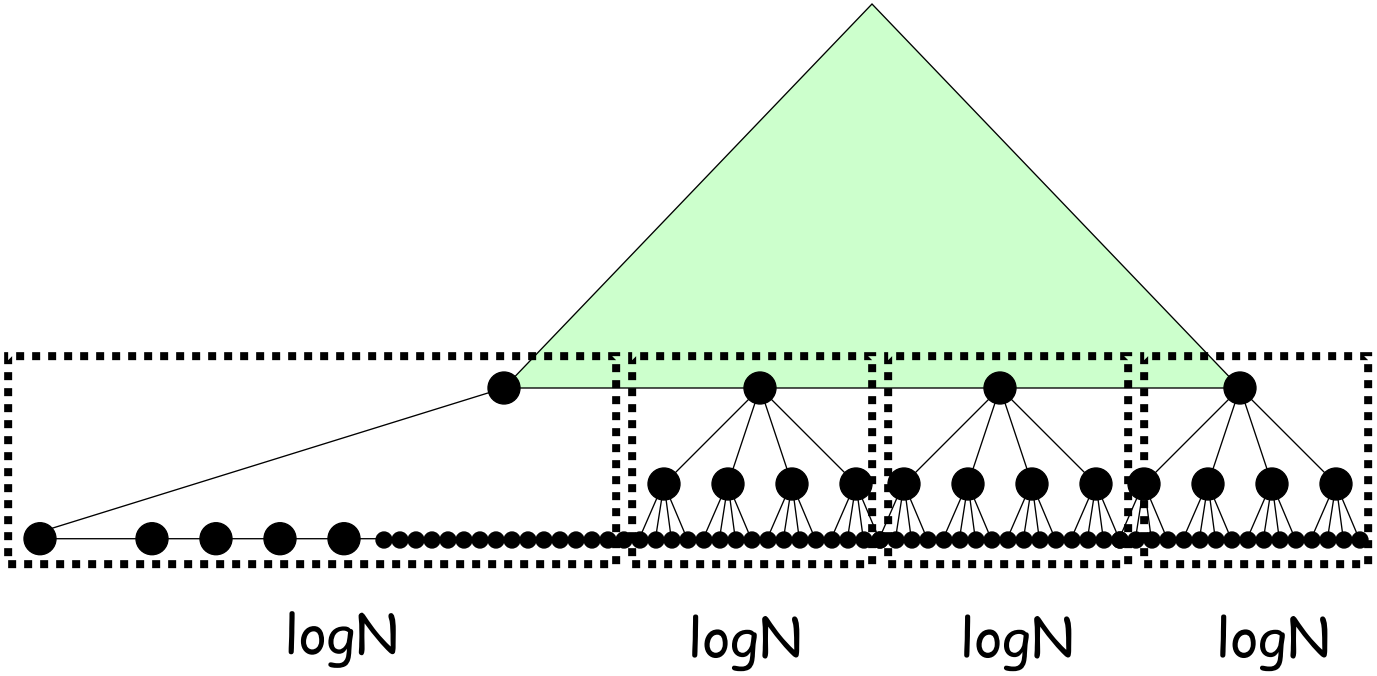
Saving on back pointer updates ?

- Idea – pack groups of leaves together beneath a single parent



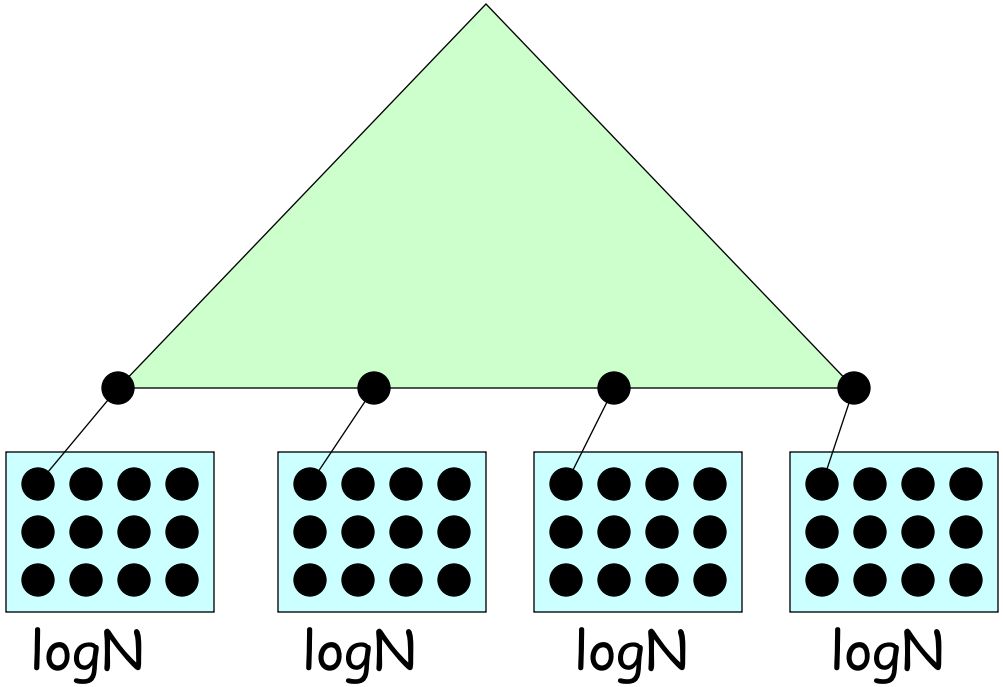
Indirection

- Idea – pack groups of leaves together beneath a single parent



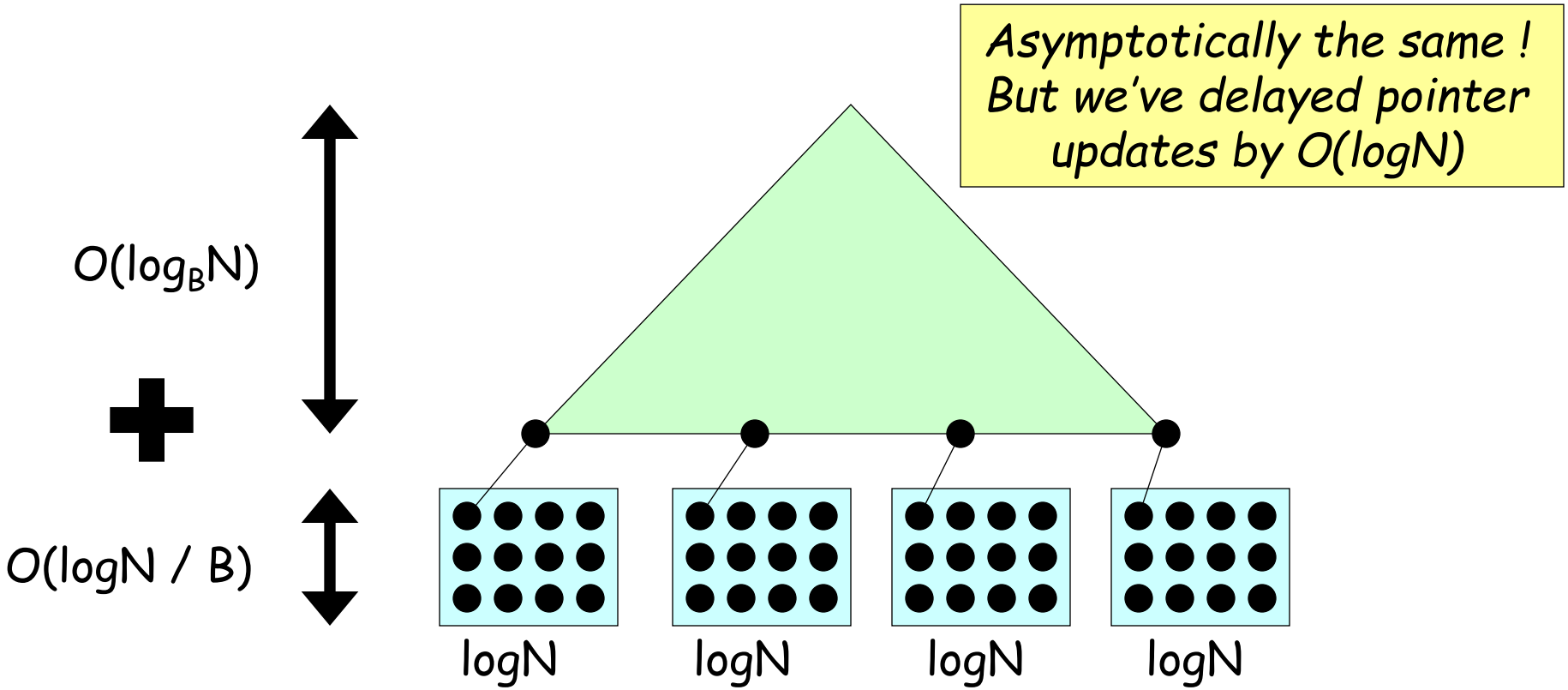
Indirection

- Idea – pack groups of leaves together beneath a single parent



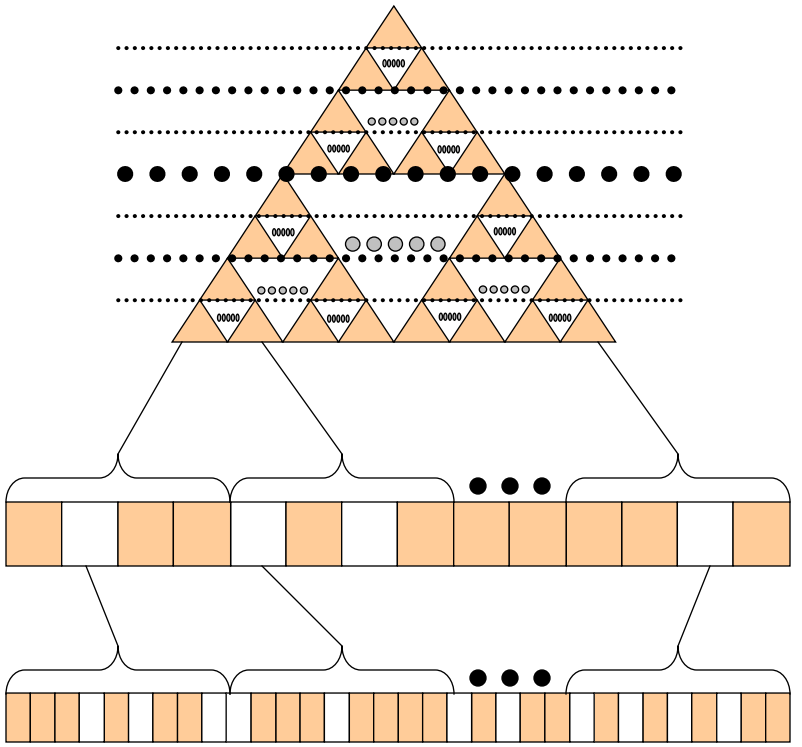
Indirection

- Searching for an element ?



Final build – two levels of indirection

- Search $O(\log_B N)$
- Scan $O(1 + k/B)$
- Updates
  - $O(\log_B N + \log^2 N / B)$



L_0 contains all the values

The end…

Presented by: Itai Lahan