1 Identification:Numerical exercise

1.1 Method 1: Equivalent DGP

This method does not involve estimation. Let the data generating process be given by

$$g\left(x_1, x_2, \epsilon|\theta\right) = \begin{pmatrix} y_1^* \\ y_2^* \\ \eta^* \\ p \end{pmatrix}.$$

To verify whether a specification is not identified I do the following exercise:

1. I draw the characteristics for say $N = 500$ upstream firms from their parametric distribution (with a fixed random seed).

$$\left\{\left(x_{1i}, x_{2i}, \epsilon_i\right)\right\}_{i=1}^{500}.$$

2. For a fixed parametric specification of the valuation functions, and for a given vector of its parameters and type distribution parameters (all summarized in θ), I find the equilibrium matches and prices

$$\left\{\left(y_{1i}^*, y_{2i}^*, \eta_i^*\right), p_i\right\}_{i=1}^{500}.$$

3. To find whether there is a different vector of parameters resulting in the same data generating process:

   (a) Compute the equilibrium matches and prices for the same draw of characteristics in (1), i.e. the same random seed, under parameter vector $\bar{\theta} \neq \theta$, denoted by

$$\left\{\left(\bar{y}_{1i}^*, \bar{y}_{2i}^*, \bar{\eta}_i^*\right), \bar{p}_i\right\}_{i=1}^{500}.$$

   (b) Search for $\bar{\theta} \neq \theta$ such that

$$\begin{pmatrix} y_{1i}^* \\ y_{2i}^* \\ \eta_i^* \\ p_i \end{pmatrix} = \begin{pmatrix} \bar{y}_{1i}^* \\ \bar{y}_{2i}^* \\ \bar{\eta}_i^* \\ \bar{p}_i \end{pmatrix},$$

   for every i. In practice, we minimize the error

$$\left(\sum_{i=1}^{500}\left[\left(y_{1i}^* - \bar{y}_{1i}^*\right)^2 + \left(y_{2i}^* - \bar{y}_{2i}^*\right)^2 + \left(\eta_i^* - \bar{\eta}_i^*\right)^2 + \left(p_i - \bar{p}_i\right)^2\right]\right)^{\frac{1}{2}}.$$

   (c) The expression in (b) is equal to zero at true parameters θ by construction. If for a given specification and its free parameters there is another vector of parameters $\bar{\theta} \neq \theta$ for which the expression in (b) is equal to zero, then that specification is not identified.

1.2 Method 2: Observationally Equivalent

We start by generating a fake dataset of equilibrium matches and prices under the true parameter values

$$\{x_{1i}, x_{2i}, y_{1i}^*, y_{2i}^*, p_i\}_{i=1}^{500}.$$

We estimate the parameter vector $\hat{\theta}^{SML}$ using the simulated maximum likelihood method and evaluate the conditional log-likehood at $\hat{\theta}^{SML}$, denoted by $L\left(\hat{\theta}^{SML}\right)$:

1. For a given vector of parameters $\hat{\theta}$, We simulate 100 markets

$$\left\{(x_{1i}, x_{2i})_{i=1}^{500}, (\epsilon_i^s)_{i=1}^{500}, (y_{1i}, y_{2i})_{i=1}^{500}, (\eta_i^s)_{i=1}^{500}\right\}_{s=1}^{100}$$

   and solve for their equilibrium matching and prices under the parameter vector $\hat{\theta}$.

$$\left\{(x_{1i}, x_{2i})_{i=1}^{500}, (y_{1i}^{*s}, y_{2i}^{*s})_{i=1}^{500}, (p_i^s)_{i=1}^{500}\right\}_{s=1}^{100}$$

   Only the unobservables ϵ and η are different across these 100 markets; x and y do not vary with s, however y^* depends on s as it is an equilibrium outcome and depends on the draw of unobservables. The observable types are fixed at the observed types in the fake dataset.

   (a) Equilibrium prices are unique up to an additive constant. In each simulation we add a constant c to all prices so that the mean of the prices in the simulation is the same as the prices in the data.

2. Evaluate the conditional log-likelihood using the 100 simulated markets in (1).

$$L(\boldsymbol{\theta}) = \frac{1}{500}\sum_{i=1}^{500}\log\left(\frac{1}{100h_{y_1}h_{y_2}h_p}\sum_{s=1}^{100}\phi\left(\frac{y_{1i}^* - y_{1i}^{*s}(\boldsymbol{\theta})}{h_{y_1}}\right)\cdot\phi\left(\frac{y_{2i}^* - y_{2i}^{*s}(\boldsymbol{\theta})}{h_{y_2}}\right).\phi\left(\frac{p_i - p_i^s(\boldsymbol{\theta})}{h_p}\right)\right).$$

3. Start the optimizer from a random starting point. Search over all possible parameter values to find a likelihood maximizer $\hat{\theta}^{SML}$.

4. Start the optimizer from the true parameters to get $\hat{\theta}^*$.

5. Compare $\hat{\theta}^{SML}$ with $\hat{\theta}^*$. If $L\left(\hat{\theta}^{SML}\right) = L\left(\hat{\theta}^*\right)$, i.e. $\hat{\theta}^{SML}$ is a global maximizer, and $\hat{\theta}^{SML} \neq \hat{\theta}^*$, the specification is not identified.

It is possible for Method 1 to not show that the model is not identified, but Method 2 shows that the model is unidentified. However, if Method 1 shows that the model is not identified, then Method 2 should also be able to show non-identification.

2 Different Specifications

2.1 Specification 1 (Identified)

Features: Non-separable production function. No coefficient on the terms including unobservables. Standard deviations of unobservables are not fixed.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+x_2y_2+x_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\left(1-\beta^u_{22}\right)x_2y_2+\epsilon\eta$$

$$\sigma^2_\epsilon=?,\sigma^2_\eta=?$$

Method 1 does not show non-identification.

Method 2 cannot find other global maximizers of the likelihood function other than the true parameters.

2.2 Specification 2(Identified)

We impose the exclusion restriction that x_2y_2 does not enter the downstream valuation function. Further, we normalize the unobservable standard deviations.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta+\beta^d_{33}\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\beta^d_{33}\epsilon\eta$$

$$\sigma^2_\epsilon=1,\sigma^2_\eta=1$$

Method 1 does not show non-identification.

Method 2 cannot find other maximizers of the likelihood function other than the true parameters.

2.3 Specification 3(Identified)

This specification replaces the $x_1\eta$ term in upstream valuation in section 2.2 with $y_1\epsilon$. It turns out that this model specification is identified. The unobservable term in upstream's valuation can be interpreted as the unobserved preference of upstream firm for y_1.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}^u x_2y_2+\beta_{31}^u y_1\epsilon+\beta_{33}^d\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^u x_1y_1+\beta_{12}^u x_1y_2+\beta_{21}^u x_2y_1+\beta_{22}^u x_2y_2+\beta_{31}^u y_1\epsilon$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^d x_1y_1+\beta_{12}^d x_1y_2+\beta_{21}^d x_2y_1+\beta_{33}^d\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1 does not show non-identification.

Method 2 cannot find other maximizers of the likelihood function other than the true parameters.

2.4 Specification 4(Not Identified)

We add $y_1\epsilon$ to the downstream valuation to the identified model in Section 2.2.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}^u x_2y_2+\beta_{13}^u x_1\eta+\beta_{31}^d y_1\epsilon+\beta_{33}^d\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^u x_1y_1+\beta_{12}^u x_1y_2+\beta_{21}^u x_2y_1+\beta_{22}^u x_2y_2+\beta_{13}^u x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^d x_1y_1+\beta_{12}^d x_1y_2+\beta_{21}^d x_2y_1+\beta_{31}^d y_1\epsilon+\beta_{33}^d\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: We can find other parameters with resulting in the same DGP

Method 2: Can find other maximizers of the log-likelihood.

The first row in the table below is the true values of models parameters, the second row are the estimates found by starting the solver from the true parameters, and the third row are the estimates when solver is started from another random point. The value of objective function at the two estimates is the same.

β_{11}^u	β_{12}^u	β_{21}^u	β_{22}^u	β_{11}^d	β_{12}^d	β_{21}^d	β_{13}^u	β_{31}^d	β_{33}^d	Log-likelihood
1	1.5	.5	2.5	2.5	-2	1	1.5	0.75	1	
0.93	1.5	0.495	2.496	2.515	-2.0	0.984	1.583	0.802	1.067	1.039943360
0.567	1.492	0.522	2.469	2.001	-1.934	0.755	1.122	0.611	0.836	1.039943360

2.5 Specification 5(Not Identified)

We change the specification in section 2.4 by moving the $y_1\epsilon$ term to the upstream valuation. The model is still not identified.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}^{u}x_2y_2+\beta_{13}^{u}x_1\eta+\beta_{31}^{u}y_1\epsilon+\beta_{33}^{d}\epsilon\eta$$

$$\Phi^{u}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{u}x_1y_1+\beta_{12}^{u}x_1y_2+\beta_{21}^{u}x_2y_1+\beta_{22}^{u}x_2y_2+\beta_{31}^{u}y_1\epsilon+\beta_{13}^{u}x_1\eta$$

$$\Phi^{d}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{d}x_1y_1+\beta_{12}^{d}x_1y_2+\beta_{21}^{d}x_2y_1+\beta_{33}^{d}\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: We can find other parameters with resulting in the same DGP
Method 2: Can find other maximizers of the log-likelihood.

2.6 Specification 6 (Not Identified)

Features: Includes coefficient on $x_1\eta$. Flexible standard deviations.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+x_2y_2+\beta_{13}^{u}x_1\eta+\epsilon\eta$$

$$\Phi^{u}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{u}x_1y_1+\beta_{12}^{u}x_1y_2+\beta_{21}^{u}x_2y_1+\beta_{22}^{u}x_2y_2+\beta_{13}^{u}x_1\eta$$

$$\Phi^{d}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{d}x_1y_1+\beta_{12}^{d}x_1y_2+\beta_{21}^{d}x_2y_1+\left(1-\beta_{22}^{u}\right)x_2y_2+\epsilon\eta$$

$$\sigma_\epsilon^2=?,\sigma_\eta^2=?$$

Method 1: $\beta_{13}^u,\sigma_\epsilon^2,\sigma_\eta^2$ are not identified, i.e. there are other combinations of $\beta_{13}^u,\sigma_\epsilon^2,\sigma_\eta^2$ different from the true values resulting in the same DGP.

Method 2: It confirms that $\beta_{13}^u,\sigma_\epsilon^2,\sigma_\eta^2$ are not identified. It can correctly estimate the other parameters.

2.7 Specification 7 (Not Identified)

Features: Coefficient on x_2y_2 in the match production function is not normalized.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+x_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^ux_1y_1+\beta_{12}^ux_1y_2+\beta_{21}^ux_2y_1+\beta_{22}^ux_2y_2+x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^dx_1y_1+\beta_{12}^dx_1y_2+\beta_{21}^dx_2y_1+\beta_{22}^dx_2y_2+\epsilon\eta$$

$$\sigma_\epsilon^2=?,\sigma_\eta^2=?$$

Method 1: β coefficients are not identified.
Method 2 : σ_ϵ^2 and σ_η^2 are correctly estimated. β coefficients are not.
Even if we fix the standard deviations, the β coefficients are not identified.

2.8 Specification 8 (Identified)

Features: coefficient on the non-separable term and flexible standard deviation of ϵ, but fixing the standard deviation of η.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+x_2y_2+\beta_{13}^ux_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^ux_1y_1+\beta_{12}^ux_1y_2+\beta_{21}^ux_2y_1+\beta_{22}^ux_2y_2+\beta_{13}^ux_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^dx_1y_1+\beta_{12}^dx_1y_2+\beta_{21}^dx_2y_1+\left(1-\beta_{22}^u\right)x_2y_2+\epsilon\eta$$

$$\sigma_\epsilon^2=?,\sigma_\eta^2=1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.9 Specification 9 (Identified)

Features: similar to specification 8, but normalizing σ_ϵ^2 instead of σ_η^2.

$$\Phi\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11} x_1 y_1 + \beta_{12} x_1 y_2 + \beta_{21} x_2 y_1 + x_2 y_2 + \beta_{13}^u x_1 \eta + \epsilon \eta$$

$$\Phi^u\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta$$

$$\Phi^d\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + \left(1 - \beta_{22}^u\right) x_2 y_2 + \epsilon \eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = ?$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.10 Specification 10 (Not Identified)

Features: similar to specification 9, but allowing for extra coefficient on $\epsilon\eta$.

$$\Phi\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11} x_1 y_1 + \beta_{12} x_1 y_2 + \beta_{21} x_2 y_1 + \beta_{22} x_2 y_2 + \beta_{13}^u x_1 \eta + \epsilon \eta$$

$$\Phi^u\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta$$

$$\Phi^d\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + \left(1 - \beta_{22}^u\right) x_2 y_2 + \left(1 - \beta_{33}^u\right) \epsilon \eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = ?$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.11 Specification 11 (Not Identified)

Features: similar to 8 adding another possible non-separable term, i.e. $y_1\epsilon$.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^{u}x_1\eta+\beta_{31}^{d}y_1\epsilon+\epsilon\eta$$

$$\Phi^{u}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{u}x_1y_1+\beta_{12}^{u}x_1y_2+\beta_{21}^{u}x_2y_1+\beta_{22}^{u}x_2y_2+\beta_{13}^{u}x_1\eta$$

$$\Phi^{d}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{d}x_1y_1+\beta_{12}^{d}x_1y_2+\beta_{21}^{d}x_2y_1+\left(1-\beta_{22}^{u}\right)x_2y_2+\beta_{31}^{d}y_1\epsilon+\epsilon\eta$$

$$\sigma_\epsilon^2=?,\sigma_\eta^2=1$$

Method 1: We can find other parameters with resulting in the same DGP
Method 2: Can find other maximizers of the log-likelihood.

2.12 Specification 12 (Identified)

Features: coefficient on all unobservable terms but fixed standard deviations.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^{u}x_1\eta+\beta_{31}^{d}y_1\epsilon+\beta_{33}^{d}\epsilon\eta$$

$$\Phi^{u}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{u}x_1y_1+\beta_{12}^{u}x_1y_2+\beta_{21}^{u}x_2y_1+\beta_{22}^{u}x_2y_2+\beta_{13}^{u}x_1\eta$$

$$\Phi^{d}\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{d}x_1y_1+\beta_{12}^{d}x_1y_2+\beta_{21}^{d}x_2y_1+\left(1-\beta_{22}^{u}\right)x_2y_2+\beta_{33}^{d}\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.13 Specification 13 (Identified)

Features: similar to 8 but fixing both standard deviations.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^{u}x_1\eta+\beta_{31}^{d}y_1\epsilon+\epsilon\eta$$

$$\Phi^u(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta^u_{11}x_1y_1 + \beta^u_{12}x_1y_2 + \beta^u_{21}x_2y_1 + \beta^u_{22}x_2y_2 + \beta^u_{13}x_1\eta$$

$$\Phi^d(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta^d_{11}x_1y_1 + \beta^d_{12}x_1y_2 + \beta^d_{21}x_2y_1 + (1-\beta^u_{22})\,x_2y_2 + \beta^d_{31}y_1\epsilon + \epsilon\eta$$

$$\sigma^2_\epsilon = 1, \sigma^2_\eta = 1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.14 Specification 14 (Not Identified)

Adding an extra coefficient on $\epsilon\eta$ term.

$$\Phi(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}x_1y_1 + \beta_{12}x_1y_2 + \beta_{21}x_2y_1 + \beta_{22}x_2y_2 + \beta^u_{13}x_1\eta + \beta^d_{31}y_1\epsilon + \beta^d_{33}\epsilon\eta$$

$$\Phi^u(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta^u_{11}x_1y_1 + \beta^u_{12}x_1y_2 + \beta^u_{21}x_2y_1 + \beta^u_{22}x_2y_2 + \beta^u_{13}x_1\eta$$

$$\Phi^d(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta^d_{11}x_1y_1 + \beta^d_{12}x_1y_2 + \beta^d_{21}x_2y_1 + (1-\beta^u_{22})\,x_2y_2 + \beta^d_{31}y_2\epsilon + \beta^d_{33}\epsilon\eta$$

$$\sigma^2_\epsilon = 1, \sigma^2_\eta = 1$$

Method 1: We can find other parameters with resulting in the same DGP

Method 2: Can find other maximizers of the log-likelihood.

2.15 Specification 15 (Not Identified)

Feature: coefficient on all unobservables.

$$\Phi(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}x_1y_1 + \beta_{12}x_1y_2 + \beta_{21}x_2y_1 + \beta_{22}x_2y_2 + \beta^u_{13}x_1\eta + \beta^d_{31}y_1\epsilon + \beta_{33}\epsilon\eta$$

$$\Phi^u(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta^u_{11}x_1y_1 + \beta^u_{12}x_1y_2 + \beta^u_{21}x_2y_1 + \beta^u_{22}x_2y_2 + \beta^u_{13}x_1\eta + \beta^u_{33}\epsilon\eta$$

$$\Phi^d(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta^d_{11}x_1y_1 + \beta^d_{12}x_1y_2 + \beta^d_{21}x_2y_1 + (1-\beta^u_{22})\,x_2y_2 + \beta^d_{33}\epsilon\eta$$

$$\sigma^2_\epsilon = 1, \sigma^2_\eta = 1$$

Method 1: We can find other parameters with resulting in the same DGP

Method 2: Can find other maximizers of the log-likelihood.

2.16 Specification 16 (Identified)

Adding an extra coefficient on $\epsilon\eta$ term on both the upstream and the downstream firms.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^u x_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^u x_1y_1+\beta_{12}^u x_1y_2+\beta_{21}^u x_2y_1+\beta_{22}^u x_2y_2+\beta_{13}^u x_1\eta+\beta_{33}^u\epsilon\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^d x_1y_1+\beta_{12}^d x_1y_2+\beta_{21}^d x_2y_1+\left(1-\beta_{22}^u\right)x_2y_2+\left(1-\beta_{33}^u\right)\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.17 Specification 17 (Identified)

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^u x_1\eta+\beta_{33}^u\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^u x_1y_1+\beta_{12}^u x_1y_2+\beta_{21}^u x_2y_1+\beta_{22}^u x_2y_2+\beta_{13}^u x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^d x_1y_1+\beta_{12}^d x_1y_2+\beta_{21}^d x_2y_1+\left(1-\beta_{22}^u\right)x_2y_2+\beta_{33}^u\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.18 Specification 18 (Not Identified)

Adding an extra non separable term $y_1\epsilon$ with a coefficient to downstream valuation.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^u x_1\eta+\beta_{31}^d y_1\epsilon+\epsilon\eta$$

$$\Phi^u(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta + \beta_{33}^u \epsilon \eta$$

$$\Phi^d(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + (1 - \beta_{22}^u) x_2 y_2 + \beta_{31}^d y_1 \epsilon + (1 - \beta_{33}^u) \epsilon \eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = 1$$

Method 1: We can find other parameters with resulting in the same DGP
Method 2: Can find other maximizers of the log-likelihood.