1 Identification:Numerical exercise

1.1 Method 1: Equivalent DGP

This method does not involve estimation. Let the data generating process be given by

$$g\left(x_1, x_2, \epsilon|\theta\right) = \begin{pmatrix} y_1^* \\ y_2^* \\ \eta^* \\ p \end{pmatrix}.$$

To verify whether a specification is not identified I do the following exercise:

1. I draw the characteristics for say $N = 500$ upstream firms from their parametric distribution (with a fixed random seed).

$$\left\{(x_{1i}, x_{2i}, \epsilon_i)\right\}_{i=1}^{500}.$$

2. For a fixed parametric specification of the valuation functions, and for a given vector of its parameters and type distribution parameters (all summarized in θ), I find the equilibrium matches and prices

$$\left\{\left(y_{1i}^*, y_{2i}^*, \eta_i^*\right), p_i\right\}_{i=1}^{500}.$$

3. To find whether there is a different vector of parameters resulting in the same data generating process:

   (a) Compute the equilibrium matches and prices for the same draw of characteristics in (1), i.e. the same random seed, under parameter vector $\bar{\theta} \neq \theta$, denoted by

   $$\left\{\left(\bar{y}_{1i}^*, \bar{y}_{2i}^*, \bar{\eta}_i^*\right), \bar{p}_i\right\}_{i=1}^{500}.$$

   (b) Search for $\bar{\theta} \neq \theta$ such that

   $$\begin{pmatrix} y_{1i}^* \\ y_{2i}^* \\ \eta_i^* \\ p_i \end{pmatrix} = \begin{pmatrix} \bar{y}_{1i}^* \\ \bar{y}_{2i}^* \\ \bar{\eta}_i^* \\ \bar{p}_i \end{pmatrix},$$

   for every i. In practice, we minimize the error

   $$\left(\sum_{i=1}^{500}\left[\left(y_{1i}^* - \bar{y}_{1i}^*\right)^2 + \left(y_{2i}^* - \bar{y}_{2i}^*\right)^2 + \left(\eta_i^* - \bar{\eta}_i^*\right)^2 + \left(p_i - \bar{p}_i\right)^2\right]\right)^{\frac{1}{2}}.$$

   (c) The expression in (b) is equal to zero at true parameters θ by construction. If for a given specification and its free parameters there is another vector of parameters $\bar{\theta} \neq \theta$ for which the expression in (b) is equal to zero, then that specification is not identified.

1.2 Method 2: Observationally Equivalent

We start by generating a fake dataset of equilibrium matches and prices under the true parameter values

$$\{x_{1i}, x_{2i}, y^*_{1i}, y^*_{2i}, p_i\}_{i=1}^{500}.$$

We estimate the parameter vector $\hat{\theta}^{SML}$ using the simulated maximum likelihood method and evaluate the conditional log-likehood at $\hat{\theta}^{SML}$, denoted by $L\left(\hat{\theta}^{SML}\right)$:

1. For a given vector of parameters $\hat{\theta}$, We simulate 100 markets

   $$\left\{(x_{1i}, x_{2i})_{i=1}^{500}, (\epsilon_i^s)_{i=1}^{500}, (y_{1i}, y_{2i})_{i=1}^{500}, (\eta_i^s)_{i=1}^{500}\right\}_{s=1}^{100}$$

   and solve for their equilibrium matching and prices under the parameter vector $\hat{\theta}$.

   $$\left\{(x_{1i}, x_{2i})_{i=1}^{500}, (y_{1i}^{*s}, y_{2i}^{*s})_{i=1}^{500}, (p_i^s)_{i=1}^{500}\right\}_{s=1}^{100}$$

   Only the unobservables ϵ and η are different across these 100 markets; x and y do not vary with s, however y^* depends on s as it is an equilibrium outcome and depends on the draw of unobservables. The observable types are fixed at the observed types in the fake dataset.

   (a) Equilibrium prices are unique up to an additive constant. In each simulation we add a constant c to all prices so that the mean of the prices in the simulation is the same as the prices in the data.

2. Evaluate the conditional log-likelihood using the 100 simulated markets in (1).

   $$L(\boldsymbol{\theta}) = \frac{1}{500}\sum_{i=1}^{500}\log\left(\frac{1}{100 h_{y_1} h_{y_2} h_p}\sum_{s=1}^{100}\phi\left(\frac{y^*_{1i} - y^{*s}_{1i}(\boldsymbol{\theta})}{h_{y_1}}\right)\cdot\phi\left(\frac{y^*_{2i} - y^{*s}_{2i}(\boldsymbol{\theta})}{h_{y_2}}\right).\phi\left(\frac{p_i - p_i^s(\boldsymbol{\theta})}{h_p}\right)\right).$$

3. Start the optimizer from a random starting point. Search over all possible parameter values to find a likelihood maximizer $\hat{\theta}^{SML}$.

4. Start the optimizer from the true parameters to get $\hat{\theta}^*$.

5. Compare $\hat{\theta}^{SML}$ with $\hat{\theta}^*$. If $L\left(\hat{\theta}^{SML}\right) = L\left(\hat{\theta}^*\right)$, i.e. $\hat{\theta}^{SML}$ is a global maximizer, and $\hat{\theta}^{SML} \neq \hat{\theta}^*$, the specification is not identified.

It is possible for Method 1 to not show that the model is not identified, but Method 2 shows that the model is unidentified. However, if Method 1 shows that the model is not identified, then Method 2 should also be able to show non-identification.

2 Different Specifications

2.1 Specification 1 (Identified)

Features: Non-separable production function. No coefficient on the terms including unobservables. Standard deviations of unobservables are not fixed.

$$\Phi\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}x_1y_1 + \beta_{12}x_1y_2 + \beta_{21}x_2y_1 + x_2y_2 + x_1\eta + \epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^u x_1y_1 + \beta_{12}^u x_1y_2 + \beta_{21}^u x_2y_1 + \beta_{22}^u x_2y_2 + x_1\eta$$

$$\Phi^d\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^d x_1y_1 + \beta_{12}^d x_1y_2 + \beta_{21}^d x_2y_1 + \left(1 - \beta_{22}^u\right) x_2y_2 + \epsilon\eta$$

$$\sigma_\epsilon^2 = ?, \sigma_\eta^2 = ?$$

Method 1 does not show non-identification.

Method 2 cannot find other global maximizers of the likelihood function other than the true parameters.

2.2 Specification 2(Identified)

We impose the exclusion restriction that x_2y_2 does not enter the downstream valuation function. Further, we normalize the unobservable standard deviations.

$$\Phi\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}x_1y_1 + \beta_{12}x_1y_2 + \beta_{21}x_2y_1 + \beta_{22}^u x_2y_2 + \beta_{13}^u x_1\eta + \beta_{33}^d \epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^u x_1y_1 + \beta_{12}^u x_1y_2 + \beta_{21}^u x_2y_1 + \beta_{22}^u x_2y_2 + \beta_{13}^u x_1\eta$$

$$\Phi^d\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^d x_1y_1 + \beta_{12}^d x_1y_2 + \beta_{21}^d x_2y_1 + \beta_{33}^d \epsilon\eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = 1$$

Method 1 does not show non-identification.

Method 2 cannot find other maximizers of the likelihood function other than the true parameters.

2.3 Specification 3(Identified)

This specification replaces the $x_1\eta$ term in upstream valuation in section 2.2 with $y_1\epsilon$. It turns out that this model specification is identified. The unobservable term in upstream's valuation can be interpreted as the unobserved preference of upstream firm for y_1.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{31}y_1\epsilon+\beta^d_{33}\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{31}y_1\epsilon$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\beta^d_{33}\epsilon\eta$$

$$\sigma^2_\epsilon=1,\sigma^2_\eta=1$$

Method 1 does not show non-identification.

Method 2 cannot find other maximizers of the likelihood function other than the true parameters.

2.4 Specification 4(Not Identified)

We add $y_1\epsilon$ to the downstream valuation to the identified model in Section 2.2.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta+\beta^d_{31}y_1\epsilon+\beta^d_{33}\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\beta^d_{31}y_1\epsilon+\beta^d_{33}\epsilon\eta$$

$$\sigma^2_\epsilon=1,\sigma^2_\eta=1$$

Method 1: We can find other parameters with resulting in the same DGP

Method 2: Can find other maximizers of the log-likelihood.

The first row in the table below is the true values of models parameters, the second row are the estimates found by starting the solver from the true parameters, and the third row are the estimates when solver is started from another random point. The value of objective function at the two estimates is the same.

β^u_{11}	β^u_{12}	β^u_{21}	β^u_{22}	β^d_{11}	β^d_{12}	β^d_{21}	β^u_{13}	β^d_{31}	β^d_{33}	Log-likelihood
1	1.5	.5	2.5	2.5	-2	1	1.5	0.75	1	
0.93	1.5	0.495	2.496	2.515	-2.0	0.984	1.583	0.802	1.067	1.039943360
0.567	1.492	0.522	2.469	2.001	-1.934	0.755	1.122	0.611	0.836	1.039943360

2.5 Specification 5(Not Identified)

We change the specification in section 2.4 by moving the $y_1\epsilon$ term to the upstream valuation. The model is still not identified.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta+\beta^u_{31}y_1\epsilon+\beta^d_{33}\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{31}y_1\epsilon+\beta^u_{13}x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\beta^d_{33}\epsilon\eta$$

$$\sigma^2_\epsilon=1,\sigma^2_\eta=1$$

Method 1: We can find other parameters with resulting in the same DGP
Method 2: Can find other maximizers of the log-likelihood.

2.6 Specification 6 (Not Identified)

Features: Includes coefficient on $x_1\eta$. Flexible standard deviations.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+x_2y_2+\beta^u_{13}x_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\left(1-\beta^u_{22}\right)x_2y_2+\epsilon\eta$$

$$\sigma^2_\epsilon=?,\sigma^2_\eta=?$$

Method 1: $\beta^u_{13},\sigma^2_\epsilon,\sigma^2_\eta$ are not identified, i.e. there are other combinations of $\beta^u_{13},\sigma^2_\epsilon,\sigma^2_\eta$ different from the true values resulting in the same DGP.

Method 2: It confirms that $\beta^u_{13},\sigma^2_\epsilon,\sigma^2_\eta$ are not identified. It can correctly estimate the other parameters.

2.7 Specification 7 (Not Identified)

Features: Coefficient on x_2y_2 in the match production function is not normalized.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+x_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\beta^d_{22}x_2y_2+\epsilon\eta$$

$$\sigma^2_\epsilon=?,\sigma^2_\eta=?$$

Method 1: β coefficients are not identified.
Method 2 : σ^2_ϵ and σ^2_η are correctly estimated. β coefficients are not.
Even if we fix the standard deviations, the β coefficients are not identified.

2.8 Specification 8 (Identified)

Features: coefficient on the non-separable term and flexible standard deviation of ϵ, but fixing the standard deviation of η.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+x_2y_2+\beta^u_{13}x_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\left(1-\beta^u_{22}\right)x_2y_2+\epsilon\eta$$

$$\sigma^2_\epsilon=?,\sigma^2_\eta=1$$

Method 1: does not show non-identification.
Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.9 Specification 9 (Identified)

Features: similar to specification 8, but normalizing σ_ϵ^2 instead of σ_η^2.

$$\Phi\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11} x_1 y_1 + \beta_{12} x_1 y_2 + \beta_{21} x_2 y_1 + x_2 y_2 + \beta_{13}^u x_1 \eta + \epsilon \eta$$

$$\Phi^u\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta$$

$$\Phi^d\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + \left(1 - \beta_{22}^u\right) x_2 y_2 + \epsilon \eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = ?$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.10 Specification 10 (Not Identified)

Features: similar to specification 9, but allowing for extra coefficient on $\epsilon\eta$.

$$\Phi\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11} x_1 y_1 + \beta_{12} x_1 y_2 + \beta_{21} x_2 y_1 + \beta_{22} x_2 y_2 + \beta_{13}^u x_1 \eta + \epsilon \eta$$

$$\Phi^u\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta$$

$$\Phi^d\left(\boldsymbol{x}, \boldsymbol{y}, \epsilon, \eta\right) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + \left(1 - \beta_{22}^u\right) x_2 y_2 + \left(1 - \beta_{33}^u\right) \epsilon \eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = ?$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.11 Specification 11 (Not Identified)

Features: similar to 8 adding another possible non-separable term, i.e. $y_1\epsilon$.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^{u}x_1\eta+\beta_{31}^{d}y_1\epsilon+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{u}x_1y_1+\beta_{12}^{u}x_1y_2+\beta_{21}^{u}x_2y_1+\beta_{22}^{u}x_2y_2+\beta_{13}^{u}x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{d}x_1y_1+\beta_{12}^{d}x_1y_2+\beta_{21}^{d}x_2y_1+\left(1-\beta_{22}^{u}\right)x_2y_2+\beta_{31}^{d}y_1\epsilon+\epsilon\eta$$

$$\sigma_\epsilon^2=?,\sigma_\eta^2=1$$

Method 1: We can find other parameters with resulting in the same DGP
Method 2: Can find other maximizers of the log-likelihood.

2.12 Specification 12 (Identified)

Features: coefficient on all unobservable terms but fixed standard deviations.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^{u}x_1\eta+\beta_{31}^{d}y_1\epsilon+\beta_{33}^{d}\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{u}x_1y_1+\beta_{12}^{u}x_1y_2+\beta_{21}^{u}x_2y_1+\beta_{22}^{u}x_2y_2+\beta_{13}^{u}x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^{d}x_1y_1+\beta_{12}^{d}x_1y_2+\beta_{21}^{d}x_2y_1+\left(1-\beta_{22}^{u}\right)x_2y_2+\beta_{33}^{d}\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: does not show non-identification.
Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.13 Specification 13 (Identified)

Features: similar to 8 but fixing both standard deviations.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^{u}x_1\eta+\beta_{31}^{d}y_1\epsilon+\epsilon\eta$$

$$\Phi^u(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta$$

$$\Phi^d(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + \left(1-\beta_{22}^u\right) x_2 y_2 + \beta_{31}^d y_1 \epsilon + \epsilon\eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = 1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.14 Specification 14 (Not Identified)

Adding an extra coefficient on $\epsilon\eta$ term.

$$\Phi(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11} x_1 y_1 + \beta_{12} x_1 y_2 + \beta_{21} x_2 y_1 + \beta_{22} x_2 y_2 + \beta_{13}^u x_1 \eta + \beta_{31}^d y_1 \epsilon + \beta_{33}^d \epsilon\eta$$

$$\Phi^u(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta$$

$$\Phi^d(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + \left(1-\beta_{22}^u\right) x_2 y_2 + \beta_{31}^d y_2 \epsilon + \beta_{33}^d \epsilon\eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = 1$$

Method 1: We can find other parameters with resulting in the same DGP

Method 2: Can find other maximizers of the log-likelihood.

2.15 Specification 15 (Not Identified)

Feature: coefficient on all unobservables.

$$\Phi(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11} x_1 y_1 + \beta_{12} x_1 y_2 + \beta_{21} x_2 y_1 + \beta_{22} x_2 y_2 + \beta_{13}^u x_1 \eta + \beta_{31}^d y_1 \epsilon + \beta_{33} \epsilon\eta$$

$$\Phi^u(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}^u x_1 y_1 + \beta_{12}^u x_1 y_2 + \beta_{21}^u x_2 y_1 + \beta_{22}^u x_2 y_2 + \beta_{13}^u x_1 \eta + \beta_{33}^u \epsilon\eta$$

$$\Phi^d(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta) = \beta_{11}^d x_1 y_1 + \beta_{12}^d x_1 y_2 + \beta_{21}^d x_2 y_1 + \left(1-\beta_{22}^u\right) x_2 y_2 + \beta_{33}^d \epsilon\eta$$

$$\sigma_\epsilon^2 = 1, \sigma_\eta^2 = 1$$

Method 1: We can find other parameters with resulting in the same DGP

Method 2: Can find other maximizers of the log-likelihood.

2.16 Specification 16 (Identified)

Adding an extra coefficient on $\epsilon\eta$ term on both the upstream and the downstream firms.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^u x_1\eta+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^u x_1y_1+\beta_{12}^u x_1y_2+\beta_{21}^u x_2y_1+\beta_{22}^u x_2y_2+\beta_{13}^u x_1\eta+\beta_{33}^u\epsilon\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^d x_1y_1+\beta_{12}^d x_1y_2+\beta_{21}^d x_2y_1+\left(1-\beta_{22}^u\right)x_2y_2+(1-\beta_{33}^u)\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.17 Specification 17 (Identified)

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^u x_1\eta+\beta_{33}^u\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^u x_1y_1+\beta_{12}^u x_1y_2+\beta_{21}^u x_2y_1+\beta_{22}^u x_2y_2+\beta_{13}^u x_1\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^d x_1y_1+\beta_{12}^d x_1y_2+\beta_{21}^d x_2y_1+\left(1-\beta_{22}^u\right)x_2y_2+\beta_{33}^u\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Method 1: does not show non-identification.

Method 2: cannot find other maximizers of the likelihood function other than the true parameters.

2.18 Specification 18 (Not Identified)

Adding an extra non separable term $y_1\epsilon$ with a coefficient to downstream valuation.

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}x_2y_2+\beta_{13}^u x_1\eta+\beta_{31}^d y_1\epsilon+\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^u_{11}x_1y_1+\beta^u_{12}x_1y_2+\beta^u_{21}x_2y_1+\beta^u_{22}x_2y_2+\beta^u_{13}x_1\eta+\beta^u_{33}\epsilon\eta$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta^d_{11}x_1y_1+\beta^d_{12}x_1y_2+\beta^d_{21}x_2y_1+\left(1-\beta^u_{22}\right)x_2y_2+\beta^d_{31}y_1\epsilon+(1-\beta^u_{33})\epsilon\eta$$

$$\sigma^2_\epsilon=1,\sigma^2_\eta=1$$

Method 1: We can find other parameters with resulting in the same DGP
Method 2: Can find other maximizers of the log-likelihood.

3 Why Specification 2.5 is not identified

Recall Specification 2.5:

$$\Phi\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}x_1y_1+\beta_{12}x_1y_2+\beta_{21}x_2y_1+\beta_{22}^u x_2y_2+\beta_{13}^u x_1\eta+\beta_{31}^u y_1\epsilon+\beta_{33}^d\epsilon\eta$$

$$\Phi^u\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^u x_1y_1+\beta_{12}^u x_1y_2+\beta_{21}^u x_2y_1+\beta_{22}^u x_2y_2+\beta_{31}^u y_1\epsilon+\beta_{13}^u x_1\eta \quad (1)$$

$$\Phi^d\left(\boldsymbol{x},\boldsymbol{y},\epsilon,\eta\right)=\beta_{11}^d x_1y_1+\beta_{12}^d x_1y_2+\beta_{21}^d x_2y_1+\beta_{33}^d\epsilon\eta$$

$$\sigma_\epsilon^2=1,\sigma_\eta^2=1$$

Preliminary numerical exercise above showed that specifications with two additive error terms in individual valuation functions are not identified. This means that we are allowed to include only one such unobserved term in each of the upstream and the downstream valuation functions. In here, the upstream valuation function includes two unobserved terms, i.e. $\beta_{31}^u y_1\epsilon$ and $\beta_{13}^u\epsilon\eta$. Alternatively, the downstream valuation can include one of these interactions of the observed and unobserved types in addition to the $\beta^d\epsilon\eta$. This would result in a non-identified model again.

There is no simple proof to show non-identification using algebra. The valuation functions affect prices through a complicated non-linear relation given by the structural function which is the solution to the optimal transport problem.

The vector of parameters to be estimated is given by

$$\boldsymbol{\theta}=\left(\beta_{11}^u,\beta_{12}^u,\beta_{21}^u,\beta_{22}^u,\beta_{11}^d,\beta_{12}^d,\beta_{21}^d,\beta_{13}^u,\beta_{31}^u,\beta_{33}^d\right).$$

I start by fixing the coefficient on the unobservable term $y_1\epsilon$, i.e. β_{31}^u ,at a value different from truth. I fix all other coefficients at the true parameter values, except for $\left(\beta_{13}^u,\beta_{33}^d\right)$. Then, we search for the maximizer of the log-likelihood function over all values of the two free parameters $\left(\beta_{13}^u,\beta_{33}^d\right)$. The goal is to find out whether we can find values for the other two coefficients on the unobservable terms that lead to an observationally equivalent model. However, the value of the log-likelihood function at the maximizers is different from the one in the unconstrained maximization. This means, there no such observationally equivalent parameterization where the parameters differ only on the coefficients of the unobserved terms.

I continued doing this by increasing the number of free parameters, i.e. searching for observationally equivalent parameterization where some of the coefficients on the observable terms differ from the truth. Value of the log-likelihood function in none of these maximization problems was equal to the unconstrained problem.

The only possible way to achieve the same likelihood value in a parameterization where β_{31}^u is fixed at a different value from the truth is to search over all the remaining 9 parameters in the model. The maximizer of the likelihood maximization problem with 9 free parameters and one parameter fixed at a value different from zero seems to be unique up to the sign of the coefficients on the unobservable, i.e. multiplying -1 by all three coefficients on the unobservable terms $\left(\beta_{13}^u, \beta_{31}^u, \beta_{33}^d\right)$ results in an observationally equivalent specification.

The table below shows three observationally equivalent parameterization. The first row is when β_{31}^u is fixed at the truth. The second and the third row are computed by increasing β_{31}^u (last column) by 0.1 compared to the row above. The likelihood value is exactly the same for all three. The superscript of θ denotes the value of β_{31}^u.

	β_{11}^u	β_{12}^u	β_{21}^u	β_{22}^u	β_{11}^d	β_{12}^d	β_{21}^d	β_{13}^u	β_{33}^d	β_{31}^u
θ	0.908	1.501	0.483	2.493	2.495	-2.0	0.984	-1.072	1.569	0.556
$\theta^{.656}$	1.066	1.506	0.468	2.506	2.731	-2.031	1.09	-1.185	1.796	0.656
$\theta^{.756}$	1.224	1.51	0.453	2.518	2.967	-2.061	1.196	-1.298	2.023	0.756

The first observation is that changing β_{31}^u changes all other parameters. The second observation is that for all $i = 1, \ldots, 10$

$$\frac{\theta_i^{.656} - \theta_i}{0.1} = \frac{\theta_i^{.756} - \theta_i^{.656}}{0.1}.$$

I continued finding observationally equivalent parameterization $\theta_i^5, \theta_i^{10}, \theta_i^{50}$ by fixing β_{31}^u at 5, 10, and 50 and searching for other parameters. For each of these values, I also found the observationally equivalent parameterizations after increasing the fixed value of β_{31}^u by 0.1. I find the rate of change seems to be constant, i.e. for each element i of the vectors of parameters

$$\frac{\theta_i^{50.1} - \theta_i^{50}}{0.1} = \frac{\theta_i^{10.1} - \theta_i^{10}}{0.1} = \frac{\theta_i^{5.1} - \theta_i^{5}}{0.1} = \frac{\theta_i^{.756} - \theta_i^{.656}}{0.1}.$$

I also experimented with increments smaller and greater than 0.1 , and the rates of change seem to be constant. This means if we varied the values of β_{31}^u and found the maximizers of the likelihood function and plotted each of the 9 parameters against β_{31}^u, we will have a straight line with different slope. Therefore, for every $c \in \mathbb{R}$, the parameterization given by

$$\boldsymbol{\theta} = \left(\beta_{11}^u, \beta_{12}^u, \beta_{21}^u, \beta_{22}^u, \beta_{11}^d, \beta_{12}^d, \beta_{21}^d, \beta_{13}^u, \beta_{33}^d, \beta_{31}^u\right)$$

is equivalent to

$$\tilde{\theta} = c\left(\gamma_1\beta_{11}^u, \gamma_2\beta_{12}^u, \gamma_3\beta_{21}^u, \gamma_4\beta_{22}^u, \gamma_5\beta_{11}^d, \gamma_6\beta_{12}^d, \gamma_7\beta_{21}^d, \gamma_8\beta_{13}^u, \gamma_9\beta_{33}^d, \beta_{31}^u\right).$$

where γ parameters are the rates of changes (slopes) as explained above, which are function of the structural parameters.

4 Why 2.2 is identified

The price equation is

$$p(x_1, x_2, \epsilon) = \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}' B \cdot T \cdot \begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix} \tag{2}$$

$$= \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}' \begin{pmatrix} \beta_{11}^d - \beta_{11}^u & \beta_{12}^d - \beta_{12}^u & -\beta_{13}^u \\ \beta_{21}^d - \beta_{21}^u & -\beta_{22}^u & 0 \\ -\beta_{31}^u & 0 & \beta_{33}^d \end{pmatrix} \cdot \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}.$$

and the matching equation is

$$\begin{pmatrix} y_1 \\ y_2 \\ \eta \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}.$$

The matrix T is a non-trivial function of the following 7 parameters

$$\left(\beta_{11}^d + \beta_{11}^u\right), \left(\beta_{12}^d + \beta_{12}^u\right), \left(\beta_{21}^d + \beta_{21}^u\right), \beta_{22}^u, \beta_{13}^u, \beta_{31}^u, \beta_{33}^d,$$

and is given by

$$T = \Sigma_Y^{1/2}\left(\Sigma_Y^{1/2} A' \Sigma_X A \Sigma_Y^{1/2}\right)^{-1/2} \Sigma_Y^{1/2} A', \tag{3}$$

where

$$\Sigma_Y = \begin{pmatrix} \sigma_{y_1} & 0 & 0 \\ 0 & \sigma_{y_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Sigma_X = \begin{pmatrix} \sigma_{x_1} & 0 & 0 \\ 0 & \sigma_{x_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} \beta_{11}^d + \beta_{11}^u & \beta_{12}^d + \beta_{12}^u & \beta_{13}^u \\ \beta_{21}^d + \beta_{21}^u & \beta_{22}^u & 0 \\ \beta_{31}^u & 0 & \beta_{33}^d \end{pmatrix}.$$

The first numerical observations is that for two observationally equivalent θ and $\tilde{\theta}$ where $\theta \neq \tilde{\theta}$, the matching matrices are equal element-wise, i.e. $T = \tilde{T}$. However, the B matrices under T and $\tilde{T}$ are not the same.

Every quadratic form $x'Dx$ for some matrix D, can be equivalently represented by a symmetric matrix given by $\frac{D+D'}{2}$. Therefore, we can rewrite the equivalent form of Equation 2 as

$$p(x_1, x_2, \epsilon) = \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}' \cdot \frac{B \cdot T + T'B'}{2} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \epsilon \end{pmatrix}. \tag{4}$$

Let us define the symmetric matrix characterizing the price quadratic form as $K \equiv \frac{B \cdot T + T'B'}{2}$. It turns out that matrix K under θ and matrix $\tilde{K}$ under

$\tilde{\theta}$ are equal. K is characterized by only 6 numbers instead of 9 since it is a symmetric matrix

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{pmatrix}.$$

If we naively look at matrices K and T without paying attention they are related we can say the data on matching and prices identify:

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{pmatrix}, \text{ and } \quad \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

Each element of matrix T and K can be rewritten as a non-linear function of the coefficients given by (3) and the definition of matrix K. This could potentially give us a system of $9 + 6 = 15$ equations with only 10 unknown parameters. However, the numerical non-identification result shows that this is not the right approach and we need to look further into the structure of these matrices. $K_{11}, K_{22}, K_{33}, K_{12}, K_{13}, K_{23}$ can be expressed as a function of matrix T and $\begin{pmatrix} \beta^d_{11} - \beta^u_{11} & \beta^d_{12} - \beta^u_{12} & -\beta^u_{13} \\ \beta^d_{21} - \beta^u_{21} & -\beta^u_{22} & 0 \\ -\beta^u_{31} & 0 & \beta^d_{33} \end{pmatrix}$ to get

$$K_{11} = 2\left(\beta^d_{11} - \beta^u_{11}\right) t_{11} + 2\left(\beta^d_{12} - \beta^u_{12}\right) t_{21} - 2\beta^u_{13} t_{31}$$

$$K_{12} = \left(\beta^d_{21} - \beta^u_{21}\right) t_{11} - \beta^u_{22} t_{21} + \left(\beta^d_{11} - \beta^u_{11}\right) t_{12} + \left(\beta^d_{12} - \beta^u_{12}\right) t_{22} - \beta^u_{13} t_{32}$$

$$K_{13} = -\beta^u_{31} t_{11} + \beta^d_{33} t_{31} + \left(\beta^d_{11} - \beta^u_{11}\right) t_{13} + \left(\beta^d_{12} - \beta^u_{12}\right) t_{23} - \beta^u_{13} t_{33}$$

$$K_{22} = 2\left(\beta^d_{21} - \beta^u_{21}\right) t_{12} - 2\beta^u_{22} t_{22}$$

$$K_{23} = -\beta^u_{31} t_{12} + \beta^d_{33} t_{32} + \left(\beta^d_{21} - \beta^u_{21}\right) t_{13} - \beta^u_{22} t_{23}$$

$$K_{33} = -2\beta^u_{31} t_{13} + 2\beta^d_{33} t_{33}$$

Data on matching already identifies T. The above system of 6 equations has 7 unknowns:

$$\left(\beta^d_{11} - \beta^u_{11}\right), \left(\beta^d_{12} - \beta^u_{12}\right), \left(\beta^d_{21} - \beta^u_{21}\right), \beta^u_{22}, \beta^u_{13}, \beta^u_{31}, \beta^d_{33}.$$

We cannot uniquely solve the system of equations for these 7 variables. This is not a proof of non-identification, but might be hint.

Let us switch to the **identified specification** in (2.2) which excludes $\beta_{31}^u y_1 \epsilon$ from the upstream valuation. The elements of matrix K imply the following 6 linear equations:

$$K_{11} = 2\left(\beta_{11}^d - \beta_{11}^u\right) t_{11} + 2\left(\beta_{12}^d - \beta_{12}^u\right) t_{21} - 2\beta_{13}^u t_{31}$$

$$K_{12} = \left(\beta_{21}^d - \beta_{21}^u\right) t_{11} - \beta_{22}^u t_{21} + \left(\beta_{11}^d - \beta_{11}^u\right) t_{12} + \left(\beta_{12}^d - \beta_{12}^u\right) t_{22} - \beta_{13}^u t_{32}$$

$$K_{13} = \beta_{33}^d t_{31} + \left(\beta_{11}^d - \beta_{11}^u\right) t_{13} + \left(\beta_{12}^d - \beta_{12}^u\right) t_{23} - \beta_{13}^u t_{33}$$

$$K_{22} = 2\left(\beta_{21}^d - \beta_{21}^u\right) t_{12} - 2\beta_{22}^u t_{22}$$

$$K_{23} = \beta_{33}^d t_{32} + \left(\beta_{21}^d - \beta_{21}^u\right) t_{13} - \beta_{22}^u t_{23}$$

$$K_{33} = 2\beta_{33}^d t_{33}$$

Since the elements of T are identified, we can uniquely solve the above system of 6 linear equations for

$$\left(\beta_{11}^d - \beta_{11}^u\right), \left(\beta_{12}^d - \beta_{12}^u\right), \left(\beta_{21}^d - \beta_{21}^u\right), \beta_{22}^u, \beta_{13}^u, \beta_{33}^d.$$

This already proves that the structural parameters $\beta_{22}^u, \beta_{13}^u, \beta_{33}^d$ are identified. However, this does not separately identify the coefficients of valuation functions, but only their differences.

The matrix T can be written as

$$\begin{aligned} T &= \overbrace{\Sigma_Y^{1/2}\left(\Sigma_Y^{1/2} A' \Sigma_X A \Sigma_Y^{1/2}\right)^{-1/2} \Sigma_Y^{1/2}}^{H} A' \\ &= \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{pmatrix} \begin{pmatrix} \beta_{11}^d + \beta_{11}^u & \beta_{21}^d + \beta_{21}^u & \beta_{13}^u \\ \beta_{12}^d + \beta_{12}^u & \beta_{22}^u & 0 \\ 0 & 0 & \beta_{33}^d \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}. \end{aligned} \tag{5}$$

Equation (5) exploits the structure of matrix T to express it as the product of a symmetric matrix H and the matrix A which is the sum of upstream and downstream coefficients. Since H is a symmetric matrix, it is only characterized by 6 parameters. Recall, that all 9 elements of matrix T are identified from data on matching, and further the three parameters $\beta_{22}^u, \beta_{13}^u, \beta_{33}^d$ are identified from the argument above using the price equation. Equation (5) defines a system of 9 equations (not all are linear):

$$\begin{aligned} H_{11}\left(\beta_{11}^{u}+\beta_{11}^{d}\right)+H_{12}\left(\beta_{12}^{u}+\beta_{12}^{d}\right) &= t_{11} \\ H_{11}\left(\beta_{21}^{u}+\beta_{21}^{d}\right)+H_{12}\beta_{22}^{u} &= t_{12} \\ H_{11}\beta_{13}^{u}+H_{13}\beta_{33}^{d} &= t_{13} \qquad (6) \\ H_{12}\left(\beta_{11}^{u}+\beta_{11}^{d}\right)+H_{22}\left(\beta_{12}^{u}+\beta_{12}^{d}\right) &= t_{21} \\ H_{12}\left(\beta_{21}^{u}+\beta_{21}^{d}\right)+H_{22}\beta_{22}^{u} &= t_{22} \\ H_{12}\beta_{13}^{u}+H_{23}\beta_{33}^{d} &= t_{23} \qquad (7) \\ H_{13}\left(\beta_{11}^{u}+\beta_{11}^{d}\right)+H_{23}\left(\beta_{12}^{u}+\beta_{12}^{d}\right) &= t_{31} \\ H_{13}\left(\beta_{21}^{u}+\beta_{21}^{d}\right)+H_{23}\beta_{22}^{u} &= t_{32} \\ H_{13}\beta_{13}^{u}+H_{33}\beta_{33}^{d} &= t_{33} \qquad (8) \end{aligned}$$

There are 9 unknowns in the above equations:

$$H_{11}, H_{12}, H_{13}, H_{22}, H_{23}, H_{33}, \left(\beta_{21}^{u}+\beta_{21}^{d}\right), \left(\beta_{12}^{u}+\beta_{12}^{d}\right), \left(\beta_{12}^{d}+\beta_{12}^{u}\right).$$

Under conditions that I have to further investigate, we can solve the system of 9 equations for the 9 unknowns listed above.

Collecting the two intermediate identification arguments above using the price equation and the matching equation, we have identified

$$\left(\beta_{21}^{u}+\beta_{21}^{d}\right), \left(\beta_{12}^{u}+\beta_{12}^{d}\right), \left(\beta_{12}^{d}+\beta_{12}^{u}\right), \left(\beta_{11}^{d}-\beta_{11}^{u}\right), \left(\beta_{12}^{d}-\beta_{12}^{u}\right), \left(\beta_{21}^{d}-\beta_{21}^{u}\right), \beta_{22}^{u}, \beta_{13}^{u}, \beta_{33}^{d}.$$

Since we have identified $\left(\beta_{21}^{u}+\beta_{21}^{d}\right), \left(\beta_{21}^{d}-\beta_{21}^{u}\right)$, we can identify $\beta_{21}^{u}, \beta_{21}^{d}$ separately. Similarly, we can identify $\beta_{12}^{u}, \beta_{12}^{d}, \beta_{11}^{u}, \beta_{11}^{d}$ separately. This completes the identification of all 9 parameters of the specification in (2.2).

5 Identification of Matching Matrix T

The equilibrium matching is linear in types, i.e.

$$\begin{aligned} y_1 &= t_{11}x_1 + t_{12}x_2 + t_{13}\epsilon, \\ y_2 &= t_{21}x_1 + t_{22}x_2 + t_{23}\epsilon, \\ \eta &= t_{31}x_1 + t_{32}x_2 + t_{33}\epsilon. \end{aligned}$$

Since $x_1, x_2 \perp \epsilon$, regressing y_1 on x_1 and x_2 identifies t_{11}, t_{12} as the coefficients, and $|t_{13}|$ as the standard deviation of the residuals. Similarly, regressing y_2 on x_1 and x_2 identifies t_{21}, t_{22} as coefficients, and $|t_{23}|$ as the standard deviation of the residuals.

$$\begin{aligned}
Cov\left(y_1,y_2\right) &= Cov\left(\begin{pmatrix}t_{11}\\t_{12}\\t_{13}\end{pmatrix}'\begin{pmatrix}x_1\\x_2\\\epsilon\end{pmatrix},\begin{pmatrix}t_{21}\\t_{22}\\t_{23}\end{pmatrix}'\begin{pmatrix}x_1\\x_2\\\epsilon\end{pmatrix}\right)\\
&= Cov\left(t_{11}x_1+t_{12}x_2+t_{13}\epsilon,t_{21}x_1+t_{22}x_2+t_{23}\epsilon\right)\\
&= t_{11}t_{21}Var\left(x_1\right)+t_{11}t_{22}Cov\left(x_1,x_2\right)+t_{11}t_{23}Cov\left(x_1,\epsilon\right)+\\
&+t_{12}t_{21}Cov\left(x_1,x_2\right)+t_{12}t_{22}Var\left(x_2\right)+t_{12}t_{23}Cov\left(x_2,\epsilon\right)+\\
&+t_{13}t_{21}Cov\left(\epsilon,x_1\right)+t_{13}t_{22}Cov\left(\epsilon,x_2\right)+t_{13}t_{23}Var\left(\epsilon\right)=\\
&= t_{11}t_{21}\sigma^2_{x_1}+\left(t_{11}t_{22}+t_{12}t_{21}\right)\sigma_{x_1x_2}+t_{12}t_{22}\sigma^2_{x_2}+\boldsymbol{t_{13}t_{23}}=0.
\end{aligned}$$

Thus,

$$t_{13}t_{23}=-\left(t_{11}t_{21}\sigma^2_{x_1}+\left(t_{11}t_{22}+t_{12}t_{21}\right)\sigma_{x_1x_2}+t_{12}t_{22}\sigma^2_{x_2}\right).$$

Since all the parameter values on the right hand side are identified, we can identify the sign of $t_{13}t_{23}$ in addition to the values of $|t_{13}|,|t_{23}|$.

Further,

$$\begin{aligned}
Var\left(\eta\right) &= Var\left(t_{31}x_1+t_{32}x_2+t_{33}\epsilon\right)\\
&= \boldsymbol{t^2_{31}}\sigma^2_{x_1}+\boldsymbol{t^2_{32}}\sigma^2_{x_2}+\boldsymbol{t^2_{33}}=1.
\end{aligned}$$

And,

$$Cov\left(y_j,\eta\right)=Cov\left(\begin{pmatrix}t_{j1}\\t_{j2}\\t_{j3}\end{pmatrix}'\begin{pmatrix}x_1\\x_2\\\epsilon\end{pmatrix},\begin{pmatrix}t_{31}\\t_{32}\\t_{33}\end{pmatrix}'\begin{pmatrix}x_1\\x_2\\\epsilon\end{pmatrix}\right)=0$$

Then, (the bold ones are unknowns in the last line).

$$\begin{gathered}
Cov\left(t_{11}x_1+t_{12}x_2+t_{13}\epsilon,t_{31}x_1+t_{32}x_2+t_{33}\epsilon\right)=\\
=t_{11}t_{31}Var\left(x_1\right)+t_{11}t_{32}Cov\left(x_1,x_2\right)+t_{11}t_{33}Cov\left(x_1,\epsilon\right)+\\
+t_{12}t_{31}Cov\left(x_1,x_2\right)+t_{12}t_{32}Var\left(x_2\right)+t_{12}t_{33}Cov\left(x_2,\epsilon\right)+\\
+t_{13}t_{31}Cov\left(\epsilon,x_1\right)+t_{13}t_{32}Cov\left(\epsilon,x_2\right)+t_{13}t_{33}Var\left(\epsilon\right)=\\
=t_{11}\boldsymbol{t_{31}}\sigma^2_{x_1}+\left(t_{11}\boldsymbol{t_{32}}+t_{12}\boldsymbol{t_{31}}\right)\sigma_{x_1x_2}+t_{12}\boldsymbol{t_{32}}\sigma^2_{x_2}+t_{13}\boldsymbol{t_{33}}=0.
\end{gathered}$$

Now, consider $Cov\left(y_2,\eta\right)$

$$\begin{gathered}
Cov\left(t_{21}x_1+t_{22}x_2+t_{23}\epsilon,t_{31}x_1+t_{32}x_2+t_{33}\epsilon\right)=\\
=t_{21}t_{31}Var\left(x_1\right)+t_{21}t_{32}Cov\left(x_1,x_2\right)+t_{21}t_{33}Cov\left(x_2,\epsilon\right)+\\
+t_{22}t_{31}Cov\left(x_1,x_2\right)+t_{22}t_{32}Var\left(x_2\right)+t_{22}t_{33}Cov\left(x_2,\epsilon\right)+\\
+t_{33}t_{31}Cov\left(\epsilon,x_1\right)+t_{33}t_{32}Cov\left(\epsilon,x_2\right)+t_{33}t_{33}Var\left(\epsilon\right)=\\
=t_{21}\boldsymbol{t_{31}}\sigma^2_{x_1}+\left(t_{21}\boldsymbol{t_{32}}+t_{22}\boldsymbol{t_{31}}\right)\sigma_{x_1x_2}+t_{22}\boldsymbol{t_{32}}\sigma^2_{x_2}+t_{23}\boldsymbol{t_{33}}=0.
\end{gathered}$$

Collecting the three equations we have

$$\begin{cases} \boldsymbol{t_{31}^2}\sigma_{x_1}^2 + \boldsymbol{t_{32}^2}\sigma_{x_2}^2 + \boldsymbol{t_{33}^2} = 1 \\ t_{11}\boldsymbol{t_{31}}\sigma_{x_1}^2 + (t_{11}\boldsymbol{t_{32}} + t_{12}\boldsymbol{t_{31}})\,\sigma_{x_1x_2} + t_{12}\boldsymbol{t_{32}}\sigma_{x_2}^2 + t_{13}\boldsymbol{t_{33}} = 0 \\ t_{21}\boldsymbol{t_{31}}\sigma_{x_1}^2 + (t_{21}\boldsymbol{t_{32}} + t_{22}\boldsymbol{t_{31}})\,\sigma_{x_1x_2} + t_{22}\boldsymbol{t_{32}}\sigma_{x_2}^2 + t_{23}\boldsymbol{t_{33}} = 0 \end{cases}$$

Let us assume $t_{33} \geq 0$, then

$$t_{33} = \sqrt{1 - t_{31}^2\sigma_{x_1}^2 - t_{32}^2\sigma_{x_2}^2}. \tag{9}$$

Then, we end up with a system of two equations (still non-linear) with two unknowns

$$\begin{cases} t_{11}\boldsymbol{t_{31}}\sigma_{x_1}^2 + (t_{11}\boldsymbol{t_{32}} + t_{12}\boldsymbol{t_{31}})\,\sigma_{x_1x_2} + t_{12}\boldsymbol{t_{32}}\sigma_{x_2}^2 + t_{13}\sqrt{1 - \boldsymbol{t_{31}^2}\sigma_{x_1}^2 - \boldsymbol{t_{32}^2}\sigma_{x_2}^2} = 0, \\ t_{21}\boldsymbol{t_{31}}\sigma_{x_1}^2 + (t_{21}\boldsymbol{t_{32}} + t_{22}\boldsymbol{t_{31}})\,\sigma_{x_1x_2} + t_{22}\boldsymbol{t_{32}}\sigma_{x_2}^2 + t_{23}\sqrt{1 - \boldsymbol{t_{31}^2}\sigma_{x_1}^2 - \boldsymbol{t_{32}^2}\sigma_{x_2}^2} = 0. \end{cases} \tag{10}$$

We can solve the system of equations above for the two unknowns.

For the parameterization which is frequently used in this document in Specification 2.2,

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \approx \begin{pmatrix} .8 & .4 & .24 \\ -.1 & .96 & -.13 \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

We can plot the two implicit functions in (5) to get a better intuition of uniqueness.

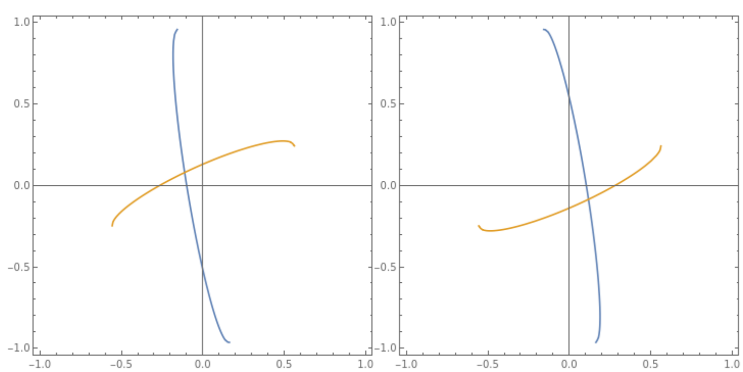

Figure 1: Plot of the equations in (5) in $t_{31} - t_{32}$ space for the specific parameterization in this document. t_{33} is then calculated via (9). Since the signs of t_{13}, t_{23} are not individually identified, the system of equations is solved separately assuming the sign of $t_{13} \cdot t_{23}$ being positive and negative. These are plotted separately. The solution to the system simply changes sign with the sign of $t_{13} \cdot t_{23}$. The intersection in the left plot is at $(-0.11, 0.82)$ and in the right plot is at $(0.11, -0.82)$.

In summary, The signs of t_{13}, t_{23} are not separately identified, however, the sign of $t_{13} \cdot t_{23}$ is identified. The sign of t_{33} is not identified.

6 Scratch Notes

$$\begin{aligned}
&\sigma_{x_1}^4 t_{11}^2 t_{31}^2 + 2\sigma_{x_1}^2 \sigma_{x_1x_2} t_{11} t_{12} t_{31}^2 + \sigma_{x_1x_2}^2 t_{12}^2 t_{31}^2 + 2\sigma_{x_1}^2 \sigma_{x_1x_2} t_{11}^2 t_{31} t_{32} + 2\sigma_{x_1x_2}^2 t_{11} t_{12} t_{31} t_{32} + \\
&\qquad + 2\sigma_{x_1}\sigma_{x_2} t_{11} t_{12} t_{31} t_{32} + 2\sigma_{x_1x_2}\sigma_{x_2}^2 t_{12}^2 t_{31} t_{32} + \sigma_{x_1x_2}^2 t_{11}^2 t_{32}^2 + 2\sigma_{x_1x_2}\sigma_{x_2}^2 t_{11} t_{12} t_{32}^2 + \\
&\qquad + \sigma_{x_2}^4 t_{12}^2 t_{32}^2 = \\
&\qquad = t_{13} - \sigma_{x_1} t_{13} t_{31}^2 - \sigma_{x_2}^2 t_{13} t_{32}^2.
\end{aligned}$$