The upstream profit is given by

$$\pi^u(\boldsymbol{x},\boldsymbol{\varepsilon}) = \max_{(\boldsymbol{y},\boldsymbol{\eta})\in\mathcal{Y}} \left\{\zeta(\boldsymbol{x},\boldsymbol{y}) + \Xi(\boldsymbol{\varepsilon},\boldsymbol{\eta}) - \pi^d(\boldsymbol{y},\eta)\right\}.$$

Then,

$$\begin{aligned}\frac{\partial \pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})}{\partial x_k} &= \left.\frac{\partial\left[\zeta(\boldsymbol{x},\boldsymbol{y}) + \xi(\boldsymbol{\varepsilon},\boldsymbol{\eta}) - \pi^d(\boldsymbol{y},\boldsymbol{\eta})\right]}{\partial x_k}\right|_{(\boldsymbol{y},\boldsymbol{\eta})=(T^u(\tilde{\boldsymbol{x}}),S^u(\tilde{\boldsymbol{x}}))} \\ &= \zeta_{x_k}(\boldsymbol{x}, T^u(\boldsymbol{x},\boldsymbol{\varepsilon})). \end{aligned} \tag{1}$$

Taking expectation of both sides of (1) with respect to the conditional distribution of $\boldsymbol{\varepsilon}$ gives

$$E_{F^u_{\varepsilon|x}}\left[\frac{\partial}{\partial x_k}\pi^u(\boldsymbol{X},\boldsymbol{\varepsilon})\,|\boldsymbol{X}=\boldsymbol{x}\right] = E_{F^u_{\varepsilon|x}}\left[\zeta_{x_k}(\boldsymbol{X},T^u(\boldsymbol{X},\boldsymbol{\varepsilon}))\,|\boldsymbol{X}=\boldsymbol{x}\right].$$

The left-hand side can be written as

$$\begin{aligned}E_{F^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}}\left[\frac{\partial\pi^u(\boldsymbol{X},\boldsymbol{\varepsilon})}{\partial x_k}|\boldsymbol{X}=\boldsymbol{x}\right] &= \int_{\mathcal{E}}\frac{\partial\pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})}{\partial x_k} f^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}(\boldsymbol{\varepsilon}|\boldsymbol{x})\;d\boldsymbol{\varepsilon} \\ &= \frac{\partial}{\partial x_k}\int_{\varepsilon\in\mathcal{E}}\pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})\, f^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}(\boldsymbol{\varepsilon}|\boldsymbol{x})\;d\boldsymbol{\varepsilon} \\ &= \frac{\partial E[\pi^u(\boldsymbol{X},\boldsymbol{\varepsilon})\,|\boldsymbol{X}=\boldsymbol{x}]}{\partial x_k}. \end{aligned} \tag{2}$$

The last equality is using the Leibniz integral rule to change the order of integration and differentiation. Thus,

$$\frac{\partial E[\pi^u(\boldsymbol{X},\boldsymbol{\varepsilon})\,|\boldsymbol{X}=\boldsymbol{x}]}{\partial x_k} = E_{F^u_{\varepsilon|x}}\left[\zeta_{x_k}(\boldsymbol{X},T^u(\boldsymbol{X},\boldsymbol{\varepsilon}))\,|\boldsymbol{X}=\boldsymbol{x}\right]. \tag{3}$$

Let

$$\zeta(x,y_1,y_2) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x y_1 + \beta_3 x y_2.$$

Then,

$$\begin{aligned}E_{F^u_{\varepsilon|x}}\left[\zeta_x(X,T^u(X,\boldsymbol{\varepsilon}))\,|X=x\right] &= E\left[\beta_1 + 2\beta_2 x + \beta_3 T^{y_1,u}(X,\boldsymbol{\varepsilon}) + \beta_4 T^{y_2,u}(X,\boldsymbol{\varepsilon})\,|X=x\right] \\ &= \beta_1 + 2\beta_2 x + \beta_3 E\left[T^{y_1,u}(X,\boldsymbol{\varepsilon})\,|X=x\right] + \beta_4 E\left[T^{y_2 u}(X,\boldsymbol{\varepsilon})\,|X=x\right] \\ &= \beta_1 + 2\beta_2 x + \beta_3\bar{y}_1^*(x) + \beta_4\bar{y}_2^*(x). \end{aligned} \tag{4}$$

Finally,

$$\frac{\partial E\left[\pi^u(X,\boldsymbol{\varepsilon})\,|X=x\right]}{\partial x} = \beta_1 + 2\beta_2 x + \beta_3\bar{y}_1^*(x) + \beta_4\bar{y}_2^*(x). \tag{5}$$

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$$\begin{aligned}E_{F^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}}\left[\frac{\partial\pi^u(\boldsymbol{X},\boldsymbol{\varepsilon})}{\partial x_k}|\boldsymbol{X}=\boldsymbol{x}\right] &= \int_{\mathcal{E}}\frac{\partial\pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})}{\partial x_k} f^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}(\boldsymbol{\varepsilon}|\boldsymbol{x})\;d\boldsymbol{\varepsilon} \\ &= \int_{\varepsilon\in\mathcal{E}}\left[\frac{\partial}{\partial x_k}\left[\pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})\, f^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}(\boldsymbol{\varepsilon}|\boldsymbol{x})\right] - \pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})\frac{\partial}{\partial x_k} f^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}(\boldsymbol{\varepsilon}|\boldsymbol{x})\right]\;d\boldsymbol{\varepsilon} \\ &= \frac{\partial}{\partial x_k}\int_{\varepsilon\in\mathcal{E}}\pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})\, f^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}(\boldsymbol{\varepsilon}|\boldsymbol{x})\;d\boldsymbol{\varepsilon} - \int_{\varepsilon\in\mathcal{E}}\pi^u(\boldsymbol{x},\boldsymbol{\varepsilon})\frac{\partial}{\partial x_k} f^u_{\boldsymbol{\epsilon}|\boldsymbol{X}}(\boldsymbol{\varepsilon}|\boldsymbol{x})\;d\boldsymbol{\varepsilon}. \end{aligned} \tag{6}$$