American Option price and Greeks approximation

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Abstract

Whaley, Ju-Zhong did American Option pricing based on geometric Brownian motion. We further analyzed the Greeks value of their models, especially Vega. Greeks value under Ju-Zhong's model are very accurate, compared to true value done through 1000-step Binomial Tree. Error is less than 1%. We also did American Option price approximation under Variance Gamma model. Main process of approximation is similar to that in Whaley & Ju-Zhong Model. The American Option price is a summation of the European Option price and an early exercise premium. Our approximation captured greater than 98% of true American Option put price implemented through finite difference method if we select $\epsilon = S_0/2000 \approx 0.7$

Keywords: Variance Gamma, Greeks, Early exercise premium

1. Introduction

Comparing to the European Option, the American Option is much harder to implement because of its path dependent property. Previously, we could only use simulation or numerical method to get its price. In the 1980's, Whaley[1] proposed analytical solution of American Option. This model performed well for short maturity and long maturity, but it didn't get an accurate price approximation for intermediate length maturity. Then in 1999 Ju-Zhong's[2] improvement on the model Whaley proposed, made American Option analytical price approximation accurate for a whole range of maturity. In Ju-Zhong's paper, it provided Greeks equation, but with several typos. We re-derived all equations in Whaley's paper, and got Greeks equations for Ju-Zhong and Whaley's methods respectively.

Ju-Zhong and Whaley's approximation were both based on the assumption that stock follows geometric Brownian motion, which is a diffusion process with no jumps. In real market, there exists jumps. Variance Gamma model is a modification on geometric Brownian motion model. Madan and Carr [3] introduced a closed-form analytical solution of European Option price under Variance Gamma. Hirsa and Madan[4] used a finite difference method, getting American Option price under Variance Gamma. Finite difference method is very accurate for price approximation, but it is computationally expensive.

We want to get an analytical approximation of American Option price under Variance Gamma using a similar structure as Ju-Zhong, where the American Option price is the summation of European Option price and early exercise premium.

2. Greeks approximation for Whaley & Ju-Zhong American option pricing model

In Appendix A, we re-derived equations in Whaley's paper, making its variable setting consistent with that in Ju-Zhong, and we derived the Greeks equations. In Appendix B, we've included all the Greeks' equations in Ju-Zhong's paper. We corrected several typos in Ju-Zhong's paper. Appendix C contains four tables with price, Delta, Gamma, Theta, Vega value under different S_0, K, T, r, δ combination. These parameters were consistent with Exhibit 3-6 in Ju-Zhong's paper.

2.1. Binomial Tree approximation

Binomial Tree was set as benchmark (True value) in Appendix C. In Ju-Zhong's paper, they used 10000-step binomial tree model to get true value of price. However, because of $\mathcal{O}(n^2)$ computational complexity, it takes very long time to calculate 10000-step binomial tree. Braunstein[5] proposed that the accuracy would be the same if we utilized the average of two trees, one with 1000 steps and one with 1001 steps. This process is significantly less burdensome, and prices more accurately than a single tree. For following paragraphs, $f^{(1000)}, f^{(1001)}$ were 1000-step and 1001-step binomial tree price, and $S^{(1000)}, S^{(1001)}$ were 1000-step and 1001-step stock price.
Delta was calculated through central differences,

$$\Delta = \frac{1}{2}\left(\frac{f_u^{(1000)} - f_d^{(1000)}}{S_u^{(1000)} - S_d^{(1000)}} + \frac{f_u^{(1001)} - f_d^{(1001)}}{S_u^{(1001)} - S_d^{(1001)}}\right)$$

Gamma was calculated through central differences. $\Delta_1^{(1000)} = \frac{f_{uu}^{(1000)} - f_{ud}^{(1000)}}{S_{uu}^{(1000)} - S^{(1000)}}$, $\Delta_2^{(1000)} = \frac{f_{du}^{(1000)} - f_{dd}^{(1000)}}{S^{(1000)} - S_{dd}^{(1000)}}$

$$\Gamma^{(1000)} = \frac{\Delta_1^{(1000)} - \Delta_2^{(1000)}}{0.5(S_{uu}^{(1000)} - S_{dd}^{(1000)})}$$

$$\Gamma = 0.5(\Gamma^{(1000)} + \Gamma^{(1001)})$$

Theta was explicitly calculated,

$$\Theta = \frac{1}{2}(\frac{f_{ud}^{(1000)} - f^{(1000)}}{2\Delta t} + \frac{f_{ud}^{(1001)} - f^{(1001)}}{2\Delta t})$$

Vega was calculated through central differences,

$$Vega = \frac{1}{2}(\frac{f_{\sigma+\Delta\sigma}^{(1000)} - f_{\sigma-\Delta\sigma}^{(1000)}}{2\Delta\sigma} + \frac{f_{\sigma+\Delta\sigma}^{(1001)} - f_{\sigma-\Delta\sigma}^{(1001)}}{2\Delta\sigma})$$

2.2. Greeks calculation under Whaley American option pricing model

Greeks equations under Whaley's model are shown in Appendix A. In Appendix C, 'Quad' columns represent the results of Whaley's method. Whaley's method's Greeks approximation is very accurate for short maturity (Appendix C.1 & C.2). However, $Vega$ approximation with intermediate length maturity shows around 5% error. It explains reason for inaccurate price approximation of Whaley's method under intermediate maturity.

2.3. Greeks calculation under Ju-Zhong American option pricing model

Greeks equations under Whaley's model are shown in Appendix B. In Appendix C, 'Mquad' columns represent the results of Ju-Zhong method. Ju-Zhong's Greeks approximation is very accurate for the whole range of maturity (Appendix C.1 - C.4).

3. American option pricing under Variance Gamma

PIDE of Variance Gamma model:

$$\frac{\partial w}{\partial \tau}(x,\tau) + \frac{1}{2}\sigma^2(\epsilon)\frac{\partial^2 w}{\partial x^2}(x,\tau) - \left(r - q + \omega(\epsilon) - \frac{1}{2}\sigma^2(\epsilon)\right)\frac{\partial w}{\partial x}(x,\tau)$$

$$+rw(x,\tau) - \int_{|y|>\epsilon} \left(w(x+y,\tau) - w(x,\tau)\right) k(y)dy = 0$$

Jump part was split into two sub-parts. When $|y| < \epsilon$, the integral formed $\sigma^2(\epsilon)$. $|y| > \epsilon$ part remains integral form.

3.1. Approximation method

• First method: American Option under VG = European Option under VG + Early exercise premium (Calculated through Ju-Zhong model)

$expint(\xi) = \int_{\xi}^{\infty} \frac{e^{-y}}{y} dy$

$\sigma^2(\epsilon) = \int_{|y|<\epsilon} y^2 k(y) dy = \frac{1}{\nu \lambda_p^2} \left(1 - e^{-\lambda_p \epsilon}(\lambda_p \epsilon + 1)\right) + \frac{1}{\nu \lambda_n^2} \left(1 - e^{-\lambda_n \epsilon}(\lambda_n \epsilon + 1)\right)$

$\omega(\epsilon) = \int_{|y|>\epsilon} (1-e^y) k(y) dy = \frac{1}{\nu} expint(\epsilon\lambda_p) - \frac{1}{\nu} expint(\epsilon(\lambda_p - 1) + \frac{1}{\nu} expint(\epsilon\lambda_n) - \frac{1}{\nu} expint(\epsilon(\lambda_n + 1)$

Ju-Zhong approximation was implemented for getting early exercise premium part of Black Scholes formula

$$\frac{\partial V(S,\tau)}{\partial \tau} - [r - (q - \omega(\epsilon))] S \frac{\partial V(S,\tau)}{\partial S} - \frac{1}{2}\sigma^2(\epsilon) \frac{\partial^2 V(S,\tau)}{\partial S^2} = -rV$$

Approx1: AmerPrice = EuroVG + EarlyExercisePremium(by Ju-Zhong). $\epsilon = S_0/1000$
Approx2: AmerPrice = EuroVG + EarlyExercisePremium(by Ju-Zhong). $\epsilon = S_0/2000$
Approx3: AmerPrice = EuroVG + EarlyExercisePremium(by Ju-Zhong). $\epsilon = S_0/8000$

• Second method: Fully using Ju-Zhong approximation (GBM) of American option pricing

For Black Scholes formula

$$\frac{\partial V(S,\tau)}{\partial \tau} - [r - (q - \omega(\epsilon))] S \frac{\partial V(S,\tau)}{\partial S} - \frac{1}{2}\sigma^2(\epsilon) \frac{\partial^2 V(S,\tau)}{\partial S^2} = -rV$$

we didn't consider European Option price under Variance Gamma here. We used Ju-Zhong's method for getting American Option price of Black Scholes PDE shown above ('JZ-Approx' in Appendix C).

3.2. Result analysis

In Approx.D, there are four tables with parameter consistent with those in Table 3 of Hirsa and Madan's paper[4], parameter settings were the same as Table 1 in Hirsa and Madan's paper[4]. Four tables in Appendix.D are all about American Put Option. For out-of-the money American Put Option, Variance Gamma model's price should be higher than that under Geometric Brownian motion. Jump leads to higher percent moneyness. However, for in the money American Put Option, jump may lead to lower percent moneyness. In Appendix D, 'JZ-Approx' under-estimated out of money American Option's prices under the Variance Gamma model, and over-priced them for in-the-money options. The key reason is explained above. 'JZ-Approx' is an approximation fully based on Ju-Zhong American Option pricing using geometric Brownian motion, which is a diffusion process.

When checking Approx 1-3, we noticed our approximation sensitivity to the ϵ we picked. 'Approx 1' and 'Approx 2' results were close, and 'Approx 3' over-priced for in-the-money option. We picked $\epsilon = S_0/8000 \to 0.1$ for 'Approx 3'. When ϵ is very small, it means $\sigma(\epsilon)$ being extreme small. Early exercise premium for in-the-money options will be larger than the true value when volatility is close to 0 and maturity isn't long enough.

Because true value of American Option price is still in progress, Appendix E's results were used for selecting best model. When getting Approx 1-3 results in Appendix D, we used FFT calculating European Put option under Variance Gamma, and it's very close to the true value implemented through finite difference (True European Put option). True American Put option = True European Put option + True early exercise premium. The error of Approx 1-3 only happens at early exercise premium estimation. In Appendix E, 4 tables all show the early exercise premium values. Approx 1-3 were calculated through early the exercise premium based on Ju-Zhong model. We noticed that Approx 2 ($\epsilon = S_0/2000 \approx 0.7$) captures the highest accuracy.

4. Conclusion

We revisited Whaley and Ju-Zhong approximation and further checked their Greeks. Both model did well on Δ, Γ and Θ. However, Whaley's model didn't perform well on $Vega$ approximation when maturity is of intermediate length. Ju-Zhong's model is robust to Greeks and price approximation for the whole range maturities.

For American Option pricing under Variance Gamma model, we used the same approximation structure as Ju-Zhong's did, and got analytical approximation. American VG was the summation of the Euro VG and the early exercise premium implemented under Ju-Zhong's method. Our answer was sensitive to the ϵ picked. When $\epsilon = S_0/2000 \approx 0.7$, our model captured highest accuracy.

Reference

[**1**]: BARONE–ADESI, G. I. O. V. A. N. N. I., and Robert E. Whaley. Efficient analytic approximation of American option values. *The Journal of Finance* 42.2 (1987): 301-320.

[**2**]: Ju, Nengjiu, and Rui Zhong. An approximate formula for pricing American options. *The Journal of Derivatives* 7.2 (1999): 31-40.

[**3**]: Madan, Dilip B., Peter P. Carr, and Eric C. Chang. The variance gamma process and option pricing. *Review of Finance* 2.1 (1998): 79-105

[**4**]: Hirsa, Ali, and Dilip B. Madan. Pricing American options under variance gamma. *Journal of Computational Finance* 7.2 (2004): 63-80

[**5**]: Braunstein, Alex. American Option Approximations. (2008)

Appendix A. Whaley American Option pricing approximation equation re-derivation and Greeks calculation

Appendix A.1. Equation re-derivation

- Equation abbreviation:

$N(x)$: cdf of standard normal; $n(x)$: pdf of standard normal

priceEuro$(S, T, 1) = c(S, T)$; priceEuro$(S, T, -1) = p(S, T)$

deltaEuro$(S, T, \phi) = \phi e^{-qT} N(\phi d_1(S))$; gammaEuro$(S, T, \phi) = \frac{e^{-qT} n(d_1(S))}{S\sigma\sqrt{T}}$

vegaEuro$(S, T, \phi) = Se^{-qT} n(d_1(S))\sqrt{T}$

thetaEuro$(S, T, \phi) = -e^{-qT} \frac{Sn(d_1(S)\sigma}{2\sqrt{T}} - \phi rKe^{-rT} N(\phi d_2(S)) + \phi qSe^{-qT} N(\phi d_1(S))$

$d_1(S) = \frac{ln(S/K)+(r-q+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$; $d_2(S) = \frac{ln(S/K)+(r-q-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$

- Equation re-derivation ($\phi = 1$: call, $\phi = -1$: put; Use the same parameter named in Ju Zhong paper)

$$h = K = 1 - e^{-r\tau}, \qquad \alpha = M = \frac{2r}{\sigma^2}, \quad \beta = N = \frac{2(r-q)}{\sigma^2}$$

$$\lambda = \frac{-(\beta - 1) + \phi\sqrt{(\beta-1)^2 + 4\frac{\alpha}{h}}}{2} \qquad \text{For } q_1 \text{ and } q_2$$

$$d_1(S^*) = \frac{ln(S^*/K) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$A = \phi\frac{S^*}{\lambda}\left[1 - e^{-q\tau} N(\phi d_1(S^*))\right] \qquad \text{For } A_1 \text{ and } A_2$$

Seed value:

$$\lambda(\infty) = \frac{-(\beta - 1) + \phi\sqrt{(\beta-1)^2 + 4\alpha}}{2} \qquad \text{For } q_1(\infty) \text{ and } q_2(\infty)$$

$$S^*(\infty) = \frac{K}{1 - \frac{1}{\lambda(\infty)}} \qquad \text{For } S^*(\infty) \text{ and } S^{**}(\infty)$$

Initial guess for S^*

$$hI = -[(r-q)\tau + 2\phi\sigma\sqrt{\tau}]\frac{K}{S^*(\infty) - K} \qquad \text{For } h_1 \text{ and } h_2$$

$$S^* = S^*(\infty) + (K - S^*(\infty))e^{hI}$$

Determine S^*

$$\phi(S^* - K) = \text{priceEuro}(S^*, \tau, \phi) + \frac{\phi S^*}{\lambda}[1 - e^{-q\tau} N(\phi d_1(S^*))$$

$$\mathrm{LHS}(S_i) = \phi(S^* - K); \quad \mathrm{RHS}(S_i) = \mathrm{priceEuro}(S^*, \tau, \phi) + \frac{\phi S^*}{\lambda}[1 - e^{-q\tau} N(\phi d_1(S^*))$$

$$b_i = \mathrm{deltaEuro}(S_i, \tau)(1 - \frac{1}{\lambda}) + \frac{\phi(1 - \phi\frac{e^{-q\tau} n(d_1(S_i))}{\sigma\sqrt{\tau}})}{\lambda}$$

$$S_{i+1} = \frac{\phi K + \mathrm{RHS}(S_i) - b_i s_i}{\phi - b_i}$$

$$V_a(S, T, \phi) = \begin{cases} \mathrm{priceEuro}(S, T, \phi) + A(\frac{S}{S^*})^\lambda & \phi(S^* - S) > 0 \\ \phi(S - K) & \phi(S^* - S) \leq 0 \end{cases}$$

Appendix A.2. Greeks equation

▷ Delta

$$\Delta(S, T, \phi) = \begin{cases} \mathrm{deltaEuro}(S, T, \phi) + \frac{A\lambda}{(S^*)^\lambda} S^{\lambda - 1} & \phi(S^* - S) > 0 \\ \phi & \phi(S^* - S) \leq 0 \end{cases}$$

▷ Gamma

$$\Gamma(S, T, \phi) = \begin{cases} \mathrm{gammaEuro}(S, T, \phi) + \frac{A\lambda(\lambda - 1)}{(S^*)^\lambda} S^{\lambda - 2} & \phi(S^* - S) > 0 \\ 0 & \phi(S^* - S) \leq 0 \end{cases}$$

▷ Theta

$$\Theta = rV_a(S, T, \phi) - \frac{1}{2}\sigma^2 S^2 \Gamma - (r - q) S \Delta$$

▷ Gamma

$$\frac{\partial \alpha}{\partial \sigma} = -2\frac{\alpha}{\sigma} \quad \frac{\partial \beta}{\partial \sigma} = -2\frac{\beta}{\sigma}$$

$$\frac{\partial \lambda}{\partial \sigma} = \frac{1}{2}\left[-\frac{\partial \beta}{\partial \sigma} + \frac{\phi}{2} \frac{2(\beta - 1)\frac{\partial \beta}{\partial \sigma} + \frac{4}{h}\frac{\partial \alpha}{\partial \sigma}}{\sqrt{(\beta - 1)^2 + \frac{4\alpha}{h}}} \right]$$

$$\frac{\partial d_1(S^*)}{\partial \sigma} = \frac{1}{\sigma\sqrt{\tau}}(\frac{1}{S^*}\frac{\partial S^*}{\partial \sigma} + \sigma\tau) - \frac{d_1 S(^*)}{\sigma} = \frac{1}{S^*\sigma\sqrt{\tau}}\frac{\partial S^*}{\partial \sigma} + (\sqrt{\tau} - \frac{d_1 S(^*)}{\sigma})$$

In order to get $\frac{\partial d_1(S^*)}{\partial \sigma}$, we need get $\frac{\partial S^*}{\partial \sigma}$ in advance. In order to get $\frac{\partial S^*}{\partial \sigma}$, we use $\phi(S^* - K) = \mathrm{priceEuro}(S^*, \tau, \phi) + \frac{\phi S^*}{\lambda}[1 - e^{-q\tau} N(\phi d_1(S^*))$

$$\phi\frac{\partial S^*}{\partial \sigma} = \mathrm{vegaEuro}(S^*, \tau, \phi) + \mathrm{deltaEuro}(S^*, \tau, \phi)\frac{\partial S^*}{\partial \sigma} + \phi\frac{S^*}{\lambda}(-e^{-q\tau})\phi n(d_1(S^*))\frac{\partial d_1(S^*)}{\partial \sigma}$$

$$+\phi[1-e^{-q\tau}N(\phi d_1(S^*))](\frac{1}{\lambda}\frac{\partial S^*}{\partial\sigma}-\frac{S^*}{\lambda^2}\frac{\partial\lambda}{\partial\sigma})$$

$$=\text{vegaEuro}(S^*,\tau,\phi)+\text{deltaEuro}(S^*,\tau,\phi)\frac{\partial S^*}{\partial\sigma}-\frac{S^*}{\lambda}e^{-q\tau}n(d_1(S^*))[\frac{1}{S^*\sigma\sqrt{\tau}}\frac{\partial S^*}{\partial\sigma}+(\sqrt{\tau}-\frac{d_1S(^*)}{\sigma})]$$

$$+(\phi-\text{deltaEuro}(S^*,\tau,\phi))(\frac{1}{\lambda}\frac{\partial S^*}{\partial\sigma}-\frac{S^*}{\lambda^2}\frac{\partial\lambda}{\partial\sigma})$$

$$\frac{\partial S^*}{\partial\sigma}=\frac{\text{vegaEuro}(S^*,\tau,\phi)-\frac{S^*}{\lambda}e^{-q\tau}n(d_1(S^*))(\sqrt{\tau}-\frac{d_1S(^*)}{\sigma})-\frac{\phi-\text{deltaEuro}(S^*,\tau,\phi)}{\lambda^2}S^*\frac{\partial\lambda}{\partial\sigma}}{\phi-\text{deltaEuro}(S^*,\tau,\phi)+e^{-q\tau}\frac{n(d_1(S^*))}{\lambda\sigma\sqrt{\tau}}-\frac{\phi-\text{deltaEuro}(S^*,\tau,\phi)}{\lambda}}$$

After getting $\frac{\partial S^*}{\partial\sigma}$, we could get $\frac{\partial d_1(S^*)}{\partial\sigma}$

$$\frac{\partial A}{\partial\sigma}=\phi[1-e^{q\tau}N(\phi d_1(S^*))](\frac{1}{\lambda}\frac{\partial S^*}{\partial\sigma}-\frac{S^*}{\lambda^2}\frac{\partial\lambda}{\partial\sigma})-e^{-q\tau}\frac{S^*}{\lambda}n(d_1(S^*))\frac{\partial d_1(S^*)}{\partial\sigma}$$

$$Vega_A(S,T,\phi)=\begin{cases}\text{vegaEuro}(S,\tau,\phi)+(\frac{S}{S^*})^\lambda\left[\frac{\partial A}{\partial\sigma}+A(\frac{\partial\lambda}{\partial\sigma}\ln(\frac{S}{S^*})-\frac{\lambda}{S^*}\frac{\partial S^*}{\partial\sigma})\right] & \phi(S^*-S)>0\\ 0 & \phi(S^*-S)\le 0\end{cases}$$

Appendix B. Ju-Zhong American Option pricing approximation Greeks calculation

Appendix B.1. Greeks equation

$$\lambda'(h) = -\frac{\phi\alpha}{h^2}\frac{1}{\sqrt{(\beta-1)^2+4\alpha/h}}$$

$$c = -\frac{(1-h)\alpha}{2\lambda+\beta-1}\left(-\frac{1}{hA(h)}\frac{\text{thetaEuro}(S^*,\tau,\phi)}{re^{-r\tau}}+\frac{1}{h}+\frac{\lambda'(h)}{2\lambda+\beta-1}\right)$$

• Greeks analysis

▷ Delta

$$\Delta = \begin{cases} \text{deltaEuro}(S,T,\phi)+\left(\frac{\lambda(h)}{S(1-\chi)}+\frac{\chi'(S)}{(1-\chi)^2}\right)(\phi(S^*-K)-V_E(S^*))\,(\frac{S}{S^*})^{\lambda(h)} & \phi(S^*-S)>0 \\ \phi & \phi(S^*-S)\leq 0 \end{cases}$$

▷ Gamma

$$\Gamma = \begin{cases} {}_{\text{gammaEuro}}(S,T,\phi)+\left(\frac{2\lambda(h)\chi'(S)}{S(1-\chi)^2}+\frac{2\chi'^2(S)}{(1-\chi)^3}+\frac{\chi''(S)}{(1-\chi)^2}+\frac{\lambda^2(h)-\lambda(h)}{S^2(1-\chi)}\right)(\phi(S^*-K)-V_E(S^*))\,(\frac{S}{S^*})^{\lambda(h)} \\ \phi(S^*-S)>0 \\ 0 \quad \phi(S^*-S)\leq 0 \end{cases}$$

▷ Theta

$$\Theta = rV_a(S,T,\phi)-\frac{1}{2}\sigma^2S^2\Gamma-(r-q)S\Delta$$

▷ Vega

$$\frac{\partial\alpha}{\partial\sigma} = -2\frac{\alpha}{\sigma} \quad \frac{\partial\beta}{\partial\sigma} = -2\frac{\beta}{\sigma}$$

$$\frac{\partial\lambda}{\partial\sigma} = \frac{1}{2}\left[-\frac{\partial\beta}{\partial\sigma}+\frac{\phi}{2}\frac{2(\beta-1)\frac{\partial\beta}{\partial\sigma}+\frac{4}{h}\frac{\partial\alpha}{\partial\sigma}}{\sqrt{(\beta-1)^2+\frac{4\alpha}{h}}}\right]$$

$$\frac{\partial\lambda'(h)}{\partial\sigma} = -\frac{\phi}{h^2}\left[\frac{1}{\sqrt{(\beta-1)^2+4\alpha/h}}\frac{\partial\alpha}{\partial\sigma}-\frac{\alpha}{2}\frac{1}{((\beta-1)^2+4\alpha/h)^{3/2}}\left(2(\beta-1)\frac{\partial\beta}{\partial\sigma}+\frac{4}{h}\frac{\partial\alpha}{\partial\sigma}\right)\right]$$

$$\frac{\partial b}{\partial\sigma} = \frac{1-h}{2(2\lambda+\beta-1)}\left[\lambda'(h)\frac{\partial\alpha}{\partial\sigma}+\alpha\frac{\partial\lambda'(h)}{\partial\sigma}-\frac{\alpha\lambda'(h)}{2\lambda+\beta-1}\left(2\frac{\partial\lambda}{\partial\sigma}+\frac{\partial\beta}{\partial\sigma}\right)\right]$$

$$\frac{\partial d_1(S^*)}{\partial\sigma} = \frac{1}{\sigma\sqrt{\tau}}(\frac{1}{S^*}\frac{\partial S^*}{\partial\sigma}+\sigma\tau)-\frac{d_1S(^*)}{\sigma} = \frac{1}{S^*\sigma\sqrt{\tau}}\frac{\partial S^*}{\partial\sigma}+(\sqrt{\tau}-\frac{d_1S(^*)}{\sigma})$$

$$\frac{\partial d_2(S^*)}{\partial\sigma} = \frac{\partial d_1(S^*)}{\partial\sigma} - \sqrt{\tau} = \frac{1}{S^*\sigma\sqrt{\tau}}\frac{\partial S^*}{\partial\sigma} - \frac{d_1 S(^*)}{\sigma}$$

In order to get $\frac{\partial d_2(S^*)}{\partial\sigma}$and $\frac{\partial d_1(S^*)}{\partial\sigma}$, we need to get $\frac{\partial S^*}{\partial\sigma}$ first. We could use equation $\phi S^* = \phi S^* e^{-q\tau}N(\phi d_1(S^*)) + \lambda[\phi(S^* - K) - V_E(S^*)]$ for getting $\frac{\partial S^*}{\partial\sigma}$

$$\phi\frac{\partial S^*}{\partial\sigma} = S^* e^{-q\tau} n(d_1(S^*))\frac{\partial d_1(S^*)}{\partial\sigma} + \text{deltaEuro}(S^*,\tau,\phi)\frac{\partial S^*}{\partial\sigma} + [\phi(S^* - K) - V_E(S^*)]\frac{\partial\lambda}{\partial\sigma}$$

$$+\lambda\left[\phi\frac{\partial S^*}{\partial\sigma} - \text{vegaEuro}(S^*,\tau,\phi) - \text{deltaEuro}(S^*,\tau,\phi)\frac{\partial S^*}{\partial\sigma}\right]$$

$$\phi\frac{\partial S^*}{\partial\sigma} = S^* e^{-q\tau} n(d_1(S^*))\left[\frac{1}{S^*\sigma\sqrt{\tau}}\frac{\partial S^*}{\partial\sigma} + (\sqrt{\tau} - \frac{d_1 S(^*)}{\sigma})\right] + \text{deltaEuro}(S^*,\tau,\phi)\frac{\partial S^*}{\partial\sigma}$$

$$+[\phi(S^* - K) - V_E(S^*)]\frac{\partial\lambda}{\partial\sigma} + \lambda\left[\phi\frac{\partial S^*}{\partial\sigma} - \text{vegaEuro}(S^*,\tau,\phi) - \text{deltaEuro}(S^*,\tau,\phi)\frac{\partial S^*}{\partial\sigma}\right]$$

$$\frac{\partial S^*}{\partial\sigma} = \frac{S^* e^{-q\tau} n(d_1(S^*))\left(\sqrt{\tau} - \frac{d_1 S(^*)}{\sigma}\right) + [\phi(S^* - K) - V_E(S^*)]\frac{\partial\lambda}{\partial\sigma} - \lambda\text{vegaEuro}(S^*,\tau,\phi)}{(1-\lambda)(\phi - \text{deltaEuro}(S^*,\tau,\phi)) - \frac{e^{-q\tau}}{\sigma\sqrt{\tau}}n(d_1(S^*))}$$

$$\frac{\partial A}{\partial\sigma} = \frac{1}{h}\left[(\phi - \text{deltaEuro}(S^*,\tau,\phi))\frac{\partial S^*}{\partial\sigma} - \text{vegaEuro}(S^*,\tau,\phi)\right]$$

$$\frac{\partial\text{thetaEuro}(S^*,T,\phi)}{\partial\sigma} = -\frac{e^{-q\tau}n(d_1(S^*))}{2\sqrt{\tau}}\left[S^* + \sigma\frac{\partial S^*}{\partial\sigma} - S^*\sigma d_1(S^*)\frac{\partial d_1(S^*)}{\partial\sigma}\right] - rKe^{-r\tau}n(d_2(S^*))\frac{\partial d_2(S^*)}{\partial\sigma}$$

$$+qS^* e^{-q\tau}n(d_1(S^*))\frac{\partial d_1(S^*)}{\partial\sigma} + q\text{deltaEuro}(S^*,\tau,\phi)\frac{\partial S^*}{\partial\sigma}$$

$$\frac{\partial c}{\partial\sigma} = -\frac{(1-h)\alpha}{2\lambda+\beta-1}[-\frac{1}{hre^{-r\tau}}\left(\frac{1}{A}\frac{\partial\text{thetaEuro}(S^*,T,\phi)}{\partial\sigma} - \frac{\text{thetaEuro}(S^*,T,\phi)}{A^2}\frac{\partial A}{\partial\sigma}\right) + \frac{1}{2\lambda+\beta-1}\frac{\partial\lambda'(h)}{\partial\sigma}$$

$$-\frac{\lambda'(h)}{(2\lambda+\beta-1)^2}(2\frac{\partial\lambda}{\partial\sigma} + \frac{\partial\beta}{\partial\sigma})] - (1-h)\left[-\frac{\text{thetaEuro}(S^*,T,\phi)}{hre^{-r\tau}A} + \frac{1}{h} + \frac{\lambda'(h)}{2\lambda+\beta-1}\right][\frac{1}{2\lambda+\beta-1}\frac{\partial\alpha}{\partial\sigma}$$

$$-\frac{\alpha}{(2\lambda+\beta-1)^2}(2\frac{\partial\lambda}{\partial\sigma} + \frac{\partial\beta}{\partial\sigma})]$$

$$\frac{\partial\chi}{\partial\sigma} = \frac{\partial b}{\partial\sigma}(\log(\frac{S}{S^*}))^2 - \left[\frac{2b}{S^*}\log(\frac{S}{S^*}) + \frac{c}{S^*}\right]\frac{\partial S^*}{\partial\sigma} + \frac{\partial c}{\partial\sigma}\log(\frac{S}{S^*})$$

$$Vega_A = \begin{cases} \text{vegaEuro}(S,\tau,\phi) + \frac{h(\frac{S}{S^*})^\lambda}{1-\chi}\left[\frac{\partial A}{\partial \sigma} + A(\frac{\partial \lambda}{\partial \sigma}\log(\frac{S}{S^*}) - \frac{\lambda}{S^*}\frac{\partial S^*}{\partial \sigma})\right] + \frac{hA(\frac{S}{S^*})^\lambda}{(1-\chi)^2}\frac{\partial \chi}{\partial \sigma} & \phi(S^*-S) > 0 \\ 0 & \phi(S^*-S) \leq 0 \end{cases}$$

Appendix C. Ju-Zhong & Whaley American Option pricing Price and Greeks Approximation comparison

Appendix C.1. VALUES AND GREEKS OF AMERICAN PUTS ($S = \$40, r = 0.0488, \delta = 0.0$)

$(K, \sigma, \tau, (yr))$	Price			Delta			Gamma			Theta			Vega		
	True	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**
(35, 0.2, 0.0833)	0.006	0.006	0.006	-0.008	-0.008	-0.008	0.009	0.010	0.009	-0.287	-0.289	-0.288	0.250	0.256	0.253
(35, 0.2, 0.3333)	0.200	0.204	0.201	-0.0900	-0.0900	-0.0900	0.036	0.035	0.035	-0.956	-0.948	-0.950	3.751	3.753	3.734
(35, 0.2, 0.5833)	0.433	0.442	0.433	-0.134	-0.134	-0.133	0.036	0.036	0.036	-0.882	-0.866	-0.874	6.577	6.567	6.532
(40, 0.2, 0.0833)	0.852	0.850	0.851	-0.469	-0.467	-0.469	0.178	0.176	0.178	-4.727	-4.681	-4.742	4.572	4.566	4.571
(40, 0.2, 0.3333)	1.580	1.577	1.576	-0.443	-0.439	-0.442	0.092	0.091	0.093	-2.012	-1.976	-2.025	8.996	8.978	8.98
(40, 0.2, 0.5833)	1.991	1.989	1.984	-0.429	-0.423	-0.427	0.072	0.071	0.072	-1.367	-1.335	-1.379	11.730	11.712	11.695
(45, 0.2, 0.0833)	5.000	5.000	5.000	-1.000	-1.000	-1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
(45, 0.2, 0.3333)	5.088	5.066	5.084	-0.881	-0.887	-0.881	0.083	0.094	0.082	-0.678	-1.039	-0.648	4.054	3.787	4.054
(45, 0.2, 0.5833)	5.267	5.236	5.260	-0.795	-0.794	-0.796	0.079	0.085	0.079	-0.709	-0.914	-0.707	7.800	7.697	7.741
(35, 0.3, 0.0833)	0.077	0.078	0.077	-0.052	-0.052	-0.052	0.031	0.031	0.031	-2.097	-2.094	-2.097	1.236	1.221	1.219
(35, 0.3, 0.3333)	0.698	0.701	0.697	-0.174	-0.174	-0.174	0.038	0.037	0.037	-2.336	-2.314	-2.325	5.905	5.912	5.907
(35, 0.3, 0.5833)	1.220	1.228	1.218	-0.213	-0.212	-0.212	0.033	0.032	0.032	-1.871	-1.843	-1.862	8.817	8.814	8.801
(40, 0.3, 0.0833)	1.310	1.308	1.309	-0.469	-0.468	-0.469	0.117	0.116	0.117	-7.447	-7.395	-7.465	4.581	4.578	4.581
(40, 0.3, 0.3333)	2.483	2.478	2.477	-0.442	-0.439	-0.441	0.060	0.059	0.060	-3.320	-3.275	-3.332	9.046	9.036	9.036
(40, 0.3, 0.5833)	3.170	3.167	3.160	-0.426	-0.422	-0.425	0.046	0.045	0.046	-2.320	-2.279	-2.333	11.828	11.819	11.804
(45, 0.3, 0.0833)	5.060	5.047	5.059	-0.923	-0.926	-0.922	0.058	0.063	0.056	-2.110	-2.514	-2.009	1.584	1.526	1.616
(45, 0.3, 0.3333)	5.706	5.679	5.699	-0.727	-0.722	-0.727	0.057	0.058	0.057	-2.42	-2.488	-2.434	7.470	7.501	7.441
(45, 0.3, 0.5833)	6.244	6.215	6.231	-0.652	-0.646	-0.652	0.049	0.049	0.049	-1.917	-1.949	-1.945	10.895	10.976	10.879
(35, 0.4, 0.0833)	0.247	0.247	0.246	-0.106	-0.106	-0.106	0.040	0.040	0.040	-4.875	-4.864	-4.868	2.122	2.115	2.114
(35, 0.4, 0.3333)	1.346	1.349	1.344	-0.226	-0.225	-0.225	0.033	0.033	0.033	-3.718	-3.687	-3.707	6.905	6.919	6.917
(35, 0.4, 0.5833)	2.155	2.162	2.150	-0.254	-0.253	-0.253	0.027	0.027	0.027	-2.842	-2.807	-2.833	9.786	9.737	9.727
(40, 0.4, 0.0833)	1.768	1.766	1.767	-0.467	-0.465	-0.467	0.087	0.087	0.087	-10.173	-10.117	-10.194	4.582	4.581	4.582
(40, 0.4, 0.3333)	3.388	3.382	3.381	-0.436	-0.434	-0.435	0.044	0.044	0.044	-4.634	-4.584	-4.646	9.048	9.043	9.039
(40, 0.4, 0.5833)	4.353	4.349	4.341	-0.417	-0.414	-0.416	0.034	0.033	0.034	-3.281	-3.233	-3.292	11.824	11.823	11.805
(45, 0.4, 0.0833)	5.287	5.273	5.288	-0.836	-0.834	-0.837	0.059	0.06	0.059	-5.667	-5.779	-5.602	2.799	2.830	2.816
(45, 0.4, 0.3333)	6.510	6.487	6.501	-0.648	-0.644	-0.648	0.044	0.044	0.044	-4.043	-4.039	-4.076	8.420	8.494	8.443
(45, 0.4, 0.5833)	7.383	7.360	7.367	-0.582	-0.577	-0.582	0.036	0.035	0.036	-3.050	-3.032	-3.085	11.691	11.768	11.694

Appendix C.2. VALUES AND GREEKS OF AMERICAN CALLS ($K = \$100, \tau = 0.5 year$)

(S, σ, r, δ)	Price			Delta			Gamma			Theta			Vega		
	True	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**
(80, 0.2, 0.03, 0.07)	0.219	0.230	0.222	0.049	0.050	0.049	0.009	0.009	0.009	-0.990	-0.990	-0.985	5.740	5.808	5.714
(90, 0.2, 0.03, 0.07)	1.387	1.405	1.386	0.208	0.208	0.207	0.023	0.023	0.023	-2.919	-2.873	-2.898	17.968	17.968	17.891
(100, 0.2, 0.03, 0.07)	4.783	4.782	4.768	0.484	0.479	0.482	0.030	0.030	0.030	-3.965	-3.88	-3.992	27.190	27.190	27.122
(110, 0.2, 0.03, 0.07)	11.098	11.041	11.079	0.772	0.767	0.773	0.026	0.027	0.026	-2.592	-2.826	-2.627	21.291	21.422	21.183
(120, 0.2, 0.03, 0.07)	20.000	20.000	20.000	0.995	1.000	1.000	0.013	0.000	0.000	-0.000	0.000	0.000	0.414	0.000	0.000
(80, 0.4, 0.03, 0.07)	2.689	2.711	2.687	0.232	0.233	0.232	0.014	0.013	0.013	-6.09	-6.045	-6.059	17.147	17.057	16.994
(90, 0.4, 0.03, 0.07)	5.723	5.742	5.711	0.377	0.376	0.376	0.015	0.015	0.015	-8.206	-8.112	-8.179	23.727	23.666	23.614
(100, 0.4, 0.03, 0.07)	10.239	10.242	10.214	0.525	0.523	0.524	0.014	0.014	0.014	-9.092	-8.966	-9.110	27.267	27.297	27.227
(110, 0.4, 0.03, 0.07)	16.183	16.152	16.146	0.660	0.656	0.659	0.012	0.012	0.012	-8.614	-8.523	-8.698	27.218	27.286	27.123
(120, 0.4, 0.03, 0.07)	23.361	23.288	23.321	0.772	0.768	0.772	0.010	0.010	0.010	-7.101	-7.175	-7.208	23.844	24.061	23.768
(80, 0.3, 0.00, 0.07)	1.037	1.062	1.040	0.133	0.134	0.133	0.013	0.013	0.013	-2.954	-2.917	-2.930	11.960	12.089	12.007
(90, 0.3, 0.00, 0.07)	3.124	3.147	3.118	0.293	0.292	0.292	0.019	0.018	0.019	-4.938	-4.829	-4.917	21.493	21.396	21.366
(100, 0.3, 0.00, 0.07)	7.036	7.0280	7.015	0.492	0.487	0.491	0.020	0.020	0.020	-5.731	-5.604	-5.775	27.139	27.113	27.066
(110, 0.3, 0.00, 0.07)	12.956	12.886	12.928	0.689	0.682	0.689	0.019	0.019	0.019	-4.800	-4.922	-4.875	25.499	25.674	25.453
(120, 0.3, 0.00, 0.07)	20.718	20.607	20.695	0.858	0.859	0.859	0.015	0.017	0.015	-2.642	-3.627	-2.600	16.343	15.972	16.277
(80, 0.3, 0.07, 0.03)	1.664	1.665	1.664	0.194	0.194	0.194	0.016	0.016	0.016	-5.147	-5.147	-5.146	15.356	15.474	15.471
(90, 0.3, 0.07, 0.07)	4.495	4.495	4.495	0.378	0.378	0.378	0.020	0.020	0.020	-8.229	-8.227	-8.226	24.018	23.943	23.937
(100, 0.3, 0.07, 0.03)	9.251	9.251	9.251	0.571	0.571	0.571	0.018	0.018	0.018	-9.811	-9.808	-9.807	27.237	27.251	27.240
(110, 0.3, 0.07, 0.03)	15.798	15.799	15.798	0.731	0.731	0.731	0.014	0.014	0.014	-9.540	-9.539	-9.537	24.655	24.776	24.758
(120, 0.3, 0.07, 0.03)	23.706	23.709	23.707	0.843	0.843	0.843	0.009	0.009	0.009	-8.093	-8.095	-8.092	18.859	19.055	19.026

Appendix C.3. VALUES AND GREEKS OF AMERICAN PUTS ($K = \$100, \tau = 3.0$ *years*, $\sigma = 0.2, r = 0.08$)

(S, δ)	**Price**			**Delta**			**Gamma**			**Theta**			**Vega**		
	True	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**
(80, 0.12)	25.658	26.245	25.725	-0.61	-0.61	-0.605	0.011	0.01	0.011	-1.304	-1.142	-1.235	33.945	35.257	33.784
(90, 0.12)	20.083	20.641	20.185	-0.506	-0.512	-0.504	0.010	0.010	0.010	-1.82	-1.739	-1.76	43.881	45.682	43.772
(100, 0.12)	15.498	15.99	15.608	-0.412	-0.42	-0.412	0.009	0.009	0.009	-2.193	-2.168	-2.154	51.196	53.593	51.430
(110, 0.12)	11.803	12.221	11.905	-0.329	-0.336	-0.33	0.008	0.008	0.008	-2.387	-2.397	-2.369	55.394	58.024	55.787
(120, 0.12)	8.886	9.235	8.974	-0.257	-0.263	-0.258	0.007	0.007	0.007	-2.414	-2.443	-2.410	56.181	58.886	56.695
(80, 0.08)	22.205	22.395	22.148	-0.688	-0.674	-0.688	0.019	0.019	0.02	-0.68	-0.582	-0.751	37.379	39.468	37.189
(90, 0.08)	16.208	16.498	16.170	-0.519	-0.512	-0.516	0.015	0.014	0.015	-1.112	-0.980	-1.132	51.835	53.465	51.787
(100, 0.08)	11.704	12.03	11.700	-0.387	-0.386	-0.384	0.012	0.011	0.012	-1.388	-1.277	-1.371	57.988	59.600	57.986
(110, 0.08)	8.368	8.687	8.390	-0.285	-0.287	-0.283	0.009	0.009	0.009	-1.499	-1.422	-1.468	58.315	59.986	58.253
(120, 0.08)	5.931	6.222	5.968	-0.206	-0.210	-0.205	0.007	0.007	0.007	-1.476	-1.430	-1.445	54.601	56.420	54.592
(80, 0.04)	20.349	20.325	20.336	-0.838	-0.836	-0.84	0.035	0.038	0.035	-0.206	-0.587	-0.223	22.374	22.233	21.398
(90, 0.04)	13.497	13.563	13.471	-0.554	-0.543	-0.554	0.023	0.022	0.023	-0.586	-0.601	-0.634	49.71	51.434	49.354
(100, 0.04)	8.944	9.108	8.931	-0.369	-0.362	-0.367	0.015	0.014	0.015	-0.807	-0.713	-0.829	57.29	58.747	56.991
(110, 0.04)	5.912	6.122	5.92	-0.246	-0.243	-0.244	0.010	0.010	0.010	-0.877	-0.774	-0.873	54.86	56.231	54.587
(120, 0.04)	3.898	4.115	3.922	-0.163	-0.163	-0.162	0.007	0.007	0.007	-0.844	-0.760	-0.825	47.951	49.466	47.776
(80, 0.0)	20.000	20.000	20.000	-1.000	-1.000	-1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
(90, 0.0)	11.695	11.634	11.705	-0.621	-0.615	-0.619	0.035	0.037	0.035	-0.268	-0.291	-0.272	41.572	42.106	41.125
(100, 0.0)	6.931	6.962	6.956	-0.358	-0.349	-0.357	0.019	0.019	0.019	-0.436	-0.439	-0.436	52.89	54.353	52.644
(110, 0.0)	4.154	4.257	4.190	-0.211	-0.206	-0.21	0.011	0.011	0.011	-0.481	-0.401	-0.47	48.489	49.892	48.292
(120, 0.0)	2.510	2.640	2.551	-0.126	-0.125	-0.125	0.006	0.006	0.006	-0.450	-0.367	-0.433	39.413	40.945	39.387

Appendix C.4. VALUES AND GREEKS OF AMERICAN CALLS ($K = \$100, \tau = 3.0$ *years*)

(S, σ, r, δ)	**Price**			**Delta**			**Gamma**			**Theta**			**Vega**		
	True	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**	**True**	**Quad**	**Mquad**
(80, 0.2, 0.03, 0.07)	2.580	2.711	2.605	0.200	0.203	0.200	0.011	0.011	0.011	-0.668	-0.615	-0.654	35.641	36.635	35.671
(90, 0.2, 0.03, 0.07)	5.168	5.301	5.181	0.321	0.319	0.320	0.013	0.013	0.013	-0.825	-0.746	-0.820	49.638	50.459	49.444
(100, 0.2, 0.03, 0.07)	9.066	9.154	9.065	0.462	0.455	0.460	0.015	0.014	0.015	-0.848	-0.783	-0.861	58.246	59.262	58.004
(110, 0.2, 0.03, 0.07)	14.443	14.444	14.430	0.616	0.606	0.615	0.016	0.016	0.016	-0.719	-0.767	-0.746	57.463	58.687	57.077
(120, 0.2, 0.03, 0.07)	21.414	21.336	21.398	0.78	0.775	0.780	0.017	0.018	0.017	-0.459	-0.802	-0.477	43.296	43.787	42.844
(80, 0.4, 0.03, 0.07)	11.327	11.625	11.336	0.404	0.406	0.403	0.007	0.007	0.007	-2.046	-1.921	-2.024	47.361	48.285	47.403
(90, 0.4, 0.03, 0.07)	15.723	16.028	15.711	0.474	0.474	0.472	0.007	0.007	0.007	-2.199	-2.046	-2.194	53.737	54.583	53.618
(100, 0.4, 0.03, 0.07)	20.794	21.084	20.760	0.539	0.537	0.537	0.006	0.006	0.006	-2.258	-2.091	-2.277	58.099	59.127	58.020
(110, 0.4, 0.03, 0.07)	26.496	26.749	26.440	0.600	0.596	0.598	0.006	0.006	0.006	-2.231	-2.073	-2.279	60.591	61.825	60.505
(120, 0.4, 0.03, 0.07)	32.782	32.982	32.709	0.657	0.650	0.655	0.005	0.005	0.005	-2.131	-2.011	-2.207	61.158	62.634	61.044
(80, 0.3, 0.00, 0.07)	5.518	5.658	5.552	0.285	0.284	0.285	0.009	0.009	0.009	-1.048	-0.953	-1.035	42.258	42.794	42.131
(90, 0.3, 0.00, 0.07)	8.842	8.947	8.868	0.381	0.375	0.379	0.010	0.009	0.010	-1.172	-1.072	-1.168	51.720	52.307	51.549
(100, 0.3, 0.00, 0.07)	13.142	13.177	13.158	0.48	0.472	0.479	0.010	0.010	0.010	-1.193	-1.139	-1.200	57.964	58.706	57.742
(110, 0.3, 0.00, 0.07)	18.453	18.394	18.458	0.582	0.572	0.581	0.010	0.010	0.010	-1.112	-1.182	-1.125	59.722	60.598	59.429
(120, 0.3, 0.00, 0.07)	24.790	24.638	24.786	0.685	0.677	0.684	0.010	0.011	0.010	-0.935	-1.247	-0.948	55.608	56.577	55.545
(80, 0.3, 0.07, 0.03)	12.146	12.282	12.176	0.48	0.485	0.481	0.009	0.009	0.009	-3.227	-3.250	-3.230	50.762	51.781	50.967
(90, 0.3, 0.07, 0.03)	17.368	17.553	17.411	0.562	0.567	0.563	0.008	0.008	0.008	-3.568	-3.593	-3.569	54.904	56.277	55.302
(100, 0.3, 0.07, 0.03)	23.348	23.586	23.402	0.632	0.637	0.633	0.006	0.006	0.006	-3.746	-3.77	-3.742	56.760	58.171	57.047
(110, 0.3, 0.07, 0.03)	29.964	30.259	30.028	0.689	0.695	0.69	0.005	0.005	0.005	-3.788	-3.804	-3.776	56.593	57.996	56.745
(120, 0.3, 0.07, 0.03)	37.104	37.459	37.176	0.737	0.743	0.738	0.004	0.004	0.004	-3.721	-3.726	-3.699	54.543	56.321	54.970

Appendix D. Variance Gamma American Option pricing Approximation comparison

Strike	T = 0.13972					
	EuroVG	**FD**	**Approx1**	**Approx2**	**Approx3**	**JZ-Approx**
1200	4.872		4.89	4.89	4.894	1.642
1220	6.24		6.268	6.268	6.276	2.733
1240	7.957		8.0	8.0	8.018	4.365
1260	10.105		10.171	10.171	10.206	6.706
1280	12.783		12.883	12.883	12.95	9.937
1300	16.107		16.256	16.257	16.379	14.237
1320	20.219		20.442	20.442	20.657	19.771
1340	25.287		25.615	25.615	25.98	26.675
1360	31.509		31.983	31.984	32.582	35.046
1380	39.121		39.796	39.797	40.745	44.934
1400	48.397		49.34	49.340	50.791	56.337

Strike	T = 0.21643					
	EuroVG	**FD**	**Approx1**	**Approx2**	**Approx3**	**JZ-Approx**
1200	10.892		10.965	10.965	10.97	6.03
1220	13.038		13.138	13.138	13.163	8.386
1240	15.561		15.699	15.699	15.761	11.404
1260	18.522		18.71	18.711	18.839	15.185
1280	21.987		22.242	22.244	22.485	19.826
1300	26.033		26.377	26.380	26.805	25.413
1320	30.748		31.208	31.212	31.928	32.017
1340	36.23		36.84	36.846	38.005	39.695
1360	42.591		43.393	43.401	45.215	48.484
1380	49.96		51.002	51.015	53.765	58.399
1400	58.48		59.821	59.838	63.887	69.437

Strike	T = 0.46575					
	EuroVG	**FD**	**Approx1**	**Approx2**	**Approx3**	**JZ-Approx**
1200	31.639		32.189	32.218	31.896	27.942
1220	35.21		35.874	35.917	35.625	32.898
1240	39.118		39.916	39.978	39.78	38.433
1260	43.388		44.343	44.43	44.432	44.57
1280	48.048		49.188	49.305	49.67	51.326
1300	53.128		54.482	54.637	55.613	58.719
1320	58.658		60.259	60.462	62.411	66.758
1340	64.671		66.557	66.817	70.254	75.452
1360	71.201		73.414	73.742	79.384	84.803
1380	78.287		80.872	81.282	90.108	94.812
1400	85.967		88.973	89.479	102.804	105.473

Strike	T = 0.56164					
	EuroVG	**FD**	**Approx1**	**Approx2**	**Approx3**	**JZ-Approx**
1200	33.397		34.108	34.137	33.832	30.891
1220	37.167		38.018	38.061	37.828	36.032
1240	41.287		42.3	42.362	42.281	41.731
1260	45.781		46.984	47.068	47.258	48.009
1280	50.676		52.099	52.212	52.846	54.882
1300	56.0		57.676	57.824	59.149	62.363
1320	61.781		63.748	63.938	66.294	70.461
1340	68.049		70.348	70.59	74.437	79.184
1360	74.835		77.512	77.815	83.767	88.533
1380	82.172		85.276	85.651	94.509	98.508
1400	90.093	95.463	93.678	94.138	106.933	109.107

Appendix E. Variance Gamma American Option pricing early exercise premium comparison

Strike	**T = 0.13972**			
	True	**Approx1**	**Approx2**	**Approx3**
1200	0.025	0.018	0.019	0.021
1220	0.070	0.028	0.029	0.036
1240	0.117	0.043	0.043	0.061
1260	0.159	0.066	0.066	0.102
1280	0.196	0.100	0.257	0.167
1300	0.239	0.151	0.151	0.273
1320	0.428	0.224	0.223	0.439
1340	0.608	0.328	0.328	0.692
1360	0.869	0.474	0.474	1.074
1380	1.103	0.676	0.676	1.625
1400	1.482	0.942	0.943	2.395

Strike	**T = 0.21643**			
	True	**Approx1**	**Approx2**	**Approx3**
1200	0.063	0.073	0.073	0.021
1220	0.124	0.100	0.101	0.036
1240	0.191	0.138	0.138	0.061
1260	0.253	0.188	0.189	0.102
1280	0.322	0.255	0.257	0.167
1300	0.392	0.344	0.347	0.273
1320	0.602	0.460	0.464	0.439
1340	0.791	0.610	0.616	0.692
1360	1.078	0.802	0.810	1.074
1380	1.374	1.043	1.054	1.625
1400	1.678	1.341	1.358	2.395

Strike	**T = 0.46575**			
	True	**Approx1**	**Approx2**	**Approx3**
1200	0.539	0.550	0.579	0.257
1220	0.677	0.663	0.707	0.415
1240	0.835	0.798	0.860	0.662
1260	0.997	0.956	1.042	1.043
1280	1.185	1.139	1.257	1.622
1300	1.388	1.354	1.509	2.485
1320	1.681	1.601	1.804	3.753
1340	2.008	1.886	2.146	5.583
1360	2.392	2.212	2.542	8.183
1380	2.795	2.585	2.995	11.821
1400	3.291	3.006	3.512	16.838

Strike	**T = 0.56164**			
	True	**Approx1**	**Approx2**	**Approx3**
1200	1.041	0.711	0.741	0.435
1220	1.250	0.851	0.894	0.661
1240	1.489	1.013	1.075	0.994
1260	1.742	1.203	1.287	1.477
1280	2.023	1.423	1.536	2.170
1300	2.339	1.676	1.824	3.149
1320	2.741	1.967	2.158	4.513
1340	3.178	2.299	2.541	6.388
1360	3.687	2.677	2.979	8.932
1380	4.241	3.104	3.479	12.337
1400	4.861	3.585	4.045	16.839