Structured Programs have Small Tree-Width and Good Register Allocation*

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Abstract

The register allocation problem for an imperative program is often modelled as the coloring problem of the interference graph of the control-flow graph of the program. The interference graph of a flow graph G is the intersection graph of some connected subgraphs of G. These connected subgraphs represent the lives, or life times, of variables, so the coloring problem models that two variables with overlapping life times should be in different registers. For general programs with unrestricted gotos, the interference graph can be any graph, and hence we cannot in general color within a factor $O(n^\varepsilon)$ from optimality unless NP=P.

It is shown that if a graph has tree-width k, we can efficiently color any intersection graph of connected subgraphs within a factor $(\lfloor k/2 \rfloor + 1)$ from optimality. Moreover, it is shown that structured ($\equiv$ goto-free) programs, including, for example, short circuit evaluations and multiple exits from loops, have tree-width at most 6. Thus, for every structured program, we can do register allocation efficiently within a factor 4 from optimality, regardless of how many registers are needed.

The bounded tree-width of structured programs imply that the many techniques for bounded tree-width may now be applied in control-flow analysis.

1 Introduction

The *register allocation problem* for an imperative program P is usually modelled as the coloring problem of the interference graph I of the control-flow graph G of P [3, 15, 16, 20]. The *control-flow graph* G is a digraph representing the flow of control between program points in the execution of the program P (see Figure 2.2, page 9). The orientation is, however, unimportant in our context, and we will hence perceive G as undirected. The *life time of a variable* is a connected subgraph of G, and the *interference graph* I is the intersection graph of the set X of all life times of variables in the program P. Thus adjacency in I denotes intersecting life times of variables. Consequently, our problem of coloring the interference graph I models that two variables with overlapping life times should be in different registers.

It should be noted that even with a good coloring, we may still run short of physical registers, in which case we have an additional *spilling problem* of simulating desired registers with the physical registers by copying to and from memory locations. The coloring then limits the amount of memory locations needed. The spilling problem is not addressed in this paper.

For general programs with unrestricted gotos, the control-flow graph can be any graph, and so can the interference graph. Hence for some fixed $\varepsilon > 0$, we cannot in polynomial time color

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within a factor $O(n^\epsilon)$ from optimality unless NP=P [30]. The current best approximation factor is $O(n(\log\log n)/(\log n)^3)$ [26]. However, in this paper we show that for structured ($\equiv$ goto-free) programs[1], including, for example, short circuit evaluations and multiple exits from loops, we can do register allocation in polynomial time within a factor 4 from optimality.

Recently Kannan and Proebsting [27] showed that if the control-flow graph of a program is series-parallel, we can color the interference graph within a factor 2 from optimality. The relevance of series-parallelism follows from the well-known fact (see e.g. [3]) that many of the structured language constructs, such as if-then-else, and while-loops, allows programs to be recursively sub-divided into basic blocks with a single entry and a single exit point. Such a recursive sub-division immediately gives a series-parallel decomposition of the flow-graph (see e.g. [33]). However, even within structured languages, there are well-known exceptions to the sub-division into basic-blocks/series-parallelism. For example [27] points to short circuit evaluation where the evaluation of a boolean expression is terminated as soon as the correct answer is found, e.g. if $e = x_1 \vee \cdots \vee x_n$ is true, then e is only evaluated till the first true x_i is found. Other exceptions include loops with multiple exits/breaks and programs/functions with multiple stop-/return-statements. In [27] the problem of dealing with these exceptions is suggested. So far, however, there has been no approach suggesting that the control-flow graphs of general structured programs should be any simpler than general graphs. The concept of reducibility [3, pp. 606–607] of control-flow graphs is associated with structured programs, but reducibility only refers to the orientation of the edges (any acyclic orientation is reducible). In our case, we are only interested in the underlying undirected graph, so the requirement of reducibility does not limit the class of graphs considered.

Our key to dealing with general structured programs is to introduce the following measure:

Definition 1 *Given a graph G, a* ($\leq k$)-complex listing *is a listing of the vertices of G such that for every vertex $v \in V$, there is a set S_v of at most k of the vertices preceding v in the listing, whose deletion from G separates v in G from all the vertices preceding v in the listing. In this case, we say that G is* ($\leq k$)-complex. *The set S_v is referred to as the* separator *of v in the listing.*

Note above that for a given listing and a vertex v, there is a unique minimal choice of the separator S_v; namely as the set of preceding vertices that can be reached by a path from v with no interior vertices preceding v in the listing.

In [27], Kannan and Proebsting implicitly show that series-parallel graphs are 2-complex. In fact this is the essential property of series-parallel graphs that they use in their algorithms. In this paper, we generalize their techniques to show

Theorem 2 *Given a k-complex listing of the vertices of a graph G, we can efficiently color the intersection graph I of any set X of connected subgraphs of G within a factor $(\lfloor k/2 \rfloor + 1)$ from optimality. If n is the number of nodes in G and ω is the maximal number of subgraphs from X intersecting a single vertex in G, the coloring is done in time $O(k\omega n + \omega^{2.5} n)$. Also, we can color I within a factor $(k+1)$ from optimality in time $O(k\omega n)$.*

Note that for $k = 2$, we get the factor 2 from [27]. Also note that the 1-complex graphs are the forests for which we get a factor 1. Hence follows the colorability of chordal graphs [22, 23].

We show the significance of Theorem 2 by showing

Theorem 3 *Assuming short circuit evaluation,*

- *Goto-free Algol [31] and Pascal [38] programs have (≤ 3)-complex control-flow graphs.*
- *All Modula-2 [39] programs have (≤ 5)-complex control-flow graphs.*

[1] Structured programs is not an well-defined agreed-upon term (see e.g. [18, 19, 29, 31, 32]). In this paper, we are referring specifically to the aspect of being goto-free.

- *Goto-free C [28] programs have (≤ 6)-complex control-flow graphs.*

Without short circuit evaluation, each of the above complexities drops by one[2]*. Control-flow graphs with listings of the above complexities are derived directly (linear time, small constants) from the parse trees of the programs.*

The reason for the gap between Algol/Pascal and Modula-2 is that Modula-2 has loops with multiple exits and multiple returns from functions. The further gap to C is due to C's continue-statement jumping to the beginning of a loop. Combining Theorems 2 and 3, we get

Corollary 4 *In time $O(\omega n + \omega^{2.5} n)$, the register allocation problem can be solved within the following factors from optimality:*

- *2 (2) for Algol/Pascal,*
- *3 (3) for Modula-2, and*
- *4 (3) for C.*

The parenthesized numbers are without short circuit evaluation. If we only want to spend time $O(\omega n)$, we can get the factors: 4 (3) for Algol/Pascal, 6 (5) for Modula-2, and 7 (6) for C.

Finally, we show that k-complexity is a short new definition of tree-width k defined in [35].

Theorem 5 *A graph is k-complex if and only if it has tree-width k. Moreover there are linear transformations between k-complex listings and tree-decompositions of width k.*

It should be noted that our work on the register allocation problem does not follow the usual pattern of deriving fast algorithms for graphs of bounded tree-width (see e.g. [5, 6, 9, 11, 17]). First of all we are studying intersection graphs of connected subgraphs rather than the graph itself. Second, the coloring problem we consider is NP-complete for any $k > 1$. The NP-completeness follows from the fact that a cycle has tree-width 2, and for a cycle the coloring problem for the intersection graph is known to be NP-complete [21].

The relationship to tree-width is relevant to goto-users. One could imagine that even goto-users have structured thoughts [29], hence that also the control-flow-graphs of their programs have simple listings/bounded tree-width. For variable k, the problem of deciding the tree-width is NP-hard [4]. For fixed k, however, there are linear time algorithms [8]. Also, for variable k, there has been work done on polynomial approximating algorithms [10]. Moreover, from Bellcore, there is a commercially available tree-width heuristic by Cook and Seymour. Our derivation of simple listings from syntax, as described in Theorem 3, is, however, much simpler than the general approaches to tree-width, so the general advice following from this paper is: you help not only yourself and your fellow humans [19, 32], but also the optimizer, by not using goto's, thus making the structure explicit from the syntax.

Detailed comparison with previous work:

- The classic approach [15, 16] to register allocation via graph coloring uses the scheme: let x be a variable/vertex in the interference graph I. Color $I \setminus \{x\}$ recursively. Color x with the least color not used by any neighboring variable in I. Typically x is chosen to be of low degree, but this does not in itself lead to any guarantees for the quality of the produced coloring.

Our $(k+1)$-approximation algorithm can produce a specific sequencing $x_1, \ldots, x_n$ of the variables so that if we choose the x in the classic scheme following this order, the largest degree encountered

[2]Standard Pascal and Algol does not have short circuit evaluation while standard Modula-2 and C does have short circuit evaluation.

becomes at most $(k+1)\omega - 1$. Here ω is the maximal number of variables live at any single point of the control-flow graph. Thus, given our sequence, the classic coloring scheme will use at most $(k+1)\omega$ colors, and since at least ω colors are needed, this is at most a factor $(k+1)$ from optimality.

- The interference graph I may be of size quadratic in the number of variables and its construction is considered a main obstacle for coloring based register allocation. Hence, for space reasons, many heuristics aim at only having parts of the I constructed at any time [14, 25, 34]. For our $(\lfloor k/2 \rfloor + 1)$-approximation algorithm the biggest sub-graphs considered are of size $O(\omega^2)$. From the k-complex listing, for each variable, our $(k+1)$-approximation algorithm identifies a small set of potential colored neighbors, but it never checks if they are actual neighbors, that is, it never checks for any two variables whether they actually interfere. Thus our $(k+1)$-approximation algorithm does not produce any part of the interference graph!
- In [25] they color straight line code optimally. By definition the control-flow graph of straight line code is a single path which is 1-complex and is hence also optimally colored by our $(\lfloor k/2 \rfloor + 1)$-approximation algorithm. In [25] they try to use this for the coloring of the straight line code in an innermost loop - which is assumed to be executed most frequently. Good coloring of the innermost loops is also the concern in [14]. However, as observed in [14], if there is more than one innermost loop, the coloring of one may negatively effect the possibility of coloring the other. The variables of different innermost loops may interfere non-trivially, so we cannot just address them independently. Now, suppose that all the most critical parts, like innermost loops, have been marked. Our approximation algorithms can then be used to first find a good coloring of the variables in the critical parts, and afterwards color the rest of the variables with different colors.
- Our algorithms are generalizations of those in [27] for series parallel control-flow graphs. If the control-flow graph is not series parallel, [27] suggest heuristics for removing a minimal set of edges so that the graph becomes a series-parallel. The removed "exception" edges requires special treatment. If the program execution goes through an exception edge, all register values may have to be reassigned. Note that with our approach we have no exceptions: the tree-width may vary, but this only affects the quality of the coloring; no special action needs to be taken.

In [27] they mention short circuit evaluation as a prime example of an obstruction to series parallelism. For example, goto-free Pascal without short circuit evaluation has series parallel control-flow graphs, but if we allow short circuit evaluation we may get exceptions to series parallelism. In fact, short circuit evaluation alone may give rise to arbitrarily many exceptions to series parallelism. However, the tree-width only grows from 2 to 3. As stated in Corollary 4, for our $(\lfloor k/2 \rfloor + 1)$-approximation algorithm, this change does not give a worse approximation factor!

Implications

- With reference to Theorem 3 and 5, it seems that tree-width of control-flow graphs offers a well-defined mathematical measure for how structured programs, or programming languages, are. Tree-width is an established measure for the computational complexity of graphs, and now it turns out to capture aspects of structured programming [18].
- Our result suggests banning gotos for the sake of optimization. Gotos have long been considered harmful to the readability of programs for humans [19, 32], and further gotos may obstruct bounded tree-width. Wirth's move from Pascal [38] to Modula-2 [39] is an excellent example of what can be done. In Modula-2 there are no gotos, but to reconcile the programmers, the language have been enriched with some extra exit structures - multiple exits from loops and multiple returns from functions. As a consequence, we get a tree-decomposition of width at most 5 as a free side-effect the parsing of any Modula-2 program.
- A substantial theory of tree-width has developed, and we contribute to this theory by showing that not only are graphs of bounded tree-width computationally simple, but so are their intersection graphs. Concretely we color intersection graphs of graphs with tree-width k within a factor $O(k)$

from optimality, while for some ε, coloring within a factor $O(n^\varepsilon)$ from optimality is NP-hard for general graphs [30]. This is the first concrete result demonstrating the computational simplicity of intersection graphs of graphs of bounded tree-width.

- For bounded tree-width, many linear time algorithms are known for problems that are otherwise NP-complete [5, 6, 11, 17] or PSPACE-complete [9]. As a consequence of Theorem 3 together with Theorem 5, this understanding may now be applied in control-flow graph analysis. The constants bounding the tree-width are truly small (≤ 6), allowing us to develop algorithms working well in both theory and practice. An example is given in [2] where bounded tree-width is used to derive a linear time algorithm for generalized dominators [24]. The best algorithm for general control-flow-graphs runs in $O(nm)$ algorithm [1].

The paper is divided as follows. Section 2 addresses Theorem 3. Section 3 shows how simple listings may be preserved under the standard optimizations from [3]. Section 4 proves Theorem 2. In Section 5, we present an efficient algorithms for computing the minimum separators of any listing, and in Section 6, we discuss a linear time heuristic for finding a good tree-decomposition directly from the three-address code generated, as in [3], from a structured program. The last two algorithms makes it easier to integrate our approach with compilers where the structure of a program has been lost by the time of register allocation Finally Section 7 presents the proof of Theorem 5.

For basic definitions for programs, grammars, and control-flow graphs, the reader is referred to [3].

2 Simplicity of structured programs

In this section, we will address Theorem 3. Our first tool, is the following simple lemma:

Lemma 6 *From a $(\leq k)$-complex listing L of a graph G, we can derive a $(\leq k)$-complex listing of G with an edge $\{v, w\}$ contracted.*

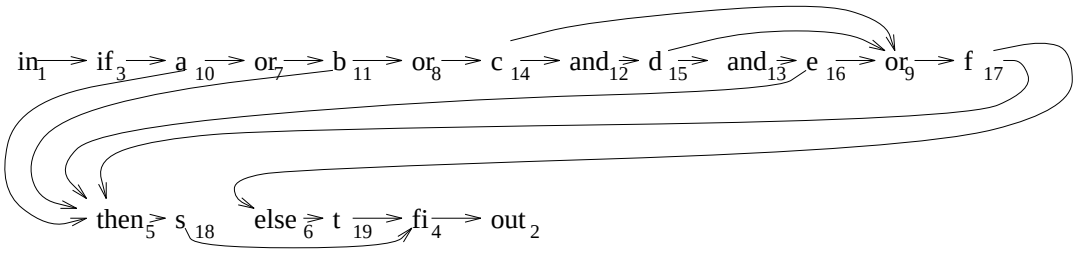
Proof: By symmetry, we may assume v comes before w. Then, we contract $\{v, w\}$ identifying both with v, and deleting w from L. ■

We will argue the correctness of Theorem 3 by studying a Modula-2 [39] inspired toy-language STRUCTURED, illustrating all the essential problems of finding good listing of control-flow graphs for structured programs. STRUCTURED is defined by the following grammar:

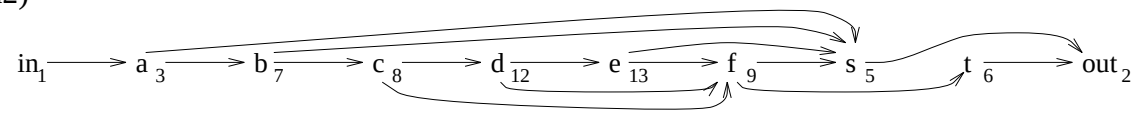
$P \rightarrow$ program S margorp,
$S \rightarrow S$; S | if B then S else S fi | if B then S fi | loop S pool | exit | stop | s | t | $\cdots$
$B \rightarrow B$ or $B \mid C$
$C \rightarrow C$ and C | a | b | $\cdots$

The grammar does not resolve the associativity of ';', 'or', 'and', but the ambiguity is irrelevant to the control-flow graphs that we will produce. In Figure 2.1 are given some examples of control-flow graphs for fragments of programs written in STRUCTURED with an 'in' and an 'out' node showing the entry and exit points. In (a1) and (b1) we have been very liberal in the use of vertices, letting every single word in the program constitute its own vertex. The indices describe 3-complex listings of the vertices. In (a2) and (b2) we have been more conservative in the use of vertices, contracting some of the edges from (a1) and (b1). However, using Lemma 6, we have inherited 3-complex listings of the vertices. Thus, because of Lemma 6, without loss of generality, we can restrict our attention to control-flow graphs with one vertex for each word of the program.

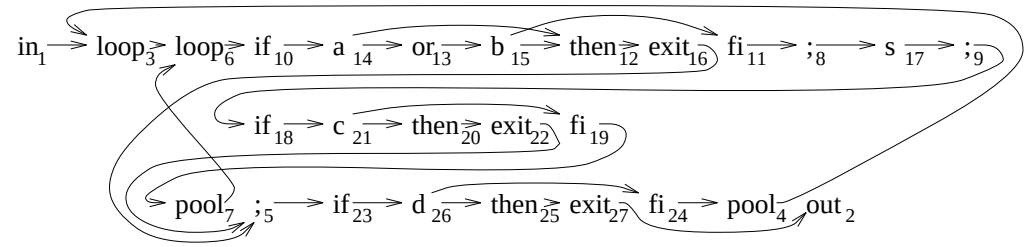
(a1)



(a2)



(b1)



(b2)

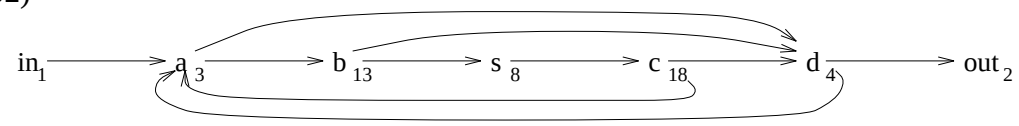


Figure 2.1: Some control-flow graphs

We shall later argue that from the viewpoint of k-complexity, STRUCTURED captures all the typical complications of control-flow graphs of imperative structured programs. First, however, we will describe an algorithm FLOW generating control-flow graphs for programs written in STRUCTURED. The vertices will be indexed according to a (≤ 5)-complex listing. An example is given in Figure 2.2.

Algorithm A: FLOW(P) returns the flow graph for P with indices indicating a (≤ 5)-complex listing of the vertices. Moreover, for each vertex v, as a side-effect it produces the separator $S(v)$ of v in the listing.

A.1. Let 'P = program S margorp'

A.2. $S(\text{program}_1) := \emptyset$; $S(\text{margorp}_2) := \{\text{program}_1\}$;

A.3. Return $\text{FLOW}_1(S, \text{program}_1, \text{margorp}_2, \text{margorp}_2, \text{margorp}_2, 3)$.

Algorithm B: $\text{FLOW}_1(S, in, out, exit, stop, next)$ returns the sub-graph for S with indices starting from $next$, and links the sub-graph with the vertices $in, out, exit, stop$. It is assumed that $in, out, exit, stop$ have indices $< next$.

B.1. If '$S = S'$; S''' then

B.1.1. $(G', next') := \text{FLOW}_1(S', in, ;_{next}, exit, stop, next + 1)$;

B.1.2. $(G'', next'') := \text{FLOW}_1(S'', ;_{next}, out, exit, stop, next')$;

B.1.3. $S(;_{next}) := \{in, out, exit, stop\}$;

B.1.4. Return $(G' \cup G'', next'')$.

B.2. If '$S =$ if B' then S'' else S''' fi' then

B.2.1. $(G', next') := \text{FLOW}_2(B', \text{if}_{next}, \text{then}_{next+2}, \text{else}_{next+3}, next + 4)$;

B.2.2. $(G'', next'') := \text{FLOW}_1(S'', \text{then}_{next+2}, \text{fi}_{next+1}, exit, stop, next')$;

B.2.3. $(G''', next''') := \text{FLOW}_1(S''', \text{else}_{next+3}, \text{fi}_{next+1}, exit, stop, next'')$;

B.2.4. $S(\text{if}_{next}) := \{in, out, exit, stop\}$;

B.2.5. $S(\text{fi}_{next+1}) := \{\text{if}_{next}, out, exit, stop\}$;

B.2.6. $S(\text{then}_{next+2}) := \{\text{if}_{next}, \text{fi}_{next+1}, exit, stop\}$;

B.2.7. $S(\text{else}_{next+3}) := \{\text{if}_{next}, \text{then}_{next+2}, \text{fi}_{next+1}, exit, stop\}$;

B.2.8. Return $(G' \cup G'' \cup G''', next''')$.

B.3. If '$S =$ if B' then S'' fi' then

B.3.1. $(G', next') := \text{FLOW}_2(B', \text{if}_{next}, \text{then}_{next+2}, \text{fi}_{next+1}, next + 3)$;

B.3.2. $(G'', next'') := \text{FLOW}_1(S'', \text{then}_{next+2}, \text{fi}_{next+1}, exit, stop, next')$;

B.3.3. $S(\text{if}_{next}) := \{in, out, exit, stop\}$;

B.3.4. $S(\text{fi}_{next+1}) := \{\text{if}_{next}, out, exit, stop\}$;

B.3.5. $S(\text{then}_{next+2}) := \{\text{if}_{next}, \text{fi}_{next+1}, exit, stop\}$;

B.3.6. Return $(G' \cup G'', next''')$.

B.4. If '$S =$ loop S' pool' then

B.4.1. $(G', next') := \text{FLOW}_1(S', \text{loop}_{next}, \text{pool}_{next+1}, out, stop, next + 2)$.

B.4.2. $S(\text{loop}_{next}) := \{in, out, stop\}$; $S(\text{pool}_{next}) := \{\text{loop}_{next}, out, stop\}$;

B.4.3. Return $(G', next')$.

B.5. If '$S =$ exit' then $S(\text{exit}_{next}) := \{in, exit\}$; Return $(\{(in, \text{exit}_{next}), (\text{exit}_{next}, exit)\}, next+1)$.

B.6. If '$S =$ stop' then $S(\text{stop}_{next}) := \{in, stop\}$;Return $(\{(in, \text{stop}_{next}), (\text{stop}_{next}, stop)\}, next+1)$.

B.7. If '$S \in \{\text{s}, \text{t}, \ldots\}$' then $S(S_{next}) := \{in, out\}$; Return $(\{(in, S_{next}), (S_{next}, out)\}, next+1)$.

Algorithm C: $\text{FLOW}_2(B, in, true, false, next)$ returns the sub-graph for B with indices starting from $next$, and links the sub-graph with the vertices $in, true, false$. It is assumed that $in, true, false$ have indices $< next$.

C.1. If '$B = B'$ or B''' then

C.1.1. $(G', next') := \text{FLOW}_2(B', in, true, \text{or}_{next}, next+1)$;

C.1.2. $(G'', next') := \text{FLOW}_2(B'', \text{or}_{next}, true, false, next')$;

C.1.3. $S(\text{or}_{next}) := \{in, true, false\}$;

C.1.4. Return $(G' \cup G'', next'')$.

C.2. If '$B = C$' then Return $\text{FLOW}_3(C, true, false, next)$.

Algorithm D: $\text{FLOW}_3(C, in, true, false, next)$ returns the sub-graph for C with indices starting from $next$, and links the sub-graph with the vertices $in, true, false$. It is assumed that $in, true, false$ have indices $< next$.

D.1. If '$C = C'$ and C''' then

D.1.1. $(G', next') := \text{FLOW}_3(C', in, \text{and}_{next}, false, next+1)$;

D.1.2. $(G'', next'') := \text{FLOW}_3(C'', \text{and}_{next}, true, false, next')$;

D.1.3. $S(\text{and}_{next}) := \{in, true, false\}$;

D.1.4. Return $(G' \cup G'', next'')$.

D.2. If '$C \in \{\text{b}, \text{c}, \ldots\}$' then

D.2.1. $S(C_{next}) := \{in, true, false\}$;

D.2.2. Return $(\{(in, C_{next}), (C_{next}, true), (C_{next}, false)\}, next+1)$.

From the algorithm, we may conclude

Theorem 7 *All control-flow graphs derived from STRUCTURED are* (≤ 5)*-complex.* ■

We will now argue that STRUCTURED captures all the types of difficulties of structured programs written in imperative languages, and in particular, that it has the same complexity as Modula-2 [39]. First it should be mentioned that we are assuming the standard that control-flow graphs are generated separately for the main program and for procedures.

Control-flow graphs for while-statements and repeat-statements would just be contractions of special cases of our general loop. Modula-2 does not have the stop-statement from STRUCTURED. However, in Modula-2 functions we can have multiple return-statements and these have the effect of the stop-statement on the control-flow graphs for functions.

Note that our if-then-fi construction is superfluous in the sense that the same control-flow can be derived an if-then-else-fi, by contraction of the else-fi part to a fi.

Our algorithm should really be seen as a generic algorithm which is easily adapted to new constructs, say more exits. This might get us beyond 5-complexity, but as long as the number of different exit types is bounded, we are still getting a very good coloring compared with general coloring. In particular, for C [28], loops both have a break corresponding to the exit in STRUCTURED, and they have a continue-statement bringing the control back to the beginning of the loop. This

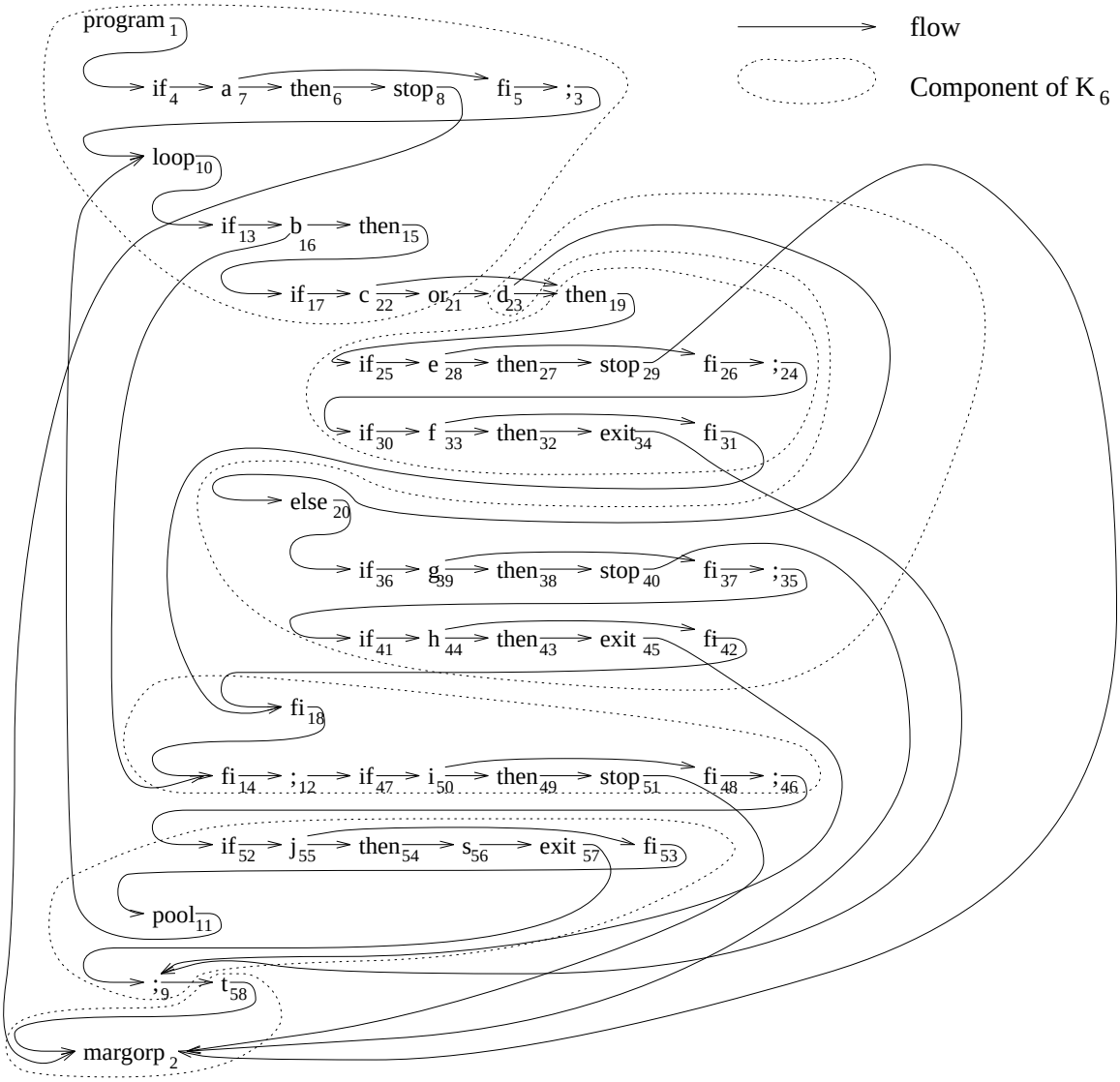


Figure 2.2: The 5-complex control-flow graph of a full program produced by FLOW. It is shown how the edges can be contracted so as to derive a clique of size 6, which is trivially 5-complex. Hence the original control-flow graph is ($\geq$ 5)-complex. The ($\leq$ 5)-complex listing produced by FLOW demonstrates that it is exactly 5-complex.

continue-statement is handled in the same fashion as the exit, forcing the complexity of goto-free C-programs up to 6.

In the introduction, it was claimed that the complexity would be decreased by one, if we did not have short circuit evaluation. Note that the only separators of size 5 are in connection with the if-then constructions. Now consider step B.2 where 'S=if B' then S'' else S''' fi'. Let B' be of the form Bb meaning that b is the last point in B'. Without short circuit evaluation, we know that the evaluation of B will go through b. Thus if we make the listing 'if$_{next}$, b_{next+1}, fi$_{next+2}$, then$_{next+3}$, else$_{next+4}$', the separators will be $S(\text{if}_{next}) = \{in, out, exit, stop\}$, $S(b_{next+1}) = \{\text{if}_{next}, out, exit, stop\}$, $S(\text{fi}_{next+2}) = \{b_{next+1}, out, exit, stop\}$, $S(\text{then}_{next+3}) = \{b_{next+1}, \text{fi}_{next+3}, exit, stop\}$, and $S(\text{else}_{next+4}) = \{b_{next+1}, \text{fi}_{next+3}, exit, stop\}$. Thus, we get a ($\leq 4$)-complex listing.

3 Robustness with respect other optimizations

In this section, we will briefly discuss how our simple listings from the previous section may be preserved under the standard optimizations mentioned in [3], hence that Theorem 3 holds despite these optimizations. For most optimizations, we just note that they can be done preserving the structural statements of a program. This holds for: redundant-instruction elimination, algebraic simplifications, use of machine idioms, elimination of global common subexpression, code motion, copy propagation, and elimination of induction variables. The only optimization from [3] that has to be done after the translation into object code, such as three-address code, is flow-of-control optimization. However, it turns out that standard flow-of-control optimization does not increase the complexity of the control-flow-graph. This is essentially because flow-of-control optimization corresponds to contraction of edges in the control-flow graph, as dealt with in Lemma 6. Take, for example, the following type of optimization from [3, p. 556] (a less contrived example is coming in Figure 3.3):

```
L0: goto L2          L0: goto L2
L1: goto L3          L1: goto L3
L2: goto L4  --->
L3: S                L3: S
L4: T                L2: T
```

The label numbers are assumed to be the numbers in the listing. The above program corresponds to a control-flow graph with edges $(0,2),(1,3),(2,4),(3,4)$. The optimization corresponds to contracting the edge $(2,4)$, giving the program point contracted to, the smaller number (2). This contraction implies that all old jumps to 4 now jumps to 2. Another example is,

```
L0: goto L2          L0: goto L2          L0: goto L2
L1: S                L1: S; goto L2       L1: S
L2: goto L4  --->                  --->   Ln: goto L2
L3: T                L3: T                L3: T
L4: U                L2: U                L2: U
```

Above, the first transformation is the same as before, but it introduces an extra statement that needs to have its own program point. This is achieved by the second transformation. Above n is understood to be the last free program point. We need to argue that this choice does not increase the complexity of our listing, but that follows from

Lemma 8 *Let L be a ($\leq k$)-complex listing L of a graph G. Suppose an edge (v,w) is split by a new vertex u (replacing (v,w) with (v,u) and (u,w)). We then get a new ($\leq k$)-complex listing if we inserting u anywhere after v and w in the listing.*

Proof: There are simply no vertices that have u in their minimum separators. ■

Lemmas 6 and 8 suffice for any flow-of-control optimization mentioned in [3]. A more extensive example of flow-of-control optimization is presented is presented in Figure 3.3.

4 Coloring the intersection graphs of k-complex graphs

Let $G = (V, E)$ be a graph and $v_1, \ldots, v_n$ a k-complex listing of its vertices. Moreover, let X be a set of connected sub-graphs of G, called *variables*, and let the *interference graph* I be the intersection graph over X. That is, two variables $v, w \in X$ are adjacent in I if and only if they they intersect in G. In this section, we study the problem of coloring I given that k is a constant. Our colors will be numbers $1, 2, \ldots$, and our aim is to minimize the maximum color used.

We use $\chi(I)$ to denote the chromatic number of I, i.e. the minimal number of colors needed to color I. For each v_i, set $P_{v_i} = \{v_1, \ldots, v_{i-1}\}$. Moreover, let S_{v_i} denote the set of $w \in P_{v_i}$ that can be reached by a path from v_i with no interior vertices in P_{v_i}. By Definition 1, $|S_{v_i}| \leq k$. Let X_{v_i} be the set of variables $x \in X$ containing v. Note that X_{v_i} is a clique in I. Finally, let $X^*_{v_i}$ denote $X_{v_i} \cup \bigcup_{w \in S_{v_i}} X_w$.

Lemma 9 *If $x \in X_v$ does not intersect P_v but intersects a variable y intersecting P_v, then y intersects S_v.*

Proof: There exists a path in $x \cup y$ from v to a vertex in P_v. The first vertex u in this path which is in P_v is in S_v. Since x does not intersect P_v, $u \in y$. ■

The following generalizes an algorithm from [27] for series-parallel graphs ($k \leq 2$):

Algorithm E: Colors I with at most $(k+1)\chi(I)$ colors.

E.1. For $i := 1, \ldots, n$,

E.1.1. Color the uncolored variables $x \in X_{v_i}$ with the smallest colors not used by variables intersecting S_{v_i}.

Correctness: From Lemma 9 it follows that the algorithm produces a proper coloring of I, i.e. that no two intersecting variables get the same color. When coloring the uncolored variables in X_{v_i}, the largest color used is at most the total number of colors used in $X^*_{v_i}$. Moreover $X^*_{v_i}$ is the union of at most $k+1$ optimally colored sets; namely the cliques $X_w, w \in S_{v_i} \cup \{v_i\}$. Thus the largest color used is $\leq (k+1)\chi(I)$. ■

Note that we color I without constructing I. The algorithm is trivially implemented to run in time $O(n\omega k)$ where $\omega = \max |X_v|$. This matches the time bound of the second algorithm from Theorem 2.

A *biclique* is a graph whose vertex set is partitioned into two cliques. Then

Lemma 10 ([27]) *Let $G = (V_1 \cup V_2, E)$ be a biclique on n vertices with the induced subgraphs on V_1 and V_2 being cliques. Then there is an $O(n^{2.5})$ algorithm that optimally colors G.*

Using the same idea as in [27], we get an improved algorithm using at most $k\chi(I)$ colors if we modify step E.1.1 as follows. If $S_{v_i} \neq \emptyset$, choose any $p(v_i) \in S_{v_i}$ and color the biclique $I|(X_{v_i} \cup X_{p(v_i)})$ using Lemma 10. Rename the colors so that the coloring of $X_{p(v_i)}$ is not changed, and such that the new colors for X_{v_i} are the smallest not used in any $X_w, w \in S_{v_i}$. If $S_{v_i} = \emptyset$ then $X^*_{v_i} = X_{v_i}$ is one optimally colored set, and if $S_{v_i} \neq \emptyset$, $X^*_{v_i}$ is the union of at most k optimally colored sets; namely

```
PROGRAM                              THREE-ADDRESS CODE          SEPARATOR

loop (1)                             L1: skip                    {0}
   S (4)                             L4: S                       {1,3}
   ; (3)                             L3: skip                    {0,1,2}
   if (5)                            L5: skip                    {0,2,3}
      A (9)                          L9: if not A then goto L4   {5,6,8}
        and (8)                      L8: skip                    {5,6,7}
            B (10)                   L10:if not B then goto L4   {6,7,8}
              then (7)               L7: skip                    {0,5,6}
      if (11)                        L11:skip                    {0,6,7}
         C (15)                      L15:if not C then goto L14  {11,13,14}
           then (13)                 L13:skip                    {0,11,12}
         T (16)                      L16:T; goto L12             {6,13}
      else (14)                      L14:skip                    {0,11,13}
         exit (17)                   L17:goto L0                 {0,14}
      fi (12)                        L12:skip                    {0,6,11}
   fi (6)                            L6: skip                    {0,2,5}
pool (2)                             L2: goto L1                 {0,1}
; (0)                                L0:                         {}


L1: S                       {0}      L1: S                       {0}
L8: if not A then goto L2   {1,2,7}  L8: if not A then goto L1   {1,7}
L7: if not B then goto L2   {0,1,2}  L7: if not B then goto L1   {0,1}
L13:if not C then goto L14  {0,2,7}  L13:if not C then goto L14  {0,1,7}
L16:T; goto L2              {2,13}   L17:T                       {13,16}
L14:goto L0                 {0,13}   L16:goto L1                 {1,13}
L2: goto L1                 {0,1}    L14:goto L0                 {0,13}
L0:                         {}       L0:                         {}

Contract (1,4,3,5),(9,8),(10,7,11),  Insert 17 between 13 and 17,
(15,13),(14,17) and (12,6,2).        Contract (2,1).


L1: S                       {0}
L8: if not A then goto L1   {1,7}
L7: if not B then goto L1   {0,1}
L13:if not C then goto L0   {0,1,7}
L17:T                       {13,14}
L16:goto L1                 {1,13}
L0:                         {}

Contract (14,0)
```

Figure 3.3: Flow-of-control optimization applied to a simple program segment.

the biclique $X_{v_i} \cup X_{p(v_i)}$ and the cliques $X_w, w \in S_{v_i} \setminus \{p(v_i)\}$. Thus the modified algorithm uses at most $k\chi(I)$ colors. In [27], $k = 2$, so the change brings their approximation factor down from 3 to 2, which is their main result.

We will now get further down to $(\lfloor k/2 \rfloor + 1)\chi(I)$ colors by carefully choosing the $p(v_i) \in S_{v_i}$. Let $p(v_i) = \perp$ denote that $p(v_i)$ is undefined. Note that any graph F with edges $\{v, p(v)\}$, where $p(v) \in S_v$, is acyclic since $v_h \in S_{v_i}$ implies $h < i$. Hence F is a forest.

Algorithm F: Colors I with at most $(\lfloor k/2 \rfloor + 1)\chi(I)$ colors.

F.1. For $i := 1, \ldots, n$,

F.1.1. Let M_i be a maximal matching in the forest on S_{v_i} with edges $\{w, p(w)\} \in S^2_{v_i}$.

F.1.2. If M_i is perfect ($\bigcup M = S_{v_i}$) then

F.1.2.1. $p(v_i) := \perp$.

F.1.2.2. Color the uncolored variables $x \in X_{v_i}$ with the smallest colors not used by variables intersecting S_{v_i}.

F.1.3. If M_i is imperfect then

F.1.3.1. Choose $p(v_i)$ from $S_{v_i} \setminus \bigcup_i M_i$.

F.1.3.2. Color the biclique $I|(X_{v_i} \cup X_{p(v_i)})$ using Lemma 10. Rename the colors so that the coloring of $X_{p(v_i)}$ is not changed, and such that the new colors for X_{v_i} are the smallest not used in any $X_w, w \in S_{v_i}$.

Correctness: For any $W \subseteq V$, let $\#(W)$ denote $|W| - |M|$ where M is a maximal matching in the forest on W with edges $\{w, p(w)\} \in W^2$. Then $\bigcup_{w \in W} X_w$ is the union of at most $\#(W)$ optimally colored sets; namely the $|M|$ bicliques $X_w \cup X_{p(w)}, \{w, p(w)\} \in M$, and the $|W| - 2|M|$ cliques $X_w, w \in W \setminus \bigcup M$. By induction on i we will show $\#(S_{v_i} \cup \{v_i\}) \leq \lfloor k/2 \rfloor + 1$.

If M_i is perfect, $\#(S_{v_i}) = |S_{v_i}|/2$, so $\#(S_{v_i} \cup \{v_i\}) \leq \lfloor k/2 \rfloor + 1$. Note that $S_{v_1} = \emptyset$, so this covers the base case of our induction.

For the case where M_i is imperfect, let h is the largest index such that $v_h \in S_{v_i}$. Then $S_{v_i} \subseteq S_{v_h} \cup \{v_h\}$. Hence $\#(S_{v_i}) \leq \#(S_{v_h} \cup \{v_h\})$. By induction, $\#(S_{v_h} \cup \{v_h\}) \leq \lfloor k/2 \rfloor + 1$. Moreover, $M_i \cup \{\{v_i, p(v_i)\}\}$ is a matching in $S_{v_i} \cup \{v_i\}$, so $\#(S_{v_i} \cup \{v_i\}) = \#(S_{v_i})$. That is,

$$\#(S_{v_i} \cup \{v_i\}) = \#(S_{v_i}) \leq \#(S_{v_h} \cup \{v_h\}) \leq \lfloor k/2 \rfloor + 1.$$

This completes the induction, thus proving that the algorithm uses at most $(\lfloor k/2 \rfloor + 1)\chi(I)$ colors. ■

Proof of Theorem 2: First note that it only takes linear time to find a maximal matching M in a forest F, as in step F.1.1. Greedily pick for M any leaf incident edge $\{v, w\}$, and recurse on $F \setminus \{v, w\}$. The leaves are found by keeping track of the degrees. Then Theorem 2 follows from Lemma 10 and the correctness of Algorithms E and F. ■

It should be noted that our improvement from k to $\lfloor k/2 \rfloor + 1$ is not an improvement for the case $k = 2$ studied in [27], but it is an improvement for any $k > 2$.

5 Finding the separators

In this section, we present an efficient algorithm for finding the minimal separators of a listing of the vertices in a graph. Such an algorithm is useful in connection with transformations like those presented in Section 3 because it allows us to forget about the separators until the we have a final

listing. Also, it will be useful in the next section, where we present a heuristic for finding good listings but where the separators are yet to be computed.

Consider a graph $G = (V, E)$ and a listing $L = v_1, \ldots, v_n$ of V. By a *separator path* from v_i to v_h, $i > h$, we mean a path $v_i v_{i_1} \cdots, v_{i_k} v_h$ such that $i_1, \ldots, i_k \geq i$. If $k = 0$, $v_i v_h$ is a separator path of length 1. Then

$$S(v) = \{u | \text{there is a separator path from } v \text{ to } u\}$$

is a minimum separator for v, contained in any other separator for v.

For technical reasons, below, we allow separator paths to be self-intersecting.

Algorithm G: Input $G = (V, E)$ and a listing $L = v_1, \ldots, v_n$ of V. Outputs the minimum separator $S(v)$ for each $v \in V$ in time $O(\sum_v S(v))$. Below $S^{-1}(w) = \{v | w \in S(v)\}$ is up-dated implicitly together with $S(\cdot)$.

G.1. For $i := n, \ldots, 1$:

G.1.1. $S(v_i) := \{v_h | (v_i, v_h) \in E, h < i\}$.

G.1.2. For all $w \in S^{-1}(v_i)$,

G.1.2.1. If $w \notin D$, for all $v_h \in S(w)$, $h < i$:

G.1.2.1.1. $S(v_i) := S(v_i) \cup \{v_h\}$.

G.1.2.1.2. $D := D \cup \{v_h\}$.

Correctness: We will prove that the following invariant is satisfied before Step G.1.1.

- For all $j > i$, $S(v_j)$ is the minimum separator for j.

Trivially, this is the case for $i = n$. Now, consider some value of $i < n$. We want to prove that v_h is inserted in $S(v_i)$ if and only if there is a "separator" path from v_i to v_h with all interior vertices in $\{v_j | j > i\}$. Step G.1.1 deals with the case where there is a separator path of length 1, hence with no interior vertices. Thus, it suffices to show that v_h is added to $S(v_i)$ in Step G.1.2.1.1 if and only if there is a separator path $v_i v_{\alpha_1} \cdots, v_{\alpha_k} v_h$ with $k \geq 1$.

$\Leftarrow$ We know we have separator paths $w v_{\alpha_1} \cdots, v_{\alpha_k} v_i$ and $w v_{\beta_1} \cdots, v_{\beta_l} v_h$. Since $w = v_j$ for some $j > i$, it follows that $v_i v_{\alpha_k} \cdots, v_{\alpha_1} w v_{\beta_1} \cdots, v_{\beta_l} v_h$ is a separator path (recall that separator paths may be self-intersecting), hence that the addition of v_h to $S(v_i)$ is correct.

$\Rightarrow$ Suppose there is a separator path of length > 1 from v_i to v_h, and let $v_i v_{\alpha_1} \cdots v_{\alpha_k} v_h$ be such a separator path where the minimal index $\mu = \min_{1 \leq \gamma \leq k} \{\alpha_\gamma\}$ of an interior vertex is as small as possible. Suppose $\alpha_\gamma = \mu$. Then $v_{\alpha_\gamma} \cdots v_{\alpha_1} v_i$ and $v_{\alpha_\gamma} \cdots v_{\alpha_k} v_h$ are separator paths, so $v_{\alpha_\gamma} \in S^{-1}(v_i)$ and $v_h \in S(v_{\alpha_\gamma})$. Hence v_h is correctly added to $S(v_i)$ in Step G.1.2.1.1 unless $v_{\alpha_\gamma} \in D$. However, if $v_{\alpha_\gamma} \in D$, it is because $v_{\alpha_\gamma} \in S^{-1}(v_j)$ for some $j > i$. Then we have a separator path $v_{\alpha_\gamma} v_{\beta_1} \cdots v_{\beta_l} v_j$. But then

$$v_i v_{\alpha_1} \cdots v_{\alpha_\gamma} v_{\beta_1} \cdots v_{\beta_l} v_j v_{\beta_l} \cdots v_{\beta_1} v_{\alpha_\gamma} \cdots v_{\alpha_k} v_h$$

is a separator path and $j < \alpha_\gamma = \mu$, contradicting the minimality of μ.

For the complexity of the algorithm, note that the loop of step G.1.2 is repeated exactly once for every $(v_i, w), v_i \in S(v)$. Also, due to step G.1.2.1.2, the condition $w \notin D$, is satisfied at most once for every w. Hence the loop of step G.1.2.1 is repeated at most once for every $(v_h, w), v_h \in S(w)$. In conclusion, the running time is $O(\sum_v S(v))$. ■

6 Simple listings from three address code

In this section, we present a heuristic for finding a simple listing when the input is in three-address code without structural statements. This allows us to integrate our register allocation with compilers where the program has been reduced to three-address code before register allocation is started. Our heuristic is much simpler and faster than the general known algorithms for finding tree-decompositions of graphs, and it will work well on three-address code produced as in [3] from structured programs. The heuristic has the advantage of producing a listing of any three-address code. In particular, it is expected to find good listings even for programs with a limited or structured use of general gotos. Similar heuristics should work for other types of intermediate code. Especially, we can take advantage of intermediate code containing more structural information than three-address code.

Intuitively, good listings have the following two characteristics: (1) program points that we exit to from many places, say from short-circuit evaluation or loops, should be listed early. (2) The entries of loops and conditional blocks should be listed before points in the body. Our heuristic will be designed with these basic goals in mind, and yet we will try to keep it very simple, not trying to identify the exact original structure of the program. With regards to (1), we essentially just try to list the statements in backwards order. For an if-then-else statement, this has the positive side-effect that we complete listing the else-part before we start listing the then-part. To satisfy (2), we introduce a general way of identifying the most important entries of various structures.

Consider a set I of pairs $(i,j) \in \{1,\ldots,n\}^2$. An *$I$-chain* from i to j, $i < j$, is a sequence of pairs $(i_1,j_1),\ldots,(i_l,j_l) \in I$ such that for all $k < l$, $i_k < i_{k+1} < j_k < j_{k+1}$. An I-chain from i to j is *maximal* if there is neither an $i' < i$ with an I-chain from i' to j, nor a $j' > j$ with an I-chain from i to j'. We shall return to the computation of maximal I-chains in Algorithm I

We are given a list $s_1 \cdots s_n$ of statements, where some contain a jump to another. Let J be the set of pairs (i,j) such that s_i contains a jump to j. Intuitively maximal J-chains are used to bring us from the beginning to the end of a conditional structure. Let S be the symmetric closure of J, i.e. $(i,j) \in S$ iff either $(i,j) \in J$, or $(j,i) \in J$. Intuitively maximal S-chains are used to bring us from loop or conditional structures to their exit points. The above intuition is very simplistic, but nevertheless, we suggest the following heuristic for generated low-complexity listings of three-address code:

Algorithm H: Finding a good listing:

H.1. $i := 0$;

H.2. For $j := n$ downto 1,

H.2.1. If s_j is not marked, mark s_j by i; $i := i+1$;

H.2.2. If there is a maximal S-chain from k to j and s_k is not marked, mark s_k by i; $i := i+1$;

H.2.3. If there is a maximal J-chain from k to j and s_k is not marked, mark s_k by i; $i := i+1$;

Although it is very techical, it can be shown that the above heurstics will give (≤ 5)-complex listings if applied to the three address code generated as in [3] from a program with structural statements from STRUCTURED. In fact, it even seems to work well after the standard optimizations discussed in 3. The working of the heuristic is illustrated in figure 6.4. First we have the three address code obtained from the program from figure 2.2 using the flow-of-control optimization from Section 3. As in Section 3, we have used the labels to indicate the listing. Also, for each three-address code statement, we have the corresponding separator. To the right is the new listing generated by Algorithm H with the corresponding new separators. All separators were found using Algorithm G. What we see is the the maximal separator is of size 5 for both the old and the new listing.

Three address code	Separators	New listing	New Separators
L1: if a then goto L2	{}	2	{1}
L10: if not b then goto L12	{1,2,9}	4	{1,2,3}
L17: if c then goto L19	{2,9,10,12}	11	{1,3,4,8,10}
L21: if not d then goto L20	{17,19,20}	15	{10,11,14}
L19: if e then goto L2	{2,9,12,17}	14	{1,10,11,13}
L30: if f then goto L9	{9,12,19}	13	{1,3,10,11,12}
L31: goto L12	{12,30}	12	{1,3,8,10,11}
L20: if g then goto L2	{2,9,12,17,19}	10	{1,3,4,8,9}
L35: if h then goto L9	{9,12,20}	9	{1,3,4,8}
L12: if i then goto L2	{2,9,10,11}	8	{1,3,4,7}
L46: if not j then goto L10	{2,11,12}	7	{1,3,4,6}
L54: s	{11,46}	6	{1,3,4,5}
L11: goto 10	{2,9,10}	5	{1,3,4}
L9: t	{1,2}	3	{1,2}
L2:	{1}	1	{}

Figure 6.4: Applying the heuristic

Algorithm I: Given I, finds the set M of pairs (i,j) with a maximal I-chain from i to j.

I.1. $M := \emptyset$;

I.2. $s := 0$; $(i_0, j_0) := (0, n+1)$;

I.3. For $i := 1$ to n,

I.3.1. If $\exists j > i : (i,j) \in I$:

I.3.1.1. Let $j := \max\{j | (i,j) \in I\}$.

I.3.1.2. While $j_s \leq i$, $M := M \cup \{(i_s, j_s)\}$; $s := s - 1$;

I.3.1.3. While $j \geq j_s > i$, $i := i_s$; $s := s - 1$;

I.3.1.4. $s := s + 1$; $(i_s, j_s) := (i, j)$;

Correctness: The essential point is to note the following invariant for our stack $(i_0, j_0) \cdots (i_s, j_s)$:

$$i_0 < i_1 < \cdots < i_s < j_s < j_{s-1} < \cdots < j_0.$$

■

The computation "$j := \max\{j | (i,j) \in I\}$" takes $O(|I|)$ total time over all i. The rest of the algorithm runs in $O(n)$ total time. Since each statement can have at most one jump, $|J| \leq n$ and $|S| \leq 2n$. Hence the total running time of Algorithm H and I is $O(n)$.

7 The complexity and tree-width

In this section, we prove the statement of Theorem 5 that graph has tree-width k if and only if it is k-complex. The notion of tree-width was introduced by Robertson and Seymour [35].

Definition 11 *A* tree-decomposition *of a graph $G = (V, E)$ is defined by a tree $T = (I, F)$ together with a family $\{W_i\}_{i \in I}$ of subsets of V such that*

(i) $\bigcup_{i \in I} W_i = V$

(ii) *for all edges* $(v, w) \in E$*, there exists an* $i \in I$ *such that* $\{v, w\} \subseteq W_i$*.*

(iii) *for all* $i, j, k \in I$*, if* j *is on the path from* i *to* k *then* $W_i \cup W_k \subseteq W_j$*.*

The width *of the decomposition is* $\max_{i \in I} |W_i| - 1$ *and the* tree-width *of* G *is the minimal width over all tree-decompositions of* G*.*

Lemma 12 *Given a* k*-complex listing* $v_1, \ldots, v_n$ *of the vertices in* G*, including the separators* S_{v_i}*, in linear time, we can construct a tree-decomposition of* G *of width* k*.*

Proof: The tree T will have the nodes $1, \ldots, n$. We build up T starting from T_1 consisting of the root 1, and setting $W_1 = \{v_1\}$. Now, for $i = 2, \ldots, n$, let h the the largest index such that $v_h \in S_{v_i}$. Set $T_i = T_{i-1} \cup \{(h, i)\}$ and $W_i = S_{v_i} \cup \{v_i\}$. Return $T = T_n$ and $\{W_i\}_{i \in I}$.

Clearly (i) is satisfied. Also (iii) is satisfied, for consider $\{v_h, v_i\} \in E$ with $h < i$. Then $v_h \in S_{v_i}$, so $\{v_h, v_i\} \in W_i$.

To prove (ii) we use induction on i. Trivially (ii) is satisfied for T_1. Let $T_i = T_{i-1} \cup \{(h, i)\}$ and suppose (ii) is satisfied for T_{i-1}. Consider any path p in T_i not in T_{i-1}. Then i is one of the end-vertices. Let k be the other end-vertex, and let j be any vertex between them. Trivially (ii) is satisfied if $j = i$, so we may assume that $j \neq i$, but then j is on the path in T_{i-1} between h and k. Thus, by induction, $W_j \supseteq W_h \cap W_k$. By definition $S_{v_i} \subseteq S_{v_h} \cup \{v_h\} = W_h$. However, $W_i = S_{v_i} \cup \{v_i\}$ and $v_i \notin W_k$, so $W_i \cap W_k \subseteq W_h \cap W_k \subseteq W_j$. ■

Lemma 13 *Given a tree-decomposition* $(T, \{W_j\}_{j \in I})$ *of* G *of width* k*, in linear time, we can construct a* k*-complex listing* $v_1, \ldots, v_n$ *of the vertices in* G *including the separators* S_{v_i}*.*

Proof: Let $T = (I, F)$. Choose any rooting of T, and identify I with $\{1, \ldots, |I|\}$ such that $1, \ldots, |I|$ is a pre-ordering of T. Let $p(j)$ denote the parent of j. Thus $\forall j \in I, p(j) < j$. Also, 1 is the root of T.

Set $n_1 = |W_1|$. Let $\{v_1, \ldots, v_{n_1}\} = W_1$ and $S_{v_i} = \{v_1, \ldots, v_{i-1}\}$ for $i \leq n_1$. Since $|W_1| \leq k + 1$, $|S_{v_i}| \leq k$. Let j run from 2 to $|I|$. Set $S_j = W_j \cap W_{p(j)}$, $U_j = W_j \setminus S_j$, and $n_j = n_{j-1} + |U_j|$. Finally, let $\{v_{n_{j-1}+1}, \ldots, v_{n_j}\} = U_j$, and $S_{v_i} = S_j \cup \{v_{n_{j-1}+1}, \ldots, v_{i-1}\}$ for $i = n_{j-1} + 1, \ldots, n_j$.

To see that the above produces a valid listing, note for $j > 1$ that (ii) and (iii) implies that S_j separates U_j from $\bigcup_{k<j} W_k$ in G. Hence $U_j = W_j \setminus \bigcup_{k<j} W_k$. Together with (i) this implies that $\{U_j\}_{j \in I}$ is a partitioning of V, hence that all vertices get listed. Also, for $i = n_{j-1} + 1, \ldots, n_j$, it implies that, indeed, $S_{v_i} = S_j \cup \{v_{n_{j-1}+1}, \ldots, v_{i-1}\}$ is a separator for v_i in the produced listing. Finally, $|S_{v_i}| \leq k$ since $S_{v_i} \subset W_j$ and $|W_j| \leq k + 1$. ■

Proof of Theorem 5: The theorem is the direct composition of Lemmas 12 and 13. ■

Definitions 1 and 11 are thus equivalent. It should be noted that alternatively, we could have proved that our k-complex listings were equivalent to partial k-trees which is a known to be equivalent to tree-decompositions width k. It is trivial to see that if a graph is a partial k-tree, it has a k-complex listing. However, the converse is not quite as trivial.

Having the established the link between structured programs and bounded tree-width, we are now ready to apply the many techniques for bounded tree-width to problems in control-flow graph analysis. Several powerful techniques are already known for problems that are generally NP hard, or even P-space hard [5, 6, 9, 11, 17]. If a graph has n vertices and tree-width k, their complexity is typically of the form $O(f(k)n)$ where $f(\cdot)$ is at least exponential.

Exploiting bounded tree-width is also interesting for problems for which we have "slow" polynomial algorithms. For generalized dominators, we have an algorithm for general graphs with

running time $O(nm)$ [1]. However, in [2] is developed an algorithm with running time $O(k^3 n)$. The polynomial dependence on k is important if the tree-width start growing because of gotos. More philosophically, in control-flow analysis, we have several problems involving an implicit computation of an expensive transitive closure. Using the bounded tree width k, we can essentially restrict our attention to n transitive closures over vertex sets of size k. Thus localizing global reasoning opens up new perspectives for the field of compiler optimization.

References

[1] S. Alstrup, J. Clausen, and K. Jørgensen, An $O(|V| * |E|)$ algorithm for finding immediate multiple-vertex dominators. Accepted for *Information Processing Letters*, 1996.

[2] S. Alstrup, P.W. Lauridsen, and M. Thorup, Generalized dominators for structured programs. Accepted for "Proc. 3rd International Static Analysis Symposium," 1996.

[3] A.V. Aho, R. Sethi, and J.D. Ullman, *Compilers: Principles, Techniques, and Tools*, Addison-Wesley, Reading, Mass., 1986.

[4] S. Arnborg, D.G. Corneil, and A. Proskorowski, Complexity of Finding Embeddings in a k-Tree, *SIAM J. Alg. Disc. Meth.* **8** (1987) 277–284.

[5] S. Arnborg, J. Lagergren, and D. Seese, Easy problems for tree-decomposable graphs, *J. Algorithms* **12** (1991) 308–340.

[6] S. Arnborg and A. Proskorowski, Linear time algorithms for NP-hard problems restricted to partial k-trees, *SIAM J. Alg. Disc. Meth.* **23** (1989) 11–24.

[7] H.L. Bodlaender, A Tourist Guide Through Treewidth, *Acta Cybernetica* **11** (1993) 1–23.

[8] H.L. Bodlaender, A Linear Time Algorithm for Finding Tree-Decompositions of Small Treewidth, *in* "Proc. 25th STOC," pp. 226–234, 1993.

[9] H.L. Bodlaender, Complexity of Path Forming Games, *Theor. Comp. Sc.* **110** (1993) 215–245.

[10] H.L. Bodlaender, J.R. Gilbert, H. Hafsteinsson, and T. Kloks, Approximating Treewidth, Pathwidth, Frontsize, and Shortest Elimination Tree, *J. Algorithms* **18**, 2 (1995) 221–237.

[11] R.B. Borie, R.G. Parker, and C.A. Tovey, Automatic Generation of Linear-Time Algorithms from Predicate Calculus Descriptions of Problems on Recursively Constructed Graph Families, *Algorithmica* **7** (1992) 555–581.

[12] P. Briggs, Register allocation via graph coloring, PhD Thesis, Rice University, 1992.

[13] P. Briggs, K.D. Cooper, K. Kennedy, and L. Torczon, Coloring heuristics for register allocation, *in* "Proc. SIGPLAN'89 Conf. Programming Language Design and Implementation," pp. 275–284, 1989.

[14] D. Callahan and B. Koblenz, Register allocation via hierarchical graph coloring, *in* "Proc. SIGPLAN'91 Conf. Programming Language Design and Implementation," pp. 192–203, 1991.

[15] G.J. Chaitin, Register Allocation and Spilling via Graph Coloring, *in* "Proc. SIGPLAN'82 Symp. Compiler Construction," pp. 98–105, 1982.

[16] G.J. Chaitin, M.A. Auslander, A.K. Chandra, J. Cocke, M.E. Hoplins, and P.W. Markstein, Register allocation via graph coloring, *Computer Languages* **6** (1981) 47–57.

[17] B. Courcelle and M. Mosbah, Monadic second-order evaluations on tree-decomposable graphs, **6** (1993) 49–82.

[18] O.J. Dahl, E.W. Dijkstra, and C.A.R. Hoare, *Structured Programming*, Academic Press, London, 1972.

[19] E.W. Dijkstra, Go To Statement Considered Harmful, *Comm. ACM* **11**, 3 (1968) 147–148.

[20] A.P. Ershov, Reduction of the problem of memory allocation in programming to the problem of colouring the vertices of a graph, *Doklady Academii Nauk SSSR* **142**, 4 (1962) 785–787. English version in *Soviet Mathematics* **3** (1962) 163–165.

[21] M.R. Garey, D.S. Johnson, G.L. Miller, and C.H. Papadimitriou, The Complexity of Coloring Circular Arcs and Chords, *SIAM J. Alg. Discr. Meth.* **1**, 2 (1980) 216–227.

[22] F. Gavril, Algorithms for Minimum Coloring, Maximum Clique, Minimum Covers by Cliques, and Maximum Independent Set of Chordal Graphs, *SIAM J. Comp.* **1**, 2 (1972) 180–187.

[23] F. Gavril, The Intersection Graph of Subtrees in Trees Are Exactly the Chordal Graphs, *J. Comb. Th. Ser. B* **16** (1974) 47–56.

[24] R. Gupta, Generalized dominators and post-dominators. *in* "Proc. POPL'92," pp. 246–257, 1992.

[25] R. Gupta, M.L. Soffa, and T. Steele, Register allocation via clique separators, *in* "Proc. SIGPLAN'89 Conf. Programming Language Design and Implementation," pp. 264–274, 1989.

[26] M. M. Halldórsson, A Still Better Performance Guarantee for Approximate Graph Coloring, *Inf. Proc. Lett.* **45** (1993) 19–23.

[27] S. Kannan and T. Proebsting, Register Allocation in Structured Programs, *in* "Proc. 6th SODA," pp. 360–368, 1995.

[28] B.R. Kernighan and D.M. Ritchie, *The C Programming Language*, Prentice-Hall, New Jersey, 1978.

[29] D.E. Knuth, Structured Programming with Go To Statements, *ACM Computing Surveys* **6**, 4 (1974) 261–301.

[30] C. Lund and M. Yannakakis, On the Hardness of Approximating Minimization Problems, *J. ACM* **41** (1994) 960–981.

[31] P. Naur, Revised Report on the Algorithmic Language Algol 60, *Comm. ACM* **6**, 1 (1963) 1–17.

[32] P. Naur, Go To Statements and Good Algol Style, *BIT* **3**, 3 (1963) 204–208.

[33] T. Nishizeki, K. Takamizawa, and N. Saito, Algorithms for detecting series-parallel graphs and D-charts, *Trans. Inst. Elect. Commun. Eng. Japan* **59**, 3 (1976) 259–260.

[34] C. Norris and L.L. Pollock, Register Allocation over the Program Dependence Graph, *in* "Proc. SIGPLAN'94 Conf. Programming Language Design and Implementation," pp. 266–277, 1994.

[35] N. Robertson and P.D. Seymour, Graph Minors I: Excluding a Forest, *J. Comb. Th. Ser. B* **35** (1983) 39–61.

[36] N. Robertson and P.D. Seymour, Graph Minors XIII: The Disjoint Paths Problem. *J. Comb. Th. Ser. B* **63** (1995) 65–110.

[37] M. Thorup, Structured Programs have Small Tree-Width and Good Register Allocation (latest full version) `http://www.diku.dk/~mthorup`, 1996.

[38] N. Wirth, The Programming Language PASCAL, *Acta Informatica* **1** (1971), 35–63.

[39] N. Wirth, *Programming in Modula-2 (3rd corr. ed.)*, Springer-Verlag, Berlin, New York, 1985.