

Effective Longest and Infinite Reduction Paths in Untyped λ -Calculi

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Abstract. A *maximal* reduction strategy in untyped λ -calculus computes for a term a longest (finite or infinite) reduction path. Some types of reduction strategies in untyped λ -calculus have been studied, but maximal strategies have received less attention. We give a systematic study of maximal strategies, recasting the few known results in our framework and giving a number of new results, the most important of which is an effective maximal strategy in $\lambda\beta\eta$. We also present a number of applications illustrating the relevance and usefulness of maximal strategies.

Keywords: λ -calculus, reduction strategies, effectiveness.

1 Introduction

Background. Considerable attention has been devoted to classification of reduction strategies in untyped λ -calculus [2, 7, 4, 3, 16, 15]. Barendregt [1, Ch.13] gives an exposition of most results. In this paper we are concerned with *effective* strategies differing in the *length* of reduction paths:

- (i) A *normalizing* strategy computes for a term a *finite* reduction path to normal form, if one exists, and otherwise some infinite reduction path;
- (ii) A *minimal* strategy computes for a term a *shortest* reduction path to normal form, if one exists, and otherwise some infinite reduction path;
- (iii) A *perpetual* strategy computes for a term an *infinite* reduction path, if one exists, and otherwise some finite reduction path to normal form;
- (iv) A *maximal* strategy computes for a term a *longest* reduction path to normal form, if all are finite, and otherwise some infinite reduction path.

So perpetual and normalizing strategies are opposite, as are maximal and minimal strategies. Minimal and normalizing strategies, and to some extent perpetual strategies, have received some attention in the literature; maximal strategies have been studied only in a few special cases.

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Curry and Feys [7] show that left-most reduction F_l is an effective normalizing strategy and Barendregt et al. [2] show that there is an effective perpetual strategy F_∞ . Since there is no effective minimal strategy [2] one might, by analogy, expect that there is no effective maximal strategy either.

However, de Vrijer [8] proves the Finite Developments Theorem by showing that a version of F_∞ contracting only labeled redexes computes the longest development. A primitive recursive function computing this length is also presented. This raises the question whether F_∞ in general computes longest reductions paths, and whether there is a simple formula, in some sense, for the length of longest (finite) reduction paths in general. Since de Vrijer's technique makes essential use of the fact that only redexes in the original term and their residuals are contracted, it does not apply to the more general setting.

Upper bounds for length of reduction paths in *typed* lambda calculi are an industry [22, 19, 10, 11, 9]. In particular, de Vrijer proves Strong Normalization of simply typed λ -calculus λ^τ by translating terms of λ^τ into recursive functionals computing the length of the longest reduction path to normal form [9], and mentions that F_∞ computes this path. This raises again the question whether F_∞ in general computes longest reductions paths in untyped lambda-calculus, and whether some formula computing upper bounds for lengths of reduction paths exists. The proof by de Vrijer makes use of the type structure of λ^τ and so does not immediately apply to the untyped setting.

A result due in part to Barendregt et al. [2] and Bergstra and Klop [4] characterizes *perpetual redexes*, i.e. redexes Δ with the property that for any context C such that $C[\Delta]$ has an infinite reduction path, $C[\Delta']$ also has an infinite reduction path, where Δ' is the contractum of R . It is natural to ask whether it is possible to give a corresponding characterization of *maximal redexes*, i.e. redexes Δ with the property that for any context C the first step in a longest reduction of $C[\Delta]$ is to contract Δ .

The preceding results all concern β -reduction. A notion of left-most reduction is normalizing for $\beta\eta$ -reduction [15], although this result is “much more complicated than for β -reduction” to cite Barendregt [1, p388]. The problem is that contracting a β -redex may create an η -redex to the left of this β -redex, as in the following example where $x \notin \text{FV}(P)$: $\lambda x.((\lambda y.P) x) x \rightarrow_\beta \lambda x.P x$. This anomaly also causes problems in finding maximal and perpetual strategies for $\lambda\beta\eta$. Indeed, no effective perpetual strategy for $\lambda\beta\eta$ seems to have been mentioned in the literature, let alone any effective maximal strategy.

Our contribution. We give a systematic study of maximal and perpetual strategies and redexes in $\lambda\beta$ and $\lambda\beta\eta$, establishing four fundamental properties, all original results:

- (i) *Maximal versus perpetual strategies and redexes* (Sect. 3 and 6). For any notion of reduction on λ -terms we show that every maximal strategy is also perpetual,² and every maximal redex is also perpetual.

² This is similar to the relationship between minimal and normalizing strategies.

- (ii) *Effective maximal and perpetual strategies in $\lambda\beta$ and $\lambda\beta\eta$* (Sect. 4). We show that a certain effective strategy is maximal and thus perpetual in $\lambda\beta\eta$, and we show that F_∞ is maximal in $\lambda\beta$ (and hence perpetual, but this is known.)
- (iii) *Bounds for length of reductions* (Sect. 5). We show that to compute an upper bound on the length of a longest β - or $\beta\eta$ -reduction path for some term, one cannot do better, in a certain sense, than simply try to reduce the term to normal form and count the number of steps along the way.
- (iv) *Maximal and perpetual redexes in $\lambda\beta$ and $\lambda\beta\eta$* (Sect. 7). We characterize maximal redexes in $\lambda\beta$ and $\lambda\beta\eta$, and we extend the well-known characterization of perpetual redexes in $\lambda\beta$ to $\lambda\beta\eta$.

This settles the unanswered questions from the background section. However, it may still not be clear from what has been said why maximal and perpetual reductions are worth studying. We therefore give a number of applications (Sect. 8) of maximal and perpetual strategies, including a *very* short proof of Church's Conservation Theorem in $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$ as well as two new results.

Related work. L. Regnier has independently shown that F_∞ is maximal in $\lambda\beta$ as an application of his σ -reductions [17]. His proof proceeds, roughly, by showing that (i) F_∞ is maximal on terms in σ -normal form; and (ii) every term reduces by σ -reductions to a term in σ -normal form with the same length of longest reduction path as the original term. Independently of Regnier and the author, F. Raamsdonk and P. Severi have later shown in a manuscript that F_∞ is maximal in $\lambda\beta$ [23], using a characterization of strongly normalizing terms.

While both of these techniques have their merits, there is an interest in seeing the theory developed without excursion via some auxiliary theory. Indeed, our presentation is very much in the style of [1, Ch. 13]. Moreover, our work extends the work of Regnier and Raamsdonk-Severi by considering the non-trivial extension of their result to $\lambda\beta\eta$. Indeed, we prove only this extended result, since the result for $\lambda\beta$ can be proved by a special case of our technique. Finally, our work extends that of Regnier and Raamsdonk-Severi by considering points (i), (iii), and (iv) above, as well as a number of new applications including a generalization of the Conservation Theorem and two new results in λ -calculus.

Z. Khasidashvili [14] studies classes of *orthogonal term rewriting systems* and gives, for some of these, effective perpetual strategies and least upper bounds for length of reductions. A formula for the length of longest developments is also given. In another paper [13], similar results are proved for *orthogonal expression reduction systems*. The case of $\beta\eta$ -reduction is not subsumed by these papers.

2 Preliminary Definitions

Most notation, terminology, and conventions are adopted from [1]. A is the set of λ -terms. We assume familiarity with substitution, the variable convention, and notions of reduction \mathbf{R} on A .

Notation 1. We denote by \rightarrow_R and \twoheadrightarrow_R the compatible closure of \mathbf{R} and the compatible, reflexive, transitive closure of \mathbf{R} , respectively. For M, N in Λ , the expression $\|M\|_x$ denotes the number of free occurrences of x in M . For any map F define $F^0(M) = M$; $F^{n+1}(M) = F(F^n(M))$. For a predicate P on the non-negative numbers, let $\mu n : P(n)$ denote the smallest n such that $P(n)$ holds; if no such n exists $\mu n : P(n)$ is undefined.

Convention 2. By a notion of reduction \mathbf{R} we shall henceforth mean one which is *finitely branching*, i.e. for which $\{M' \mid M \rightarrow_R M'\}$ is finite.

Definition 3. Let \mathbf{R} be a notion of reduction. An *R-reduction path* from a term M_0 is a (possibly infinite) sequence $M_0 \rightarrow_R M_1 \rightarrow_R M_2 \rightarrow_R \dots$. If the sequence is finite it *ends* in the last term M_n and has *length* n .

Definition 4. Let \mathbf{R} be a notion of reduction.

- (i) $\infty_R(M) \Leftrightarrow$ there is an infinite R -reduction path from M .
- (ii) $k_R(M) \Leftrightarrow$ there is an R -reduction path from M of length k .
- (iii) $M \twoheadrightarrow_R^k N \Leftrightarrow$ there is an R -reduction path from M of length k ending in N .
- (iv) $\text{NF}_R(M) \Leftrightarrow$ there is no R -reduction path of length 1 or more from M .
- (v) $\text{SN}_R(M) \Leftrightarrow$ all R -reduction paths from M are finite.
- (vi) $\text{WN}_R(M) \Leftrightarrow$ there is an R -reduction from M ending in N with $\text{NF}_R(N)$.

Convention 5. The index R will almost always be implicit from the context; this also holds for similar notions introduced below. The predicates NF_R, SN_R are sometimes regarded as sets, so we write $M \in \text{NF}_R$ rather than $\text{NF}_R(M)$.

Elements of $\text{NF}_R, \text{SN}_R, \text{WN}_R$ are *R-normal forms*, *R-strongly normalizing*, and *R-weakly normalizing*, respectively.

Definition 6. Let \mathbf{R} be a notion of reduction.

- (i) A *one-step R-reduction strategy* F is a mapping $F : \Lambda \rightarrow \Lambda$ such that $M \rightarrow_R F(M)$ if $M \notin \text{NF}_R$, and $F(M) = M$ otherwise.
- (ii) Let F be a one-step R -reduction strategy. The *F-reduction path* from M is the finite or infinite sequence

$$M \rightarrow_R F(M) \rightarrow_R F^2(M) \rightarrow_R \dots \rightarrow_R F^n(M) \rightarrow_R \dots$$

$L_F(M) = \mu n : F^n(M) \in \text{NF}_R$ is the length of the F -reduction path from M .

The details of the next lemma are left out since they are not hard and mostly well-known. In (i) one uses König's Lemma, and (ii) is [1, 15.1.5]. For (iii) one uses (i), a refinement of η -postponement [1, 15.1.6], and the fact that $\forall M : \text{SN}_\eta(M)$.

Lemma 7.

- (i) $\infty_R(M) \Leftrightarrow \forall n \geq 0 : n_R(M)$.
- (ii) $\text{WN}_\beta(M) \Leftrightarrow \text{WN}_{\beta\eta}(M)$.
- (iii) $\text{SN}_\beta(M) \Leftrightarrow \text{SN}_{\beta\eta}(M)$.

3 Maximal and Perpetual Reduction Strategies

Definition 8. Let \mathbf{R} be a notion of reduction, F a one-step R -reduction strategy.

- (i) (**Bergstra and Klop [4]**) F is *R -perpetual* if $\forall M : \infty_R(M) \Rightarrow \infty_R(F(M))$.
- (ii) (**Regnier [17]**) F is *R -maximal* if $\forall M \forall n \geq 1 : n_R(M) \Rightarrow (n-1)_R(F(M))$.

Remark. The rationale behind this definition can be seen from the properties:

- (i) if F is R -perpetual and $\infty(M)$ then the F -reduction path is infinite;
- (ii) if F is R -maximal and $\text{SN}(M)$ then no reduction path is longer than the F -reduction path.

Indeed, (i) is the definition of perpetual strategy in [2].

Proposition 9. Let \mathbf{R} be a notion of reduction, F a one-step R -reduction strategy.

- (i) If F is R -maximal then F is R -perpetual.
- (ii) Not vice versa.

Proof.

- (i) Assume $\infty_R(M)$. Then for all $n \geq 1$, $n_R(M)$, and since F is R -maximal, $(n-1)_R(F(M))$. Then for all $m \geq 0$, $m_R(F(M))$. By Lemma 7, $\infty_R(F(M))$.
- (ii) Define $F_{bk} : \Lambda \rightarrow \Lambda$ by: If $M \in \text{NF}_\beta$ then $F_{bk}(M) = M$; otherwise let $M \equiv C[(\lambda x.P)Q]$ where $(\lambda x.P)Q$ is the left-most β -redex in M and

$$F_{bk}(M) = \begin{cases} C[P\{x:=Q\}] & \text{if } \text{SN}_\beta(Q). \\ C[(\lambda x.M)F_{bk}(N)] & \text{if } \infty_\beta(Q). \end{cases}$$

F_{bk} is a one-step perpetual β -reduction strategy [4], but F_{bk} is not β -maximal since $F_{bk}((\lambda x.z) (\mathbf{I} \mathbf{I})) = z$, while the longest reduction uses two steps. \square

Remark. It is easy to find an *effective* perpetual, non-maximal one-step strategy.

4 Effective Maximal and Perpetual Strategies: $\lambda\beta$ and $\lambda\beta\eta$

We first review the situation in $\lambda\beta$.³

Definition 10 (Barendregt et al. [2]). Define $F_\infty : \Lambda \rightarrow \Lambda$ by: If $M \in \text{NF}$ then $F_\infty(M) = M$; otherwise let $M \equiv C[(\lambda x.P)Q]$ where $(\lambda x.P)Q$ is the left-most redex in M and

$$F_\infty(M) = \begin{cases} C[P\{x:=Q\}] & \text{if } x \in \text{FV}(P). \\ C[P] & \text{if } x \notin \text{FV}(P) \text{ and } Q \in \text{NF}. \\ C[(\lambda x.P)F_\infty(Q)] & \text{if } x \notin \text{FV}(P) \text{ and } Q \notin \text{NF}. \end{cases}$$

³ Such remarks will be made in the remainder to make clear the index of notions like NF, see Convention 5.

The proof of the next theorem is a simplification of the similar proof for $\beta\eta$ -reduction. Since the extension to $\lambda\beta\eta$ is not a priori obvious, see the remarks in Sect. 1, it is better to give the extended proof and leave out the restricted one, rather than vice versa.

Theorem 11 (Regnier [17]). F_∞ is an effective maximal one-step strategy.

Corollary 12 (Barendregt et al. [2]). F_∞ is perpetual.

Proof. By Proposition 9(i). □

We now turn to $\lambda\beta\eta$. The following development resembles [1, 13.4.4-6], but extended to count lengths of reductions and take η -reductions into account. The technique for taking η -reductions into account seems to be of independent interest; for instance, it is also used in the proof of Proposition 33.

Definition 13. Suppose $x \notin \text{FV}(P)$.

- (i) In $(\lambda x.P x) Q$ the β -redex is left-most.
- (ii) In $\lambda x.(\lambda y.P) x$ the η -redex is left-most.

Definition 14. Define $F_\infty : \Lambda \rightarrow \Lambda$ by: If $M \in \text{NF}$ then $F_\infty(M) = M$; otherwise let $M \equiv C[\Delta]$ where Δ is the left-most redex in M and define

$$F_\infty(M) = \begin{cases} C[P\{x:=Q\}] & \text{if } \Delta \equiv (\lambda x.P) Q \text{ and } x \in \text{FV}(P). \\ C[P] & \text{if } \Delta \equiv (\lambda x.P) Q \text{ and } x \notin \text{FV}(P) \text{ and } Q \in \text{NF}. \\ C[(\lambda x.P) F_\infty(Q)] & \text{if } \Delta \equiv (\lambda x.P) Q \text{ and } x \notin \text{FV}(P) \text{ and } Q \notin \text{NF}. \\ C[P] & \text{if } \Delta \equiv \lambda x.P x \text{ and } x \notin \text{FV}(P). \end{cases}$$

The proof of the next lemma is omitted. It is similar to the proof of [1, 3.1.16].

Lemma 15.

- (i) If $M \twoheadrightarrow^n M'$ then $K\{x:=M\} \twoheadrightarrow^{l \cdot n} K\{x:=M'\}$ where $l = \|K\|_x$.
- (ii) If $M \twoheadrightarrow^n M'$ then $M\{x:=K\} \twoheadrightarrow^n M'\{x:=K\}$.

Notation 16. If $M \equiv C[\Delta]$ where Δ is a β - or η -redex and no β -redex is to the left of Δ in M , then we write $\Delta \leq_\beta M$.

Lemma 17.

- (i) Suppose $\lambda x.P x \leq_\beta M$ where $x \notin \text{FV}(P)$ and

$$C[\lambda x.P x] \equiv M \rightarrow M'.$$

Then either

$$M' \equiv C[P];$$

or

$$M' \equiv C'[\lambda x.P' x],$$

where $P \twoheadrightarrow^{n_1} P'$, $C[U] \twoheadrightarrow^{n_2} C'[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}(\lambda x.P' x)$, $n_1 + n_2 = 1$, and $\lambda x.P' x \leq_\beta M'$.

(ii) Suppose $(\lambda x.P) Q \leq_\beta M$ and

$$C[(\lambda x.P) Q] \equiv M \rightarrow M' .$$

Then either

$$M' \equiv C[P\{x:=Q\}] ;$$

or

$$M' \equiv C'[(\lambda x.P') Q'] ,$$

where $P \twoheadrightarrow^{n_1} P'$, $Q \twoheadrightarrow^{n_2} Q'$, $C[U] \twoheadrightarrow^{n_3} C'[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}((\lambda x.P') Q')$, $n_1 + n_2 + n_3 = 1$ and $(\lambda x.P') Q' \leq_\beta M'$.

Proof. By induction on $M \rightarrow M'$. □

Main Lemma 18. Let $M_0 \equiv C_0[(\lambda x.P_0) Q_0]$ where $(\lambda x.P_0) Q_0$ is the left-most redex in M_0 and $x \notin \text{FV}(P_0)$. Assume $n(M_0)$ and $m = \max\{k | k(Q_0)\} < n$. Then $(n - m - 1)(C_0[P_0])$.

Proof. Let $M_0 \twoheadrightarrow^n M_n$ and consider two cases, according to Lemma 17(ii).

(a)

$$M_0 \equiv C_0[(\lambda x.P_0) Q_0] \twoheadrightarrow^n C_n[(\lambda x.P_n) Q_n] \equiv M_n ,$$

where $P_0 \twoheadrightarrow^{n_1} P_n$, $Q_0 \twoheadrightarrow^{n_2} Q_n$, $C_0[U] \twoheadrightarrow^{n_3} C_n[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}((\lambda x.P_n) Q_n)$, and $n_1 + n_2 + n_3 = n$, $n_2 \leq m$. Then

$$C_0[P_0] \twoheadrightarrow^{n_1+n_3} C_n[P_n] .$$

Since

$$\begin{aligned} n_1 + n_3 &= n - n_2 \\ &\geq n - m \\ &> n - m - 1 , \end{aligned}$$

it holds that $(n - m - 1)(C_0[P_0])$.

(b)

$$M_0 \equiv C_0[(\lambda x.P_0) Q_0] \twoheadrightarrow^k C_k[(\lambda x.P_k) Q_k] \rightarrow C_k[P_k] \twoheadrightarrow^{n-k-1} M_n ,$$

where $P_0 \twoheadrightarrow^{k_1} P_k$, $Q_0 \twoheadrightarrow^{k_2} Q_k$, $C_0[U] \twoheadrightarrow^{k_3} C_k[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}((\lambda x.P_k) Q_k)$, and $k_1 + k_2 + k_3 = k$, $k_2 \leq m$. Then

$$C_0[P_0] \twoheadrightarrow^{k_1+k_3} C_k[P_k] \twoheadrightarrow^{n-k-1} M_n .$$

Since

$$\begin{aligned} k_1 + k_3 + n - k - 1 &= k - k_2 + n - k - 1 \\ &= n - k_2 - 1 \\ &\geq n - m - 1 , \end{aligned}$$

it holds that $(n - m - 1)(C_0[P_0])$. □

Theorem 19. F_∞ is an effective maximal one-step strategy.

Proof. Obviously, F_∞ is an effective one-step strategy. To show that F_∞ is also maximal, let $M_0 \equiv C_0[\Delta_0]$ where Δ_0 is the left-most redex in M_0 , and suppose $n(M_0)$, i.e. $M_0 \rightarrow^n M_n$, for some $n \geq 1$. We proceed by induction on the length of M_0 , splitting into cases according to the definition of F_∞ .

- (1) $\Delta_0 \equiv (\lambda x.P_0) Q_0$ and $x \in \text{FV}(P_0)$. Then $F_\infty(M_0) = C_0[P_0\{x := Q_0\}]$. We consider two subcases, according to Lemma 17(ii).

(a)

$$M_0 \equiv C_0[(\lambda x.P_0) Q_0] \rightarrow^n C_n[(\lambda x.P_n) Q_n] \equiv M_n ,$$

where $P_0 \rightarrow^{n_1} P_n$, $Q_0 \rightarrow^{n_2} Q_n$, $C_0[U] \rightarrow^{n_3} C_n[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}((\lambda x.P_n) Q_n)$, and $n_1 + n_2 + n_3 = n$. By Lemma 15,

$$C_0[P_0\{x := Q_0\}] \rightarrow^{l \cdot n_2} C_0[P_0\{x := Q_n\}] \rightarrow^{n_1 + n_3} C_n[P_n\{x := Q_n\}] ,$$

where $l = \|P_0\|_x \geq 1$, so

$$\begin{aligned} n_1 + l \cdot n_2 + n_3 &\geq n \\ &> n - 1 , \end{aligned}$$

and so $(n - 1)(C_0[P_0\{x := Q_0\}])$.

(b)

$$M_0 \equiv C_0[(\lambda x.P_0) Q_0] \rightarrow^k C_k[(\lambda x.P_k) Q_k] \rightarrow C_k[P_k\{x := Q_k\}] \rightarrow^{n-k-1} M_n ,$$

where $P_0 \rightarrow^{k_1} P_k$, $Q_0 \rightarrow^{k_2} Q_k$, $C_0[U] \rightarrow^{k_3} C_k[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}((\lambda x.P_k) Q_k)$, and $k_1 + k_2 + k_3 = k$. By Lemma 15,

$$C_0[P_0\{x := Q_0\}] \rightarrow^{l \cdot k_2} C_0[P_0\{x := Q_k\}] \rightarrow^{k_1 + k_3} C_k[P_k\{x := Q_k\}] \rightarrow^{n-k-1} M_n ,$$

where $l = \|P_0\|_x \geq 1$, so

$$\begin{aligned} n - k - 1 + k_1 + l \cdot k_2 + k_3 &\geq n - k - 1 + k \\ &\geq n - 1 , \end{aligned}$$

and so $(n - 1)(C_0[P_0\{x := Q_0\}])$.

- (2) $\Delta_0 \equiv (\lambda x.P_0) Q_0$, $x \notin \text{FV}(P_0)$. We consider two subcases.

- (i) $Q_0 \in \text{NF}$. Then $F_\infty(M_0) = C_0[P_0]$ and by Main Lemma 18 (with $m = 0$) $(n - 1)(F_\infty(M_0))$.

- (ii) $Q_0 \notin \text{NF}$. Then $F_\infty(M_0) = C_0[(\lambda x.P_0) F_\infty(Q_0)]$, and we consider two situations.

- (a) $\neg n(Q_0)$. Let $m = \max\{k \mid k(Q_0)\} < n$. Main Lemma 18 implies that $(n - m - 1)(C_0[P_0])$ and by the induction hypothesis $(m - 1)(F_\infty(Q_0))$. Then clearly for some Q', M'

$$C_0[(\lambda x.P_0) (F_\infty(Q_0))] \rightarrow^{m-1} C_0[(\lambda x.P_0) Q'] \rightarrow C_0[P_0] \rightarrow^{n-m-1} M' .$$

Since

$$m - 1 + 1 + n - m - 1 = n - 1 ,$$

also $(n - 1)(C_0[(\lambda x.P_0) (F_\infty(Q_0))])$.

- (b) $n(Q_0)$. By the induction hypothesis it holds that $(n-1)(F_\infty(Q_0))$ and thereby $(n-1)(C_0[(\lambda x.P_0)(F_\infty(Q_0))])$.
- (3) $\Delta_0 \equiv \lambda x.P_0 x, x \notin \text{FV}(P_0)$. Then $F(M_0) = C_0[P_0]$, and we consider two subcases, according to Lemma 17(i):

(a)

$$M_0 \equiv C_0[\lambda x.P_0 x] \twoheadrightarrow^n C_n[\lambda x.P_n x] \equiv M_n ,$$

where $P_0 \twoheadrightarrow^{n_1} P_n$, $C_0[U] \twoheadrightarrow^{n_2} C_n[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}(\lambda x.P_n x)$, and $n_1 + n_2 = n$. Then

$$C_0[P_0] \twoheadrightarrow^{n_1+n_2} C_n[P_n] .$$

Since

$$\begin{aligned} n_1 + n_2 + l &= n \\ &> n-1 , \end{aligned}$$

also $(n-1)(C_0[P_0])$.

(b)

$$M_0 \equiv C_0[\lambda x.P_0 x] \twoheadrightarrow^k C_k[\lambda x.P_k x] \rightarrow C_k[P_k] \twoheadrightarrow^{n-k-1} M_n ,$$

where $P_0 \twoheadrightarrow^{k_1} P_k$, and $C_0[U] \twoheadrightarrow^{k_2} C_k[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}(\lambda x.P_k x)$, and $k_1 + k_2 = k$. Then

$$C_0[P_0] \twoheadrightarrow^{k_1+k_2} C_k[P_k] \twoheadrightarrow^{n-k-1} M_n .$$

Since

$$\begin{aligned} n - k - 1 + k_1 + k_2 &= n - 1 - k + k \\ &= n - 1 , \end{aligned}$$

also $(n-1)(C_0[P_0])$. □

Corollary 20. F_∞ is perpetual.

Proof. By Proposition 9(i). □

5 Upper Bounds for Length of β - and $\beta\eta$ -Reductions

We first present the results for β . The first proposition gives the most obvious way of counting the number of steps in a longest reduction to normal form.

Proposition 21. *There is a partial effective $l : \Lambda \rightarrow \mathbb{N}$ such that*

$$\forall M \in \text{SN} : l(M) = L_{F_\infty}(M) .$$

Proof. Since F_∞ is effective, and one can determine effectively whether a term is in normal form, one can choose $l : \Lambda \rightarrow \mathbb{N}$, $l(M) = \mu n : F_\infty^n(M) \in \text{NF}$. □

If $\infty(M)$ then $l(M)$ is undefined. It is natural to ask whether there is a “simple formula” f such that $f(M)$ is the length of the longest reduction from M when $\text{SN}(M)$, and $f(M)$ is some unpredictable number when $\infty(M)$. One could hope that the freedom to return arbitrary values on non-SN terms could give a simple formula on SN terms. A reasonable formalization of “simple formula” is the notion of a primitive recursive function. The following proposition, which answers a more general question, shows that our hopes are in vain.

Proposition 22. *There is no total effective $l : \Lambda \rightarrow \mathbb{N}$ such that*

$$\forall M \in \text{SN} : l(M) \geq L_{F_\infty}(M) .$$

Proof. Suppose such an l existed and consider the following function $c : \Lambda \rightarrow \mathbb{N}$:

$$c(M) = \begin{cases} 0 & \text{if } F_\infty^{l(M)}(M) \in \text{NF} . \\ 1 & \text{if } F_\infty^{l(M)}(M) \notin \text{NF} . \end{cases}$$

Here c is total and effective. Then by perpetuality of F_∞ and the definition of l :

$$\begin{aligned} c(M) = 0 &\Rightarrow F_\infty^{l(M)}(M) \in \text{NF} \Rightarrow M \in \text{SN} . \\ c(M) = 1 &\Rightarrow F_\infty^{l(M)}(M) \notin \text{NF} \Rightarrow M \notin \text{SN} . \end{aligned}$$

So c gives a procedure to decide for any M whether $\text{SN}(M)$, which is known to be impossible, a contradiction. \square

Corollary 23. *Let $\# : \Lambda \rightarrow \mathbb{N}$ be some effective coding of Λ . There does not exist a primitive recursive $l : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall M \in \text{SN} : l(\#M) \geq L_{F_\infty}(M)$.*

Proof. By Proposition 22 since a primitive recursive function is total. \square

Remark. Proposition 22 is related to a result in Recursion Theory [18, Thm. 2-II] stating that there are partial recursive functions which cannot be extended to total recursive functions. That result is proved by a diagonalization showing that the partial recursive function $\lambda x. \varphi_x(x) + 1$ differs from every total recursive function φ_y at y .

The corresponding results for $\lambda\beta\eta$ are entirely similar. In Proposition 22 one uses undecidability of $\text{SN}_{\beta\eta}$ which follows from undecidability of SN_β by Lemma 7(iii).

6 Maximal and Perpetual Redexes

Definition 24. Let \mathbf{R} be a notion of reduction, Δ a redex with contractum Δ' .

- (i) (**Bergstra and Klop [4]**) Δ is R -perpetual if $\forall C : \infty_R(C[\Delta]) \Rightarrow \infty_R(C[\Delta'])$.
- (ii) Δ is R -maximal if $\forall n \geq 1 \forall C : n_R(C[\Delta]) \Rightarrow (n-1)_R(C[\Delta'])$.

Remark. A strategy that always contracts perpetual redexes is perpetual, but perpetual strategies may also contract non-perpetual redexes. The reason is that a *strategy* is confronted with a redex in a given context, and needs only to make sure that contracting the redex in this particular context preserves the possibility, if present, of an infinite reduction. A perpetual *redex*, on the other hand, must preserve the existence of infinite reduction paths in *all* contexts.

Proposition 25. *Let \mathbf{R} be a notion of reduction.*

- (i) *A redex which is \mathbf{R} -maximal is also \mathbf{R} -perpetual.*
- (ii) *Not vice versa.*

Proof.

- (i) Given \mathbf{R} -maximal redex Δ with contractum Δ' , and a context C , assume $\infty_{\mathbf{R}}(C[\Delta])$. To prove $\infty_{\mathbf{R}}(C[\Delta'])$ it suffices by Lemma 7 to show that $n_{\mathbf{R}}(C[\Delta'])$ for any n . Since $\infty_{\mathbf{R}}(C[\Delta])$ we have by Lemma 7 for any n , $n_{\mathbf{R}}(C[\Delta])$ and thereby $(n+1)_{\mathbf{R}}(C[\Delta])$. Thus $n_{\mathbf{R}}(C[\Delta'])$ for any n by maximality.
- (ii) Any redex $(\lambda x.P)Q$ with $x \in \text{FV}(P)$ is β -perpetual (see below), but $\Delta \equiv \mathbf{I} \mathbf{I}$ with contractum \mathbf{I} is not β -maximal: with $C \equiv (\lambda xy.y \ x \ x) \ \square$, $C[\Delta]$ has a reduction of length 3, but the longest reduction of $C[\Delta']$ has length 1. \square

7 Maximal and Perpetual Redexes: $\lambda\beta$ and $\lambda\beta\eta$

We start with perpetual redexes in $\lambda\beta$.

Theorem 26. *Let $\Delta \equiv (\lambda x.P) Q$.*

- (i) **(Barendregt et al. [2]).** *If $x \in \text{FV}(P)$ then Δ is perpetual.*
- (ii) **(Bergstra and Klop [4]).** *If $x \notin \text{FV}(P)$ then Δ is perpetual iff for every substitution θ of strongly normalizing terms for variables: $\infty(Q\theta) \Rightarrow \infty(P\theta)$.*

Given the preceding theorem, the following proposition completely characterizes perpetual redexes in $\lambda\beta\eta$. The proof is omitted since (i) follows directly from Lemma 7(iii), and (ii) is [15, IV.4.9].

Proposition 27.

- (i) *$(\lambda x.P) Q$ is $\beta\eta$ -perpetual iff it is β -perpetual.*
- (ii) *$\lambda x.P \ x$ with $x \notin \text{FV}(P)$ is $\beta\eta$ -perpetual.*

We now proceed to maximal redexes in $\lambda\beta$. The intuition is as follows. Given a redex Δ with contractum Δ' , we can conceive a context C which is such that $C[\Delta]$ can duplicate Δ . Therefore the longest reduction path from $C[\Delta]$ is obtained only if we do not contract Δ until it has been duplicated. But then Δ is not maximal. The only escape is when the reduct of Δ has an infinite reduction path. Then $C[\Delta']$ has arbitrarily long reduction paths, so Δ is maximal.

Proposition 28. *Let Δ be a redex with contractum Δ' . Δ is maximal iff $\infty(\Delta')$.*

Proof.

\Leftarrow : If $\infty(\Delta')$ then for any $n > 0$ and context C , $(n-1)(C[\Delta'])$.

\Rightarrow : We assume $\text{SN}(\Delta')$ and prove that Δ is not maximal by finding an n such that $n(C[\Delta])$ but not $(n-1)(C[\Delta'])$.

Since $\text{SN}(\Delta')$ there is by König's Lemma an m such that $(m-1)(\Delta')$ and not $m(\Delta')$. Then $m(\Delta)$. So for $C \equiv (\lambda x y. y \ x \ x) []$ we have for some Q

$$C[\Delta] \rightarrow \lambda y. y \ \Delta \ \Delta \twoheadrightarrow^{2m} \lambda y. y \ Q \ Q ;$$

that is, $(2m+1)(C[\Delta])$.

On the other hand, any reduction of $C[\Delta']$ has form

$$C[\Delta'] \twoheadrightarrow^k C[Q'] \rightarrow \lambda y. y \ Q' \ Q' \twoheadrightarrow^{2l} \lambda y. y \ Q'' \ Q''$$

for some $Q'' \in \text{NF}$, where $k+l \leq m-1$, and therefore $k+1+2l < 2m$. So, not $(2m)(C[\Delta'])$. \square

The same property holds for $\lambda\beta\eta$ with a similar proof.

8 Applications

In this section we give three examples illustrating the usefulness of maximal and perpetual reductions. In the first example we give a *very* short proof of Church's Conservation Theorem for $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$;⁴ we have not seen the latter result in the literature. In the second example we generalize this result, showing that in $\lambda\mathbf{A}\beta$ and $\lambda\mathbf{A}\beta\eta$ all reduction paths are equally long. Finally, in the third example we mention a recent result by the author.

First the short proof of the Conservation Theorem in $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$.

Definition 29.

- (i) In $\lambda\beta$ define $F_l : A \rightarrow A$ by: $F(M) = M$ if $M \in \text{NF}$; otherwise let $M \equiv C[\Delta]$ where Δ is the left-most redex in M and Δ has contractum Δ' , and define $F(M) = C[\Delta']$.
- (ii) The definition in $\lambda\beta\eta$ is literally the same.

Remark. Both F_l and F_∞ in $\lambda\beta$ can be viewed as β -reduction strategies in $\lambda\mathbf{I}$ and $\lambda\mathbf{A}$ since the set of $\lambda\mathbf{I}$ -terms and $\lambda\mathbf{A}$ -terms are closed under β -reduction. Similarly for $\lambda\beta\eta$.

Lemma 30 (Regnier [17]). *In $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$, F_l and F_∞ are identical.*

⁴ $\lambda\mathbf{I}$ is the restriction of λ -calculus obtained by requiring that for all abstractions $\lambda x. M$, x occur free in M . $\lambda\mathbf{A}$ is the restriction of λ -calculus obtained by requiring that for all abstractions $\lambda x. M$, x occur free in M exactly once, see e.g. [12].

Proof. Obvious: all abstractions in $\lambda\mathbf{I}$ have form $\lambda x.M$ where $x \in \text{FV}(M)$. \square

Proposition 31 (Curry and Feys [7], Klop [15]). *In $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$, F_l is normalizing.*

Corollary 32 (Church [6]).

- (i) *In $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$: $\text{WN}(M) \Leftrightarrow \text{SN}(M)$.*
- (ii) *In $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$ all strategies are perpetual.*

Proof.

- (i) follows by the following equivalences that hold both in $\lambda\mathbf{I}\beta$ and $\lambda\mathbf{I}\beta\eta$ due to Corollary 12, Lemma 30, and Proposition 31:

$$\text{WN}(M) \Leftrightarrow \exists n : F_l^n(M) \in \text{NF} \Leftrightarrow \exists n : F_\infty^n(M) \in \text{NF} \Leftrightarrow \text{SN}(M) .$$

- (ii) follows from (i). \square

Remark. Not all strategies are maximal in $\lambda\mathbf{I}\beta$ or $\lambda\mathbf{I}\beta\eta$; for instance the strategy which always contracts the right-most redex is not maximal, as the example $(\lambda xy.y \ x \ x) (\mathbf{I} \ \mathbf{I}) \rightarrow_\eta^3 \lambda y.y \ \mathbf{I} \ \mathbf{I}$ shows.

Next we prove the generalization of the Conservation Theorem: in $\lambda\mathbf{A}\beta$ and $\lambda\mathbf{A}\beta\eta$ all reduction paths have the same length.

Proposition 33.

- (i) *In $\lambda\mathbf{A}\beta$ and $\lambda\mathbf{A}\beta\eta$, F_∞ is minimal.*
- (ii) *In $\lambda\mathbf{A}\beta$ and $\lambda\mathbf{A}\beta\eta$, all reduction paths have the same length.*

Proof.

- (i) We prove the assertion for $\lambda\mathbf{A}\beta\eta$. Let $M_0 \equiv C_0[\Delta_0]$ where Δ_0 is the left-most redex in M_0 , and let $M_0 \rightarrow^n M_n$ be a reduction path of minimal length to normal form.
- (a) $\Delta_0 \equiv (\lambda x.P_0) Q_0$. Since $x \in \text{FV}(P_0)$, $F_\infty(M_0) = C_0[P_0\{x:=Q_0\}]$, so we have to prove that the first step in a shortest reduction path is to contract Δ_0 . By Lemma 17(ii), since $M_n \in \text{NF}$,

$$M_0 \equiv C_0[(\lambda x.P_0)Q_0] \rightarrow^k C_k[(\lambda x.P_k)Q_k] \rightarrow C_k[P_k\{x:=Q_k\}] \rightarrow^{n-k-1} M_n,$$

where $P \rightarrow^{k_1} P_k$, $Q \rightarrow^{k_2} Q_k$, and $C[U] \rightarrow^{k_3} C'[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}((\lambda x.P_k) Q_k)$, and $k_1 + k_2 + k_3 = k$. Since $\|P_0\|_x = 1$, by Lemma 15

$$M_0 \equiv C_0[(\lambda x.P_0)Q_0] \rightarrow C_0[P_0\{x:=Q_0\}] \rightarrow^k C_k[P_k\{x:=Q_k\}] \rightarrow^{n-k-1} M_n,$$

and this reduction has the same length as the first one.

- (b) $\Delta_0 \equiv \lambda x.P_0 x$, $x \notin \text{FV}(P_0)$. We have to prove that the first step in a shortest reduction path is to contract Δ_0 . By Lemma 17(i), since $M_n \in \text{NF}$,

$$M_0 \equiv C_0[\lambda x.P_0 x] \twoheadrightarrow^k C'[\lambda x.P_k x] \rightarrow C_k[P_k] \twoheadrightarrow^{n-k-1} M_n ,$$

where $P_0 \twoheadrightarrow^{k_1} P_k$, $C_0[U] \twoheadrightarrow^{k_2} C_k[U]$ for all U with $\text{FV}(U) \subseteq \text{FV}(\lambda x.P_k x)$, and $k_1 + k_2 = k$. Then

$$M_0 \equiv C_0[\lambda x.P_0 x] \rightarrow C_0[P_0] \twoheadrightarrow^k C_k[P_k] \twoheadrightarrow^{n-k-1} M_n ,$$

and this reduction has the same length as the first one.

- (ii) F_∞ is maximal in $\lambda\mathbf{A}\beta$ and $\lambda\mathbf{A}\beta\eta$, and minimal by (i). Since the longest and shortest reduction path have the same length, *all* reduction paths have the same length. \square

Finally we mention one last application. When one proves $\infty(M) \Rightarrow P(M)$, for some predicate P , one often utilizes some property of infinite reduction paths to prove $P(M)$. By using a perpetual strategy one knows from the assumption $\infty(M)$ not just that *some* reduction path is infinite, but that the F -reduction path is infinite, and the latter has some useful properties that might make it easier to prove $P(M)$. For instance this technique is essential in the author's proof [21] that any term with an infinite β -reduction path must have $\Omega \equiv (\lambda x.x x) (\lambda x.x x)$ embedded, i.e. must have form

$$\cdots (\lambda x. \cdots x \cdots x \cdots) \cdots (\lambda x. \cdots x \cdots x \cdots) \cdots$$

9 Conclusion

We have given a systematic study of perpetual and maximal reduction strategies and redexes, recasting the few well-known results as well as adding the following results which, to the best of our knowledge, are original:

- (i) The relationship between maximal and perpetual strategies and redexes;
- (ii) A maximal and perpetual strategy in $\lambda\beta\eta$ (and a maximal strategy in $\lambda\beta$);
- (iii) Maximal redexes in $\lambda\beta\eta$ and $\lambda\beta$, and perpetual redexes in $\lambda\beta\eta$;
- (iv) Non-effectiveness of upper bounds for lengths of β - and $\beta\eta$ -reductions;
- (v) Applications, including a short proof of the Conservation Theorem.

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