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Matching workers to firms facing budget constraints

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ABSTRACT

We study a many-to-one matching model with salaries in which firms face budget constraints. Mongell and Roth (1986) show that when firms face a budget constraint, a stable matching may not exist. We introduce an algorithm to find a strong stable matching by changing the budget of firms such that the total budget remains the same and each firm's budget change is bounded by the value of at most one worker for that firm.

1. Introduction

We study a many-to-one matching problem with salaries taking into account the budgetary constraints faced by firms. Kelso and Crawford (1982) show that a stable matching always exists if workers are regarded as gross substitutes. But Mongell and Roth (1986) show that when firms face a budget constraint, workers are not gross substitutes from a firm's point of view; therefore, a stable matching may not exist. We study the same problem that Mongell and Roth (1986) researched under a special form of firms' valuation of workers: the multiplicative functional form. We provide an example in this case such that the assumption of gross substitution fails, and Kelso and Crawford (1982)'s algorithm for finding a stable matching does not work. We show that changing the budgets of firms such that the total budget of all firms remains the same and each firm's budget change is bounded by, at most, one worker's value, allows us to find a stable matching.

Nguyen and Vohra (2018) and Nguyen and Vohra (2022) take the same approach as our paper, but in different contexts. Similar to our paper, they change the resource constraint to prove the existence of an equilibrium.[1] In the context of the national resident matching program with couples, Nguyen and Vohra (2018) show that perturbing the capacity of hospitals ensures that a stable matching exists. They prove that this perturbation is less than 2 for each hospital's capacity and less than 4 for aggregate capacities.

In recent work, Jagadeesan and Teytelboym (2022) study the existence of stable matching in the presence of budget constraints.[2] However, their net substitutability condition is not satisfied in our setting.[3] Moreover, their definition of stable matching, which is the solution concept that they work with, is weaker than our solution concept.

2. Model

There is a finite set of workers W, and a finite set of firms F. Each firm $f \in F$ has a budget $b_f \in \mathbb{R}_+$; denote the budget of all firms as $B = (b_f)_{f \in F}$. Each firm can hire (match) zero, one, or more workers and has to pay a non-negative salary to the hired workers.[4] A worker cannot work in two different firms. Formally, a many-to-one matching is an assignment of workers to firms with specified salaries, $\mu : W \to (F \times S) \cup \{(\emptyset, 0)\}$, where each worker is either matched with one firm or is unmatched with a salary equal to zero. The set of all possible salaries is represented as $S = \mathbb{R}_+$. A firm can hire more than one worker; denote the set of workers assigned to firm f by $\mu^o(f) := \{(w,s) | (f,s) = \mu(w) \text{ for some } w \in W\}$.

Each worker $w \in W$ has an intrinsic value (type) V_w. Let the technology (type) of firm $f \in F$ be V_f, where $V_1 < V_2 < \cdots < V_{|F|}$. We consider the case in which the value of the worker w for firm f is $V_w.V_f$, i.e., there is complementarity between firms and workers. Each

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1 Gul et al. (2024) show an existence result for a pseudo-market using fiat currency that has value only within that market, utilizing explicit randomization in their proposed mechanism.

2 The first draft of our paper was publicly available in 2020, before their work was publicly available.

3 See example 1 in the online appendix.

4 We assume the outside option of firms and workers to be zero.

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firm has a utility function $u_f : \cup_{A \subset W}(A \times S^A) \to \mathbb{R}_+$, which under matching μ is equal to

$$u_f(\mu^o(f)) = \sum_{(w,s_w)\in\mu^o(f)} V_w.V_f - s_w,$$

if $\sum_{(w,s_w)\in\mu^o(f)} s_w \le b_f$. Otherwise, the utility is $-\infty$; firms face a hard budget constraint. Denote worker w's utility with $u_w : (F \cup \{\emptyset\}) \times S \to \mathbb{R}_+$, which is equal to the worker's salary if he/she is employed and zero if he/she is unemployed.

A matching μ is *individually rational*, if (1) for each $f \in F$, $u_f(\mu^o(f)) \ge u_f(C)$ for all $C \subset \mu^o(f)$; and (2) $u_w(\mu(w)) \ge u_w(\emptyset, 0)$. A matching μ is *strongly stable* if it is individually rational and, for any firm f and $A \subset W$, if there is a feasible salary vector $(s_w)_{w\in A}$ with $u_f(\{(w, s_w)|w \in A\}) > u_f(\mu^o(f))$, then there exists a $w \in A$ such that $u_w(\mu(w)) > u_w(f, s_w)$.[5]

We say salaries are *fair* when the salaries that workers receive for each unit of productivity are the same for different workers. Formally, for all $w, w' \in W$:

$$\frac{s_w}{V_w} = \frac{s_{w'}}{V_{w'}}.$$

We introduce a *fair salary algorithm*, which results in a strongly stable matching by changing the budget of firms so that the total budget remains the same and each firm's budget change is bounded by the value of at most one worker to that firm. The *fair salary algorithm* has two stages: in the first stage we find salaries, then, in the second stage we introduce the matching mechanism.

Before we introduce the algorithm, we need to define a function that helps us decide which firm will hire a worker in the algorithm. Define

$$\alpha(\vec{x}) = \frac{\sum_{f\in F} x_f b_f}{\sum_{w\in W} V_w},$$

where x_f is the fth array of vector $\vec{x} \in [0, 1]^{|F|}$. Let $y \in [0, 1]$, and define $\overrightarrow{y_f} \in [0, 1]^{|F|}$ for $f \in F$ a vector such that all the elements before f are zero, all the elements after f are one, and the fth element is equal to y, i.e.,

$$\overrightarrow{y_f}.e_i = \begin{cases} 1 & \text{if } i > f, \\ 0 & \text{if } i < f, \\ y & \text{if } i = f. \end{cases}$$

Observe that if $y = 1$ and $f = 1$, then $\overrightarrow{y_f} = \vec{1}$. Intuitively, $\alpha(\vec{x})$ is an auxiliary variable that helps us determine the optimal salary-to-productivity ratio, α^*. This optimal ratio categorizes firms into three groups: those with technology levels below the ratio, those with technology levels above the ratio, and at most one firm with productivity equal to the ratio. Additionally, the vector $\overrightarrow{y_f}$ ensures that the budgets of firms with technology levels strictly higher than the ratio are fully utilized, the budget of the firm with productivity equal to the ratio is partially utilized, and firms with lower technology levels are excluded from the market.

Lemma 1. *If $\alpha(\vec{1}) > V_1$:*

1. *Either there exists $f^* \in F$ and $y \in [0, 1]$ such that:*

   $$V_{f^*} = \alpha(\overrightarrow{y_{f^*}}),$$

2. *or $f^* \in F$ exists such that*

   $$V_{f^*-1} < \alpha(\overrightarrow{1_{f^*}}) \le V_{f^*}.$$

[5] Our definition of a strong stable match is equivalent to the strong core; moreover, it is equivalent to the Jagadeesan and Teytelboym (2022) definition of competitive equilibrium, which may not exist in their setting even if the net substitutability condition is satisfied. Note that most of the literature on matching with transfers works with the concept of stable matching which is weaker than our definition of a strong stable match.

Proof:

By definition, $\alpha(\vec{1}) = \alpha(\overrightarrow{1_1}) > V_1$. If $\alpha(\overrightarrow{1_2}) \le V_1$, then by continuity, $y \in [0, 1]$ exists such that

$$V_1 = \alpha(\overrightarrow{y_1}).$$

If $\alpha(\overrightarrow{1_2}) > V_1$, and $\alpha(\overrightarrow{1_2}) \le V_2$, then part 2 of Lemma 1 is satisfied. Otherwise, $\alpha(\overrightarrow{1_2}) > V_2$. If $\alpha(\overrightarrow{1_3}) \le V_2$, then by continuity, $y \in [0, 1]$ exists such that

$$V_2 = \alpha(\overrightarrow{y_2}).$$

By repeating this argument, we prove either Lemma 1 or $\alpha(\overrightarrow{1_{|F|}}) > V_{|F|}$. By definition, $\alpha(\overrightarrow{0_{|F|}}) = \alpha(\vec{0}) = 0$. Hence, by continuity, if $\alpha(\overrightarrow{1_{|F|}}) > V_{|F|}$, then $y \in [0, 1]$ exists such that

$$V_{|F|} = \alpha(\overrightarrow{y_{|F|}}).$$

Stage one:

Define $\alpha^* = \alpha(\vec{1})$, if $\alpha(\vec{1}) \le V_1$. If $\alpha(\vec{1}) > V_1$, then define $\alpha^* = \alpha(\overrightarrow{y_{f^*}})$, where $f^* \in F$, and $y \in [0, 1]$ satisfies Lemma 1.[6] Choose $s_w = \alpha^* V_w$ for all $w \in W$.

In the first stage of the algorithm, we begin with a vector of ones $(\vec{1})$ and gradually reduce each element from 1 to 0. This process allows the algorithm to identify the least efficient firm, denoted as f^*, and exclude all firms that are less efficient than f^*. The optimal salary-to-productivity ratio (α^*) is then calculated by dividing the total budget of the sufficiently efficient firms (those with $f \ge f^*$, adjusting b_{f^*} to $y_{f^*} b_{f^*}$) by the total productivity of their workers. Observe that when $\alpha(\vec{1}) \le V_1$, then α^*, the ratio of the salary to productivity, is such that all firms want to hire workers, because $V_w V_f - \alpha^* V_w = (V_f - \alpha^*)V_w \ge 0$, for all $f \in F$ and $w \in W$. If $\alpha(\vec{1}) > V_1$, then Lemma 1 states that if $s_w = \alpha^* V_w$, then f^* exists such that $V_{f^*-1} < \alpha^* \le V_{f^*}$. This means that firms $f' \in F$ such that $V_{f'} < V_{f^*}$ do not hire any worker at this wage, because $V_f V_w - \alpha^* V_w = (V_f - \alpha^*)V_w < 0$. Firms $f'' \in F$ such that $V_{f''} \ge V_{f^*}$ have incentive to hire workers, because $V_f V_w - \alpha^* V_w = (V_f - \alpha^*)V_w \ge 0$.

Next, we introduce the second stage of the fair salary algorithm which finds a strongly stable matching with fair salaries by changing the budgets of firms.

Stage two, the matching mechanism:

1. Remove firms that are not efficient; $V_f < \alpha^*$ (if these firms exist).
2. Change b_{f^*} to $b'_{f^*} = y_{f^*} b_{f^*}$.
3. Among firms that are efficient enough ($V_f \ge \alpha^*$), choose one firm, and match workers as much as possible until the budget constraint does not allow the matching of even one more worker. Choose another firm and match some worker/workers (or zero workers) from the rest of the workers until the budget constraint does not allow the matching of more workers. Continue until there are no more firms in this group of firms ($V_f \ge \alpha^*$). Remove all the firms that have exhausted their budgets completely.
4. Choose one worker among the unmatched workers (if one exists), and match that worker with one of the remaining firms. Update the budget of the firm to the sum of the salaries that it has to pay, and remove both the firm and the worker. Repeat the same process for another unmatched worker. Continue until no unmatched worker remains.
5. Add $b_{f^*} - b'_{f^*}$ to the budget of firm f^*.

Theorem 1. *Using the fair salary algorithm results in a strongly stable matching with fair salaries in which*

(i) The budget of firm $f \in F$ changes at most $\max_{w\in W}\{\alpha^* V_w\}$,
(ii) The sum of the budgets of all the firms remains the same.

[6] If there are multiple $f^* \in F$, and $y \in [0, 1]$, which satisfy Lemma 1, select the firm with highest value as f^*.

Proof:
Observe that:
$$\frac{S_w}{V_w} = \frac{S_{w'}}{V_{w'}} = \alpha^*, \forall w, w' \in W.$$

Firms do not have any incentive to change their allocations at these wages: In step two, for not efficient firms, all $f \in F$ such that $V_f < V_{f^*}$, hiring a worker at these wages generates a negative profit, so the algorithm removes these firms from the market. In step three, if $b_f \neq b_{f'}$, then $y_{f^*} < 1$, which means $V_{f^*} = \alpha^*$. In this case, f^* makes a zero profit, so f^* is indifferent between hiring and not hiring workers. In step four, efficient firms ($V_f \geq \alpha^*$) that have exhausted their budgets completely hire zero or positive numbers of workers. These firms are indifferent between workers because the value of spending one dollar for hiring a worker is the same among different workers and is
$$\frac{V_f V_w - S_w}{S_w} = \frac{V_f V_w - \alpha^* V_w}{\alpha^* V_w} = \frac{V_f - \alpha^*}{\alpha^*}.$$
These firms do not have any incentive to change their allocations. After step four and before starting step five, there remain some unmatched workers and some firms, which are efficient, but still have not finished spending their budgets. It is not possible to match all workers without spending all budgets; in other words, it is not possible to exhaust all budgets without matching all workers, because salaries are such that $\sum_{w\in W} S_w = \sum_{w\in W} \alpha^* V_w = b'_{f^*} + \sum_{f>f^*} b_f$.[7] In step five, after matching workers and updating budgets, all firms $f \in F$ such that $V_f > \alpha^*$ have reached their budgets, and do not have any incentive to change their allocations, as they are indifferent between workers. In step six, changing the budget of firm f^* does not create an incentive for f^* to change the allocation. This is due to the fact that if $b_f \neq b_{f'}$, then f^* is indifferent between hiring workers or not hiring them. Note that if an efficient firm strictly prefers another vector of salaries to the current one, then it has to pay a salary less than the current one to at least one of the workers, because all efficient firms have exhausted their budgets completely.

(i) Consider a firm f such that $V_f > V_{f^*}$: In step five, when the algorithm matches w to f, firm f's remaining budget is less than S_w; therefore, it increases at most $S_w = \alpha^* V_w$, because the algorithm does not match two or more workers to a firm at step five. After matching all workers, there are some firms that do not receive any worker at step five. The algorithm decreases their budgets. Their budgets do not change by more than the salary of a worker, because in step four they could not hire any worker. Consider f^*: In step two, the algorithm changes the budget from b_{f^*} to b'_{f^*}, but in step six, it adds the difference back into its budget. In step five, the same as it does at other efficient firms, the budget b'_{f^*} changes by, at most, $\max_{w\in W} \alpha^* V_w$. Hence, after step six, budget b_{f^*} does not change by more than $\max_{w\in W} \alpha^* V_w$.

(ii) The sum of budgets at the end of step five is equal to the sum of salaries. This is because the algorithm hires all workers at the specified salaries. The sum of all the salaries at the end of step five is equal to the sum of the budgets at step four; i.e.,$\sum_{w\in W} S_w = \sum_{w\in W} \alpha^* V_w = b'_{f^*} + \sum_{f>f^*} b_f$. Finally, in step six, by adding $b_{f^*} - b'_{f^*}$ to the budget of firm f^*, the sum of all these firms' budgets will be $b_{f^*} + \sum_{f>f^*} b_f$.

Given a proposed budget vector, the fair salary algorithm is a method for finding a nearby budget vector with a strongly stable matching in which the total of all the budgets is the same.[8] For markets where a central authority assigns budgets, this mechanism is very helpful. For example, when a state or federal government assigns budgets to schools or other branches of the government, this algorithm, without changing the total budgets, results in a strongly stable matching.

3. Conclusion

We study the problem of finding stable matching in cases in which firms hire workers with a specified salary while facing a budget constraint. We introduce an algorithm that results in a strong stable matching with fair salaries by changing the firms' budgets. The budget change for each firm is bounded by the value of at most one worker to that firm; moreover, the sum of all new budgets equals the sum of all previous budgets, i.e., there is no need for additional funds to find a strong stable matching.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.econlet.2024.112048.

Data availability

No data was used for the research described in the article.

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[7] Note that the number of unmatched workers is less than or equal to the number of firms that still have some amount left in their budgets (the remaining firms).

[8] See example 2 & 3 in the online appendix.