Phase Retrieval

ECE 554 Final Presentation

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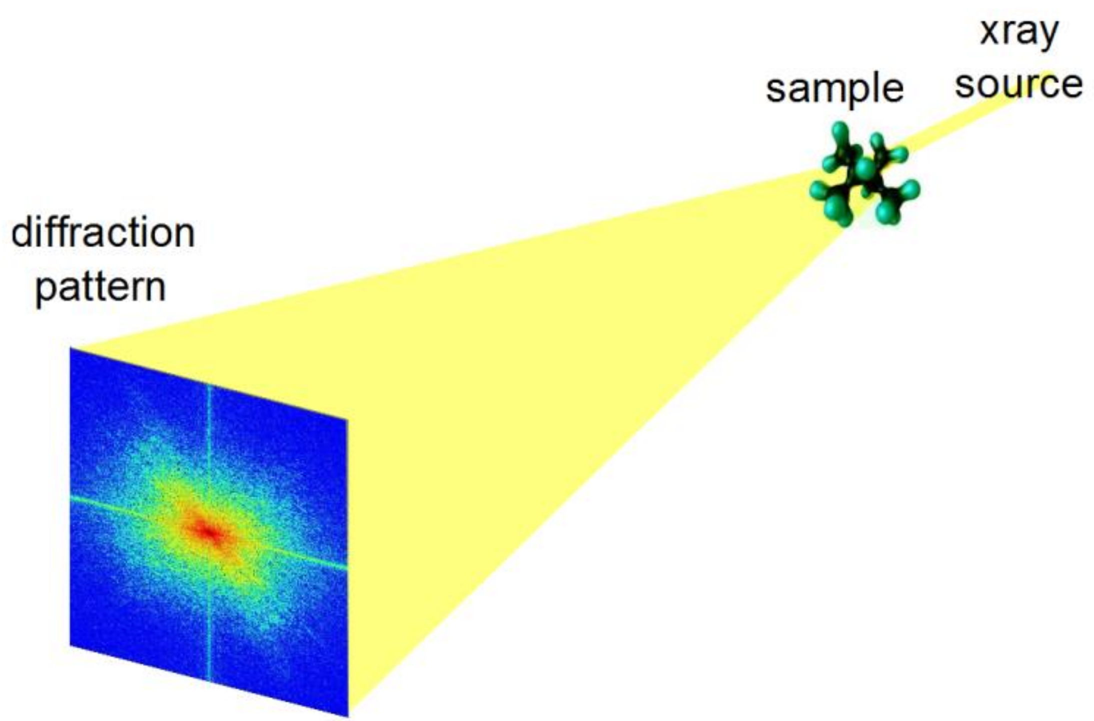
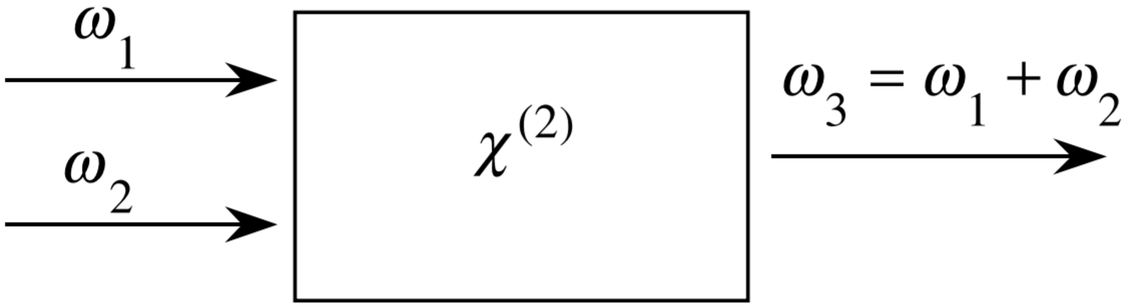
Outline

- Why Does Phase Matter?
- When Is Phase Lost?
- How Difficult Is Phase Retrieval?
- Algorithms for Phase Retrieval
- Current Research



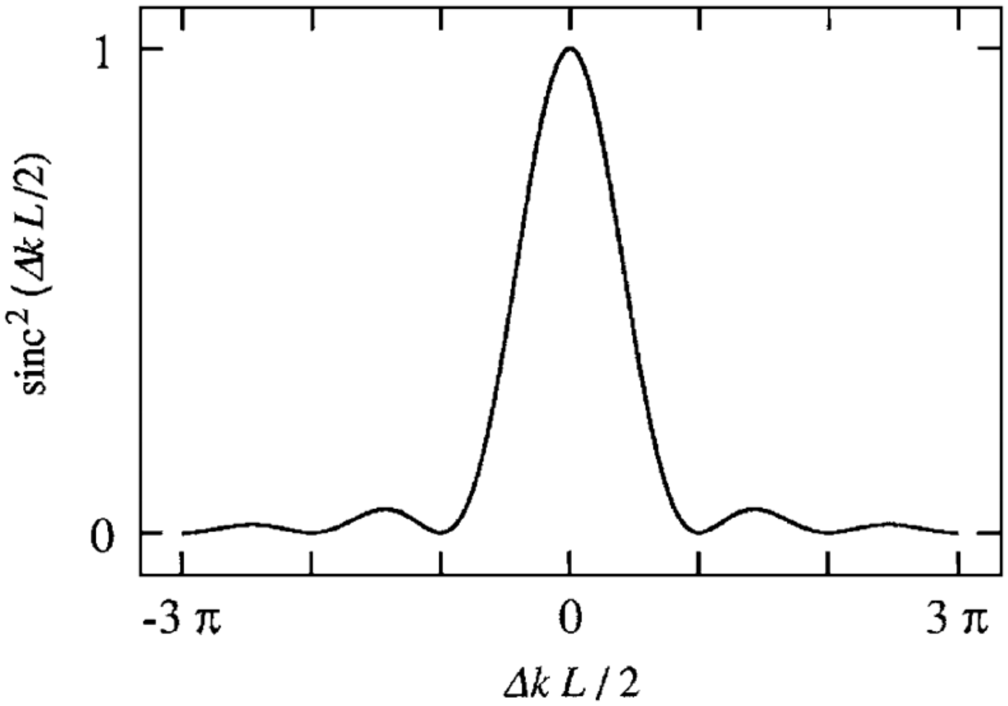
Relevance: Phase Matching



$$I_3 = I_3^{(\max)} \left[\frac{\sin(\Delta k L/2)}{(\Delta k L/2)} \right]^2 \longrightarrow$$

Intensity of generated wave

$$\Delta k = k_1 + k_2 - k_3$$



- Significant reduction in sum-frequency generation efficiency occurs when $\Delta k \neq 0$. (No perfect phase-matching)

Ref: Nonlinear Optics, Robert W. Boyd, Chapter 2

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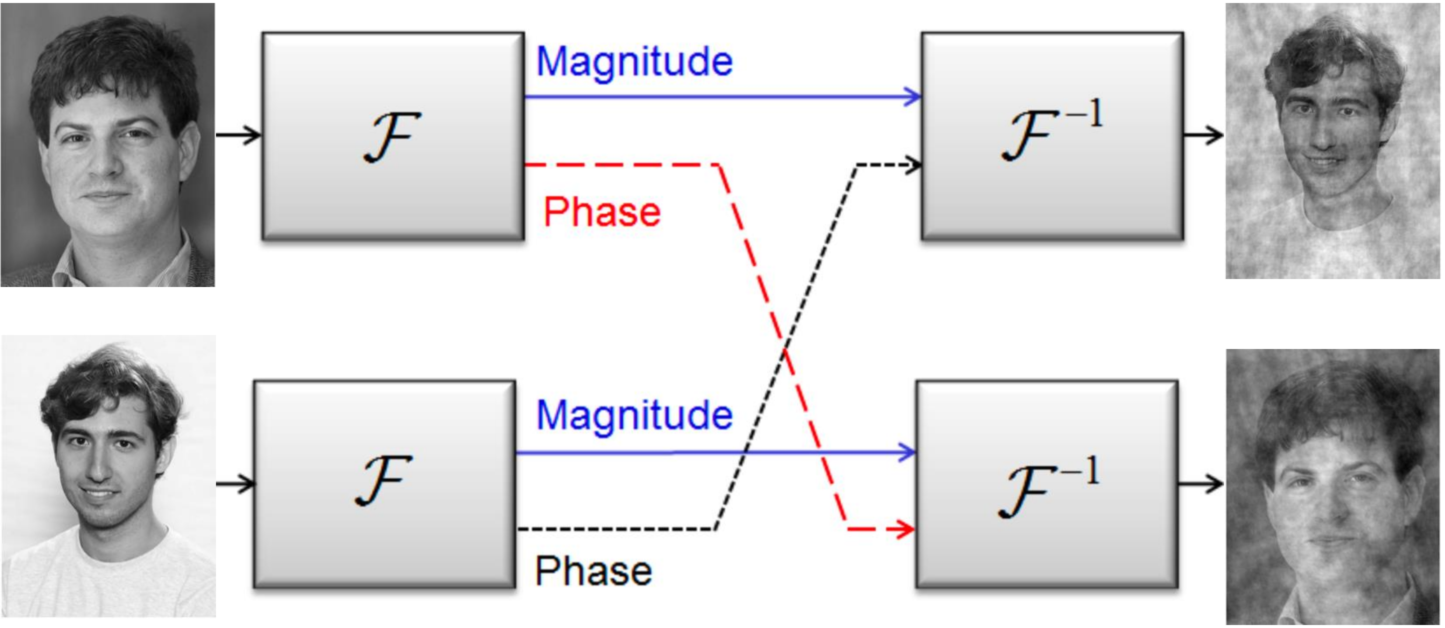
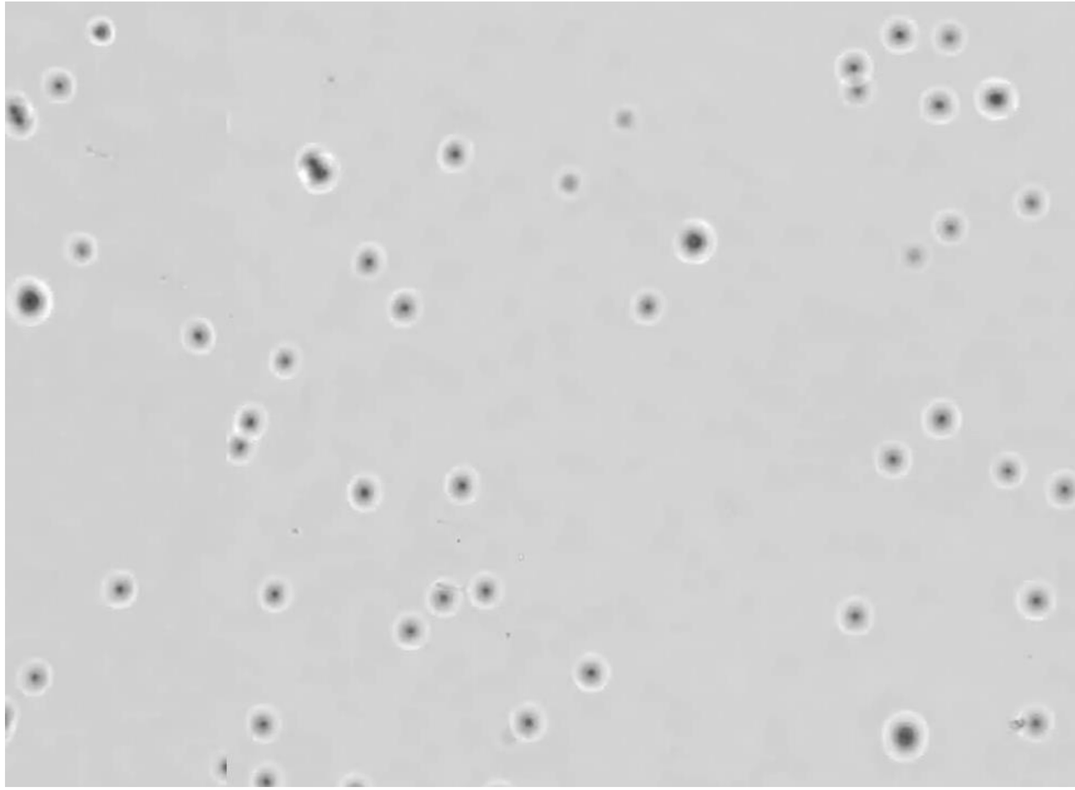
Why does Phase Matter?



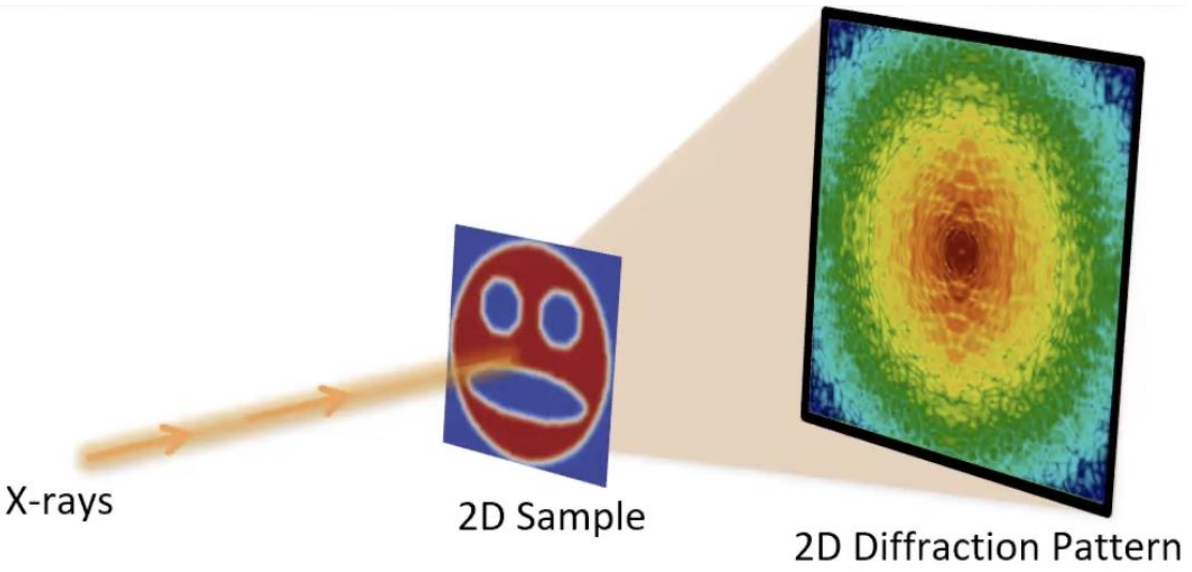


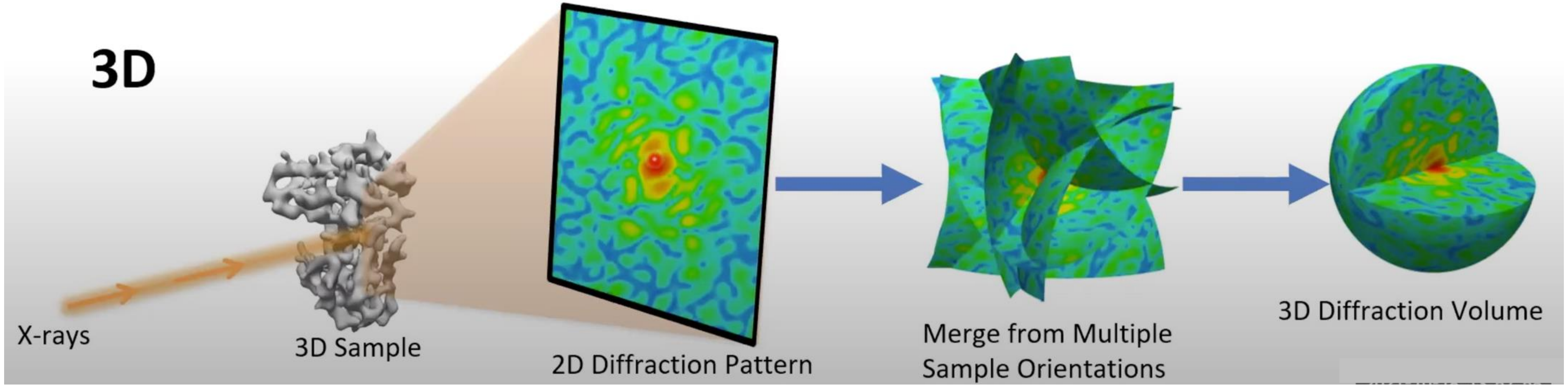
- More information in phase of Fourier Transform of an image!

- Intensity of light doesn’t change so much in transparent materials!
- Phase Contrast Imaging: Enhances contrast by detecting phase shifts in light.

Phase Problem Example: Coherent Diffraction Imaging (CDI)

- In CDI, a highly coherent beam of wavelike particle (X-rays, electrons, photons) is incident on an object.
- Diffraction pattern = Magnitude of the Fourier transform of the object (In far-field)
- Phase is not captured!





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Notation

	Fourier Transform	**Inverse Fourier Transform**
Continuous	$\hat{\rho}(\boldsymbol{q}) = \int \rho(\boldsymbol{r}) e^{-2\pi i(\boldsymbol{r}.\boldsymbol{q})} d\boldsymbol{r}$	$\rho(\boldsymbol{r}) = \int \hat{\rho}(\boldsymbol{q}) e^{-2\pi i(\boldsymbol{r}.\boldsymbol{q})} d\boldsymbol{q}$

Sample Electron Density

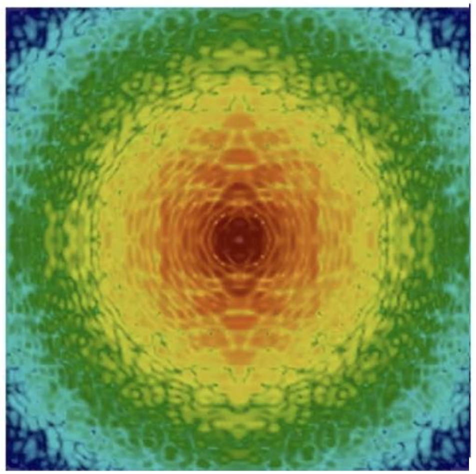
$$\rho(\boldsymbol{r})$$

2D Sample

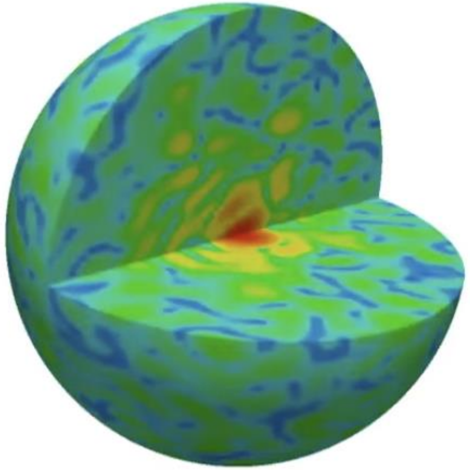
3D Sample

Diffraction Intensity

$$I(\boldsymbol{q}) = |\hat{\rho}(\boldsymbol{q})|^2$$

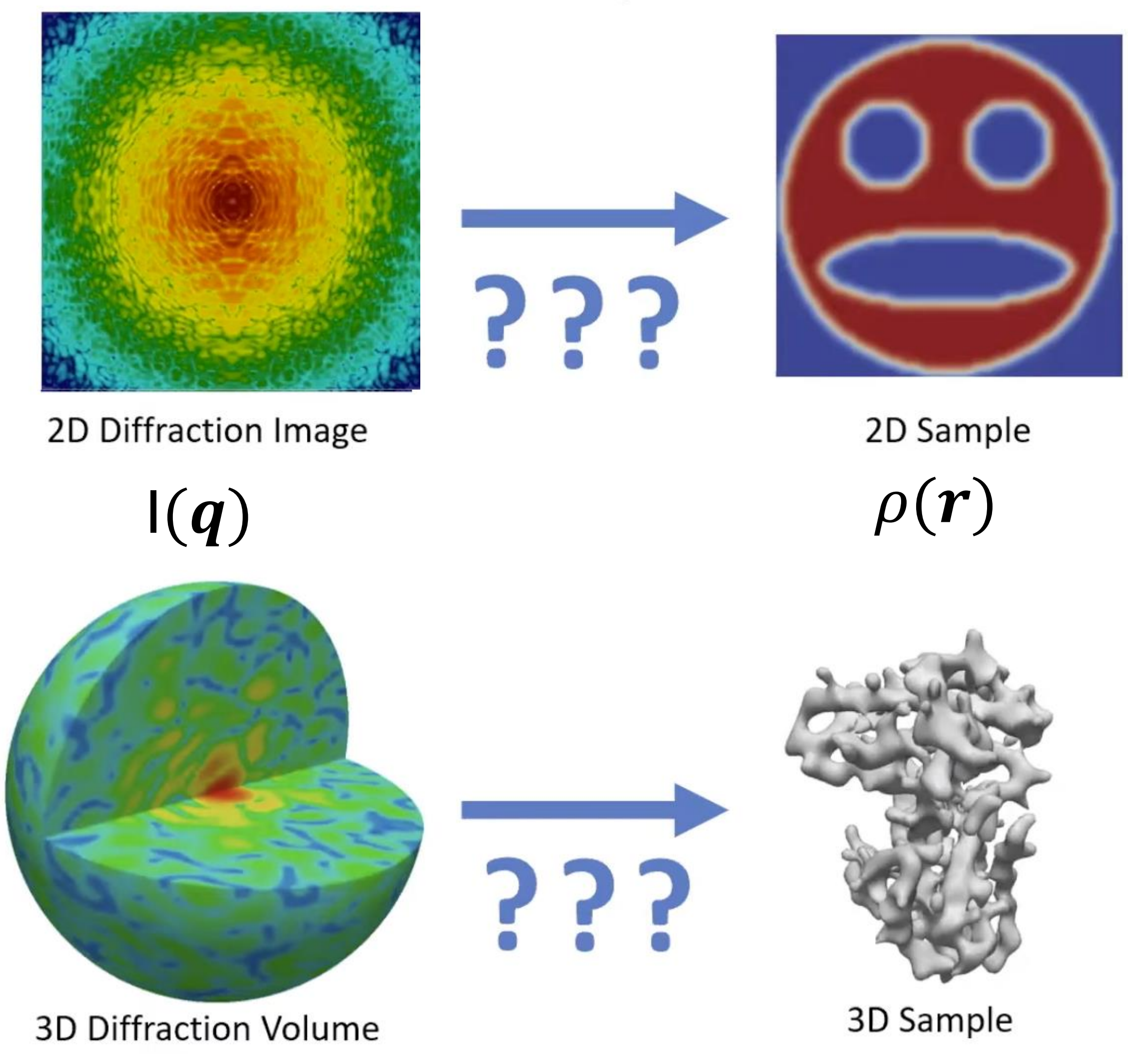


2D Diffraction Image



3D Diffraction Volume

What is Phase Retrieval problem?



$$\hat{\rho}(\boldsymbol{q}) = \underbrace{\sqrt{I(\boldsymbol{q})}}_{\substack{\text{Magnitude}\\ \text{Measured}}}\ \underbrace{e^{i(\phi(\boldsymbol{q}))}}_{\substack{\text{Phase}\\ \text{Not Measured}}}$$

Goal: Reconstruct the density $\rho(\boldsymbol{r})$ from the measured diffraction intensities $I(\boldsymbol{q})$

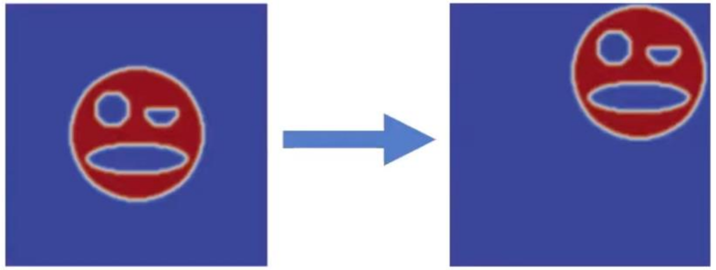
Challenge: The complex phases $\phi(\boldsymbol{q})$ are not measured and need to be retrieved.

Requirement: Additional constraints are needed to determine the phases (support constraint)

How hard is Phase Retrieval?

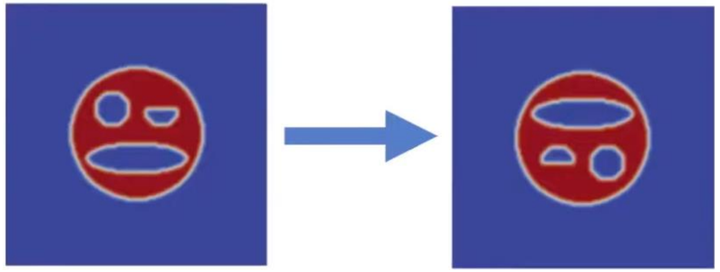
Inverse Problem

Translation



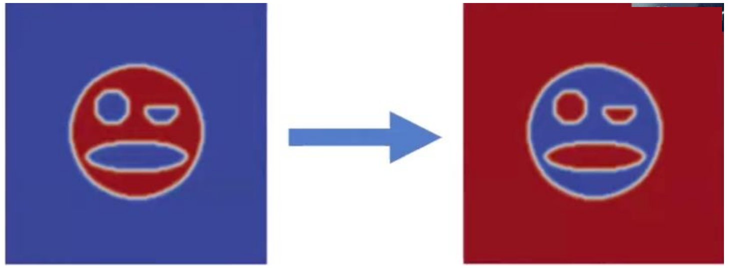
$$\rho(\boldsymbol{r}) \leftrightarrow \hat{\rho}(\boldsymbol{q})$$
$$\rho(\boldsymbol{r}+\boldsymbol{\tau}) \leftrightarrow \hat{\rho}(\boldsymbol{q})e^{2\pi i(\boldsymbol{\tau}.\boldsymbol{q})}$$
$$\left|\hat{\rho}(\boldsymbol{q})e^{2\pi i(\boldsymbol{\tau}.\boldsymbol{q})}\right| = |\hat{\rho}(\boldsymbol{q})|$$

Inversion



$$\rho(\boldsymbol{r}) \leftrightarrow \hat{\rho}(\boldsymbol{q})$$
$$\rho(-\boldsymbol{r}) \leftrightarrow \overline{\hat{\rho}(\boldsymbol{q})}$$
$$\left|\overline{\hat{\rho}(\boldsymbol{q})}\right| = |\hat{\rho}(\boldsymbol{q})|$$

Global Phase Factor

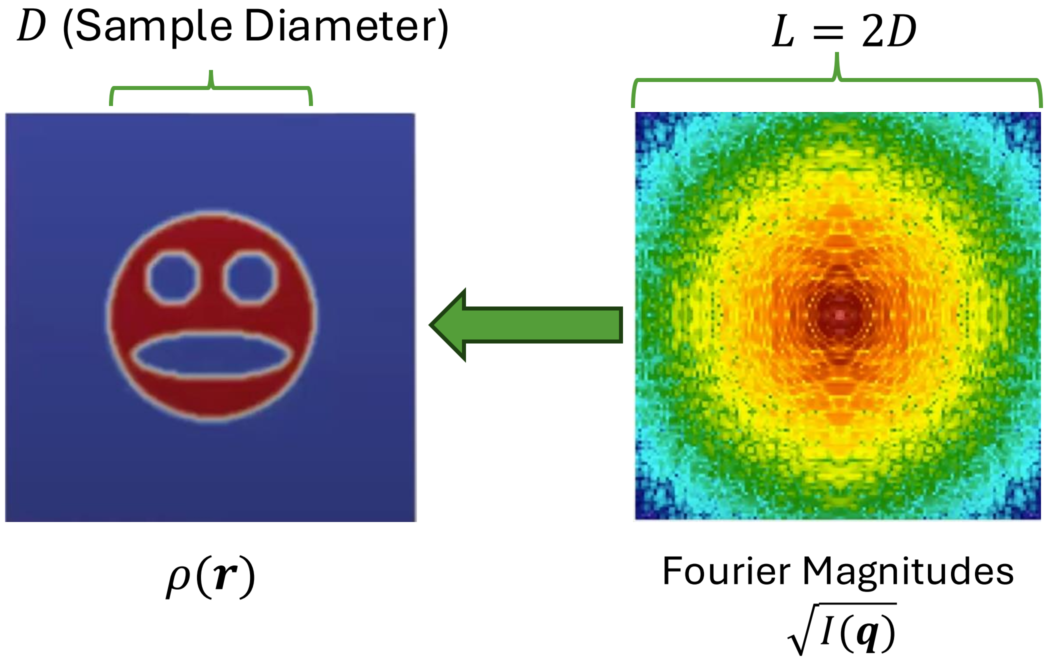


$$\rho(\boldsymbol{r}) \leftrightarrow \hat{\rho}(\boldsymbol{q})$$
$$\rho(\boldsymbol{r})e^{i\boldsymbol{\phi}} \leftrightarrow \hat{\rho}(\boldsymbol{q})e^{i\boldsymbol{\phi}}$$
$$\left|\hat{\rho}(\boldsymbol{q})e^{i\boldsymbol{\phi}}\right| = \hat{\rho}(\boldsymbol{q})$$

Phase Retrieval Solvability Criterion

Theorem: If $I(\boldsymbol{q}) = |\hat{\rho}(\boldsymbol{q})|^2$ then $\rho(\boldsymbol{r})$ can be uniquely reconstructed up to translation, multiplication by a global phase factor, and inversion if it satisfies certain conditions:

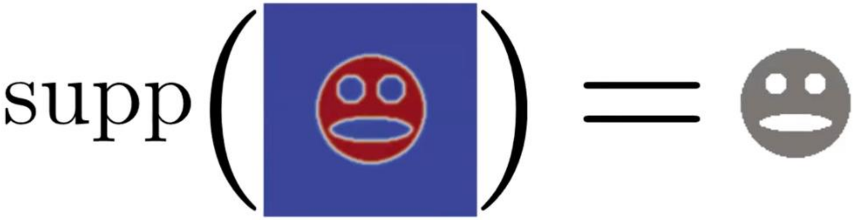
1. The Fourier magnitudes are oversampled by a factor of at least 2, i.e. $L \geq 2D$ (D is Sample Diameter)



2. The sample should not be homometric. However, you have no control over it, and it's rare! [Additional Slides]

Phase Retrieval with support constraint

Support: The support of an object is the region where it is nonzero



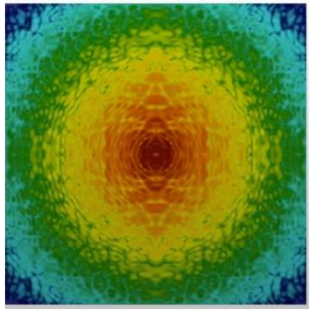
Goal: Given an intensity $I(\boldsymbol{q})$ and finite support region S find $\rho(\boldsymbol{r})$ that satisfies:

Magnitude Constraint

$$I(\boldsymbol{q}) = |\,\hat{\rho}(\boldsymbol{q})|^2$$

$\rho(\boldsymbol{r})$

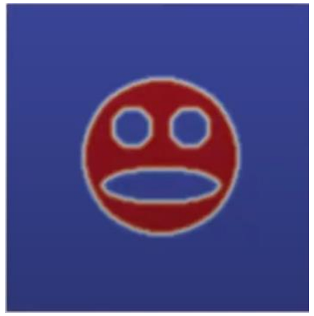
$$\left|\sum_{r} \rho(\boldsymbol{r}) e^{-2\pi i(\boldsymbol{r}.\boldsymbol{q})}\right|^2$$



$I(\boldsymbol{q})$

Support Constraint

$supp(\rho)$ is contained within S



$\rho(\boldsymbol{r})$

$\xrightarrow{supp(\rho)}$

Error Reduction Algorithm (ER)

$\rho(\boldsymbol{r})$ $\xrightarrow{\text{Fourier Transform}}$ $\hat{\rho}(\boldsymbol{q})$

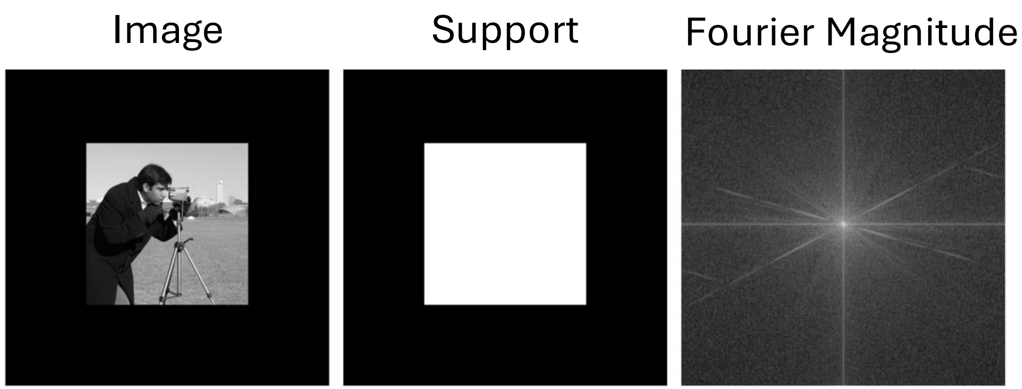
Magnitude Constraint

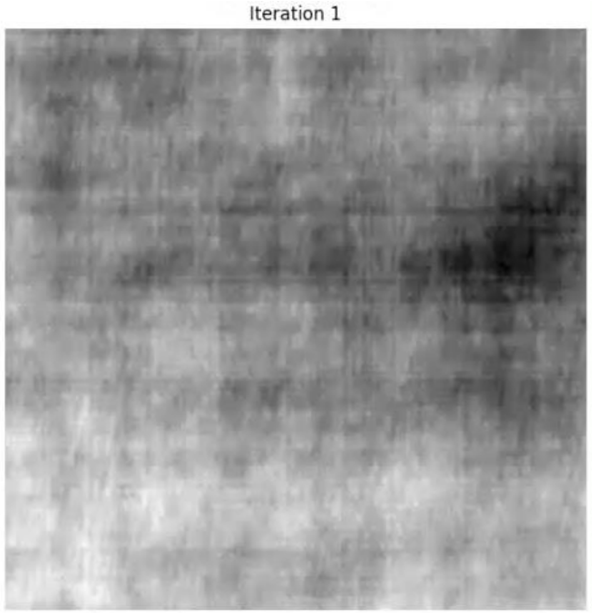
$$|\hat{\rho}(\mathbf{q})|^2 = \mathrm{I}(\mathbf{q})$$

$\hat{\rho}'(\boldsymbol{q})$ $\xrightarrow{\text{Inverse Fourier Transform}}$ $\acute{\rho}(\boldsymbol{r})$

Support Constraint

$$\rho(\boldsymbol{r}) = \begin{cases} \acute{\rho}(\boldsymbol{r}), & r \in S \\ 0, & r \notin S \end{cases}$$





Hybrid Input-Output Algorithm (HIO)

$$\rho(\mathbf{r}) = \begin{cases} \acute{\rho}(\mathbf{r}), & r \in S \\ \rho_{old}(\mathbf{r}) - \beta\acute{\rho}(r), & r \notin S \end{cases}$$

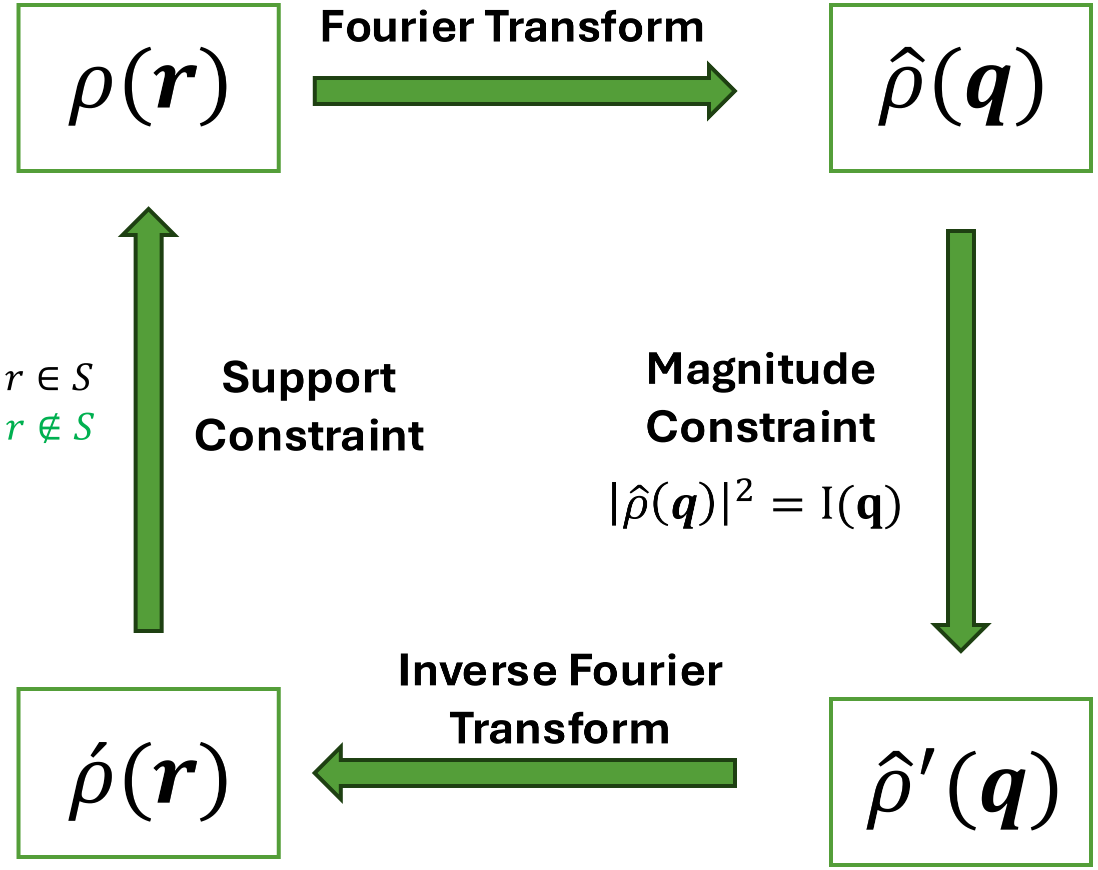
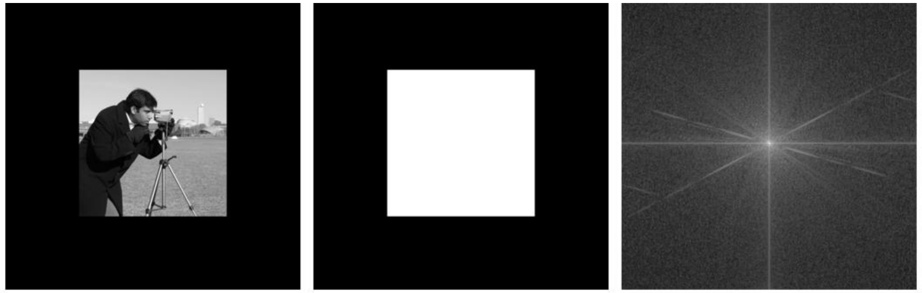
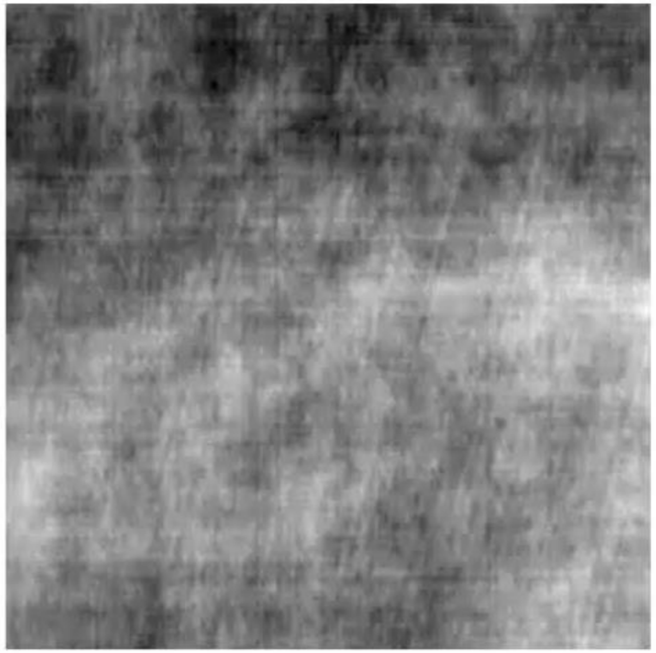


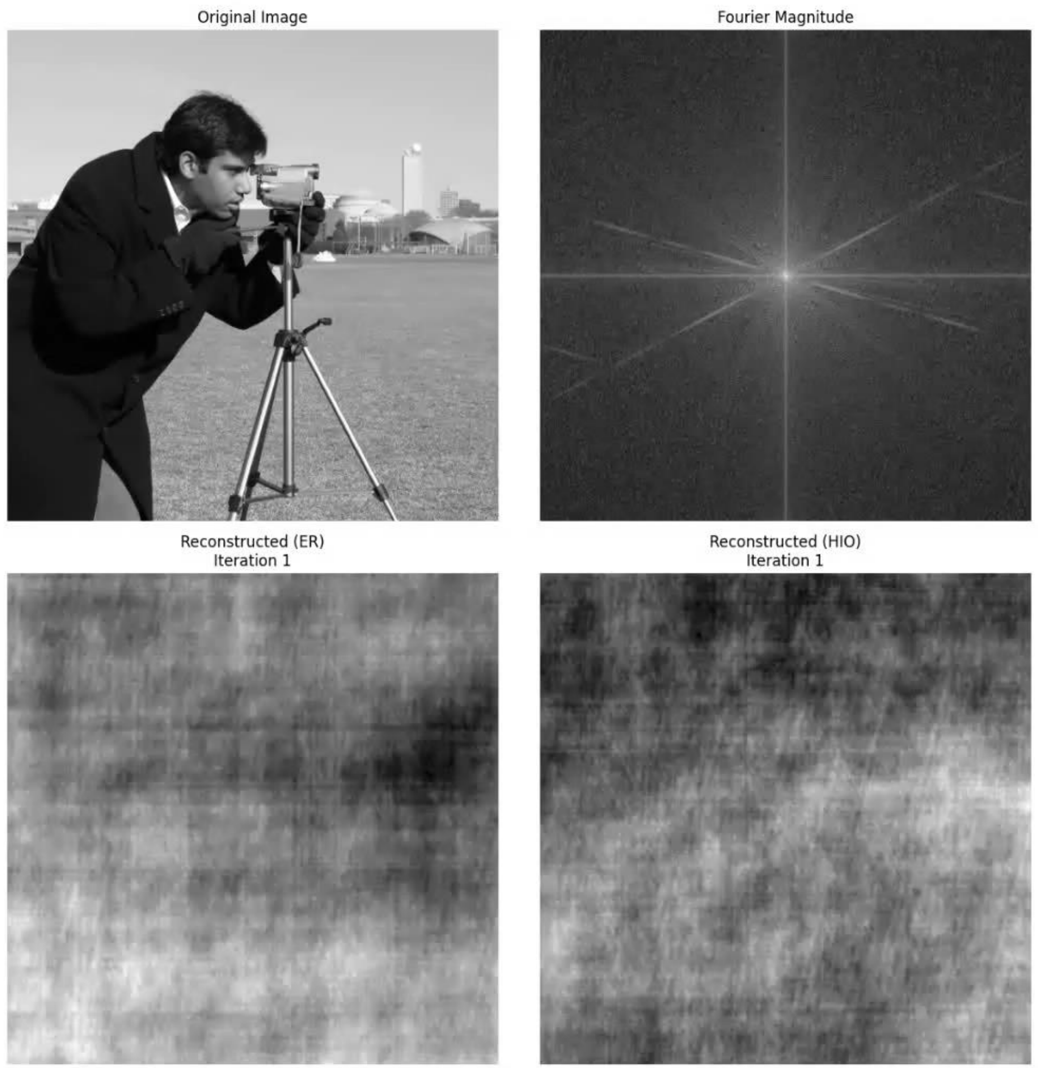
Image Support Fourier Magnitude



Iteration 1



Comparison of HIO and ER Algorithms



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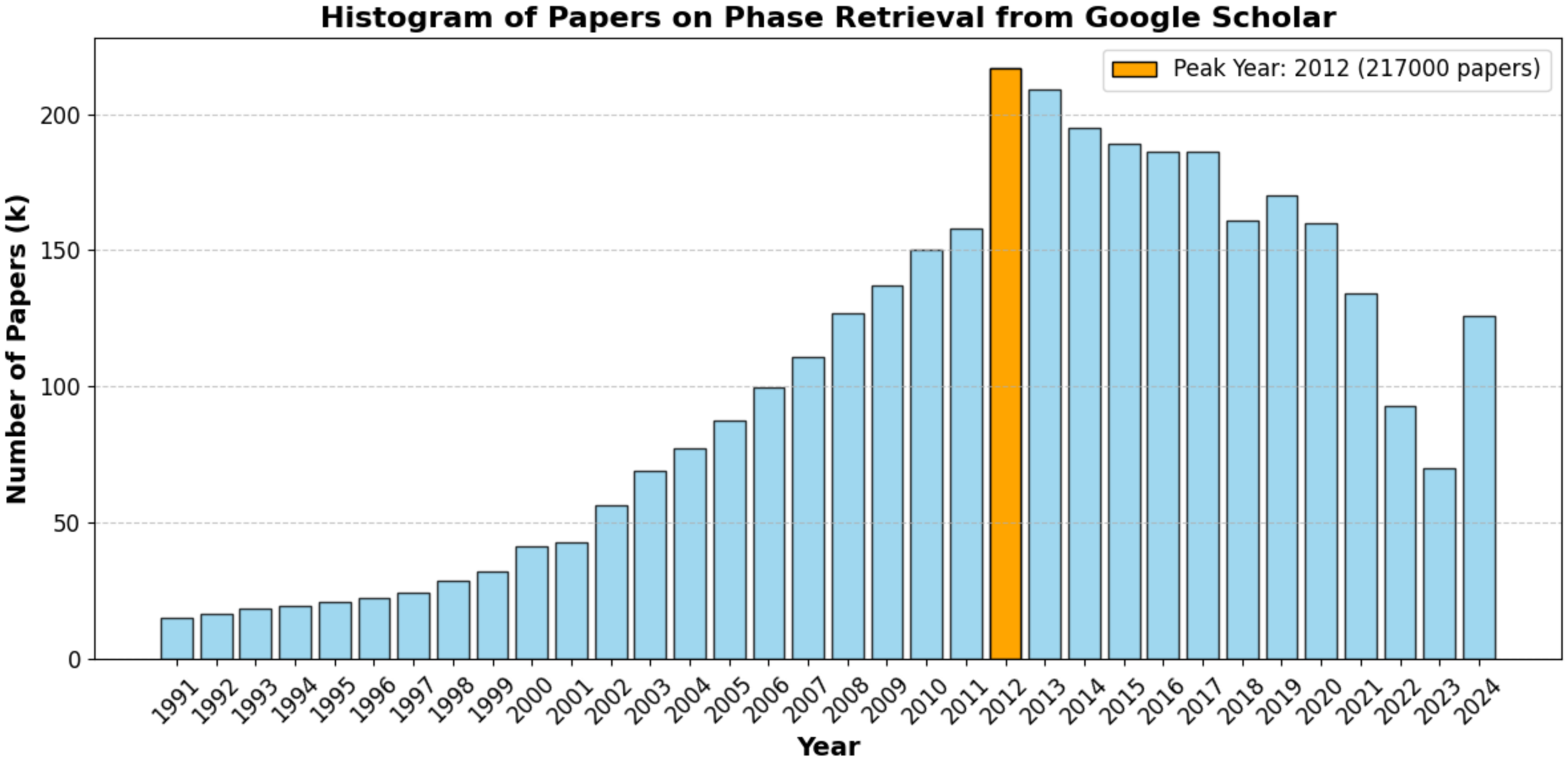
Current Research



2012: Beginning of the deep learning era with the AlexNet paper, authored by Geoffrey E. Hinton, a winner of the 2024 Nobel Prize in Physics!

[1]: Diffusion Posterior Sampling for General Noisy Inverse Problems

[2]: DOLPH: Diffusion Models for Phase Retrieval

[3]: prDeep: Robust Phase Retrieval with a Flexible Deep Network

[4]: Phase Retrieval: From Computational Imaging to Machine Learning

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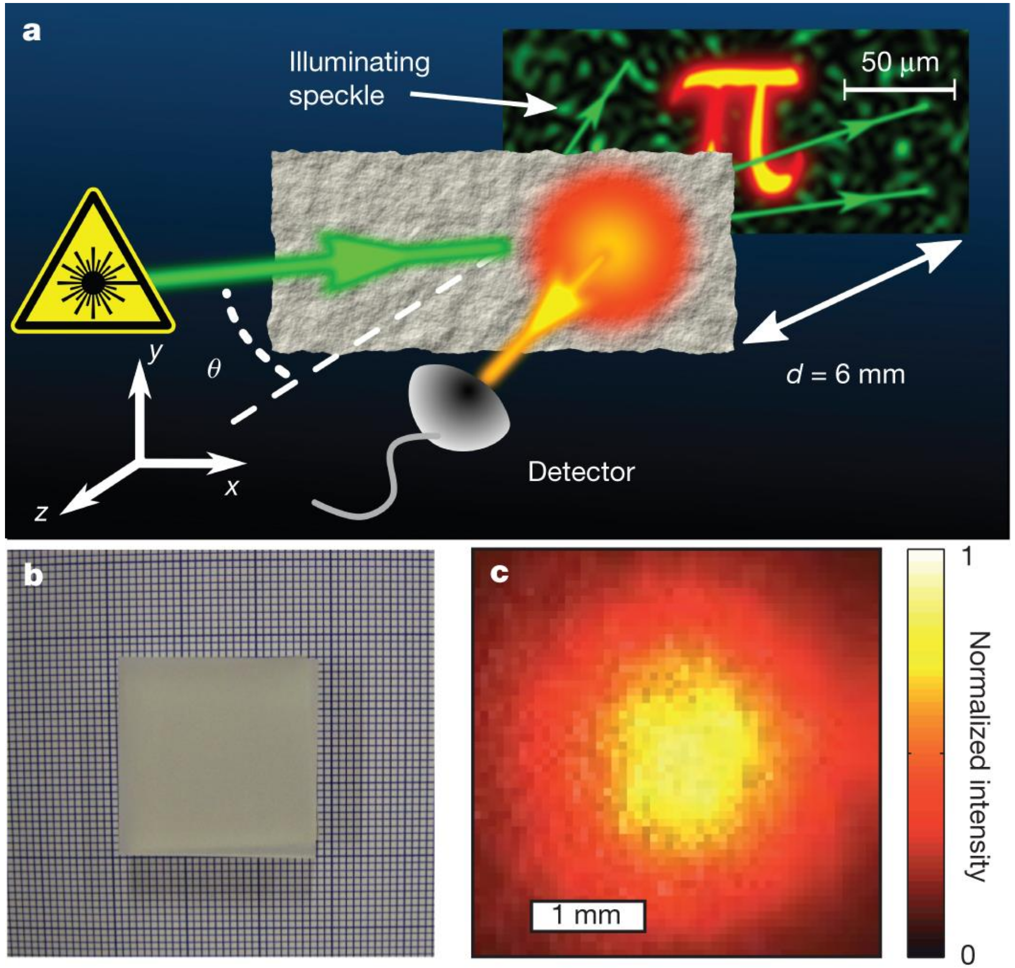
Thanks for your attention!

Additional Slides: Imaging Through Thin Scattering Media



- **Angular memory effect**: Rotating the incident beam over small angles θ does not significantly change the resulting speckle pattern; rather, it only translates it over a distance $\Delta r \approx \theta d$

$$I(\theta) = \int_{-\infty}^{\infty} O(r) S(r - \theta d)\, \mathrm{d}^2 r = [O * S](\theta)$$

- **Theorem:** Operations of convolution and autocorrelation can be interchanged.

$$\langle I \star I \rangle (\Delta\theta) = \langle O * S \rangle \star \langle O * S \rangle$$

$$= \langle O \star O \rangle * \langle S \star S \rangle = [O \star O] * \underbrace{\langle S \star S \rangle}_{\approx\, \delta(x)}$$

Speckle scale must be smaller than object size → $\approx \delta(x)$

$$|\mathcal{F}\{I\}|^2 = |\mathcal{F}\{0\}|^2 \;\rightarrow$$ Phase is lost!

Ref: Non-invasive imaging through opaque scattering layers, Nature 2012

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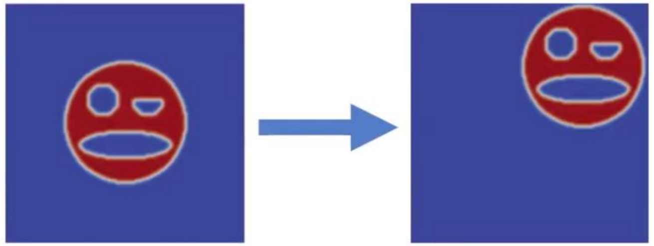
Additional Slides: Homometric Structure

Translation

$$\rho(\boldsymbol{r}) \leftrightarrow \hat{\rho}(\boldsymbol{q})$$
$$\rho(\boldsymbol{r}+\boldsymbol{\tau}) \leftrightarrow \hat{\rho}(\boldsymbol{q})e^{2\pi i(\boldsymbol{\tau}.\boldsymbol{q})}$$
$$\left|\hat{\rho}(\boldsymbol{q})e^{2\pi i(\boldsymbol{\tau}.\boldsymbol{q})}\right| = \hat{\rho}(\boldsymbol{q})$$



Inversion

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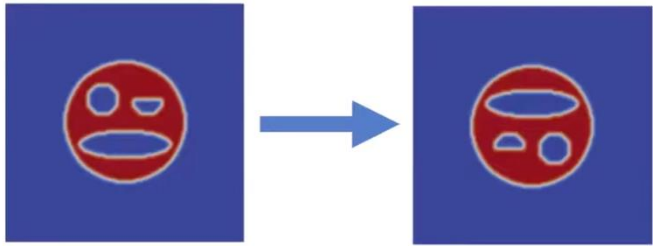
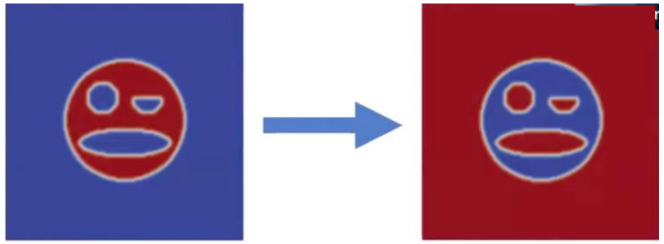
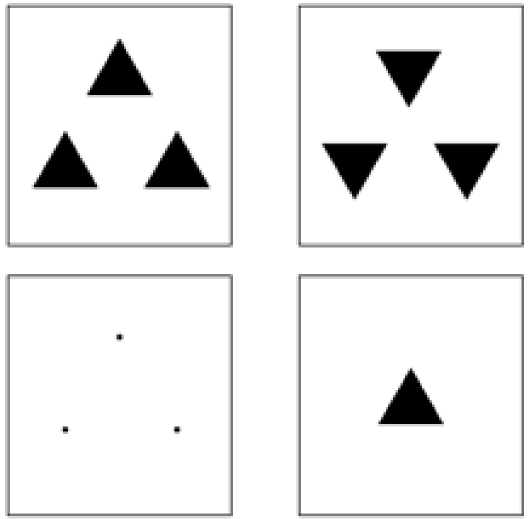


Global Phase Factor

$$\rho(\boldsymbol{r}) \leftrightarrow \hat{\rho}(\boldsymbol{q})$$
$$\rho(\boldsymbol{r})e^{\boldsymbol{i\phi}} \leftrightarrow \hat{\rho}(\boldsymbol{q})e^{\boldsymbol{i\phi}}$$





Homometric Structure: Example of two homometric structures (top) formed by the convolution between two non-centrosymmetric structures (bottom) with different orientations. [$f(r)$ is non-centro-symmetric if $f(\boldsymbol{r}+\tau) \neq f(-\boldsymbol{r}+\tau)$ for all τ]

1D: Almost all 1D functions have homometric structure
2D: Homometric structures are very rare in 2 and higher dimensions

Ref: Homometric structures, Wikipedia

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