Machine learning and vision laboratory

Dr H.Mohammad Zadeh

department: Electrical Engineering

Amirreza Velae	400102222
github	repository

Pre-Lab 1

October 14, 2024

Machine learning and vision laboratory

Pre-Lab 1

Amirreza Velae	400102222
github	repository

Correlation, Causality, and Independence

We have three points $(1,1),(2,2),(3,3)$ in a two-dimensional space.

- If we want to project these data into a one-dimensional space, find the dominant eigenvalue and the corresponding eigenvector.

solution

As a rule of thumb, it's trivial to see that these 3 points lie perfectly on $y = x$ line. It's not necessery to compute eighen values, cause one is 0 (cause the rank of matrix will be 1) and one is $\begin{bmatrix}1\\1\end{bmatrix}$ cause all points lie on this. However, for the sake of Pre-Labe, I've done the typical algorithm.
The mean vector μ is calculated as:

$$\mu = \frac{1}{3}\left(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3\right) = \frac{1}{3}\begin{bmatrix}1+2+3\\1+2+3\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$$

Now, subtract the mean vector from each data point to obtain the centered data:

$$\mathbf{y}_i = \mathbf{x}_i - \mu$$

Thus,

$$\mathbf{y}_1 = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix}3\\3\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

where $n = 3$ is the number of data points. Calculating each outer product:

$$\mathbf{y}_1\mathbf{y}_1^\top = \begin{bmatrix}-1\\-1\end{bmatrix}\begin{bmatrix}-1 & -1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$\mathbf{y}_2\mathbf{y}_2^\top = \begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\mathbf{y}_3\mathbf{y}_3^\top = \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

Summing these:

$$\sum_{i=1}^{3} \mathbf{y}_i \mathbf{y}_i^\top = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Thus, the covariance matrix is:

$$\mathbf{C} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{y}_i \mathbf{y}_i^\top = \frac{1}{3} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

To find the eigenvalues λ and eigenvectors $\mathbf{v}$ of $\mathbf{C}$, solve the characteristic equation:

$$\det(\mathbf{C} - \lambda \mathbf{I}) = 0$$

where $\mathbf{I}$ is the identity matrix.
Compute $\mathbf{C} - \lambda \mathbf{I}$:

$$\mathbf{C} - \lambda \mathbf{I} = \begin{bmatrix} \frac{2}{3} - \lambda & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} - \lambda \end{bmatrix}$$

The determinant is:

$$\det(\mathbf{C} - \lambda \mathbf{I}) = \left(\frac{2}{3} - \lambda\right)^2 - \left(\frac{2}{3}\right)^2 = \frac{4}{9} - \frac{4}{3}\lambda + \lambda^2 - \frac{4}{9} = \lambda^2 - \frac{4}{3}\lambda = 0$$

Solving for λ:

$$\lambda(\lambda - \frac{4}{3}) = 0 \quad \Rightarrow \quad \lambda = 0 \quad \text{or} \quad \lambda = \frac{4}{3}$$

Thus, the eigenvalues are 0 and $\frac{4}{3}$. The dominant eigenvalue is the largest eigenvalue, which is $\lambda = \frac{4}{3}$.
To find the corresponding eigenvector $\mathbf{v}$, solve:

$$(\mathbf{C} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$$

Substituting $\lambda = \frac{4}{3}$:

$$\begin{bmatrix} \frac{2}{3} - \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} - \frac{4}{3} \end{bmatrix} \mathbf{v} = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \mathbf{v} = \mathbf{0}$$

This simplifies to the system:

$$-\frac{2}{3}v_1 + \frac{2}{3}v_2 = 0 \quad \Rightarrow \quad v_1 = v_2$$

Thus, the eigenvector is any non-zero scalar multiple of:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For normalization, we can choose:

$$\mathbf{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Project the data into a one-dimensional space and calculate the reconstruction error.

Soloution

Projection Formula:

$$p_i = \mathbf{v}_1^\top \mathbf{y}_i$$

Calculations:

$$p_1 = \frac{1}{\sqrt{2}}(-1) + \frac{1}{\sqrt{2}}(-1) = -\sqrt{2}$$

$$p_2 = 0$$

$$p_3 = \frac{1}{\sqrt{2}}(1) + \frac{1}{\sqrt{2}}(1) = \sqrt{2}$$

Reconstruction Formula:

$$\mathbf{y}_i' = p_i \mathbf{v}_1$$

Reconstructed Points:

$$\mathbf{y}_1' = -\sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{y}_2' = 0 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{y}_3' = \sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Error for Each Point:

$$\mathbf{e}_i = \mathbf{y}_i - \mathbf{y}_i' = \mathbf{0} \quad \forall\, i = 1, 2, 3$$

Total Reconstruction Error:

$$\text{Total Error} = \sum_{i=1}^{3} \|\mathbf{e}_i\|^2 = 0$$

All data points lie perfectly along the principal component direction $\mathbf{v}_1$, resulting in zero reconstruction error. The projection captures all the variance in the data.

End of Pre-Lab 1