Bandit-style Geometric decision algorithm against an Adaptive adversary

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Outline

Online Geometric Optimization in the Bandit Setting

- **Paper Title**: Online Geometric Optimization in the Bandit Setting Against an Adaptive Adversary
- **Authors :** H. Brendan McMahan & Avrim Blum
- **Published in:** 2004
- **Conference:** Learning Theory, Springer Berlin Heidelberg

Introduction to Online Optimization

Definition: Online optimization involves making a sequence of decisions based on incoming data, where each decision is made without knowledge of future data.

Two fundamental assumptions in online optimization:

1. **Bunded Losses:** The losses determined by an adversary should not be allowed to be unbounded.
2. **Bounded Decision Set:** The decision set must be somehow bounded and/or structured, though not necessarily finite.

Prediction from Expert Advice

- **Prediction from Expert Advice** is a framework in online learning and decision-making.
- Involves making sequential predictions by aggregating advice from a set of experts.
- The goal is to perform nearly as well as the best expert in hindsight.

Key Components

1. **Experts:** Sources that provide predictions or advice.
2. **Learner (Main Character):** Aggregates expert advice to make decisions.
3. **Feedback:** Information about the actual outcome to update future predictions.
4. **Objective:** Minimize the difference between the learner's performance and the best expert's performance.

Family Members as Experts

- Main character decides whether to take an umbrella each day.
- Three family members act as **experts** providing daily weather predictions, the father, mother, and brother.

Figure: Family Members as Experts

Example Scenario

Table: Daily Weather Predictions and Outcomes

Day	Father	Mother	Brother	Actual Weather
1	No Rain	No Rain	Rain	Rain

Example Scenario

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2	Rain	No Rain	Rain	No Rain

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Example Scenario

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Day	**Father**	**Mother**	**Brother**	**Actual Weather**
1	No Rain	No Rain	Rain	Rain
2	Rain	No Rain	Rain	No Rain
3	Rain	No Rain	No Rain	Rain
4	No Rain	Rain	No Rain	No Rain
5	Rain	No Rain	No Rain	Rain
Cost	2	4	3	-

Weighted Majority Algorithm

- **Mechanism:**
  - Assigns a weight to each expert based on their past performance.
  - Aggregates predictions by considering these weights.
  - Updates weights multiplicatively based on the correctness of experts' predictions.
- **Applications:**
  - Ensemble learning in machine learning.
  - Financial decision-making.

How Weighted Majority Works

Algorithm Steps:

1. **Initialization:** Assign equal weights to all experts.
2. **Prediction**

$$\text{Prediction} = \text{sign}\left(\sum_{i=1}^{N} w_t(i) \cdot \text{Prediction}_t(i)\right)$$

3. **Update Weights:** After observing the outcome, update weights:

$$w_{t+1}(i) = \begin{cases} w_t(i) & \text{if expert } i \text{ was correct} \\ w_t(i) \cdot \varepsilon & \text{if expert } i \text{ was incorrect} \end{cases}$$

4. **Iteration:** Repeat the prediction and update steps for each round.

Parameters:

- N: Number of experts.
- $\varepsilon \in (0, 1)$: Penalty factor for incorrect experts.

Regret Bound for Weighted Majority

Lemma : Denote by M_t the number of mistakes the algorithm makes until time t, and by $M_t(i)$ the number of mistakes made by expert i until time t. Then, for any expert $i \in [N]$ we have

$$M_T \leq 2(1+\varepsilon)M_T(i) + \frac{2\log N}{\varepsilon}$$

Corollary : The regret of the WM algorithm is bounded by

$$M_T \leq 2M_T(i^*) + O(\sqrt{M_T(i^*) logN})$$

where i^* is the best expert.

Proof: Just let $\varepsilon^* = \sqrt{\frac{logN}{M_T(i^*)}}$.

Proof:

- Let $\phi_t = \sum_{i=1}^{N} w_i(t)$. Note that $\phi_1 = N$.
- If the prediction is wrong, then $\phi_{t+1} \leq \frac{1}{2}\phi_t(1-\varepsilon) + \frac{1}{2}\phi_t$.
- Thus $\phi_t \leq \phi_1(1-\varepsilon)^{M_t} = N(1-\varepsilon)^{M_t}$.
- By definition, $w_T(i) = (1-\varepsilon)^{M_T(i)}$. Also $w_t(i) \leq \phi_t$.
- $(1-\varepsilon)^{M_T(i)} \leq N(1-\varepsilon)^{M_T} \rightarrow M_T(i) log(1-\varepsilon) \leq logN + M_T log(1-\varepsilon)$.
- Using the fact that $-x - x^2 \leq log(1-x) \leq -x$ for $x \in (0,1)$, we get:

$$-M_T(i)(\varepsilon + \varepsilon^2) \leq logN - M_T\frac{\varepsilon}{2} \rightarrow M_T \leq 2(1+\varepsilon)M_T(i) + \frac{2logN}{\varepsilon}$$

Randomized Weighted Majority Algorithm

- **Algorithm Steps:**
  1. **Initialization:** Set $w_1(i) = 1$ for all experts.
  2. **Probability Assignment:**

$$P_t(i) = \frac{w_t(i)}{\sum_{j=1}^{N} w_t(j)}$$

  3. **Expert Selection:** Choose expert i with probability $P_i(t)$.
  4. **Prediction and Update:** Make prediction based on selected expert and update weights as in Weighted Majority.
- **Better Regret Bound:**

$$\mathbb{E}[M_T] \leq (1+\varepsilon)M_T(i^*) + \frac{logN}{\varepsilon}$$

RWM Regret Bound Proof

Proof: Let $\phi_t = \sum_{i=1}^{N} w_i(t)$ and $\tilde{m}_t$ be an indicator variable for the event that the prediction is wrong at time t and $\tilde{m}_t(i) = 1$ if expert i is wrong at time t.

- Note that

$$\phi_{t+1} = \sum_{i=1}^{N} w_i(t)(1 - \varepsilon \tilde{m}_t(i)) = \phi_t(1 - \varepsilon \sum_{i=1}^{N} P_t(i)\tilde{m}_t(i))$$
$$= \phi_t(1 - \varepsilon \mathbb{E}[\tilde{m}_t]) \leq \phi_t e^{-\varepsilon \mathbb{E}[\tilde{m}_t]}$$

- With the same argument as in the WM proof, we get:

$$(1-\varepsilon)^{M_T(i)} \leq N e^{-\varepsilon M_T}$$
$$\to M_T(i) log(1-\varepsilon) \leq log N - \varepsilon \mathbb{E}[M_T]$$
$$\to -M_T(\varepsilon + \varepsilon^2) \leq log N - \varepsilon \mathbb{E}[M_T]$$
$$\to \mathbb{E}[M_T] \leq (1+\varepsilon) M_T(i^*) + \frac{log N}{\varepsilon}$$

Introduction to Multi-Armed Bandit Algorithms

- **Definition:** A framework for making a sequence of decisions under uncertainty, aiming to maximize cumulative rewards.
- **Key Concepts:**
  - **Exploration:** Trying different actions to gather more information.
  - **Exploitation:** Selecting the best-known action to maximize immediate reward.
- **Types of Bandit Problems:**
  - **Stochastic Bandits:** Rewards are drawn from fixed probability distributions.
  - **Contextual Bandits:** Incorporates contextual information to make more informed decisions.
- **Applications:**
  - Recommendation Systems
  - Clinical Trials
  - Online Advertising

Bandit Algorithms in Online Advertising

- **Use Case:** Optimizing Ad Selection to Maximize CTR
- **How It Works:**
  - Each ad variant is considered an arm of the bandit.
  - The algorithm dynamically selects which ad to display based on past performance.
  - Balances exploration (trying new ads) with exploitation (showing top-performing ads).

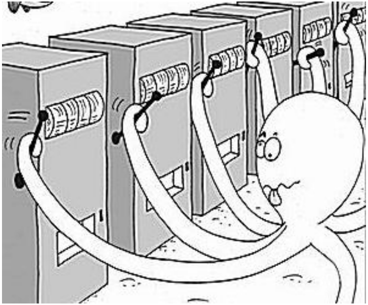


Figure: Bandit Setting

Online Optimization vs. Bandit Algorithms

Both online optimization and bandit algorithms involve sequential decisions, but differ in feedback and objectives.

Online Optimization

- **Full Feedback:** Receives complete information about all possible actions after each decision.
- **Objective:** Minimize cumulative loss compared to the best fixed decision in hindsight.

Bandit Algorithms

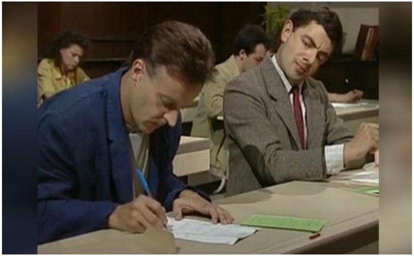
- **Partial Feedback:** Only receives feedback for the action actually taken, not for all possible actions.
- **Objective:** Balance exploration and exploitation to maximize cumulative rewards.

Oblivious Adversary

- **Definition:** Fixes the sequence of events beforehand.
- **Traits:**
  - Non-responsive.
  - Simpler to analyze.

Adaptive Adversary

- **Definition:** Adjusts based on algorithm's past actions.
- **Traits:**
  - Responsive and dynamic.
  - Harder to counter.



Introducing Follow the Perturbed Leader (FPL)

- **Overview:**
  - FPL is a randomized online algorithm that minimizes regret in adversarial settings by adding noise to each expert's cumulative loss and selecting the leader with the lowest perturbed loss.
- **Algorithm Steps:**
  1. **Initialization:** Set cumulative loss $L_i(0) = 0$ for all experts i.
  2. **For each round** $t = 1, 2, \ldots, T$:
     1. **Perturbation:** Draw a random perturbation γ_i for each expert i from a specified distribution (e.g., exponential).
     2. **Leader Selection:** Choose the expert i^* with the minimum $L_i(t-1) + \gamma_i$.
     3. **Decision:** Follow the prediction of expert i^*.
     4. **Update:** Update the cumulative loss $L_i(t) = L_i(t-1) + \ell_i(t)$ for each expert.

Why Perturbation?

Table: Daily Weather Predictions, Outcomes, and Probability of Mistake using Random Weighted Majority

Day	Father	Mother	P of Mistake	Actual Weather
1	Rain	No Rain	$\frac{1}{2}$	Rain
2	Rain	No Rain	$\frac{1}{2}$	No Rain
3	Rain	No Rain	$\frac{1}{2}$	Rain
4	Rain	No Rain	$\frac{1}{2}$	No Rain
5	Rain	No Rain	$\frac{1}{2}$	Rain

Regret Bound for GEX

Lemma: Let $S \subseteq \mathbb{R}^n$ be a set of (not necessarily positive) decisions, and $k^t = [\mathbf{c}^1, \ldots, \mathbf{c}^T]$ a set of cost vectors on those decisions, such that $|\mathbf{c}^t \cdot \mathbf{x}| \leq R$ for all $\mathbf{x} \in S$ and $\mathbf{c}^t \in k^t$. Then, there is an algorithm $\mathcal{A}(\varepsilon)$ that achieves

$$E[\mathrm{loss}(\mathcal{A}(\varepsilon), k^t)] \leq \mathrm{OPT}(k^t) + \varepsilon(4n+2)R^2 T + \frac{4n}{\varepsilon}$$

Use your Maximum Power

In general, oblivious adversaries are easier to handle than adaptive adversaries. However, we can convert an oblivious adversary to an adaptive by being pessimistic.

Figure: An evil adversary in machine learning using its maximum power

Regret Bound for Any Adversary

Lemma: Fix T, let H^* be the set of decision histories of length 0 to $T-1$, and let K^* be the set of all cost histories of length 0 to $T-1$. Then, fix a decision algorithm $\mathcal{A} : K^* \to \Delta(S)$, where $\Delta(S)$ is the set of probability distributions on the set S of possible decisions. Define

$$R(\mathcal{A}, \mathcal{V}) = \mathbb{E}_{\mathcal{A},\mathcal{V}}\left[\sum_{t=1}^{T} \mathbf{c}^t \cdot \mathbf{x}^t - \min_{\mathbf{x}\in S} \sum_{t=1}^{T} \mathbf{c}^t \cdot \mathbf{x}\right].$$

Let $\mathcal{V}$ be an arbitrary adversary. Then, there exists an oblivious adversary $\mathcal{V}'$ such that

$$R(\mathcal{A}, \mathcal{V}') \geq R(\mathcal{A}, \mathcal{V}).$$

Exploration and Exploitation

Now that we have a way to convert an oblivious adversary to an adaptive one, we can focus on using online optimization algorithms to solve bandit problems.

- **Exploration:** Use the perturbation or other methods to explore different experts and their predictions.
- **Exploitation:** Follow the leader to exploit the best-performing expert.

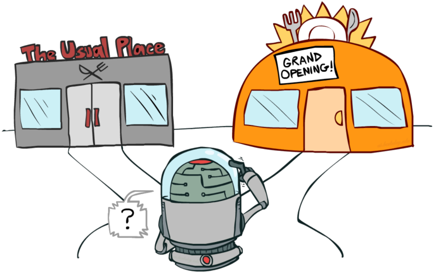


Figure: Exploration vs. Exploitation in Bandit Algorithms

Usage of Basis

- **Definition:** A basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a set of linearly independent vectors that span a vector space.
- **Advantages:**
  - Reduces the problem of exploring an infinite set to exploring a finite set.
  - Simplifies the decision-making process.
- **Dimention:** At most n.

Problem Formulation

Table: Summary of notation

$S \subseteq \mathbb{R}^n$	set of decisions, a compact subset of $\mathbb{R}^n$
$D \in \mathbb{R}$	L_1 bound on diameter of S, $\forall \mathbf{x}, \mathbf{y} \in S$, $\lvert \mathbf{x} - \mathbf{y} \rvert_1 \le D$
$n \in \mathbb{N}$	dimension of decision space
$\mathcal{V} : H^* \to \mathbb{R}^n$	adversary, function from decision histories to cost vectors
$\mathcal{A}$	an online optimization algorithm
$\mathbf{c}^t \in \mathbb{R}^n$	cost vector on time t
$\hat{\mathbf{c}}^t \in \mathbb{R}^n$	BGA's estimate of the cost vector on time t
$M \in \mathbb{R}^+$	bound on single-iteration cost, $\lvert \mathbf{c}^t \cdot \mathbf{x}^t \rvert \le M$
$B \subseteq S$	sampling basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$
$\ell^t \in [-M, M]^n$	vector, $\ell_i^t = \mathbf{c}^t \cdot \mathbf{b}_i$ for $\mathbf{b}_i \in B$
$\hat{\ell}^t \in \mathbb{R}^n$	BGA's estimate of ℓ^t
$T \in \mathbb{N}$	end of time, index of final iteration
$\mathbf{x}^t \in S$	BGA's decision on time t
$\tilde{\mathbf{x}}^t \in S$	decision recommended by GEX on time t
$\chi^t \in \{0, 1\}$	indicator, $\chi^t = 1$ if BGA explores on t, 0 otherwise
$\gamma \in [0, 1]$	the probability BGA explores on each timestep
$\tilde{z}^t \in [-R, R]$	loss of GEX, $\tilde{z}^t = \hat{\mathbf{c}}^t \cdot \tilde{\mathbf{x}}^t$

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Algorithm BGA

Choose parameters γ and ϵ, **where** ϵ **is a parameter of GEX**
$t = 1$
Fix a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq S$
while playing **do**
  Let $\chi^t = 1$ **with probability** γ **and** $\chi^t = 0$ **otherwise**
  if $\chi^t = 0$ **then**
    Select $\mathbf{x}^t$ **from the distribution** $\mathrm{GEX}(\hat{\mathbf{c}}^1, \ldots, \hat{\mathbf{c}}^{t-1})$
    Incur cost $z^t = \mathbf{c}^t \cdot \mathbf{x}^t$
    $\hat{\mathbf{c}}^t = 0 \in \mathbb{R}^n$
  else
    Draw j **uniformly at random from** $\{1, \ldots, n\}$
    $\mathbf{x}^t = \mathbf{b}_j$
    Incur cost and observe $z^t = \mathbf{c}^t \cdot \mathbf{x}^t$
    Define $\hat{\ell}^t$ **by** $\hat{\ell}_i^t = 0$ **for** $i \neq j$ **and** $\hat{\ell}_j^t = (n/\gamma) z^t$
    $\hat{\mathbf{c}}^t = (B^\top)^{-1} \hat{\ell}^t$
  end if
  $\hat{\mathbf{c}}^{1:t} = \hat{\mathbf{c}}^{1:t-1} + \hat{\mathbf{c}}^t$
  $t = t + 1$
end while

How BGA Works - Part 1

- **Initialization:**
  - Choose parameters γ and ϵ, where ϵ is a parameter of GEX.
  - Set $t \leftarrow 1$.
  - Fix a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq S$.

How BGA Works - Part 1

- **Initialization:**
  - Choose parameters γ and ϵ, where ϵ is a parameter of GEX.
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- **Main Loop: While** playing **do**
  - Let $\chi^t = 1$ with probability γ and $\chi^t = 0$ otherwise.

How BGA Works - Part 1

- **Initialization:**
  - Choose parameters γ and ϵ, where ϵ is a parameter of GEX.
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  - Fix a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq S$.
- **Main Loop: While** playing **do**
  - Let $\chi^t = 1$ with probability γ and $\chi^t = 0$ otherwise.
  - **If** $\chi^t = 0$ **then**
    - Select $\mathbf{x}^t$ from the distribution $\mathrm{GEX}(\hat{\mathbf{c}}^1, \ldots, \hat{\mathbf{c}}^{t-1})$.
    - Incur cost $z^t = \mathbf{c}^t \cdot \mathbf{x}^t$.
    - Set $\hat{\mathbf{c}}^t = \mathbf{0} \in \mathbb{R}^n$.

How BGA Works - Part 2

- **Else**
  - Draw j uniformly at random from $\{1, \ldots, n\}$.
  - Set $\mathbf{x}^t = \mathbf{b}_j$.
  - Incur cost and observe $z^t = \mathbf{c}^t \cdot \mathbf{x}^t$.
  - Define $\hat{\ell}^t$ by:
    - $\hat{\ell}_i^t = 0$ for all $i \neq j$.
    - $\hat{\ell}_j^t = \left(\frac{n}{\gamma}\right) z^t$.
  - Set $\hat{\mathbf{c}}^t = (B^\top)^{-1}\hat{\ell}^t$.

How BGA Works - Part 2

- **Else**
  - Draw j uniformly at random from $\{1, \ldots, n\}$.
  - Set $\mathbf{x}^t = \mathbf{b}_j$.
  - Incur cost and observe $z^t = \mathbf{c}^t \cdot \mathbf{x}^t$.
  - Define $\hat{\ell}^t$ by:
    - $\hat{\ell}_i^t = 0$ for all $i \neq j$.
    - $\hat{\ell}_j^t = \left(\frac{n}{\gamma}\right) z^t$.
  - Set $\hat{\mathbf{c}}^t = (B^\top)^{-1} \hat{\ell}^t$.
- Update cumulative costs:
  - $\hat{\mathbf{c}}^{1:t} = \hat{\mathbf{c}}^{1:t-1} + \hat{\mathbf{c}}^t$.
  - Increment time step: $t \leftarrow t + 1$.

Regret Bound for BGA

The regret of the BGA algorithm is bounded by

$$E[\text{loss(BGA)}] \leq (1-\gamma)E[\text{loss(GEX)}] + \gamma MT$$

`Intuition:` BGA explores with probability γ and exploits with probability $1-\gamma$.
Order of Regret: It can be shown that the regret of BGA is of order $O(T^{3/4}\sqrt{lnT})$.

Future Work

1. **Exploring while Exploiting:** I think the exploitation phase has some information that can be used to explore better.
2. **Hyperparameter Tuning:** The parameters γ can be tuned to improve the performance of the algorithm.
3. **Game Theory:** The algorithm can be viewed as an Stackelberg game. Thus there is a possibility of using game theory.
4. **Experiments:** The algorithm hasn't tested on real-world data. It would be interesting to see how it performs in practice.
5. **Bound Tightening:** Investigate whether the current bounds of $O(T^{3/4}\sqrt{\ln T})$ against adaptive adversaries and $O(T^{2/3})$ against oblivious adversaries can be improved to $O(\sqrt{T})$.

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