Online Geometric Optimization in the Bandit Setting Against an Adaptive Adversary

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Introduction

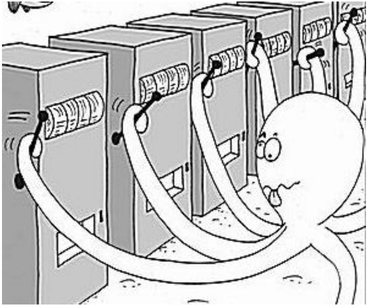
- Overview of online optimization in adversarial environments.
- Importance of studying bandit settings with limited feedback.
- Addressing challenges posed by adaptive adversaries.

Context: Online Optimization with an Adversary

- Online optimization involves making sequential decisions.
- Adversaries can influence the environment by selecting cost vectors.
- Applications in machine learning, economics, and network routing.

Objective: Solving the Bandit Version

- Transition from full-information to bandit feedback.
- Challenges due to limited information after each decision.
- Goal: Develop algorithms that perform well despite these limitations.



Description of the Online Optimization Problem

- Set of feasible points $S \subset \mathbb{R}^n$.
- Sequential decision-making over T rounds.
- At each round t, select $x_t \in S$ and incur cost $c_t \cdot x_t$.

Mathematical Example

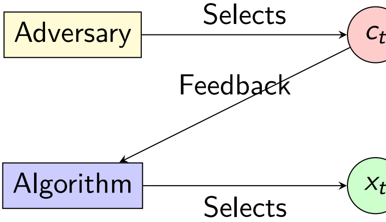
Example: Let S be the unit simplex in $\mathbb{R}^n$, i.e.,

$$S = \left\{ x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i = 1 \right\}$$

This setup is common in online portfolio selection.

Adversary and Algorithm Interactions

- Adversary selects cost vectors $c_t \in \mathbb{R}^n$.
- Algorithm selects decisions $x_t \in S$ without knowing c_t in advance.
- Objective is to minimize cumulative cost over T rounds.



Set $S \subset \mathbb{R}^n$

- S represents all possible feasible decisions.
- Geometric properties of S influence algorithm design.
- Examples: Convex sets, polytopes, or combinatorial structures.

Mathematical Example

Convex Set: Let $S = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$ be the unit ball in $\mathbb{R}^n$. This convex set allows the use of gradient-based optimization methods due to its smooth boundary.

Cost Incurred: $c_t \cdot x_t$

- Inner product representing the cost for decision x_t.
- c_t is the cost vector chosen by the adversary.
- Objective: Minimize the sum $\sum_{t=1}^{T} c_t \cdot x_t$.

$$\text{Total Cost} = \sum_{t=1}^{T} c_t \cdot x_t$$

The Role of an Oracle for Offline Optimization

- Oracle solves $\min_{x \in S}(c \cdot x)$ efficiently.
- Assumes full knowledge of cost vectors.
- Serves as a benchmark for online algorithms.

Mathematical Example

Offline Optimization: Given all cost vectors $\{c_1, c_2, \ldots, c_T\}$ in advance, the oracle solves:

$$\min_{x \in S} \sum_{t=1}^{T} c_t \cdot x$$

This provides the best possible cumulative cost.

Objective: $\min_{x \in S} (c \cdot x)$

- Offline problem assumes all cost vectors are known in advance.
- Provides the optimal benchmark for comparison.
- Online algorithms aim to perform nearly as well without this knowledge.

$$\text{Optimal Offline Cost} = \min_{x \in S} \sum_{t=1}^{T} c_t \cdot x$$

Observing Only the Cost: $c_t \cdot x_t$

- Limited feedback: Only the incurred cost is observed.
- No access to the full cost vector c_t.
- Necessitates exploration to gather information.

Concrete Example

Example: In online advertising, selecting an ad (decision x_t) and only observing the click-through rate (incurred cost) without knowing the underlying user preferences (cost vector c_t).

Example: Online Shortest Path Problem

- Decision set S consists of all possible paths in a network.
- Cost vectors represent edge weights chosen by the adversary.
- Feedback is the total cost of the chosen path only.

Concrete Example

Consider a network with 4 nodes (A, B, C, D) and 5 edges. At each round, an adversary assigns weights to the edges. The algorithm selects a path from node A to node D and only observes the sum of the weights on the chosen path.

Difference Between Oblivious and Adaptive Adversaries

- **Oblivious Adversary**: Chooses all cost vectors in advance.
- **Adaptive Adversary**: Chooses c_t based on past algorithm decisions.
- Adaptive adversaries are more powerful and challenging.

Adversary Type	**De**
Oblivious	Chooses a
Adaptive	Chooses c_t

Table: Comparison of Adversary Types

Impact of Adversary's Strategy on the Algorithm

- Adaptive strategies can exploit algorithm's weaknesses.
- Necessitates robust algorithms that can handle changing environments.
- Importance of maintaining low regret despite adversary's adaptability.

Formal Definition of Regret

- **Regret** measures the performance gap between the algorithm and the best offline decision.
- Mathematically:

$$\text{Regret} = \mathbb{E}\left[\sum_{t=1}^{T} c_t \cdot x_t\right] - \mathbb{E}\left[\min_{x \in S} \sum_{t=1}^{T} c_t \cdot x\right]$$

  - $\mathbb{E}$: Expectation over the algorithm's randomness.
  - c_t: Cost vector at round t.
  - x_t: Decision made by the algorithm at round t.
- Goal is to achieve sublinear regret: $\text{Regret} = o(T)$.

Goal: Minimize Regret

- Achieve sublinear regret:
  Regret $= o(T)$.
- Ensures average regret per round goes to zero as T increases.
- Fundamental objective in online learning and optimization.

Description of the Kalai-Vempala Algorithm

- General algorithm for online convex optimization.
- Utilizes the exponential weights framework.
- Assumes full feedback: Observes entire cost vector c_t after each decision.

Mathematical Formulation

The Kalai-Vempala algorithm updates the weight $w_t(x)$ for each decision $x \in S$ as:

$$w_{t+1}(x) = w_t(x) \exp\left(-\eta c_t \cdot x\right)$$

where η is the learning rate.

Assumption: Adversary Selects Cost Vectors After Decision

- The algorithm selects x_t without knowing c_t.
- After selection, c_t is revealed.
- Enables the use of gradient-based updates in the algorithm.

$$x_t = \frac{w_t(x)}{\sum_{x' \in S} w_t(x')}$$

Transition to the Bandit Setting

- In the bandit setting, only the incurred cost $c_t \cdot x_t$ is observed.
- No access to the full cost vector c_t.
- Requires estimation techniques to infer c_t from limited feedback.

Estimation Technique

Importance Sampling: To estimate the cost vector, the algorithm can use techniques like importance sampling where:

$$\hat{c}_t = \frac{c_t \cdot x_t}{P(x_t)} e_{x_t}$$

where $P(x_t)$ is the probability of selecting x_t and e_{x_t} is the standard basis vector corresponding to x_t.

Key Challenge: Working with Partial Feedback

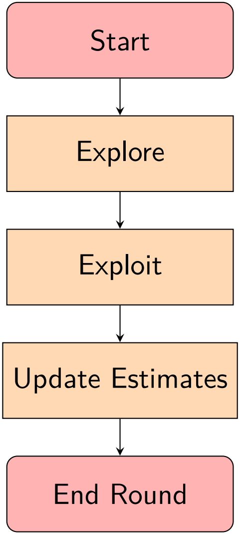
- Limited information makes it difficult to update cost estimates accurately.
- Balancing exploration (gathering information) and exploitation (using current knowledge).
- Ensuring robust performance against adaptive adversaries.
- Incorporating additional mathematical tools for estimation.

Description of the Bandit-style Geometric Decision Algorithm (BGA)

- Designed for bandit feedback in online geometric optimization.
- Combines exploration and exploitation phases.
- Leverages geometric properties of the decision set S.

Alternating Between Exploration and Exploitation

- **Exploitation**: Use current estimates to make informed decisions.
- **Exploration**: Sample randomly from a basis to gather new information.
- Ensures that the algorithm learns the cost structure over time.

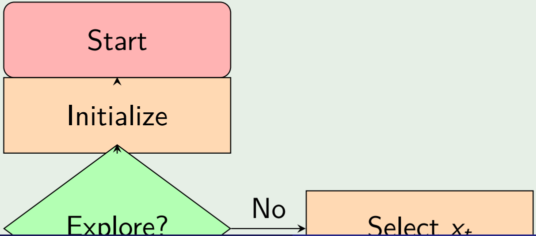


Pseudocode for BGA Algorithm

BGA Algorithm

1. Initialize estimates and sampling basis $B \subset S$.
2. For each round $t = 1$ to T:
   1. With probability γ, **explore** by selecting a random basis vector $b \in B$.
   2. Otherwise, **exploit** by selecting x_t using the Kalai-Vempala strategy on estimated costs.
   3. Observe the incurred cost $c_t \cdot x_t$.
   4. Update cost estimates based on the observed cost.

Flowchart Representation



Explanation of the Sampling Basis $B \subset S$

- B is a carefully chosen subset of S that facilitates exploration.
- Ensures coverage of different regions in the decision space.
- Basis vectors are used to approximate the cost structure.
- Geometric properties of B influence the efficiency of exploration.

Mathematical Insight

If B forms a **basis** in the linear algebra sense, any decision $x \in S$ can be expressed as a linear combination of vectors in B. This allows efficient reconstruction and estimation of the cost vector c_t.

Decision Making Based on Basis B

- During exploration, a basis vector b is selected uniformly at random.
- Provides unbiased estimates of the cost vector c_t.
- Facilitates efficient updating of cost estimates.

$$\hat{c}_t = \frac{c_t \cdot b}{P(b)} e_b$$

where $P(b)$ is the probability of selecting basis vector b, and e_b is the corresponding basis vector.

Exploration Probability γ

- γ controls the frequency of exploration.
- Higher γ leads to more exploration, improving cost estimates.
- Lower γ emphasizes exploitation, leveraging current knowledge.

Balancing γ

The optimal choice of γ depends on problem parameters such as T and n. Typically, γ is set to decrease over time to balance exploration and exploitation effectively.

Trade-off Between Exploration and Exploitation

- Balancing γ is crucial for minimizing regret.
- Too much exploration can waste resources, while too little can lead to poor estimates.
- Optimal γ depends on problem parameters like T and n.

Update Rule for Cost Vector Estimates

- Use observed costs to refine estimates of c_t.
- Employ matrix inversion techniques to handle correlated estimates.
- Ensures accurate approximation of the true cost vectors over time.
- Utilizes geometric properties of the decision set for efficient updates.

$$\hat{c}_{t+1} = \hat{c}_t - \eta \nabla L_t(x_t)$$

where η is the learning rate and $L_t(x_t)$ is the loss at round t.

Introduction to the Mathematical Analysis of BGA

- Formalize the algorithm's update rules and decision-making process.
- Define assumptions and properties of the cost vectors and decision set.
- Set the stage for deriving regret bounds.

Assumptions

1. The cost vectors c_t are bounded, i.e., $\|c_t\| \leq C$ for some constant C.
2. The decision set S is convex and compact.
3. The adversary is adaptive, choosing c_t based on past decisions $x_1, \ldots, x_{t-1}$.

Regret Bounds and Performance Guarantees

- Analyze the cumulative regret over T rounds.
- Provide theoretical guarantees under adaptive adversary models.
- Compare performance with existing algorithms like Kalai-Vempala.

Theoretical Insight

Under the adaptive adversary model, BGA maintains a regret bound of:

$$\text{Regret} = O\left(T^{3/4}\sqrt{\ln T}\right)$$

This ensures that the algorithm performs competitively even in dynamic environments.

Theoretical Regret Result

- BGA achieves regret $O\left(T^{3/4}\sqrt{\ln T}\right)$.
- Sublinear growth ensures that average regret per round diminishes.
- Demonstrates effectiveness in the bandit setting against adaptive adversaries.

Mathematical Proof Sketch

The regret bound is derived using:

1. Concentration inequalities to bound estimation errors.
2. Matrix inversion to handle dependencies in cost estimates.
3. Balancing exploration and exploitation through parameter tuning.

Impact of Parameters γ, ϵ, and T

- γ: Balances exploration and exploitation.
- ϵ: Determines precision of cost estimates.
- T: Number of rounds influences overall regret.
- Optimal tuning of parameters is essential for best performance.

$$\gamma = T^{-1/4}, \quad \epsilon = T^{-1/2}$$

High-Probability Bounds on Cost Vector Estimates

- Establish bounds that hold with high probability.
- Use concentration inequalities to ensure reliable estimates.
- Critical for guaranteeing low regret in adversarial settings.

$$\Pr\left(\|\hat{c}_t - c_t\| \geq \delta\right) \leq \exp(-k\delta^2)$$

where k is a constant depending on T and n.

Using Martingale Inequalities for Estimating c_t

- Apply martingale-based techniques to handle dependencies over time.
- Ensure that estimates remain unbiased and concentrated around true values.
- Facilitates robust analysis against adaptive adversaries.

Theorem (Azuma-Hoeffding Inequality)

Let $\{X_t\}$ be a martingale with bounded differences $|X_t - X_{t-1}| \leq c_t$. Then, for any $\lambda > 0$,

$$\Pr\left(X_T - X_0 \geq \lambda\right) \leq \exp\left(-\frac{\lambda^2}{2\sum_{t=1}^{T} c_t^2}\right)$$

Random Exploration for Estimating True Cost Vectors

- Randomly selecting basis vectors aids in uncovering the cost structure.
- Ensures diverse coverage of the decision space.
- Reduces bias in cost estimates by providing varied perspectives.

$$\mathbb{E}[\hat{c}_t] = c_t$$

Unbiased Estimation

The estimator $\hat{c}_t$ is unbiased because:

$$\mathbb{E}[\hat{c}_t] = \sum_{b \in B} P(b) \cdot \frac{c_t \cdot b}{P(b)} e_b = c_t$$

Importance of Unbiased Estimates

- Unbiased estimates are crucial for accurate decision-making.
- Prevents systematic errors that could be exploited by the adversary.
- Enhances the reliability and performance of the BGA algorithm.

$$\mathbb{E}[\hat{c}_t] = c_t$$

Consequence

Ensures that the algorithm's decisions are based on accurate representations of the cost vectors, leading to effective optimization over time.

Detailed Analysis of Expected Regret for BGA

- Derive bounds on the expected cumulative regret.
- Show how BGA maintains low regret despite adaptive adversaries.
- Compare theoretical performance with empirical observations.

Expected Regret Bound

The expected regret of BGA satisfies:

$$\mathbb{E}[\text{Regret}] \leq O\left(T^{3/4}\sqrt{\ln T}\right)$$

- The bound holds under the adaptive adversary model.
- Demonstrates sublinear growth, ensuring diminishing average regret.

Summary of Performance Guarantees

- BGA achieves sublinear regret in the bandit setting.
- Provides robustness against adaptive adversaries.
- Maintains competitive performance compared to full-information algorithms.

Key Takeaways

- Sublinear regret ensures that the algorithm becomes more effective over time.
- Robustness against adaptive adversaries makes BGA suitable for dynamic environments.
- The bandit setting presents unique challenges that BGA effectively addresses.

Comparison with Kalai-Vempala Algorithm

- Kalai-Vempala assumes full feedback, achieving different regret bounds.
- BGA extends these ideas to the bandit setting with limited feedback.
- Demonstrates comparable performance with additional challenges.

Algorithm	Feedback	Regret Bound
Kalai-Vempala	Full Information	$O(\sqrt{T})$
BGA	Bandit	$O(T^{3/4}\sqrt{\ln T})$

Table: Comparison of Regret Bounds

Recap of the Main Contribution

- Introduction of BGA for online geometric optimization in bandit settings.
- Achieves low-regret performance against adaptive adversaries.
- Bridges the gap between full-information and bandit feedback models.

Implications

BGA provides a robust framework for decision-making in environments where feedback is limited and adversaries can adapt, making it applicable to various real-world scenarios.

Theoretical Bounds for Performance Against an Adaptive Adversary

- Established $O(T^{3/4}\sqrt{\ln T})$ regret bound.
- Guarantees hold under general adversarial conditions.
- Highlights the algorithm's robustness and efficiency.

Future Directions

Further research can explore tightening these bounds and extending BGA to other optimization settings.

Discussion of Open Questions and Potential Future Work

- Can regret bounds be further improved for adaptive adversaries?
- Exploration of different sampling strategies for basis selection.
- Extensions to other types of online optimization problems.
- Incorporating additional feedback mechanisms to enhance performance.

Improving Regret Bounds for Adaptive Adversaries

- Investigate tighter analyses and alternative algorithms.
- Explore the impact of different geometric properties of S.
- Consider hybrid models combining full and bandit feedback.
- Study the interplay between exploration rates and adversary adaptability.

Potential Approaches

1. Utilize advanced concentration inequalities.
2. Incorporate adaptive learning rates.
3. Leverage multi-armed bandit techniques.

Key References

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ithms.

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Example

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