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سوال 1)

الف)

1. $I(X;Y|Z) < I(X;Y)$ 2. $I(X;Y|Z) > I(X;Y)$

$I(X;Y|Z) - I(X;Y) = H(X|Z) - H(X|Y;Z) - H(X) + H(X|Y)$

1: $H(X|Z) = 0$ iff X is completely determined by Z. Thus if $X=Y=Z$

$I(X,Y|Z) - I(X;Y) = H(X|Z) - H(X|Y;Z) - H(X) + H(X|Y) = -H(X) < 0$ for all X's

2: X, Z, Y are independent and X is conditionally independent of Z given Y:

$H(X|Z) = H(X)$

$H(X|Y) = H(X)$

$H(X|Y;Z) = H(X|Y) = H(X)$

$\Rightarrow I(X,Y|Z) = I(X;Y)$

So let $Z = X+Y$ and X and Y independent. we have:

$I(X,Y|Z) - I(X;Y) = H(X|Z) - H(X|Y;Z) - H(X) + H(X|Y) = H(Y|Z) \geq 0$

($H(X|Y;Z) = 0$, $H(X|Y) = H(X)$)

So $I(X,Y|Z) > I(X;Y)$ iff $X \perp Y$ and $Z = X+Y$ and $X \perp Z$ → from two given conditions, this always satisfies.

Ex: let $X, Y \sim Ber(1/2)$: $H(X|Z) = -\sum_3 H(X|Z=3) \cdot \sum_3 p(3) \sum_k p(k) \log p(X=3|Z=3)$

$= -\left(p(3=1) \sum_{k=\{0,1\}} p(k) \log p(X=k|Z=3) + p(3=2) \sum_{k=\{0,1\}} p(k) \log p(X=k|Z=3) + p(3=3) \sum_{k=\{0,1\}} p(k) \log p(X=k|Z=3)\right)$

$= -\left(\frac{1}{4} \times \log(1) + \frac{1}{2} \log\left(\frac{1}{2}\right) + \frac{1}{4} \log(1)\right) = -\frac{1}{2} \log\left(\frac{1}{2}\right) = \frac{1}{2} \log(2) = \frac{1}{2}$

ب)

1. $H(X,Y,Z) - H(X,Y) = H(Z|Y,X) + H(Y|X) + H(X) - H(Y|X) - H(X) = H(Z|Y,X)$

$H(X,Z) - H(X) = H(Z|X) + H(X) - H(X) = H(Z|X)$

Define $A = Z|X \Rightarrow H(X,Y,X) - H(X,Y) - H(X,Z) + H(X) = H(Z|X,Y) - H(Z|X) = H(A|Y) - H(A)$

$= -I(Y;A) \geq 0$

2. $I(X;Z|Y) = I(Z;Y|X) + I(Z;Y) - I(X;Z)$ • $I(X;Z) = H(X) - H(X|Z)$

$= H(X) - H(X|Z|Y) - H(Z) + H(Z|Y|X) + H(Z) - H(Z|Y) - H(X) + H(X|Z)$

• $H(Y|X) = H(X|Y) - H(X) + H(Y) \rightsquigarrow H(Z|Y|X) = H(X|Z|Y) + H(Z|Y) - H(X)$

$\rightsquigarrow H(Z|Y) - H(X) - H(Z|Y) + H(X|Z) = H(X|Z) - H(X) \le 0$ $H(X|Z) - H(X)$ iff Y, Z are independent.

1) $P(Z=A|Y=y) = (1+\exp(-\frac{y-5}{2}))^{-1} \rightsquigarrow \log_2 P(Z=A|Y=y) = -\log_2(1+\exp(-\frac{y-5}{2})) = x$ (C

$1 - P(Z=A|Y=y) = 1 - (1+\exp(-\frac{y-5}{2}))^{-1} \rightsquigarrow \log_2 1 - P(Z=A|Y=y) = \log_2\left(\frac{\exp(-\frac{y-5}{2})}{1+\exp(-\frac{y-5}{2})}\right) = \frac{-y-5}{2}\log_2 e + \log_2(1+\exp(-\frac{y-5}{2}))$

$P(Z) = E_y[Z, Y=y] = \sum_y P(Z|Y=y)P(y) = \frac{1}{4}\sum_y P(Z|Y=y) = \frac{1}{4}$, $H(Z) = 1$

$H(Z|Y) = \sum_y P(y) \sum_z P(Z|Y=y) \log(P(Z|Y=y)) = \frac{1}{4}\left(x\log(x) + \frac{1-x}{2}\log\frac{1-x}{2} + \frac{1-x}{3}\log\frac{1-x}{3} + \frac{1-x}{6}\log\frac{1-x}{6}\right)$

$= \frac{1}{4}\left(x\log x + \frac{1-x}{2}\log 1-x - \frac{1-x}{2} + \frac{1-x}{3}\log 1-x - \frac{1-x}{3}\log(3) + \frac{1-x}{6}\log\frac{1-x}{6} - \frac{1-x}{6}(1+\log(3))\right)$

$= \frac{1}{4}\left(x\log x + \log(1-x) - (1-x) - \frac{1-x}{6}\log(3)\right) = \frac{1}{4}\left(x\log x - \frac{y-5}{2}\log e + \log_2(x) + x - 1 + \frac{1-x}{6}\log(3)\right)$

2) $I(X;Y)$ represents the mutual information between RV X and Y. Thus if $I(X;Y)$ is big/small, it means with observing X, we can have more/less information about Y which represent the value or utility we get by choosing route Z.

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$D_{KL}(P(X,Y,Z) \| P(X)P(Y)P(Z)) = \int_x\int_y\int_z P(x,y,z)\log\left(\frac{P(x,y,z)}{P(x)P(y)P(z)}\right)dz\,dy\,dx$

$= \int_x\int_y\int_z P(x,y,z)\log P(x,y,z)\,dz\,dy\,dx - \int_x\int_y\int_z P(x,y,z)\log P(x)\,dz\,dy\,dx - \int_x\int_y\int_z P(x,y,z)\log P(y)\,dz\,dy\,dx$

$- \int_x\int_y\int_z P(x,y,z)\log P(z)\,dz\,dy\,dx$

$= -H(X,Y,Z) - \int_x \log P(x)\int_y\int_z P(x,y,z)\,dz\,dy\,dx - \int_y \log P(y)\int_x\int_z P(x,y,z)\,dz\,dx\,dy$

$- \int_z \log P(z)\int_x\int_y P(x,y,z)\,dy\,dx\,dz$

$\int_{x_n}\int_{x_{n-1}} P(x_1, x_2, \dots, x_n)\,dx_{n-1}\cdots dx_1 = P(x_n)$

$$\therefore H(x,y,z) = -\int_x p(x)\log p(x)\,dx - \int_y p(y)\log p(y)\,dy - \int_z p(z)\log p(z)\,dz$$

$$= H(x,y,z) + H(x) + H(y) + H(z)$$

$D_{KL}(p(x,y,z)\,||\,p(x)p(y)p(z)) = 0$ iff $p(x,y,z) = p(x)p(y)p(z) \Rightarrow x, y, z$ are independent.

or

$H(x,y,z) = H(z|x,y) + H(y|x) + H(x) = H(z) + H(y) + H(x)$ iff z is independent from y and x; y is independent from x $\Rightarrow x \perp y \perp z$

(3)

$$I(X_1;X_4) + I(X_2;X_3) - I(X_1;X_3) - I(X_2;X_4)$$

$$= H(X_1) - H(X_1|X_4) + H(X_2) - H(X_2|X_3) - H(X_1) + H(X_1|X_3) - H(X_2) + H(X_2|X_4)$$

$$= H(X_1|X_3) + H(X_2|X_4) - H(X_2|X_3) - H(X_4|X_1)$$

from bayes rule we have: $H(X_1,X_2|X_3) = H(X_1|X_3) + H(X_2|X_1,X_3)$

thus we have:

$$= H(X_1,X_2|X_3) - H(X_2|X_1,X_3) + H(X_2,X_1|X_4) - H(X_1|X_2,X_4) + H(X_1|X_2,X_3) - H(X_2,X_1|X_3) + H(X_2|X_1,X_4) - H(X_1,X_2|X_4)$$

$$= H(X_1|X_2,X_3) + H(X_2|X_1,X_4) - H(X_1|X_2,X_4) - H(X_2|X_1,X_3)$$

we know that in markov process, $H(X_i|X_{i+1},\cdot) = 0$. Thus:

$$= H(X_2|X_1,X_4) - H(X_2|X_1,X_3,X_4) = I(X_2;X_3|X_1,X_4)$$

also: $I(X_2;X_3|X_1,X_4) = 0$ iff $X_2 \perp X_3 | X_1, X_4$

سوال 2

آ

$f(x)$ is convex iff for $0 \le \lambda \le 1$ and $x_1, x_2 \in R$: $f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$

or by first order condition: $f(x_2) \ge f(x_1) + \nabla f(x_1)(x_2 - x_1)$

if x_1 is a critical point for $f(x)$, then $\nabla f(x_1) = 0$ Thus: $f(x_2) \ge f(x_1)$ and x_1 is a global minimum of $f(x)$

in reverse, $f(x_1) = 0$ if x_1 is a global minima, because $f(x_2) = f(x_1) + \nabla f(x_1)(f(x_2) - f(x_1)) + O(n^2)$

then if $\nabla f(x_1) \neq 0$, let $x_2 = \eta + \nabla f(x_1) = x_1$ where $\eta \ll 1$, we have $f(x_2) < f(x_1)$ and this is a conflict

ب

$$L(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z) = x^2 + y^2 + z^2 - \lambda_1 (z^2 - x^2 - y^2) - \lambda_2 (z - x - y - 1)$$

$$\rightarrow \begin{cases} \frac{\partial L}{\partial x} = 2x + 2\lambda_1 x + \lambda_2 = 0 \\ \frac{\partial L}{\partial y} = 2y + 2\lambda_1 y + \lambda_2 = 0 \end{cases} \xrightarrow{\text{subtract}} 2(x - y) + 2\lambda_1 (x - y) = 0 \Rightarrow \begin{cases} x = y \\ 1 + \lambda_1 = 0 \end{cases} \quad * \lambda_1 \neq 0$$

$$\rightarrow \begin{cases} \frac{\partial L}{\partial z} = 2z - 2\lambda_1 z - \lambda_2 = 0 \\ \frac{\partial L}{\partial \lambda_1} = z^2 - x^2 - y^2 = 0 \Rightarrow z^2 - 2x^2 = 0 \\ \frac{\partial L}{\partial \lambda_2} = z - x - y - 1 = 0 \Rightarrow z - 2x - 1 = 0 \end{cases} \Rightarrow z^2 = (2x + 1)^2 = 2x^2 \rightsquigarrow 4x^2 + 4x + 1 = 2x^2 \rightsquigarrow 2x^2 + 4x + 1 = 0 \rightarrow x = -1 \pm \frac{\sqrt{2}}{2}$$
$$\rightarrow z = \pm\sqrt{2} - 1$$

Critical points on $y_1 = 0$ and $y_2 = 0$ for f: $f(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, -\sqrt{2} - 1) = 2(\sqrt{2} - 1)^2 = 2(3 - 2\sqrt{2}) = 6 - 4\sqrt{2}$ minimum

$f(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, \sqrt{2} - 1) = 2(-\sqrt{2} - 1)^2 = 2(3 + 2\sqrt{2}) = 6 + 4\sqrt{2}$ maximum

ج

$L(\mu, \lambda) = \inf_x \{f(x) + \mu^T g(x) + \lambda^T h(x)\}$

Let x^* be the feasible point; then we have: $h(x^*) = 0$, $\mu^T g(x^*) \le 0 \Rightarrow L(\mu, \lambda) = \inf_x \{f(x) + \mu^T g(x) + \lambda^T h(x)\}$

$\le f(x^*) + \mu^T g(x^*) + \lambda^T h(x^*) \le f(x^*)$

Considering the point that x^* is any feasible point, then if x_0 is optimal solution for $\min_x f(x)$, x_0 is also a feasible point. Then we have: $L(\mu, \lambda) = \inf_x \{f(x) + \mu^T g(x) + \lambda^T h(x)\} \le f(x_0) = \min_x f(x)$

let $\min_{\text{s.t.}} e^{-x}$ $x, y > 0$, $\frac{x^2}{y} \le 0$ $\Rightarrow L(\lambda) = \inf_{x,y>0} \{e^{-x} + \lambda \frac{x^2}{y}\} = \begin{cases} 0 & \lambda \ge 0 \\ -\infty & \lambda < 0 \end{cases} \Rightarrow d^* = \max_\lambda 0 : \lambda \ge 0$

however: $e^{-\varepsilon}, \varepsilon > 0 = 1$ Thus $\max_\lambda L(\lambda) = 0 < \min_x f(x) = 1$

د

minimize $-\sum_{i=1}^{n} \log(a_i + x_i)$ (Also known as water filling problem)

s.t. $x \ge 0$, $1^T x = 1$

first, we clarify our feasible set. x is optimal if $x \geq 0$ and $\sum x_i = 1$. If we form the lagrangian dual problem, we have:

$$L(\mu, \lambda) = -\sum_{i=1}^{n} \log(a_i + x_i) + \lambda(1^T x - 1) + \mu^T x = -\sum_{i=1}^{n} \log(a_i + x_i) + \lambda\left(\sum_{i=1}^{n} x_i - 1\right) + \sum_{i=1}^{n} \mu_i x_i$$

$$\frac{\partial L}{\partial x_i} = -\frac{1}{a_i + x_i} + \lambda_i + \mu_i = 0 \Rightarrow \mu_i = \frac{1}{a_i + x_i} - \lambda_i$$

From KKT conditions, we know that $\mu^T x = 0$. Thus: $x_i\left(\frac{1}{a_i + x_i} - \lambda_i\right) = 0$. Also from the fact that $\mu \geq 0$, we have: $\lambda_i \geq \frac{1}{a_i + x_i}$. let's break it into 2 conditions:

$x_i \neq 0 \Rightarrow \lambda_i = \frac{1}{a_i + x_i} \Rightarrow x_i = \frac{1}{\lambda_i} - a_i$ or $x_i = 0 \Rightarrow x_i^* = \max\left\{0, \frac{1}{\lambda_i} - a_i\right\}$ such that $\sum_i x_i^* = 1$

first we form the lagrangian:

$$L(\rho, \lambda, \mu, \sigma) = f(x) - \lambda^T g(x) = -\int \rho(x) \log \rho(x) - \lambda_0\left(\int \rho(x)\, dx - 1\right) - \lambda_1\left(\int (x-\mu)^2 \rho(x)\, dx - \sigma^2\right)$$

$$= -\int \rho(x)\left[\log \rho(x) - \lambda_0 - \lambda_1 (x-\mu)^2\right] dx + \lambda_0 + \lambda_1 \sigma^2$$

Using Calculus of Variations, we can find critical points of $L(\rho, \lambda, \mu, \sigma^2)$ by solving following system of equations:

$$\frac{d}{dx}\frac{\partial L}{\partial \rho'} - \frac{\partial L}{\partial \rho} = 0 \quad \left(\left(\frac{\partial L}{\partial \rho'} = 0\right)\right) \Rightarrow \frac{\partial L}{\partial \rho} = -\log \rho(x) - 1 - \lambda_0 - \lambda_1 (x-\mu)^2 = 0 \Rightarrow \rho(x) = e^{-1-\lambda_0-\lambda_1(x-\mu)^2}$$

now, since $\rho(x)$ is probability distribution, we have: $\int \rho(x)\, dx = 1 \Rightarrow \int e^{-1-\lambda_0-\lambda_1(x-\mu)^2} dx = 1 \Rightarrow e^{-1-\lambda_0} \int e^{-\lambda_1(x-\mu)^2} dx = 1$

Let $t = \sqrt{\lambda_1}(x-\mu) \Rightarrow dt = \sqrt{\lambda_1}\, dx \Rightarrow \int e^{-\lambda_1(x-\mu)^2} dx = \frac{1}{\sqrt{\lambda_1}} \int e^{-t^2} dt = \frac{\sqrt{\pi}}{\sqrt{\lambda_1}} \Rightarrow e^{-1-\lambda_0} = \frac{\sqrt{\lambda_1}}{\sqrt{\pi}} \Rightarrow \lambda_0 = -1 + \frac{1}{2}\log \pi - \frac{1}{2}\log \lambda_1$

from second constraint, we have:

$$\int (x-\mu)^2 \rho(x)\, dx = \int (x-\mu)^2 e^{-1-\lambda_0-\lambda_1(x-\mu)^2} dx = \sigma^2$$

again, let $t = \sqrt{\lambda_1}(x-\mu)$. we have: $\sigma^2 = \int \frac{t^2}{\lambda_1} e^{-1-\lambda_0-t^2} \frac{dt}{\sqrt{\lambda_1}} = \frac{e^{-1-\lambda_0}}{\lambda_1 \sqrt{\lambda_1}} \int t^2 e^{-t^2} dt = \frac{e^{-1-\lambda_0}}{\lambda_1 \sqrt{\lambda_1}} \frac{\sqrt{\pi}}{2}$ $\left(e^{-1-\lambda_0} = \frac{\sqrt{\lambda_1}}{\sqrt{\pi}}\right) \Rightarrow \sigma^2 = \frac{\sqrt{\lambda_1} \times \sqrt{\pi}}{2\lambda_1 \sqrt{\lambda_1} \times \sqrt{\pi}}$

$= \frac{1}{2\lambda_1} \Rightarrow \lambda_1 = \frac{1}{2\sigma^2}$

Thus we have: $\rho(x) = e^{-1-\lambda_0} e^{-\lambda_1(x-\mu)^2} = \frac{\sqrt{\lambda_1}}{\sqrt{\pi}} e^{-\lambda_1(x-\mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \rightsquigarrow \rho(x) \sim \mathcal{N}(\mu, \sigma^2)$

سوال ۳

الف)

The chain is A-periodic and irreducible, Thus we have $\pi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix}$

$$\lim_{n\to\infty} P^{(n)} = \begin{bmatrix} \cdots\cdots \pi \cdots\cdots \\ \cdots\cdots \pi \cdots\cdots \\ \vdots \\ \cdots\cdots \pi \cdots\cdots \end{bmatrix}$$

using the probability of transition $\begin{cases} p_{ii} = 1 - \frac{|N(i)|}{M} \\ p_{ij} = \begin{cases} \frac{1}{M} & j \in N(i) \\ 0 & j \notin N(i) \end{cases} \end{cases}$

we know that $M \geq \arg\max_{x \in \mathcal{L}} |N(x)|$; Thus $p_{ij} = p_{ji} = \frac{1}{M}$. Thus $\underline{P}$ is symmetric.

using the fact that for a node i, we have $\sum_{\substack{j \in \mathcal{L} \\ i \neq j}} p_{ij} = |N(i)| \times \frac{1}{M} + 1 - \frac{|N(i)|}{M} = 1$; columns and rows of $\underline{P}$ sum to be 1.

we know that if $\underline{P}$'s columns sum $\underline{t}$ 1, then the stationary distribution π defines as $\pi_i = \frac{1}{n}$.

ب)

Main problem in this counting is that we are counting one sample multiple times, i.e., the sampling is uniform; however we can take alot more information via a better sampling. Also, there is a 2^n multiplier when there is no need to count the possible outcomes of X in the numerator when we have uniform sampling, i.e. a better way could be initializing $X = (x_1, \ldots, x_n)$ 2^n times non-uniformly and then check the constrains; however given approach in C is better and uses almost all of driven data.

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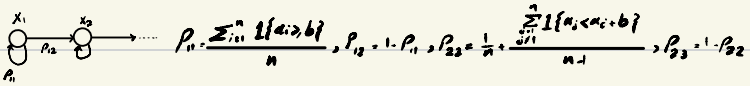
1. let $X_i = (0, 0, \ldots, 0)$. then we have:



from part A, we know that if a markov chain is A-periodic and irreducible, it has a stationary distribution π over Nodes. Given Markov chain is obviously irreducible, and it's A-periodic (since biggest period has length 1). Thus the Markov-process has a stationary distribution π which tells us after infinitly step Random walk, which nodes are most possible to be found our agent at.

2. we just have to check when the MC stops, i.e., after $T \to \infty$ passes, MC stops at X_z, where z is all possible X's where $\sum_i a_i x_i < b$ constrain is satisfied. Thus z is all possible valid outcomes.

سوال 4

الف

$p(s|same) = \alpha_{ss} e^{s\lambda_s}$, $p(s|different) = \alpha_{ds} e^{-s\lambda_d}$, $s=[0,1]$

The probability distributions seem rational, because if two pictures are same, we expect a higher score proportional to s. i.e., $\frac{dp(s|same)}{ds} = \lambda_s \alpha_{ss} s e^{s\lambda_s} > 0$ and vice versa for different pictures: $\frac{dp(s|different)}{ds} = -\lambda_d \alpha_{sd} s e^{-s\lambda_d} < 0$

α_{ss} and α_{ds} are normalizing constants to make $p(s|same)$ and $p(s|different)$ a probability distribution over some interval s. for example, for $s=[0,1]$; we have:

$$\int_0^1 p(s|same)\,ds = \int_0^1 \alpha_{ss} e^{s\lambda_s} ds = \frac{\alpha_{ss}}{\lambda_s} e^{s\lambda_s}\Big|_0^1 = \alpha_{ss}\frac{e^{\lambda_s}-1}{\lambda_s} = 1 \Rightarrow \alpha_{ss} = \frac{\lambda_s}{e^{\lambda_s}-1}$$

$$\int_0^1 p(s|different)\,ds = \int_0^1 \alpha_{ds} e^{s\lambda_d} ds = \frac{\alpha_{ds}}{\lambda_d} e^{s\lambda_d}\Big|_0^1 = \alpha_{ds}\frac{e^{\lambda_d}-1}{\lambda_d} = 1 \Rightarrow \alpha_{ds} = \frac{\lambda_d}{e^{\lambda_d}-1}$$

ب

Let A_j = the picture j is for person j. Then we have:

$$p(A_j|s_j) = \frac{p(s_j|A_j)\,p(A_j)}{p(s_j)} = \frac{p(s_j|A_j)\,p(A_j)}{\sum_i p(s_i|A_i)\,p(A_i)}$$

Let's assume the condition of truely assigning pictures to someone is uniform and uncorrelated; i.e. $\forall i,j: p(A_i) = p(A_j)$

Thus: $$p(A_j|s_j) = \frac{p(s_j|A_j)\,p(A_j)}{\sum_i p(s_j|A_i)\,p(A_i)} = \frac{p(s_j|A_j)}{p(s_j|A_j) + \sum_{i\neq j} p(s_j|A_i)} = \frac{\alpha_{ss} e^{s\lambda_s}}{(n-1)\alpha_{ds} e^{-s\lambda_d} + \alpha_{ss} e^{s\lambda_s}} = \frac{\frac{\lambda_s}{e^{\lambda_s}-1} e^{s\lambda_s}}{\frac{(n-1)\lambda_d}{e^{\lambda_d}-1} e^{s\lambda_d} + \frac{\lambda_s}{e^{\lambda_s}-1} e^{s\lambda_s}}$$

ج

$$p(A|max) = \frac{p(max|A)\,p(A)}{p(max|A)\,p(A) + p(max|A^c)\,p(A^c)} = \frac{p(max|A)}{p(max|A) + p(max|A^c)}$$

So if we have a rating policy, we should baid it based on two different condition. one, takes the hypotesis that given picture for every one is right, i.e. $p(max|A)$. And the other one takes the hypoteis that our indicator thinks that picture is not mapped quite correctly. Then, if we take the assumption that $p(A) = p(A^c)$, we will have a short formula like given above. Secondly, if we consider the case wher A_i's are independent, we will have: $p(A|max) = \frac{p(max|A)}{(n-1)p(max|A^c) + p(max|A)}$ which almost a common sense.

Then we have: $p_{s_{max}}(max) = p(A_j, i\neq j \leq max)\,p(max) = \left(\int_0^{max} p(x)\,dx\right)^{n-1} p(max)$

$\rightsquigarrow$ from part b: $p_{s_{max}} = \int_0^{max} \frac{1}{n}\left(p(x|A) + p(x|A^c)\right)^{n-1} dx$

also note that $p(A) = \int_0^1 p(A|max)\,p(max)\,d\,max = \int_0^1 \frac{\alpha_{ss} e^{s\lambda_s}}{\alpha_{ss} e^{s\lambda_s} + (n-1)\alpha_{ds} e^{-s\lambda_d}} \left(\alpha_{ss} e^{s\lambda_s} + (n-1)\alpha_{ss} e^{-s\lambda_d}\right)\left(\frac{\alpha_{ss}}{\lambda_s} e^{s\lambda_s} - 1 - (n-1)\frac{\alpha_{ds}}{\lambda_d}(e^{s\lambda_s}-1)\right) ds$

$$= \frac{1}{n^2}\int_0^1 \alpha_{ss} e^{s\lambda_s}\left(\frac{\alpha_{ss}}{\lambda_{ss}}(e^{s\lambda_s}-1) + 1 - (1-n)\frac{\alpha_{ds}}{\lambda_d}(e^{-s\lambda_d}-1)\right)^{n-1} ds$$

سوال ۵

(آ)

$$\hat{\theta}_{MLE} = \arg\max_{\theta} P(X_{1:n}|\theta) = \arg\max_{\theta} L(\theta; X) \quad , P(x_i|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$$L(\mu,\sigma) = \prod_{i=1}^{n} p(x_i|\mu,\sigma^2) = \frac{1}{\sigma^n (2\pi)^{n/2}} \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)\right]$$

$$\rightsquigarrow \log L(\mu,\sigma) = -n\log\sigma - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_i (x_i-\mu)^2$$

$$\rightsquigarrow \begin{cases} \frac{\partial \log(\mu,\sigma)}{\partial \mu} = 0 \rightsquigarrow \frac{-2\sum(x_i-\mu)}{2\sigma^2} = 0 \Rightarrow \hat{\mu} = \frac{\sum x_i}{n} \\ \frac{\partial \log(\mu,\sigma)}{\partial \sigma} = 0 \rightsquigarrow -\frac{n}{\sigma} + \frac{\sum_i (x_i-\mu)^2}{\sigma^3} = 0 \Rightarrow \hat{\sigma}^2 = \frac{\sum_i (x_i-\bar{x})^2}{n} = \frac{1}{n}\sum_i x_i^2 - \frac{1}{n^2}\sum_i\sum_j x_i x_j \end{cases}$$

let $\delta_i = \mu - x_i$ $\rightsquigarrow \hat{\sigma}^2 = \frac{1}{n}\sum_i (\mu-\delta_i)^2 - \frac{1}{n^2}\sum_i\sum_j (\mu-\delta_i)(\mu-\delta_j)$

$\rightsquigarrow E[\hat{\sigma}^2] = \frac{1}{n}\sum_i E[(\mu-\delta_i)^2] - \frac{1}{n^2}\sum_i\sum_j E(\mu-\delta_i)(\mu-\delta_j)$

$= \frac{1}{n}\sum_i \left(E[\mu^2] - 2\mu E[\delta_i] + E[\delta_i^2]\right) - \frac{1}{n^2}\sum_i\sum_j \left(E[\mu^2] - \mu E[\delta_i+\delta_j] + E[\delta_i\delta_j]\right)$

($E[\delta_i] = 0$, $E[\delta_i^2] = \sigma^2$, $E[\delta_i+\delta_j] = 0$, $E[\delta_i\delta_j] = \sigma^2$)

$= \frac{1}{n}(n\mu^2 + n\sigma^2) - \frac{1}{n^2}(n^2\mu - n\sigma^2) = \mu^2 + \sigma^2 - \mu^2 - \frac{1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2$

biased!

(ب)

$\theta \longmapsto \tau(\theta)$: $\theta_{mle} = \arg\max_{\theta} p(x|\theta=\hat{\theta}) = \arg\max_{\theta} p(x|\tau(\theta))$ where $\hat{\tau}(\theta) = \tau(\hat{\theta})$. Thus we have:

$$\begin{cases} \hat{\mu}_2 = \tau(\hat{\mu}) = \frac{1}{1+\hat{\mu}^2} = \frac{1}{1+(\frac{\sum x_i}{n})^2} \\ \hat{\sigma} = \sqrt{\frac{\sum(x_i-\hat{\mu})^2}{n}} \end{cases}$$

(ج)

1. $\hat{\mu}_{IS} = \frac{1}{n}\sum_{i=1}^{n} f(x_i)\frac{p(x_i)}{q(x_i)}$ $\rightsquigarrow E_q[\hat{\mu}_{IS}] = \frac{1}{n}\sum_{i=1}^{n} E_{x\sim q}\left[f(x_i)\frac{p(x_i)}{q(x_i)}\right] = \frac{1}{n}\sum_{i=1}^{n}\int_{x\sim q} f(x_i)\frac{p(x_i)}{q(x_i)} q(x_i)\,dx_i = \frac{1}{n}\sum_{i=1}^{n}\int_{x\sim q} f(x_i)p(x_i)\,dx$

$= \frac{1}{n}\sum_{i=1}^{n} E_p[f(x_i)]$

$\rightsquigarrow \hat{\mu}_{IS} = \frac{1}{n}\sum_{i=1}^{n} E_{x\sim p}[f(x_i)] = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu \Rightarrow IS$ is un biased

2. $Var(\hat{\mu}_{IS}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\right)$ where $g(x_i) = f(x_i)\frac{p(x_i)}{q(x_i)} \Rightarrow \sigma_{IS}^2 = \frac{1}{n^2} Var\left(\sum_i g(x_i)\right) = \frac{1}{n} Var(g(x_i)) = \frac{1}{n}\left[E[g(x_i)^2] - \hat{\mu}_{IS}^2\right]$

$= \frac{1}{n}\left[\int f(x_i)^2 \frac{p(x_i)^2}{q(x_i)^2} q(x_i)\,dx_i - \hat{\mu}_{IS}^2\right] = \frac{1}{n}\left[\int f(x_i)^2 \frac{p(x_i)^2}{q(x_i)} dx_i - \mu^2\right]$

for minimum $Var(\hat{\mu}_{IS})$, we can choose $f(x) = \frac{q(x)}{p(x)} \times \mu$ which gives us zero variance.

Also, lets assume $f(x) = \frac{1}{p(x)\,q(x)}$. Then $\sigma^2 = \infty$ if $q(x) = 0$ for some x.

3. $\hat{\mu}_{N\text{-}IS} = \frac{\sum_{i=1}^{n} w_i\, f(x_i)}{\sum_{i=1}^{n} w_i} = \frac{\sum_{i=1}^{n} \frac{p(x_i)}{q(x_i)}\, f(x_i)}{\sum_{i=1}^{n} \frac{p(x_i)}{q(x_i)}}$

for $n \to \infty$, like $n=1$ we have: $E\{\hat{\mu}_{N\text{-}IS}\} = E\left[\frac{\frac{p(x)}{q(x)} f(x)}{\frac{p(x)}{q(x)}}\right] = E[f(x_1)]$ which is not necessarily μ.

for $n \to \infty$: $\lim_{n\to\infty} E\{\hat{\mu}_{N\text{-}IS}\} = \lim_{n\to\infty} E\left[\frac{\sum_{i=1}^{n} w_i f(x_i)}{n}\right] E\left[\frac{n}{\sum_{i=1}^{n} w_i}\right] \cdot \operatorname{cov}\left(\frac{\sum_{i=1}^{n} w_i f(x_i)}{n}, \frac{n}{\sum_{i=1}^{n} w_i}\right)$

for $f(x) = 1$ we have:

$\lim_{n\to\infty} E\{\hat{\mu}_{N\text{-}IS}\} = \mu \times 1 + \lim_{n\to\infty} \operatorname{cov}\left(\frac{\sum_{i=1}^{n} w_i f(x_i)}{n}, \frac{n}{\sum_{i=1}^{n} w_i}\right) = \mu$

So $\hat{\mu}_{N\text{-}IS}$ is unbiased for $n \to \infty$

4. $m \le f(x) \le M \Rightarrow$

$\hat{\mu}_{N\text{-}IS} = \frac{\sum_{i=1}^{n} w_i f(x_i)}{\sum_{i=1}^{n} w_i} \le \frac{\sum_{i=1}^{n} w_i M}{\sum_{i=1}^{n} w_i} = M$

$\hat{\mu}_{N\text{-}IS} = \frac{\sum_{i=1}^{n} w_i f(x_i)}{\sum_{i=1}^{n} w_i} \ge \frac{\sum_{i=1}^{n} w_i m}{\sum_{i=1}^{n} w_i} = m$

$\Rightarrow m \le \hat{\mu}_{N\text{-}IS} \le M$

$\sigma^2 = E[\hat{\mu}_{N\text{-}IS}^2] - \mu^2$ also $E[(M - \hat{\mu}_{N\text{-}IS})(\hat{\mu}_{N\text{-}IS} - m)] = -E[\hat{\mu}_{N\text{-}IS}^2] - mM + (M+m)\mu \ge 0 \Rightarrow E[\hat{\mu}_{N\text{-}IS}^2] \le -mM + (M+m)\mu$

Since $M - \hat{\mu}_{N\text{-}IS}$ and $\hat{\mu}_{N\text{-}IS} - m$ both are positive values with positive expected value

$\Rightarrow \sigma^2 \le (M+m)\mu - mM - \mu^2 = (M-\mu)(\mu-m) \le \frac{(M+m)^2}{4}$

5.

$H = \int_2^{\infty} \exp\left(-\frac{x^2}{2}\right) dx = \sqrt{2\pi}\,(1 - \varphi(2)) \approx 0{,}0577$

(الف)

$\hat{H}_{MC} = \frac{1}{n} \sum \frac{f(x_i)}{p(x_i)}\, 1\{x_i \ge 2\} = 0$ because all x_i's are less than 2

(ب)

$\hat{H}_{IS} = \frac{1}{n} \sum_i \frac{\sqrt{2\pi}\, p(x_i)}{q(x_i)}$ where $p \sim N(0,1)$ and $q \sim N(3,1)$

because samples now are from $N(3,1)$, we shift all of samples by 3. and then we have:

$\hat{H}_{IS} = \frac{1}{|S_1|} \sum_{x \in S_1} \frac{\sqrt{2\pi}\, e^{-\frac{x^2}{2}}}{e^{-\frac{(x-3)^2}{2}}} = \frac{1}{|S_1|} \sum_{x \in S_1} e^{\frac{2-6x}{2}} \approx 0{,}0293$

we see that IS estimator is a better estimator than MC; because of its low variance

سوال 6

(آ

$D_{KL}(p\|q) = \int p(x) \log \frac{p(x)}{q(x)} dx$ ← Jensen inequality: $f(\sum_{i=1}^{n} \lambda_i x_i) \le \sum_{i=1}^{n} \lambda_i f(x_i)$ for: $f \to$ convex $\{\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \ge 0$

from Jensen ineq., we have: $\log(E_x[g(x)]) \ge E_x[\log(g(x))]$

$-D_{KL}(p\|q) = -\int p(x)\left(-\log \frac{q(x)}{p(x)}\right) dx = \int p(x) \log \frac{q(x)}{p(x)} dx \le \log \int p(x) \frac{q(x)}{p(x)} dx = \log \int q(x) dx$

$= \log 1 = 0$

$\rightsquigarrow D_{KL}(p\|q) \ge 0$

Now, let $q(x) = p(x)$, we have: $D_{KL}(p\|p) = \int p(x) \log \frac{p(x)}{p(x)} dx = 0$ (→ 0)

So minimum of D_{KL} is when $q(x) = p(x)$ where KL divergence is zero.

(ب

$p(z|x) = \frac{p(z,x)}{p(x)}$

$D_{KL}(q(z)\|p(z|x)) = \int q(z) \log \frac{q(z)\,p(x)}{p(z,x)} dz = \int q(z)[\log q(z) + \log p(x) - \log p(x,z)] dz$

$= \int q(z)[\log q(z) - \log p(z,x)] dz + \int q(z) \log p(x) dz$ (I)

I: $\int q(z) \log p(x) dz = \log p(x) \int q(z) dz = \log p(x)$

$\rightsquigarrow D_{KL}(q(z)\|p(z|x)) = \int q(z) \log q(z) - \log p(x,z) + \log p(x)$

$E_q[\log q(z) - \log p(x,z)]$

$\rightsquigarrow D_{KL}(q(z)\|p(z|x)) = \log p(x) + E_q[\log q(z) - \log p(x,z)]$

$= \log p(x) - (E_q[\log p(x,z)] - E_q[\log q(z)])$

$= \log p(x) - L(q)$

$L(Q) = E_q[\log p(x,z)] - E_q[\log Q(z)]$, $Q(z) = \prod_{i=1} q_i(z)$

$p(z_1, z_2, \dots, z_n, x_1, x_2, \dots, x_m) = p(x_{1:m}) \prod_{i=1}^{n} p(z_i | z_1, z_2, \dots, z_{i-1}, x_{1:m})$

$\rightsquigarrow L(Q) = \sum_{j=1}^{n} E[\log(p(z_i | z_1, \dots, z_{i-1}, x)] - E_j[\log q(z_i)]$

$\rightsquigarrow L(q_k) = E[\log p(z_k | z_{-k}, x) - E_j[\log q(z_k)] + C$

Define: $F = \int q(z_k) E_{-k}[\log p(z_k | z_{-k}, x) dz_k - \int q(z_k) \log q(z_k) dz_k$

$= \int q(z_k) (E_{-k}[\log p(z_k | z_{-k}, x) - q(z_k)) dz_k$

using calculus of variation, we have $\frac{d}{dz_k} \frac{\partial F}{\partial \dot{q}(z_k)} = \frac{\partial F}{\partial q(z_k)}$ to maximize $L(q_k)$. Also it's obvious that if all of q_k's are optimal for $L(q_k)$, the $L(q_k)$ is optimal.

$\rightsquigarrow \frac{d}{dz_k} \frac{\partial F}{\partial \dot{q}(z_k)} = \frac{\partial F}{\partial q(z_k)} \Rightarrow \frac{\partial F}{\partial q(z_k)} = E_{-k}[\log p(z_k | z_{-k}, x)] - \log q(z_k) - 1 = 0 \Rightarrow \log q_k(z_k) = E_{q_{-k}}[\log p(x,z)] + C$

$\ln q^*(z) = E_{\pi,\mu,\Lambda}[\ln p(x, z, \pi, \mu, \Lambda)] + C = E_{\pi,\mu,\Lambda}[\ln p(z|\pi) p(x|z, \mu, \Lambda)] + C$

$= E_\pi[\ln p(z|\pi)] + E_{\mu,\Lambda}[\ln p(x|z, \mu, \Lambda)] + C$

$= E_\pi[\ln \prod_{n=1}^{N} \prod_{k=1}^{K} \pi_k^{z_{nk}}] + E_{\mu,\Lambda}[\ln \frac{1}{(2\pi)^{D/2} |\Lambda_k|^{-1/2}} \exp(-\frac{1}{2}(x-\mu_k)^T \Lambda_k^{-1}(x-\mu_k))] + C$

$= \sum_{n=1}^{N} \sum_{k=1}^{K} z_{nk} [E_\pi[\pi_k] + \frac{1}{2} E[\ln|\Lambda_k|] - \frac{D}{2} \ln(2\pi) - \frac{1}{2} E_{\mu_k,\Lambda_k}[(x-\mu_k)^T \Lambda_k^{-1} (x-\mu_k)]] + C$

Let $\rho_{nk} =$ we have: $\ln q^*(z) = \sum_{n=1}^{N} \sum_{k=1}^{K} z_{nk} \rho_{nk} \rightsquigarrow q^*(z) \propto \prod_{n=1}^{N} \prod_{k=1}^{K} \rho_{nk}^{z_{nk}}$

Let $r_{nk} = \frac{\rho_{nk}}{\sum_j \rho_{nj}}$ which normalizes ρ_{nk} such that $q^*(z)$ becomes a categorical probability distribution

whit $E[z_{nk}] = r_{nk}$

Due to structual property of our model, we have: $q(\pi, \mu, \Lambda) = q(\pi) q(\mu, \Lambda)$

$= q(\pi) \prod_{k=1}^{K} q(\mu_k, \Lambda_k)$

Then:

- $\log q^*(\pi) = E_z[\log p(\pi)p(z|\pi)] + C = E_z[\log p(\pi) + \log p(z|\pi)] + C = \log p(\pi) + E_z[\log p(z|\pi)] + C$

$= (\alpha_0 - 1)\sum_{k=1}^{K}\pi_k + \sum_{n=1}^{N}\sum_{k=1}^{K}\pi_k E_z[z_{nk}] = \sum_{k=1}^{K}\pi_k(\alpha_0 + \sum_{n=1}^{N} r_{nk} - 1)$

$E_z[z_{nk}] \rightsquigarrow r_{nk}$

Thus $q^*(\pi)$ has dirichlet distribution with $\alpha = \alpha_0 + \sum_{n=1}^{N} r_{nk}$

- $\ln q^*(\mu_k, \Lambda_k) = E_z[\log p(\mu_k, \Lambda_k)p(z|\mu_k, \Lambda_k)] + C = \log p(\mu_k, \Lambda_k) + E_z[\log p(x|z, \mu_k, \Lambda_k)p(z|\mu_k, \Lambda_k)] + C$

$= \log p(\mu_k, \Lambda_k) + E_z[\log \prod_{n=1}^{N}\mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})^{z_{nk}}] + C$

$= \log p(\mu_k, \Lambda_k) + E_z[\sum_{n=1}^{N} z_{nk}\log\mathcal{N}(x_n|\mu_k, \Lambda_k^{-1})] + C$

$= \log p(\mu_k, \Lambda_k) + \sum_{n=1}^{N} E_z[z_{nk}]\log\mathcal{N}(x_n|\mu_k, \Lambda_k^{-1}) + C$

$= \log\mathcal{N}(\mu_k|m_0, (\beta_0\Lambda_k)^{-1})\mathcal{W}(\Lambda_k|W_0, \nu_0) + \sum_{n=1}^{N} r_{nk}\log\mathcal{N}(x_n|\mu_k, \Lambda_k^{-1}) + C$

Given Normal-wishart properties, we have: $q^*(\mu_k, \Lambda_k) = \mathcal{N}(\mu_k|m_k, (\beta_k\Lambda_k)^{-1})\mathcal{W}(\Lambda_k|W_k, \nu_k)$

where: $N_k = \sum_{n=1}^{N} r_{nk}$, $\bar{x}_k = \frac{1}{N_k}\sum_{n=1}^{N} r_{nk}x_n$, $S_k = \frac{1}{N_k}\sum_{n=1}^{N} r_{nk}(x_n - \bar{x}_k)(x_n - \bar{x}_k)^T$, $\nu_k = \nu_0 + N_k$

$\beta_k = \beta_0 + N_k$, $m_k = \frac{1}{\beta_k}(\beta_0 m_0 + \bar{x}_k N_k)$, $W_k^{-1} = W_0^{-1} + N_k S_k + \frac{\beta_0 N_k}{\beta_0 + N_k}(\bar{x}_k - m_0)(\bar{x}_k - m_0)^T$

To calculate ρ_{nk} which gives us r_{nk}, we have to calculate $E_{\mu_k, \Lambda_k}[(x_n - \mu_k)^T\Lambda_k(x_n - \mu_k)]$, $E[\ln\pi_k]$, $E[\ln|\Lambda_k|]$

$E_{\mu_k, \Lambda_k}[(x_n - \mu_k)^T\Lambda_k(x_n - \mu_k)] = D\beta_k^{-1} + \nu_k(x_n - m_k)^T W_k(x_n - m_k)$

$E[\ln|\pi_k|] = \ln\tilde{\pi}_k$, $E[\ln|\Lambda_k|] = \ln\tilde{\Lambda}_k$

to calculate r_{nk}, we need rest of parameter, and vice-versa. to solve this problem we can use estimation-maximization to estimate r_{nk} and then update μ_k, Λ_k to maximize $\log q^*(z)$, which gives us minimum D_{KL}.

$Q(\theta|\theta^{(t)}) = E_{z\sim p(z|x,\theta^{(t)})}[\log(x, z|\theta)]$

$\theta^{(t+1)} = \arg\max_{\theta} Q(\theta|\theta^{(t)})$

where $\theta^{(t)} = \{\mu^{(t)}, \Lambda^{(t)}, \pi^{(t)}\}$