Cheeger's Inequality

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1 Introduction

This paper provides the reader with an introduction to Spectral Graph Theory and concludes with a proof of the (discrete) Cheeger Inequality using eigenvalues of the Laplacian matrix. The idea and properties of undirected graphs as well as important properties of the Laplacian matrix will be explored. Calculations of the Laplacian matrix for certain examples will be provided. Most of this material has been sourced from [1], [2], [3], [4], [5].

2 Fundamentals of Graph Theory

Definition: An undirected graph G is an ordered pair $G = (V, E)$ of sets V, E where

$$E \subset \{(x, y) : x, y \in V, x \neq y\} \subset V \times V. \tag{1}$$

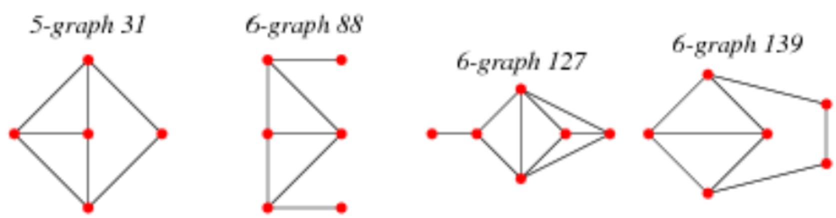

Figure 1: Examples of graphs. Credit: https://mathworld.wolfram.com/LineGraph.html

From here on, all graphs in this paper are finite, meaning that V and E are finite sets, and are undirected.

2.1 Characterizations of Graph Elements

Elements of V are called vertices and elements of E are called edges. Two vertices $v_i, v_j \in V$ are called adjacent to one another if $(v_i, v_j) \in E$. A graph G is called connected if for any vertex, there exists a union of edges that connects it to any other vertex in the graph. From this point on, we will only be dealing with connected graphs.

The degree $d(v)$ of a vertex $v \in V(G)$ is the number of vertices $v \in V(G)$ that are adjacent to v. A vertex $v \in V(G)$ is incident with an edge $(v_i, v_j) \in E(G)$ if $v = v_i$ or $v = v_j$.

3 Laplacian Matrix

Definition: Suppose $G = (V, E)$ with $V = \{1, 2, ..., n\}$. For an edge $\{u, v\} \in E$, define an $n \times n$ matrix $L_{G_{\{u,v\}}}$ by

$$L_{G_{\{u,v\}}}(i, j) := \begin{cases} 1, & i = j \text{ and } i \in \{u, v\}, \\ -1, & i = j \text{ and } j = v, \text{or vice versa}, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

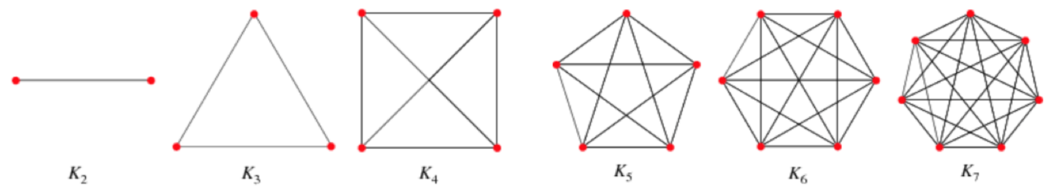

Figure 2: Connected graphs. Credit: https://mathworld.wolfram.com/CompleteGraph.html

The Laplacian Matrix L_G of a graph G is defined by

$$\boxed{L_G := \sum_{\{u,v\}\in E} L_{G_{\{u,v\}}}}. \tag{3}$$

3.1 Examples of the Laplacian Matrix

Let J denote the graph below.

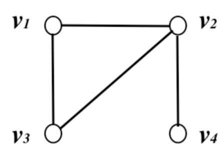

Figure 3: Unconnected graph. Credit: https://csustan.csustan.edu/ tom/Clustering/GraphLaplacian-tutorial.pdf

We note that $V(J) = (v_1, v_2, v_3, v_3)$ and $E(J) = (\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\})$ and the Laplacian matrix is

$$L_J = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

with eigenvalues (0,4,4,4). We observe that, as by definition, the diagonal entries correspond to the number of edges connecting each vertex. The rows of each diagonal, only paying attention to the entries to the right of the diagonals due to the symmetry of the definition, show the specific connections. For instance, v_1 has two edges, $\{(v_1, v_2), (v_1, v_3)\}$ and therefore $L_{1,1} = 2$ and $L_{1,2}, L_{1,3} = -1$, indicating a connection between the appropriate vertices. After a simple computation, we note for future reference that the eigenvalues of L_J are $(0, 1, 3, 4)$. We will now calculate the Laplacian matrix of a connected graph, K^4 shown in Figure 2:

$$L_{K_4} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

with eigenvalues $(0, 4, 4, 4)$. Note that the smallest non-zero eigenvalue of the Laplacian matrix of the connected graph K^4 is larger than that of J. We will explore the meaning of this after proving general facts about the Laplacian matrix.

3.2 Lemma - L_G is self-adjoint

By the definition of L_G, the entries of L_G are real and it is clear that $L_G^T = L_G$ since

$$L_G^T = \sum_{\{u,v\}\in E} L_{G_{\{u,v\}}}^T = \sum_{\{u,v\}\in E} L_{G_{\{u,v\}}} = L_G \tag{4}$$

A matrix M is self-adjoint if $M^\dagger = M$ where $M^\dagger$ is the conjugate-transpose of M. This requirement reduces to $M^T = M$ if the entries of M are real. Therefore, the result follows that L_G is self-adjoint.

3.3 Lemma - All eigenvalues of L_G are real

If λ is an eigenvalue of L_G then $\lambda \in \mathbb{R}$ since L_G is self-adjoint (Linear Algera Done Right, 7.13).

3.4 Lemma: L_G is positive semi-definite

A matrix M is positive semi-definite if $x^T M x \geq 0 \; \forall x \in \mathbb{R}^n$.

$$x^T L_{G_{\{u,v\}}} x = (x_n - x_v)^2 \; \forall x \in \mathbb{R}^n$$

$$x^T L_G x = \sum_{\{u,v\}\in E} x^T L_{G_{\{u,v\}}} x = \sum_{\{u,v\}\in E} (x_u - x_v)^2 \geq 0. \tag{5}$$

3.5 Theorem - Eigenvalues of L_G are non-negative

Suppose λ is an eigenvalue of L_G and $x \in \mathbb{R}^n$ is the corresponding eigenvector. We have that

$$L_G x = \lambda x \implies x^T L_G x = x^T \lambda x = \lambda x^T x. \tag{6}$$

Since we proved in the previous lemma that $x^T L_G x \geq 0$ and we have that $x^T x \geq 0$, $\lambda \geq 0$.

3.6 Corollary - $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of L_G

We have proven that L_G is self-adjoint and has real entries, then by the Real Spectral Theorem, $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of L_G. Order the eigenvalues λ_i of L_G by $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m$.

3.7 Lemma - For any graph G, $\lambda_1 = 0$

Consider $x = (1, 1, ..., 1) \in \mathbb{R}^n$. The i^{th} entry of the vector $M = L_G x$ is

$$m_i = \sum_{k=1}^{n} L_{i,k} = 0 \tag{7}$$

since the sum of any row of L_G is 0.

$$L_G x = \mathfrak{o} \implies 0 \text{ is an eigenvalue of } L_G \implies \lambda_1 = 0. \tag{8}$$

3.8 Lemma - $\lambda_2 > 0$ for connected graphs

If $G = (V, E)$ is a connected graph then $\lambda_2 > 0$. This lemma is equivalent to proving that the multiplicity of λ_1 is 1 (i.e. dim $E(\lambda_1, L_G) = 1$)
Since 0 is an eigenvalue of L_G, we have that $\exists \mathsf{o} \neq \gamma \in \mathbb{R}^n$ eigenvector of λ_1.

$$\gamma^T L_G \gamma = \gamma^T(\mathsf{o}) = \mathsf{o} \implies \gamma^T L_G \gamma = \sum_{\{u,v\} \in E} (\gamma_u - \gamma_v)^2 = \mathsf{o} \tag{9}$$

This implies that $\forall \{u, v\} \in E$ we have that $\gamma_u = \gamma_v$ and therefore $\gamma_i = \gamma_j \; \forall i, j \in V$ by the assumption that G is connected. $\gamma = \alpha(1, 1, ..., 1)$ such that $\alpha \in \mathbb{R}$ and $E(1, L_G) = \text{span}\{(1, 1, ..., 1)\} \implies \text{dim}E(1, L_G) = 1$. Hence, the multiplicity of λ_1 is 1, which implies that $\lambda_2 > 0$. λ_2 is called the spectral gap of L_G, which is defined as a matrix's smallest non-zero eigenvalue.

4 Rayleigh Quotient

4.1 Definition of the Rayleigh Quotient

Let $\Phi \in \mathbb{R}^{n \times n}$ be a symmetric matrix ($\Phi^T = \Phi$). A quadtratic form is a polynomial that only has second-order terms. One such quadtratic form on Φ is the function $\Omega : \mathbb{R}^n \to \mathbb{R}$ defined by $\Omega x = x^T \Phi x$ where $x \in \mathbb{R}^n$.

$$\Omega(x) = x^T \Phi x = \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij} x_i x_j$$

For a matrix A, the Rayleigh quotient is defined

$$R(x) = \frac{x^T A x}{x^T x}. \tag{10}$$

Note that $R(x)$ is independent of $\|x\|$ since for any $\lambda \in \mathbb{R}$, we have that

$$R(\lambda x) = \frac{\lambda x^T A \lambda x}{\lambda x^T \lambda x} = \frac{\lambda^2}{\lambda^2} R(x) = R(x).$$

Therefore, we naturally redefine the Rayleigh Quotient as

$$\boxed{R(x) = x^T A x \text{ such that } \|x\| = 1}. \tag{11}$$

We can see that $R(x)$ is a normalized quadratic form.

4.2 Maximum and minimum eigenvalues and the Rayleigh Quotient

For a matrix $A \in \mathbb{R}^{n,n}$ we have the following characterization of the eigenvalues of A:

$$\boxed{\min_{x \in \mathbb{R}^n : \|x\| = 1} R(x) = \lambda_2}, \tag{12}$$

$$\boxed{\max_{x \neq \mathsf{o} \in \mathbb{R}^n : \|x\| = 1} R(x) = \lambda_n} \tag{13}$$

where λ_2 is the smallest non-zero eigenvalue and λ_n is the largest eigenvalue of A.

Proof:
Let Φ be a symmetric matrix. Let $\Phi = \Gamma \Lambda \Gamma^T$ be the spectral decomposition, where the basis Γ is

an orthonormal basis and Λ has the eigenvalues of Φ $(\lambda_1, ..., \lambda_n)$ on its diagonal and 0 elsewhere. Rearrange the basis list of Γ to ensure that the eigenvalues on the diagonal of Λ are in decreasing order. For any unit vector x, we have

$$x^T \Phi x = x^T(\Gamma\Lambda\Gamma x) = (x^T\Gamma)\Lambda(\Gamma^T x).$$

Let $y = \Gamma^T x$. Note that

$$\|y\|^2 = y^T y = (\Gamma^T x)^T(\Gamma^T x) = x^T\Gamma\Gamma^T x = x^T x = 1$$

and therefore $\|y\| = 1$. We then have

$$x^T\Phi x = y^T\Lambda y.$$

This redefines our Rayleigh Quotient:

$$R(y) = y^T\Lambda y.$$

Since Λ is diagonal we have

$$y^T\Lambda y = \sum_{i=1}^{n} \lambda_i y_i^2.$$

Recall that $\|y\| = \sum_{j=1}^{n} y_j^2 = 1$ and that $\lambda_1 \geq ... \geq \lambda_n$. This directly implies that $\sum_{i=1}^{n} \lambda_i y_i^2$ is maximized when $y_1 = 1$ and $y_k = 0$ $\forall k \neq 1$ which implies that

$$\min_{y\in\mathbb{R}^n:\|y\|=1} y^T\Lambda y = \lambda_2$$

and occurs when $y = (1, 0, ..., 0)$. Similarly,

$$\max_{y\in\mathbb{R}^n:\|y\|=1} y^T\Lambda y = \lambda_n$$

and occurs when $y = (0, 0, ..., 1)$. This completes the proof.

4.3 Rayleigh Quotient of the Laplacian Matrix

We will now apply the results in the previous section to the Laplacian Matrix. As we had found previously, we know that

$$g^T L_G g = \sum_{\{u,v\}\in E} (g(u) - g(v))^2$$

for any vector $g \in \mathbb{R}^n$. Further, with some thought of the Laplacian matrix, we can see that

$$g^T g = \sum_{u} (g(u))^2 d_u$$

where we recall that d_u is simply the number of edges connected to vertex u. Putting this together and recalling the definition of the Rayleigh Quoetient from the previous section, we now define the Rayleigh Quotient of the Laplacian matrix

$$\boxed{R(g) := \frac{\sum_{\{u,v\}\in E}(g(u) - g(v))^2}{\sum_u (g(u))^2 d_u}}. \tag{14}$$

5 Cheeger's Constant

5.1 Sparsest Cut Problem

One of the most important characteristics of the graph is its ability to be divided into two seperate subgraphs, a bipartition, minimizing the number of edges across the cut while maximizing the number of vertices contained in each partition. This ability is mathematically characterized by a function called the Cheeger Constant. From an application perspective, a network of computers can be seen as a graph. Bipartitioning this network prevents communication between the two sets of computers and because each partition set is large, the overall communication of the network is greatly hindered. The Cheeger Constant takes into account the number of edges that must be broken in order to form such a bipartition, therefore, weighing the practicality of such a cut. If the Cheeger Constant is high, then the network is susceptible to be bipartitioned and cause immense damage with a relatively low number of disconnections. On the other hand, if the Cheeger Constant is low, then it would require a large number of disconnections to bipartion the network into two sets of large sizes. We will now go on to mathematically define this function.

A collection of verticies is simply a subset of the vertex set $V(G)$ of a graph G, $S \subset V(G)$. A bipartion, or cut (S, K) is defined to be the partition of a graph into two disjoint subsets S, K. In essence, if we have that G is a graph, then S, K form a bipartition of G if $S \cap K = \emptyset$, $S \cup K = G$. The bipartition (S, K) of a graph G defines the edge boundary of a collection of vertices S:

$$\partial S := \{\{x, y\} \in E(G) : x \in S, y \in V(G)\backslash S = K\}. \tag{15}$$

Let S be a collection of vertices. The Cheeger ratio of S is defined

$$h_S := \frac{|\partial S|}{\mathrm{vol}(S)} \tag{16}$$

where $|\partial S|$ denotes the cardinality of ∂S and $\mathrm{vol} S = \sum_{v \in S} d(v)$ where we recall that $d(v)$ is the degree of vertex v, or the number of edges connecting v to another vertex. The Cheeger constant of a graph G is defined

$$\boxed{h_G := \min_{S \subsetneq V(G)} \left\{ h_S : 0 < |S| \leq \frac{1}{2}|V(G)| \right\}} \tag{17}$$

and is utilized to solve the Sparsest Cut Problem by finding the bipartition that is most vulnerable to be disconnected and cause a significant stop in communication between the partitioned sets and relating it to the spectral gap.

6 Cheeger's Inequality

For an undirected, connected graph G with a Laplacian matrix $L_G \in \mathbb{R}^n$, the Cheeger constant of G (h_G) and the spectral gap of G (λ_2) obey the inequality

$$\boxed{2h_G \geq \lambda_2 \geq \frac{h_G^2}{2}}. \tag{18}$$

6.1 Proof of the Upper-Bound

Let χ_S denote the $(0, 1)$ indicator vector

$$\chi_S(v) = \begin{cases} 1, & v \in S \\ 0, & \text{otherwise} \end{cases}$$

Suppose $h_G = h_S$ for some $S \subsetneq V(G)$. Consider

$$g = \chi_S - \frac{\text{vol}(S)}{\text{vol}(G)}\mathbf{1}$$

where $\mathbf{1} = (1, 1, ..., 1)$ is simply the 0-eigenvector found in 3.8. From this, we have that

$$\begin{aligned} R(g) &= \frac{\sum_{\{u,v\}\in E}(g(u) - g(v))^2}{\sum_u (g(u))^2 d_u} \\ &= \frac{\sum_{\{u,v\}\in E}((\chi_S - \frac{\text{vol}(S)}{\text{vol}(G)}\mathbf{1})(u) - (\chi_S - \frac{\text{vol}(S)}{\text{vol}(G)}\mathbf{1})(v))^2}{\sum_u ((\chi_S - \frac{\text{vol}(S)}{\text{vol}(G)}\mathbf{1})(u))^2 d_u} \\ &= \frac{\sum_{\{u,v\}\in E}((\chi_S(\mathbf{1}u) - \frac{\text{vol}(S)}{\text{vol}(G)}(\mathbf{1}u)) - (\chi_S(v) - \frac{\text{vol}(S)}{\text{vol}(G)}(\mathbf{1}u))^2}{\sum_u (\chi_S(\mathbf{1}u) - \frac{\text{vol}(S)}{\text{vol}(G)}(\mathbf{1}u))^2 d_u} \\ &\leq 2\frac{|\partial S|}{\text{vol}\, S} \end{aligned} \tag{19}$$

where the first three equalities come from basic definitions of the Rayleigh Quotient and g. The final line comes from noticing that the sum in the denominator (in line 3) is greater than or equal to $1/2 \sum_u d_u = 1/2 \,\text{vol}\, S$ by noting that $\frac{\text{vol}(S)}{\text{vol}\, G} \leq 1/2$ by construction. Further, we have that the sum in the numerator (in line 4) is less than or equal to the cardinality of the edge-boundary by noting that the indicator function is only positive for both u and v (elements of S), if there is an edge connecting them together. Therefore, putting it all together, we have that

$$\boxed{\lambda_2 \leq R(g) \leq 2h_G = 2h_S = 2\frac{|\partial S|}{\text{vol} S}}$$

where $\lambda_2 \leq R(g)$ is a direct consequence of the definition of the Rayleigh Quotient. Note that $2h_G$ is an upper-bound for λ_2. Having proved the upper-bound, the proof reduces to producing a lower-bound for λ_2 in terms of Cheeger ratios.

6.2 Proof of the Lower-Bound

Let $\lambda_2 = R(g)$, implying that g is a λ_2-eigenvector and consequently, $\sum_v g(v)d_v = 0$ by noting our calculation in section 4.3 and applying that g is an eigenvector.

We then arrange the vertices such that we achieve

$$g(v_1) \geq g(v_2) \geq ... \geq g(v_n).$$

Let $S_i = \{v_1, ..., v_i\}$ and define the variable $\alpha_G = \min_i h_{S_i}$, the minimum cheeger constant of the set $\{S_i\}_{i\in(1,...,n)}$. Let $r \in (1, ..., n)$ be the largest integer such that $\text{vol}S_r \leq \text{vol}(G)/2$. Due to our assumption that $\sum_v g(v)d_v = 0$, we have that

$$\sum_v g(v)^2 d_v \leq \sum_v (g(v) - g(v_r))^2 d_v$$

where the inequality comes from noting that $g(v_r)$ is non-negative by construction. Note that summing index is over $\{v_1, ..., v_r\}$. For the sake of simplicity, we will define the positive and negative part of $g - g(v_r)$,

$$g_+(v) = \begin{cases} g(v) - g(v_r) & g(v) \geq g(v_r) \\ 0 & \text{otherwise,} \end{cases}$$

$$g_-(v) = \begin{cases} |g(v) - g(v_r)| & g(v) \leq g(v_r) \\ 0 & \text{otherwise.} \end{cases}$$

By our assumption that $\lambda_2 = R(g)$, we have that

$$\begin{aligned} \lambda_2 &= \frac{\sum_{\{u,v\}\in E}(g(u) - g(v))^2}{\sum_v g(v)^2 d_v} \\ &\geq \frac{\sum_{\{u,v\}\in E}(g(u) - g(v))^2}{\sum_v (g(v) - g(v_r))^2 d_v} \\ &\geq \frac{\sum_{\{u,v\}\in E}((g_+(u) - g_+(v))^2 + (g_-(u) - g_-(v))^2)}{\sum_v (g_+(v)^2 + g_-(v)^2) d_v} \end{aligned} \tag{20}$$

where the first equality comes from the definition of the Rayleigh Quotient, the first inequality comes from our result that $\sum_v g(v)^2 d_v \leq \sum_v (g(v) - g(v_r))^2 d_v$, and the second inequality comes from rewriting $g(u), g(v)$ in terms of their positive and negative parts as defined above, expanding and dropping non-negative terms.
Define the following terms

$$a = (g_+(u) - g_+(v))^2, b = (g_-(u) - g_-(v))^2, c = (g_+(v)^2 d_v), d = (g_-(v)^2) d_v.$$

Without loss of generality, assume $R(g_+) \leq R(g_-)$.
Notice that $R(g_+) = a/c$ and $R(g_-) = b/d$ and consider the fact that

$$\frac{a+b}{c+d} \geq \min\{\frac{a}{c}, \frac{b}{d}\}$$

which implies that

$$\boxed{\lambda_2 \geq \frac{a+b}{c+d} = \min\{\frac{a}{c}, \frac{b}{d}\} \geq R(g_+)}$$

by our assumption.
We introduce the notation

$$\tilde{\text{vol}(S)} = \min\{\text{vol}(S), \text{vol}(G) - \text{vol}(S)\}$$

to make the following derivation less clustered. Recall that $\alpha_G = \min_i h_{S_i}$, which combined with the definition above, leads us to the inequality

$$\alpha_G \leq \frac{|\partial S_i|}{\tilde{\text{vol}}(S_i)}$$

with equality when S_i achieves the minimum Cheeger ratio of the set $\{S_i\}_{i\in(1,\ldots,n)}$.

$$\begin{aligned} \lambda_2 &\geq R(g_+) \\ &= \frac{\sum_{\{u,v\}\in E}(g_+(u) - g_+(v))^2}{\sum_u g_+^2(u) d_u} \\ &= \frac{\sum_{\{u,v\}\in E}(g_+(u) - g_+(v))^2)(\sum_{\{u,v\}\in E}(g_+(u) + g_+(v))^2)}{\sum_u (g_+^2(u) d_u \sum_{\{u,v\}\in E}(g_+(u) + g_+(v))^2} \end{aligned} \tag{21}$$

where we simply multiplied by $\frac{\sum_{\{u,v\}\in E}(g_+(u)+g_+(v))^2}{\sum_{\{u,v\}\in E}(g_+(u)+g_+(v))^2}$.

$$\geq \frac{(\sum_{\{u,v\}\in E}(g_+(u)^2 - g_+(v)^2))^2}{2(\sum_u g_+(u)^2 d_u)^2} \tag{22}$$

The previous line follows from noting that $(a+b)^2 = a^2 + 2ab + b^2 \leq 2(a^2+b^2)$ which implies that $\frac{1}{(a+b)^2} \geq \frac{1}{2(a^2+b^2)}$. By letting $a = g_+(v)$, $b = g_-(u)$, this leads us to the inequality $\sum_{\{u,v\}\in E}(g(u) + g(v))^2 \leq 2\sum_{\{u,v\}\in E}(x(u)^2 + x(v)^2)$ which then equals $2\sum_u = d_u g(u)^2$ by changing the index and noting the definition of the degree d_u as the number of edges in contact with u. We can then get the inequality $\sum_{\{u,v\}\in E}(g_+(u)^2 + g_+(v)^2))^2 \geq \sum_{\{u,v\}\in E}(g_+(u)^2 - g_+(v)^2))^2$ trivially by noting the change in sign. This completes the previous step, we now continue:

$$= \frac{(\sum_i \left|g_+(v_i)^2 - g_+(v_{i+1})^2\right| |\partial S_i|)^2}{2(\sum_u g_+(u)^2 d_u)^2} \tag{23}$$

The previous line follows from noting that $\sum_{u\sim v}(g_+(u)^2 - g_+(v)^2) = \sum_i \left|g_+(v_i)^2 - g_+(v_{i+1})^2\right| |\partial(S_i)|$ through a change of index which becomes more clear by noting that $|\partial(S_i)| = \partial(S_i)$ since it is a positive number, and it serves to account for the number of boundary edges naturally counted for in the previous summing index.

$$\geq \frac{(\sum_i \left|g_+(v_i)^2 - g_+(v_{i+1})^2\right| \alpha_G \left|\tilde{\text{vol}}(S_i)\right|)^2}{2(\sum_u g_+(u)^2 d_u)^2} \tag{24}$$

The previous line follows from recalling that $|\partial S_i| \geq |\alpha_G| \tilde{\text{vol}}(S_i)$ and simply substituing for $|\partial S_i|$.

$$= \frac{\alpha_G^2}{2} \frac{(\sum_i g_+(v_i)^2)(\left|\tilde{\text{vol}}(S_i) - \tilde{\text{vol}}(S_{i+1})\right|))^2}{(\sum_u g_+(u)^2 d_u)^2} \tag{25}$$

The previous line follows from pulling out α_G from the sum since it is independent from the summing index. Further, we note that $|g_+(v_i)^2 - g_+(v_{i+1})^2| \left|\tilde{\text{vol}}(S_i)\right| = g_+(v_i)^2 \left|\tilde{\text{vol}}\, S_i - \text{vol}(\tilde{S}_{i+1})\right|$ which is a clear and direct result from distributing and observing the indices.

$$\begin{aligned} &= \frac{\alpha_G^2}{2} \frac{(\sum_i g_+(v_i)^2 d_{v_i})^2}{(\sum_u g_+(u)^2 d_u)^2} \\ &= \frac{\alpha_G^2}{2} \text{ by cancelling out the second fraction} \end{aligned} \tag{26}$$

By recalling that $\alpha_G = \min_i h_{S_i}$, which does not necessarily achieve the Cheeger constant of G (unless one of the S_i is the appropriate set), we have that $\frac{\alpha_G^2}{2} \geq \frac{h_G^2}{2}$. By the above we found that $\lambda_2 \geq \frac{\alpha_G^2}{2}$ and therefore we have proven the lower-bound for λ_2. Hence we now have

$$\boxed{2h_G \geq \lambda_2 \geq \frac{h_G^2}{2}},$$

proving the Cheeger Inequality (eq. 18).

7 References

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