Gauge Theory in the Ising Model

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Spontaneous symmetry breaking is characterized by an order parameter which describes the quantity which breaks the symmetry under consideration. In the Ising model, the order parameter is the local magnetization. The emergence of order in the Ising model is due to the spontaneous breaking of the global $\mathbb{Z}_2$ symmetry. If the global symmetry is replaced with a local symmetry, spontaneous symmetry breaking becomes forbidden. We explore an Ising-like gauge theory and describe its phases in the absence of spontaneous symmetry breaking of a local order parameter. This paper also serves as a brief introduction to lattice gauge theories which are extensively studied in high energy physics and quantum information science.

I. GAUGE STRUCTURE

In the Ising model (IM), the spin variables are defined on the vertices of a square lattice. Instead, we place the spin variables on the edges (links) and define the gauge transformation at a particular site (vertex) as the operation which flips all spins linked to the site, amounting to $2d$ spins in d dimensions (see Figures 1 and 2). We will refer to this model as the Ising gauge theory and use the language of 2 dimensions, although the formalism developed here holds for arbitrary dimensions. We group the spins located on a unit square into a so-called plaquette and define the gauge-invariant Hamiltonian [1–3]

$$H = -J \sum_p G_p, \qquad G_p = \prod_{i \in p} \sigma_i \tag{1}$$

where G_p is the product of spins located in plaquette p. Notice that H is gauge-invariant as each G_p is gauge-invariant. We introduce a uniform magnetic field which acts on all spins. In the absence of a magnetic field, there are infinitely-many ground states related by a gauge transformation. One such ground state has all spins aligned, but we can gauge transform this ground state by flipping the spins of an arbitrary number of plaquettes with no cost to the energy.

If the expectation value of the magnetization is non-vanishing, then there is spontaneous symmetry breaking of the gauge structure since the magnetization is not gauge-invariant. To reach the opposite ground state in the IM, all spins must flip, which carries an extensive energy cost. In other words, the two ground states are energetically far away from each other since they are related by a global transformation. In the case of a gauge structure, there is an extensive number of ground states. They are related by a gauge transformation and therefore the energy cost in switching ground states is finite. Consequently, the system can traverse the entire ground state manifold, and we obtain a vanishing expectation value for the magnetization.

The above observation is an instance of a much more general result which states that gauge structures cannot be spontaneously broken. The key principle that distinguishes a gauge structure and a global symmetry, such as

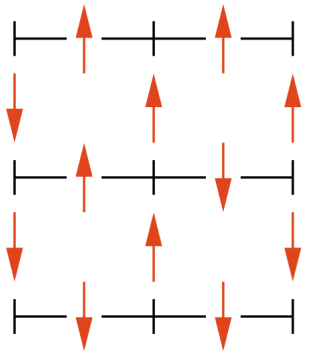

FIG. 1. Spins are defined on the edges of a d-dimensional grid. A plaquette is defined as the set of spins living on a square (or cube/hypercube in higher dimensions).

$\mathbb{Z}_2$ symmetry in the IM or SO(3) symmetry in Euclidean space, is that a gauge structure is a *local* symmetry (or redundancy). As we saw above, locality gives the system more freedom to traverse the state space which forbids symmetry breaking. This result is known as Elitzur's theorem which implies that for an observable to obtain a non-zero expectation value, it must be gauge-invariant [4]. Local observables such as the magnetization are no longer relevant in characterizing phase transitions since they are not gauge-invariant, and therefore always evaluate to zero.

II. PHASE TRANSITION

In light of Elitzur's theorem, we turn our attention to gauge-invariant observables. In the XY-model, the phase transition is of topological origin and is characterized by a correlation function. It is therefore natural to investigate correlation functions, and one of central importance is the Wilson loop [1–3]

$$\langle W_C \rangle = \left\langle \prod_{l \in C} \sigma(l) \right\rangle \tag{2}$$

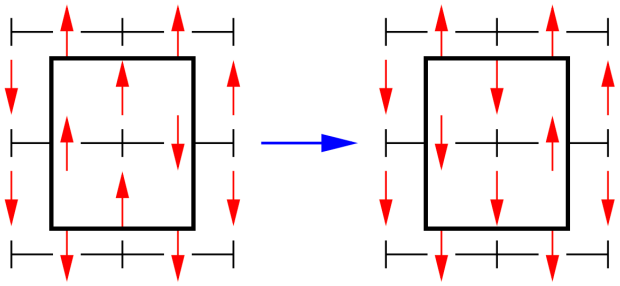

FIG. 2. A gauge transformation is defined with respect to a particular vertex. All spins linked to this vertex are flipped while the remaining spins on the lattice are left unchanged. Visually, it is evident that plaquette operators are invariant under this transformation.

defined as a product of spin variables along a closed contour C. There are no constraints on the geometry of this contour (i.e. it may self-intersect). The Wilson loop is a discretized version of the line integral of a gauge potential over a closed contour. In electromagnetism, we have a U(1) gauge potential, and its integral over a closed contour equals the magnetic flux through that contour. For the reader who is more familiar with condensed matter physics, the Berry potential also plays the role of a gauge potential where the flux is the Berry phase. The Wilson loop in the current consideration has a similar flux analogy which we discuss below. To see that this quantity is gauge-invariant, observe that for each vertex the contour intersects, it must enter the vertex through some link and leave through some link. A gauge transformation centered at this vertex then changes two spin variables in the product, leaving it invariant. At generic temperature, it is challenging to compute this observable and so we will instead compute the behavior in the low and high temperature regimes.

To investigate the high temperature regime ($\beta \to 0$), we Taylor expand the Boltzmann weight

$$\begin{aligned} e^{\beta G_p} &= \sum_{n=0}^{\infty} \frac{1}{n!} (\beta G_p)^n \\ &= \cosh\beta + \sinh\beta \cdot G_p \\ &= \cosh\beta (1 + \tanh\beta \cdot G_p) \end{aligned} \tag{3}$$

where we used that $G_p^2 = 1$, and so we obtain

$$\langle W_C \rangle \propto \sum_{\{\sigma\}} \prod_{l \in C} \sigma(l) \prod_p (1 + \tanh\beta \cdot G_p). \tag{4}$$

Note that

$$\begin{aligned} \sum_{\{\sigma\}} \prod_{l \in C} \sigma(l) &= \sum_{\{\sigma \in C\}} \prod_{l \in C} \sigma(l) \sum_{\{\sigma \notin C\}} 1 \\ &= \prod_{l \in C} \sum_{\sigma(l) = \pm 1} \sigma(l) \sum_{\{\sigma \notin C\}} 1 = 0 \end{aligned} \tag{5}$$

where the last equality arises from the vanishing sum over $\sigma(l)$. To have non-vanishing contributions to the expectation value, we must have even powers of the spin variables on the contour since $\sigma^2 = 1$. This can be achieved by including terms from the factor $\prod_p (1 + \tanh\beta \cdot G_p)$. Since we are in the high temperature regime, we keep only the lowest powers of $\tanh\beta$ that contribute. For a spin variable σ_c on the contour, we can multiply it by the spin variables of a plaquette p on the interior of the contour with link σ_c. However, this also multiplies all the other spin variables on the plaquette, and therefore we must also find matches for those using other plaquettes so that all spin variables are squared to 1. This amounts to including all plaquettes in the interior of the contour, and since we want the lowest order contribution, we need the minimal surface bounded by C. In $d = 2$, the interior surface of C is the only surface but for higher dimensions, there are many contributing surfaces. We are not interested in plaquettes outside of the contour since they cannot be squared by a finite number of plaquettes (at least in $d = 2$). Therefore, we obtain $\langle W_C \rangle \propto (\tanh\beta)^{N_p}$ where $N_p \propto A_C$ is the number of plaquettes inside the minimal surface of area A_C bounded by C, or equivalently [2]

$$\langle W_C \rangle_{\beta \to 0} \propto e^{-f(\beta) A_C}, \tag{6}$$

for some well-behaved function $f(\beta)$. The expectation value follows an area law.

For the low temperature expansion ($\beta \to \infty$), we consider excitations (droplets) of the ground state. A single spin flip will frustrate $2(d-1)$ plaquettes (i.e. they are in an energetically disfavored state), and the change in energy per plaquette is $4(d-1)J\beta$. Assuming there are N_l links, we have the low-order expansion of the partition function

$$Z = e^{JN_p\beta} \cdot \left(1 + N_l e^{-4(d-1)J\beta} + \dots\right) \tag{7}$$

where the first term corresponds to the ground state and N_p is the number of plaquettes in the system. The numerator of the Wilson loop expectation value can be similarly expanded

$$\begin{aligned} Z \langle W_C \rangle = e^{JN_p\beta} \cdot \Big(1 &+ (N_l - L_C) e^{-4(d-1)J\beta} \\ &- L_C e^{-4(d-1)J\beta} + \dots\Big) \end{aligned} \tag{8}$$

where the $N_l - L_C$ term describes the combinatorial factor in which the droplet is not on the contour (and so $W_C = 1$ in the numerator for that spin configuration), and the $-L_C$ term denotes when the droplet is on the contour (and so $W_C = -1$). Here, L_C denotes the length of the contour. We generalize Z to include n independent (i.e. not adjacent) droplets, which has combinatorial factor $\binom{N_l}{n} \approx N_l^n / n!$ in the thermodynamic limit. The numerator can be generalized to obtain

$$\langle W_C\rangle = \frac{1+(N_l-2L_C)e^{-4(d-1)J\beta}+\frac{1}{2}(N_l-2L_C)^2e^{-8(d-1)J\beta}+\dots}{1+N_le^{-4(d-1)J\beta}+\frac{1}{2}N_l^2e^{-8(d-1)J\beta}+\dots} = \exp\Big(-2L_Ce^{-4(d-1)\beta}\Big) \tag{9}$$

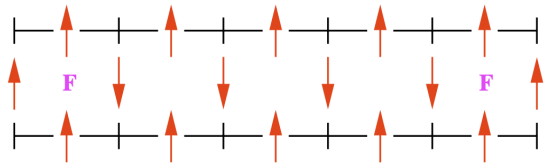

FIG. 3. In a background ground state (all states spin-up), an excitation is introduced where the spins in the middle row are flipped. The energy of this excitation is localized to the ends of the row where they are identified with so-called frustrated (F) plaquettes which are in an energetically disfavored state. The domain wall is only at the end points of a line of excitations, and is therefore 0-dimensional.

and hence [2]

$$\langle W_C\rangle_{\beta\to\infty} \propto e^{-g(\beta)L_C} \tag{10}$$

for some well-behaved function $g(\beta)$. The expectation value follows a perimeter law.

To summarize, a phase transition occurs at some temperature for all dimensions of the Ising gauge theory (IGT), characterized by the Wilson loop expectation value. This operator is equal to the product of plaquettes located in any 2-dimensional surface bounded by the contour (there is no need to do a temperature expansion to see this). In the high temperature regime, the Wilson loop operator is exponentially suppressed by the minimal area bounded by the contour, and consequently there is limited proliferation of the plaquette flux, defined as the sum of the plaquette operators, $\sum_{p\in A_C} G_p$. In the low temperature regime, a perimeter law is obeyed (weaker), and so there is stronger proliferation of plaquette flux. In other words, the plaquettes are more favored to align in a given contour. In the language of electromagnetism, there is proliferation of magnetic flux in the low temperature regime, while in the high temperature regime, this proliferation is suppressed.

The Ising gauge theory in $d = 2$ is a special case as there is no phase transition. In other words, the phase transition *occurs* at $\beta = \infty$. To see why there is no phase transition at non-zero temperature, we consider a Peierls-type argument. If we consider excitations from the ferromagnetic ground state, the domain walls are 0-dimensional, and therefore, using the same argument as the 1D IM, the proliferation of droplets is favored which prevents order (see Figure 3).

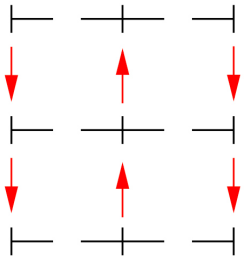

FIG. 4. The lattice following the gauge choice in which we freeze the spin variables along the x-axis. Each row is a copy of the 1D IM, and distinct rows do not interact.

III. DUALITY

The connection between the 2D IGT and the 1D IM extends even further as they are dual [1–3]. Consider the gauge $\sigma_n^{\hat{x}} = 1$ where we set all the links along the $\hat{x}$-axis to 1. This is a gauge choice in that every configuration of spins can be gauge transformed to a configuration in which all the spins along the $\hat{x}$-axis are just 1. The Hamiltonian then becomes [1–3]

$$H = -J\sum_{n,\mu=\hat{y},\hat{\tau}} \sigma_n^{\hat{y}}\sigma_{n+\hat{\mu}}^{\hat{y}} \tag{11}$$

where we set the two $\hat{x}$-spin variables in each plaquette operator to 1. This is exactly (many copies of) the Hamiltonian of the 1D IM, and therefore, the 2D IGT is in the universality class of the 1D IM.

Further, it can be shown that the 3D IGT is dual to the 3D IM [1–3]. At first, it might seem strange that they are dual since the IM is characterized by a local order parameter, while Elitzur's theorem forbids describing the IGT by such a parameter. However, this is not an issue since the duality transformation is non-local, and so their coexistence in the same universality class does not imply that the phase transition in the IGT is driven by spontaneous symmetry breaking of a local order parameter. Under this transformation, the magnetization of the IM is mapped to the so-called kink operator which acts on the ferromagnetic ground state by flipping all $\hat{x}$-spins linked to and below the site n. Such excitations are topological defects called kinks, similar to vortices in the Kosterlitz-Thouless transition, which proliferate in the high temperature regime. This duality is generalized to higher dimensions [3].

IV. RELATION TO OTHER FIELDS

Lattice gauge theories are of considerable importance in high energy physics where they are used to model gauge theories on discrete space-time lattices. In particular, they are used to study non-Abelian gauge theories such as Quantum Chromodynamics (theory of strong interaction), particularly the confinement of quarks [1]. The discretization of space-time allows for the use of numerical methods more suitable for lattice models. Lattice Quantum Chromodynamics is an inherently non-perturbative theory in the low energy regime and relies on complex numerical simulations to compute properties of quarks and gluons [5]. In quantum information science, lattice models are used as platforms for encoding qubits that are robust against noise, so-called stabilizer codes. The qubit is encoded in topological objects like kinks [6]. Their topological nature may enable noise-resilient quantum computation.

In both high energy physics and quantum information science, lattice gauge theories are quantized (the spin variables are elevated to non-commuting operators). In addition, a matter field is included that describes some fundamental constituent of matter which couples to the gauge field. These fields are incredibly rich and contain many open problems related to lattice gauge theories, including studying Yang-Mills theories on lattices.

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