DRP: Lie Algebra

Amir Shapour Mohammadi

1 Introduction

This paper delves into the nature of groups which are endowed with the structure of a differentiable manifold, focusing on the connection between group elements infinitesimally close to the identity and the group itself. The idea and properties of a Lie group and its corresponding Lie algebra will be explored. Examples of deriving Lie algebras and other calculations of interest are provided. In particular, the Lie groups of $SO(3)$ and $SU(2)$ will be explored in depth and their connection will be demonstrated. Most of this material has been sourced from [1], [2], [3].

2 Lie Group

A Lie Group is a group which has the structure of a differentiable manifold such that multiplication and inverse are smooth maps. Equivalently, Lie Groups are continuous (and non discrete) groups which have differentiable structure. For instance, the group of rotations of a circle along the $\mathbb{R}^2$ plane is a continuous group, denoted $SO(2)$ in group notation and S^1 in manifold notation. On the other hand, the group of rotations of a square along the $\mathbb{R}^2$ plane is not continuous since its the elements of the rotational symmetry group are of the form $n\frac{\pi}{2}$ where $n \in \mathbb{Z}/4$ and the group is therefore discrete. The most common Lie groups are matrix Lie groups which are isomorphic to matrix groups. For the rest of this paper, Lie groups will be assumed to be matrix groups unless specified otherwise.

2.1 Examples of Matrix Groups

A matrix group is a set of matrices that satisfies the group axioms with the operation of matrix multiplication.

$$M_n(\mathbb{R}) = \{X \text{ is an } n \times n \text{ matrix: } x_{ab} \in \mathbb{R}\ \forall a, b\}$$

$$M_n(\mathbb{C}) = \{X \text{ is an } n \times n \text{ matrix: } x_{ab} \in \mathbb{C}\ \forall a, b\} \supseteq M_n(\mathbb{R})$$

where x_{ab} denotes the elements of X

$$GL(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) : \det X \neq 0\} \subseteq M_n(\mathbb{R})$$

$$GL(n, \mathbb{C}) = \{X \in M_n(\mathbb{C}) : \det X \neq 0\} \subseteq M_n(\mathbb{C})$$

$$Sp(2n, \mathbb{R}) = \{X \in M_{2n}(\mathbb{R}) : X^T \Omega X = \Omega\} \subseteq M_{2n}(\mathbb{R})$$

where $\Omega = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$.

3 Exponential Map

The exponential map for matrices is defined as:

$$\boxed{\exp(X) = \lim_{n \mapsto \infty} (I + \frac{X}{n})^n = \sum_{n=0}^{\infty} \frac{X^n}{n!}}$$

where X is a matrix and I denoted the identity element of the Lie group. Analogous to the definition of the exponential function for real numbers, the exponential function for matrices is based on the idea of generating all elements from infinitesimal elements (near the identity of the continuous group). By focusing on the infinitesimal elements of a continuous group (the tangent space), all elements of the group can be generated through the exponential map (function mapping the tangent space, defined below, to the Lie group). While the exponential map in this section is defined for matrices and matrix groups, it can be defined for all Lie groups using more general manifold concepts.

4 Tangent Space

The tangent space of a Lie group is the vector space at the identity of the Lie group ($\theta = 0$ for $SO(2)$) and has the same dimension as the Lie group. Building upon the notion of the exponential map, the tangent space contains the elements of the Lie group that are infinitesimally close to the identity. Since every Lie group is a differentiable manifold, there exists parameterized curves on the manifold with origin at the group identity:

$$\boxed{T_I G = \{\gamma'(0) : \gamma(0) = I\}}$$

where γ is a smooth parameterized curve. If $G \subseteq GL_n(\mathbb{F})$, then $\gamma'(0) \in M_n(\mathbb{F})$. Further, if G is a matrix group, then $T_I G$ is a vector space of matrices.

5 Lie Bracket for Matrix Lie Groups

The tangent space with the additional structure of the Lie bracket operation forms the Lie algebra of the Lie group. The Lie bracket is a fundamental notion of measuring the anti-commutivity of a Lie group.

$$\boxed{[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}}$$

While the Lie bracket in this section is defined for matrices and matrix groups, it can be defined for all Lie groups using more general manifold concepts. For matrix Lie groups, the Lie bracket is defined:

$$\boxed{[X, Y] = XY - YX \in \mathfrak{g}}$$

The Lie bracket satisfies the following axioms: Let $a, b \in \mathbb{F}$, and $x, y, z \in \mathfrak{g}$

1. Billinearity

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

2. Alternitivity

$$[x, x] = 0 \ \forall x \in \mathfrak{g}$$

3. Jacobi Identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \ \forall x, y, z \in \mathfrak{g}$$

4. Anti-commutivity

$$\text{Bilinearity, alternativity} \implies [x, y] = -[y, x] \ \forall x, y \in \mathfrak{g}$$

See definition 2.11 of [1] for more detailed properties of Lie brackets.

As previosuly stated, the Lie bracket measures the anti-communitivity of a Lie group. For instance, in $SO(2)$, the group of rotations in $\mathbb{R}^2$, rotations are communitive. Rotating an object in $\mathbb{R}^2$ is communitive in that $\theta \circ \phi = \phi \circ \theta$ (where $\theta, \phi \in SO(2)$) and the Lie bracket reveals this property in that $[\theta, \phi] = \theta\phi - \phi\theta = 0$. Hence θ and ϕ are said to commute. On the other hand, in $SO(3)$, the group of rotations in $\mathbb{R}^3$, rotations are not communitive ($[\omega, \tau] = \omega\tau - \tau\omega \neq 0$ such that $\omega, \tau \in SO(3)$) [2].

5.1 Proposition: Closure of the Matrix Lie Bracket

Let $u(s)$ and $v(t)$ be paramterized curves along the Lie group G such that $u(0) = v(0) = I$ ($u(s), v(t) \in T_I G$). By fixing s, one can form another parameterized curve $w_s(t)$ such that

$$w_s'(t) = u'(s)v(t)u(s)^{-1}$$

Taking the derivative at $t = 0$ yields another curve, $w_s'(0) \in T_I G$:

$$w_s'(t) = u(s)v'(t)u(s)^{-1}$$

$$w_s'(0) = u(s)v'(0)u(s)^{-1}$$

$w_s'(0)$ is a tangent vector at I for each s. By letting s vary:

$$x(s) = u(s)v'(0)u(s)^{-1}$$

$$\left.\frac{d}{ds}\right|_{s=0} x(s) = \left.\frac{d}{ds}\right|_{s=0} (u(s)v'(0)u(s)^{-1}) = u'(0)v'(0)u(0)^{-1} + u(0)v'(0)(-u'(0)) = UV - VU$$

by letting $V = v'(0)$ and $U = u'(0)$.

6 Lie Algebra

The Lie algebra is found by using the defining equation of a Lie group to find the set of curves in the Lie group that are in the tangent space. Applying the exponential map to elements of the Lie algebra produces elements of the Lie group.

$$\boxed{\exp : \mathfrak{g} \mapsto G}$$

Theorem: For a matrix Lie group G, the Lie algebra $\mathfrak{g} = T_I(G)$ is isomorphic to the vector space of matrices X:

$$\boxed{\mathfrak{g} \cong \{X : \exp(tX) \in G \ \forall t \in \mathbb{R}\}}$$

An example of finding a Lie algebra is worked out in Section 8. See p.93 of [2] for more details regarding the notion that the Lie algebra is the set of matrices which is mapped to the Lie group using the exponential map.

7 Ideals

7.1 Definitions

Normal subgroup: $H \lhd G := \{H \subset G : ghg^{-1} \in H \text{ for any fixed } h \in H, \forall g \in G\}$

A group is called simple if the normal subgroups it contains is only itself and the identity.

G is simple: $\nexists\, H \subseteq G$ such that $ghg^{-1} \in H, H \neq I$

An ideal of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is closed under Lie brackets with arbitrary elements of $\mathfrak{g}$

$\mathfrak{h}$ **is an ideal of** $\mathfrak{g}$**:** $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[a, b] \in \mathfrak{h}\ \forall a \in \mathfrak{g}, \forall b \in \mathfrak{h}$

A Lie algebra is called simple if the ideals is contains is only itself.

$\mathfrak{g}$ **is simple:** $\nexists\, \mathfrak{h} \subseteq \mathfrak{g}$ such that $[a, b] \in \mathfrak{h}\ \forall a \in \mathfrak{g}, \forall b \in \mathfrak{h}$

A common method of proving the simplicity of a Lie algebra is to show that all elements of a basis for the Lie algebra vector space produces the entire basis when Lie bracketed with other elements of the Lie algebra in an appropriate fashion.

7.2 Connection between Normal Subgroups and Ideals

7.2.1 Statement

The tangent space of a normal subgroup of a Lie group is an ideal of the Lie algebra.

7.2.2 Proof

a. $T_I(H)$ is a subspace of $T_I(G)$
Let $H \subseteq G$ be a normal subgroup of matrix group G
$\forall a \in T_I(H)$, $a \in T_I(G)$ since $H \subseteq G$, all tangent vectors of H are tangent vectors of $G \implies T_I(H)$ is a subspace of $T_I(G)$
b. $T_I(H)$ is closed under Lie brackets with arbitary memberrs of $T_I(G)$ Let H be a normal subgroup of the Lie group G and let $X = a'(0)$, $Y = b'(0)$.

$$a \in G, b \in H \implies aba^{-1} \in H$$

Following a similar structure to the derivation of the matrix Lie bracket, $\exists$ a parametized curve:

$$C_s(t) = a(s)b(t)a(s)^{-1}$$

$$C'_s(t) = ab'(t)a(s)^{-1}$$

$$C'_s(0) = a(s)b'(0)a(s)^{-1}$$

Letting s vary, $D(s) = C'_s(0)$ is a smooth path in $T_I(H)$

$$D'(s) = a'(s)b'(0)a(s)^{-1} + a(s)b'(0)a'(s)^{-1}$$

$$D'(0) = a'(0)b'(0)a(0)^{-1} + a(0)b'(0)a'(0)^{-1} = XY - YX = [X, Y] \in T_I(H)$$

7.3 Kernel of a Lie Algebra Homomorphism is an Ideal

Statement: if $\phi : \mathfrak{g} \mapsto \mathfrak{g}'$ is a Lie algebra homomorphism (a homomorphism between Lie algebras) and

$$\mathfrak{h} = \{X \in \mathfrak{g} : \phi(X) = \mathbf{0}\}$$

is its kernel *implies* $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.
Proof that $\mathfrak{h}$ is a subspace of the Lie algebra vector space $\mathfrak{g}$:

$$X_1, X_2 \in \mathfrak{h} \implies \phi(X_1) = \mathbf{0}, \phi(X_2) = \mathbf{0}$$

$$\implies \phi(X_1 + X_2) = \mathbf{0}$$

$$\implies X_1 + X_2 \in \mathfrak{h}$$

$$X \in \mathfrak{h} \implies \phi(X) = \mathbf{0}$$

$$c\phi(X) = \mathbf{0}$$

$$\phi(cX) = \mathbf{0}$$

$$cX \in \mathfrak{h}$$

due to the properties of a homomorphism.
Proof that $\mathfrak{h}$ is closed under Lie brackets with members of $\mathfrak{g}$:

$$X \in \mathfrak{h} \implies \phi(X) = \mathbf{0}$$

$$\implies \phi([X, Y]) = [\phi(X), \phi(Y)] = [\mathbf{0}, \phi(Y)] = \mathbf{0} \ \forall Y \in \mathfrak{g}$$

$$\implies [X, Y] \in \mathfrak{h} \ \forall Y \in \mathfrak{g}$$

Q.E.D. Therefore, a Lie algebra is not simple if it admits a non-trivial homomorphism. There is great difficulty in finding classical simple Lie groups. On the other hand, proving the simplicity of a Lie algebra proves the simplicity of the Lie group, and thus the problem gets shifted to finding simple Lie groups, which is much less difficult.

7.4 Ideal of $\mathfrak{gl}(n, \mathbb{C})$

7.4.1 Normal Subgroup of $GL(n, \mathbb{C})$

Consider the map:

$$\det : GL(n, \mathbb{C}) \mapsto \mathbb{C}^{\times}$$

$$\ker(\det(GL(n, \mathbb{C}))) = SL(n, \mathbb{C}) \text{ since } \det(\Gamma) = 1 \text{ such that } \Gamma \in SL(n, \mathbb{C})$$

Since the map is a group homomorphism, it follows that $SL(n, \mathbb{C})$ is a normal subgroup of $GL(n, \mathbb{C})$ for any n. Hence by the statement proved in this section, $\mathfrak{gl}(n, \mathbb{C})$ is not a simple Lie algebra for any n.

7.4.2 Ideal of $\mathfrak{gl}(n, \mathbb{C})$

The ideal of $\mathfrak{gl}(n, \mathbb{C})$ will now be explicitly calculated
Note that $\mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$ where $M_n(\mathbb{C})$ is the group of complex matrices of dimension $n \times n$
Consider the map:

$$\mathrm{Tr} : M_n\mathbb{C} \mapsto \mathbb{C}$$

The kernel of this map is $\mathfrak{sl}(n, \mathbb{C})$ since $\mathfrak{sl}(n, \mathbb{C}) = \{\sigma \in M_n(\mathbb{C}) : \mathrm{Tr}(\sigma) = 0\}$
What remains is to show that the trace map is a Lie algebra homomorphism.
The trace map is certainly a vector space homomorphism since

$$\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B), \mathrm{Tr}(zA) = z\,\mathrm{Tr}(A) \forall z \in \mathbb{C}$$

$$\begin{aligned} \mathrm{Tr}([X, Y]) &= \mathrm{Tr}(XY - YX) \\ &= \mathrm{Tr}(XY) - \mathrm{Tr}(YX) \\ &= 0 \text{ since } \mathrm{Tr}(XY) = \mathrm{Tr}(YX) \\ &= [\mathrm{Tr}(X), \mathrm{Tr}(Y)] \end{aligned}$$

Therefore, the trace map is a Lie bracket homomorphism and thus, a Lie algebra homomorphism. Further, $\mathfrak{sl}(n, \mathbb{C})$ is an ideal of $\mathfrak{gl}(n, \mathbb{C})$.

8 Lie Algebra, Generators, Elements and Ideals of the Special Orthogonal Group of Dimension 3: $SO(3)$

8.1 Definition

$SO(3)$ represents the group of rotations in $\mathbb{R}^3$.

$$SO(3) = \{X \in SL(3, \mathbb{R}) : X^T X = I\}$$

8.2 Finding the Lie Algebra

$$\exists \text{ a smooth parameterized curve on } SO(3)$$

$$\text{Let } \alpha \text{ denote such a curve: } \alpha(a, b) \mapsto SO(3) \text{ such that } \alpha(0) = I$$

$$\boxed{\left.\frac{d}{dt}\right|_{t=0} (\alpha(t)^T \alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} (I)}$$

$$\alpha'(0)^T \alpha(0) + \alpha(0)^T \alpha'(0) = 0 \text{ since } I \text{ is a constant matrix}$$

$$\alpha'(0)^T = -\alpha'(0) \text{ since } \alpha(0) = I$$

$$\mathfrak{so}(3) \subseteq \{X \in M_3(\mathbb{R}) : X^T = -X\}$$

Proving the other direction, if $X \in \{X \in M_3(\mathbb{R}) : X^T = -X\}$, for $\epsilon > 0$ consider the curve

$$\alpha : (-\epsilon, \epsilon) \mapsto M_3(\mathbb{R}); \alpha(t) = \exp(tX)$$

$$\implies \alpha(t)^T \alpha(t) = \exp(tX)^T \exp(tX)$$
$$= \exp(tX^T)\exp(tX)$$
$$= \exp(-tX)\exp(tX) = \exp(0) = I$$

Therefore, $\alpha : (-\epsilon, \epsilon) \mapsto SO(3)$. Since $\alpha'(0) = X \implies$

$$\{X \in M_3(\mathbb{R}) : X^T = -X\} \subseteq \mathfrak{so}(3)$$

Hence,

$$\boxed{\mathfrak{so}(3) = \{X \in M_3(\mathbb{R}) : X^T = -X\}}$$

The generators for $\mathfrak{so}(3)$ are

$$\boxed{R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, R_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

8.3 Finding the Lie Brackets of the Generators

$$\boxed{[R_x, R_y] = R_z, [R_y, R_z] = R_x, [R_z, R_x] = R_y}$$

proving that the Lie algebra is indeed closed under the Lie bracket and therefore is a vector space (the other vector space properties are trivial).

8.4 Finding the Elements of the Lie Group

By applying the exponential map to the generators, the elements of the Lie group can be retrieved.

$$\boxed{\mathbb{R}_{\overline{n}}(\theta) = \exp(\theta(n_1 R_x + n_2 R_y + n_3 R_z))}$$

where $\overline{n} = (n_1, n_2, n_3)$ such that $||n|| = 1$ is an arbitrary axis of rotation and θ is the angle of rotation For instance, the rotation about the y-axis would be:

$$\mathbb{R}_{\overline{y}}(\theta) = \exp(\theta(R_y))$$

9 Lie Algebra, Generators, Elements and Ideals of the Special Unitary Group of Dimension 2: $SU(2)$

9.1 Definition

$SU(2)$ is isomorphic to the unit quaternions $\mathbb{H}$ and represents the group of rotations in $\mathbb{R}^3$

$$SU(2) = \{X \in SL(3, \mathbb{C}) : X^\dagger X = I\}$$

where $X^\dagger$ is the transpose of the complex-conjugate of X.

9.2 Finding the Lie Algebra

$$\exists \text{ a smooth parameterized curve on } SU(2)$$

$$\text{Let } \alpha \text{ denote such a curve: } \alpha(a,b) \mapsto SU(2) \text{ such that } \alpha(0) = I$$

$$\boxed{\left.\frac{d}{dt}\right|_{t=0} (\alpha(t)^\dagger \alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} (I)}$$

$$\alpha'(0)^\dagger \alpha(0) + \alpha(0)^\dagger \alpha'(0) = 0 \text{ since } I \text{ is a constant matrix}$$

$$\alpha'(0)^\dagger = -\alpha'(0) \text{ since } \alpha(0) = I$$

$$\mathfrak{su}(2) \subseteq \{X \in M_2\mathbb{C} : X^\dagger = -X\} = \{\begin{pmatrix} ia & -\overline{z} \\ z & -ia \end{pmatrix} : a \in \mathbb{R}, z \in \mathbb{C}\}$$

Proving the other direction is similar to the process shown for $SO(3)$. In addition, the $\det X = 1$ condition requires X to be traceless:

$$X \in \mathfrak{su}(2) \implies \det \exp(tX) = 1$$

$$\det \exp(tX) = \exp \operatorname{Tr}(X) \implies \exp \operatorname{Tr}(X) = 1 \implies \operatorname{Tr}(X) = 0$$

$$\boxed{\mathfrak{su}(2) = \{X \in M_2(\mathbb{C}) : X^\dagger = -X, \operatorname{Tr}(X) = 0\}}$$

The generators for $\mathfrak{su}(2)$ are

$$\boxed{u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}$$

9.3 Finding the Lie Brackets of the Generators

$$\boxed{[u_1, u_2] = u_3, [u_3, u_1] = u_2, [u_2, u_3] = u_1}$$

proving that the Lie algebra is indeed closed under the Lie bracket and therefore is a vector space (the other vector space properties are trivial).
Note that the Lie brackets of the generators of $SO(3)$ and $SU(2)$ are equal.

9.4 Finding the Elements of the Lie Group

Similar to the process for finding the elements of $SO(3)$, the elements of $SU(2)$ are of the form:

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$. Note that $SU(2) \cong \mathbb{H}$ [2].

10 Relationship between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$

Let $v \in \mathbb{R}^3$ be a vector/point and $q \in \mathbb{H} \cong SU(2)$ such that $|q| = 1$ (unit quaternion).

$$v \mapsto qvq^{-1} \text{ is a rotation of } v \text{ in } \mathbb{R}^3$$

$$\boxed{(-q)v(-q^{-1}) = qvq^{-1}}$$

Therefore, transformations of v using q and $-q$ result in the same rotation.
Recall that using $SO(3)$, a rotation in $\mathbb{R}^3$ can be represented by the matrix:

$$\Psi = \exp\left(\theta(n_1 R_x + n_2 R_y + n_3 R_z)\right)$$

where $\overline{n} = (n_1, n_2, n_3)$ such that $||n|| = 1$ is the axis of rotation and θ is the angle of rotation.
Using $SU(2)$, a rotation in $\mathbb{R}^3$ can be represented by the matrix:

$$\Phi = \begin{pmatrix} \alpha & \beta \\ -\overline{\alpha} & \overline{\beta} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$

$$\exists \text{ 2-1 homomorphism } \sigma : SU(2) \mapsto SO(3)$$
$$\exists\ \rho : SU(2)/\{\pm I\} \xrightarrow{\sim} SO(3)$$

From a topological perspective, $SO(3) \cong \mathbb{RP}(3)$ and $SU(2) \cong S^3$. In addition,

$$\dim T_I \mathbb{RP}(3) = \dim T_I \mathbb{S}^3 = 3.$$

Further, since the Lie algebra vector space is the tangent space, the anti-pode to the identity is not an element of the Lie algebra. Intuitively,

$$\boxed{\exists\ d\sigma : T_I S^3 \xrightarrow{\sim} T_I \mathbb{RP}(3) \equiv \exists\ \omega : \mathfrak{su}(2) \xrightarrow{\sim} \mathfrak{so}(3)}$$

The contrast between the Lie group homomorphism σ to the Lie algebra isomorphism ω is due to the identical behavior of elements of $SO(3)$ and $SU(2)$ infinitesimally close to the identity, in the tangent space. As demonstrated, the Lie algebra uncovers properties of the Lie group by focusing on elements infinitesimally close to the identity.

11 Bibliography

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