Topological Band Theory

Amir Shapour Mohammadi

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Princeton University

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Anomalous Quantum Hall Effect

Electron equation of motion

$$\frac{\mathrm{d}\mathsf{r}}{\mathrm{d}t} = \frac{1}{\hbar}\nabla_{\mathsf{k}} E_{\mathsf{k}} - \frac{\mathrm{d}\mathsf{k}}{\mathrm{d}t} \times \mathcal{B}(\mathsf{k}), \frac{\mathrm{d}\mathsf{k}}{\mathrm{d}t} = -\frac{e}{\hbar}\mathsf{E} \tag{1}$$

Current density

$$\mathsf{j} = \sigma \mathsf{E} + \sigma_{AH} \hat{\mathsf{n}} \times \mathsf{E} \tag{2}$$

Quantized anomalous Hall current (for band insulator)

$$\sigma^n_{AH}\hat{\mathsf{n}} = \frac{\sigma_0}{2\pi}\int_{BZ} d^2\mathsf{k}\ \mathcal{B}_n(\mathsf{k}) := \sigma_0 C_n \hat{\mathsf{n}}. \tag{3}$$

Symmetries of Chern Number

Assuming time-reversal ($\Theta \left| \mathsf{k} \right\rangle = \left| -\mathsf{k} \right\rangle$) symmetry, we have

$$\mathcal{B}(\mathsf{k}) = \Theta^{\dagger}\mathcal{B}(\mathsf{k})\Theta = -\mathcal{B}(-\mathsf{k}) \text{ (odd)}. \tag{4}$$

Assuming (space) inversion symmetry ($\mathcal{I} \left| \mathsf{r} \right\rangle = \left| -\mathsf{r} \right\rangle, \mathcal{I} \left| \mathsf{k} \right\rangle = \left| -\mathsf{k} \right\rangle$), we have

$$\mathcal{B}(\mathsf{k}) = \mathcal{I}^{\dagger}\mathcal{B}(\mathsf{k})\mathcal{I} = \mathcal{B}(-\mathsf{k}) \text{ (even)}. \tag{5}$$

Θ + inversion symmetry implies $\mathcal{B}(\mathsf{k})$ vanishes identically.

Haldane Model (1988)

Goal: craft a Chern insulator (time-reversal breaking topological insulator), not the simplest model.
Recall the continuum model for graphene

$$H = v_F(\tau_z \sigma_x k_x + \sigma_y k_y) \tag{6}$$

where σ acts on sublattice space. We must break Θ and/or chiral (sublattice, $\mathcal{C}$) symmetry to open a band gap.

$$m_R \sigma_z \ (\mathcal{C} \text{ broken, } \Theta \text{ obeyed}) \tag{7}$$

To yield non-zero Chern number, we must break Θ symmetry (i.e. make the Berry curvature have the same sign at both Dirac points)

$$m_H \tau_z \sigma_z \ (\mathcal{C}, \Theta \text{ broken}) \tag{8}$$

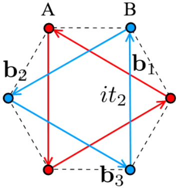
where τ acts on the valley space.

Haldane Model (tight-binding realization)

Haldane did this by adding an imaginary, or directional, (recall $\Theta \propto \mathcal{K}$) next-to-nearest neighbor hopping induced by a magnetic field ($\Phi_{net} = 0$)

$$H = \Delta \sum_{i} (-1)^{\tau_i} c_i^\dagger c_i + t_1 \sum_{NN} (c_i^\dagger c_j + h.c.) + t_2 \sum_{NNN} (i c_i^\dagger c_j + h.c.). \quad (9)$$

The new term is restricted to $\alpha\alpha$ sublattice hoppings.



Thus, we have a Chern insulator by modifying the graphene model. Can we find a topological insulator which obeys Θ symmetry (i.e. not a Chern insulator)?

Kane-Mele Model (2005), $\mathbb{Z}_2$ Index

Essentially, we consider two copies of Haldane's model, one per spin, induced by graphene's spin-orbit coupling

$$\lambda_{SO} s_z \tau_z \sigma_z \tag{10}$$

which is Θ invariant ($C = 0$). We use the Chern number of each spin system C_σ to define a topological index

$$\nu = \frac{1}{2}(C_\uparrow - C_\downarrow) \mod 2 \in \mathbb{Z}_2 \tag{11}$$

This model leads to the QSHE where there are counter-propagating electron spins on each edge (to obey Θ). However, ν is not well-defined when we include Rashba spin-orbit coupling, and in general terms which are s_z non-conserving. We now aim at deriving a $\mathbb{Z}_2$ index without the use of spin Chern numbers.

Modern Theory of (Charge) Polarization

Polarization is hard to define. Total charge polarization (for 1D system)

$$\frac{\mathrm{dP}(t)}{\mathrm{d}t} = \mathrm{j}(t), \mathrm{P} = \sum_{\text{n filled}} \int dx \langle W_{n0}(x)| \hat{x} |W_{n0}(x)\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \mathcal{A}(k) \tag{12}$$

where $\mathcal{A}(k)$ is the summed over all filled bands. P only defined modulo $e\mathrm{R}/\Omega$ (lattice vector), but $\Delta\mathrm{P}$ (due to smooth change in parameters) is gauge-invariant

$$H(k_y = -\pi) \to H(k_y = \pi) \tag{13}$$

$$\Delta\mathrm{P} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk_y \int_{-\pi}^{\pi} dk \mathcal{B}(k, k_y) = C \tag{14}$$

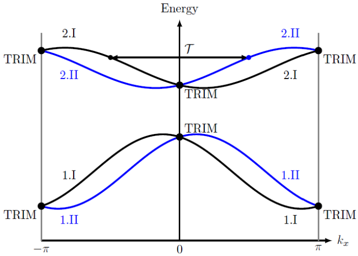
given a closed loop in parameter space. Note that $\Delta\mathrm{P}$ disappears in Θ symmetry.

Kramer's Degeneracy

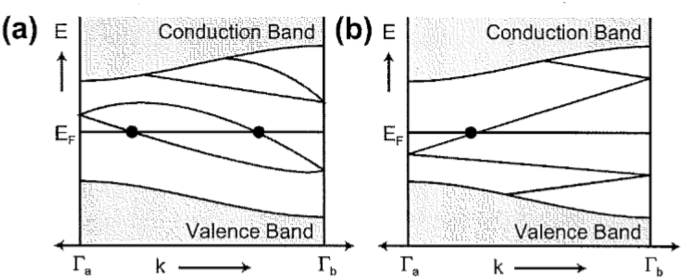
In d dimensions, there are 2^d Θ invariant momenta at the center and boundary of the BZ ($\mathrm{k}_i = -\mathrm{k}_i + G$). If our Hamiltonian is Θ symmetric then

$$H|n\rangle = E|n\rangle \rightarrow H(\Theta|n\rangle) = E(\Theta|n\rangle) \tag{15}$$

In general, $\Theta|n\rangle \neq |n\rangle$ (e.g. $\perp$ for fermionic systems). We can apply this to get a picture of what we expect the band structure of a Θ symmetric system to look like.



Intuition for $\mathbb{Z}_2$ Invariant



We define the time-reversal polarizations

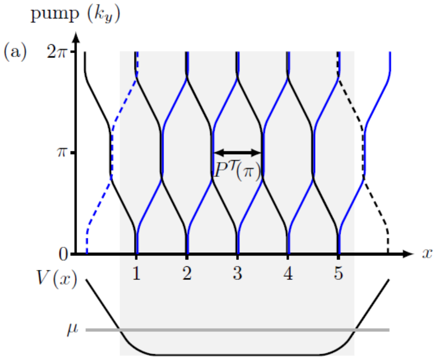
$$\mathrm{P}^s = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk A^s(k) \tag{16}$$

where $s = I, II$. Due to Θ symmetry, $\mathrm{P} = \mathrm{P}^I + \mathrm{P}^{II}$ will vanish, but the difference $\mathrm{P}^\Theta = \mathrm{P}^I - \mathrm{P}^{II}$ is generally non-zero and can be expressed

$$\mathrm{P}^\Theta = \frac{1}{i\pi}\log\left(\frac{\sqrt{\det(B(\pi))}}{\mathrm{Pf}(B(\pi))}\frac{\mathrm{Pf}(B(0))}{\sqrt{\det(B(0))}}\right) \mod 2 \tag{17}$$

where $B_{mn}(k) = \langle\phi_m(-k)|\,\Theta\,|\phi_n(k)\rangle$ is the sewing matrix. Note that P^Θ is only dependent on the Θ symmetric momenta. The expression simplifies greatly in the presence of inversion symmetry.

Pair-switching

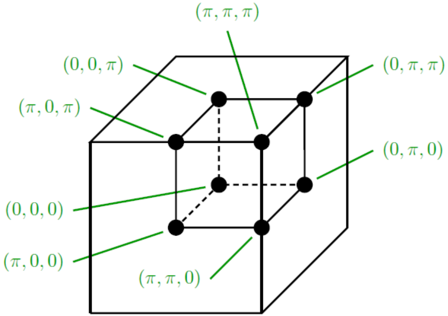


$$\nu = \prod_{l=1}^{2} \frac{\sqrt{\det B(\mathrm{k}_i)}}{\mathrm{Pf}(B(\mathrm{k}_i))}, \mathrm{v}_k = \frac{1}{\hbar}\frac{\partial E_k}{\partial k}\hat{\mathrm{y}} \tag{18}$$

Generalization to Higher Dimension

In 3D (and higher dimensions), we can define weak and strong $\mathbb{Z}_2$ topological indices respectively

$$\boldsymbol{\nu} = (\nu_s; \nu_x, \nu_y, \nu_z). \tag{19}$$



Classification of Topological Insulators

Time-reversal (Θ), charge-conjugation (Ξ), chiral symmetry (Π).

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Works Cited

Bernevig, B. Andrei, and Taylor L. Hughes. Topological Insulators and Topological Superconductors. Princeton University Press, 2013.

Links to lecture notes:

- https://www.physics.upenn.edu/~kane/pubs/chap1.pdf
- https://ethz.ch/content/dam/ethz/special-interest/phys/theoretical-physics/cmtm-dam/documents/tqn/05.pdf