Geometric Criteria for Fractional Chern Insulators

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Thesis Defense

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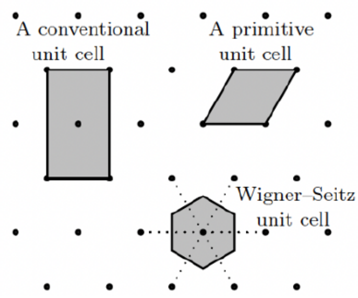
Table of Contents

Electrons in Lattices

Solids have lattice structure



Periodic structure, so we should work in momentum space

Bloch's theorem (eigenstates of periodic potential)

$$H(\mathbf{r}) = \frac{\hbar^2 \mathbf{k}^2}{2m} + U_{ion}(\mathbf{r}) \quad \rightarrow \quad f_{n,\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{N}} e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,\mathbf{k}}(\mathbf{r}) \tag{1}$$

Quantum Geometry

Quantum geometric tensor captures geometry of energy bands

$$Q_{\mu\nu} = \langle \partial_\mu \psi | \left(\mathbf{1} - |\psi\rangle \langle \psi| \right) |\partial_\nu \psi\rangle \tag{2}$$

Quantum metric, Berry curvature

$$g_{\mu\nu} = \mathrm{Re}(Q_{\mu\nu}), \qquad \Omega_{\mu\nu} = -2\mathrm{Im}(Q_{\mu\nu}) \tag{3}$$

Berry phase (time-independent) around closed loop $\gamma = \partial\Gamma$

$$\phi_\gamma = \int_\Gamma d^2k \cdot \Omega_{xy}(k) \tag{4}$$

similar to Aharonov-Bohm effect

TKNN Invariant (Chern number)

Evolution of operators in interaction picture

$$\begin{aligned} \langle A(t) \rangle_{\psi(t)} &= \langle \tilde{A}(t) \rangle_{\phi(t)} \\ &= \langle \tilde{A}(t) \rangle_0 + \frac{i}{\hbar} \int_0^t dt' \cdot \left\langle \left[\tilde{H}_1(t'), \tilde{A}(t) \right] \right\rangle_0 \end{aligned} \tag{5}$$

Kubo formula

$$\sigma_{xy}(\omega) = \frac{\text{vol}}{\hbar\omega} \int_{\mathbb{R}_{\geq 0}} dt \cdot e^{i\omega t} \langle 0 | \left[j_y, \tilde{j}_x(t) \right] | 0 \rangle \tag{6}$$

Contribution of band n to Hall conductivity

$$\sigma_{xy}^{(n)} = \sigma_0 C_n, \qquad C_n \equiv \frac{1}{2\pi} \int_{\text{BZ}} d^2\mathbf{k} \cdot \Omega_{xy}^{(n)}(\mathbf{k}) \in \mathbb{Z} \tag{7}$$

Bulk-Edge Correspondence

Chern number is a bulk property of the valence (occupied) bands

Topology is robust!

Topology changes implies gap closes, valence bands meet Fermi level at edge, and topological valence band becomes edge state (delocalized state in conduction band)

$\#$ edge states $= \Delta\mathcal{C}$ (number of band-crossings)

Edge states are chiral

Planar Electron in Magnetic Field: Landau Levels and Integer Quantum Hall Effect

Planar electron with magnetic field applied perpendicular to plane

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 = \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right) \tag{8}$$

Kinetic energy is quenched

Landau level geometry is simple (constant)

Landau level topology: $\mathcal{C} = 1$

Integer quantum Hall effect (IQHE): quantized Hall conductivity and chiral edge states in Landau level due to non-trivial topology (**single-particle effect**)

Chern Insulator: Lattice Analogue of IQHE

To have $\mathcal{C} \neq 0$, we must break time-reversal symmetry (TRS); we need a lattice analogue of the magnetic field

Haldane model: complex hoppings on honeycomb lattice to break TRS

Checkerboard model: similar idea but on a checkerboard lattice (square lattice with sublattice)

Complex hoppings mimic Aharanov-Bohm phase induced by magnetic field but can be intrinsically present from strong spin-orbit coupling

Fractional Quantum Hall Effect

Interactions play a dominant role since kinetic energy is quenched!
Need fractional filling (i.e. $\nu = 1/m$, m odd) to see effect of interactions (**many-body effect**)

Interesting physics including fractional braiding statistics and fractional charge

Fractional Chern Insulator: Lattice Analogue of FQHE

Is there a lattice-analogue of FQHE?

Chern insulator + flat bands + interactions = fractional Chern insulator (FCI)?

Recreate algebra of LLL-projected density operators (GMP algebra); **Geometric Stability Hypothesis**:

1. Berry curvature is constant,
2. Trace inequality is saturated (**trace condition**, $\operatorname{tr} g \geq |\Omega_{12}|$).

Flat bands of Haldane model, checkerboard model

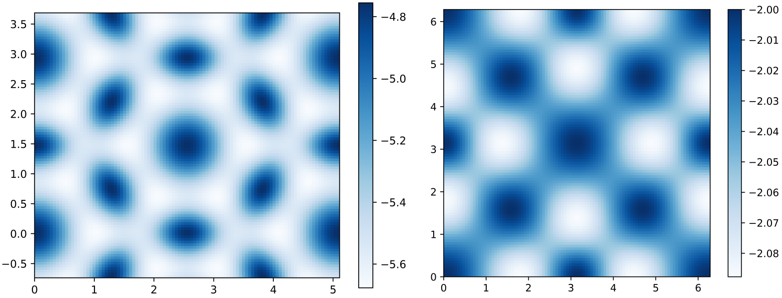


Figure: Band ratios: 0.3762, 0.0374.

Quantum geometry of Haldane model

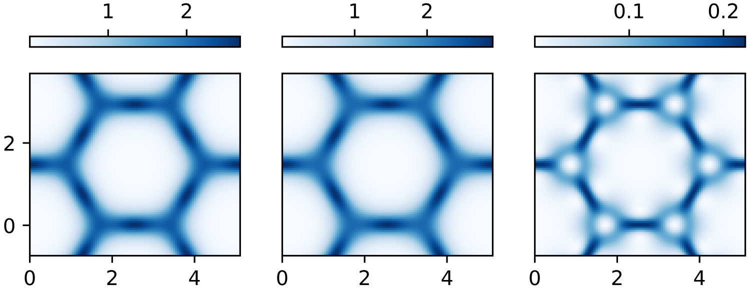


Figure: Ω_{xy}, $\mathrm{tr}\, g$, $|\mathrm{tr}\, g - |\Omega_{xy}||$. $\mathcal{C} = 0.9883$, $B = 2.3812$, $T = 0.6395$.

Quantum geometry of checkerboard model

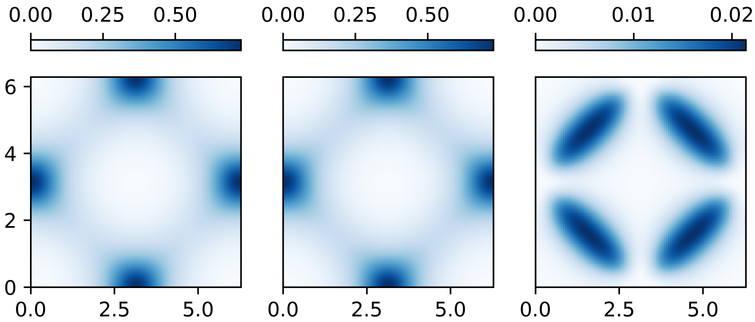


Figure: Ω_{xy}, $\operatorname{tr} g$, $|\operatorname{tr} g - |\Omega_{xy}||$. $\mathcal{C} = 1.0060$, $B = 6.1631$, $T = 0.5014$.

Geometry-independence

Model is defined on a graph with weights t_{ij}, not a lattice which also defines position.

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + h.c. \tag{9}$$

Transform lattice embedding: $\delta \mathbf{x}_\alpha = \mathbf{x}_\alpha^X - \mathbf{x}_\alpha^Y$ (unitary transformation of Hamiltonian)

Berry curvature is geometry-dependent

$$\Omega^Y(\mathbf{k}) = \Omega^X(\mathbf{k}) + (\nabla_{\mathbf{k}} \times \overline{\delta \mathbf{x}_n}(\mathbf{k})). \tag{10}$$

so it cannot be criteria or an observable

Models of FCIs

Mimic Coulomb interaction in topological flat bands with ideal band geometry

$$H = H_0 + U \sum_{\langle i,j \rangle} n_i n_j + V \sum_{\langle\langle i,j \rangle\rangle} n_i n_j \tag{11}$$

Implemented flat-band projection to reduce size of Hilbert space

Wrote own code for exact-diagonalization of flat-band projected model using Jordan-Wigner transformation; code supports any two-band model of fermions or bosons

Conclusions

Role of geometry in FCI is an active area of research

Numerical simulations are paramount to researching FCIs since perturbative methods do not work

On-going project to perform the exact-diagonalization computation for the two-body interacting Haldane model and checkerboard model, projected onto the flat bands

Future work: fix numerical simulations to study FCI phases in nearly ideal band geometry

Backup slides

Quantum Geometry II

Distance

$$\begin{aligned} ds^2 &\equiv |\psi(\lambda_\mu + d\lambda_\mu) - \psi(\lambda_\mu)|^2 \\ &= |d\lambda^\mu \cdot |\partial_\mu \psi\rangle|^2 \\ &= \langle \partial_\mu \psi | \partial_\nu \psi \rangle \cdot d\lambda^\mu d\lambda^\nu \equiv (\gamma_{\mu\nu} + i p_{\mu\nu}) \cdot d\lambda^\mu d\lambda^\nu \end{aligned} \tag{12}$$

Quantum metric

$$|\langle \psi(\lambda) | \psi(\lambda_\mu + d\lambda_\mu) \rangle| = 1 - \frac{1}{2} g_{\mu\nu} \cdot d\lambda^\mu d\lambda^\nu \tag{13}$$

Jordan-Wigner Transformation

Representation of fermions as delocalized, hard bosons in 1 dimension

$$c_j^\dagger = J_j \sigma_j^+, \qquad c_j = J_j \sigma_j^- \tag{14}$$

JW string

$$J_j = \prod_{k=1}^{j-1} (-\sigma_k^z) \tag{15}$$

Locality breaks down in higher dimensions, but can still apply JW transformation for numerical results

Difference between gauge and embedding transformation

Transformation of Bloch state under change of embedding

$$u^{Y}_{n\alpha}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\delta\mathbf{x}_\alpha} u^{X}_{n\alpha}(\mathbf{k}) \tag{16}$$

Gauge transformations (in momentum space) can be expressed as $e^{-i\Lambda(\mathbf{k})}$ for some well-behaved function $\Lambda(\mathbf{k})$. Gauge transformations cannot depend on sublattice

Perturbation theory does not apply to FQHE

First term creates Landau levels, after which the only term in the Hamiltonian is the interaction

$$H = \frac{1}{2m} \sum_{j=1}^{N} \mathbf{\Pi}_j^2 + \sum_{j=1}^{N} \sum_{k<j} \frac{e^2}{4\pi\epsilon_0 |\mathbf{x}_j - \mathbf{x}_k|}. \tag{17}$$

Perturbation theory requires comparing two terms, and we only have one