Dairy Bike Task2.2

DB 1021

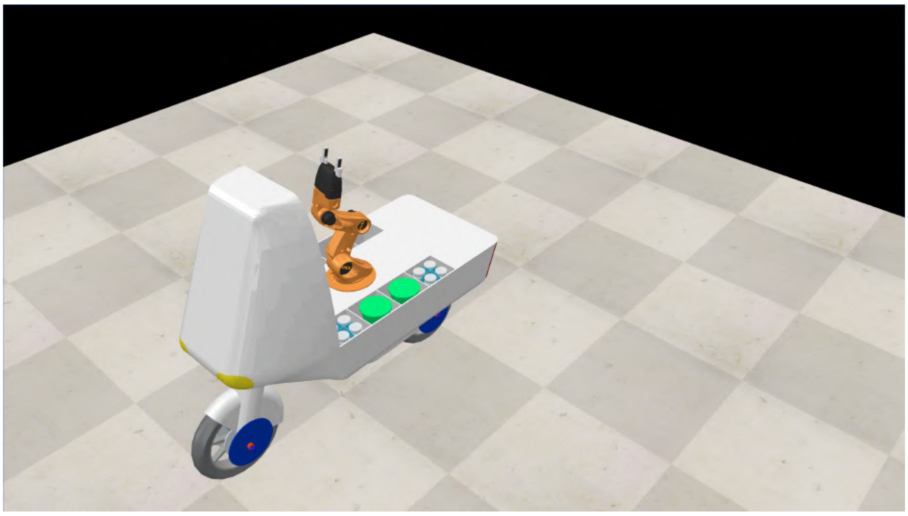
Question 1

Import the labeled JPEG image of your Dairy Bike (DB) as designed in Task 2.1. Describe various parts of the DB and brief reasoning for the selection of your design. Describe the various components of DB with their masses and Moment of Inertia.

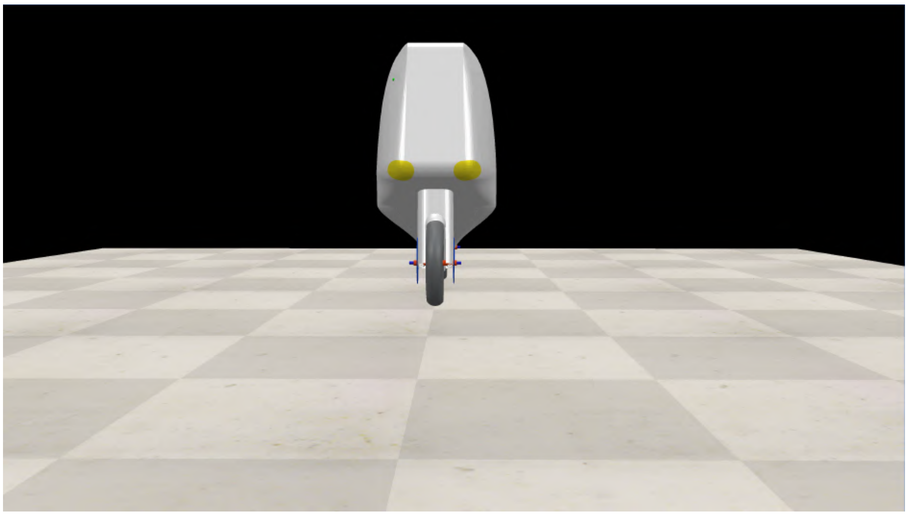
The Isometric view, front view and side view of the Dairy Bike is shown in Figure 1. The chassis(dynamic and respondable) of the Dairy Bike is shown in Figure (2).It consists of two parts , a Mainframe(Figure 3) and a Steering Fork(Figure 4). Chassis is designed in SolidWorks and imported as urdf file in Coppeliasim. Steering Fork is connected to front wheel through front motor, and it is connected to Main Frame through a revolute joint which acts as Steering Mechanism.The Main Frame have a platform for mounting manipulator, a projection for attaching Reaction Wheel and 4 containers on each side for holding dairy products.

Main Frame is designed like a truss in order to provide sufficient strength under loading (Manipulator and packages) with less material. Reaction wheel is placed at a high position and the compartments side-wise, to ensure efficient balancing. Manipulator is placed at center so that it can reach to all compartments easily. In order to provide stability during steering, Steering Fork is designed at an angle to the vertical. Body design is done with streamline and aerodynamics in mind. Mud-guards for wheels and provision for headlamp and tail-lamp are provided.

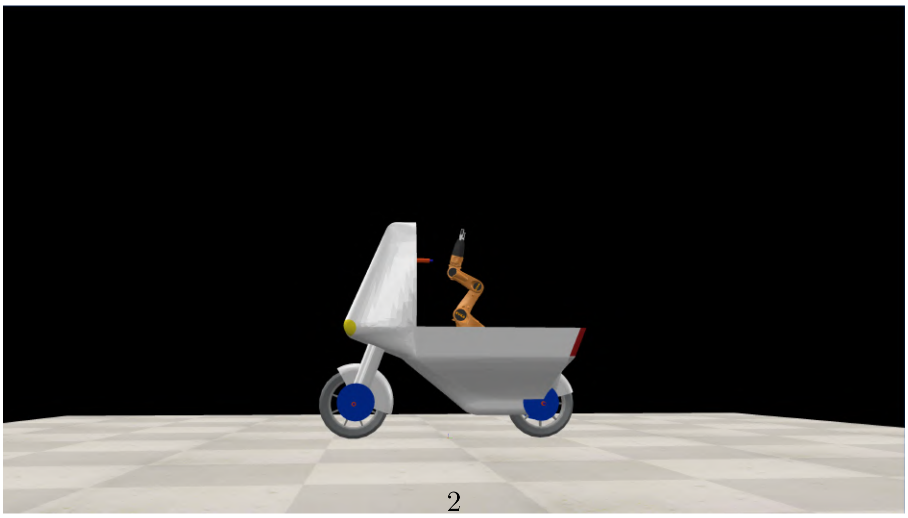
Table 1 contains mass and moment of inertia of each part.



(a) Isometric View



(b) Front View



(c) Side View

Figure 1: Dairy Bike

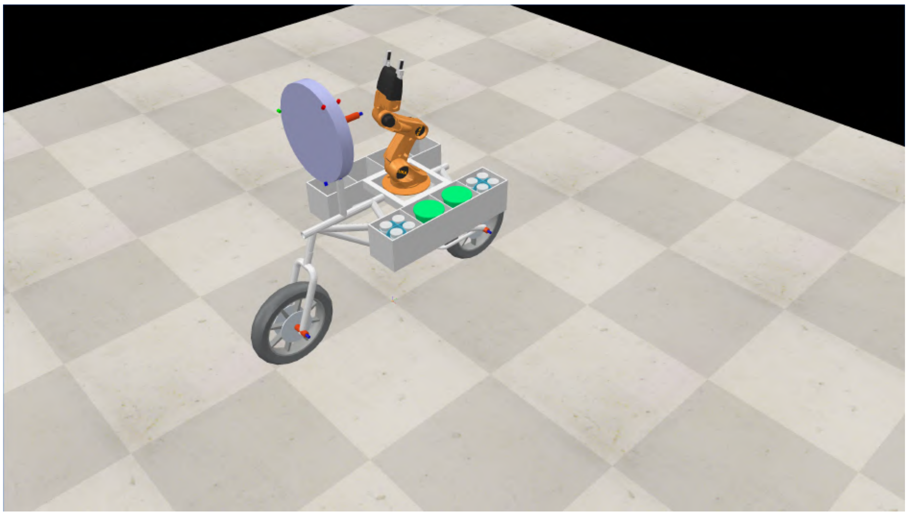
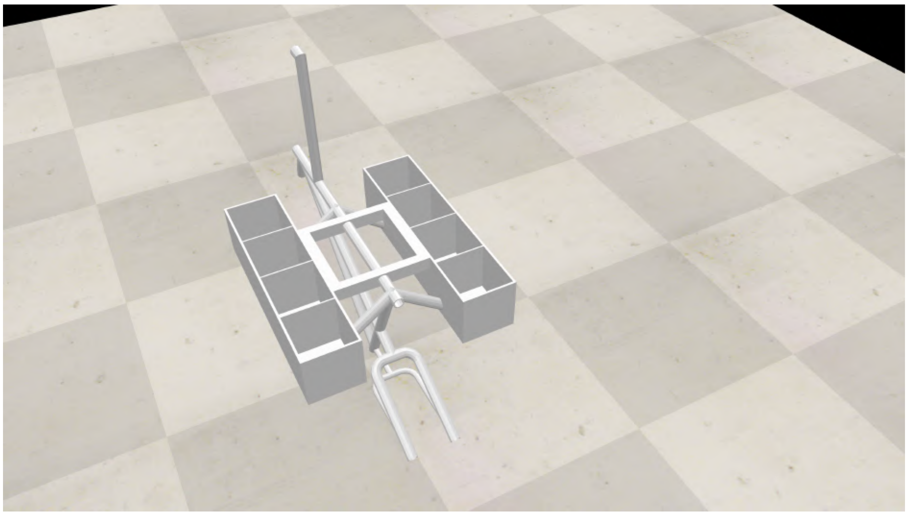


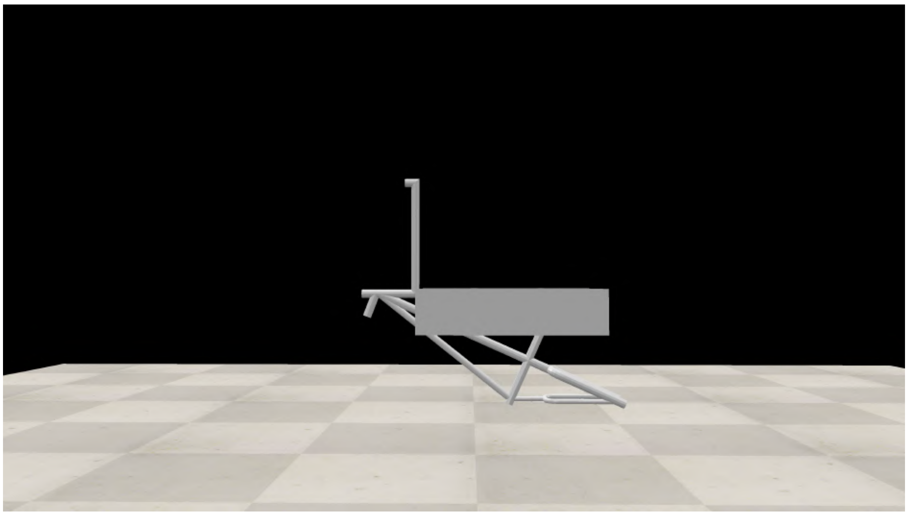
Figure 2: Chassis

Part Name	Mass (Kg)	Centre of Mass Height(m)	Moment of Inertia(Kgm^2)	
			Centroidal Axis	Bottom Axis
Main Frame	6.359	0.4032	0.1799597	1.21374385616
Steering Fork	0.1554	0.2673	0.0011975124	0.0123
Reaction Wheel	4.952	0.7818	0.27855	3.3052680
Front Wheel	1.238	0.15	0.006384366	0.0342393
Rear Wheel	1.238	0.15	0.0063843667	0.0342393
Manipulator	5.290	0.700360	0.1560074	3.0584196

Table 1: Mass and Moment of Inertia of Parts



(a) Isometric View



(b) Front View

Figure 3: Main Frame

Figure 4: Steering Fork

Question 2

Find out the Kinetic Energy and Potential Energy of the overall system. Also specify the chosen states for your system.

Let the bottom plane be the reference for gravitational potential.Let m be the total mass, z be the height of the centre of mass of the overall system from the reference plane , I_1 be the moment of inertia of overall system about axis passing through the points at which wheels touch the reference plane , I_2 be the moment of inertial of Reaction Wheel about its centroidal axis parallel to reference plane, θ be the angle made by the perpendicular to reference plane with an axis passing through the centroid of the overall system perpendicular to the reference plane(as shown in the Figure 5) , α be the orientation of reaction wheel w.r.t z axis of the overall system (ccw rotation taken to be positive). Now the Kinetic energy and potential energy of the overall system is(assuming wheels are not rotating about its axis)

$$T = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\alpha} + \dot{\theta})^2 \tag{1}$$

$$V = mgzcos(\theta) \tag{2}$$

I_1 , I_2 ,m and z can be obtained from Table 1 as

$$I_1 = 7.37(Kgm^2) \tag{3}$$

$$I_2 = 0.27855(Kgm^2) \tag{4}$$

$$z = 0.54872325(m) \tag{5}$$

$$m = 19.2324(Kg) \tag{6}$$

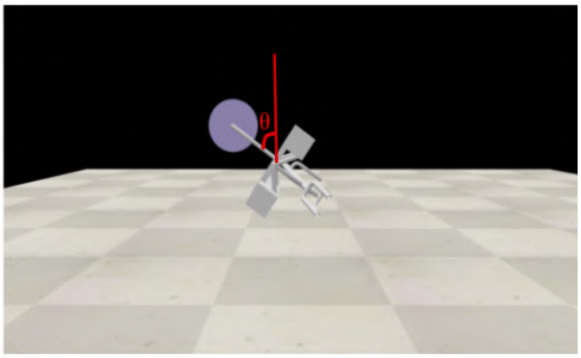


Figure 5: Angle Reference

Question 3

Find out the Lagrangian function and write down the governing dynamical equations using Euler-Lagrange mechanics

Lagrangian of the system is calculated using (1) and (2)

$$\begin{aligned}\mathcal{L} &= T - V \\ &= \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\alpha}+\dot{\theta})^2 - mgzcos(\theta) \qquad (7)\end{aligned}$$

The Lagrangian equations for two generalised coordinates θ and α with a non-conservative torque T acting are ,

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad (8)$$

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}) - \frac{\partial \mathcal{L}}{\partial \alpha} = T \qquad (9)$$

The partial derivatives for (8) and (9) are obtained using (7) as ,

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_1\dot{\theta} + I_2(\dot{\theta}+\dot{\alpha}) \qquad (10)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mgzsin(\theta) \qquad (11)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = I_2(\dot{\theta}+\dot{\alpha}) \qquad (12)$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 0 \qquad (13)$$

Substituting these into (8) and (9) yields the governing dynamical equations of the dairy bike,

$$I_1\ddot{\theta} + I_2(\ddot{\theta} + \ddot{\alpha}) - mgzsin(\theta) = 0 \tag{14}$$

$$I_2(\ddot{\theta} + \ddot{\alpha}) = T \tag{15}$$

Equations (14) and (15) represents the governing dynamical equations of the entire system with terms defined in Question 2.

Question 4

Find out all the equilibrium points and comment on the stability of each of them

Let the state variables be

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = \alpha$$
$$x_4 = \dot{\alpha}$$

Derivatives of state variables are obtained by rearranging (8) and (9)

$$\dot{x_1} = x_2 \tag{16}$$
$$\dot{x_2} = \frac{mgzsin(x_1)}{I_1} - \frac{T}{I_1} \tag{17}$$
$$\dot{x_3} = x_4 \tag{18}$$
$$\dot{x_4} = -\frac{mgzsin(x_1)}{I_1} + T(\frac{1}{I_1} + \frac{1}{I_2}) \tag{19}$$

Now at obtain equilibrium points (without application of torque , T = 0), derivatives of state variables are zero. Equating (16) , (17) , (18) and (19) to zero yields ,

$$x_2 = 0 \tag{20}$$
$$sin(x_1) = 0 \tag{21}$$
$$x_4 = 0 \tag{22}$$
$$-sin(x_1) = 0 \tag{23}$$

Solving (20) , (21) ,(22) and (23) for x_1 , x_2 , x_3 and x_4 yields the **equilibrium points** $(x_1, x_2, x_3, x_4)_1$ = **(0,0,0,0)** and $(x_1, x_2, x_3, x_4)_2$ = **(π,0,0,0).** To find out stability of these points, Jacobian of (16) , (17) ,(18) and (19) is found.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial(16)}{\partial x_1} & \frac{\partial(16)}{\partial x_2} & \frac{\partial(16)}{\partial x_3} & \frac{\partial(16)}{\partial x_4} \\ \frac{\partial(17)}{\partial x_1} & \frac{\partial(17)}{\partial x_2} & \frac{\partial(17)}{\partial x_3} & \frac{\partial(17)}{\partial x_4} \\ \frac{\partial(18)}{\partial x_1} & \frac{\partial(18)}{\partial x_2} & \frac{\partial(18)}{\partial x_3} & \frac{\partial(18)}{\partial x_4} \\ \frac{\partial(19)}{\partial x_1} & \frac{\partial(19)}{\partial x_2} & \frac{\partial(19)}{\partial x_3} & \frac{\partial(19)}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{mgz}{I_1}cos(x_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgz}{I_1}cos(x_1) & 0 & 0 & 0 \end{bmatrix}$$

Then Jacobian at equilibrium point (0,0,0,0) and (π,0,0,0) are

$$\mathbf{J}_{(0,0,0,0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{mgz}{I_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgz}{I_1} & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{J}_{(\pi,0,0,0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{mgz}{I_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{mgz}{I_1} & 0 & 0 & 0 \end{bmatrix}$$

Substituting (3) , (4) , (5) and (6) yields Jacobian at two equilibrium positions as,

$$\mathbf{J}_{(0,0,0,0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 14.047 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -14.047 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{J}_{(\pi,0,0,0)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -14.047 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 14.047 & 0 & 0 & 0 \end{bmatrix}$$

Eigen values of J_1 and J_2 are obtained from octave to be (0,0,3.7479,-3.7479) and (0 + 0i,0 + 0i ,0 + 3.7479i , 0 - 3.7479i) respectively. Clearly the **upright position (**J_1 **,(0,0,0,0)) is unstable** due to positive real part in the eigen value and **vertically down position (**J_2 **, (**π**,0,0,0)) is marginally stable** as it has simple poles (non repeated) on imaginary axis.

Question 5

Find out the Jacobian Matrices around the unstable equilibrium point and specify the state matrix A and input matrix B for the robot designed.

Jacobian of (16) , (17) ,(18) and (19) is,

$$\mathbf{J}_1 = \begin{bmatrix} \frac{\partial(16)}{\partial x_1} & \frac{\partial(16)}{\partial x_2} & \frac{\partial(16)}{\partial x_3} & \frac{\partial(16)}{\partial x_4} \\ \frac{\partial(17)}{\partial x_1} & \frac{\partial(17)}{\partial x_2} & \frac{\partial(17)}{\partial x_3} & \frac{\partial(17)}{\partial x_4} \\ \frac{\partial(18)}{\partial x_1} & \frac{\partial(18)}{\partial x_2} & \frac{\partial(18)}{\partial x_3} & \frac{\partial(18)}{\partial x_4} \\ \frac{\partial(19)}{\partial x_1} & \frac{\partial(19)}{\partial x_2} & \frac{\partial(19)}{\partial x_3} & \frac{\partial(19)}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{mgz}{I_1} cos(x_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgz}{I_1} cos(x_1) & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{J}_2 = \begin{bmatrix} \frac{\partial(16)}{\partial T} \\ \frac{\partial(17)}{\partial T} \\ \frac{\partial(18)}{\partial T} \\ \frac{\partial(19)}{\partial T} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-1}{I_1} \\ 0 \\ \frac{1}{I_2} + \frac{1}{I_1} \end{bmatrix}$$

Unstable equilibrium position $(x_1, x_2, x_3, x_4) = (0,0,0,0)$ is substituted in J_1 and J_2 to obtain state matrix A and input matrix B about this unstable equilibrium position ,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{mgz}{I_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgz}{I_1} & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ \frac{-1}{I_1} \\ 0 \\ \frac{1}{I_2} + \frac{1}{I_1} \end{bmatrix} \tag{25}$$

Substituting (3) , (4) , (5) and (6) gives state matrix A and input matrix B for the robot designed as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 14.047 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -14.047 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ -0.135685 \\ 0 \\ 3.7257 \end{bmatrix} \tag{27}$$

Question 6

Find the controllability matrix [C] and comment on the controllability of the designed system.

Controllability matrix calculated from A and B as

$$\mathbf{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{I_1} & 0 & \frac{-mgz}{I_1^2} \\ \frac{-1}{I_1} & 0 & \frac{-mgz}{I_1^2} & 0 \\ 0 & \frac{1}{I_1}+\frac{1}{I_2} & 0 & \frac{mgz}{I_1^2} \\ \frac{1}{I_1}+\frac{1}{I_2} & 0 & \frac{mgz}{I_1^2} & 0 \end{bmatrix}$$

Substituting (3) , (4) , (5) and (6) in $\mathbf{C}$ gives the controllability matrix as,

$$\mathbf{C} = \begin{bmatrix} 0 & -0.1356852 & 0 & -1.90599 \\ -0.1356852 & 0 & -1.90599 & 0 \\ 0 & 3.7257 & 0 & 1.90599 \\ 3.7257 & 0 & 1.90599 & 0 \end{bmatrix}$$

$$Rank(\mathbf{C}) = 4$$

Rank of $\mathbf{C}$ is 4 which is equal to the number of state variables of the system. Therefore the system is fully controllable.

Question 7

Convert your model from continuous time system to discrete time system and comment on the importance of this step.

Linearised model of the dairy bike in state space is

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \tag{28}$$

A and B are given by (24) and (25) respectively.Equation 28 can be solved analytically as

$$\begin{aligned} \dot{x} - \mathbf{A}x &= \mathbf{B}u \\ e^{(-\mathbf{A}t)}(\dot{x} - \mathbf{A}x) &= e^{(-\mathbf{A}t)}\mathbf{B}u \\ \frac{d}{dt}(e^{(-\mathbf{A}t)}x) &= e^{(-\mathbf{A}t)}\mathbf{B}u \end{aligned} \tag{29}$$

Let t_k and t_{k+1} be two consecutive sampling times and $T = t_{k+1}$ - t_k. Integrating (29) from t_k to t_{k+1} under the assumption of Zero Order Hold (ZOH) (u(t) = u(t_k) for $t_k <$ t $< t_{k+1}$) yields

$$x_{k+1} = \mathbf{A}_d x_k + \mathbf{B}_d u_k \tag{30}$$

Where,

$$\mathbf{A}_d = e^{(\mathbf{A}T)} = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{A}T)^k \tag{31}$$

$$\mathbf{B}_d = (\int_0^T e^{(\mathbf{A}t)}\, dt)\mathbf{B} = (\sum_{k=1}^{\infty} \frac{1}{k!}\mathbf{A}^{k-1}T^k)\mathbf{B} \tag{32}$$

Substituting (26) ,(27) and a suitable time step T into (31) and (32) and taking first few(which is a good approximation for real world systems)terms of Taylor series expansion gives the dicretized state space representation.Considering T = 50 ms(default smpling time in Coppeliasim), discretized state space model of the dairy bike is,

$$\mathbf{x_{k+1}} = \begin{bmatrix} 1.01761 & 0.05029 & 0.00000 & 0.00000 \\ 0.70647 & 1.01761 & 0.00000 & 0.00000 \\ -0.01761 & -0.00029 & 1.00000 & 0.05000 \\ -0.70647 & -0.01761 & 0.00000 & 1.00000 \end{bmatrix} x_k + \begin{bmatrix} -0.00017010 \\ -0.00682403 \\ 0.00465762 \\ 0.18632478 \end{bmatrix} u_k$$

When we practically implement a controller on micro controller it does not gives desired output if the controller was designed in continuous time domain. The reason being that data we recieve from sensor are discrete in nature and hence cannot be implemented using continuous model.So we need to convert our model into discrete time model assuming a fair sampling time.

Question 8

Find out the gain K for the system using LQR function in the octave and comment on the stability of the modified matrix.

Continuous model:

For state matrix A and input matrix B given by (26) and (27) with Q = [5000 0 0 0;0 10 0 0;0 0 1000 0;0 0 0 0.01] and R=[2000] yields(using lqr in octave)

$$K_{continuos} = [-369.8430; -99.5210; -0.7071; -1.0070] \tag{33}$$

Eigen values of the modified matrix **A** - **B**K are obtained from octave as -1.1310 + 1.1234i , -1.1310 - 1.1234i , -3.7449 + 0.0305i , -3.7449 - 0.0305i. Real part of all the eigen values are negative , hence the modified system matrix is stable.

Discrete model:

For discrete state matrix A_d and input matrix B_d with Q = [5000 0 0 0;0 10 0 0;0 0 1000 0;0 0 0 0.01] and R=[2000] yields (using dlqr in octave),

$$K_{discrete} = [-320.1800; -86.1051; -0.5541; -0.8041] \tag{34}$$

Eigen values of the modified matrix **A** - **B**K are obtained from octave as 0.9435 + 0.0531i , 0.9435 - 0.0531i , 0.8292 + 0.0013i , 0.8292 - 0.0013i. All the eigen values lies inside the unit circle, hence the modified matrix is stable.