

**Nonlinear Control of Underactuated Mechanical  
Systems with Application to Robotics and  
Aerospace Vehicles**

by

Reza Olfati-Saber

Submitted to the Department of Electrical Engineering and Computer  
Science

in partial fulfillment of the requirements for the degree of  
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## **Abstract**

This thesis is devoted to nonlinear control, reduction, and classification of underactuated mechanical systems. Underactuated systems are mechanical control systems with fewer controls than the number of configuration variables. Control of underactuated systems is currently an active field of research due to their broad applications in Robotics, Aerospace Vehicles, and Marine Vehicles. The examples of underactuated systems include flexible-link robots, mobile robots, walking robots, robots on mobile platforms, cars, locomotive systems, snake-type and swimming robots, acrobatic robots, aircraft, spacecraft, helicopters, satellites, surface vessels, and underwater vehicles. Based on recent surveys, control of general underactuated systems is a major open problem.

Almost all real-life mechanical systems possess kinetic symmetry properties, i.e. their kinetic energy does not depend on a subset of configuration variables called external variables. In this work, I exploit such symmetry properties as a means of reducing the complexity of control design for underactuated systems. As a result, reduction and nonlinear control of high-order underactuated systems with kinetic symmetry is the main focus of this thesis. By “reduction”, we mean a procedure to reduce control design for the original underactuated system to control of a lower-order nonlinear or mechanical system. One way to achieve such a reduction is by transforming an underactuated system to a cascade nonlinear system with structural properties. If all underactuated systems in a class can be transformed into a specific class of nonlinear systems, we refer to the transformed systems as the “normal form” of the corresponding class of underactuated systems.

Our main contribution is to find explicit change of coordinates and control that transform several classes of underactuated systems, which appear in robotics and aerospace applications, into cascade nonlinear systems with structural properties that are convenient for control design purposes. The obtained cascade normal forms are three classes of nonlinear systems, namely, systems in strict feedback form, feedforward form, and nontriangular linear-quadratic form. The names of these three classes are due to the particular lower-triangular, upper-triangular, and nontriangular structure in which the state variables appear in the dynamics of the corresponding nonlinear systems. The triangular normal forms of underactuated systems can be controlled using existing backstepping and feedforwarding procedures. However, control of the

nontriangular normal forms is a major open problem. We address this problem for important classes of nontriangular systems of interest by introducing a new stabilization method based on the solutions of fixed-point equations as stabilizing nonlinear state feedback laws. This controller is obtained via a simple recursive method that is convenient for implementation. For special classes of nontriangular nonlinear systems, such fixed-point equations can be solved explicitly.

As a result of the reduction process, one obtains a reduced nonlinear subsystem in cascade with a linear subsystem. For many classes of underactuated systems, this reduced nonlinear subsystem is physically meaningful. In fact, the reduced nonlinear subsystem is itself a Lagrangian system with a well-defined lower-order configuration vector. In special cases, this allows construction of Hamiltonian-type Lyapunov functions for the nonlinear subsystem. Such Lyapunov functions can be then used for robustness analysis of normal forms of underactuated systems with perturbations. The Lagrangian of the reduced nonlinear subsystem is parameterized by the shape variables (i.e. the complement set of external variables). It turns out that “a control law that changes the shape variables” to achieve stabilization of the reduced nonlinear subsystem, plays the most fundamental role in control of underactuated systems.

The key analytical tools that allow reduction of high-order underactuated systems using transformations in explicit forms are “normalized generalized momentums and their integrals” (whenever integrable). Both of them can be obtained from the Lagrangian of the system. The difficulty is that many real-life and benchmark examples do not possess integrable normalized momentums. For this reason, we introduce a new procedure called “momentum decomposition” which uniquely represents a non-integrable momentum as a sum of an integrable momentum term and a non-integrable momentum-error term. After this decomposition, the reduction methods for the integrable cases can be applied. The normal forms for underactuated systems with non-integrable momentums are perturbed versions of the normal forms for the integrable cases. This perturbation is the time-derivative of the momentum-error and only appears in the equation of the momentum of the reduced nonlinear subsystem.

Based on some basic properties of underactuated systems as actuation/passivity of shape variables, integrability/non-integrability of appropriate normalized momentums, and presence/lack of input coupling; I managed to classify underactuated systems to 8 classes. Examples of these 8 classes cover almost all major applications in robotics, aerospace systems, and benchmark systems. In all cases, either new control design methods for open problems are invented, or significant improvements are achieved in terms of the performance of control design compared to the available methods. Some of the applications of our theoretical results are as the following: i) trajectory tracking for flexible-link robots, ii) (almost) global exponential tracking of feasible trajectories for an autonomous helicopter, iii) global attitude and position stabilization for the VTOL aircraft with strong input coupling, iv) automatic calculation of differentially flat outputs for the VTOL aircraft, v) reduction of the stabilization of a multi-link planar robot underactuated by one to the stabilization of the Acrobot, or the Pendubot, vi) semiglobal stabilization of the Rotating Pendulum, the Beam-and-Ball system, using fixed-point state feedback, vii) global stabilization of the 2D and 3D Cart-Pole systems to an equilibrium point, viii) global asymptotic

stabilization of the Acrobot and the Inertia-Wheel Pendulum.

For underactuated systems with nonholonomic velocity constraints and symmetry, we obtained normal forms as the cascade of the constraint equation and a reduced-order Lagrangian control system which is underactuated or fully-actuated. This depends on whether the sum of the number of constraints and controls is less than, or equal to the number of configuration variables. This result allows reduction of a complex locomotive system called the snakeboard. Another result is global exponential stabilization of a two-wheeled mobile robot to an equilibrium point which is  $\epsilon$  far from the origin ( $\epsilon \ll 1$ ), using a smooth dynamic state feedback.

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# Chapter 1

## Introduction

### 1.1 Mechanics and Control Theory

Control of mechanical systems is currently among one of the most active fields of research due to the diverse applications of mechanical systems in real-life. Though, the study of mechanical systems goes back to Euler and Lagrange in the 1700's, it was not until 1850's that *mechanical control systems* came to the picture in regulation of steam engines. During the past century, a series of scientific, industrial, and military applications motivated rigorous analysis and control design for mechanical systems. On the other hand, theoretically challenging nature of analysis of the behavior of nonlinear dynamical systems attracted many mathematicians to study control systems. As a result, the efforts of engineers and scientists together led to creation of *Linear Control*, *Optimal Control*, *Adaptive Control*, and *Nonlinear Control* theories. More recently, *Robust Control* theory was born and added to the above picture because of an inevitable need to deal with the presence of uncertainties in real-life control systems. Before the creation of most of these theories, humans managed to fulfill their long-time dream to travel in the space and land on the moon. This is one of the greatest accomplishments of control scientists and engineers of the first half of the past century. Aerospace Vehicles and Robotics Systems with ability to explore the surface of other planets are still among the most complex machines built by humans. For the past 50 years, Aerospace and Robotics applications remained as some of the most influential sources of motivation for rigorous analysis and control of both mechanical systems and nonlinear systems. All along the way, advances achieved by researchers in *Mechanics* and *Nonlinear Control Theory* mutually affected and enhanced each other. Control of Multi-body Systems [53] in mechanics and rigorous analysis (e.g. controllability and observability) and control design (e.g. stabilization and output tracking) for nonlinear systems affine in control [36, 65], which shaped the field of nonlinear control theory, were just two theoretical consequences of this effort.

## 1.2 Nonlinear Control Systems

In this section, we provide an overview of the past research on classes of nonlinear control systems and mechanical systems that are relevant to this work. In this thesis, our main focus is on control design for nonlinear systems that arise from control of an important and broad class of mechanical systems known as underactuated systems. *Underactuated Systems* are mechanical control systems with fewer actuators (i.e. controls) than configuration variables. One of the main contributions of this thesis is explicit transformation of high-order underactuated systems into cascade nonlinear systems with both triangular and nontriangular structural properties. This transformation is performed using a global/semiglobal change of coordinates obtained from the Lagrangian of the system in closed-form. After applying this transformation, control of the original (possibly) high-order underactuated system, reduces to control of lower-order nonlinear systems (possibly) non-affine in control. This motivated us to develop new control design methods for both *cascade nonlinear systems with nontriangular structures* and classes of nonlinear systems non-affine in control, i.e. *highly nonlinear systems*. As a result, the main body of this thesis involves the following topics:

- i) Dynamics, Reduction, and Control of Underactuated Systems
- ii) Control of Highly Nonlinear Systems
- iii) Nonlinear Control of Cascade Systems with Nontriangular Structures

In the following, we present the state-of-the-art of research in each of the above topics.

### 1.2.1 Underactuated Systems

Underactuated mechanical systems are systems that have fewer control inputs than configuration variables. Underactuated systems appear in a broad range of applications including *Robotics*, *Aerospace Systems*, *Marine Systems*, *Flexible Systems*, *Mobile Systems*, and *Locomotive Systems*. The “underactuation” property of underactuated systems is due to the following four reasons: i) dynamics of the system (e.g. aircraft, spacecraft, helicopters, underwater vehicles, locomotive systems without wheels), ii) by design for reduction of the cost or some practical purposes (e.g. satellites with two thrusters and flexible-link robots), iii) actuator failure (e.g. in a surface vessel or aircraft), iv) imposed artificially to create complex low-order nonlinear systems for the purpose of gaining insight in control of high-order underactuated systems (e.g. the Acrobot, the Pendubot, the Beam-and-Ball system, the Cart-Pole system, the Rotating Pendulum, the TORA system (all are described in chapter 3)).

The main control methods applied to examples of inverted-pendulum type underactuated systems is based on swing-up of the pendulum from its downward position and then switching to a balancing controller that is designed using a linearization technique or gain scheduling to balance the pendulum [94]. This includes swing up control using energy-based methods [90] for the Acrobot (i.e. a two-link planar robot

with an actuator at the elbow) [86], the Pendubot (i.e. a two-link planar robot with an actuator at the shoulder) [92], the cart-pole system [21], a triple-link inverted pendulum [90], and the rotating pendulum [6]. The balancing controller for the Acrobot using spline functions can be found in [13, 14].

Due to its complexity, the beam-and-ball has been the focus of study among researchers with diverse interests including approximate feedback linearization methods by Hauser *et al.* [34], using small nested saturations for stabilization of feedforward cascade nonlinear systems due to Teel [102] and stabilization by output feedback [103]. Moreover, stabilization of the beam-and-ball by construction of Lyapunov functions is addressed in [80]. This is done based on the original work due to Mazenc and Praly in [56]. Recently, global stabilization of the beam-and-ball system is achieved in [79].

Passivity-based method is mostly used for swing-up control design of underactuated systems with inverted-pendulums [28, 90]. Moreover, a passivity-based approach was employed for a special example of a pendulum on a cart (i.e. the TORA system) that was transformed to a cascade form [37]. The main drawback of these passivity-based methods is their narrow range of applications. In fact, to the best of my knowledge, so far no applications of energy-based methods in control of real-life underactuated systems in Robotics, Aerospace, and Marine applications is known.

The VTOL (vertical take off and landing) aircraft is another example of an underactuated system that has been extensively used as a test-bed for different methods of trajectory tracking and configuration stabilization. This includes tracking for slightly non-minimum phase systems [35] and hybrid/switching based control methods [58].

Exponential stabilization of examples of underwater vehicles and surface vessels that are underactuated was achieved in [27] and [75] using appropriate coordinate transformations and analysis of a time-varying linear system. A similar type result for attitude control of an underactuated spacecraft is given in [62].

The role of second-order nonholonomic constraints in the necessity of the use of discontinuous stabilizing feedback laws for stabilization of underactuated systems is discussed in [111]. This is mainly based on the famous condition of Brockett on stabilizability of nonlinear systems using time-invariant continuously differentiable state feedbacks [15]. In addition to this issue, accessibility of classes of underactuated mechanical systems has been recently addressed in a recent important work by Reyhanoglu *et al.* [78]. This is based on a framework applied to analysis of controllability of nonholonomic systems [11] and a famous controllability theorem due to Sussmann [97]. An example of a discontinuous stabilizing feedback for a system with an internal unactuated degree of freedom is given in [77].

Adaptive control [33] and sliding mode control techniques [96] have been also applied to underactuated mechanical systems for rather limited applications.

Flexible-link robots are an important class of underactuated systems that are appropriate for space applications due to their lightweight and fast execution of commands. The Euler-Bernoulli model for a flexible arm is an infinite dimensional system [25]. A truncated modal analysis can be used to obtain a finite dimensional state-space model for flexible robots [25, 107]. Trajectory tracking for flexible robots is rather complicated and common measurements like the angle of rotation or the position of the tip, respectively, lead to a poor performance and non-minimum phase zero dy-

namics (see [20, 24] and references therein). In [20], a noncollocated minimum-phase output is proposed based on an analysis of the initial infinite dimensional model and then a finite-order compensator is designed for trajectory tracking. Here, we obtain a nonlinear noncollocated minimum-phase output for a flexible one-link robot arm from a finite-order Euler-Lagrangian equations of the system.

The method of controlled Lagrangians (i.e. applying a control input that preserves the Lagrangian structure of a mechanical system) has been recently applied to local stabilization of the cart-pole system and the rotating pendulum to an equilibrium manifold [10]. However, so far this method has been unable to stabilize the Rotating Pendulum or more general underactuated systems to an equilibrium point.

In addition to more traditional methods, recently, hybrid and switching-based control methods are finding their way in control of underactuated mechanical systems [7, 41] and bipedal locomotion of walking robots [88, 89, 91].

In conclusion, apart from linearization-based techniques, control of underactuated mechanical systems has been mainly along the line of stabilization of special examples of cascade nonlinear systems and using energy-based methods combined with a supervisory-based switching control. The state-of-the-art of research in control of underactuated systems is currently very far from our goal to find control design methods for broad classes of high-order underactuated systems that are effective for Robotics and Aerospace applications.

### 1.2.2 Highly Nonlinear Systems

To describe what we mean by “highly nonlinear systems”, first we present an evolution of control systems from linear systems to nonlinear systems. We start with linear time-invariant control systems in the form

$$\dot{x} = Ax + Bu, \quad y = Cx \quad (1.1)$$

where  $x \in \mathbb{R}^n, y \in \mathbb{R}^m, u \in \mathbb{R}^p$ . Questions regarding controllability, observability, stabilization, and tracking for this system using state or output feedback has been quite well-understood for a long time. However, adding *constraints* or further specifications to the description of the system might make the control design for the system rather complicated. For example, if the system is controllable with bounded control inputs, is there any static or dynamic state feedback that asymptotically stabilizes the system? or what if  $A, B, C$  are known up to an uncertainty (i.e.  $A = A_0 + \Delta$  where  $A_0$  is known and  $\Delta$  is unknown but norm-bounded). None of these problems can be addressed as simple as the control problems for the original linear system (1.1). One can observe that minor deviations from the standard problem of stabilization of an LTI control system and additional constraints make the system rather complex. As a further step, consider the following linear system in feedback connection with a memoryless nonlinearity  $\phi$

$$\dot{x} = Ax + Bu, \quad y = \phi(Cx) \quad (1.2)$$

where  $\phi(0) = 0$  and  $\exists c_2 > c_1 \geq 0 : c_1 z^2 \leq z \cdot \phi(z) \leq c_2 z^2, \forall z \in \mathbb{R}$ . We call these systems *slightly nonlinear systems*. Control design and analysis for systems in the form (1.2) with different types of nonlinearities led to Absolute Stability theory in 60's. An *input-output* stability approach or *a frequency domain analysis* are the dominant tools in dealing with slightly nonlinear systems. In mid-eighties, this notion of feedback interconnection of a linear system and a nonlinearity was generalized to feedback of an LTI system and an uncertainty (or operator) that has bounded gain. This led to Robust Stability theory [23] and more recently Integral Quadratic Constraints (IQC's) [61] due to Megretski and Rantzer. Though, these methods are successful in dealing with linear systems, uncertain linear systems, and slightly nonlinear systems, they are not applicable to truly nonlinear systems (i.e. systems with nonlinear time-evolution of the state) that do not include any basic linear parts. To be more precise, a modification of (1.2) as

$$\dot{x} = \sigma(Ax + Bu), \quad y = Cx \quad (1.3)$$

called a *recurrent neural network* with a saturation-type (i.e. sigmoidal) nonlinearity  $\sigma$ , has no fundamental similarities in terms of controllability and observability to an LTI control system or a slightly nonlinear system. In addition, the linearity of the output does not simplify analysis of the system due to the fact that time-evolution of the system follows a nonlinear law. Dynamic neural networks are examples of highly nonlinear systems and only recently around mid-nineties conditions for controllability (only the discrete-time case) and observability of these systems were introduced by Albertini and Sontag [2, 3]. The analysis method employed in the preceding work was a rather involved *time-domain analysis*. Obviously, due to the fact that a frequency domain analysis can only deal with systems that have linear time-evolution of the state.

A rather standard form for nonlinear systems affine in control in analogy to (1.1) is

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) \quad (1.4)$$

where  $f, g, h$  are nonlinear smooth functions. By a *highly nonlinear system*, we mean a system in the following form

$$\dot{x} = f(x, u), \quad y = h(x) \quad (1.5)$$

where  $f$  is a nonlinear function of  $(x, u)$  (regardless of linearity or nonlinearity of  $h(x)$ ) such that there exist no diffeomorphism  $(z, v) = (T_1(x), T_2(x, u))$  and matrices  $A, B$  satisfying  $\dot{z} = Az + Bu$ . A comprehensive local theory regarding controllability, observability, stabilization, tracking, and disturbance decoupling for nonlinear systems in (1.4) can be found in [36] (see references therein for further details). The main tools to address these control problems were differential-geometry and Lie theory that became very common in the literature around late 70's and early 80's. Though these methods were rather successful in local analysis of nonlinear systems affine in control they usually fail to work for a global analysis and nonlinear systems that are non-

affine in control. Moreover, lie algebraic conditions are not robust to uncertainties in  $f, g, h$ .

*Input-to-State Stability (ISS)* theory, essentially developed and introduced by Sontag [82, 84], combines both Absolute Stability and Robust Stability theories in one for highly nonlinear systems in the general form (1.5). The main tools in this theory for robustness analysis to disturbances are control Lyapunov functions (CLF's). The problem is that in general it is not easy to construct CLF's for highly nonlinear systems.

In many control applications, a global/semiglobal control design and analysis is required. In addition, after applying certain nonlinear coordinate transformation to the dynamics of nonlinear systems affine in control, the transformed system or its subsystems could be nonlinear systems that are non-affine in control. This motivated us to consider global/semiglobal stabilization and analysis of highly nonlinear systems that arise from the study of underactuated mechanical systems and nonholonomic systems. This topic is covered in chapters 4, 5, 6, and 7.

### 1.2.3 Cascade Nonlinear Systems

Cascade nonlinear systems arise in many control applications either naturally after some change of coordinates, or due to design of an output feedback or a dynamic state feedback. In general, they are in the following form

$$\begin{aligned}\dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi, u)\end{aligned}\tag{1.6}$$

The most well-known results for cascade systems are related to nonlinear systems in *triangular forms*. Namely, nonlinear systems in *strict feedback forms* and *feedforward forms*. The backstepping procedure has proven to be successful for systems in strict feedback form [57, 36, 40, 80]. Control of feedforward cascade nonlinear systems is also well-studied. The simplest example of a feedforward nonlinear system is a perturbed chain of integrators that can be stabilized using small nested saturations [102]. Then, more general classes of feedforward systems were either controlled using a nonlinear small-gain theorem [101] or construction of Lyapunov functions [56]. However, control of *cascade nonlinear systems with nontriangular structures* has proven to be rather problematic. This is due to a counterexample by Sussmann [98] and existence of the *peaking phenomenon* [100]. It turns out that if some coupling terms in the dynamics of cascade nonlinear systems satisfy appropriate growth conditions then it is possible to stabilize them using low-gain or high-gain feedback laws [80], [79]. In chapter 5, we show that *the Rotating Pendulum and the Pendubot are examples of underactuated systems that have nontriangular structures* and they do not satisfy any of the aforementioned growth conditions. This suggests developing new stabilization methods for cascade nonlinear systems with nontriangular structures. This stabilization method is one of the main contributions of this work which will be presented in chapter 7.

### 1.3 Statement of the Contributions

In this section, we briefly present the main contributions of this thesis. More detailed statement of the contributions are given in the subsequent sections.

The main focus in this thesis is on nonlinear control of underactuated mechanical systems. This is motivated by broad applications of underactuated systems and theoretically challenging problems that they have to offer. Though, control of mechanical systems with nonholonomic first-order (or velocity) constraints is very well-studied, there is very little known about control of underactuated systems with nonholonomic second-order (or acceleration) constraints. In fact, based on two recent surveys [78, 87], control of general underactuated mechanical systems is currently considered a major open problem.

Our main contribution in this thesis is to provide analytical tools that allow us to translate and solve control problems for broad classes of high-order underactuated systems to equivalent control problems for lower-order nonlinear systems. We call this systematic translation procedure *reduction of underactuated systems*. Such a reduction process requires *an explicit transformation*, i.e. a global change of coordinates (or diffeomorphism) in closed-form. We introduce these explicit change of coordinates for broad classes of underactuated systems including both real-life examples in robotics and aerospace applications and benchmark nonlinear systems. Under such a diffeomorphism, the original underactuated system transforms into a nonlinear system that is referred to as a *normal form*. It turns out that underactuated systems with symmetry properties can be reduced to normal forms that are *structured cascade nonlinear systems*. These cascade normal forms are nonlinear systems in *strict feedback form* [57], *feedforward form* [102], and *nontriangular linear-quadratic form* [69] (see section 3.8 for definitions). This in turn implies that these normal forms can be controlled using backstepping procedure [57], feedforwarding [102, 101, 56, 80], and fixed-point state feedback laws [67] (presented in chapter 7), respectively. Therefore, this thesis includes a complete package for control design and analysis of both low-order and high-order underactuated systems with symmetry by providing

- i) Classification of underactuated systems,
- ii) Reduction of each class to its normal form using an explicit transformation,
- iii) Control design for each reduced normal form (or class).

All three types of aforementioned cascade normal forms are special classes of *Byrnes-Isidori normal form with a double-integrator linear part* [36]. However, the special structure of each type of normal form allows effective use of the existing and recently developed nonlinear control design methods. This is very important, since even stabilization of the general Byrnes-Isidori normal form with a double-integrator linear part is currently an open problem in nonlinear control theory.

Nearly all current applications of underactuated systems possess certain kinetic symmetry properties. By *kinetic symmetry*, we mean invariance of the kinetic energy under the action of a cyclic group. Of course, the reader should notice that this

is a different notion than the famous classical symmetry in mechanics [1, 54] which in a simpler form was familiar to Lagrange. The *classical symmetry* is defined as the invariance of the Lagrangian of a system under a symmetry group action. In simple words, mechanical systems that their inertia matrix is independent of a subset of configuration variables possess kinetic symmetry property w.r.t. this subset of variables. We call this subset of variables *external variables*. The compliment of the set of *external variables* are called *shape variables*. Thus, shape variables are configuration variables that appear in the inertia matrix of a (simple) mechanical system. *The key tools in reduction of high-order underactuated systems are normalized generalized momentums conjugate to external or shape variables together with their integrals* (see section 2.8 for the definitions). It turns out that actuation or lack of actuation of the shape variables of an underactuated system and also integrability of the normalized momentums play the fundamental role in the structure of the corresponding normal form of the system. These two properties together with another two basic properties of underactuated systems led us to *classification of underactuated mechanical systems with kinetic symmetry to 8 classes*. For each class, we provide its associated normal form, control design method, and examples from real-life and benchmark nonlinear systems. A surprising result is that the reduced-order subsystem of the normal form, for several classes of underactuated systems is a Lagrangian system satisfying forced (or unforced) Euler-Lagrange equations. In other words, I obtain *Byrnes-Isidori normal forms for underactuated systems with zero-dynamics that are well-defined lower-order Lagrangian systems*. In certain cases, this allows us to use the Hamiltonian/Lagrangian of the reduced system as a Lyapunov function.

Several examples of real-life and benchmark-type high-order underactuated systems possess *non-integrable normalized momentums* (e.g. a flexible-link robot and a 3D Cart-Pole system). We introduce a method called *Momentum Decomposition* which uniquely decomposes a non-integrable normalized momentum as the summation of an integrable momentum and a non-integrable momentum. This class of underactuated systems with non-integrable momentums can be then reduced to *cascade normal forms plus structured perturbation terms*. A possible way to deal with this problem is to design a nonlinear controller for the nominal unperturbed system that renders the desired equilibrium point of the underactuated system asymptotically stable and makes the perturbation term exponentially vanishing (the details are provided in chapter 4). Then, after performing a *Lyapunov-based Robustness Analysis*, the stability of the perturbed system in closed-loop with the nominal nonlinear controller can be verified. A Lyapunov function candidate for this analysis can be calculated via the Lyapunov function of the core reduced system of the unperturbed system.

In chapter 5, we apply our theoretical results in reduction and control of high-order underactuated systems to a number of complex underactuated systems. Here is a list of contributions on these applications:

- Global asymptotic stabilization of the Acrobot to any arbitrary equilibrium point of the system including the upright equilibrium using a nonlinear state feedback.

- Global asymptotic stabilization of the TORA system using a nonlinear state feedback.
- Global asymptotic stabilization of the Inertia-Wheel Pendulum using a nonlinear state feedback.
- Almost global exponential stabilization of the Cart-Pole system, and the Rotating Pendulum to their upright relative equilibrium point.
- Global asymptotic and local exponential stabilization of the 2D and 3D Cart-Pole systems to their upright equilibrium point.
- Semiglobal asymptotic stabilization of the Cart-Pole system using a fixed-point state feedback law.
- Partial Semiglobal asymptotic stabilization of the Rotating Pendulum and the Beam-and-Ball system using a fixed-point state feedback law.
- Global exponential stabilization of a nonholonomic two-wheeled mobile robot to a point  $\epsilon \ll 1$  far from the origin using a smooth dynamic state feedback.
- Trajectory tracking for a flexible one-link robot arm using a nonlinear noncollocated output.
- Reduction of stabilization of a planar  $n$ -link robot underactuated by one to stabilization of the Acrobot or the Pendubot.
- Global position and attitude stabilization for the VTOL aircraft with strong input coupling.
- Almost global exponential tracking of the feasible trajectories of an autonomous helicopter.
- Automatic calculation of differentially flat outputs for the VTOL aircraft as a by-product of the decoupling of the effects of the body torque in the translational dynamics of the VTOL.
- Automatic calculation of weakly differentially flat outputs for a relatively accurate model of an autonomous helicopter as a by-product of the approximate decoupling of the effects of the body torque in the translational dynamics of the helicopter.

### 1.3.1 Normal Forms for Underactuated Systems

Underactuated systems as mechanical systems with fewer inputs than configuration variables with a nonholonomic second-order (or acceleration) constraint are not fully (i.e. exact) feedback linearizable. In [90], Spong showed that actuated configuration variables of an underactuated system with non-interacting inputs (defined in section 3.4) can be globally linearized using an invertible change of control. This is called

*collocated partial feedback linearization.* This is an important initial step in some of our results and proves that a broad class of underactuated systems can be partially linearized. However, the main difficulty arises after applying this change of control. Since the new control appears both in the actuated and unactuated subsystems of the original underactuated system. In chapter 3, we propose an appropriate structure for a global change of coordinates that decouples these two subsystems w.r.t. the new control. Sufficient conditions are provided for the existence of this diffeomorphism. The condition is a simple algebraic test on the inertia matrix of the system. After applying this transformation, one obtains a cascade normal form which is a special class of Byrnes-Isidori normal form with a double-integrator linear part [36]. The weakness of this result is that to obtain the change of coordinates, based on Frobenius's theorem, it is required to solve a first-order partial differential equation (PDE). Solving this PDE, in turn, reduces to solving a finite number of ODE's which constitute the characterization equations of the PDE. For low-order underactuated systems with 2 DOF, this PDE can be explicitly solved. Solving this PDE explicitly for high-order underactuated is not a feasible approach. Instead, in chapter 4, we present a new method to resolve this issue. The method mainly relies on the use of normalized generalized momentums and their integrals (wherever possible).

Partial feedback linearization is an initial simplifying step for reduction and control of underactuated, regardless of the method used for decoupling of the actuated and unactuated subsystems. For this purpose, we extended the partial feedback linearization result in [90] to two other cases: i) linearization of the dynamics of the unactuated configuration variables which is called *noncollocated partial feedback linearization*, and ii) linearization of actuated configuration variables for underactuated systems with *input coupling*. The former one is used for partial feedback linearization of underactuated systems with unactuated shape variables (e.g. a flexible-link robot or the Rotating Pendulum). The latter one is very useful in aerospace applications, particularly, for decoupling of the effects of the body torque in the translational dynamics of a VTOL aircraft or a helicopter.

### 1.3.2 Reduction and Control of Low-Order Underactuated Systems

In chapter 3, combination of a partial feedback linearization step and an explicit decoupling change of coordinates is used for reduction of underactuated systems with two degrees of freedom. Derivation of explicit normal forms for low-order underactuated systems is important, because we later show that control of special classes of high-order underactuated systems reduces to control of a number of low-order (e.g. 2 DOF) systems. As an example, in chapter 4, we prove that stabilization of an  $n$ -link ( $n > 2$ ) planar robot with  $(n - 1)$  controls reduces to stabilization of the Acrobot or the Pendubot regardless of  $n$ .

There is a simple way to classify underactuated systems with 2 DOF and a single shape variable. Either their shape variable is actuated, or it is unactuated. We call these two classes of underactuated systems *Class-I* and *Class-II* systems, respectively.

Examples of Class-I systems are the Acrobot, the Inertia-Wheel Pendulum, and the TORA system. We proved that all Class-I underactuated systems can be transformed into a cascade nonlinear system in strict feedback form and (possibly) globally asymptotically stabilized using backstepping procedure. In contrast, all Class-II underactuated systems can be globally/semiglobally transformed into cascade systems in nontriangular quadratic normal form (definition 3.8.5). Examples of Class-II systems include flexible-link robots, the Cart-Pole system, the Rotating Pendulum, the Pendubot, and the Beam-and-Ball system. Stabilization of general partially-linear non-triangular cascade nonlinear systems is currently an open problem [36]. We address this problem for important classes of nontriangular linear-quadratic normal forms and more general nontriangular forms in chapter 7.

We provide sufficient conditions so that under a second change of coordinates, Class-II systems can be reduced to cascade nonlinear systems in feedforward forms. Among all the stated examples, the Cart-Pole system is the only one that falls within this class and can be globally asymptotically stabilized using a state feedback in the form of nested saturations due to Teel [102]. In chapter 5, we present detailed control design for all the aforementioned examples of underactuated systems with 2 DOF.

### 1.3.3 Reduction and Control of High-Order Underactuated Systems

Most of real-life underactuated systems are high-order systems. An aircraft or a helicopter has 6 DOF and 4 controls. A flexible  $n$ -link robot with  $m$  deformation modes at each link has  $n(m+1)$  DOF,  $n$  controls, and  $nm$  unactuated degrees of freedom (e.g. for  $n = 2, m = 3$  the system has 8 DOF and 2 controls). This motivated us to focus on reduction of high-order underactuated systems (in chapter 4). The kinetic symmetry property of underactuated systems and the degree in which the shape variables are actuated are two main factors in reduction of high-order underactuated systems. Our key analytical tools here are the normalized generalized momentums conjugate to actuated/unactuated configuration variables and their integrals (wherever possible). It turns out that the integrability of these normalized momentums plays a fundamental role in the structure of the normal forms for high-order underactuated systems.

An important property of normal forms for high-order underactuated systems is that they are physically meaningful. To be more precise, consider a class of underactuated systems with  $n$  degrees of freedom,  $m$  actuated shape variables with non-interacting inputs. Assume the normalized momentums conjugate to the  $(n-m)$  external variables of this system are integrable. We call this class of high-order underactuated systems *Class-I* systems. We prove that there exists a global diffeomorphism obtained from the Lagrangian of the system that transforms the dynamics of a Class-I underactuated system into a reduced-order system with a new configuration vector of dimension  $(n-m)$  and a well-defined Lagrangian function parameterized by the shape variables which satisfies the unforced Euler-Lagrange equations. This normal form is in fact a cascade nonlinear system in strict feedback form with a zero-dynamics (associated with the shape variables as the output) which is a Lagrangian function

itself. We refer to this Lagrangian zero-dynamics system as the *core reduced system*. In chapter 4, we provide a method for stabilization of this core reduced system using a nonlinear static state feedback. As a by-product, we obtain an energy-type Lyapunov function for the closed-loop core reduced system. In certain cases, the physical property of the zero-dynamics subsystem allows using Hamiltonian/Lagrangian of the core reduced system as a valid Lyapunov function. These Lyapunov functions can be later used for robustness analysis of the perturbed normal forms of Class-I underactuated systems.

Now, consider the class of underactuated systems with integrable normalized momentums conjugate to shape variables and non-interacting inputs. We call them *Class-II* underactuated systems. After the reduction process, Class-II underactuated systems can be explicitly transformed into nontriangular quadratic normal forms. In this case, still the reduced-order subsystem (i.e. the zero-dynamics) is a well-defined Lagrangian system (under minor conditions). But it satisfies a forced Euler-Lagrange equation with a rather complex input force with a nonlinear dependence on both the configuration and momentum of the reduced Lagrangian system. The complexity of the structure of this input force is one of the main sources of extreme difficulties in control design for Class-II underactuated systems.

We provide simple algebraic sufficient conditions on the inertia matrix and potential energy of the system such that the nontriangular normal form of a Class-II underactuated system can be transformed into a feedforward system using a second explicit change of coordinates. The subclass of Class-II systems that satisfies these sufficient conditions are called *Class-III* systems.

The second important factor in reduction of underactuated systems, particularly in aerospace applications and locomotive systems, is the presence of input coupling between actuated and unactuated subsystems. For flat underactuated systems including aircraft and helicopters, the reduction is rather straightforward. In chapter 4, we address reduction of non-flat underactuated systems with input coupling. This class of real-life systems can potentially have very high degrees of freedom. An example is a flying bird as a three-body system with 14 DOF and 8 controls. In general, many locomotive systems in the nature fall within this class. Other examples include a swimming fish and a walking human being. All of these three examples have actuated shape variables and they locomote in their environment using the coupling between their translational dynamics and shape variables. Therefore, the presence of input coupling is very crucial for the purpose of locomotion of these underactuated systems. We call these new classes of non-flat and underactuated systems with input coupling *Class-IV* and *Class-V* systems, respectively.

For the reduction of Class-I and Class-II underactuated systems, we assumed the normalized momentums are integrable. A natural question to ask is whether it is possible to reduce these systems if their normalized momentums are non-integrable. The answer to this question is positive. The integrability property of the normalized momentums can be considered as the third most important factor in reduction of high-order underactuated systems. To address reduction underactuated systems with non-integrable momentums, we introduce a new procedure called *Momentum Decomposition*. This method uniquely decomposes a non-integrable momentum as

the summation of an integrable momentum and a non-integrable momentum. Let us call the underactuated systems with non-integrable normalized momentums corresponding to Class-I,II,II, respectively, *Class-VI*, *Class-VII*, and *Class-VIII* systems. We prove that normal forms of Class-VI,VII,VIII underactuated systems with non-integrable momentums are perturbed versions of the normal forms of Class-I,II,III systems with a structured perturbation that only appears in the dynamics of the momentum of the reduced subsystem.

In conclusion, we obtain 8 classes of underactuated systems. Namely, Class-I through Class-VIII systems which are precisely defined with their subclasses in section 4.6. The result of classification of underactuated systems is summarized in Table 4.1. For each class, related examples and an appropriate control design method are given. Our main contribution is that the methods for reduction and control of high-order systems presented in chapter 4, applies to the majority of important examples of robots, aerospace systems, and benchmark nonlinear systems.

### 1.3.4 Reduction and Control of Underactuated Systems with Nonholonomic Velocity Constraints

In chapter 6, we focus on reduction of underactuated systems with kinetic symmetry and nonholonomic velocity constraints. Our main result on reduction of nonholonomic underactuated systems with symmetry is that under certain assumptions, the reduced system can be represented as the cascade of the constraint equation and a lower-order underactuated (or fully-actuated) Lagrangian system. If the sum of the number of controls and constraint equations is less than the dimension of the configuration vector, the reduced system is underactuated. Otherwise, it is fully-actuated. The *snakeboard* example (Fig. 6-4) is a mobile robot with 6 degrees of freedom, 3 controls, and 2 nonholonomic contraints [50]. Our reduction method, provides a systematic way for calculation of the reduced system of the snakeboard. Furthermore, we prove that the kinematic model of a snakeboard and a car are diffeomorphic. This implies that the snake-board can be transformed into a first-order chained-type nonholonomic system (equation 2.8) using an explicit change of coordinates.

Based on Brockett's result [15], the class of mechanical systems with first-order nonholonomic constraints can not be stabilized to the origin using a smooth static state feedback. An example of this class is a simple two-wheeled mobile robot. We introduce a new class of parameterized diffeomorphisms that are called *near-identity diffeomorphisms*. An example of a near-identity diffeomorphism is  $\psi(x; \theta, \epsilon) = x + \epsilon v(\theta)$  where  $v(\theta)$  is a unit vector and for a fix  $\theta$  and  $\epsilon$ ,  $\psi$  is clearly a diffeomorphism which is equal to  $id(x) = x$  for  $\epsilon = 0$ . The point  $x_\epsilon = x + \epsilon v(\theta)$  is called an  $\epsilon$ -nearby point of  $x$ . We use a near-identity change of coordinates for global exponential stabilization/tracking of a nonholonomic two-wheeled mobile robot to an  $\epsilon$ -nearby point/trajectory of a desired position/trajectory using a smooth dynamic state feedback ( $\epsilon \ll 1$ ). This motivated us to formally define the notions of  $\epsilon$ -stabilization and  $\epsilon$ -tracking for nonlinear systems.

### **1.3.5 Applications in Robotics and Aerospace Vehicles**

In chapter 5, we provide detailed control design for several examples of low-order and high-order underactuated systems in robotics and aerospace applications. A list of some of the examples and their associated control task is stated in the last part of the introduction of section 1.3.

### **1.3.6 Control of Nontriangular Cascade Nonlinear Systems**

Stabilization of nontriangular cascade nonlinear systems is currently considered a major open problem [36]. In chapter 7, we address this stabilization problem for important classes of nontriangular cascade nonlinear systems, particularly, systems in nontriangular linear-quadratic normal form. This includes normal forms for Class-II, Class-IV, and Class-VII underactuated systems.

## **1.4 Outline**

The outline of this thesis is as follows. Chapter 1, provides an introduction and statement of the contributions. In chapter 2, we give some background on classes of mechanical control systems and define some basic notions in mechanics. Chapter 3 is devoted to the dynamics of underactuated system, several examples of underactuated systems, partial feedback linearization techniques, normal forms for underactuated systems, and classification of low-order underactuated systems. Our main contributions are presented in chapter 4. This includes reduction, classification, and control of high-order underactuated systems. In chapter 5, we present detailed control design for several robotics and aerospace applications. In chapter 6, we address reduction and control of underactuated systems with nonholonomic velocity constraints and symmetry properties. In chapter 7, we introduce fixed-point state feedback laws for stabilization of nontriangular cascade nonlinear systems. In chapter 8, we make concluding remarks and state possible directions of future research.

# Chapter 2

## Mechanical Control Systems

### 2.1 Introduction

This chapter provides some background on mechanical control systems. We start by defining simple mechanical control systems and introduce Legendre normal form of a simple mechanical system. Then, we discuss fully-actuated mechanical systems and their transformation into a set of double-integrators using a change of control. We define underactuated mechanical systems in contrast to fully-actuated mechanical systems and demonstrate why control of underactuated systems is a challenging open problem. A special class of mechanical systems called flat mechanical systems is described next. We present a result on controllability limitations of underactuated flat mechanical systems in lack of gravity terms based on the Legendre normal form of a flat underactuated system. We briefly review state-of-the-art of research in modeling and control of mechanical systems with nonholonomic velocity (i.e. first-order) constraints and clarify their close relations to underactuated mechanical systems that contain nonholonomic acceleration constraints. Symmetry in mechanics plays a crucial role in some of the main results of this thesis regarding reduction of underactuated mechanical systems. Therefore, the role of symmetry and conservation laws in mechanics and their relation through Noether's theorem is explained in the final section.

### 2.2 Simple Lagrangian Systems

In this section, we state forced Euler-Lagrange equations of motion as the model of control mechanical systems and introduce Legendre normal form for simple mechanical systems.

*Simple mechanical systems* are systems that their Lagrangian is in the form of the difference between a (positive semidefinite) kinetic energy and a potential energy

$$\mathcal{L}(q, \dot{q}) = K - V = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q)$$

where  $q \in Q$  denotes the *configuration vector* that belongs to an  $n$ -dimensional *configuration manifold*  $Q$ ,  $M(q)$  is the *inertia matrix* which is a positive definite symmetric

matrix.  $K$  is the kinetic energy and  $V(q)$  is the potential energy of the system. Let  $f_i(q) : Q \rightarrow \mathbb{R}^n, i = \{1, \dots, m\}$  be  $m$  linearly independent external forces applied to the system. The Euler-Lagrange equation for this *mechanical control system* is as the following

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = F(q)u \quad (2.1)$$

where  $u \in \mathbb{R}^m$  and  $F(q) = (f_1(q), \dots, f_m(q))$  denotes the matrix of external forces. The equations of motion for this mechanical system can be derived as the following

$$\sum_j m_{kj}(q) \ddot{q}_j + \sum_{i,j} \Gamma_{ij}^k(q) \dot{q}_i \dot{q}_j + g_k(q) = e_k^T F(q)u, \quad k = 1, \dots, n$$

where  $e_k$  is the  $k$ th standard basis in  $\mathbb{R}^n$ ,  $g_k(q) = \partial_{q_k} V(q)$ , and  $\Gamma_{ij}^k(q)$  are called *Christoffel symbols* and can be defined as

$$\Gamma_{ij}^k(q) = \frac{1}{2} \left( \frac{\partial m_{kj}(q)}{\partial q_i} + \frac{\partial m_{ki}(q)}{\partial q_j} - \frac{\partial m_{ij}(q)}{\partial q_k} \right).$$

in the vector form, we have

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = F(q)u \quad (2.2)$$

$$c_{ij} = \sum_{k=1}^n \Gamma_{kj}^i(q) \dot{q}_k$$

is an element of  $C(q, \dot{q})$ . The term  $C(q, \dot{q})\dot{q}$  contains two types of terms involving  $\dot{q}_i \dot{q}_j$  that are called *Centrifugal* terms ( $i = j$ ) and *Coriolis* terms ( $i \neq j$ ). Also,  $G(q)$  contains the gravity terms (see [95] for more details and examples). The interesting relationship between the matrices  $M$  and  $C$  is that  $S_0 = M(q) - 2C(q, \dot{q})$  is a skew symmetric matrix. In other words, because  $M$  is symmetric, one gets  $\dot{M}(q) = C(q, \dot{q}) + C^T(q, \dot{q})$ . Taking this property into account and noting that  $M(q)$  is a positive definite matrix, the *Legendre transform* with respect to  $\dot{q}$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = M(q)\dot{q}$$

is invertible and the dynamics of the mechanical system in (2.2) can be rewritten in the canonical form

$$\begin{cases} \dot{q} &= M^{-1}(q)p \\ \dot{p} &= -G(q) + \tilde{C}^T(q, p)M^{-1}(q)p + F(q)u \end{cases} \quad (2.3)$$

where  $\tilde{C}(q, p) = C(q, M^{-1}p)$ . We call (2.3) *Legendre Normal Form* of a simple mechanical system. Defining the variables  $x_1 = q$ ,  $x_2 = p$ , (2.3) can be rewritten as

$$\begin{aligned}\dot{x}_1 &= M^{-1}(x_1)x_2 \\ \dot{x}_2 &= -G(x_1) + x_2^T Q(x_1)x_2 + F(x_1)u\end{aligned}\tag{2.4}$$

or

$$\dot{x} = f(x) + g(x)u\tag{2.5}$$

which is a nonlinear system affine in control with state  $x = \text{col}(x_1, x_2)$  and vector fields

$$\begin{aligned}f(x) &= \text{col}(M^{-1}(x_1)x_2, -G(x_1) + x_2^T \sigma(x_1)x_2) \\ g(x) &= \text{col}(0, F(x_1))\end{aligned}$$

with the property that  $f(x)$  has a mixed linear-quadratic structure w.r.t.  $x_2$  and  $g(x)$  is independent of  $x_2$ . Note that  $\sigma$  is a cubic matrix such that  $x_2^T Q(x_1)x_2 \in \mathbb{R}^n$ . The main advantage of the form (2.5) is that many analytical tools are available for controllability and observability analysis and control design of nonlinear systems affine in control [36].

*Remark 2.2.1.* For mechanical systems, equations (2.2) and (2.3) are two equivalent forms. However, The Legendre normal form is a first-order ODE and (2.2) is a second-order ODE. Later, we will see that from control point of view (2.3) is more appropriate and can be directly used for controllability analysis of a mechanical system.

*Remark 2.2.2.* Defining the Hamiltonian of a simple Lagrangian system as

$$H(q, p) = \frac{1}{2}p^T M^{-1}(q)p + V(q)$$

the Legendre normal form in (2.3) is equivalent to the following forced Hamiltonian equations

$$\begin{aligned}\dot{q} &= \frac{\partial H(q, p)}{\partial p} \\ \dot{p} &= -\frac{\partial H(q, p)}{\partial q} + F(q)u\end{aligned}$$

However, the Hamiltonian form is not appropriate for direct controllability analysis and control design compared to the Legendre normal form.

In the following sections, we describe several important classes of mechanical control systems.

## 2.3 Fully-actuated Mechanical Systems

Consider the control mechanical system in (2.1). We call (2.1) a *fully-actuated mechanical system* if  $m = \text{rank } F(q) = n$ , i.e.  $F(q)$  is an invertible matrix. For fully-

actuated systems the number of control inputs is equal to the dimension of their configuration manifold. Therefore, fully-actuated mechanical systems are *exact feedback linearizable* (i.e. they have no zero-dynamics [36]). This can be proved by applying the following change of control

$$u = F(q)^{-1}(M(q)v + C(q, \dot{q})\dot{q} + G(q))$$

and redefining the variables as  $x_1 = q$ ,  $x_2 = \dot{q}$ , to obtain

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= v\end{aligned}$$

which is clearly a (vector) double integrator. This is why most of problems for fully-actuated mechanical systems can be reduced to equivalent problems for linear systems. This fact suggests that control of mechanical systems is challenging either in the presence of uncertainties (e.g. parametric uncertainties in  $M, C, G$ ), or when the number of control inputs is less than  $n$  which in both cases exact feedback linearization is not possible. Moreover, under any holonomic or nonholonomic constraint (both to be defined), control and analysis of mechanical systems could potentially be rather complicated. The same is true if the control input of a mechanical system is required to be bounded.

## 2.4 Underactuated Mechanical Systems

A control mechanical system with configuration vector  $q \in Q$  and Lagrangian  $\mathcal{L}(q, \dot{q})$  satisfying the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = F(q)u \quad (2.6)$$

is called an *Underactuated Mechanical System (UMS)* if  $m = \text{rank } F(q) < n = \dim(Q)$  (see the definitions for equation (2.1)). In other words, underactuated systems are mechanical systems that have fewer actuators than configuration variables. This restriction of the control authority does not allow exact feedback linearization of underactuated systems. As a special case, assume  $F(q) = (0, I_m)^T$ . Then, the first  $(m - n)$  equations in (2.6) can be expressed as a second-order dynamic equation

$$\varphi(q, \dot{q}, \ddot{q}) = 0 \quad (2.7)$$

which contains Coriolis, centerifugal, and gravity terms that can be highly nonlinear. If there exists no function  $h$  such that  $\ddot{h} = \varphi(q, \dot{q}, \ddot{q})$ , equation (2.7) is called a *second-order nonholonomic constraint*, or a *nonholonomic acceleration constraint* (e.g. see section A.8). We assume all underactuated systems throughout this work have second-order nonholonomic constraints unless otherwise is stated.

For the special case of  $F(q) = (0, I_m)^T$ , we refer to the first  $(n - m)$  equations of (2.6) as the *unactuated subsystem* and to the last  $m$  equations of (2.6) as the *actuated*

*subsystem.* In [90], Spong has shown that the actuated subsystem of a general underactuated system with  $F(q) = (0, I_m)^T$  can be linearized using an invertible change of control. This procedure is called *partial feedback linearization*. However, after partial linearization, the unactuated subsystem of (2.6) still remains as a nonlinear system that is coupled with the linearized actuated subsystem through both the new control input and other nonlinear terms. This highly complicates control design for underactuated systems. One of the main contributions of this thesis is to decouple the actuated and unactuated subsystem of the underactuated system in (2.6) with respect to the new control (see section 3.7). The main focus of this thesis is control design and analysis of underactuated mechanical systems. A comprehensive treatment of control of underactuated system will be presented in the subsequent chapters.

## 2.5 Flat Mechanical Systems

Mechanical systems that their inertia matrix is constant are called *Flat Mechanical Systems*. One of the characters of a flat mechanical system is that the Christoffel symbols associated with their inertia matrix vanishes identically and therefore the matrix  $C(q, \dot{q})$  vanishes as well. Now, assume that for a mechanical system with inertia matrix  $M(q)$  there exists a diffeomorphism  $z = \psi(q)$  such that  $d\psi(q) = M^{1/2}(q)dq$ , then we have

$$\dot{z} = M^{1/2}\dot{q}$$

and the Lagrangian of the system can be written as

$$\mathcal{L}(z, \dot{z}) = \frac{1}{2}\dot{z}^T I \dot{z} - V(\psi^{-1}(z))$$

Thus,  $M(z) = I$  is constant and in new coordinates the mechanical system is flat.

The Legendre normal form for a flat mechanical system with  $B(q) = I$  is as the following

$$\begin{aligned}\dot{q} &= M^{-1}p \\ \dot{p} &= \tau - G(q)\end{aligned}$$

This is due to the fact that  $\tilde{C}(q, p) = C(q, M^{-1}(q)p) = 0$ . The following proposition reveals a major controllability limitation of flat underactuated mechanical systems.

**Proposition 2.5.1.** *Consider an underactuated flat mechanical system with the gravity term  $G(q) = (g_1(q), \dots, g_n(q))^T$ . Suppose for some  $1 \leq i \leq n$ ,  $q_i$  is unactuated and  $g_i(q) = 0$ , then the system is not controllable/stabilizable.*

**Proof.** Based on the assumptions,  $\dot{p}_i = 0$  and  $p_i$  is a conserved quantity, i.e.  $p_i(t) = p_i(0), \forall t \geq 0$ . If  $p_i(0) \neq 0$  then the system cannot be stabilized to any equilibrium point/manifold with  $p = 0$ .  $\square$

## 2.6 Nonholonomic Mechanical Systems

A Lagrangian mechanical system with  $m < n$  velocity constraints

$$W^T(q)\dot{q} = 0$$

( $W$  is an  $m \times n$  matrix) that are non-integrable (i.e.  $\nexists h(t) : \dot{h} = W^T(q)\dot{q}$ ) is called a *Mechanical System with First-order Nonholonomic Constraints*. Control of nonholonomic mechanical systems with velocity constraints has been extensively studied in the recent years by numerous researchers. The history of research in this field goes back to early 1900's. Formulation of the dynamics of nonholonomic systems can be found in [64]. Controllability and stabilization of nonholonomic systems is considered in [11]. In an important work, Bloch and Crouch addressed modeling of nonholonomic systems on Riemannian manifolds [8]. More recently, many advances have been made on controllability of nonholonomic systems by Lewis and Murray [49], locomotion and control design for nonholonomic systems by Ostrowski in [74], and reduction of nonholonomic systems with symmetry due to Bloch *et al.* [9].

Following the line of formulation in [11] and based on [64], the dynamics of nonholonomic systems with velocity constraints can be described as the following:

$$\begin{aligned} M(q)\ddot{q} + C(q, \dot{q}) + G(q) &= W(q)\lambda + B(q)u \\ W^T(q)\dot{q} &= 0 \end{aligned}$$

where  $\lambda \in R^m$  is the vector of Lagrange multipliers.

*Remark 2.6.1.* In the literature [111], [78], sometimes in contrast to nonholonomic systems with first-order (or velocity) constraints, *underactuated mechanical systems* are called *systems with second-order (or acceleration) nonholonomic constraints*. This is due to the fact that if  $B(q) = (0, I)^T$  is partitioned according to  $q = (q_1, q_2)^T$ , then regardless of the control input we have

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = 0$$

or

$$m_{11}(q)\ddot{q}_1 + m_{21}(q)\ddot{q}_2 + h_1(q, \dot{q}) = 0$$

where  $m_{ij}$ 's are block matrices of the inertia matrix associated with  $q = (q_1, q_2)^T$ . However, this terminology is somewhat misleading since the Lagrangian of an underactuated mechanical system satisfies Euler-Lagrange equations of motions without any external differential-algebraic constraints that requires use of Lagrangian multipliers in a variational setting as in [8] for nonholonomic systems with first-order constraints.

Control design for broad classes of nonholonomic mechanical systems can be re-

duced to control of chained-type systems as the following

$$\begin{aligned} x_1^{(k_1)} &= u_1 \\ x_2^{(k_2)} &= u_2 \\ x_3^{(k_3)} &= x_2 u_1 \\ \vdots \\ x_n^{(k_n)} &= x_{n-1} u_1 \end{aligned} \tag{2.8}$$

where  $k_i \geq 1, i = 1, \dots, n$  are integers. Mobile robots, car-like vehicles, and multi-trailer systems [104] are examples of chained-form systems in (2.8). For  $k_i = 1$ , system (2.8) is nonholonomic with first-order constraints. Recently, in an important work [47], (quasi) exponential stabilization of higher-order chained-type systems was addressed by Laiou and Astolfi based on the original work of Astolfi [4]. A discontinuous dynamic-state feedback was introduced for stabilization. Traditionally, stabilization of chained-form systems with at most first-order nonholonomic constraints has been done using sinusoids [63] or homogeneous time-varying state feedback [60], [59] due to M'Closkey and Murray. Later, the theoretical results in [60] were applied to local exponential stabilization of the attitude of a spacecraft with nonholonomic constraints [62] and configuration stabilization of a nonholonomic underwater vehicle [27].

## 2.7 Symmetry in Mechanics

In this section, we define symmetry in mechanics, the connection between symmetry and conservation laws (i.e. Noether's theorem), and the role of symmetry in reduction of the dynamics of a mechanical system. Formally, the symmetry in mechanics is defined as the invariance of the Lagrangian of a system under the action of a left (or right) invariant Lie group. For a complete treatment of symmetry see [54] (also [1]), for definitions of Lie groups, left invariant Lie groups, and the action of Lie groups on manifolds see Warner and Boothby.

Almost all real-life mechanical systems possess certain symmetry properties. For example, the Lagrangian of a helicopter, a car, or a Satellite is independent of their position. This gives rise to symmetries (i.e. invariance of the Lagrangian) w.r.t. translation.

Here, we only consider symmetry of Lagrangian systems in a Euclidean space  $Q = \mathbb{R}^n$ . We say the Lagrangian  $\mathcal{L}(q, \dot{q})$  is symmetric w.r.t. the configuration variable  $q_i$  iff

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad i \in \{1, \dots, n\}$$

Denote the  $i$ th generalized momentum by  $p_i$  as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

and consider the unforced Euler-Lagrange equations of motion

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

It immediately follows that *the symmetry of  $\mathcal{L}$  w.r.t.  $q_i$  implies that the  $i$ th momentum  $p_i$  is conserved (i.e.  $\dot{p}_i = 0$ ) and vice versa.* The last theorem in its most general form is called Noether's theorem [1] which establishes a one-to-one equivalence between the existence of symmetries and conservation laws. Lagrange himself was aware of this fact and he called  $q_i$  an *ignorable coordinate*. This is due to the fact that the degree of the second-order equation of motion (i.e. Hamiltonian form) for the mechanical system reduces by 2. Therefore, *the existence of symmetry properties leads to reduction of the mechanical systems.*

For the special case of a simple mechanical systems that its Lagrangian is symmetric w.r.t.  $q_i$ , the condition  $\dot{p}_i = 0$  is equivalent to a first-order constraint

$$W^T(q)\dot{q} = p_i(0)$$

where

$$W(q) = (m_{i1}(q), \dots, m_{in}(q))^T$$

is the  $i$ th row of the inertia matrix  $M(q)$ . If this constraint is non-integrable, the analysis of the system reduces to the analysis of a mechanical system with a first-order nonholonomic constraint. Therefore, in addition to the conservation laws, the existence of symmetries gives rise to holonomic/nonholonomic velocity constraints for mechanical systems.

In this thesis, we use a different notion of symmetry called Kinetic Symmetry that is rather similar to classical symmetry. By *Kinetic Symmetry*, we mean the kinetic energy of the system is invariant w.r.t.  $q_i$ , i.e.

$$\frac{\partial K}{\partial q_i} = 0$$

The kinetic symmetry is equivqlent to classical symmetry in lack of potential energy. However, the majority of systems that we consider here have nonzero (i.e. non-constant) potential energies and the notions of kinetic symmetry and classical symmetry do not coincide. It is important to notice that *the existence of kinetic symmetries (in presence of a potential field) does not lead to the existence of conserved quantities.* later, we will see that the fact that the generalized momentums are not conserved has a crucial role in controllability of large classes of underactuated mechanical systems. In addition, we will show that the existence of kinetic symmetry properties leads to reduction of underactuated Lagrangian systems regardless of the lack of conserved quantities.

## 2.8 Normalized Momentums and Integrability

In this work, *normalized momentums* are the most important analytical tools in reduction of underactuated systems. Consider an underactuated system with configuration vector  $q = (q_1, q_2)$  and Lagrangian

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - V(q)$$

The generalized momentum conjugate to  $q_i$  can obtained via partial Legendre transform

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = m_{i1}(q)\dot{q}_1 + m_{i2}(q)\dot{q}_2, \quad i = 1, 2$$

The normalized momentums conjugate to  $q_1$  and  $q_2$  w.r.t.  $q_1$  are defined as the following

$$\pi_1 := m_{11}^{-1}(q)p_1 = \dot{q}_1 + m_{11}^{-1}(q)m_{12}(q)\dot{q}_2, \quad \pi_2 := m_{21}^{-1}(q)p_2 = \dot{q}_1 + m_{21}^{-1}(q)m_{22}(q)\dot{q}_2$$

where  $\pi_1$  is always well-defined and  $\pi_2$  is defined wherever  $m_{21}(q)$  is an invertible square matrix. We say  $\pi = \pi(q, \dot{q})$  is an integrable normalized momentum, if there exists a *generalized configuration function*  $h = h(q)$  such that  $\dot{h} = \pi$  where the function  $h$  is defined as

$$\dot{h}(q, \dot{q}) = \nabla h(q) \cdot \dot{q}$$

We call  $h(q)$  the *integral* of  $\pi$ . Whenever  $\dot{h} : h = \pi$ , we say  $\pi$  is *non-integrable*.

# Chapter 3

## Normal Forms for Underactuated Systems

### 3.1 Introduction

Underactuated mechanical systems appear in a variety of real-life control applications including *Robotics* (e.g. lightweight flexible-link robots, nonholonomic mobile robots, and walking robots), *Aerospace Vehicles* (e.g. helicopters, aircraft, spacecraft, and satellites), *Underwater Vehicles*, and *Surface Vessels*. Due to their broad range of applications, control of underactuated systems is extremely important. The restriction of the control authority for underactuated systems causes major difficulties in control design for these systems. During the past decade, the challenging nature of analysis and control of underactuated systems has attracted numerous researchers with interests in nonlinear control theory, robotics and automation, control of autonomous vehicles, and control of flexible structures. Control of general underactuated mechanical systems is currently considered a major open problem based on recent surveys [78, 87].

The reasons behind the complexity of control design for underactuated systems can be summarized as follows. Underactuated systems are not fully feedback linearizable. Moreover, many recent and traditional methods of nonlinear control design including backstepping [36, 57], forwarding [56, 80, 101, 102], high-gain/low-gain designs [80, 79], and sliding mode control [40] are not directly applicable to underactuated systems with the exception of a few special cases (e.g. the beam-and-ball system and the cart-pole system). This is due to the fact that a method for transforming underactuated systems into cascade nonlinear systems with upper/lower triangular or nontriangular structural properties has not yet been discovered. The classes of nonlinear systems of interest include Byrnes-Isidori normal forms (equation (3.14)) [36], strict feedback forms (definition 3.8.2) [36, 57], and feedforward forms (definition 3.8.3) [102, 101, 56].

In this chapter, our main contribution is to obtain the aforementioned *cascade normal forms for underactuated systems* by applying global/semiglobal nonlinear change of coordinates. The structural properties of the new normal forms allow effective use of many existing control design methods. In addition, a new class of underac-

tuated systems is found that leads to a cascade normal form with a nontriangular quadratic structure (see definition 3.8.5). Due to the nontriangular nature of this normal form, backstepping/forwarding procedures [80] are not applicable to it. Stabilization of this class of nontriangular nonlinear systems is equivalent to control of a special Byrnes-Isidori normal form with a double-integrator linear part that is a major open problem in nonlinear control theory [36, 98, 99]. In chapter 7, we address important special cases of this stabilization problem including the cases that appear in control of underactuated mechanical systems.

To attain insight in dealing with control design for higher-order underactuated systems (e.g. a helicopter with 6 DOF and 4 controls), first it is very useful to consider calculation of cascade normal forms for lower-order underactuated systems. For doing so, we focus on the class of underactuated systems with two degrees of freedom and kinetic symmetry (defined in section 2.7). We prove that for this special class of underactuated systems both the decoupling change of coordinates and the corresponding cascade normal form can be obtained in explicit forms. The structure of the obtained normal forms for underactuated systems with two degrees of freedom naturally leads to their classification in two classes. The first class includes systems like the Acrobot, the Inertia Wheel Pendulum, and the TORA system (see section 3.3 for the definitions) which can be transformed into cascade nonlinear systems in strict feedback form (definition 3.8.2). The second class includes the Pendubot, the Cart-Pole system, the Rotating Pendulum, and the Beam-and-Ball system (see section 3.3 for the definitions) which can be transformed into a nontriangular quadratic normal form (definition 3.8.5). In addition, the cart-pole system can be transformed into a feedforward form after another transformation (definition 3.8.3).

As an initial step, our main result on normal forms for underactuated systems partly relays on a method called *collocated partial feedback linearization* due to Spong [90]. Spong showed that the dynamics of the actuated configuration variables of any underactuated mechanical system can be globally linearized by applying a change of control. This allowed application of passivity-based/energy-based methods to solving swing-up problem for special examples of underactuated systems that involve inverted pendulums. However, after this change of control, the new control appears both in the actuated subsystem and the unactuated subsystem of the original system. This coupling w.r.t. the input highly complicates control design for underactuated systems. To resolve this issue, we introduce an appropriate structure for a global change of coordinates that decouples the linear actuated and nonlinear unactuated subsystems w.r.t. the new control. This global change of coordinates transforms the dynamics of the system into a cascade nonlinear system in Byrnes-Isidori normal form with a double-integrator linear part [36]. In addition, characterization of the cases where this normal form reduces to (strict) feedback/feedforward forms is provided.

We introduce a procedure to linearize the dynamics of the unactuated configuration variables using a change of control. This is applicable under the condition that the number of unactuated variables is less than or equal to the number of actuated variables. We call this procedure *noncollocated partial feedback linearization*. This new partial linearization method is used in obtaining the nontriangular quadratic normal form for a class of underactuated systems with kinetic symmetry.

The outline of this chapter is as follows. First, we present standard models for underactuated systems in section 3.2. Then, we provide several examples of underactuated systems including both benchmark systems and real-life applications in section 3.3. In sections 3.4 and 3.5, we explain collocated and noncollocated partial feedback linearization methods. Our main results on normal forms for underactuated systems are presented in section 3.7. In section 3.9, we focus on normal forms and classification of underactuated systems with two degrees of freedom and kinetic symmetry.

## 3.2 Dynamics of Underactuated Systems

The Euler-Lagrange equations of motion for an underactuated system are as the following

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = F(q)\tau$$

where  $\tau \in \mathbb{R}^m$  is the control and  $F(q) \in \mathbb{R}_{n \times m}$  is a non-square matrix of external forces with  $m < n$  and full column rank. Here,  $m$  denotes the number of control inputs that is less than the number of configuration variables  $n$ . For the case of simple Lagrangian systems, the equations of motion for an underactuated mechanical system can be expressed as the following

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = F(q)\tau \quad (3.1)$$

Assuming  $F(q) = [0, I_m]^T$ , the configuration vector can be partitioned as  $q = (q_1, q_2) \in \mathbb{R}^{(n-m)} \times \mathbb{R}^m$  according to  $F(q)$  where  $q_1$  and  $q_2$  denote the *actuated* and *unactuated* configuration vectors, respectively [90]. After partitioning the inertia matrix  $M(q)$  accordingly, the dynamics of an underactuated system takes the form

$$\begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h_1(q, \dot{q}) \\ h_2(q, \dot{q}) \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \quad (3.2)$$

with  $\tau \in \mathbb{R}^m$ . Due to the lack of control in the first equation of (3.2), it is not possible to fully linearize this underactuated system using a change of control. However, it is still possible to partially linearize the system such that the dynamics of  $q_2$  transforms into a double integrator [90]. We demonstrate this procedure in sections 3.4 and 3.5.

## 3.3 Examples of Underactuated Systems

In this section, we present several examples of underactuated systems. These examples include the Acrobot, the Pendubot, the Cart-Pole system, the Rotating Pendulum, the Beam-and-Ball system, the TORA (translational rotational actuator) system, the Inertia-Wheel Pendulum, and the VTOL aircraft (vertical take-off and landing). All the examples are chosen due to the complexity of their control design and the fact that for analysis and control purposes they are of high interest in the

literature. We briefly introduce each example with its related control design task and later use them as applications of our theoretical results by providing control design and analysis for each case in chapter 5. The state-of-the-art of research in control design for each example is provided as well. The inertia matrix for each system is given for the future use. The examples of this section only include underactuated systems with two or three degrees of freedom. Later in chapter 5, we give examples of higher-order underactuated systems that appear in Robotics and Aerospace applications.

### 3.3.1 The Acrobot and the Pendubot

The *Acrobot* [13, 86] is a two-link planar robot with revolute joints and one actuator at the elbow as shown in Figure 3.3.1 (a). The *Pendubot* shown in Figure 3.3.1 (b) is also a two-link planar robot with revolute joints and an actuator at the shoulder. The

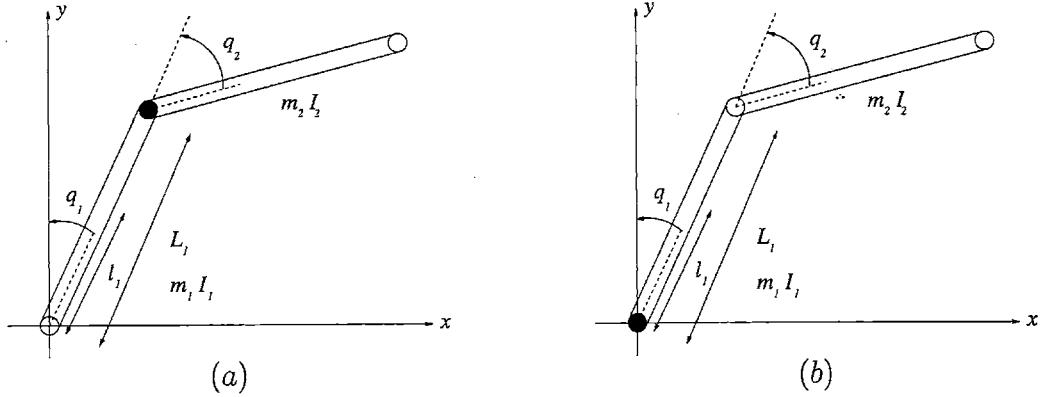


Figure 3-1: (a) The Acrobot, (b) The Pendubot.

Acrobot and the Pendubot graphically seem to be very similar (i.e. they share the exact same inertia matrix). Though, later we will see that the difference in the location of their single actuator causes a major difference in their standard representation (i.e. normal form) and control design. The inertia matrix for both the Acrobot and the Pendubot is the same and has the following form

$$M(q_2) = \begin{bmatrix} a_1 + a_2 \cos(q_2) & a_3 + \frac{1}{2}a_2 \cos(q_2) \\ a_3 + \frac{1}{2}a_2 \cos(q_2) & a_4 \end{bmatrix}$$

where the  $a_i$ 's are positive constants. A possible control task is to stabilize the upright equilibrium point for the Acrobot or the Pendubot from any initial condition. A more complicated task is to stabilize the Acrobot or the Pendubot to any of their arbitrary infinite equilibrium points. In the past, the former task has been done using swinging up these double inverted pendulums from their downward initial positions and bringing them close to their upright position, then switching to a linear controller around the upright equilibrium point [13, 86, 90, 94, 28]. The latter task has never

been done before to the best of our knowledge. In this work, we perform both of these tasks for the Acrobot and the former task for the Pendubot using a single state feedback.

### 3.3.2 The Cart-Pole System and the Rotating Pendulum

The *Cart-Pole system* shown in Figure 3.3.2 (a) consists of an inverted pendulum on a cart. Figure 3.3.2 (b) shows the *Rotating Pendulum* [112, 6] which is an inverted pendulum on a rotating arm. The inertia matrix for the Cart-Pole system is

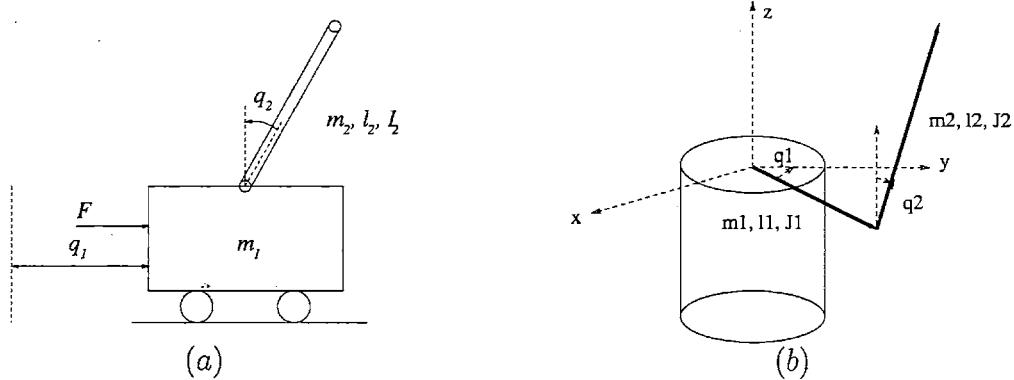


Figure 3-2: (a) The Cart-Pole system, (b) The Rotating Pendulum.

$$M(q_2) = \begin{bmatrix} a_1 & a_2 \cos(q_2) \\ a_2 \cos(q_2) & a_3 \end{bmatrix}$$

and for the Rotating Pendulum the inertia matrix is

$$M(q_2) = \begin{bmatrix} b_1 + c_1 \sin^2(q_2) & b_2 \cos(q_2) \\ b_2 \cos(q_2) & b_3 \end{bmatrix}$$

where all the  $a_i, b_i, c_i > 0$  are constants. Clearly, the only difference between the inertia matrices of these two mechanical systems is in the first element  $m_{11}(q_2)$ . Another similarity between the Cart-Pole system and the Rotating Pendulum is that both have the same form of the potential energy

$$V(q_2) = a_4 \cos(q_2)$$

with  $a_4 > 0$ . Swing-up control design for the inverted pendulum in the Cart-Pole system [21, 90] and the Rotating pendulum [112, 6] has been done by several researchers. The drawback of the existing swing-up design methods is that they are relatively slow. In this work, we introduce aggressive swing-up control designs for both the Cart-Pole system and the Rotating Pendulum using bounded control inputs. In addition, we

address asymptotic stabilization of both of these systems to their upright equilibrium points using nonlinear state feedback.

### 3.3.3 The Beam-and-Ball System

The *Beam-and-Ball system* illustrated in Figure 3-3 consists of a beam and a ball on it. The task is to bring the ball to the center of the beam by applying a torque  $\tau$  to the beam. The vertical distance between the center of mass of the ball and the location of the external torque (i.e. control) is shown by  $d$ . The inertia matrix for the beam and ball system takes the following form

$$M(q_2) = \begin{bmatrix} a_1 + a_2 q_2^2 & -a_3 d \\ -a_3 d & a_4 \end{bmatrix}$$

where all the  $a_i$ 's are positive constants. Due to complexity of this system, stabiliza-

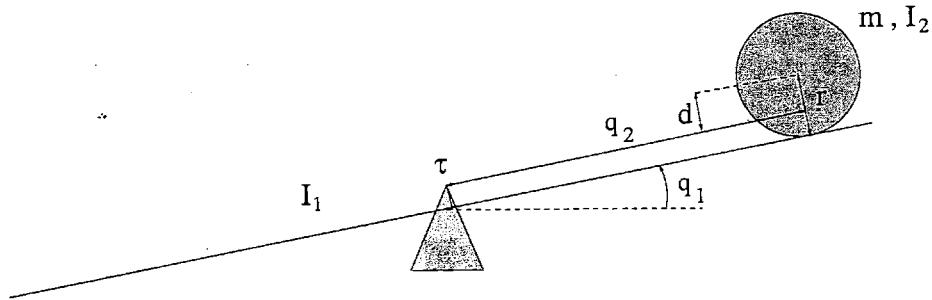


Figure 3-3: The Beam-and-Ball System.

tion and tracking for the beam-and-ball system using state or output feedback has been considered by many researchers [34, 102, 103, 80, 79]. In [34], tracking for the beam-and-ball system was considered using approximate input-output linearization. Semiglobal stabilization of the beam-and-ball system using small nested saturations (i.e. state feedback) was addressed by Teel in [102]. Also, stabilization of this system using output feedback is due to Teel and Praly [103]. In [80], global stabilization of the beam-and-ball system with friction was achieved using a numeric version of the method of construction of Lyapunov functions with cross-terms which is originally due to Mazenc and Praly [56]. Moreover, global stabilization of the beam-and-ball system as a perturbed chain of integrators with nontriangular (definition 3.8.4) homogeneous higher-order perturbations was achieved in [79]. In all the aforementioned works, the model of the beam-and-ball system is taken from [34] with the assumption that  $d = 0$ . In this work, we consider aggressive stabilization of the beam-and-ball system with  $d = 0$  in chapter 7 and introduce the challenges in dealing with the case  $d \neq 0$ .

### 3.3.4 The TORA System

The TORA (Translational Oscillator with Rotational Actuator) system was first introduced in [109]. Figure 3-4 illustrates the *TORA system* consisting of a translational oscillating platform with mass  $m_1$  that is controlled via a rotational eccentric mass  $m_2$ . The inertia matrix for the system is in the form

$$M(q_2) = \begin{bmatrix} a_1 & a_2 \cos(q_2) \\ a_2 \cos(q_2) & a_3 \end{bmatrix}$$

where the  $a_i > 0$  are constants and the potential energy is given by

$$V(q_1, q_2) = \frac{1}{2}k_1 q_1^2 + m_2 g r \cos(q_2)$$

Due to the fact that the TORA system is not fully feedback linearizable an an easy

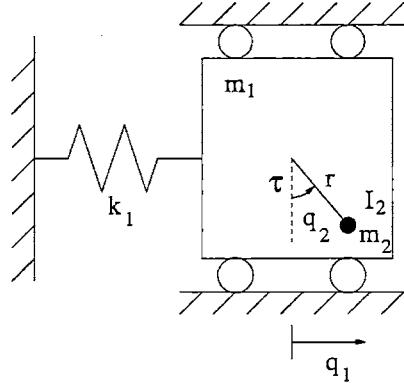


Figure 3-4: The TORA system.

change of coordinates was available to transform it to a cascade nonlinear system [109], it received a tremendous attention by several researchers [37, 80, 38, 39]. In most of these works the TORA system was used as a benchmark example for passivity-based techniques and stabilization/tracking using output feedback in zero gravity, i.e.  $g = 0$ . Here, we are interested in global stabilization of the TORA system in the presence of gravity, i.e.  $g \neq 0$ .

### 3.3.5 The Inertia-Wheel Pendulum

The Inertia-Wheel Pendulum was first introduced by Spong *et al.* in [93]. Figure 3-5 shows the inertia wheel pendulum that consists of a pendulum with a rotating uniform inertia-wheel at its end. The pendulum is unactuated and the system has to be controlled via the rotating wheel. The task is to stabilize the pendulum in its upright equilibrium point while the wheel stops rotating. The specific angle of rotation of the wheel is not important. *The inertia wheel pendulum is the first example of a flat underactuated mechanical system with two degrees of freedom and a single actuator.* This is due to the constant inertia matrix of the Inertia-Wheel Pendulum.

In [93], an energy-based method is used for the swing-up of the pendulum and then

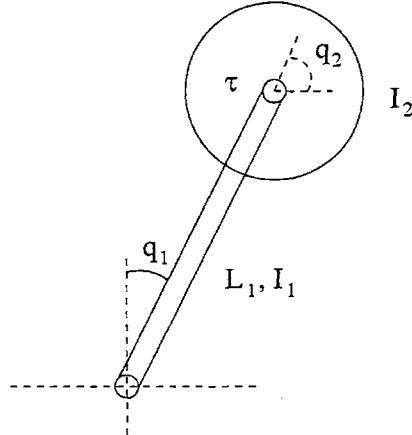


Figure 3-5: The Inertia Wheel Pendulum

a supervisory-based switching strategy is employed to switch to a stabilizing local nonlinear controller with a relatively large region of attraction. Here, we achieve global stabilization of the upright equilibrium point of the inertia wheel pendulum using a single nonlinear state feedback in analytically explicit form.

### 3.3.6 The VTOL Aircraft

The VTOL aircraft depicted in Figure 3-6 is a simplified planar model of a real vertical take off and landing plane (e.g. the Harrier). The dynamics of the VTOL aircraft is

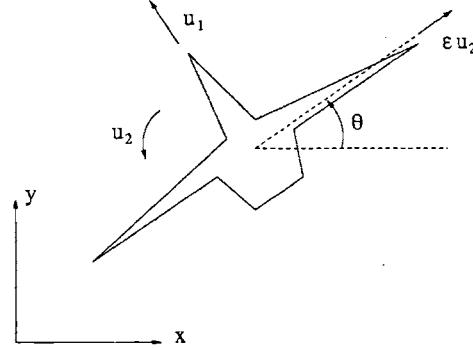


Figure 3-6: The VTOL aircraft.

given in [35, 55] as the following

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -u_1 \sin(\theta) + \epsilon u_2 \cos(\theta) \\
 \dot{y}_1 &= y_2 \\
 \dot{y}_2 &= u_1 \cos(\theta) + \epsilon u_2 \sin(\theta) - g \\
 \dot{\theta} &= \omega \\
 \dot{\omega} &= u_2
 \end{aligned} \tag{3.3}$$

The VTOL aircraft is an underactuated system with three degrees of freedom and two control inputs. In [35], it is assumed that  $|\epsilon|$  is relatively small and the VTOL aircraft is treated as a slightly nonminimum phase system. Then, using approximate input-output linearization, bounded tracking is achieved. In section 5.10, we discuss global configuration stabilization for the VTOL aircraft with strong input coupling (i.e.  $|\epsilon| \gg 1$ ). In this case, the system is strongly non-minimum phase and stabilization or tracking for system (3.3) is a challenging problem.

### 3.4 Collocated Partial Feedback Linearization

The actuated configuration vector for the underactuated mechanical system in (3.2) is  $q_2$ . The procedure of linearization of the dynamics of  $q_2$  is called *collocated partial linearization* which is due to Spong [90]. Spong has shown that all underactuated systems in the form (3.2) can be globally partially linearized using a change of control. Here, we restate this result as the following with our own phrasing due to applications in this work.

**Proposition 3.4.1.** (*Spong [90]*) *There exists a global invertible change of control in the form*

$$\tau = \alpha(q)u + \beta(q, \dot{q})$$

*that partially-linearizes the dynamics of (3.2) as the following*

$$\begin{aligned}\dot{q}_1 &= p_1 \\ \dot{p}_1 &= f_0(q, p) + g_0(q)u \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\tag{3.4}$$

*where  $\alpha(q)$  is an  $m \times m$  positive definite symmetric matrix and*

$$g_0(q) = -m_{11}^{-1}(q)m_{12}(q)$$

**Proof.** From the first line of (3.2), we have

$$\ddot{q}_1 = -m_{11}^{-1}(q)h_1(q, \dot{q}) - m_{11}^{-1}(q)m_{12}(q)\ddot{q}_2$$

which proves the expression for  $g_0(q)$ . Substituting this in the second line of (3.2), we get

$$(m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q))\ddot{q}_2 + h_2(q, \dot{q}) - m_{11}^{-1}(q)h_1(q, \dot{q}) = \tau$$

Thus, defining

$$\begin{aligned}\alpha(q) &= m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q) \\ \beta(q, \dot{q}) &= h_2(q, \dot{q}) - m_{21}(q)m_{11}^{-1}(q)h_1(q, \dot{q})\end{aligned}$$

and observing that  $\alpha(q)$  is positive definite and symmetric finishes the proof.  $\square$

The main property of the underactuated system in (3.4) is that after partial feedback linearization, the new control  $u$  appears in the dynamics of both  $(q_1, p_1)$ -subsystem (i.e. nonlinear subsystem) and  $(q_2, p_2)$ -subsystem (i.e. linear subsystem). This is one of the main sources of the complexity of control design for underactuated systems. In section 3.7, we introduce a global change of coordinates that decouples these two subsystems while leaves the linear subsystem invariant.

*Remark 3.4.1.* Defining  $x = (q_1, p_1, q_2, p_2)$ , the dynamics of (3.4) can be rewritten as

$$\dot{x} = f(x) + g(x)u$$

with obvious definitions of  $f(x)$  and  $g(x)$ . In [78], using explicit expressions of  $f(x)$  and  $g(x)$ , sufficient conditions are provided for small-time local controllability (STLC) [97] of a special class of underactuated mechanical systems in the form (3.4).

### 3.5 Noncollocated Partial Feedback Linearization

In this section, we present a partial feedback linearization procedure for underactuated systems that linearizes the dynamics of the unactuated configuration variables. We show this is possible if the number of the control inputs is greater or equal to the number of unactuated configuration variables. We call this procedure *noncollocated partial feedback linearization*. To be more precise, consider the following underactuated system

$$\begin{bmatrix} m_{00}(q) & m_{01}(q) & m_{02}(q) \\ m_{10}(q) & m_{11}(q) & m_{12}(q) \\ m_{20}(q) & m_{21}(q) & m_{22}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_0 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h_0(q, \dot{q}) \\ h_1(q, \dot{q}) \\ h_2(q, \dot{q}) \end{bmatrix} = \begin{bmatrix} \tau_0 \\ \tau_1 \\ 0 \end{bmatrix} \quad (3.5)$$

where  $q = (q_0, q_1, q_2) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  with  $n_1 = n_2 = m$  and  $n_0 = n - 2m \geq 0$ .

*Remark 3.5.1.* For the special case where there are equal number of actuated and unactuated variables  $n = 2m$  and  $n_0 = 0$ . Thus,  $q_0$  has dimension zero or  $q = (q_1, q_2)$ .

**Proposition 3.5.1.** Consider the underactuated mechanical system in (3.5). Then, there exists a change of control in the form

$$\tau = \alpha_1(q)u + \beta_1(q, \dot{q})$$

where  $\tau = \text{col}(\tau_0, \tau_1)$ ,  $u = \text{col}(u_0, u_1)$  and

$$u_1 = \alpha_0(q)u_0 + \alpha_2(q)u_2 + \beta_2(q, \dot{q})$$

that partially linearizes the dynamics of (3.5) as the following

$$\begin{aligned}\dot{q}_0 &= p_0 \\ \dot{p}_0 &= u_0 \\ \dot{q}_1 &= p_1 \\ \dot{p}_1 &= f_0(q, p) + g_0(q)u_0 + g_2(q)u_2 \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u_2\end{aligned}\tag{3.6}$$

over the set

$$U = \{q \in \mathbb{R}^n \mid \det(m_{21}(q)) \neq 0\}$$

where

$$\begin{aligned}f_0(q, p) &= -m_{21}^{-1}(q)h_2(q, p) \\ g_0(q) &= -m_{21}^{-1}(q)m_{20}(q) \\ g_2(q) &= -m_{21}^{-1}(q)m_{22}(q)\end{aligned}$$

**Proof.** Apparently, based on collocated feedback linearization (proposition 3.4.1) the dynamics of the actuated configuration vector  $(q_0, q_1)$  can be linearized using a change of control  $\tau = \alpha_1(q)u + \beta_1(q, \dot{q})$  that gives

$$\begin{aligned}\ddot{q}_0 &= u_0 \\ \ddot{q}_1 &= u_1\end{aligned}$$

Over the set  $U$ ,  $m_{21}(q)$  is an invertible matrix and from the last equation in (3.5), we get

$$\ddot{q}_1 = -m_{21}^{-1}(q)h_2(q, \dot{q}) - m_{21}^{-1}(q)m_{20}(q)\ddot{q}_0 - m_{21}^{-1}(q)m_{22}(q)\ddot{q}_2$$

Thus, after a second change of control as

$$u_1 = -m_{21}^{-1}h_2(q, \dot{q}) - m_{21}^{-1}(q)m_{20}(q)u_0 - m_{21}^{-1}(q)m_{22}(q)u_2$$

where  $u_2$  is the new control, the result follows.  $\square$

## 3.6 Partial Feedback Linearization Under Input Coupling

Consider the underactuated mechanical control system

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = F(q)\tau$$

with configuration  $q \in \mathbb{R}^n$  and control  $\tau \in \mathbb{R}^m$  and  $\text{rank } F(q) = m < n$ . Without loss of generality,  $F(q)$  can be written as

$$F(q) = \begin{bmatrix} F_1(q) \\ F_2(q) \end{bmatrix}$$

such that  $F_2(q)$  is an invertible  $m \times m$  matrix and  $q$  can be decomposed to  $(q_1, q_2) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$  according to  $F(q)$ . By *input coupling*, we mean  $F_1(q) \not\equiv 0$  for all  $q$ . In the following, we provide the conditions which allow partial feedback linearization for an underactuated (simple) Lagrangian system.

**Proposition 3.6.1.** *Consider the following underactuated system with input coupling (i.e.  $F_1(q) \not\equiv 0, \det(F_2(q)) \neq 0$  for all  $q$ )*

$$\begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h_1(q, \dot{q}) \\ h_2(q, \dot{q}) \end{bmatrix} = \begin{bmatrix} F_1(q) \\ F_2(q) \end{bmatrix} \tau \quad (3.7)$$

and assume the following matrix is invertible for all  $q$

$$\Lambda(q) = F_2(q) - m_{21}(q)m_{11}^{-1}(q)F_1(q) \quad (3.8)$$

Then, there exists a change of control in the form

$$\tau = \alpha(q)u + \beta(q, \dot{q})$$

with

$$\begin{aligned} \alpha(q) &= \Lambda^{-1}(q)[m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q)] \\ \beta(q, \dot{q}) &= \Lambda^{-1}(q)[h_2(q, \dot{q}) - m_{21}(q)m_{11}^{-1}(q)h_1(q, \dot{q})] \end{aligned}$$

that partially linearizes (3.7) as the following

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= f_0(q, p) + g_0(q)u \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned}$$

where

$$\begin{aligned} f_0(q, p) &= m_{11}^{-1}(q)[F_1(q)\beta(q, \dot{q}) - h_1(q, \dot{q})] \\ g_0(q) &= m_{11}^{-1}(q)[F_1(q)\alpha(q) - m_{12}(q)] \end{aligned}$$

**Proof.** From the first equation in (3.7), we get

$$\ddot{q}_1 = -m_{11}^{-1}m_{12}\ddot{q}_2 - m_{11}^{-1}h_1 + m_{11}^{-1}F_1$$

substituting this in the second line of (3.7) gives

$$(m_{22} - m_{21}m_{11}^{-1}m_{12})\ddot{q}_2 + h_2 - m_{11}^{-1}h_1 = (F_2 - m_{21}m_{11}^{-1}F_1)\tau = \Lambda(q)\tau$$

Thus, applying  $\tau = \alpha(q)u + \beta(q, \dot{q})$  with  $\alpha, \beta$  as defined in the question partially linearizes the system and the first equation of (3.7) reduces to

$$\ddot{q}_1 = m_{11}^{-1}(q)[F_1(q)\beta(q, \dot{q}) - h_1(q, \dot{q})] + m_{11}^{-1}(q)[F_1(q)\alpha(q) - m_{12}(q)]u$$

and the result follows.  $\square$

*Remark 3.6.1.* The partial feedback linearization procedure in proposition 3.6.1 is particularly useful for autonomous vehicles in  $SE(3)$  like an aircraft or a helicopter with six degrees of freedom.

### 3.7 Normal Forms for Underactuated Systems

The complexity of control design for underactuated mechanical systems is partly due to the fact that the control input  $u$  appears in the dynamics of both the unactuated subsystem (i.e.  $(q_1, p_1)$ -subsystem) and the actuated subsystem (i.e.  $(q_2, p_2)$ -subsystem) in (3.4). In this section, we provide a method to decouple these two subsystems w.r.t. the control input  $u$  using a global change of coordinates. The following result provides an appropriate structure and sufficient conditions for the existence of a decoupling change of coordinates.

**Theorem 3.7.1.** Consider an underactuated mechanical system with an inertia matrix  $M(q) = \{m_{ij}(q)\}; i, j = 1, 2$  where  $q = (q_1, q_2)$  and  $q_1 = \{q_1^i\} \in \mathbb{R}^n$  and  $q_2 = \{q_2^j\} \in \mathbb{R}^m$  denote the unactuated and actuated configuration variables, respectively. Denote

$$g_0(q) = -m_{11}^{-1}(q)m_{12}(q)$$

and

$$g(q) = \begin{bmatrix} g_0(q) \\ I_{m \times m} \end{bmatrix}$$

where  $g_0(q) = (g_0^1(q), \dots, g_0^m(q))$  with  $g_0^j(q) \in \mathbb{R}^n$ ,  $j = 1, \dots, m$  and  $I_{m \times m}$  is the identity matrix. Define the following distribution

$$\Delta(q) = \text{span}\{\text{columns of } g(q)\}$$

that has full column rank and thus is globally nonsingular. Then, a necessary and sufficient condition for the distribution  $\Delta(q)$  to be globally involutive (i.e. completely

integrable) is that

$$\frac{\partial g_0^j(q)}{\partial q_1} g_0^i(q) - \frac{\partial g_0^i(q)}{\partial q_1} g_0^j(q) + \frac{\partial g_0^j(q)}{\partial q_2^i} - \frac{\partial g_0^i(q)}{\partial q_2^j} = 0, \forall i, j = 1, \dots, m \quad (3.9)$$

In addition, if condition (3.9) holds, there exists a global change of coordinates given by

$$\begin{aligned} z_1 &= \Phi(q_1, q_2) \\ z_2 &= (D_{q_1}\Phi(q)) \cdot p_1 + (D_{q_2}\Phi(q)) \cdot p_2 \\ \xi_1 &= q_2 \\ \xi_2 &= \dot{q}_2 \end{aligned}$$

that transforms the dynamics of the system into the normal form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \quad (3.10)$$

*Remark 3.7.1.* Normal form (3.10) is a special case of the famous Byrnes-Isidori normal form [36] with a double integrator as the following

$$\begin{aligned} \dot{z} &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \quad (3.11)$$

*Remark 3.7.2.* The main advantage of the normal form (3.10) is that the control input of the actuated subsystem of the original system does not appear in the unactuated subsystem. This simplifies control design for this underactuated system by reducing the control of the original higher order system into control of its lower-order nonlinear unactuated subsystem. This is due to a recent result by the author in [67] on control design for systems in normal forms (3.10) and (3.11) which is explained in chapter 7.

*Remark 3.7.3.* The main disadvantage of theorem 3.7.1 is that it is not always possible to find  $\Phi(q_1, q_2)$  in an analytical explicit form. Later, we show that under kinetic symmetry properties of the system it is possible to calculate  $\Phi(q_1, q_2)$  in closed-form.

**Proof.**(Theorem 3.7.1) Note that  $\Delta(q)$  globally has a full column rank of  $m$  and is therefore a globally nonsingular distribution (see [36] for definitions and notations in

this proof). Calculating the Lie bracket of the  $i$ th and  $j$ th columns of  $g(q)$ , we get

$$\begin{aligned} [g^i(q), g^j(q)] &= \begin{bmatrix} \frac{\partial g_0^j(q)}{\partial q_1} & \frac{\partial g_0^j(q)}{\partial q_2} \\ \frac{\partial e_j}{\partial q_1} & \frac{\partial e_j}{\partial q_2} \\ \frac{\partial g_0^i(q)}{\partial q_1} & \frac{\partial g_0^i(q)}{\partial q_2} \end{bmatrix} \cdot \begin{bmatrix} g_0^i(q) \\ e_i \end{bmatrix} - \begin{bmatrix} \frac{\partial g_0^i(q)}{\partial q_1} & \frac{\partial g_0^i(q)}{\partial q_2} \\ \frac{\partial e_i}{\partial q_1} & \frac{\partial e_i}{\partial q_2} \\ \frac{\partial g_0^j(q)}{\partial q_1} & \frac{\partial g_0^j(q)}{\partial q_2} \end{bmatrix} \cdot \begin{bmatrix} g_0^j(q) \\ e_j \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial g_0^j(q)}{\partial q_1} g_0^i(q) - \frac{\partial g_0^i(q)}{\partial q_1} g_0^j(q) + \frac{\partial g_0^j(q)}{\partial q_2} e_i - \frac{\partial g_0^i(q)}{\partial q_2} e_j \\ 0_{m \times 1} \end{bmatrix} \end{aligned}$$

which based on condition (3.9) implies  $[g^i(q), g^j(q)] = 0$  for all  $i, j, q$ . Therefore  $\Delta(q)$  is globally involutive. To prove the converse, assume  $\Delta(q)$  is globally involutive. Then, for all  $i, j, q$ ,  $[g^i(q), g^j(q)]$  can be expressed as a linear combination of  $g^k(q)$ 's. But the lower  $m \times 1$  block of  $[g^i(q), g^j(q)]$  is identically zero and linearly independent of  $e_k$ 's. This means that  $[g^i(q), g^j(q)] = 0$  and thus condition (3.9) holds. Now, we prove the rest of the theorem. Based on Frobenius theorem (see [36]), because  $\Delta(q)$  is a globally nonsingular and involutive distribution, the following equation

$$\frac{\partial \phi}{\partial q} g(q) = 0$$

has  $d = (n - m)$  linearly independent solutions  $\phi_k(q_1, q_2), k = 1, \dots, n$ . Denoting  $\Phi(q_1, q_2) = (\phi_1, \dots, \phi_d)$ ,  $\Phi$  satisfies the property

$$\frac{\partial \Phi}{\partial q_1} g_0(q) + \frac{\partial \Phi}{\partial q_2} = 0.$$

After applying the change of coordinates

$$z_1 = \Phi(q_1, q_2), z_2 = \dot{z}_1$$

we get

$$z_1 = \Phi(q_1, q_2), z_2 = \frac{\partial \Phi}{\partial q_1} p_1 + \frac{\partial \Phi}{\partial q_2} p_2$$

and because  $\partial \Phi / \partial q_1$  is globally nonsingular (due to the proof of Frobenius theorem), based on implicit mapping theorem, there exists a smooth function  $\Psi$  such that

$$q_1 = \Psi(z_1, \xi_1), p_1 = \left( \frac{\partial \Phi}{\partial q_1} \Big|_{q=(\Psi(z_1, \xi_1), \xi_1)} \right)^{-1} (z_2 - \xi_2 \cdot \frac{\partial \Phi}{\partial q_2} \Big|_{q=(\Psi(z_1, \xi_1), \xi_1)})$$

(we drop the substitution  $q = (\Psi(z_1, \xi_1), \xi_1)$  in  $\partial \Phi / \partial q_i$  due to the simplicity of notation). Calculating  $\dot{z}_2$  as the following

$$\dot{z}_2 = \frac{\partial \Phi}{\partial q_1} f_0(q, p) + \frac{\partial^2 \Phi}{\partial q_1^2} \left( \frac{\partial \Phi}{\partial q_1} \right)^{-2} (z_2 - \frac{\partial \Phi}{\partial q_1} \xi_2)^2 + \frac{\partial^2 \Phi}{\partial q_2^2} \xi_2^2 + \left( \frac{\partial \Phi}{\partial q_1} g_0(q) + \frac{\partial \Phi}{\partial q_2} \right) u \quad (3.12)$$

and noting that the coefficient of  $u$  in the last equation is identically zero, we get

$$\dot{z}_2 = f(z, \xi_1, \xi_2)$$

where  $f$  is the right hand side of (3.12). Therefore, the dynamics of the system in new coordinates is in normal form (3.10) and this finishes the proof.  $\square$

Here is an important corollary of theorem 3.7.1:

**Corollary 3.7.1.** *All underactuated mechanical systems with a single actuator that are globally partial feedback linearizable can be transformed into normal form (3.10) using a global invertible change of coordinates.*

**Proof.** For this case  $m = 1$  and  $i = j = 1$ . By symmetry w.r.t  $i$  and  $j$  indices, condition (3.9) globally holds for all  $q$ .  $\square$

## 3.8 Classes of Structured Nonlinear Systems

In this section, we define important classes of structured nonlinear systems that appear frequently in this thesis.

**Definition 3.8.1.** (cascade system) We say a nonlinear system is in *cascade form* if it has the following structure

$$\begin{aligned}\dot{z} &= f(z, \xi) \\ \dot{\xi} &= g(\xi, u)\end{aligned}\tag{3.13}$$

where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $g : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ ,  $(z, \xi)$  is the composite state, and  $u$  is the control input. When the second equation is a linear time-invariant system, i.e.  $\dot{\xi} = A\xi + Bu$ , (3.13) is called a *partially linear cascade nonlinear system*.

**Definition 3.8.2.** (feedback form) We say a nonlinear system is in *strict feedback form* [57], if it has the following triangular structure

$$\begin{aligned}\dot{z} &= f(z, \xi_1), \\ \dot{\xi}_1 &= \xi_2, \\ &\dots \\ \dot{\xi}_m &= u,\end{aligned}$$

**Definition 3.8.3.** (feedforward form) We say a nonlinear system is in *feedforward form* [102], if it possesses the following triangular structure

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi_1(x_2, \dots, x_n, u) \\ \dot{x}_2 &= x_3 + \varphi_2(x_3, \dots, x_n, u) \\ &\dots \\ \dot{x}_n &= u + \varphi_n(x_n, u)\end{aligned}$$

where  $x \in \mathbb{R}^n$  and the  $\varphi_i$ 's are at least quadratic in  $(x, u)$ .

**Definition 3.8.4.** (nontriangular form) We say a partially linear cascade nonlinear system is in *nontriangular form*, if it has the following structure

$$\begin{aligned}\dot{z} &= f(z, \xi_1, \xi_2, \dots, \xi_m), \\ \dot{\xi}_1 &= \xi_2, \\ &\quad \dots \\ \dot{\xi}_m &= u,\end{aligned}\tag{3.14}$$

where  $z \in \mathbb{R}^n$  and  $u \in \mathbb{R}^p$ . The nonlinear system in (3.14) is also called *Byrnes-Isidori normal form* [36].

**Definition 3.8.5.** (nontriangular linear-quadratic form) We say a cascade nonlinear system is in *nontriangular linear-quadratic form*, if it has the following structure

$$\begin{aligned}\dot{z}_1 &= \mu(z_1)z_2 + \eta(\xi_1)\xi_2 \\ \dot{z}_2 &= \phi(z_1, \xi_1) + \Sigma(\xi_1, z_2, \xi_2) \\ \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= u,\end{aligned}\tag{3.15}$$

where  $z_1, z_2 \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^p$ ,  $\mu(z_1)$  is a positive definite matrix,  $\phi : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ , and  $\Sigma : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  has the following quadratic form in  $(z_2, \xi_2)$

$$\Sigma(\xi_1, z_2, \xi_2) = \begin{bmatrix} z_2 \\ \xi_2 \end{bmatrix}^T \Pi(\xi_1) \begin{bmatrix} z_2 \\ \xi_2 \end{bmatrix}$$

where  $\Pi = (\Pi^1, \dots, \Pi^n)$  is a cubic matrix with elements in  $\mathbb{R}^{n \times n}$  and for  $v \in \mathbb{R}^n$ ,  $v^T \Pi v := (v^T \Pi^1 v, \dots, v^T \Pi^n v)^T \in \mathbb{R}^n$ . If  $\eta \equiv 0$ , (3.15) is called a *nontriangular quadratic form*, or in case  $\Sigma \equiv 0$ , (3.15) is called *nontriangular linear form*. In all cases, the quadratic or linearity properties are implicitly w.r.t.  $\xi_2$ .

*Remark 3.8.1.* Clearly, if  $\Sigma(\xi_1, z_2, \xi_2) = 0$ , then the nonlinear system in (3.15) reduces to a strict feedback system. In addition, if  $\mu(z_1) = I_n$ ,  $f(z_1, \xi_1) = \xi_1$ , and  $\Pi_{(1,1)}(\xi_1) = 0$  ( $\Pi_{(1,1)}$  is the first cubic partition of  $\Pi$  according to  $(z_2, \xi_2)$ ), then (3.15) reduces to a nonlinear system in strict feedforward form. Thus, the famous feedback/feedforward triangular forms are special cases of the quadratic nontriangular form.

*Remark 3.8.2.* In the special case where  $z_1, z_2, \xi_1, \xi_2$  are all scalars,  $\Pi(\xi_1)$  in definition 3.8.5 is a  $2 \times 2$  matrix (i.e. not a cubic matrix) which is in fact the case for underactuated mechanical systems with two DOF. However, we need the generality of definition 3.8.5 for further applications related to normal forms of higher-order underactuated systems.

### 3.9 Normal Forms for Underactuated Systems with 2 DOF

In this section, we introduce explicit cascade normal forms for underactuated mechanical systems with two degrees of freedom and kinetic symmetry w.r.t. one (actuated or unactuated) configuration variable. The importance of these normal forms is due to the fact that many examples of benchmark problems in nonlinear control design including the cart-pole system [101, 56, 94], the beam-and-ball system [34, 102, 103, 80, 79], the acrobot [13, 86], the pendubot [92], the rotating pendulum[6, 10], the inertia-wheel pendulum [93], and the TORA example [41, 37] all fall within the class of underactuated systems with two degrees of freedom and kinetic symmetry. Nevertheless, so far no (global) cascade normal forms are known for the majority of these nonlinear systems except for the beam-and-ball system, and the TORA example. This motivated us to provide a (control-oriented) classification of underactuated systems with two DOF based on their associated cascade normal forms. These cascade forms are systems in strict feedback form [57], feedforward form [102], and nontriangular quadratic form [69] (to be defined). The main benefit of this classification and the associated normal form is that if the system has feedback/feedforward triangular structure (under appropriate conditions) it can be globally asymptotically stabilized using backstepping [36, 57, 80] or forwarding procedures [102, 101, 56, 80]. The only open ended case is stabilization of systems in nontriangular quadratic form that will be addressed in this thesis in chapter 7.

Now, consider an underactuated mechanical system with two DOF and configuration vector  $q = (q_1, q_2)^T$ . Assume the inertia matrix of this system only depends on  $q_2$ , i.e.  $M = M(q_2)$ . Then, the system has kinetic symmetry w.r.t.  $q_1$ . The Lagrangian of this underactuated system can be expressed as the following

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \dot{q}^T \begin{bmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{bmatrix} \dot{q} - V(q) \quad (3.16)$$

Here is our main result on normal forms for underactuated systems with two DOF and kinetic symmetry:

**Theorem 3.9.1.** *All underactuated mechanical systems with two degrees of freedom  $(q_1, q_2)$  and kinetic symmetry w.r.t.  $q_1$  have a Lagrangian given by (3.16) and can be (possibly globally) transformed into quadratic nontriangular normal form (3.15) (or (3.10)) using a change of coordinates in analytically explicit form.*

The proof of this theorem relays on the corresponding proofs for two cases where either  $q_2$  is actuated, or  $q_2$  is unactuated. This provides a natural classification of underactuated systems to two classes as the following.

**Definition 3.9.1.** (Class-I,II underactuated systems) We call an underactuated system with Lagrangian (3.16) *Class-I underactuated system*, iff  $q_2$  is actuated. We call it a *Class-II underactuated system*, iff  $q_2$  is unactuated.

The Euler-Lagrange equations of motion for this underactuated system are

$$\begin{aligned}\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} &= \tau_1 \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} &= \tau_2\end{aligned}\quad (3.17)$$

where for a Class-I system  $\tau_1 = 0$  and for a Class-II system  $\tau_2 = 0$ . Equation (3.17) can be rewritten as

$$\begin{aligned}m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + m'_{11}(q_2)\dot{q}_1\dot{q}_2 + m'_{12}(q_2)\dot{q}_2^2 - g_1(q_1, q_2) &= \tau_1 \\ m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 - \frac{1}{2}m'_{11}(q_2)\dot{q}_1^2 + \frac{1}{2}m'_{22}(q_2)\dot{q}_2^2 - g_2(q_1, q_2) &= \tau_2\end{aligned}\quad (3.18)$$

where  $g_i(q_1, q_2) = -\partial V(q)/\partial q_i$ ,  $i = 1, 2$  and  $'$  denotes  $d/dq_2$ . The following proposition states that all Class-I underactuated systems can be globally and explicitly transformed into cascade nonlinear systems in strict feedback form.

**Proposition 3.9.1.** (*Class-I normal form*) Consider a Class-I underactuated system with two DOF. Then, the following global change of coordinates obtained from the Lagrangian of the system

$$\begin{aligned}z_1 &= q_1 + \gamma(q_2) \\ z_2 &= m_{11}(q_2)p_1 + m_{12}(q_2)p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} \\ \xi_1 &= q_2 \\ \xi_2 &= p_2\end{aligned}\quad (3.19)$$

where

$$\gamma(q_2) = \int_0^{q_2} m_{11}^{-1}(\theta)m_{12}(\theta)d\theta \quad (3.20)$$

transforms the dynamics of the system into a cascade nonlinear system in strict feedback form

$$\begin{aligned}\dot{z}_1 &= m_{11}^{-1}(\xi_1)z_2 \\ \dot{z}_2 &= g_1(z_1 - \gamma(\xi_1), \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\quad (3.21)$$

where  $u$  is the new control from collocated partial feedback linearization and  $g_1(q_1, q_2) = -\partial V(q)/\partial q_1$ .

**Proof.** By definition of  $z_1$  and  $z_2$ , we have  $\dot{z}_1 = m_{11}^{-1}(q_2)z_2$ . Noting that  $z_2 = \partial \mathcal{L}/\partial \dot{q}_1$ , from the first line of (3.17),  $\tau_1 = 0$  and we get

$$\dot{z}_2 = \frac{\partial \mathcal{L}}{\partial q_1} = \frac{\partial K}{\partial q_1} - \frac{\partial V}{\partial q_1} = g_1(q_1, q_2)$$

The last equality follows from kinetic symmetry property of the system w.r.t.  $q_1$  (i.e.  $\partial K / \partial q_1 = 0$ ).  $\square$

*Remark 3.9.1.* After a second change of coordinates

$$y_1 = z_1, \quad y_2 = m_{11}^{-1}(\xi_1)z_2$$

the dynamics of the system in (3.21) transforms into normal form (3.10).

**Corollary 3.9.1.** *The Acrobot, the Inertia-Wheel pendulum, and the TORA example are all Class-I underactuated systems and can be globally transformed into normal form (3.21) using an explicit global change of coordinates.*

**Proposition 3.9.2.** *(Class-II normal form) Consider a Class-II underactuated system with two DOF. Then, the following change of coordinates obtained from the Lagrangian of the system*

$$\begin{aligned} z_1 &= q_1 + \gamma(q_2) \\ z_2 &= m_{21}(q_2)p_1 + m_{22}(q_2)p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} \\ \xi_1 &= q_2 \\ \xi_2 &= p_2 \end{aligned} \tag{3.22}$$

where

$$\gamma(q_2) = \int_0^{q_2} m_{21}^{-1}(\theta)m_{22}(\theta)d\theta \tag{3.23}$$

is defined over

$$U = \{q_2 \mid m_{21}(q_2) \neq 0\}$$

transforms the dynamics of the system into a cascade nonlinear system in quadratic nontriangular form where  $u$  is the new control from noncollocated partial feedback linearization.

**Proof.** By definition of  $z_1$  and  $z_2$ , it follows that  $\dot{z}_1 = z_2/m_{21}(q_2)$ . From the second line of (3.16),  $\tau_2 = 0$  and we get

$$\begin{aligned} \dot{z}_2 &= \partial \mathcal{L} / \partial q_2 \\ &= \partial K / \partial q_2 - \partial V(q) / \partial q_2 \quad (\partial K / \partial q_2 \neq 0) \\ &= g_2(q_1, q_2) + \frac{1}{2}m'_{11}(q_2)p_1^2 + m'_{21}(q_2)p_1p_2 + \frac{1}{2}m'_{22}(q_2)p_2^2 \end{aligned}$$

Hence, after substituting

$$\begin{aligned} q_1 &= z_1 - \gamma(q_2) \\ p_1 &= (z_2 - m_{22}(q_2)p_2) / m_{21}(q_2) \end{aligned}$$

in the last equation, we obtain

$$\begin{aligned}\dot{z}_2 &= g_2(z_1 - \gamma(q_2), q_2) + \frac{m'_{11}(q_2)}{2m_{21}^2(q_2)}[z_2 - m_{22}(q_2)p_2]^2 \\ &\quad + \frac{m'_{21}(q_2)}{m_{21}(q_2)}[z_2 - m_{22}(q_2)p_2]p_2 + \frac{m'_{22}(q_2)}{2}p_2^2\end{aligned}$$

which gives the following quadratic nontriangular normal form for Class-II underactuated systems

$$\begin{aligned}\dot{z}_1 &= m_{21}^{-1}(q_2)z_2 \\ \dot{z}_2 &= g_2(z_1 - \gamma(q_2), q_2) \\ &\quad + \frac{m'_{11}(q_2)}{2m_{21}^2(q_2)}z_2^2 + \left\{ \frac{m'_{21}(q_2)}{m_{21}(q_2)} - \frac{m_{22}(q_2)m'_{11}(q_2)}{2m_{21}^2(q_2)} \right\} z_2p_2 \\ &\quad + \left\{ \frac{m'_{22}(q_2)}{2m_{21}^2(q_2)}m'_{11}(q_2) - \frac{m_{22}(q_2)}{m_{21}(q_2)}m'_{21}(q_2) + \frac{1}{2}m'_{22}(q_2) \right\} p_2^2 \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\tag{3.24}$$

□

**Corollary 3.9.2.** *The Pendubot, the Cart-Pole system, the Rotating Pendulum, and the Beam-and-Ball system are all Class-II underactuated systems with two degrees of freedom and can be transformed into a cascade nonlinear system in quadratic nontriangular form over the set  $U = \{q_2 \mid m_{21}(q_2) \neq 0\}$  using a change of coordinates in explicit form.*

**Proof.** The proof is by direct substitution of the elements of the inertia matrix of each system in equation (3.24). □

**Proposition 3.9.3.** *(feedforward normal form) Consider a Class-II underactuated system with two degrees of freedom over the set  $U = \{q_2 \mid m_{21}(q_2) \neq 0\}$  and assume the following conditions hold:*

- i)  $g_2(q_1, q_2)$  is independent of  $q_1$ , i.e.  $D_{q_1}D_{q_2}V(q) = 0$ .
- ii)  $m_{11}$  is constant.
- iii)  $\psi(q_2) = g_2(q_2)/m_{21}(q_2)$  satisfies  $\psi'(0) \neq 0$ .

Then, applying the change of coordinates

$$y_1 = z_1, \quad y_2 = z_2/m_{21}(q_2)$$

transforms the normal form (3.24) for Class-II underactuated systems with two de-

grees of freedom into the following cascade nonlinear system in feedforward form

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \psi(q_2) + \left\{ \frac{m'_{22}(q_2)}{2m_{21}(q_2)} - \frac{m_{22}(q_2)}{m_{21}^2(q_2)} m'_{21}(q_2) \right\} p_2^2 \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\tag{3.25}$$

In addition, the origin for this feedforward system can be globally asymptotically stabilized using nested saturations [102].

**Proof.** The normal form (3.25) follows from the proof of proposition 3.9.2 by direct calculation and conditions *i*), *ii*). The stabilization using nested saturations follows from condition *iii*) and [102].  $\square$

**Corollary 3.9.3.** *The Cart-Pole system can be transformed into a cascade nonlinear system in feedforward form using the change of coordinates in proposition 3.9.3 and can be globally asymptotically stabilized to its upright equilibrium point over  $q_2 \in (-\pi/2, \pi/2)$  using nested saturations.*

**Proof.** For the cart-pole system,  $g_2 = g_2(q_2)$  and  $m_{11}$  is constant. In addition,  $\psi(q_2) = g \tan(q_2)$  ( $g$  is the gravity constant) that implies  $\psi'(0) = g \neq 0$ . Thus, the cart-pole system satisfies all three conditions in proposition 3.9.3. Also, because  $m_{21}(q_2) = a \cos(q_2)$  ( $a > 0$  is a constant), the stabilization result over  $q_2 \in (-\pi/2, \pi/2)$  follows (see section 5.4 for further details on control design for the cart-pole system).  $\square$

# Chapter 4

## Reduction and Control of High-Order Underactuated Systems

In this chapter, we address reduction of high-order underactuated mechanical systems with kinetic symmetry. By reduction, we mean control of the original higher-order underactuated system reduces to control of a lower-order nonlinear system (we precisely define reduction later on). It turns out that actuation or lack of actuation of a subset of configuration variables that somehow represent the “shape” of a mechanical system plays an important role in reduction of underactuated mechanical systems. The details regarding the notion of “shape variables” for Lagrangian systems and “kinetic symmetry” are discussed in the following section.

Our main contribution in this chapter is to obtain structured cascade normal forms for higher-order underactuated systems in explicit form and present an appropriate control design method for each normal form. These cascade normal forms have structural properties that allow effective use of existing and recently developed systematic nonlinear control design methods. Types of cascade normal forms that are explicitly obtained for underactuated systems include nonlinear systems in *strict feedback form*, *feedforward form*, and *nontriangular linear/quadratic form* (see section 3.8 for the definitions).

### 4.1 Shape Variables and Kinetic Symmetry

Consider a simple Lagrangian system with configuration vector  $q \in Q$ . The variables that appear in the inertia matrix of the system are called *shape variables*. If a configuration variable  $q_j$  does not appear in the inertia matrix i.e.  $\partial M(q)/\partial q_j = 0$ , it is called an *external variable*. This implies that for an external variable  $q_j$  the following identity holds

$$\frac{\partial K(q, \dot{q})}{\partial q_j} = 0$$

In other words, a simple Lagrangian system has kinetic symmetry w.r.t. external variables. A more general way to define this symmetry is via action of a group on a manifold. Let  $q_j \in G$  where  $G$  is a cyclic group, e.g.  $\mathbb{R}$  (a translation group), or  $S_1$  (a rotation group). Decompose the configuration manifold as  $Q = B \times G$ . The action of  $G$  on  $Q$  is a mapping  $\Phi : Q \times G \rightarrow Q$ . For a fix  $j \in \{1, \dots, n\}$ , let  $\alpha \in G$  and  $q \in Q$ , define  $\Phi(q, \alpha) = (q_1, \dots, q_j + \alpha, \dots, q_n)$ . Formally, we call  $q_j$  an *external variable*, if  $M(q) = M(\Phi(q, \alpha))$ . Let  $Q_x = G_1 \times \dots \times G_n$  be the configuration manifold of all external variables where  $G_i$ 's are one-dimensional cyclic groups. Then  $Q = Q_x \times Q_s$  where  $Q_s$  is called the *shape space*, i.e. the configuration manifold of the shape variables. The notion of shape variables or internal variables originally appeared in the control literature by the study of interconnected mechanical systems and multi-body problems [44, 45, 46, 51].

In the classical sense, symmetry in mechanics is defined as the invariance of the Lagrangian under the action of a group. This is fundamentally different from the invariance of the kinetic energy. The classical symmetry gives raise to the existence of conserved quantities while the kinetic symmetry does not give raise to the conservation of the generalized momentums. (unless both  $\tau_j$  and  $\partial V(q)/\partial q_j$  (i.e. the  $j$ th gravity term) vanish in which case  $\partial \mathcal{L}/\partial \dot{q}_j$  is a conserved quantity). The fact that the generalized momentum  $p_j = \partial \mathcal{L}/\partial \dot{q}_j$  is not a conserved quantity in the presence of the kinetic symmetry plays a vital role in controllability and stabilization of broad classes of underactuated mechanical systems (see remark 4.2.4).

Throughout this chapter, we consider underactuated mechanical control systems with kinetic symmetry that have the following Lagrangian

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q_s) \dot{q} - V(q) \quad (4.1)$$

where  $q = (q_x, q_s) \in Q = Q_x \times Q_s$  is the configuration vector decomposed to external and shape variables. The forced Euler-Lagrange equations of motion for this system is

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_x} - \frac{\partial \mathcal{L}}{\partial q} &= F_x(q)\tau \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_x} - \frac{\partial \mathcal{L}}{\partial q} &= F_s(q)\tau \end{aligned} \quad (4.2)$$

where  $\tau \in \mathbb{R}^m$  and  $F(q) = \text{col}(F_x(q), F_s(q))$  is the *force matrix* with the underactuation property

$$\text{rank } F(q) = m < n = \dim(Q)$$

In analysis and control design for the underactuated system (4.2), a number of cases arise depending on whether the shape configuration vector  $q_s$  is fully-actuated, partially-actuated, or unactuated and the presence or lack of any input couplings due to the force matrix  $F(q)$ . This leads to analysis of a finite number of cases. For each

case, we provide a method for reduction of the given underactuated system and global transformation of the system into a cascade normal form. Eventually, this leads to classification of underactuated control systems based on four basic properties (to be mentioned later). It turns out that there are eight different classes of underactuated systems and three types of obtained normal forms. Namely, cascade nonlinear systems in strict feedback form, strict feedforward form, and nontriangular linear-quadratic form.

## 4.2 Underactuated Systems with Noninteracting Inputs and Integrable Momentums

In this section, we address reduction of high-order underactuated systems with integrable normalized momentums for the noninteracting input case. There could be three cases: i) underactuated systems with fully-actuated shape variables, ii) underactuated systems with unactuated shape variables, and iii) underactuated systems with partially-actuated shape variables. We address reduction of cases i) and ii) in the following. Case iii) is rather similar to Case ii) and will not be discussed in this section. Later, we present reduction of underactuated systems with non-integrable momentums in case iii).

### 4.2.1 Underactuated Systems with Actuated Shape Variables

First, we consider underactuated mechanical systems in (4.2) with fully-actuated shape variables and noninteracting inputs i.e.  $F_x(q) = 0, F_s(q) = I_m$ .

**Theorem 4.2.1.** *Consider an underactuated system with kinetic symmetry w.r.t.  $q_x$  and fully-actuated shape variables  $q_s$*

$$\begin{aligned} m_{xx}(q_s)\ddot{q}_x + m_{xs}(q_s)\ddot{q}_s + h_x(q, \dot{q}) &= 0 \\ m_{sx}(q_s)\ddot{q}_x + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) &= \tau \end{aligned} \quad (4.3)$$

*Let  $\tau = \alpha(q_s)u + \beta(q, \dot{q})$  be the collocated partially linearizing change of control for (4.3). Assume all the elements of*

$$\omega = m_{xx}^{-1}(q_s)m_{xs}(q_s)dq_s \quad (4.4)$$

*are exact one-forms and let  $\omega = d\gamma(q_s)$ . Then, there exists a global change of coordinates (i.e. diffeomorphism) obtained from the Lagrangian of the system*

$$\begin{aligned} q_r &= q_x + \gamma(q_s) =: \Phi(q_x, q_s) \\ p_r &= m_{xx}(q_s)\dot{q}_x + m_{xs}(q_s)\dot{q}_s = \frac{\partial \mathcal{L}}{\partial \dot{q}_x} \end{aligned} \quad (4.5)$$

*that transforms the dynamics of the underactuated system in (4.3) into a cascade*

normal form in strict feedback form

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u\end{aligned}\tag{4.6}$$

In addition, the  $(q_r, p_r)$ -subsystem is a Lagrangian system with configuration vector  $q_r$  that belongs to the reduced manifold  $Q_r = \Phi(Q_x \times Q_s)$  with reduced Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2}q_r^T m_r(q_s)\dot{q}_r - V_r(q_r, q_s)\tag{4.7}$$

that satisfies the unforced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = 0$$

with

$$\begin{aligned}m_r(q_s) &:= m_{xx}(q_s) \\ V_r(q_r, q_s) &:= V(q_r - \gamma(q_s), q_s) \\ g_r(q_r, q_s) &:= -\partial V_r(q_r, q_s)/\partial q_r\end{aligned}$$

**Proof.** By definition of  $q_r, p_r$ , we have

$$\dot{q}_r = m_{xx}^{-1}(q_s)p_r = m_r^{-1}(q_s)p_r$$

From the first line of Euler-Lagrange equation in (4.2), we get

$$\dot{p}_r = \frac{\partial \mathcal{L}}{\partial q_x} = \frac{\partial K}{\partial q_x} - \frac{\partial V(q_x, q_s)}{\partial q_x} =: g_x(q_x, q_r)$$

due to kinetic symmetry w.r.t.  $q_x$ ,  $\partial K/\partial q_x = 0$ . But

$$g_x(q_x, q_s) = -\frac{\partial V(q_r - \gamma(q_s), q_s)}{\partial q_r} \frac{\partial q_r}{\partial q_x} = g_r(q_r, q_s) \times I_{n-m} = g_r(q_r, q_s)$$

and the second line of (4.6) follows. By direct calculation

$$p_r = \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r}, \quad \dot{p}_r = \frac{\partial \mathcal{L}_r}{\partial q_r}$$

which proves the reduced Lagrangian  $\mathcal{L}_r$  satisfies the unforced Euler-Lagrange equation.  $\square$

*Remark 4.2.1.* Stabilization of the reduced system

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s)\end{aligned}\tag{4.8}$$

is addressed in section 4.7 of this chapter.

*Remark 4.2.2.* After renaming the variables in (4.6) as  $(z_1, z_2) = (q_r, p_r)$  and  $(\xi_1, \xi_2) = (q_s, p_s)$ , one obtains

$$\begin{aligned}\dot{z} &= f(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.9}$$

which is a special case of *Byrnes-Isidori normal form* [36] associated to the output  $\xi_1 = q_s$  which has global uniform relative degree two and the following *Lagrangian zero-dynamics*

$$\begin{aligned}\dot{z}_1 &= m_r^{-1}(\xi_1)z_2 \\ \dot{z}_2 &= g_x(z_1 - \gamma(\xi_1), \xi_1)\end{aligned}$$

*Remark 4.2.3.* Based on standard backstepping procedure, control design for the nonlinear system in (4.3) reduces to control design for its  $(q_r, p_r)$ -subsystem with control input  $q_s$ . For this reason, we call  $(q_r, p_r)$ -subsystem the *reduced system* with reduced configuration vector  $q_r$ . Explicit transformation of the system into the normal form (4.6) reduces control of the original underactuated nonlinear system with  $n$  second-order subsystems to control of the reduced system with  $(n - m)$  second-order subsystems. This is a tremendous reduction in complexity of control design for (possibly) high-order underactuated systems.

*Remark 4.2.4.* If the potential energy  $V(q)$  is independent of the external variable  $q_x$ , i.e.  $\partial V(q_x, q_s)/\partial q_x = 0$ , then  $g_x = 0$  and the generalized momentum  $p_r$  is a conserved quantity. Therefore, the nonlinear system in (4.6) is not controllable or stabilizable to any equilibrium points for initial conditions with  $p_r(0) \neq 0$ . For example, neither the Acrobot, nor the Pendubot are controllable/stabilizable in lack of gravity.

The fact that the reduced system with configuration vector  $q_r$  is a simple Lagrangian system that satisfies the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = 0$$

without any input forces means that the system must be controlled via its potential force (or energy) that is parameterized by  $q_s$ . Therefore the shape vector  $q_s$  plays the role of the control input for the dynamics of the reduced system in  $Q_r$ . In addition,

the reduced system is a *double-integrator gradient system* parameterized with  $q_s$  as

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= -\frac{\partial V_r(q_r, q_s)}{\partial q_r}\end{aligned}$$

which admits a *reduced Hamiltonian* parameterized by  $q_s$  as the following

$$H_r(q_r, p_r, q_s) = \frac{1}{2}p_r^T m_r^{-1}(q_s)p_r + V_r(q_r, q_s) \quad (4.10)$$

that satisfies

$$\dot{H}_r = \frac{\partial H_r(q_r, p_r, q_s)}{\partial q_s} p_s \quad (4.11)$$

Based on (4.11) the reduced Hamiltonian  $H_r$  is not necessarily a conserved quantity. To see this, first we need the following definition.

**Definition 4.2.1.** (vector sigmoidal function)  $\vec{\sigma}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called a *vector sigmoidal function* if it satisfies the following properties.

- i)  $\exists L > 0 : \forall x \in \mathbb{R}^n, \|\vec{\sigma}(x)\| \leq L$ .
- ii)  $\sigma(0) = 0$  and  $x\vec{\sigma}(x) > 0, \forall x \neq 0$ .
- iii)  $y^T(\vec{\sigma}(x) - \vec{\sigma}(x+y)) < 0, \forall y \neq 0$ . Examples of vector sigmoidal functions are  $\vec{\sigma}(x) = (\sigma_1(x_1), \dots, \sigma_n(x_n))^T$  where  $\sigma_i$  are one-dimensional sigmoidal functions like  $\arctan(z)$  and  $\tanh(z)$ .

Now, consider the following cases:

- i)  $p_s = 0$ , or  $p_s$  satisfying  $\nabla_{q_s} H_r \cdot p_s = 0$  conserves the energy  $H_r$ .
- ii)  $p_s = -\vec{\sigma}(\nabla_{q_s} H_r)$  decreases the energy of the reduced system.
- iii)  $p_s = \vec{\sigma}(\nabla_{q_s} H_r)$  increases the energy of the reduced system.

In other words, *the energy is not necessarily conserved for the reduced system in the parameterized Hamiltonian form*

$$\begin{aligned}\dot{q}_r &= \frac{\partial H_r(q_r, p_r, q_s)}{\partial p_r} \\ \dot{p}_r &= -\frac{\partial H_r(q_r, p_r, q_s)}{\partial q_r}\end{aligned}$$

**Remark 4.2.5.** The fact that the total energy of the reduced system can be increased or decreased is useful for swing-up control design and stabilization of an underactuated system to an equilibrium manifold/point, respectively.

## 4.2.2 Underactuated Systems with Unactuated Shape Variables

In this section, we address reduction of underactuated mechanical systems in (4.2) that all of their shape variables are unactuated, have equal number of external and shape variables, and their inputs are noninteracting, i.e.  $F_x(q) = I_m, F_s(q) = 0$ .

**Theorem 4.2.2.** *Consider an underactuated mechanical system with fully-actuated external variables  $q_x$  and unactuated shape variable  $q_s$  and assume  $\dim(Q_x) = \dim(Q_s) = m$ . The Euler-Lagrange equations of motion for this system is as the following*

$$\begin{aligned} m_{xx}(q_s)\ddot{q}_x + m_{xs}(q_s)\ddot{q}_s + h_x(q, \dot{q}) &= \tau \\ m_{sx}(q_s)\ddot{q}_x + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) &= 0 \end{aligned} \quad (4.12)$$

Let  $\tau = \alpha(q_s)u + \beta(q, \dot{q})$  be the noncollocated partially linearizing change of control for (4.12) over

$$U = \{q_s \in Q_s \mid \det(m_{xs}(q_s)) \neq 0\}$$

Assume all the elements of

$$\omega = m_{sx}^{-1}(q_s)m_{ss}(q_s)dq_s$$

are exact one-forms over  $U$  and let  $\omega = d\gamma(q_s)$ . Then, there exists a global change of coordinates (i.e. diffeomorphism) obtained from the Lagrangian of the system

$$\begin{aligned} q_r &= q_x + \gamma(q_s) \\ p_r &= m_{sx}(q_s)\dot{q}_x + m_{ss}(q_s)\dot{q}_s = \frac{\partial \mathcal{L}}{\partial \dot{q}_s} \end{aligned} \quad (4.13)$$

that transforms the dynamics of the underactuated system (4.12) into a cascade nonlinear system in nontriangular quadratic form

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \quad (4.14)$$

where  $\Sigma = \nabla_{q_s}K$  ( $K$  is the kinetic energy) is a quadratic form in  $(p_r, p_s)$

$$\Sigma(q_s, p_r, p_s) = \begin{bmatrix} p_r \\ p_s \end{bmatrix}^T \Pi(q_s) \begin{bmatrix} p_r \\ p_s \end{bmatrix}$$

with a cubic weight matrix  $\Pi(q_s)$  and

$$\begin{aligned} m_r(q_s) &:= m_{sx}(q_s) \\ g_r(q_r, q_s) &:= -[\nabla_{q_s}V_r(q_x, q_s)]_{q_x=q_r-\gamma(q_s)} \end{aligned}$$

In addition, if  $V(q) = V(q_x)$ , then  $g_r \equiv 0$  and the  $(q_r, p_r)$ -subsystem is a Lagrangian system with configuration vector  $q_r$  and reduced Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2} \dot{q}_r^T m_r(q_s) \dot{q}_r$$

that satisfies the forced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = \Sigma(q_s, p_r, p_s)$$

**Proof.** By definition of  $q_r$  and  $p_r$ , we have  $\dot{q}_r = m_r^{-1}(q_s)p_r$ . Calculating  $\dot{p}_r$ , we get

$$\begin{aligned} \dot{p}_r &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_s} = \frac{\partial \mathcal{L}}{\partial q_s} \\ &= -\frac{\partial V(q_x, q_s)}{\partial q_s} + \frac{\partial K}{\partial q_s} \\ &= g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s) \end{aligned}$$

Noting that  $\partial \mathcal{L}_r / \partial q_r = 0$ , one obtains

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = \dot{p}_r = \Sigma$$

and the forced Euler-Lagrange equation in the question follows. It remains to calculate  $K(q_s, p_r, p_s)$  and show it has a quadratic form with a cubic weight matrix. For doing so, note that

$$\begin{bmatrix} \dot{q}_x \\ \dot{q}_s \end{bmatrix} = W(q_s) \begin{bmatrix} p_r \\ p_s \end{bmatrix}$$

where  $W$  is an  $n \times n$  matrix given by

$$W(q_s) = \begin{bmatrix} m_{sx}^{-1}(q_s) & -m_{sx}^{-1}(q_s)m_{ss}(q_s) \\ 0 & I_m \end{bmatrix}$$

Thus

$$K(q_s, p_r, p_s) = \frac{1}{2} \begin{bmatrix} p_r \\ p_s \end{bmatrix}^T N(q_s) \begin{bmatrix} p_r \\ p_s \end{bmatrix}$$

with

$$N(q_s) = W^T(q_s) M(q_s) W(q_s)$$

Setting the  $i$ th layer of  $\Pi(q_s)$  to  $dN(q_s)/dq_s^i$  (i.e. element-wise derivative of  $N(q_s)$ )

w.r.t. the  $i$ th component of  $q_s$ ) finishes the proof.  $\square$

**Example 4.2.1.** Consider an underactuated system with configuration vector  $q = (x_1, x_2, \theta_1, \theta_2)$  that belongs to  $Q = \mathbb{R} \times \mathbb{R} \times S_1 \times S_1$ . The system has the following Lagrangian and  $(x_1, x_2)$  are actuated with external forces  $F = (F_1, F_2)^T$

$$\mathcal{L} = \frac{1}{2} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & \cos \theta_1 & 0 \\ 0 & 1 & 0 & \cos \theta_2 \\ \cos \theta_1 & 0 & 1 & 0 \\ 0 & \cos \theta_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} - \cos \theta_1 \cos \theta_2$$

Apparently,  $q_s = (\theta_1, \theta_2)$  is the vector of shape variables and  $q_x = (x_1, x_2)$  is the vector of actuated external variables of this system. We have

$$m_{sx}(q_s) = \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix}$$

and  $\det(m_{sx}) = \cos \theta_1 \cos \theta_2$ . Thus, over  $U = \mathbb{R}^2 \times (-\pi/2, \pi/2)^2$ ,  $m_{sx}(q_s)$  is invertible and the system can be partially linearized using a noncollocated partially linearizing change of control  $F = \alpha(q_s)u + \beta(q_s, \dot{q})$ . Moreover, the vector of one-forms

$$\omega = m_{sx}^{-1}(q_s)m_{ss}dq_s = \begin{bmatrix} \frac{1}{\cos \theta_1} & 0 \\ 0 & \frac{1}{\cos \theta_2} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$

has exact elements  $\omega_1 = (1/\cos \theta_1)d\theta_1$  and  $\omega_2 = (1/\cos \theta_2)d\theta_2$  which means

$$\gamma(\theta_1, \theta_2) = (\gamma_0(\theta_1), \gamma_0(\theta_2))^T$$

with

$$\gamma_0(\theta) = \int_0^\theta \frac{ds}{\cos s} = \log \left( \frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)} \right), \quad \theta \in (-\pi/2, \pi/2)$$

Now, based on theorem 4.2.2 after applying the change of coordinates

$$q_r = \begin{bmatrix} x_1 + \gamma_0(\theta_1) \\ x_2 + \gamma_0(\theta_2) \end{bmatrix}, \quad p_r = \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

the dynamics of the system transforms into normal form (4.14).

Before presenting our next result, we need the following definition and lemma.

**Definition 4.2.2.** We say a square matrix function  $m(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n^2}$  has *differentially symmetric rows*, if the  $i^{th}$  row of the matrix  $m_{i*}(x)$  satisfies

$$\frac{\partial m(x)}{\partial x_i} = \frac{\partial m_{i*}(x)}{\partial x} \quad (4.15)$$

for  $i = 1, \dots, n$ .

*Remark 4.2.6.* Apparently, in the scalar case with  $n = 1$ , the condition of the differential symmetry of rows trivially holds. This is already used in reduction of a planar Cart-Pole system in corollary 3.9.3. In addition, any linear combination of a constant matrix and diagonal matrices in the form  $\text{diag}(f_1(x_1), \dots, f_n(x_n))$  where  $f_i$ 's are scalar functions has differentially symmetric rows as well. The following matrix is an example of a non-diagonal matrix with differentially symmetric rows

$$m(x_1, x_2) = \begin{bmatrix} \cos x_1 & \sin x_1 \sin x_2 \\ 0 & \cos x_1 \cos x_2 \end{bmatrix}$$

where  $x_1, x_2 \in \mathbb{R}$ .

**Lemma 4.2.1.** Suppose  $m(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n^2}$  is a matrix with differentially symmetric rows. Let  $\dot{x}(t) = v(t), w(t) \in \mathbb{R}^n$  be functions of time. Then, the following identity holds

$$\dot{m}(x, \dot{x})w(t) = \nabla_x \{v(t)^T m(x)w(t)\}_{x=x(t)} \quad (4.16)$$

where  $\dot{m}$  is the element-wise time-derivative of the matrix  $m(x(t))$ .

**Proof.** By definition of  $\dot{m}$ , we have

$$\begin{aligned} \dot{m}(x, \dot{x})w(t) &= \text{col}(\dot{m}_{1*}w(t), \dots, \dot{m}_{n*}w(t)) \\ &= \text{col}(\dot{x}^T \nabla_x m_{1*}(x)w(t), \dots, \dot{x}^T \nabla_x m_{n*}(x)w(t)) \\ &= \text{col}(\dot{x}^T \nabla_{x_1} m(x)w(t), \dots, \dot{x}^T \nabla_{x_n} m(x)w(t)) \\ &= \nabla_x \{v(t)^T m(x)w(t)\}_{x=x(t)} \end{aligned}$$

□

**Proposition 4.2.1.** Assume all the conditions in theorem 4.2.2 hold. In addition, the underactuated system (4.12) satisfies the following conditions

- i)  $m_{xx}(q_s)$  is constant.
- ii)  $m_{sx}(q_s)$  has differentially symmetric rows, i.e.

$$\frac{\partial m_{sx}(q_s)}{\partial q_s^i} = \frac{\partial m_{sx}^{i*}}{\partial q_s}, \quad i = 1, \dots, m$$

where  $m_{sx}^{i*}(q_s)$  is the  $i$ th row of  $m_{sx}(q_s)$ .

- iii)  $V(q) = V(q_s)$ .

Then, applying the change of coordinates

$$z_1 = q_r, \quad z_2 = m_r^{-1}(q_s)p_r$$

(where  $(q_r, p_r)$  are defined in (4.13)) transforms the original system (4.12) into a cascade system in feedforward form as the following

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \psi(q_s) + p_s^T \Pi(q_s) p_s \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u\end{aligned}\tag{4.17}$$

where  $\Pi(q_s)$  is a cubic matrix and  $\psi : Q_s \rightarrow \mathbb{R}^m$  is defined as

$$\psi(q_s) = -m_r^{-1}(q_s) \nabla_{q_s} V(q_s)$$

Moreover, if  $\psi(0) = 0$  and  $\psi(q_s)$  has an invertible Jacobian  $\nabla_{q_s} \psi(q_s)$  at  $q_s = 0$ , the origin for (4.17) (and (4.12)) can be globally asymptotically and locally exponentially stabilized over  $U$  using a state feedback in explicit form as nested saturations.

**Proof.** We prove that under assumptions in the question  $\dot{p}_r$  only contains a quadratic form in  $p_s$  plus  $g_r(q_s) = -\nabla_{q_s} V(q_s)$  and is independent of  $p_r$ . For doing so, recall that

$$\Sigma(q_s, p_r, p_s) = \nabla_{q_s} K$$

in equation (4.14). Since  $m_{xx}$  is constant and

$$K = \frac{1}{2} p_x^T m_{xx} p_x + p_s^T m_{sx}(q_s) p_x + \frac{1}{2} p_s^T m_{ss}(q_s) p_s$$

we get

$$\nabla_{q_s} K = \nabla_{q_s} \{p_s^T m_{sx}(q_s) p_x\} + \nabla_{q_s} \left\{ \frac{1}{2} p_s^T m_{ss}(q_s) p_s \right\}$$

Clearly, only the first term depends on  $p_x$  (and thus  $p_r$ ). Based on lemma 4.2.1, condition ii) implies  $m_{sx}(q_s)$  satisfies the following property

$$\dot{m}_{sx} p_x = \nabla_{q_s} \{p_s^T m_{sx}(q_s) p_x\} \tag{4.18}$$

Notice that the time derivative in  $\dot{m}_{sx}$  is taken element-wise. Calculating  $\dot{z}_2$ , we obtain

$$\dot{z}_2 = \frac{d}{dt} \{m_{sx}^{-1}(q_s)\} p_r + m_{sx}^{-1}(q_s) \dot{p}_r$$

but  $m_{sx}^{-1}(q_s) m_{sx}(q_s) = I_m$  and after differentiating both sides w.r.t. time, we get

$$\frac{d}{dt} m_{sx}^{-1}(q_s) = -m_{sx}^{-1}(q_s) \dot{m}_{sx} m_{sx}^{-1}(q_s)$$

Therefore,  $\dot{z}_2$  can be expressed as

$$\dot{z}_2 = m_{sx}^{-1}(q_s) \{-\dot{m}_{sx}[p_x + m_{sx}^{-1}(q_s)m_{ss}(q_s)p_s] + \dot{p}_r\}$$

or

$$\begin{aligned}\dot{z}_2 &= m_{sx}^{-1}(q_s) [-\dot{m}_{sx}p_x + \nabla_{q_s}\{p_s^T m_{sx}(q_s)p_x\} - \dot{m}_{sx}m_{sx}^{-1}(q_s)m_{ss}(q_s)p_s \\ &+ g_r(q_s) + \nabla_{q_s}\{\frac{1}{2}p_s^T m_{ss}(q_s)p_s\}] \\ &= m_{sx}^{-1}(q_s)g_r(q_s) + p_s^T \Pi(q_s)p_s\end{aligned}$$

where  $\Pi(q_s)$  satisfies

$$-m_{sx}^{-1}(q_s)\dot{m}_{sx}m_{sx}^{-1}(q_s)m_{ss}(q_s)p_s + m_{sx}^{-1}(q_s)\nabla_{q_s}\{\frac{1}{2}p_s^T m_{ss}(q_s)p_s\} = p_s^T \Pi(q_s)p_s$$

and the feedforward normal form in (4.17) follows. The stabilization of feedforward systems with higher-order perturbations using nested saturations is due Teel [102]. To obtain the stabilizing state feedback law explicitly, define

$$z_3 = q_s, z_4 = p_s$$

where  $z_i \in \mathbb{R}^m$  and let

$$A = [\nabla_{q_s} \psi(q_s)]_{q_s=0}$$

be the invertible matrix in the question. The dynamics of (4.17) can be written as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= Az_3 + \varphi(z_3, z_4) \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= u\end{aligned}$$

where  $\varphi(z_3, z_4) = z_4^T \Pi(z_3)z_4 + \psi(z_3) - Az_3$  is quadratic in  $(z_3, z_4)$ . Following [102], define the change of coordinates and control

$$\begin{aligned}\eta_1 &= A^{-1}z_1 + 2A^{-1}z_2 + 2z_3 + z_4 \\ \eta_2 &= A^{-1}z_2 + z_3 + z_4 \\ w &= z_3 + z_4 + u\end{aligned}$$

we get

$$\begin{aligned}\dot{\eta}_1 &= \eta_2 + w + 2A^{-1}\varphi(z_3, z_4) \\ \dot{\eta}_2 &= w\end{aligned}$$

Setting

$$w = -\vec{\sigma}_1(\eta_2 + \vec{\sigma}_2(\eta_1))$$

or

$$u = -z_3 - z_4 + \vec{\sigma}_1(\eta_2) + \vec{\sigma}_2(\eta_1)$$

globally asymptotically stabilizes the origin for this system where  $\vec{\sigma}_i$ 's are saturation functions that operate component-wise on an  $m$ -dimensional vector and have sufficiently small thresholds and magnitudes (see [102] for further details).  $\square$

A restriction of underactuated systems with unactuated shape variables which are considered in this section is that they have equal number of external and shape variables. This is in fact not necessary, if one uses a collocated partially linearizing feedback that linearizes the dynamics of external variables, or a noncollocated nonlinear combination of  $q_x, q_s$ . The following result provides a global normal form for a class of underactuated systems with unactuated shape variables. This class is important particularly due to its application in tracking control for flexible-link robots.

**Theorem 4.2.3.** Consider an underactuated mechanical system with fully-actuated external variables  $q_x \in \mathbb{R}^{n-m}$  and unactuated shape variable  $q_s \in \mathbb{R}^m$  which is augmented with an integrator at the input and satisfies the following equations of motion

$$\begin{aligned} m_{xx}(q_s)\ddot{q}_x + m_{xs}(q_s)\ddot{q}_s + h_x(q, \dot{q}) &= \tau \\ m_{sx}(q_s)\ddot{q}_x + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) &= F_d(q_s, \dot{q}_s) \\ \dot{\tau} &= v \end{aligned} \quad (4.19)$$

where  $F_d(q_s, \dot{q}_s)$  is an internal damping force acting on shape variables (e.g.  $F_d = -D(q_s)\dot{q}_s$  with  $D(q_s) = D^T(q_s) \geq 0$ ). Assume the potential energy of the system only depends on shape variables, i.e.  $V(q) = V(q_s)$ . Suppose all the elements of the vector

$$\omega = m_{xx}^{-1}(q_s)m_{xs}(q_s)dq_s$$

are exact one-forms and let  $\omega = d\gamma(q_s)$ . Then, there exists a nonlinear output

$$y = h(q) = q_x + \gamma(q_s)$$

which has global relative degree 3 w.r.t.  $v$  so that after applying the global change of coordinates

$$\begin{aligned} z_1 &= q_s \\ z_2 &= p_s \\ \xi_1 &= y = q_x + \gamma(q_s) \\ \xi_2 &= \dot{y} = \dot{q}_x + m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s \\ \xi_3 &= \ddot{y} = \ddot{q}_x + m_{xx}^{-1}(q_s)m_{xs}(q_s)\ddot{q}_s + \frac{d}{dt}\{m_{xx}^{-1}(q_s)m_{xs}(q_s)\}\dot{q}_s \end{aligned} \quad (4.20)$$

the system in (4.19) transforms into Byrnes-Isidori normal form with a triple-integrator

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z, \xi_1, \xi_2, \xi_3) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= u\end{aligned}\tag{4.21}$$

where

$$f(z, \xi_1, \xi_2, \xi_3) = M_s^{-1}(z_1)[g(z_1) + F_d(z_1, z_2) + \Sigma(z_1, z_2, \xi_2) + m_{sx}(z_1)\xi_3]$$

and

$$\begin{aligned}M_s(q_s) &= m_{ss}(q_s) - m_{sx}(q_s)m_{xx}^{-1}(q_s)m_{xs}(q_s) \\ g(q_s) &= -\nabla V(q_s)\end{aligned}$$

Also,  $\Sigma$  is quadratic in  $(z_2, \xi_2)$  with a cubic weight matrix  $\Pi(z_1)$  and

$$u = m_{xx}^{-1}(q_s)v + \beta(q, \dot{q}, \tau)$$

is an invertible change of control in  $v$ . In addition, the zero-dynamics associated with the output  $y = h(q)$  is a simple Lagrangian system with reduced shape Lagrangian

$$\mathcal{L}_s(q_s, \dot{q}_s) = \frac{1}{2}\dot{q}_s^T M_s(q_s) \dot{q}_s - V(q_s)$$

that satisfies the forced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_s}{\partial \dot{q}_s} - \frac{\partial \mathcal{L}_s}{\partial q_s} = F_d(q_s, \dot{q}_s)$$

Moreover, if the following conditions hold

- i)  $V(q_s)$  is a positive definite proper function with  $V(0) = 0$ .
- ii)  $p_s^T F_d(q_s, p_s) < 0$  for all  $q_s \in \mathbb{R}^m, p_s \neq 0$ .

Then, the Lagrangian zero-dynamics given by

$$\begin{aligned}\dot{q}_s &= M_s^{-1}(q_s)\pi_s \\ \dot{\pi}_s &= -\nabla V(q_s) + F_d(q_s, \dot{q}_s)\end{aligned}$$

is globally minimum-phase with a reduced Hamiltonian

$$\mathcal{H}_s(q_s, \pi_s) = \frac{1}{2}\pi_s^T M_s^{-1}(q_s)\pi_s + V(q_s)$$

which is a valid Lyapunov function for the zero-dynamics, i.e.  $\mathcal{H}_s$  is smooth positive definite and proper function satisfying  $\dot{\mathcal{H}}_s \leq 0$  (where  $\pi_s$  is the generalized momentum  $\partial \mathcal{L}_s / \partial \dot{q}_s$  of the reduced system conjugate to  $q_s$ ).

**Proof.** By definition of  $\xi_2$ , we have

$$\dot{\xi}_2 = \xi_3 = m_{xx}^{-1}(q_s)(\tau - h_x(q_s, \dot{q})) + \frac{d}{dt}\{m_{xx}^{-1}(q_s)m_{xs}(q_s)\}\dot{q}_s$$

or simply

$$\xi_3 = m_{xx}^{-1}(q_s)\tau + \beta_1(q_s, \dot{q})$$

which means  $\xi_3$  is an invertible change of variable in  $\tau$ . Differentiating the last equation in time gives  $\dot{\xi}_3 = u$  with a new control  $u$  as defined in the question. To prove the equation of  $\dot{z}_2$  in (4.21), let us substitute for  $\dot{q}_x$  from the definition of  $\xi_3$  in the second line of (4.19) to get

$$M_s(q_s)\ddot{q}_s + m_{sx}(q_s)\xi_3 - m_{sx}(q_s)\frac{d}{dt}\{m_{xx}^{-1}(q_s)m_{xs}(q_s)\}\dot{q}_s + h_s = F_d$$

or

$$\ddot{q}_s = M_s^{-1}(q_s)[-m_{sx}(q_s)\xi_3 + m_{sx}(q_s)\frac{d}{dt}\{m_{xx}^{-1}(q_s)m_{xs}(q_s)\}\dot{q}_s + F_d(q_s, \dot{q}_s) - h_s(q, \dot{q})]$$

that is in the form of second line of (4.21). To obtain the specific structure of  $f(z, \xi)$  in (4.21), we need to calculate  $h_s$  more explicitly as

$$h_s(q_s, \dot{q}) = \dot{m}_{sx}\dot{q}_x + \dot{m}_{ss}\dot{q}_s - \nabla_{q_s}K - g_s(q_s)$$

where  $g_s(q_s) = -\nabla V(q_s)$  and  $\nabla_{q_s}K$  is quadratic in  $(p_x, p_s) = (\dot{q}_x, \dot{q}_s)$ . But

$$\begin{bmatrix} p_x \\ p_s \end{bmatrix} = \begin{bmatrix} I & -m_{xx}^{-1}(q_s)m_{xs}(q_s) \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_2 \\ z_2 \end{bmatrix}$$

thus after substituting for  $(p_x, p_s)$  in  $\nabla_{q_s}K$ , we obtain

$$\nabla_{q_s}K = \begin{bmatrix} \xi_2 \\ z_2 \end{bmatrix} \Pi_K(q_s) \begin{bmatrix} \xi_2 \\ z_2 \end{bmatrix}$$

where  $\Pi_K(q_s)$  is a cubic matrix. Defining

$$\Sigma := \begin{bmatrix} \xi_2 \\ z_2 \end{bmatrix} \Pi_k(q_s) \begin{bmatrix} \xi_2 \\ z_2 \end{bmatrix} + m_{sx}(q_s)\frac{d}{dt}\{m_{xx}^{-1}(q_s)m_{xs}(q_s)\}\dot{q}_s - \dot{m}_{sx}\dot{q}_x - \dot{m}_{ss}\dot{q}_s$$

or

$$\Sigma := \begin{bmatrix} \xi_2 \\ z_2 \end{bmatrix} \Pi_k(q_s) \begin{bmatrix} \xi_2 \\ z_2 \end{bmatrix} - \dot{m}_{sx}\xi_2 + M_s\dot{q}_s$$

gives the desired structure of  $f(z, \xi)$  in (4.21) (the last two terms are both quadratic in  $(z_2, \xi_2)$ ). To obtain the equations of the zero-dynamics, let  $\xi_1 = \xi_2 = \xi_3 \equiv 0$  and

notice that

$$\xi_2 = \dot{q}_x + m_{xx}^{-1}(q_s)m_{ss}(q_s)\dot{q}_s = 0$$

is a holonomic velocity constraint and under such a constraint the new Lagrangian of the system can be expressed as

$$\mathcal{L}_s = \mathcal{L}(q_s, \dot{q}_x, \dot{q}_s) + \lambda_1^T(q_x + \gamma(q_s)) + \lambda_2^T(\dot{q}_x + m_{xx}^{-1}(q_s)m_{ss}(q_s)\dot{q}_s)$$

where  $\lambda_1, \lambda_2 \in \mathbb{R}^{n-m}$  are the vectors of Lagrangian multipliers. Under the constraints  $y = 0$  and  $\dot{q}_x = -m_{xx}^{-1}(q_s)m_{ss}(q_s)\dot{q}_s$ ,  $\mathcal{L}_s$  can be expressed as

$$\mathcal{L}_s = \frac{1}{2}\dot{q}_s^T \begin{bmatrix} -m_{xx}^{-1}(q_s)m_{ss}(q_s) \\ I_m \end{bmatrix} \begin{bmatrix} m_{xx}(q_s) & m_{xs}(q_s) \\ m_{sx}(q_s) & m_{ss}(q_s) \end{bmatrix} \begin{bmatrix} -m_{xx}^{-1}(q_s)m_{ss}(q_s) \\ I_m \end{bmatrix} \dot{q}_s - V(q_s)$$

that after simplification takes the form

$$\mathcal{L}_s(q_s, \dot{q}_s) = \frac{1}{2}\dot{q}_s^T(m_{ss}(q_s) - m_{sx}(q_s)m_{xx}^{-1}(q_s)m_{ss}(q_s))\dot{q}_s - V(q_s)$$

or

$$\mathcal{L}_s(q_s, \dot{q}_s) = \frac{1}{2}\dot{q}_s^T M_s(q_s)\dot{q}_s - V(q_s)$$

Therefore, the zero-dynamics is a simple Lagrangian system with inertia matrix  $M_s(q_s)$  and its Lagrangian  $\mathcal{L}_s$  satisfies the Euler-Lagrange equation with input damping force  $F_d(q_s, \dot{q}_s)$ . The equation of the zero-dynamics in shape space can be equivalently written as

$$M_s(q_s)\ddot{q}_s + C_s(q_s, \dot{q}_s)\dot{q}_s = g(q_s) + F_d(q_s, \dot{q}_s)$$

where  $\dot{M}_s = C_s(q_s, \dot{q}_s) + C^T(q_s, \dot{q}_s)$ . Apparently,  $\mathcal{H}_s$  by definition is smooth proper positive definite function. Calculating  $\dot{\mathcal{H}}_s$ , we get

$$\begin{aligned} \dot{\mathcal{H}}_s &= \dot{q}_s^T M_s(q_s)\ddot{q}_s + \frac{1}{2}\dot{q}_s^T \dot{M}_s(q_s)\dot{q}_s + \dot{q}_s^T \nabla V(q_s) \\ &= -\dot{q}_s^T C \dot{q}_s + \frac{1}{2}\dot{q}_s^T (C + C^T)\dot{q}_s + \dot{q}_s^T g(q_s) + \dot{q}_s^T \nabla V(q_s) + \dot{q}_s^T F_d(q_s, \dot{q}_s) \\ &= \dot{q}_s^T F_d(q_s, \dot{q}_s) + \frac{1}{2}\dot{q}_s^T (C^T - C)\dot{q}_s \end{aligned}$$

but  $C^T - C$  is a skew-symmetric matrix and for all  $\dot{q}_s = p_s \neq 0$

$$p_s^T (C^T - C)p_s = [p_s^T (C^T - C)p_s]^T = -p_s^T (C^T - C)p_s = 0$$

Thus

$$\dot{\mathcal{H}}_s = p_s^T F_d(q_s, p_s) < 0$$

for all  $p_s \neq 0$ . Based on LaSalle's invariance principle [40], all the solutions of the zero-dynamics system asymptotically converge to the largest invariant set in

$\{(q_s, p_s) \mid \mathcal{H}_s = 0\} = \{(q_s, p_s) \mid p_s = 0\}$  which is equal to  $\{0\}$ . Therefore, the origin is globally asymptotically stable for the zero-dynamics, i.e. the zero-dynamics is globally minimum-phase, and  $\mathcal{H}_s$  is a valid Lyapunov function for the Lagrangian zero-dynamics.  $\square$

**Example 4.2.2.** (Flexible link-robots) Consider a flexible one-link robot arm with a single actuator at the joint  $\theta$  as shown in Figure 4-1. Modeling the link as an

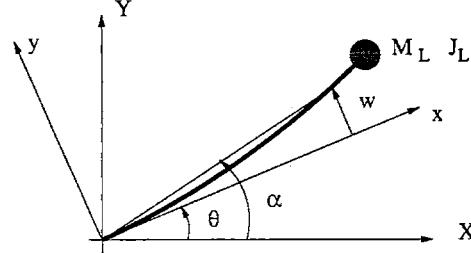


Figure 4-1: A flexible one-link robot arm.

Euler-Bernoulli beam and using truncated modal analysis with  $m$  modes that have amplitudes  $\delta = (\delta_1, \dots, \delta_m)^T$  [25], the dynamics of this flexible link is an underactuated system as the following [25]

$$\begin{bmatrix} m_{\theta\theta}(\delta) & m_{\theta\delta} \\ m_{\delta\theta} & m_{\delta\delta} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} h_\theta(\delta, \dot{\delta}, \dot{\theta}) \\ h_\delta(\delta, \dot{\theta}) \end{bmatrix} = \begin{bmatrix} \tau \\ -D\dot{\delta} \end{bmatrix}$$

with a scalar control  $\tau$ . The external variable  $q_x = \theta$  is actuated and the  $m$ -dimensional shape variable  $q_s = \delta$  is unactuated but has natural springs and dampers with constants  $k_j, d_j > 0$  at each deformation mode such that

$$V(\delta) = \frac{1}{2}\delta^T k\delta, F_d(\dot{\delta}) = -D\dot{\delta}$$

with  $k = \text{diag}(k_1, \dots, k_m)$  and  $D = \text{diag}(d_1, \dots, d_m)$ . This flexible one-link arm is an example of a high-order underactuated system with kinetic symmetry and many unactuated shape variables. In section 5.9, based on theorem 4.2.3, we prove that there exists a nonlinear noncollocated minimum-phase output in the form

$$y = \theta + \sigma(c^T \delta)$$

that has global relative degree 3 w.r.t. the control  $v$  of the augmented system with an integrator  $\dot{v} = v$ . Here,  $c \in \mathbb{R}^m$  is a constant and  $\sigma(\cdot)$  is a scalar sigmoidal function (see [71] for details on trajectory tracking control design).

## 4.3 Underactuated Systems with Input Coupling

In this section, we focus on reduction of a class of underactuated systems with input coupling. By *input coupling*, we mean there exists no permutation of rows of

the force (or input) matrix  $F(q)$  with full column rank  $m$  that transforms it into  $\text{col}(F_1(q), F_2(q))$  such that  $F_1 \equiv 0$  and  $F_2$  is an  $m \times m$  invertible matrix. In the dynamics of aerospace vehicles like aircraft and helicopter, the effect of the body torque appears in the translational dynamics of the vehicle. This motivated us to consider eliminating this coupling effect for the class of underactuated systems with kinetic symmetry and input coupling via a global change of coordinates that decouples the dynamics of the external variables and shape variables w.r.t. to the control applied to the shape variables. The VTOL aircraft has a constant inertia matrix, while we will see in chapter 5 that the inertia matrix of an aircraft or a helicopter depends on the pitch and roll angles and is not constant.

On the other hand, in the nature, a flying bird, a swimming fish, and a walking human being are all examples of mechanical systems that their position is unactuated and their locomotion is due to the changes in their physical shape (see [74, 9, 49, 104] for simpler examples). Moreover, the body of all three examples contains fully-actuated joints. In other words, a bird, a fish, and a human are examples of non-flat underactuated systems with actuated shape variables and unactuated external variables. Roughly speaking, the non-flatness property of these systems is due to the fact that any changes in the shape variables of any of them affects the physical shape and thus their corresponding inertia matrix. Therefore, for the sake of generality of our analysis, we first consider the class of non-flat underactuated systems with fully-actuated shape variables and input coupling.

**Theorem 4.3.1.** *Consider an underactuated system with kinetic symmetry w.r.t.  $q_x$  and fully-actuated shape variables. Assume the inertia matrix of this system is block diagonal, i.e.  $M(q_s) = \text{diag}(m_{xx}(q_s), m_{ss}(q_s))$ , and the force matrix is independent of  $q_x$ , i.e.  $F(q) = F(q_s)$ . The dynamics of the system can be expressed in the form*

$$\begin{aligned} m_{xx}(q_s)\ddot{q}_x + h_x(q, \dot{q}) &= F_x(q_s)\tau \\ m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) &= F_s(q_s)\tau \end{aligned} \quad (4.22)$$

where  $\tau \in \mathbb{R}^m$  and  $F_s(q_s)$  is an invertible  $m \times m$  matrix. Due to  $m_{xs}(q_s) = 0$ , system (4.22) is globally partially feedback linearizable using an invertible change of control

$$\tau = F_s^{-1}(q_s)[m_{ss}(q_s)u + h_s(q, \dot{q})]$$

Suppose all the elements of

$$\omega = m_{xx}^{-1}(q_s)F_x(q_s)F_s^{-1}(q_s)m_{ss}(q_s)dq_s$$

are exact one-forms and let  $\omega = d\gamma(q_s)$ . Then, the following global change of coordinates (i.e. diffeomorphism)

$$\begin{aligned} q_r &= q_x - \gamma(q_s) \\ p_r &= m_{xx}(q_s)p_x - F_x(q_s)F_s^{-1}(q_s)m_{ss}(q_s)p_s \end{aligned} \quad (4.23)$$

with  $(p_x, p_s) = (\dot{q}_x, \dot{q}_s)$  transforms (4.22) into the following cascade system in nontri-

angular quadratic normal form

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u\end{aligned}\tag{4.24}$$

In addition, the  $(q_r, p_r)$ -subsystem is a Lagrangian system with configuration vector  $q_r$  and the reduced Lagrangian parameterized by  $q_s$

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2} \dot{q}_r^T m_r(q_s) \dot{q}_r - V_r(q_r, q_s)$$

that satisfies the following forced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = -F_x(q_s) F_s^{-1}(q_s) g_s(q_r + \gamma(q_s), q_s) + \Sigma(q_s, p_r, p_s)$$

where

$$\begin{aligned}m_r(q_s) &:= m_{xx}(q_s) \\ V_r(q_r, q_s) &:= V(q_r + \gamma(q_s), q_s) \\ g_r(q_r, q_s) &:= [g_x(q_x, q_s) - F_x(q_s) F_s^{-1}(q_s) g_s(q_x, q_s)]_{q_x=q_r+\gamma(q_s)} \\ g_x(q_x, q_s) &:= -\nabla_{q_x} V(q_x, q_s) \\ g_s(q_x, q_s) &:= -\nabla_{q_s} V(q_x, q_s)\end{aligned}$$

and

$$\Sigma(q_s, p_r, p_s) = \begin{bmatrix} p_r \\ p_s \end{bmatrix}^T \Pi(q_s) \begin{bmatrix} p_r \\ p_s \end{bmatrix}$$

is a quadratic form in  $(p_r, p_s)$  with a cubic weight matrix  $\Pi(q_s)$ .

**Proof.** Consider the following generalized momentums

$$\pi_x := \frac{\partial \mathcal{L}}{\partial \dot{q}_x} = m_{xx}(q_s) \dot{q}_x, \quad \pi_s := \frac{\partial \mathcal{L}}{\partial \dot{q}_s} = m_{ss}(q_s) \dot{q}_s$$

By definition of  $q_r, p_r$ , we have  $\dot{q}_r = m_r^{-1}(q_s)p_r$  and

$$p_r = \pi_x - F_x(q_s) F_s^{-1}(q_s) \pi_s$$

Based on Euler-Lagrange equation,  $\pi_x, \pi_s$  satisfy

$$\dot{\pi}_x = \frac{\partial \mathcal{L}}{\partial q_x} + F_x(q_s) \tau, \quad \dot{\pi}_s = \frac{\partial \mathcal{L}}{\partial q_s} + F_s(q_s) \tau$$

Thus, we have

$$\begin{aligned}\dot{p}_r &= \dot{\pi}_x - F_x(q_s)F_s^{-1}(q_s)\dot{\pi}_s - \frac{d}{dt}\{F_x(q_s)F_s^{-1}(q_s)\}\pi_s \\ &= \frac{\partial \mathcal{L}}{\partial q_x} - F_x(q_s)F_s^{-1}(q_s)\frac{\partial \mathcal{L}}{\partial q_s} - p_s^T\pi_F(q_s)p_s\end{aligned}$$

where  $\pi_F(q_s)$  is a cubic matrix with  $(n-m)$  layers that are  $m \times m$  matrices satisfying the relation

$$p_s^T\pi_F(q_s)p_s = \frac{d}{dt}\{F_x(q_s)F_s^{-1}(q_s)\}m_{xx}(q_s)p_s$$

Note that  $\tau$  is eliminated from the equation of  $\dot{p}_r$ . By direct calculation, we have

$$\frac{\partial \mathcal{L}}{\partial q_x} = g_x(q_x, q_s), \quad \frac{\partial \mathcal{L}}{\partial q_s} = g_s(q_x, q_s) + \frac{\partial K}{\partial q_s}$$

Thus, noting that  $\partial K/\partial q_s$  is a vector quadratic form in  $(p_x, p_s)$  with a cubic weight matrix as a function of  $q_s$  and setting

$$g_r(q_r, q_s) = [g_x(q_x, q_s) - F_x(q_s)F_s^{-1}(q_s)g_s(q_x, q_s)]_{q_x=q_r+\gamma(q_s)}$$

it follows that

$$\dot{p}_r = g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s)$$

where

$$\Sigma(q_s, p_r, p_s) = -F_x(q_s)F_s^{-1}(q_s)\frac{\partial K}{\partial q_s} - p_s^T\pi_F(q_s)p_s$$

which proves equation (4.24). To obtain an explicit expression for  $\Sigma$ , let the cubic matrix  $\pi_K(q_s)$  satisfy

$$F_x(q_s)F_s^{-1}(q_s)\frac{\partial K}{\partial q_s} = \begin{bmatrix} p_x \\ p_s \end{bmatrix}^T \pi_K(q_s) \begin{bmatrix} p_x \\ p_s \end{bmatrix}$$

and note that each layer of  $\pi_K(q_s)$  is block diagonal according to the partition of  $q = (q_x, q_s)$  (because  $M(q_s)$  is block diagonal). By definition of  $p_r$ , we get

$$\begin{bmatrix} p_x \\ p_s \end{bmatrix} = W(q_s) \begin{bmatrix} p_r \\ p_s \end{bmatrix}$$

where  $W$  is an  $n \times n$  matrix given by

$$W(q_s) = \begin{bmatrix} m_{xx}^{-1}(q_s) & m_{xx}^{-1}(q_s)F_x(q_s)F_s^{-1}(q_s)m_{ss}(q_s) \\ 0 & I_m \end{bmatrix}$$

This means

$$\Pi(q_s) = -W^T(q_s)\pi_K(q_s)W(q_s) - \text{diag}(0, \pi_F(q_s))$$

where the last term is a cubic matrix with block diagonal layers of 0 (as  $n$  layers of  $0_{n-m}$ ) and  $\pi_F(q_s)$ . The part that the reduced Lagrangian satisfies the forced Euler-Lagrange equation given in the question follows from the equation of  $\dot{p}_r$  and the identity

$$\frac{\partial \mathcal{L}_r}{\partial q_r} = \left[ \frac{\partial \mathcal{L}}{\partial q_x} \right]_{q_x=q_r+\gamma(q_s)} \cdot \frac{\partial q_x}{\partial q_r} = [g_x(q_x, q_s)]_{q_x=q_r+\gamma(q_s)}$$

□

**Example 4.3.1.** (A Flying Bird) Figure 4-2 shows the simplified model of a flying bird. The position of the center of the body of the bird and the orientation of the body is denoted by  $(x, R_0) \in \mathbb{R}^3 \times SO(3)$ . In addition, each wing has its own orientation  $R_i \in SO(3)$ ,  $i = 1, 2$  which is fully-actuated. The orientation of the tail is parameterized by a pitch angle  $\phi_3$  and roll angle  $\theta_3$  which both are actuated. Therefore, *this bird is an underactuated system with 14 DOF and 8 control inputs*. It is clear that if

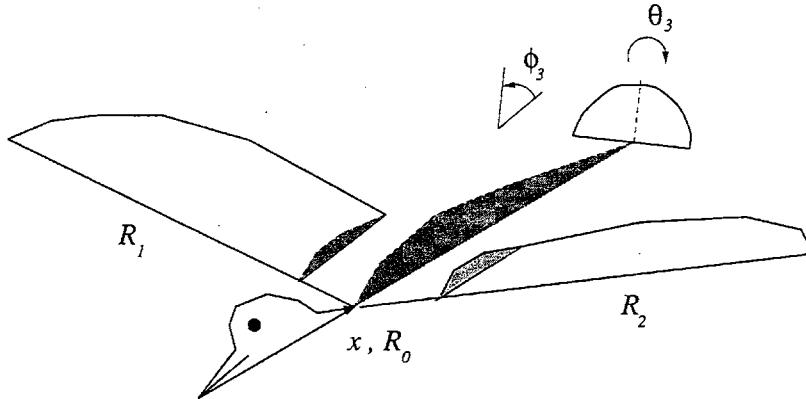


Figure 4-2: A flying bird.

$R_0, R_1, R_2, \phi_3, \theta_3$  are fixed, then the physical shape of a bird does not change under translation of the center of mass of its body. This means that the vector of external variables is  $q_x = x$  and the vector of shape variables is  $q_s = (R_0, R_1, R_2, \phi_3, \theta_3)$ . Due to the variable physical shape of the bird, the inertia matrix for this system is not constant and depends on  $q_s = (R_0, R_1, R_2, \phi_3, \theta_3)$ .

The following result provides classes of underactuated systems in theorem 4.3.1 that can be globally transformed into nonlinear systems in feedforward form. In addition, sufficient conditions are given such that these feedforward forms are globally asymptotically stabilizable using a state feedback in explicit form.

**Proposition 4.3.1.** *Assume all the conditions in theorem 4.3.1 hold and in addition*

- i)  $m_r = m_{xx}$  is constant.
- ii)  $V(q) = k_0^T q_x + V(q_s)$  where  $k_0$  is a constant vector.

Then, the global change of coordinates in (4.23) transforms the dynamics of the underactuated system (4.22) into a cascade nonlinear system in feedforward form as the following

$$\begin{aligned}\dot{q}_r &= m_r^{-1} p_r \\ \dot{p}_r &= g_r(q_s) + p_s^T \Pi(q_s) p_s \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u\end{aligned}\tag{4.25}$$

where  $\Pi(q_s)$  is a cubic matrix and

$$g_r(q_s) = -k_0 + F_x(q_s) F_s^{-1}(q_s) \nabla_{q_s} V(q_s)$$

Moreover, if the following conditions are satisfied

- iii)  $\dim(q_x) = \dim(q_s)$ .
- iv)  $g_r(0) = 0$  and the Jacobian matrix  $\nabla_{q_s} g_r(q_s)$  is invertible at  $q_s = 0$ .

Then, the origin  $(q_r, p_r, q_s, p_s) = 0$  for the nonlinear system in (4.25) can be globally asymptotically stabilized using a state feedback in explicit form as nested saturations.

**Proof.** Following the proof of theorem 4.3.1, because  $m_{xx}$  is constant  $\partial K / \partial q_s$  is a quadratic form in  $p_s$  and  $\Sigma$  takes the following form

$$\Sigma = \Sigma(q_s, p_s) = p_s^T \pi(q_s) p_s$$

which is independent of  $p_r$ . In addition, we have

$$g_r(q_r, q_s) = g_r(q_s) = -k_0 - F_x(q_s) F_s^{-1}(q_s) g_s(q_s)$$

and (4.25) holds. The result on stabilization of the origin follows from the method of nested saturations for feedforward systems due to Teel [102]. (see the proof of proposition 4.2.1 for further details on control design).  $\square$

**Theorem 4.3.2.** Consider a flat underactuated system with configuration vector  $q = (q_x, q_s)$  and a force matrix that is independent of  $q_x$ . Assume the potential energy of the system is independent of  $q_s$ , i.e.  $V(q) = V(q_x)$ . The Euler-Lagrange equations of motion for this system are as the following

$$\begin{aligned}m_{xx} \ddot{q}_x - g_x(q_x) &= F_r(q_s) \tau_r + F_x(q_s) \tau \\ m_{ss} \ddot{q}_s &= F_s(q_s) \tau\end{aligned}\tag{4.26}$$

where  $g_x(q_x) = -\nabla_{q_x} V(q_x)$ ,  $\tau_r \in \mathbb{R}$ ,  $F_r(q_s) : \mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$  is a unit vector that is onto over a unit ball in  $\mathbb{R}^{n-m}$ ,  $\tau \in \mathbb{R}^m$ ,  $m_{xx}$  and  $m_{ss}$  are constant, and  $F_s(q_s)$  is an  $m \times m$  invertible matrix. Let

$$\tau = F_s(q_s)^{-1} m_{ss} u$$

be the partially linearizing change of control. Assume all the elements of

$$\omega = m_{xx}^{-1} F_x(q_s) F_s^{-1}(q_s) m_{ss} dq_s$$

are exact one-forms and let  $\omega = d\gamma(q_s)$ . Then, the following global change of coordinates

$$\begin{aligned} q_r &= q_x - \gamma(q_s) \\ p_r &= m_{xx} p_x - F_x(q_s) F_s^{-1}(q_s) m_{ss} p_s \end{aligned} \tag{4.27}$$

with  $(p_x, p_s) = (\dot{q}_x, \dot{q}_s)$  transforms the dynamics of (4.26) into the following form

$$\begin{aligned} \dot{q}_r &= m_r^{-1} p_r \\ \dot{p}_r &= g_r(q_r + \gamma(q_s)) - p_s^T \pi_F(q_s) p_s + F_r(q_s) \tau_r \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \tag{4.28}$$

where  $\pi_F(q_s)$  is a cubic matrix satisfying

$$\frac{d}{dt} \{F_x(q_s) F_s^{-1}(q_s)\} m_{ss} p_s = p_s^T \pi_F(q_s) p_s$$

In addition, the  $(q_r, p_r)$ -subsystem is a flat Lagrangian system with configuration vector  $q_r$  and reduced Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2} \dot{q}_r^T m_r \dot{q}_r - V(q_r + \gamma(q_s))$$

satisfying the forced Euler-Lagrangian equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = -p_s^T \pi_F(q_s) p_s + F_r(q_s) \tau_r$$

where  $m_r = m_{xx}$ . Moreover, if the following conditions hold:

- i)  $p_s^T \pi_F(q_s) p_s = (p_s^T Q p_s) F_r(q_s)$  with  $Q \in \mathbb{R}^{m^2}$ .
- ii)  $V(q_x)$  is a linear function.

Then, the reduced Lagrangian system is a fully-actuated flat system that satisfies

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = F_r(q_s) \tilde{\tau}_r$$

where

$$\tilde{\tau}_r = \tau_r - p_s^T Q p_s$$

is the new scalar control.

**Proof.** The proof is by direct calculation.  $\square$

**Example 4.3.2.** (VTOL aircraft) The VTOL aircraft discussed in section 3.3.6 satisfies all the conditions of proposition 4.3.2 and after applying a global change of coordinates that decouples the position and orientation dynamics of the VTOL aircraft w.r.t. the input torque  $u_2$  in equation (5.41), the reduced dynamics of the position is a fully-actuated flat mechanical system. The configuration of the VTOL aircraft can be globally asymptotically stabilized to the origin using a smooth static state feedback (see [70], or section 5.10 for more details on control design procedure). Later on, we will show a similar result applies to an autonomous helicopter.

**Corollary 4.3.1.** Assume all the conditions in theorem 4.3.2 hold. In addition, suppose  $F_x, F_s$  are constant and  $\tau_r = 0$ . Then, the nonlinear system in (4.28) is in strict feedback form.

**Example 4.3.3.** (Inertia-Wheel Pendulum) The Inertia-Wheel Pendulum satisfies all the conditions of the preceding corollary and can be globally asymptotically stabilized using a state feedback in explicit form (see section 5.3 for more details).

## 4.4 Underactuated Systems with Non-integrable Momentums

In this section, we consider the class of underactuated systems with kinetic symmetry that their normalized momentums are not-integrable. By *normalized momentums*, we mean

$$\pi_x = \dot{q}_x + m_{xx}^{-1}(q_s) m_{xs}(q_s) \dot{q}_s = m_{xx}^{-1}(q_s) \frac{\partial \mathcal{L}}{\partial \dot{q}_x} \quad (4.29)$$

$$\pi_s = \dot{q}_x + m_{sx}^{-1}(q_s) m_{ss}(q_s) \dot{q}_s = m_{sx}^{-1}(q_s) \frac{\partial \mathcal{L}}{\partial \dot{q}_s} \quad (4.30)$$

where  $\pi_x, \pi_s$  are the normalized momentums conjugate to  $q_x, q_s$ , respectively. The normalized momentum  $\pi_x$  (or  $\pi_s$ ) is called *non-integrable*, if  $\nexists h = h(q_x, q_s) : \dot{h} = \pi_x$  (or  $\pi_s$ ). It turns out that in this case, underactuated systems can be reduced to *kinematic systems* driven by the shape velocity vector  $p_s$  as the control input. First, we consider underactuated systems with actuated shape variables and non-integrable normalized momentums.

**Theorem 4.4.1.** Consider the underactuated system in (4.3) and suppose all the assumptions in theorem (4.2.1) hold except for the exactness property of the elements

of  $\omega$ . In other words, assume given

$$\mu(q_s) = m_{xx}^{-1}(q_s)m_{xs}(q_s)$$

$\mu(q_s)p_s$  is not integrable. Then, after applying the global change of coordinates

$$\begin{aligned} q_r &= q_x \\ p_r &= m_{xx}(q_s)p_x + m_{xs}(q_s)p_s \end{aligned}$$

with  $(p_x, p_s) = (\dot{q}_x, \dot{q}_s)$ , the dynamics of the system transforms into the following nontriangular normal form

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_s)p_r - \mu(q_s)p_s \\ \dot{p}_r &= g_r(q_r, q_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \tag{4.31}$$

that is affine in the shape velocity  $p_s$  as the control input of  $(q_r, p_r, q_s)$ -subsystem.

**Proof.** The equation of  $\dot{p}_r$  follows from the proof of theorem (4.2.1) and the first equation in (4.31) is a direct result of the definition of  $p_r$ .  $\square$

**Remark 4.4.1.** Defining  $x = (q_r, p_r, q_s)$  and  $v = p_s$ , the dynamics of the  $(q_r, p_r, q_s)$ -subsystem can be written as a nonlinear system affine in control

$$\dot{x} = f(x) + g(x)v$$

where  $g(x) = \text{col}(-\mu(q_s), I_m)$ . This system has  $m$  (i.e. number of control inputs) first order differential equations less than the original system and both the control design and controllability analysis of this system can be carried out in a lower-order space  $\mathbb{R}^{3m}$  instead of the original space  $\mathbb{R}^{4m}$ .

**Remark 4.4.2.** Recall that in the integrable case for underactuated systems with actuated shape variables, it is possible to use backstepping procedure due to strict feedback form of the transformed system. Here, due to nontriangular structure of the normal form in (4.31) the backstepping procedure is not applicable. Control design for (4.31) will be treated later in chapter 7.

**Example 4.4.1.** (three-link planar robot) Consider a planar three-link robot arm with revolute joints  $(q_1, q_2, q_3)$  and two actuators at  $q_2, q_3$  as shown in Figure 4-3. The dynamics of this robot is given in section A.2 (Appendix A). It can be shown that the inertia matrix of this triple-link robot has the following structure

$$M(q) = M(q_2, q_3) = \begin{bmatrix} m_{11}(q_2, q_3) & m_{12}(q_2, q_3) & m_{13}(q_2, q_3) \\ m_{21}(q_2, q_3) & m_{22}(q_3) & m_{23}(q_3) \\ m_{31}(q_2, q_3) & m_{32}(q_3) & m_{33} \end{bmatrix}$$

Thus,  $q_x = q_1$  is the external variable and  $q_s = (q_2, q_3)$  is the vector of shape variables. Moreover, both shape variables are actuated. However, by direct calculation, it can

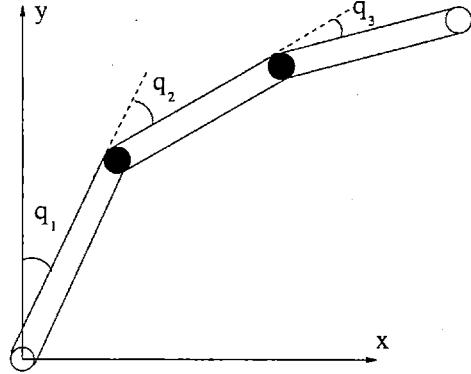


Figure 4-3: A triple-link robot arm with two actuators.

be shown that the normalized momentum

$$\pi_1 = \dot{q}_1 + \frac{m_{12}(q_2, q_3)}{m_{11}(q_2, q_3)} \dot{q}_2 + \frac{m_{13}(q_2, q_3)}{m_{11}(q_2, q_3)} \dot{q}_3$$

is non-integrable (see lemma A.2.1). Therefore, based on theorem 4.4.1, a three-link robot can be transformed into a nontriangular kinematic normal form with  $p_s = (\dot{q}_2, \dot{q}_3)$  as the input.

Now, we present the case of underactuated systems with actuated external variables and non-integrable normalized momentums.

**Theorem 4.4.2.** Consider the underactuated mechanical system in (4.12) and suppose all the assumptions in theorem (4.2.2) hold except for the exactness property of the elements of  $\omega$ . In other words, assume given

$$\mu(q_s) = m_{sx}^{-1}(q_s)m_{ss}(q_s)$$

$\mu(q_s)p_s$  is non-integrable. Then, after applying the change of coordinates

$$\begin{aligned} q_r &= q_x \\ p_r &= m_{sx}(q_s)p_x + m_{ss}(q_s)p_s \end{aligned}$$

with  $(p_x, p_s) = (\dot{q}_x, \dot{q}_s)$ , the dynamics of the system transforms into the following nontriangular linear-quadratic normal form

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_s)p_r - \mu(q_s)p_s \\ \dot{p}_r &= g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \tag{4.32}$$

over

$$U = \{q_s \in Q_s \mid \det(m_{sx}(q_s)) \neq 0\}$$

with  $p_s$  as the control input for the  $(q_r, p_r, q_s)$ -subsystem where

$$\begin{aligned} m_r(q_s) &:= m_{sx}(q_s) \\ g_r(q_r, q_s) &:= \nabla_{q_s} V(q_r, q_s) \\ \Sigma(q_s, p_r, p_s) &:= \nabla_{q_s} K \end{aligned}$$

**Proof.** The proof for the equation of  $\dot{p}_r$  in (4.32) is the same as in theorem 4.2.2. By definition of  $p_r$ , we have

$$\dot{q}_x = m_{sx}^{-1}(q_s)p_r - m_{sx}^{-1}(q_s)m_{ss}(q_s)p_s = m_r^{-1}(q_s)p_r - \mu(q_s)p_s$$

and the result follows. Based on the structure of normal form (4.32), it is clear that the  $(q_r, p_r, q_s)$ -subsystem is kinematic (i.e. has a control input  $p_s$  that is the shape velocity vector).  $\square$

*Remark 4.4.3.* The normal form in (4.32) is very similar to a normal form previously found for Lagrangian systems with nonholonomic velocity constraints and classical symmetry due to Bloch *et al.* [9]. However, the formulation of theorem 4.4.2 is fundamentally different from the setup studied in [9].

**Proposition 4.4.1.** *Assume all the assumptions in theorem 4.4.2 hold. In addition, the underactuated system (4.12) satisfies the following conditions*

- i)  $m_{xx}(q_s)$  is constant.
- ii)  $m_{sx}(q_s)$  has differentially symmetric rows, i.e.

$$\frac{\partial m_{sx}(q_s)}{\partial q_s^i} = \frac{\partial m_{sx}^{i*}}{\partial q_s}, \quad i = 1, \dots, m$$

where  $m_{sx}^{i*}(q_s)$  is the  $i$ th row of  $m_{sx}(q_s)$ .

- iii)  $V(q) = V(q_s)$ .
- iv)  $\mu(q_s)$  is analytic at  $q_s = 0$  (element-wise).

Then, applying the change of coordinates

$$z_1 = q_r + \mu(0)q_s, \quad z_2 = m_r^{-1}(q_s)p_r$$

(where  $(q_r, p_r)$  are defined in (4.13)) transforms the underactuated system (4.12) into a cascade system in feedforward form as the following

$$\begin{aligned} \dot{z}_1 &= z_2 + \varphi_1(q_s, p_s) \\ \dot{z}_2 &= \psi(q_s) + p_s^T \Pi(q_s) p_s \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \tag{4.33}$$

where  $\varphi_1(q_s, p_s) = (\mu(0) - \mu(q_s))p_s$  is at least quadratic in  $(q_s, p_s)$ ,  $\Pi(q_s)$  is a cubic matrix, and  $\psi : Q_s \rightarrow \mathbb{R}^m$  is defined as

$$\psi(q_s) = -m_r^{-1}(q_s)\nabla_{q_s}V(q_s)$$

Moreover, if  $\psi(0) = 0$  and  $\psi(q_s)$  has an invertible Jacobian  $\nabla_{q_s}\psi(q_s)$  at  $q_s = 0$ , the origin for (4.33) can be globally asymptotically and locally exponentially stabilized over  $U$  using a state feedback in the form of nested saturations.

**Proof.** Following the proofs of proposition 4.2.1 and theorem 4.4.2, after applying the change of coordinates

$$\bar{z}_1 = q_r, z_2 = m_r^{-1}(q_s)p_r$$

we obtain

$$\begin{aligned}\dot{\bar{z}}_1 &= z_2 - \mu(q_s)p_s \\ \dot{z}_2 &= \psi(q_s) + p_s^T\pi(q_s)p_s \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u\end{aligned}$$

Noting that  $\mu(q_s)$  is element-wise analytic at  $q_s = 0$  and

$$\mu(q_s)p_s = \mu(0)p_s - \varphi_1(q_s, p_s)$$

where

$$\varphi_1(q_s, p_s) = O(\|(q_s, p_s)\|^2)$$

by setting  $z_1 = \bar{z}_1 + \mu(0)q_s$  the result follows. The rest of the proof is rather similar to the proof of proposition 4.2.1.  $\square$

**Example 4.4.2. (3D Cart-Pole System)** Consider an inverted pendulum mounted on a moving platform via an unactuated 2 DOF joint  $(\theta, \phi)$  with two external forces as shown in Figure 4-4. The angles  $\theta, \phi$  denote rotation around  $x_2$ -axis and  $x_1$ -axis, respectively. Let  $(x_1, x_2)$  denote the position of the center of mass of the platform  $M$ . Then, the kinetic energy of the 3D Cart-Pole system (given in section A.7) is in the form

$$K = \frac{1}{2} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}^T \begin{bmatrix} M+m & 0 & ml \cos \theta & 0 \\ 0 & M+m & -ml \sin \theta \sin \phi & ml \cos \theta \cos \phi \\ ml \cos \theta & -ml \sin \theta \sin \phi & ml^2 & 0 \\ 0 & ml \cos \theta \cos \phi & 0 & ml^2 \cos^2 \theta \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \quad (4.34)$$

Clearly, the inertia matrix for this system only depends on  $(\theta, \phi)$ . Therefore,  $(\theta, \phi)$  are the shape variables and  $(x_1, x_2)$  are the external variables of the 3D Cart-Pole system. The system is an underactuated system with four DOF  $(x_1, x_2, \theta, \phi)$  and two

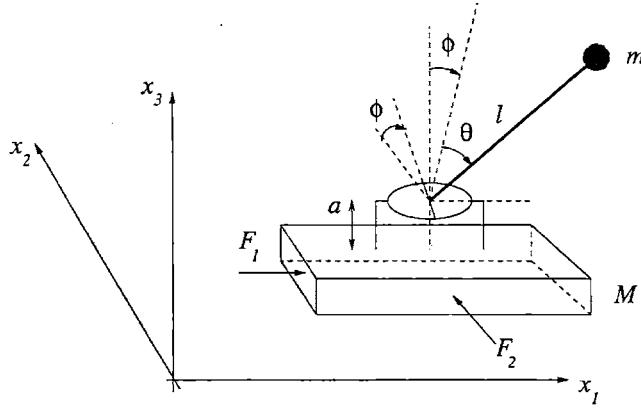


Figure 4-4: An inverted pendulum on a moving platform with 4 DOF and 2 controls.

actuated external variables  $(x_1, x_2)$ . For the matrix

$$m_{sx}(q_s) = \begin{bmatrix} ml \cos \theta & -ml \sin \theta \sin \phi \\ 0 & ml \cos \theta \cos \phi \end{bmatrix}$$

we have

$$\det(m_{sx}(q_s)) = m^2 l^2 \cos^2 \theta \cos \phi$$

Thus,  $m_{sx}$  is nonsingular over  $U = \mathbb{R}^2 \times (-\pi/2, \pi/2)^2$ . The vector of one-forms is  $\omega = \mu(q_s)p_s$  where

$$\mu(q_s) = m_{sx}^{-1}(q_s)m_{ss}(q_s) = l \begin{bmatrix} \frac{1}{\cos \theta} & \frac{\sin \theta \sin \phi}{\cos \phi} \\ 0 & \frac{\cos \phi}{\cos \theta} \\ 0 & \frac{1}{\cos \phi} \end{bmatrix}$$

Since the following one-forms are not exact

$$\begin{aligned} \omega_1 &= \frac{1}{\cos \theta} d\theta + \frac{\sin \theta \sin \phi}{\cos \phi} d\phi \\ \omega_2 &= \frac{\cos \theta}{\cos \phi} d\phi \end{aligned}$$

Theorem 4.2.2 is not applicable and the normalized momentum  $\pi_s$  of the 3D Cart-Pole system is non-integrable. On the other hand, the dynamics of the 3D Cart-Pole system satisfies all four conditions of proposition 4.4.1. Therefore, it can be transformed into a feedforward system over  $U$ .

**Proposition 4.4.2.** *The 3D Cart-Pole system (in example 4.4.2) can be transformed into a nonlinear system in feedforward form using the change of coordinates (in 4.4.1)*

$$z_1 = q_x + lq_s, z_2 = m_{sx}^{-1}(q_s)p_r$$

In addition, the origin for the system can be globally asymptotically and locally exponentially stabilized using nested saturations.

**Proof.** Transformation into a feedforward system is already proven in example 4.4.2. To prove the stabilization using nested saturations, we need to prove

$$\psi(q_s) = -m_{sx}^{-1}(q_s)\nabla_{q_s}V(q_s) = \frac{g}{\cos\phi} \begin{bmatrix} \sin\theta \\ \sin\phi \end{bmatrix}$$

has an invertible Jacobian at  $q_s = 0$ . Calculating this Jacobian, we have

$$\frac{\partial\psi}{\partial(\theta,\phi)} = g \begin{bmatrix} \frac{\cos\theta}{\cos\phi} & \frac{\sin\theta\sin\phi}{\cos^2\phi} \\ 0 & \frac{1}{\cos^2\phi} \end{bmatrix}_{(\theta,\phi)=0} = gI_{2\times 2} \neq 0$$

which is clearly nonsingular and the stabilization result follows.  $\square$

## 4.5 Momentum Decomposition for Underactuated Systems

In this section, we consider reduction of underactuated systems with non-integrable normalized momentums. We introduce a method to decompose a non-integrable normalized momentum as an integrable momentum term called *locked momentum* and a non-integrable term called *error momentum*. We call this procedure *Momentum Decomposition*. This decomposition is unique for a fix choice of a *locked configuration* (to be defined later). This procedure is described in the following.

Consider an underactuated mechanical system with kinetic symmetry and configuration vector  $(q_x, q_s)$  where  $q_x$  and  $q_s$  denote the vector of external and shape variables, respectively. The normalized momentums for this system are given by

$$\pi_x := m_{xx}^{-1}(q_s) \frac{\partial \mathcal{L}}{\partial \dot{q}_x} = \dot{q}_x + m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s \quad (4.35)$$

$$\pi_s := m_{sx}^{-1}(q_s) \frac{\partial \mathcal{L}}{\partial \dot{q}_s} = \dot{q}_x + m_{sx}^{-1}(q_s)m_{ss}(q_s)\dot{q}_s \quad (4.36)$$

which both can be expressed as the following

$$\pi = \dot{q}_x + \mu(q_s)\dot{q}_s \quad (4.37)$$

Assume  $\pi$  is non-integrable, i.e.  $\#h : h = \pi$ . Our goal is to find an *integrable approximation* of  $\pi$  which allows us to apply our previous reduction procedures for underactuated systems with integrable normalized momentums to the present case. We refer to  $\mu(q_s)$  as the *shape inertia matrix*. The following theorem demonstrates how every non-integrable normalized momentum can be decomposed as an explicitly integrable momentum plus a non-integrable momentum term.

**Theorem 4.5.1. (Momentum Decomposition)** Let  $\pi = \dot{q}_x + \mu(q_s)\dot{q}_s$  be a non-integrable normalized momentum of a mechanical control system with kinetic symmetry w.r.t.  $q_x$  ( $\eta(q_s)$  is a  $d \times m$  matrix). Denote the shape vector by  $q_s = (q_s^1, \dots, q_s^m)^T$  and the normalized momentum by  $\pi = (\pi^1, \dots, \pi^d)^T$ . Fix  $\bar{q}_s$  and define the locked shape inertia matrix  $\mu^l(q_s)$  by its elements

$$\mu_{ij}^l(q_s^j) = [\mu_{ij}(q_s)]_{q_s^k=\bar{q}_s^k, k=1, \dots, m; k \neq j} \quad (4.38)$$

and the locked momentum as

$$\pi^l = \dot{q}_x + \mu^l(q_s)\dot{q}_s \quad (4.39)$$

Then, the locked momentum is integrable in an explicit form

$$q_r = q_x + \gamma(q_s) \quad (4.40)$$

i.e.  $\dot{q}_r = \pi^l$ , where  $\gamma(q_s) = (\gamma_1(q_s), \dots, \gamma_d(q_s))^T$  and

$$\gamma_i(q_s) = \sum_{j=1}^m \int_0^{q_s^j} \mu_{ij}^l(\theta^j) d\theta^j, \quad i = 1, \dots, d \quad (4.41)$$

In addition, denoting

$$\mu^e(q_s) := \mu(q_s) - \mu^l(q_s) \quad (4.42)$$

the normalized error momentum can be defined as

$$\pi^e := \mu^e(q_s)\dot{q}_s \quad (4.43)$$

and the non-integrable momentum  $\pi$  can be uniquely decomposed as

$$\pi = \pi^l + \pi^e \quad (4.44)$$

where  $\pi^l$  is integrable and  $\pi^e$  is non-integrable. Moreover,  $\pi^e$  is independent of  $(q_x, \dot{q}_x)$  and vanishes at  $q_s = \bar{q}_s$ .

**Proof.** The  $i$ th element of  $\pi$  can be expressed as

$$\begin{aligned} \pi^i &= \dot{q}_x^i + \sum_{j=1}^m \mu_{ij}(q_s)\dot{q}_s^j \\ &= \dot{q}_x^i + \sum_{j=1}^m [\mu_{ij}^l(q_s^j) + \mu_{ij}^e(q_s)]\dot{q}_s^j \\ &= \dot{q}_x^i + \sum_{j=1}^m \mu_{ij}^l(q_s^j)\dot{q}_s^j + \sum_{j=1}^m \mu_{ij}^e(q_s)\dot{q}_s^j \end{aligned}$$

or

$$\pi = \pi^l + \pi^e$$

But the  $i$ th element of  $\pi^l$  is

$$\dot{q}_x^i + \sum_{j=1}^m \mu_{ij}^l(q_s^j)\dot{q}_s^j = \dot{q}_x^i + \dot{\gamma}_i$$

Thus, setting  $q_r = q_x + \gamma(q_s)$ ,  $\dot{q}_r = \pi^l$  and  $\pi^l$  is explicitly integrable. Since  $\pi^e = \eta^e(q_s)\dot{q}_s$ , it is independent of  $(q_x, \dot{q}_x)$ . Also, by definition of  $\mu^l(q_s)$ ,  $\mu^l(\bar{q}_s) = \mu(\bar{q}_s)$  and therefore both  $\mu^e(q_s)$  and  $\pi^e$  vanish at  $q_s = \bar{q}_s$ .  $\square$

In the following, we apply the momentum decomposition method introduced in theorem 4.5.1 to underactuated systems with kinetic symmetry for two cases: i) underactuated systems with actuated shape variables, and ii) underactuated systems with unactuated shape variables.

**Theorem 4.5.2.** *Consider the following underactuated system with kinetic symmetry and actuated shape variables*

$$\begin{aligned} m_{xx}(q_s)\ddot{q}_x + m_{xs}(q_s)\ddot{q}_s + h_x(q, \dot{q}) &= 0 \\ m_{sx}(q_s)\ddot{q}_x + m_{ss}(q_s)\ddot{q}_s + h_s(q, \dot{q}) &= \tau \end{aligned} \quad (4.45)$$

Let  $\tau = \alpha(q_s)u + \beta(q, \dot{q})$  be the global collocated partially linearizing change of control for (4.45). Assume the following normalized momentum with  $\mu_x(q_s) = m_{xx}^{-1}(q_s)m_{xs}(q_s)$  is non-integrable

$$\pi_x = \dot{q}_x + \mu_x(q_s)\dot{q}_s$$

Let

$$\pi_x = \pi_x^l + \pi_x^e$$

be the momentum decomposition of  $\pi_x$  as an integrable locked momentum  $\pi_x^l = \dot{q}_x + \mu_x^l(q_s)\dot{q}_s$  associated with the locked shape  $q_s = \bar{q}_s$  and a non-integrable normalized error momentum  $\pi_x^e$ . Then, there exists a function  $\gamma(q_s)$  in explicit form satisfying  $\nabla_{q_s}\gamma(q_s) = \mu_x^l(q_s)$  such that the global change of coordinates

$$\begin{aligned} q_r &= q_x + \gamma(q_s) \\ p_r &= m_{xx}(q_s)(\dot{q}_x + \mu_x^l(q_s)\dot{q}_s) \end{aligned} \quad (4.46)$$

transforms the dynamics of the underactuated system in (4.45) into a cascade non-linear system in strict feedback form

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s) + \delta \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \quad (4.47)$$

with a perturbation  $\delta = -\dot{p}_e$  where  $p_e = m_r(q_s)\pi_x^e$  is the error momentum and

$$\begin{aligned} m_r(q_s) &:= m_{xx}(q_s) \\ V_r(q_r, q_s) &:= V(q_x, q_s)|_{q_x=q_r-\gamma(q_s)} \\ g_r(q_r, q_s) &:= -\nabla_{q_r}V_r(q_r, q_s) \end{aligned}$$

The perturbation  $\delta$  is in the form

$$\delta = m_r(q_s)\mu_x^e(q_s)u + p_s^T \Pi_\delta(q_s)p_s$$

which is affine in  $u$ , quadratic in  $p_s$  with cubic weight matrix  $\Pi_\delta(q_s)$ , and vanishes uniformly in  $u$  at  $(q_s, p_s) = (\bar{q}_s, 0)$ . In addition, the reduced system is a Lagrangian system with reduced parameterized Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2}\dot{q}_r^T m_r(q_s) \dot{q}_r - V_r(q_r, q_s)$$

that satisfies the forced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = \delta \quad (4.48)$$

**Proof.** By definition of  $q_r$ ,  $p_r$ , and  $m_r(q_s)$ ,  $\dot{q}_r = m_r^{-1}(q_s)p_r$ . Due to kinetic symmetry, we have

$$\frac{\partial \mathcal{L}}{\partial q_x} = -\nabla_{q_x} V(q_x, q_s)$$

which means in new coordinates

$$\frac{\partial \mathcal{L}_r}{\partial q_r} = g_r(q_r, q_s)$$

On the other hand, by definition of  $\pi_x$  and based on momentum decomposition theorem, we have

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_x} = m_r(q_s)\pi_x = m_r(q_s)(\pi_x^l + \pi_x^e) = p_r + p_e$$

Hence

$$\dot{p}_r + \dot{p}_e = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_x} = \frac{\partial \mathcal{L}}{\partial q_x} = g_r(q_r, q_s)$$

or

$$\dot{p}_r = g_r(q_r, q_s) - \dot{p}_e$$

The forced Euler-Lagrange equation with input  $\delta = -\dot{p}_e$  follows from the last equation.  $\square$

*Remark 4.5.1.* Theorem 4.5.2 shows that control design for the original underactuated system with a non-integrable normalized momentum reduces to control design for system (4.47) which is in strict feedback form plus a nonlinear perturbation  $\delta$ . One possible way to stabilize the nonlinear system in (4.47) is to design a controller for the unperturbed system which reduces to control of the  $(q_r, p_r)$ -subsystem combined

with a backstepping stage. Then, perform a Lyapunov-based robustness analysis w.r.t. the nonlinear perturbation  $\delta$  which vanishes at a desired equilibrium point with shape configuration and velocity  $(q_s, p_s) = (\bar{q}_s, 0)$ .

*Remark 4.5.2.* If the normalized momentum  $\pi_x$  is integrable, then  $\pi^e \equiv 0$  which means  $\delta, p_e \equiv 0$  and thus theorem 4.5.2 reduces to theorem 4.2.1 stated earlier for underactuated systems with actuated shape variables and integrable normalized momentums.

*Remark 4.5.3.* The main difference between normal forms (4.47) and (4.31) for underactuated systems with non-integrable normalized momentums is that in the former one, the perturbation term  $\delta$  appears in the equation of  $\dot{p}_r$  and a reduced Lagrangian system with configuration vector  $q_r$  and input force  $\delta$  is identifiable (as in theorem 4.5.2), but in the latter case, the perturbation  $\mu(q_s)p_s$  appears in the equation of  $\dot{q}_r$  and no Lagrangian reduced system can be identified.

**Example 4.5.1.** (three-link planar robot with actuated shape variables) Consider the three-link planar robot arm with two actuated shape variables  $(q_2, q_3)$  as shown in Figure 4-5 (this system was also discussed earlier in example 4.4.1). The dynamics of the actuated shape variables of this system can be globally linearized using a collocated partially-linearizing change of control as the following

$$\ddot{q}_2 = u_2, \ddot{q}_3 = u_3$$

The normalized momentum of this system conjugate to  $q_1$  is non-integrable (see lemma A.2.1). Note that for this three-link robot

$$\mu(q_s) = m_{11}^{-1}(q_2, q_3) [ \begin{array}{cc} m_{12}(q_2, q_3) & m_{13}(q_2, q_3) \end{array} ]$$

After locking  $q_2, q_3$  at  $(\bar{q}_2, \bar{q}_3) = (0, 0)$ , the locked approximation of  $\pi_1$  can be calcu-

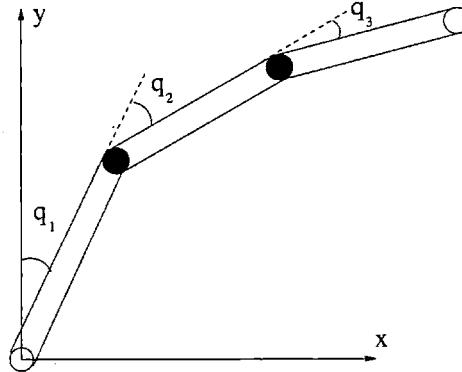


Figure 4-5: A triple-link robot arm with actuated shape variables.

lated as

$$\pi_1^l = \dot{q}_1 + \frac{m_{12}(q_2, 0)}{m_{11}(q_2, 0)} \dot{q}_2 + \frac{m_{13}(0, q_3)}{m_{11}(0, q_3)} \dot{q}_3$$

which gives explicit forms for  $\gamma_1$  and  $\gamma_2$  as the following

$$\gamma_1(q_2) = \int_0^{q_2} \frac{m_{12}(s, 0)}{m_{11}(s, 0)} ds, \quad \gamma_2(q_3) = \int_0^{q_3} \frac{m_{12}(0, s)}{m_{11}(0, s)} ds$$

After applying the change of coordinates

$$\begin{aligned} q_r &= q_1 + \gamma(q_2) + \gamma(q_3) \\ p_r &= m_{11}(q_2, q_3)[p_1 + \frac{m_{12}(q_2, 0)}{m_{11}(q_2, 0)} p_2 + \frac{m_{13}(0, q_3)}{m_{11}(0, q_3)} p_3] \end{aligned}$$

the system transforms into normal form (4.47). Calculating the error momentum  $p_e = m_r(q_s)\pi_1^e$ , we get

$$p_e = (m_{12}(q_2, q_3) - m_{12}(q_2, 0))\dot{q}_2 + (m_{13}(q_2, q_3) - m_{13}(0, q_3))\dot{q}_3$$

and thus  $\delta = -\dot{p}_e$  is given by

$$\delta = -(m_{12}(q_2, q_3) - m_{12}(q_2, 0))u_2 - (m_{13}(q_2, q_3) - m_{13}(0, q_3))u_3 + \lambda(q_2, \dot{q}_2, q_3, \dot{q}_3)$$

where  $\delta$  can be made exponentially vanishing by applying a linear control  $u_3 = K_3(q_3, p_3) = -c_1q_3 - c_2p_3$  with  $c_1, c_2 > 0$ . Keeping this in mind, the dynamics of the reduced system for the three-link robot can be rewritten as

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_2, 0)p_r + \varphi_1 \\ \dot{p}_r &= g_r(q_r, q_2, 0) + \varphi_2 \end{aligned}$$

where the nonlinear perturbations  $\varphi_1, \varphi_2$  are given by

$$\varphi_1 = (m_r^{-1}(q_2, q_3) - m_r(q_2, 0))p_r, \quad \varphi_2 = (g_r(q_r, q_2, q_3) - g_r(q_r, q_2, 0)) + \delta$$

Both  $\varphi_1$  and  $\varphi_2$  vanish exponentially as  $(q_3, p_3)$  converges exponentially fast to zero. In other words, control of the three-link planar robot which is a sixth order nonlinear system reduces to stabilization of the second-order unperturbed reduced system

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_2, 0)p_r \\ \dot{p}_r &= g_r(q_r, q_2, 0) \end{aligned}$$

But this normal form is exactly the same as the reduced system of an Acrobot with configuration vector  $(q_1, q_2)$  and the following inertia matrix and potential energy

$$M_{Acro}(q_2) = \begin{bmatrix} m_{11}(q_2, 0) & m_{12}(q_2, 0) \\ m_{21}(q_2, 0) & m_{22}(0) \end{bmatrix}, \quad V_{Acro}(q_r, q_2) = [V(q_1, \dot{q}_2, 0)]_{q_1=q_r-\gamma_1(q_2)}$$

The shape variable of this Acrobot is  $q_2$  which is actuated. Since the Acrobot can be globally asymptotically stabilized to its upright equilibrium point using a smooth state feedback  $u_2 = K_2(q_r, p_r, q_2, p_2)$  [68] (under minor conditions), after a rather elementary perturbation analysis it follows that the origin for a three-link planar

robot can be globally asymptotically stabilized using (static) state feedback laws  $u_2 = K_2(q_r, p_r, q_2, p_2)$  and  $u_3 = K_3(q_3, p_3)$ .

*Remark 4.5.4.* The result of example 4.5.1 is in agreement with our intuition that if one sets the joint angle  $q_3$  identically to zero, a three-link planar robot graphically looks the same as an Acrobot with an actuated shape variable  $q_2$ . Keep in mind that this choice is not unique and one could have started by exponential stabilization of  $(q_2, p_2)$ -subsystem to zero and then stabilize an Acrobot with an external variable  $q_1$  and shape variable  $q_3$ .

Next, we consider the case of underactuated systems with partially-actuated shape variables and non-integrable momentums.

**Theorem 4.5.3.** Consider the following underactuated system with kinetic symmetry and partially-actuated shape variables  $q_s = \text{col}(q_{s_1}, q_{s_2}) \in Q_{s_1} \times Q_{s_2}$

$$\begin{aligned} m_{xx}(q_s)\ddot{q}_x + m_{xs_1}(q_s)\ddot{q}_{s_1} + m_{xs_2}(q_s)\ddot{q}_{s_2} + h_x(q, \dot{q}) &= \tau_1 \\ m_{s_1x}(q_s)\ddot{q}_x + m_{s_1s_1}(q_s)\ddot{q}_{s_1} + m_{s_1s_2}(q_s)\ddot{q}_{s_2} + h_{s_1}(q, \dot{q}) &= 0 \\ m_{s_2x}(q_s)\ddot{q}_x + m_{s_2s_1}(q_s)\ddot{q}_{s_1} + m_{s_2s_2}(q_s)\ddot{q}_{s_2} + h_{s_2}(q, \dot{q}) &= \tau_2 \end{aligned} \quad (4.49)$$

Assume  $\dim(Q_x) = \dim(Q_{s_1})$  and  $\dim(Q_{s_2}) \geq 0$ . Let  $\tau = \alpha(q_s)u + \beta(q, \dot{q})$  be the noncollocated partially linearizing change of control for (4.49) over

$$U = \{q_s \in Q_s \mid \det(m_{s_1x}(q_s)) \neq 0\}$$

which linearizes the dynamics of the shape variables as  $\ddot{q}_s = u$ . Assume the following normalized momentum is non-integrable

$$\pi_{s_1} = \dot{q}_x + \mu_{s_1}(q_s)\dot{q}_s$$

with

$$\mu_{s_1}(q_s) = m_{s_1x}^{-1}(q_s) [ \begin{array}{cc} m_{s_1s_1}(q_s) & m_{s_1s_2}(q_s) \end{array} ]$$

Let

$$\pi_{s_1} = \pi_{s_1}^l + \pi_{s_1}^e$$

be the momentum decomposition of  $\pi_{s_1}$  as an integrable locked momentum  $\pi_{s_1}^l = \dot{q}_x + \mu_{s_1}^l(q_s)\dot{q}_s$  associated with the locked shape  $q_s = \bar{q}_s$  and a non-integrable normalized error momentum  $\pi_{s_1}^e$ . Then, there exists a function  $\gamma(q_s)$  in explicit form satisfying  $\nabla_{q_s}\gamma(q_s) = \mu_{s_1}^l(q_s)$  such that the following change of coordinates over  $U$

$$\begin{aligned} q_r &= q_x + \gamma(q_s) \\ p_r &= m_{s_1x}(q_s)(\dot{q}_x + \mu_{s_1}^l(q_s)\dot{q}_s) \end{aligned} \quad (4.50)$$

transforms the dynamics of the underactuated system in (4.49) into a cascade non-

linear system in nontriangular quadratic normal form

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_{s_1}, q_{s_2}) p_r \\ \dot{p}_r &= g_r(q_r, q_{s_1}, q_{s_2}) + \Sigma(q_{s_1}, q_{s_2}, p_r, p_s) + \delta \\ \dot{q}_{s_1} &= p_{s_1} \\ \dot{p}_{s_1} &= u_1 \\ \dot{q}_{s_2} &= p_{s_2} \\ \dot{p}_{s_2} &= u_2\end{aligned}\tag{4.51}$$

with a perturbation  $\delta = -\dot{p}_e$  where  $p_e = m_r(q_s)\pi_{s_1}^e$  is the error momentum and

$$\begin{aligned}m_r(q_s) &:= m_{s_1x}(q_s) \\ V_r(q_r, q_s) &:= V(q_x, q_s)|_{q_x=q_r-\gamma(q_s)} \\ g_r(q_r, q_x) &:= -\nabla_{q_{s_1}} V_r(q_r, q_{s_1}, q_{s_2}) \\ \Sigma(q_{s_1}, q_{s_2}, p_r, p_s) &:= \nabla_{q_{s_1}} K\end{aligned}$$

The perturbation  $\delta$  is in the form

$$\delta = m_r(q_s)\mu_{s_1}^e(q_s)u + p_s^T \Pi_\delta(q_s)p_s$$

which is affine in  $u$ , quadratic in  $p_s$  with a cubic weigh matrix  $\Pi_\delta(q_s)$ , and vanishes uniformly in  $u$  at  $(q_s, p_s) = (\bar{q}_s, 0)$ . In addition, if  $V(q) = V(q_x)$ , the reduced system is a Lagrangian system itself with reduced parameterized Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2} \dot{q}_r^T m_r(q_s) \dot{q}_r$$

that satisfies the forced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = \Sigma(q_s, p_r, p_s) + \delta\tag{4.52}$$

**Proof.** By definition of  $q_r$  and  $p_r$ ,  $\dot{q}_r = m_{s_1x}^{-1}(q_s)p_r$ . From the second line of (4.49), we have

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_{s_1}} = \frac{\partial \mathcal{L}}{\partial q_{s_1}} = -\nabla_{q_{s_1}} V(q_x, q_s) + \nabla_{q_{s_1}} K$$

which after substituting  $q_x = q_r - \gamma(q_s)$  and  $\dot{q}_x = m_{s_1x}^{-1}(q_s)p_r - \mu_{s_1}^l(q_s)p_s$ , one obtains

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_{s_1}} = g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s)$$

Based on momentum decomposition theorem

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_{s_1}} = m_r(q_s)(\pi_{s_1}^l + \pi_{s_1}^e) = p_r + p_e$$

Hence  $\dot{p}_r = g_r(q_r, q_s) + \Sigma(q_s, p_r, p_s) - \dot{p}_e$ . The forced Euler-Lagrange equation follows from the last equation in a straightforward fashion.  $\square$

*Remark 4.5.5.* Assuming  $\bar{q}_{s_2} = 0$ , the normal form in (4.51) can be rewritten as the following

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_{s_1}, 0)p_r + \varphi_1 \\ \dot{p}_r &= g_r(q_r, q_{s_1}, 0) + \Sigma(q_{s_1}, 0, p_r, p_{s_1}, 0) + \varphi_2 \\ \dot{q}_{s_1} &= p_{s_1} \\ \dot{p}_{s_1} &= u_1 \\ \dot{q}_{s_2} &= p_{s_2} \\ \dot{p}_{s_2} &= u_2\end{aligned}\tag{4.53}$$

where

$$\begin{aligned}\varphi_1 &= (m_r^{-1}(q_{s_1}, q_{s_2}) - m_r(q_{s_1}, 0))p_r \\ \varphi_2 &= g_r(q_r, q_{s_1}, q_{s_2}) - g_r(q_r, q_{s_1}, 0) + \Sigma(q_{s_1}, q_{s_2}, p_r, p_{s_1}, p_{s_2}) - \Sigma(q_{s_1}, 0, p_r, p_{s_1}, 0) + \delta\end{aligned}$$

where  $\varphi_1$  and the first two terms of  $\varphi_2$  vanish at  $(q_{s_2}, p_{s_2}) = 0$ . If  $\delta$  vanishes at  $(q_{s_2}, \dot{q}_{s_2}, u_2) = 0$ , then both nonlinear perturbations  $\varphi_1, \varphi_2$  can be made exponentially vanishing over compact domains by applying the linear control  $u_2 = K_2(q_{s_2}, p_{s_2}) = -c_1 q_{s_2} - c_2 p_{s_2}$  with  $c_1, c_2 > 0$ . This reduces the control design for the overall system in (4.51) to control of its  $(q_r, p_r, q_{s_1})$ -subsystem with control input  $p_{s_1}$

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_{s_1}, 0)p_r \\ \dot{p}_r &= g_r(q_r, q_{s_1}, 0) + \Sigma(q_{s_1}, 0, p_r, p_{s_1}, 0) \\ \dot{q}_{s_1} &= p_{s_1}\end{aligned}\tag{4.54}$$

Later, we show that under certain conditions, stabilization of the last nonlinear system reduces to stabilization of the following reduced system

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_{s_1}, 0)p_r \\ \dot{p}_r &= g_r(q_r, q_{s_1}, 0)\end{aligned}\tag{4.55}$$

with control input  $q_{s_1}$ . Since  $q_r$  and  $q_{s_1}$  have the same dimensions, this stabilization problem can be addressed (see section 4.7 of this chapter).

**Example 4.5.2.** (three-link planar robot) In this example, we consider a three-link planar robot as shown in Figure 4-6 with two controls. Both the external variable  $\theta_1$  and one of the two shape variables  $\theta_2, \theta_3$  are actuated. Notice that this is different than the case discussed in example 4.5.1 where both shape variables are actuated. The inertia matrix and dynamics of this robot are given in section A.2. The inertia matrix of this robot has the following structure

$$M = M(\theta_2, \theta_3) = \begin{bmatrix} m_{11}(\theta_2, \theta_3) & m_{12}(\theta_2, \theta_3) & m_{13}(\theta_2, \theta_3) \\ m_{21}(\theta_2, \theta_3) & m_{22}(\theta_3) & m_{23}(\theta_3) \\ m_{31}(\theta_2, \theta_3) & m_{32}(\theta_3) & m_{33} \end{bmatrix}$$

All of the normalized momentums of this three-link planar robot are non-integrable

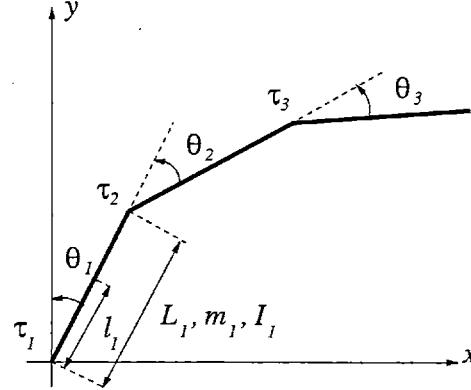


Figure 4-6: A planar three-link robot arm.

(see lemma A.2.1). In the following, we consider two cases: i)  $\theta_2$  is unactuated, and ii)  $\theta_3$  is unactuated. In each case, based on theorem 4.5.3 and using momentum decomposition procedure we obtain reduced normal forms of the system.

Case (i):  $\theta_2$  is unactuated. Set  $(q_x, q_{s_1}, q_{s_2}) = (\theta_1, \theta_2, \theta_3)$ . Assuming  $m_{12}(\theta_2, \theta_3) > 0$  for all  $\theta_2, \theta_3$ , using a global noncollocated partially linearizing feedback law, the dynamics of the shape variables can be transformed into

$$\begin{aligned}\dot{\theta}_2 &= \omega_2 \\ \dot{\omega}_2 &= u_1 \\ \dot{\theta}_3 &= \omega_3 \\ \dot{\omega}_3 &= u_2\end{aligned}$$

Then, calculating the normalized momentum conjugate to  $\theta_2$ , we get

$$\pi_{\theta_2} = \dot{\theta}_1 + \frac{m_{22}(\theta_3)}{m_{21}(\theta_2, \theta_3)} \dot{\theta}_2 + \frac{m_{23}(\theta_3)}{m_{21}(\theta_2, \theta_3)} \dot{\theta}_3$$

Thus

$$\mu_{\theta_2}(\theta_2, \theta_3) = \left[ \begin{array}{cc} \frac{m_{22}(\theta_3)}{m_{21}(\theta_2, \theta_3)} & \frac{m_{23}(\theta_3)}{m_{21}(\theta_2, \theta_3)} \end{array} \right]$$

Now, according to the locked shape  $(\bar{\theta}_2, \bar{\theta}_3) = (0, 0)$ , the locked shape inertia matrix  $\mu_{\theta_2}^l$  is given by

$$\mu_{\theta_2}^l(\theta_2, \theta_3) = \left[ \begin{array}{cc} \frac{m_{22}(0)}{m_{21}(0, 0)} & \frac{m_{23}(0)}{m_{21}(0, 0)} \end{array} \right]$$

Defining  $\gamma_1$  and  $\gamma_2$  as the integral of the elements of  $\mu_{\theta_2}^l$

$$\gamma_1(\theta_2) = \int_0^{\theta_2} \frac{m_{22}(0)}{m_{21}(s, 0)} ds, \quad \gamma_2(\theta_3) = \int_0^{\theta_3} \frac{m_{23}(0)}{m_{21}(0, s)} ds,$$

and applying the global change of coordinates

$$\begin{aligned}\theta_r &= \theta_1 + \gamma_1(\theta_2) + \gamma_2(\theta_3) \\ \omega_r &= m_{12}(\theta_2, \theta_3)(\omega_1 + \frac{m_{22}(0)}{m_{21}(\theta_2, 0)}\omega_2 + \frac{m_{23}(\theta_3)}{m_{21}(0, \theta_3)}\omega_3)\end{aligned}$$

the dynamics of the three-link robot with unactuated shape variable  $\theta_2$  transforms into the following nontriangular quadratic normal form

$$\begin{aligned}\dot{\theta}_r &= m_{12}(\theta_2, \theta_3)\omega_r \\ \dot{\omega}_r &= -g_r(\theta_r, \theta_2, \theta_3) + \Sigma(\theta_2, \theta_3, \omega_r, \omega_2, \omega_3) - \dot{p}_e \\ \dot{\theta}_2 &= \omega_2 \\ \dot{\omega}_2 &= u_1 \\ \dot{\theta}_3 &= \omega_3 \\ \dot{\omega}_3 &= u_2\end{aligned}\tag{4.56}$$

where the error momentum  $p_e$  is given by

$$p_e = (\frac{m_{22}(\theta_3)}{m_{21}(\theta_2, \theta_3)} - \frac{m_{22}(0)}{m_{21}(\theta_2, 0)})\omega_2 + (\frac{m_{23}(\theta_3)}{m_{21}(\theta_2, \theta_3)} - \frac{m_{23}(0, \theta_3)}{m_{21}(0, \theta_3)})\omega_3$$

Clearly,  $p_e$  vanishes at  $(\theta_3, \omega_3) = (0, 0)$ . This implies the perturbation  $\delta = -\dot{p}_e$  vanishes at  $(\theta_3, \omega_3, u_2) = (0, 0, 0)$  as well. Following the procedure in remark 4.5.5, stabilization of (4.56) reduces to stabilization of its fourth-order subsystem as

$$\begin{aligned}\dot{\theta}_r &= m_{12}(\theta_2, 0)\omega_r \\ \dot{\omega}_r &= -g_r(\theta_r, \theta_2, 0) + \Sigma(\theta_2, 0, \omega_r, \omega_2, 0) \\ \dot{\theta}_2 &= \omega_2 \\ \dot{\omega}_2 &= u_1\end{aligned}\tag{4.57}$$

which is exactly the normal form for the Pendubot with inertia matrix and potential energy

$$M_{Pen}(q_1) = \begin{bmatrix} m_{11}(\theta_2, 0) & m_{12}(\theta_2, 0) \\ m_{21}(\theta_2, 0) & m_{22}(0) \end{bmatrix}, V_{Pen}(\theta_1, \theta_2) = V(\theta_1, \theta_2, 0)$$

and an unactuated shape variable  $\theta_2$ . Later, we show that the Pendubot can be stabilized using Fixed Point Controllers [67].

Case (ii):  $\theta_3$  is unactuated. Take  $(q_x, q_{s_1}, q_{s_2}) = (\theta_1, \theta_3, \theta_2)$ . After a similar argument to Case (i), it can be shown that the control of the overall system reduces to control of the Pendubot with inertia matrix and potential energy

$$M_{Pen}(q_1) = \begin{bmatrix} m_{11}(0, \theta_3) & m_{13}(0, \theta_3) \\ m_{31}(0, \theta_3) & m_{33} \end{bmatrix}, V_{Pen}(\theta_1, \theta_3) = V(\theta_1, 0, \theta_3)$$

Notice that this Pendubot has an unactuated shape variable  $\theta_3$ . In this case, the perturbations  $\varphi_1, \varphi_2$  in (4.53) vanish exponentially over a compact domain by applying

the linear feedback  $u_2 = K_2(\theta_2, \omega_2) = -c_1\theta_2 - c_2\omega_2$  with  $c_1, c_2 > 0$ .

*Remark 4.5.6.* In conclusion, based on Figure 4-7, control of the middle robot reduces to control of the Acrobot in example 4.5.1 and control of the other two robots reduces to control of the Pendubot as discussed in Cases (i) and (ii) of example 4.5.2. This fact can be intuitively understood by the following graphical procedure without any calculations.

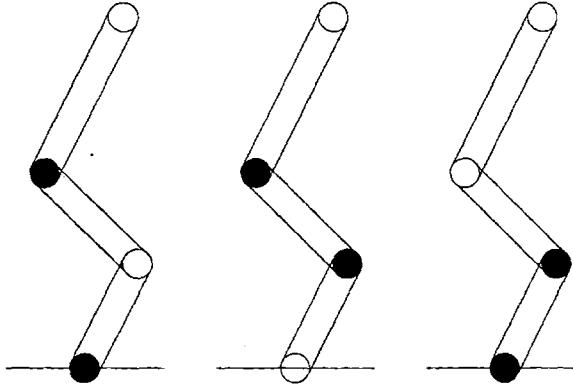


Figure 4-7: Three possible actuation configuration for a planar three-link robot arm.

*Graphical Reduction Procedure for an  $n$ -link underactuated planar robot:*

- 1) Set  $j = n$ .
- 2) If  $j$  is the last actuated joint, QUIT.
- 3) If joint  $j$  is actuated, lock the joint angle  $\theta_j = 0$  and merge the  $j$ th link and the  $(j-1)$ th link as one. Set  $j := j - 1$  and GOTO step 2.
- 4) If joint  $j$  is unactuated, set  $j := j - 1$  and GOTO step 2.

This algorithm ends up with the Acrobot or the Pendubot as the final underactuated system with a single actuator and two remaining links.

**Proposition 4.5.1.** Consider the 3D Cart-Pole system with configuration vector  $(q_x, q_s) = (x_1, x_2, \theta, \phi)$  and unactuated shape variables  $(\theta, \phi)$ . Then, the following change of coordinates over  $U = \mathbb{R}^2 \times (-\pi/2, \pi/2)^2$

$$\begin{aligned} z_1 &= x + \gamma(\theta, \phi) \\ z_2 &= \dot{x} + (l\dot{\theta}/\cos\theta, l\dot{\phi}/\cos\phi)^T \end{aligned} \quad (4.58)$$

with  $\gamma(\theta, \phi) = (\gamma_0(\theta), \gamma_0(\phi))^T$  and  $\gamma_0$  defined as

$$\gamma_0(\theta) = \int_0^\theta \frac{l}{\cos s} ds = l \log \left( \frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)} \right), \quad \theta \in (-\pi/2, \pi/2)$$

transforms the dynamics of the 3D Cart-Pole System into a cascade nonlinear system in feedforward form

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \psi(q_s) + p_s^T \Pi(q_s) p_s + \varphi(q_s) u \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u\end{aligned}\tag{4.59}$$

where

$$\psi(q_s) = -m_{sx}^{-1}(q_s) \nabla_{q_s} V(q_s)$$

has a nonsingular Jacobian at  $q_s = 0$ ,  $\Pi(q_s)$  is a cubic matrix, and  $\varphi(q_s) = -\mu_s^e(q_s)$  satisfies  $\varphi(0) = 0_{2 \times 2}$  and  $\sigma_{\max}(\varphi(q_s)) = O(|q_s|^2)$ . In addition, the origin for (4.59) can be globally asymptotically and locally exponentially stabilized using a state feedback in the form of nested saturations.

**Proof.** We use the method of momentum decomposition to reduce the dynamics of the 3D Cart-Pole system. This system has equal number of external and shape variables. Thus, following the notation of theorem 4.5.3,  $\dim(Q_{s2}) = 0$  and the external and shape configuration vectors of the 3D Cart-Pole system are  $q_x = (x_1, x_2)$  and  $q_s = q_{s1} = (\theta, \phi)$ , respectively. From the kinetic energy of the 3D Cart-Pole system in (4.34), the block matrices  $m_{sx}(q_s)$  and  $m_{ss}(q_s)$  for this system can be identified as

$$m_{sx}(q_s) = \begin{bmatrix} ml \cos \theta & -ml \sin \theta \sin \phi \\ 0 & ml \cos \theta \cos \phi \end{bmatrix}, \quad m_{ss}(q_s) = \begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \cos^2 \theta \end{bmatrix}$$

In example 4.4.2, it was shown that the normalized momentum

$$\pi_s = \dot{q}_x + \mu_s(q_s) \dot{q}_s$$

is non-integrable over  $U = \mathbb{R}^2 \times (-\pi/2, \pi/2)^2$  where

$$\mu_s(q_s) = m_{sx}^{-1}(q_s) m_{ss}(q_s) = \begin{bmatrix} \frac{l}{\cos \theta} & \frac{l \sin \theta \sin \phi}{\cos \phi} \\ 0 & \frac{l \cos \theta}{\cos \phi} \end{bmatrix}$$

Hence, after choosing the lock configuration  $(\bar{\theta}, \bar{\phi}) = (0, 0)$  and setting  $\phi = 0$  and  $\theta = 0$ , respectively, in the first and second columns of  $\mu_s$ , the normalized locked shape inertia and error inertia for the system can be obtained as

$$\mu_s^l(\theta, \phi) = \begin{bmatrix} \frac{l}{\cos \theta} & 0 \\ 0 & \frac{l}{\cos \phi} \end{bmatrix}, \quad \mu_s^e(\theta, \phi) = \begin{bmatrix} 0 & \frac{l \sin \theta \sin \phi}{\cos \phi} \\ 0 & \frac{l(\cos \theta - 1)}{\cos \phi} \end{bmatrix}$$

Take

$$\gamma_0(z) = \int_0^z \frac{l}{\cos s} ds, \quad z \in (\pi/2, \pi/2)$$

Then,  $\gamma(\theta, \phi) = (\gamma_0(\theta), \gamma_0(\phi))^T$  satisfies  $\nabla_{q_s} \gamma(q_s) = \mu_s^l(q_s)$ . The change of coordinates in (4.50) can be explicitly calculated as

$$\begin{aligned} q_r &= x + \gamma(\theta, \phi) \\ p_r &= m_{sx}(\theta, \phi)(\dot{x} + \mu_s^l(\theta, \phi)\dot{q}_s) \end{aligned} \quad (4.60)$$

After simplification, we obtain

$$\begin{aligned} q_r^1 &= x_1 + \gamma_0(\theta) \\ p_r^1 &= ml \cos(\theta) \dot{x}_1 - ml \sin(\theta) \sin(\phi) \dot{x}_2 + ml^2 \dot{\theta} - ml^2 \sin(\theta) \tan(\phi) \dot{\phi} \\ q_r^2 &= x_2 + \gamma_0(\phi) \\ p_r^2 &= ml \cos(\theta) \cos(\phi) \dot{x}_2 + ml^2 \cos(\theta) \dot{\phi} \end{aligned} \quad (4.61)$$

where  $q_r = (q_r^1, q_r^2)^T$  is the configuration vector of the reduced system and  $p_r = (p_r^1, p_r^2)^T$  is its conjugate momentum. The nonlinear change of coordinates in (4.61) transforms the dynamics of the 3D Cart-Pole system into

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(\theta, \phi)p_r \\ \dot{p}_r &= g_r(\theta, \phi) + \Sigma(q_s, p_r, p_s) + \delta \\ \dot{\theta} &= \omega_1 \\ \dot{\omega}_1 &= u_1 \\ \dot{\phi} &= \omega_2 \\ \dot{\omega}_2 &= u_2 \end{aligned} \quad (4.62)$$

where  $m_r = m_{sx}$  and

$$g_r(\theta, \phi) = -\nabla_{q_s} V(\theta, \phi) = \begin{bmatrix} mg l \sin \theta \cos \phi \\ mg l \cos \theta \sin \phi \end{bmatrix}$$

Notice that  $m_{xx}$  is constant for this system and  $\Sigma$  is linear in  $\dot{q}_x$ , thus

$$\Sigma(q_s, p_r, p_s) = \nabla_{q_s} K = \nabla_{q_s} \{ \dot{q}_s^T m_{sx}(q_s) \dot{q}_x \} + \nabla_{q_s} \{ \frac{1}{2} \dot{q}_s^T m_{ss}(q_s) \dot{q}_s \}$$

Following the line of proposition 4.4.1, apply a second change of coordinates as the following

$$\begin{aligned} z_1 &= q_r \\ z_2 &= m_r^{-1}(\theta, \phi)p_r \end{aligned} \quad (4.63)$$

We prove that the last change of coordinates transforms the dynamics of the 3D

Cart-Pole system into the feedforward normal form in (4.59). First, observe that

$$\begin{aligned} z_1 &= x + \gamma(\theta, \phi) \\ z_2 &= \dot{x} + \mu_s^l(\theta, \phi)\dot{q}_s \end{aligned} \quad (4.64)$$

and therefore  $\dot{z}_1 = z_2$ . In addition, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{q}_s} &= m_{sx}(q_s)(\dot{q}_x + \mu_s(q_s)\dot{q}_s) \\ &= m_{sx}(q_s)(\dot{q}_x + \mu_s^l(q_s)\dot{q}_s + \mu_s^e(q_s)\dot{q}_s) \\ &= m_{sx}(q_s)z_2 + p_e \end{aligned}$$

where

$$p_e = m_{sx}(q_s)\mu_s^e(q_s)\dot{q}_s$$

But from Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_s} = \frac{\partial \mathcal{L}}{\partial q_s} = g_r(q_s) + \Sigma$$

and we obtain

$$\begin{aligned} m_{sx}(q_s)\dot{z}_2 &= g_r(q_s) + \Sigma - \dot{m}_{sx}z_2 - \dot{p}_e \\ &= g_r(q_s) + \nabla_{q_s} \left\{ \frac{1}{2}\dot{q}_s^T m_{ss}(q_s)\dot{q}_s \right\} \\ &\quad + \nabla_{q_s} \left\{ \dot{q}_s^T m_{sx}(q_s)\dot{q}_x \right\} - \dot{m}_{sx}\dot{q}_x - \dot{m}_{sx}\mu_s^l(q_s)\dot{q}_s - \dot{p}_e \end{aligned}$$

But similar to condition ii) of proposition 4.4.1,  $m_{sx}(q_s)$  block of the inertia matrix of the 3D Cart-Pole system has differentially symmetric rows that implies

$$\nabla_{q_s} \left\{ \dot{q}_s^T m_{sx}(q_s)\dot{q}_x \right\} = \dot{m}_{sx}\dot{q}_x$$

Therefore, we obtain the following equations for the  $z$ -subsystem

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \psi(q_s) + \tilde{\Sigma}(q_s, p_s) + \varphi(q_s)u \end{aligned} \quad (4.65)$$

where

$$\begin{aligned} \psi(q_s) &= m_{sx}^{-1}(q_s)g_r(q_s) \\ \tilde{\Sigma}(q_s, p_s) &= m_{sx}^{-1}(q_s)[\nabla_{q_s} \left\{ \frac{1}{2}\dot{q}_s^T m_{ss}(q_s)\dot{q}_s \right\} - \dot{m}_{sx}\mu_s^l(q_s)\dot{q}_s - \frac{d}{dt} \{m_{sx}(q_s)\mu_s^e(q_s)\}\dot{q}_s] \\ \varphi(q_s) &= -\mu_s^e(q_s) \end{aligned}$$

and  $\tilde{\Sigma}(q_s, p_s) = p_s^T \Pi(q_s)p_s$  is quadratic in  $p_s$ . Notice that from the explicit expression

of  $\mu_s^e(q_s)$ ,  $\varphi(0) = 0_{2 \times 2}$  and  $\sigma_{max}(\varphi(q_s)) = O(|(\theta, \phi)|^2)$ . The stabilization result using nested saturations [102] follows from the fact that  $\bar{\Sigma}$  is quadratic and  $[\nabla_{q_s} \psi(q_s)]_{q_s=0} = gI_{2 \times 2}$  is non-singular (where  $g$  is the gravity constant).  $\square$

## 4.6 Classification of Underactuated Systems

In this section, we classify underactuated systems based on possession or lack of some or all of the following four properties:

- (a) Actuated shape variables
- (b) Non-interacting inputs (i.e. lack of input coupling)
- (c) Integrable normalized momentums
- (d) Extra required conditions (as described in the definition of each class in the following).

This leads at most to sixteen possible classes. However, due to redundancy and the fact that some of these cases never have appeared in applications, we only focus on a subset of these sixteen classes that frequently appear in control of underactuated mechanical systems. This subset includes eight different classes.

**Note:** In case an underactuated system is flat (i.e. has constant inertia matrix), we call the variables that appear in the force matrix  $F(q)$  *shape variables*.

Table 4.1 provides definition and examples of different classes of underactuated systems. The following remarks have to be taken into account regarding Table 4.1:

- a “Yes” or a “No” means the corresponding property holds or does not hold, respectively.
- “Q”, “L”, and “LQ” in front of “Nontriangular” refer to “Quadratic”, “Linear”, and “Linear-Quadratic” structure of the normal form with respect to  $p_s$  (i.e. shape velocity).
- a “Yes” for property (d) means that certain extra conditions are required that can be found in the description of the corresponding class.
- a controlled VTOL aircraft means the dynamics of the VTOL in closed loop with a feedback  $u_1 = u_1(x, y, \dot{x}, \dot{y})$  and a single remaining control  $u_2$ .

In the following, we rewrite the normal form corresponding to each class of underactuated systems in table 4.1 by partitioning the state vector as  $\text{col}(z, \xi) \in \mathbb{R}^n \times \mathbb{R}^r$ . Throughout this section,  $(z_1, z_2, \xi_1, \xi_2)$  denotes  $(q_r, p_r, q_s, p_s)$  unless otherwise is stated. Furthermore, the term  $f_0$  is due to the potential energy of the system. Here is the description of each class:

Table 4.1: Classification of Underactuated Systems

Class	(a)	(b)	(c)	(d)	Normal Form	Control Design	example(s)
I	Yes	Yes	Yes	No	Strict Feedback	Backstepping	Acrobot (2D,1C), TORA, Inertia-Wheel Pendulum
IIa	No	Yes	Yes	No	Nontriangular Q	Fixed-Point Law	Rotating Pendulum Pendubot, Beam-n-Ball
IIb	No	Yes	Yes	Yes	Nontriangular L	Fixed-Point Law	$\approx$ Flexible-link Robots
III	No	Yes	Yes	Yes	Feedforward	Nested Saturations	2D Cart-Pole (2D,1C)
IVa	Yes	No	Yes	No	Nontriangular Q	Fixed-Point Law	Controlled VTOL (3D,1C)
IVb	Yes	No	Yes	Yes	Nontriangular Q	Fixed-Point Law	Controlled VTOL (3D,1C)
V	Yes	No	Yes	Yes	Strict Feedback	Backstepping	VTOL aircraft (3D,2C) $\approx$ Aircraft (6D,4C) $\approx$ Helicopter (6D,4C)
VIa	Yes	Yes	No	No	Nontriangular L	Fixed-Point Law	Three-link arm (3D,2C)
VIb	Yes	Yes	No	Yes	Feedback+ $\Delta$	Backstepping	Three-link arm (3D,2C)
VIIa	No	Yes	No	No	Nontriangular LQ	Fixed-Point Law	Three-link arm (3D,2C)
VIIb	No	Yes	No	Yes	Nontriangular+ $\Delta$	Fixed-Point Law	Three-link arm (3D,2C)
VIII	No	Yes	No	Yes	Feedforward	Nested Saturations	3D Cart-Pole (4D,2C)

- **Class-I:** underactuated systems with actuated shape variables, integrable momentums, and non-interacting inputs. From equation (4.6), the normal form of Class-I underactuated systems can be rewritten as the following

$$\begin{aligned}\dot{z}_1 &= N(\xi_1)z_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.66}$$

where  $N(\xi_1)$  is a symmetric and positive definite matrix. System (4.66) is in *strict feedback form* with a vector field *non-affine in  $\xi_1$* .

- **Class-IIa:** underactuated systems with unactuated shape variables, integrable normalized momentums, non-interacting inputs, and equal number of external and shape variables. From equation (4.14), the normal form of Class-IIa underactuated systems can be rewritten as the following

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) + z_2^T g_{11}(\xi_1)z_2 + z_2^T g_{12}(\xi_1)\xi_2 + \xi_2^T g_{22}(\xi_1)\xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.67}$$

where  $z_2 = m_r^{-1}(q_s)$  and  $g_{ij}$ 's are cubic matrices with the property  $g_{12} = g_{21}^T$ . System (4.67) is in *nontriangular quadratic form* with a vector field *non-affine in  $(\xi_1, \xi_2)$* .

- **Class-IIb:** a subclass of Class-IIa underactuated systems with unequal number of external and shape variables. From equation (4.21), the normal form of Class-

IIb underactuated systems can be rewritten as the following

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(z_1) + f_d(z_1, z_2) + z_2^T g_{11}(\xi_1) z_2 + z_2^T g_{12}(\xi_1) \xi_2 + \xi_2^T g_{22}(\xi_1) \xi_2 + g_3(z_1) \xi_3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= u\end{aligned}\tag{4.68}$$

where  $z, \xi$  are defined in (4.20) and  $F_d$  is a force due to dissipation satisfying  $z_2^T f_d(z_1, z_2) < 0, \forall z_1, \forall z_2 \neq 0$ . The  $g_{ij}$ 's are cubic matrices with the property  $g_{12} = g_{21}^T$ . System (4.68) is in *nontriangular form* with a vector field *non-affine in*  $(\xi_1, \xi_2)$  and *affine in*  $\xi_3$ .

- **Class-III:** a subclass of Class-IIa underactuated systems satisfying conditions of proposition 4.2.1. From equation (4.17) the normal form of Class-III underactuated systems can be rewritten as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(\xi_1) + \xi_2^T g(\xi_1) \xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.69}$$

where  $z_2 = m_r^{-1}(q_s)p_r$  and  $g(\xi)$  is a cubic matrix. System (4.69) is in *strict feedforward form* with a vector field *non-affine in*  $(\xi_1, \xi_2)$ .

- **Class-IVa:** underactuated systems with actuated shape variables, integrable momentums, and input coupling. From equation (4.24), taking  $z_2 = m_r^{-1}(q_s)p_r$ , the normal form for Class-IVa underactuated systems can be written as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) + z_2^T g_{11}(\xi_1) z_2 + z_2^T g_{12}(\xi_1) \xi_2 + \xi_2^T g_{22}(\xi_1) \xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.70}$$

The  $g_{ij}$ 's are cubic matrices with the property  $g_{12} = g_{21}^T$ . System (4.70) is in *nontriangular quadratic form* with a vector field *non-affine in*  $(\xi_1, \xi_2)$ . Notice that normal forms (4.70) and (4.67) represent the same class of cascade nonlinear systems, but two different classes of underactuated systems.

- **Class-IVb:** a subclass of Class-IVa underactuated systems with constant  $m_{xx}$  and  $V(q) = V(q_s)$ . From equation (4.25), the normal form for Class-IVb un-

deractuated systems can be written as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(\xi_1) + \xi_2^T g(\xi_1) \xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.71}$$

where  $z_2 = m_r^{-1} p_r$  and  $g(\xi_1)$  is a cubic matrix. System (4.71) is in *strict feed-forward form* with a vector field *non-affine in*  $(\xi_1, \xi_2)$ .

- **Class-Va:** underactuated systems with actuated shape variables, integrable momentums, input coupling, and an extra control in the dynamics of the external variables with the property that their  $m_{xx}$  is constant and  $V(q) = V(q_x)$ . From equation (4.28), the normal form for Class-Va underactuated systems can be written as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) + g_0(\xi_1) T + \xi_2^T g(\xi_1) \xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.72}$$

where  $z_2 = m_r^{-1} p_r$ ,  $g(\xi_1)$  is a cubic matrix,  $g_0(\xi_1) : \mathbb{R}^p \rightarrow \mathbb{R}^n$  is defined by

$$g_0(\xi_1) := \frac{1}{m_{xx}} R(\xi_1) \vec{e},\tag{4.73}$$

$\vec{e}$  is a fixed unit vector in  $\mathbb{R}^n$  and  $R(\xi_1) \in SO(n)$  is a rotation matrix in  $\mathbb{R}^n$ . The control  $T \in \mathbb{R}$  (or thrust) is the magnitude of a force applied to the dynamics of the vector of external variables,  $q_x \in \mathbb{R}^n$ , in the direction  $\vec{e}$ . Assuming that  $T = K(z, \xi_1)$ , system (4.72) is in *nontriangular quadratic* normal form with a vector field *non-affine in*  $(\xi_1, \xi_2)$ .

- **Class-Vb:** a subclass of Class-Va underactuated systems with the property that there exists a positive definite and symmetric matrix  $Q(\xi_1) \in \mathbb{R}^{p \times p}$  such that

$$\xi_2^T g(\xi_1) \xi_2 = [\xi_2^T Q(\xi_1) \xi_2] g_0(\xi_1)\tag{4.74}$$

and  $V(q_x)$  is a linear function. From theorem 4.3.2 and equation (4.72), the normal form for Class-Vb underactuated systems can be given as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0 + g_0(\xi_1) \bar{T} \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.75}$$

where  $f_0 \in \mathbb{R}^n$  is a constant vector and  $\bar{T}$  in a new control defined as

$$\bar{T} = T + \xi_2^T Q(\xi_1) \xi_2$$

Assuming that the state feedback  $\bar{T} = K(z, \xi_1)$  globally asymptotically stabilizes the equilibrium  $z = z_0$  for the  $z$ -subsystem of (4.75), the dynamics of (4.75) is in *strict feedback form* with a vector field *non-affine in  $\xi_1$* .

- **Class VIa:** underactuated systems with actuated shape variables, non-integrable normalized momentums, and non-interacting inputs. From equation (4.31), the normal form for Class-VIa underactuated systems can be expressed as

$$\begin{aligned}\dot{z}_1 &= N(\xi_1)z_2 + g_0(\xi_1)\xi_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.76}$$

System (4.76) is in *nontriangular linear* normal form with a vector field *affine in  $\xi_2$* . Note that  $\#h = h(\xi_1) : \mathbb{R}^p \rightarrow \mathbb{R}^n : Dh(\xi_1) = g_0(\xi_1)$ .

- **Class VIb:** a subclass of Class-VIa with the property that there exists a partition of  $q_s = \text{col}(q_{s_1}, q_{s_2})$  with an  $n$ -dimensional vector  $q_{s_1}$  such that the perturbation  $\delta$  in (4.47) vanishes identically at  $(q_{s_2}, p_{s_2}, u_2) = 0$  where  $\dot{q}_{s_2} = p_{s_2}, \dot{p}_{s_2} = u_2$ . Let us partition  $\xi_i$  and  $u$  as  $\text{col}(\eta_i, \mu_i)$ ,  $i = 1, 2$  and  $\text{col}(u_1, u_2)$ , respectively. Then, from equation (4.47), the normal form for Class-VIb underactuated systems can be obtained as

$$\begin{aligned}\dot{z}_1 &= N(\eta_1)z_2 + \varphi_1(z_2, \eta_1, \mu_1) \\ \dot{z}_2 &= f_0(z_1, \eta_1) + \varphi_2(z_1, \eta, \mu, u) \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= u_1 \\ \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= u_2\end{aligned}\tag{4.77}$$

where  $N(\eta_1) = m_r^{-1}(q_{s_1}, 0)$  and the perturbation terms  $\varphi_1, \varphi_2$  vanish identically at  $(\mu_1, \mu_2, u_2) = 0$ . The  $(z, \eta)$ -subsystem of (4.77) is a *perturbation of a strict feedback form*.

- **Class VIIa:** underactuated systems with unactuated shape variables, non-integrable normalized momentums, and non-interacting inputs. From equation (4.32), the normal form for Class-VIIa underactuated systems can be given as

$$\begin{aligned}\dot{z}_1 &= z_2 + g_0(\xi_1)\xi_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) + z_2^T g_{11}(\xi_1)z_2 + z_2^T g_{12}(\xi_1)\xi_2 + \xi_2^T g_{22}(\xi_1)\xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.78}$$

where  $z_2 = m_r^{-1}(q_s)p_r$  and the  $g_{ij}$ 's are cubic matrices satisfying  $g_{12} = g_{21}^T$ . System (4.78) is in *nontriangular linear-quadratic* normal form with a vector field *non-affine in*  $(\xi_1, \xi_2)$ .

- **Class VIIb:** underactuated systems with partially unactuated shape variables, non-integrable normalized momentums, and non-interacting inputs. In addition, there exists a partition of  $q_s = \text{col}(q_{s_1}, q_{s_2})$  with an  $n$ -dimensional vector  $q_{s_1}$  such that the perturbation  $\delta$  in (4.51) vanishes identically at  $(q_{s_2}, p_{s_2}, u_2) = 0$ . Following the notation used in the definition of Class-VIb, based on equation (4.51), the normal form for Class-VIIb underactuated systems can be written as

$$\begin{aligned}\dot{z}_1 &= z_2 + \varphi_1(z_2, \eta_1, \mu_1) \\ \dot{z}_2 &= f_0(z_1, \eta_1) + z_2^T g_{11}(\eta_1)z_2 + z_2^T g_{12}(\eta_1)\eta_2 + \eta_2^T g_{22}(\eta_1)\eta_2 + \varphi_2(z, \eta, \mu, u) \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= u_1 \\ \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= u_2\end{aligned}\tag{4.79}$$

where the  $g_{ij}$ 's are cubic matrices and  $\varphi_1, \varphi_2$  vanish identically at  $(\mu_1, \mu_2, u_2) = 0$ . The  $(z, \eta)$ -subsystem of (4.79) is a *perturbation of a nontriangular quadratic normal form* (see the normal form of Class-IIa).

- **Class VIII:** a subclass of Class-VIIa underactuated systems satisfying the conditions of proposition 4.4.1. From equation (4.33), the normal form for Class-VIII underactuated systems can be expressed as

$$\begin{aligned}\dot{z}_1 &= z_2 + \varphi_1(\xi_1, \xi_2) \\ \dot{z}_2 &= f_0(\xi_1) + \xi_2^T g(\xi_1)\xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_1 &= u\end{aligned}\tag{4.80}$$

where

$$z_1 = q_r + \mu(0)q_s, \quad z_2 = m_r^{-1}(q_s)p_r$$

and  $\varphi_1 \sim O(|(\xi_1, \xi_2)|^2)$  as defined in proposition (4.4.1). System (4.80) is in *strict feedforward form* with a vector field *non-affine in*  $(\xi_1, \xi_2)$ .

Let us make the following definition.

**Definition 4.6.1.** (core reduced system) Define *the core reduced system* as the dynamics of the  $z$ -subsystem without the following types of terms:

- terms containing  $\xi_2$ ,
- terms containing  $z_2$  in the equation of  $\dot{z}_2$ ,

iii) high-order perturbation terms  $\varphi_i$ .

The following corollary creates a connection in terms of the dynamics of the  $z$ -subsystem among all the above classes of underactuated system.

**Corollary 4.6.1.** *According to definition 4.6.1, for all the aforementioned classes of underactuated mechanical systems, the core reduced system is in the normal form*

$$\begin{aligned}\dot{z}_1 &= N(v)z_2 \\ \dot{z}_2 &= f(z_1, v)\end{aligned}\tag{4.81}$$

where  $v$  is the control input for (4.81) and  $N(v)$  is a positive definite symmetric matrix. (Note that for several classes  $N(v) = I$ ).

Global asymptotic stabilization of nonlinear systems in normal form (4.81) is addressed in theorem 4.7.1 for the case where  $z_1, z_2, v \in \mathbb{R}^n$ .

*Remark 4.6.1.* Based on proposition 5.11.4. The normal form of the first-level approximate model of a helicopter is in the form

$$\begin{aligned}\dot{z} &= f(z) + g_1(\xi_1)T + g_2(\xi_1, \xi_2, \epsilon) + g_3(\xi_1, \epsilon)u \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{4.82}$$

where  $z = \text{col}(x, v) \in \mathbb{R}^6$ ,  $\xi_1 = \text{col}(\phi, \theta, \psi)$  (i.e. three Euler angles),  $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^T \in \mathbb{R}^3$ , and  $g_2, g_3$  are linear with respect to the matrix

$$E(\epsilon) = \begin{bmatrix} 0 & \epsilon_2 & 0 \\ \epsilon_1 & 0 & \epsilon_3 \\ 0 & 0 & 0 \end{bmatrix}; \quad \epsilon_1 > 0, \epsilon_2 < 0, \epsilon_3 > 0$$

Thus,  $g_2, g_3$  vanish identically at  $\epsilon = (0, 0, 0)$ . The vector field of (4.82) is quadratic in  $\xi_2$  and affine in control  $u$ . Under the assumption that  $\epsilon = (0, 0, 0)$ , (4.82) reduces to the normal form of a Class-VIb underactuated system in (4.75).

## 4.7 Nonlinear Control of the Core Reduced System

The fundamental property of the three basic cascade normal forms for underactuated systems is that *the unforced and unperturbed reduced system for all classes of underactuated systems in section 4.6 is the same and is given by*

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s)\end{aligned}\tag{4.83}$$

We refer to (4.83) as the *Core Reduced System* (also, see definition 4.6.1). This section is devoted to stabilization of the core reduced system for all underactuated systems

discussed in this chapter. The main property of the core reduced system is that it is a simple Lagrangian system without any external force. Thus, the shape vector  $q_s$  plays the role of the control input for the core reduced system. Here is our main result on global asymptotic stabilization of the core reduced system (4.83). (For simplicity of the notation, in the following theorem the subscript “r” is dropped).

**Theorem 4.7.1.** *Consider the following nonlinear system non-affine in control*

$$\begin{aligned}\dot{q} &= N(u)p \\ \dot{p} &= g(q, u)\end{aligned}\tag{4.84}$$

where  $q, p, u \in \mathbb{R}^n$ ,  $g(q, u) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a smooth function with  $g(0, 0) = 0$ ,  $N(u)$  is an invertible matrix for all  $u$ , and  $M(u) = N^{-1}(u)$  is a positive definite and symmetric inertia matrix. Suppose there exists an isolated root  $u = \alpha(q)$  of  $g(q, u) = 0$  with the property  $\alpha(0) = 0$  such that

$$\det\left(\frac{\partial g}{\partial u}(q, \alpha(q))\right) \neq 0$$

Set

$$\psi(q, v) := [M(u)g(q, u)]_{u=\alpha(q)+v}$$

Then, for all  $q \in \mathbb{R}^n$ ,  $w = \psi(q, v)$  is a local diffeomorphism around a neighborhood of  $v = 0$ . Assume there exists an open ball  $B_r(0)$  around  $w = 0$  and a function  $\beta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$\psi(q, \beta(q, w)) = w, \quad \forall w \in B_r(0) \subset \mathbb{R}^n$$

uniformly in  $q$ . Let  $\vec{\sigma}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the vector sigmoidal function

$$\vec{\sigma}(x) = \frac{x}{(1 + \|x\|^2)^{\frac{1}{2}}}$$

Then, for all  $c_1, c_2 \in (0, r/2]$ , the static state feedback  $u = K(q, p)$  defined as the following

$$\begin{aligned}u &= K(q, p) := \alpha(q) + \beta(q, w), \\ w &= K_b(q, p) := -c_1\vec{\sigma}(q) - c_2\vec{\sigma}(p);\end{aligned}\tag{4.85}$$

globally asymptotically stabilizes the origin  $(q, p) = (0, 0)$  for the nonlinear system in (4.84).

**Proof.** Calculating  $\nabla_v \psi(q, v)$  at  $v = 0$  (or  $u = \alpha(q)$ ), we get

$$\nabla_v \psi(q, v) = M(\alpha(q)) \cdot [\nabla_v g(q, \alpha(q)) + v]_{v=0}$$

By assumption, both  $M(\alpha(q))$  and  $\nabla_u g(q, \alpha(q))$  are invertible matrices. This means that  $\nabla_v \psi(q, v)$  is invertible at  $v = 0$  and thus  $\psi(q, v)$  is a local diffeomorphism for all

$q$ . Consider the invertible change of control  $w = \psi(q, v)$  with an inverse  $v = \beta(q, w)$  defined over  $\mathbb{R}^n \times B_r(0)$  so that  $\psi(q, \beta(q, w)) = w$ . Then, the dynamics of (4.84) can be written as

$$\begin{aligned}\dot{q} &= N(u)p \\ \dot{p} &= N(u)w\end{aligned}\tag{4.86}$$

with  $u = \alpha(q) + \beta(q, w)$ . Fix  $c_1, c_2 \in (0, r/2]$ . We have

$$\|w\| = \|K_b(q, p)\| \leq c_1\|\vec{\sigma}(q)\| + c_2\|\vec{\sigma}(p)\| < c_1 + c_2 \leq r$$

and for all  $(q, p)$ ,  $K_b(q, p) \in B_r(0)$ . Applying the state feedback  $w = K_b(q, p)$  in the question, the closed-loop system in (4.86) takes the following form

$$\begin{aligned}\dot{q} &= N(u)p \\ \dot{p} &= -c_1N(u)\vec{\sigma}(q) - c_2N(u)\vec{\sigma}(p)\end{aligned}\tag{4.87}$$

with  $u = K(q, p)$  (defined in (4.85)). Consider the following positive definite Lyapunov function candidate

$$H(q, p) = c_1\phi(q) + \frac{1}{2}\|p\|^2$$

where the positive semidefinite potential function  $\phi(q) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is defined as

$$\phi(q) = (1 + \|q\|^2)^{\frac{1}{2}} - 1$$

and satisfies the property  $(\nabla\phi(q))^T = \vec{\sigma}(q)$ . Calculating  $\dot{H}$  along the solutions of the closed-loop system (4.87), we obtain

$$\begin{aligned}\dot{H} &= \frac{\partial H(q, p)}{\partial q} \cdot \dot{q} + \frac{\partial H(q, p)}{\partial p} \cdot \dot{p} \\ &= c_1\vec{\sigma}(q)^T N(u)p - c_1p^T N(u)\vec{\sigma}(q) - c_2p^T N(u)\vec{\sigma}(p), \quad (N^T = N) \\ &= -c_2p^T N(u)\vec{\sigma}(p) < 0, \quad \forall p \neq 0\end{aligned}$$

The last inequality is due to positive definiteness of the term

$$p^T N(u)\vec{\sigma}(p) = \frac{p^T N(u)p}{(1 + \|p\|^2)^{\frac{1}{2}}} \geq 0$$

for  $p \neq 0$  and all  $u$ , particularly,  $u = K(q, p)$ . Based on LaSalle's invariance principle [40], all the solutions of the closed-loop system converge to the largest invariant set  $\Omega$  in  $\{(q, p) : \dot{H} = 0\} = \{(q, 0)\}$ . But for system (4.87),  $\Omega = \{(0, 0)\}$ . Therefore, the origin  $(q, p) = (0, 0)$  is globally asymptotically stable for (4.87) (and (4.84)). Furthermore,  $H(q, p)$  is a smooth positive definite and proper Lyapunov function for the system.  $\square$

**Definition 4.7.1.** (bi-saturated PD controller) We call  $w = K_b(q, p)$  in equation

(4.85), a bi-saturated PD controller.

**Corollary 4.7.1.** Assume all the conditions in theorem 4.7.1 hold and let

$$w = k_b(q, p) := -c_0 \vec{\sigma}(q + p), \quad c_0 \in (0, r]$$

Then,  $(q, p) = (0, 0)$  is GAS for (4.84).

**Proof.** We prove that the following smooth and positive definite function

$$H(q, p) = c_0 \phi(q + p) + \frac{1}{2} \|p\|^2$$

is a valid Lyapunov function for the closed-loop system

$$\begin{aligned} \dot{q} &= N(u)p \\ \dot{p} &= -c_0 N(u) \vec{\sigma}(q + p) \end{aligned} \tag{4.88}$$

We have

$$\begin{aligned} \dot{H} &= c_0 \vec{\sigma}(q + p)^T (\dot{q} + \dot{p}) - c_0 p^T N(u) \vec{\sigma}(q + p) \\ &= -c_0 \|\vec{\sigma}(q + p)\|^2 \end{aligned}$$

Thus, all the solutions converge to the largest invariant set  $\Omega$  in  $\{(q, p) : q + p = 0\}$ . By inspection,  $\Omega = \{(0, 0)\}$  and the origin is GAS for (4.88) and thus (4.84).  $\square$

The following result shows that under further structure in the dynamics of the reduced system it is possible to find a static state feedback in closed-form that globally asymptotically stabilizes the origin for the reduced system.

**Theorem 4.7.2.** Consider the reduced system in (4.83) and assume that  $\dim(Q_r) = \dim(Q_s) = d$  and the reduced inertia matrix is diagonal, i.e.

$$m_r(q_s) = \text{diag}(m_{r1}(q_s), \dots, m_{rd}(q_s))$$

In addition, assume that (4.83) is in the following form

$$\begin{aligned} \dot{q}_r^j &= m_{rj}^{-1}(q_s^j) p_r^j \\ \dot{p}_r^j &= g_{rj}(q_r^j, q_s^j) \end{aligned} \tag{4.89}$$

for  $j = 1, \dots, d$ . Let  $q_s^j = \alpha_j(q_r^j)$  be an isolate root of  $g_{rj}(q_r, q_s)$  for  $j = 1, \dots, d$ , i.e.

$$g_{rj}(q_r, \alpha_j(q_r^j)) = 0, \quad \left[ \frac{\partial g_{rj}}{\partial q_s^j} \right]_{q_s^j=\alpha_j(q_r^j)} > 0$$

Then, the following state feedback

$$q_s = \alpha(q_r) - \vec{\sigma}(q_r + p_r)$$

globally asymptotically stabilizes the equilibrium point  $(q_r, p_r) = (0, 0)$  of (4.89) where

$$\alpha(q_r) = (\alpha_1(q_r^1), \dots, \alpha_d(q_r^d))^T, \vec{\sigma}(z) = (\sigma_1(z_1), \dots, \sigma_n(z_i))$$

and all  $\sigma_i$ 's are one dimensional sigmoidal functions.

The proof of this theorem relies on the following theorem for the one-dimensional case.

**Theorem 4.7.3.** Consider the following system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = f(p) + g(q, u), \end{cases} \quad (4.90)$$

where  $f$  and  $g$  are  $C^1$  functions with  $f(0) = 0$  and  $g(0, 0) = 0$ ,  $f$  is decreasing ( $f' \leq 0$ ), and  $q, p, u \in \mathbb{R}$ . Suppose that zero is not a critical value for  $g(q, u)$  and

$$\frac{\partial g(q, u)}{\partial u} \neq 0$$

on the manifold

$$M = \text{Ker}(g) = \{(q, u) \in \mathbb{R}^2 : g(q, u) = 0\}$$

Then,  $g(q, u)$  has an isolated root  $\alpha(q)$  such that

$$g(q, \alpha(q)) = 0$$

and there exists a continuously differentiable state feedback law in the following form

$$u = \alpha(q) - \sigma(q + p), \quad (4.91)$$

that globally asymptotically stabilizes the origin for (4.90) ( $\sigma$  is a sigmoidal function).

**Proof.** Because zero is not a critical value for  $g$ ,  $M = g^{-1}(0) = \text{Ker}(g)$  is a one dimensional differentiable manifold in  $\mathbb{R}^2$ . In other words,  $M$  is a two-dimensional curve in plane that divides  $\mathbb{R}^2$  into two connected regions:

$$A^+ = \{(q, u) \in \mathbb{R}^2 : u > \alpha(q)\}$$

and

$$A^- = \{(q, u) \in \mathbb{R}^2 : u < \alpha(q)\}.$$

But  $g_u(q, u) \neq 0$  on  $M$ , based on implicit mapping theorem it follows that  $g(q, u) = 0$  has an isolated root  $u = \alpha(q)$  that is a  $C^1$  function and  $M$  can be parameterized as

$$M = \{(q, \alpha(q)) : q \in R\}.$$

Also, on  $M$  either  $g_u(q, u) > 0$  or  $g_u(q, u) < 0$  (otherwise, at some point on  $M$   $g_u(q, u) = 0$ ). Without loss of generality assume

$$\frac{\partial g(q, u)}{\partial u}|_{u=\alpha(q)} > 0$$

by continuity of  $g_u(q, u)$  there exists a tubular neighborhood of  $M$

$$N_\delta(M) = \{(q, u) \in \mathbb{R}^2 : |u - \alpha(q)| < \delta, \delta > 0\}$$

such that  $g_u(q, u) > 0$  on  $N_\delta(M)$ . Suppose  $|\sigma(s)| < \delta, \forall s$  where  $\sigma$  is a sigmoidal function and  $\delta > 0$ . Then, given

$$u = k(q, p) = \alpha(q) - \sigma(q + p),$$

$(q, k(q, p)) \in N_\delta(M), \forall q, p$  and therefore

$$g_u(q, k(q, p)) > 0$$

for all  $q, p$ . Now, consider the closed loop system (4.90) with control input  $u = k(q, p)$  as the following

$$\begin{cases} \dot{q} = p, \\ \dot{p} = f(p) + g(q, \alpha(q) - \sigma(q + p)), \end{cases} \quad (4.92)$$

we prove that the origin  $(q, p) = (0, 0)$  is globally asymptotically stable for (4.92). Define

$$h(q, p) = -f(p) - g(q, \alpha(q) - \sigma(q + p)),$$

and

$$H(q) = \int_0^q h(s, 0) ds$$

Noting that  $sh(s, 0) > 0, \forall s \neq 0$ ,  $H(q)$  is positive semidefinite with  $H(0) = 0$  and  $H(q) > 0, \forall q \neq 0$ . Consider the following Lyapunov function candidate for (4.92)

$$V(q, p) = H(q) + \frac{1}{2}p^2,$$

we have

$$\dot{V} = p(h(q, 0) - h(q, p)),$$

but

$$h_p(q, p) = -f'(p) + \sigma'(q + p) \cdot g_u(q, k(q, p)) > 0,$$

therefore,  $\dot{V} \leq 0, \forall q, p$ . But in  $\dot{V}(q, p) = 0$  or  $\{(q, p) : p = 0\}$  the only invariant set is the origin  $(q, p) = (0, 0)$ . From LaSalle's invariance principle, it follows that the origin is globally asymptotically stable for (4.92) and the result follows.  $\square$

# Chapter 5

## Applications to Robotics and Aerospace Vehicles

In this chapter, detailed control design procedures are provided for several challenging examples of underactuated mechanical systems with kinetic symmetry. In each case, distinctive comparisons are made between our approach and available control methods (if any) that clarify the advantages of the reduction procedures introduced in chapter 4.

All the examples presented in this section are physically meaningful systems. The examples are chosen either based on the importance of their applications in robotics and aerospace, or due to their challenging nature and significance as benchmark systems in nonlinear control theory. The systems of the former class are namely a flexible-link robot, a multi-link underactuated arm, the VTOL aircraft, and an autonomous helicopter. The latter category include the Acrobot, the Inertia-Wheel Pendulum, the TORA system, the Pendubot, the Cart-Pole system, the Rotating Pendulum, and the Beam-and-Ball system. The aforementioned examples of underactuated systems include both high-order and low-order systems. In this section, the examples appear from low-order systems to high-order systems.

Our control design approach for each example consists of three steps:

- i) Reduction to a cascade nonlinear system with structural properties.
- ii) Control of the low-order nonlinear part of the cascade system.
- iii) Reconstruction of the controller for the overall system using the controller of the nonlinear part obtained in step ii).

For nonlinear systems in strict feedback form, step iii) is known as the standard backstepping procedure. The third step is a rather routine procedure and the main difficulty in control design for an underactuated system is passing through steps i) and ii). These three steps are precisely demonstrated for each example.

Some of our main results include global asymptotic stabilization of the Acrobot, the Cart-Pole system, the Inertia-Wheel Pendulum, a three-link planar pendulum with actuated shape variables, and the VTOL aircraft. In addition, (almost) global

exponential output tracking/stabilization of feasible trajectories of an autonomous helicopter is achieved. The closest available result to this work is global asymptotic output tracking in [31].

Among the aforementioned nonlinear control problems, some of them are known to be open in the literature. As an example, global stabilization of the Acrobot (Figure 5-1) using a smooth static state feedback in analytically explicit form has been an open problem in nonlinear control for near a decade. One of our contributions is to address this problem. More precisely, we present a nonlinear state feedback in closed-form that globally asymptotically stabilizes any arbitrary equilibrium point of the Acrobot including the upright position. The main available result on asymptotic stabilization of the Acrobot to the upright position can be found in [13, 86]. The method in [13] consists of two stages: i) a slow swing-up of the Acrobot using a passivity-based approach that brings it close to the upright position, ii) switching to a balancing linear controller that locally asymptotically stabilizes the pendulum around its upright equilibrium point. The main drawback of this two-stage method is the poor performance of both swing-up and balancing controllers in terms of time (i.e. both controllers are slow). This is due to the fact that a gain-scheduled controllers has to keep the state of the system close to a set of equilibrium points. However, since the vector field of a closed-loop system vanishes at the set of equilibrium points, the state moves slowly close to this set of equilibrium points. This is the fundamental weakness of the gain scheduling method. The nonlinear state feedback that we use does not require a swing-up stage and has a large region of attraction (i.e.  $\mathbb{R}^4$ ), therefore the nonlinear controller does not need to keep the state close to the set of equilibrium points. As a result, from the simulation results, the speed of convergence to the state to zero, is significantly better than a switching-based controller that uses a slow swing-up and balancing. Traditionally, a rather similar two-stage switching-based approach has been applied to control of the Pendubot [92, 90], the Cart-Pole system [94, 28], and the Rotating Pendulum [112, 6]. From now on, we avoid repeating the weaknesses of the passivity-based swing-up methods and the gain-scheduling approach in nonlinear control, for each of these examples throughout this section.

Another contribution is that we introduce aggressive swing-up control design methods for benchmark-type underactuated systems that involve inverted-pendulums. Namely, the Cart-Pole system, and the Rotating Pendulum. Formally, a swing-up problem is equivalent to stabilization of a system to a relative equilibrium point. By aggressive swing-up control, we mean (almost) exponential stabilization of an inverted-pendulum system to an equilibrium manifold corresponding to the upright position of the pendulum.

## 5.1 The Acrobot

The *Acrobot* is a two-link revolute planar robot with one actuator at the elbow, as shown in Figure 5-1. In this section, we provide a detailed control design procedure for global asymptotic stabilization of the Acrobot to any of its arbitrary equilibrium points, particularly, the upright equilibrium point. The inertia matrix for the Acrobot

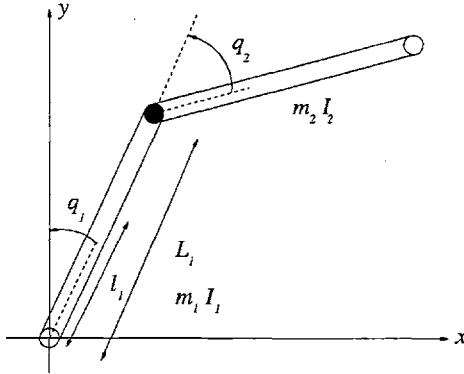


Figure 5-1: The Acrobot

is given by (see section A.3)

$$\begin{aligned} m_{11}(q_2) &= a + b \cos(q_2) \\ m_{12}(q_2) &= c + (b/2) \cos(q_2) \\ m_{22}(q_2) &= c \end{aligned}$$

with

$$\begin{aligned} a &= m_1 l_1^2 + m_2 (L_1^2 + l_2^2) + I_1 + I_2 \\ b &= 2m_2 L_1 l_2 \\ c &= m_2 l_2^2 + I_2 \end{aligned}$$

where  $q_i$ ,  $m_i$ ,  $L_i$ ,  $l_i$ , and  $I_i$  denote the angle, the mass, the length of the center of mass, and the inertia of the  $i$ th link, respectively. Clearly,  $q_2$  is the shape variable of the Acrobot. Since this shape variable is actuated the Acrobot is a Class-I underactuated system. Also, the potential energy for the Acrobot is as the following

$$V(q_1, q_2) = (m_1 l_1 + m_2 L_1) g_0 \cos(q_1) + m_2 l_2 g_0 \cos(q_1 + q_2)$$

which depends on both  $q_1, q_2$ . Thus, the Acrobot does not possess symmetry in the classical sense, but it possesses kinetic symmetry with respect to  $q_1$ . This shows that kinetic symmetry is a less restrictive property than classical symmetry as the invariance of the Lagrangian under symmetry group actions.

Based on theorem 4.2.1 (or proposition 3.9.1), the following global change of coordinates

$$\begin{aligned} q_r &= q_1 + \gamma(q_2) \\ p_r &= m_{11}(q_2)p_1 + m_{12}(q_2)p_2 \end{aligned}$$

transforms the dynamics of the Acrobot to a cascade nonlinear system in strict feed-

back form

$$\begin{aligned}\dot{q}_r &= p_r/m_{11}(q_2) \\ \dot{p}_r &= g_r(q_r, q_2) \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\tag{5.1}$$

where  $g_r(q_r, q_2) = -\nabla_{q_1} V(q_1, q_2)|_{q_1=q_r-\gamma(q_2)}$  is given by

$$g_r(q_r, q_2) = (m_1 l_1 + m_2 L_1) g_0 \sin(q_r - \gamma(q_2)) + m_2 l_2 g_0 \sin(q_r - \gamma(q_2) + q_2) \tag{5.2}$$

and the function  $\gamma(q_2)$  can be explicitly calculated as the following

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{12}(s)}{m_{11}(s)} ds = \frac{q_2}{2} + \frac{(2c-a)}{\sqrt{a^2-b^2}} \arctan\left(\sqrt{\frac{a-b}{a+b}} \tan\left(\frac{q_2}{2}\right)\right) \tag{5.3}$$

for  $q_2 \in [-\pi, \pi]$ . The function  $\gamma(q_2)$  is plotted in Figure 5-2. Observe that  $\gamma(q_2)$  is a nonlinear function for  $(2c-a) \neq 0$ . The special case of  $2c-a=0$  that gives a linear change of coordinates had been previously known. Stabilization of the Acrobot

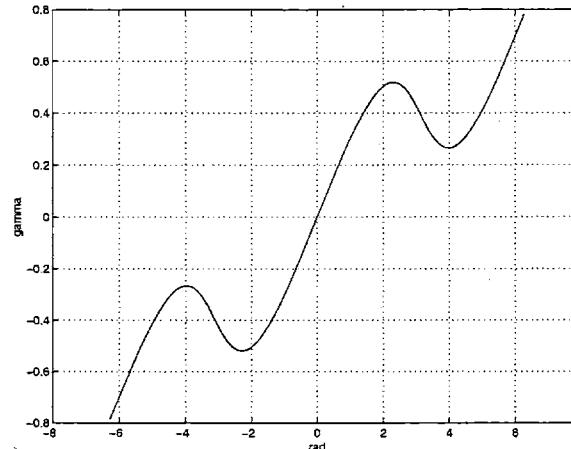


Figure 5-2: The function  $\gamma(q_2)$  for  $a = 20/3, b = 4, c = 4/3$ .

reduces to stabilization of the following second-order nonlinear system non-affine in control

$$\begin{aligned}\dot{q}_r &= m_{11}^{-1}(v)p_r \\ \dot{p}_r &= g_r(q_r, v)\end{aligned}\tag{5.4}$$

with control input  $v = q_2$ . This stabilization problem is extremely complicated even for a second-order system due to the highly nonlinear way that the control  $v$  appears in the dynamics of (5.4). Here is our main result for the Acrobot:

**Proposition 5.1.1.** *Assume the parameters of the Acrobot satisfy either of the following conditions*

i)  $A < B$ .

ii)  $B < \rho A$ ,  $a - 2c > 0$ , and  $c - b/2 \geq 0$ .

where

$$\begin{aligned} A &= m_1 l_1 + m_2 L_1 \\ B &= m_2 l_2 \\ \rho &= \frac{\psi(\pi)}{1 - \psi(\pi)} < \frac{1}{2} \\ \psi(q_2) &= \frac{m_{12}(q_2)}{m_{11}(q_2)} \end{aligned}$$

Then, there exists a state feedback in the form

$$v = K_r(q_r, p_r) = \alpha(q_r) - \sigma(q_r + p_r)$$

that globally asymptotically stabilizes the origin for the reduced nonlinear system of the Acrobot in (5.4) where  $\alpha(q_r)$  is a smooth function that satisfies

$$g_r(q_r, \alpha(q_r)) = 0$$

and  $\sigma(\cdot)$  is a scalar sigmoidal function. In addition, there exists a state feedback in explicit form

$$u = K(q_r, p_r, q_s, p_s)$$

that globally asymptotically stabilizes the origin  $(q_r, p_r, q_s, p_s) = 0$  for the composite dynamics of the Acrobot in (5.1).

**Proof.** Based on theorems 4.7.2 and 4.7.3, if  $g_r(q_r, v) = 0$  has an isolated root  $v = \alpha(q_r)$  that satisfies

$$\det \left( \frac{\partial g_r(q_r, v)}{\partial v} \right) \Big|_{v=\alpha(q_r)} \neq 0$$

then, there exists a static state feedback in the explicit form

$$v = K_r(q_r, p_r) = \alpha(q_r) \pm \sigma(q_r + p_r)$$

which globally asymptotically stabilizes any equilibrium point  $(\bar{q}_r, 0)$  of the core reduced system (5.4) (the sign  $\pm$  in state feedback  $v = K_r$  depends on whether  $\nabla_v g_r(q_r, \alpha(q_r)) < 0$ , or  $> 0$ , respectively). In order to find the conditions under which the state feedback  $K_r$  exists, it suffices to determine a set of sufficient conditions that

guarantees the following equations have no solutions in  $v$

$$\begin{aligned} g_r(q_r, v) &= 0 \\ \nabla_v g_r(q_r, v) &= 0 \end{aligned} \quad (5.5)$$

This set of nonlinear equations can be rewritten as

$$\begin{aligned} A \sin(q_r - \gamma(v)) + B \sin(q_r - \gamma(v) + v) &= 0 \\ A \frac{m_{12}(v)}{m_{11}(v)} \cos(q_r - \gamma(v)) + B \left( \frac{m_{12}(v)}{m_{11}(v)} - 1 \right) \cos(q_r - \gamma(v) + v) &= 0 \end{aligned} \quad (5.6)$$

where

$$A = (m_1 l_1 + m_2 L_1), \quad B = m_2 l_2$$

Setting  $q_1 = q_r - \gamma(v)$ , from the property  $\sin^2(q_1 + v) + \cos^2(q_1 + v) = 1$ , it follows that

$$\left( \frac{A \sin(q_1)}{B} \right)^2 + \left( \frac{Am_{12} \cos(q_1)}{B(m_{11} - m_{12})} \right)^2 = 1$$

Keep in mind that  $a > b$  and  $c > b/2$  imply  $m_{11}(q_2), m_{12}(q_2) > 0$  for all  $q_2$ . Solving for  $\sin^2(q_1)$  from the last equation, we obtain

$$\sin^2(q_1) = \frac{B^2(m_{11} - m_{12})^2 - A^2 m_{12}^2}{A^2(m_{11} - m_{12})^2 - A^2 m_{12}^2} \quad (5.7)$$

Thus, if  $A < B$  the right hand side of (5.7) is greater than 1 and the left hand side of (5.7) is less than or equal to 1. This means the equations in (5.5) have no solutions in  $v$ . On the other hand, (5.7) can be rewritten as

$$\sin^2(q_1) = \frac{[B(m_{11} - m_{12}) - Am_{12}][B(m_{11} - m_{12}) + Am_{12}]}{A^2 m_{11} (m_{11} - 2m_{12})}$$

By assumption,  $m_{12}(q_2) \geq c - b/2 \geq 0$ . Assuming  $A \geq B$ , we get

$$B(m_{11} - m_{12}) + Am_{12} = (A - B)m_{12} + Bm_{11} > 0$$

Also,  $m_{11} - 2m_{12} = a - 2c > 0$ . Thus, by taking

$$B(m_{11} - m_{12}) - Am_{12} < 0$$

the right hand side of the last equation is negative, while the left hand side is positive. In other words, the equations in (5.5) have no real solutions in  $v$ . The last condition can be expressed as

$$\frac{B}{A + B} < \frac{m_{12}(v)}{m_{11}(v)} =: \psi(v) \quad (5.8)$$

It is elementary to show that  $\psi(v)$  takes its minimum at  $v = \pm\pi$  which is equal to

$$\psi(\pi) = \frac{c - b/2}{a - b}$$

Note that  $\psi(q_2) \leq m_{12}/(a - 2c + 2m_{12}) < 1/2$  for all  $q_2$ . Therefore, if  $B < \psi(\pi)(A+B)$ , or

$$B < \frac{\psi(\pi)}{1 - \psi(\pi)} A$$

where  $\rho := \frac{\psi(\pi)}{1 - \psi(\pi)} < 1$ , equation (5.7) has no real solution in  $q_1$  (and thus in  $v$ ). This finishes the proof of the first part. The existence and closed-form expression for the state feedback

$$u = K(q_r, p_r, q_s, p_s)$$

follows from the backstepping procedure.  $\square$

*Remark 5.1.1.* Assume both links of the Acrobot have the same uniform mass density. If  $m_1 = m_2$  and  $l_2 > 3l_1$ , then  $A < B$  and condition i) in proposition 5.1.1 is satisfied. Also, if  $l_2 > 1.5l_1$ , then  $c - b/2 > 0$ . Let us take  $m_1 = \lambda m_2$  and  $l_2 = 2l_1$ . Then,  $a - 2c > 0$ , and  $\psi(\pi) = 5/2(\lambda + 1)$ . This gives  $\rho = 5/(2\lambda - 3)$  and  $A, B$  can be calculated as the following

$$A = m_1 l_1 + m_2 L_1 = (\lambda + 2)m_2 l_1, \quad B = 2m_2 l_1$$

Therefore, the condition  $B < \rho(\lambda)A$  is satisfied for all  $\lambda > \lambda^* = 1.5$  and condition ii) of proposition 5.1.1 holds.

*Remark 5.1.2.* To find the isolated root  $v = \alpha(q_r)$  of  $g_r(q_r, v) = 0$ , let us differentiate the last equation with respect to parameter  $s \in \mathbb{R}$ . We get

$$\nabla_{q_r} g_r(q_r, v) \cdot \frac{dq_r}{ds} + \nabla_v g_r(q_r, v) \cdot \frac{dv}{ds} = 0$$

or

$$\begin{aligned} \frac{dq_r}{ds} &= 1 \\ \frac{dv}{ds} &= -\frac{\nabla_{q_r} g_r(q_r, v)}{\nabla_v g_r(q_r, v)} \end{aligned}$$

The solution of this ODE with zero initial conditions both positive and negative in time gives a parameterization  $(q_r, \alpha(q_r))$  of the isolated root of  $g_r(q_r, v) = 0$ .

For the Acrobot, it is possible to solve the equation  $g_r(q_r, v) = 0$  in  $q_r$  explicitly as a function of  $v$ . This gives a numeric look up table of pairs  $(q_r, v)$  as the solution of equation  $g_r(q_r, v) = 0$ . To show this, notice that  $g_r(q_r, v) = 0$  is equivalent to

$$A \sin(q_1) + B \sin(q_1 + v) = 0$$

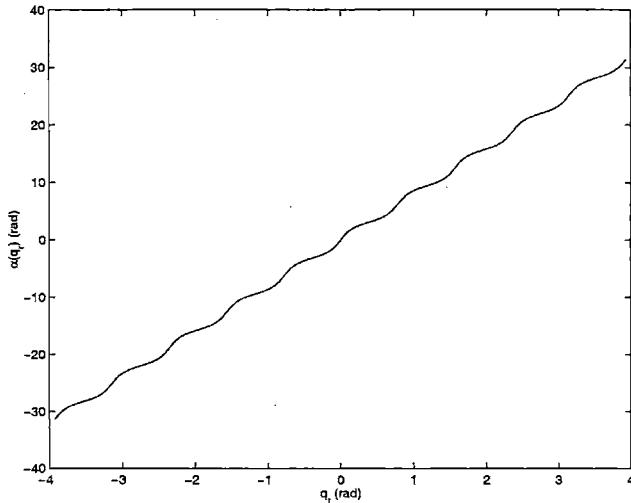


Figure 5-3: The function  $\alpha(q_r)$  for  $m_1 = m_2 = 1$  and  $l_1 = l_2 = 1$ .

or

$$(A + B \cos(v)) \sin(q_1) + B \sin(v) \cos(q_1) = 0$$

Hence, the last equation can be solved in  $q_1$  as

$$q_1 = -\arctan\left(\frac{B \sin(v)}{A + B \cos(v)}\right)$$

but  $q_1 = q_r - \gamma(v)$ , thus

$$q_r = \gamma(v) - \arctan\left(\frac{B \sin(v)}{A + B \cos(v)}\right) =: \phi(v)$$

In other words, solving the equation  $g_r(q_r, v) = 0$  in  $v$  is equivalent to inverting the function  $\phi(v)$  which is available in closed-form. This inversion can be numerically done using a piece-wise linear approximation, splines, neural networks, or other convenient curve fitting approaches. In this work, we use a piece-wise linear approximation method constructed from a look up table of pairs  $(\phi(v), v)$  as the solution of equation  $g_r(q_r, v) = 0$ . Figures 5-4 through 5-9 show simulation results for the Acrobot for the choice of parameters  $m_1 = m_2 = 1$ ,  $l_1 = l_2 = 1$ , and  $I_1 = I_2 = 1/3$  (the mass densities of both links are assumed to be uniform and equal). The simulation results demonstrate effective stabilization of the upright position for the Acrobot from initial conditions with large deviations of  $q_1, q_2$  from zero.

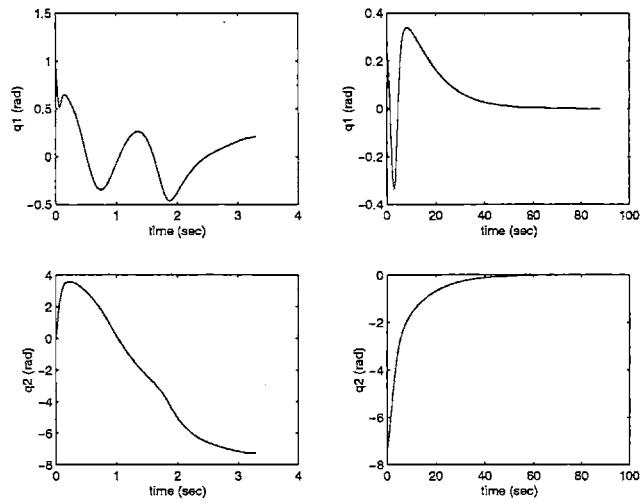


Figure 5-4: Trajectory of  $(q_1, q_2)$  for the Acrobot with initial condition  $(q_1, q_2, p_1) = (\pi/3, 0, 0)$  and control input  $p_2$ .

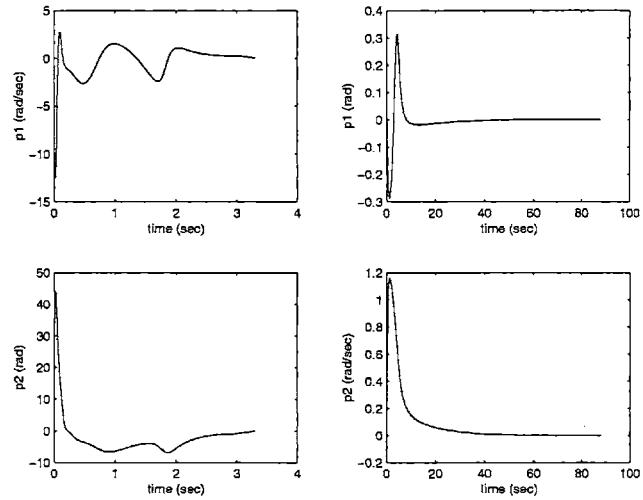


Figure 5-5: Trajectory of  $(p_1, p_2)$  for the Acrobot with initial condition  $(q_1, q_2, p_1) = (\pi/3, 0, 0)$  and control input  $p_2$ .

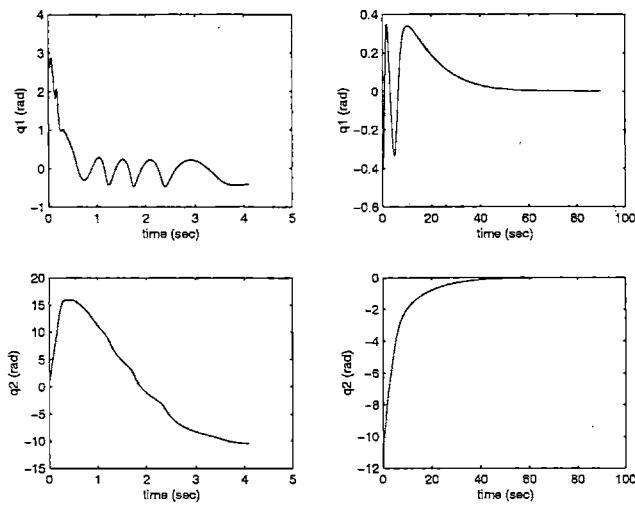


Figure 5-6: Trajectory of  $(q_1, q_2)$  for the Acrobot with initial condition  $(q_1, q_2, p_1) = (\pi, 0, 0)$  and control input  $p_2$ .

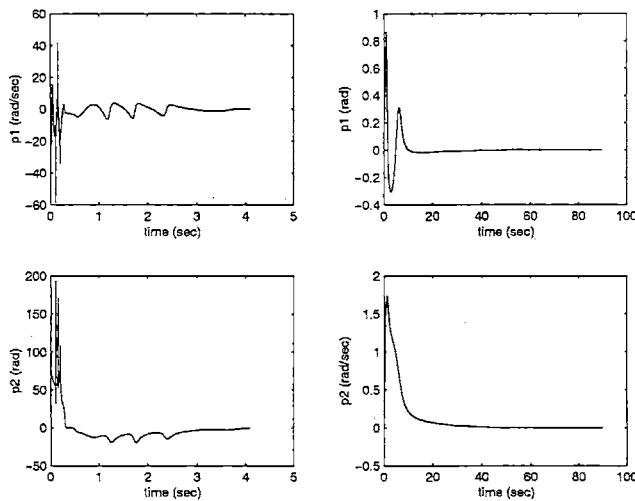


Figure 5-7: Trajectory of  $(p_1, p_2)$  for the Acrobot with initial condition  $(q_1, q_2, p_1) = (\pi, 0, 0)$  and control input  $p_2$ .

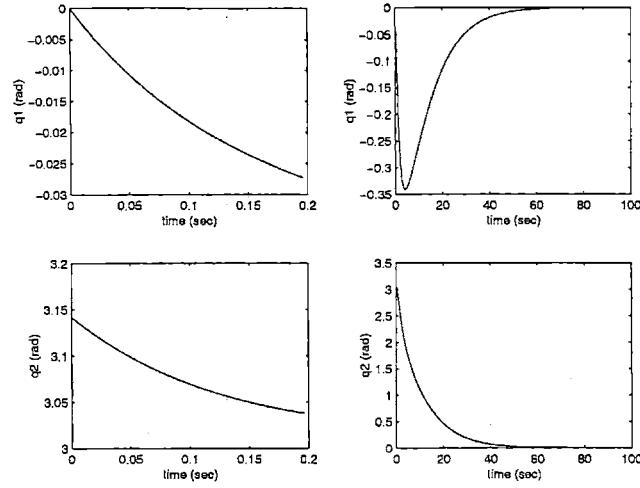


Figure 5-8: Trajectory of  $(q_1, q_2)$  for the Acrobot with initial condition  $(q_1, q_2, p_1) = (0, \pi, 0)$  and control input  $p_2$ .

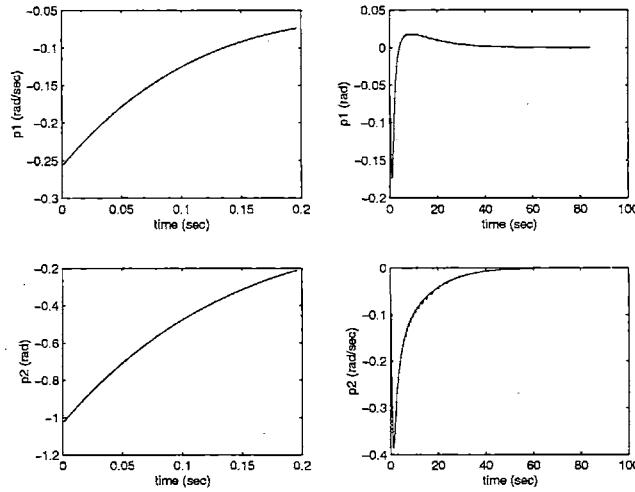


Figure 5-9: Trajectory of  $(p_1, p_2)$  for the Acrobot with initial condition  $(q_1, q_2, p_1) = (0, \pi, 0)$  and control input  $p_2$ .

## 5.2 The TORA System

The TORA<sup>1</sup> system depicted in Figure 5-10 consists of a translational oscillating platform which is controlled via a rotational eccentric mass. The TORA system has been

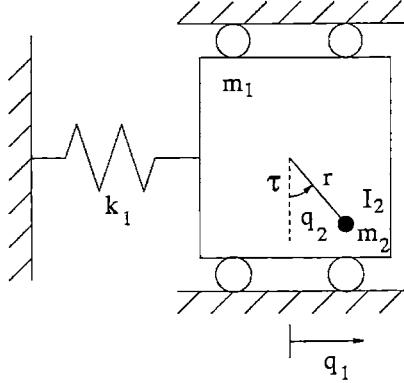


Figure 5-10: The TORA system.

mainly a benchmark example for passivity-based approaches in control of nonlinear systems and stabilization methods for nonlinear cascade systems [109, 37, 80, 38, 39]. Global stabilization of the TORA system using state feedback and backstepping procedure has been known due to Wan *et al.* [109]. More recently, global output tracking for the TORA system was addressed in [39]. Here, we provide a new globally asymptotically stabilizing state feedback for the TORA system with a physically meaningful Lyapunov function. The advantage of using a Lagrangian-based change of coordinates is that the reduced system (i.e. zero-dynamics) has a meaningful Lagrangian structure which simplifies the control design. It turns out that for the TORA system, the reduced system is an undamped one-dimensional mass-spring system.

The Lagrangian of the TORA system is as the following

$$\begin{aligned} \mathcal{L}(q, \dot{q}) = & \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2[(\dot{q}_1 + r \cos(q_2)\dot{q}_2)^2 + (r \sin(q_2)\dot{q}_2)^2] \\ & + \frac{1}{2}(m_2r^2 + I_2)\dot{q}_2^2 - V(q_1, q_2) \end{aligned} \quad (5.9)$$

where the potential energy is equal to

$$V(q_1, q_2) = \frac{1}{2}k_1q_1^2 + m_2gr \cos(q_2)$$

The Lagrangian of the TORA system can be rewritten as

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} m_1 + m_2 & m_2r \cos(q_2) \\ m_2r \cos(q_2) & m_2r^2 + I_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - V(q_1, q_2) \quad (5.10)$$

---

<sup>1</sup>Translational Oscillator with Rotational Actuator

From the equation of the Lagrangian in (5.10), it is clear that  $q_2$  is the shape variable and  $q_1$  is the external variable of the TORA system. Since  $q_2$  is actuated, the TORA system is a Class-I underactuated system that can be globally transformed into a cascade system in strict feedback form. The Euler-Lagrange equations of motion for the TORA system are given by

$$\begin{aligned}(m_1 + m_2)\ddot{q}_1 + m_2r \cos(q_2)\ddot{q}_2 - m_2r \sin(q_2)\dot{q}_2^2 + k_1 q_1 &= 0 \\ m_2r \cos(q_2)\ddot{q}_1 + (m_2r^2 + I_2)\ddot{q}_2 + m_2gr \sin(q_2) &= \tau\end{aligned}\quad (5.11)$$

After collocated partial feedback linearization using a change of control

$$\tau = \alpha(q_2)u + \beta(q, \dot{q})$$

with

$$\alpha(q_2) = (m_2r_2^2 + I_2) - \frac{(m_2r \cos(q_2))^2}{(m_1 + m_2)} > 0, \quad \forall q_2 \in [-\pi, \pi]$$

the dynamics of the shape variable  $q_2$  reduces to

$$\dot{q}_2 = p_2, \quad p_2 = u$$

The normalized momentum conjugate to  $q_1$  is

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = (m_1 + m_2)\dot{q}_1 + m_2r \cos(q_2)\dot{q}_2$$

Observe that both  $p_r$  and the normalized momentum  $\pi_r = m_{11}^{-1}p_r$  are trivially integrable. The function  $\gamma(q_2)$  can be calculated explicitly as

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{12}(s)}{m_{11}} ds = \frac{m_2r \sin(q_2)}{m_1 + m_2}$$

Thus, based on theorem 4.2.1, the global change of coordinates

$$\begin{aligned}q_r &= q_1 + (m_2r \sin(q_2))/(m_1 + m_2) \\ p_r &= (m_1 + m_2)p_1 + m_2r \cos(q_2)p_2\end{aligned}\quad (5.12)$$

transforms the dynamics of the TORA system into a cascade nonlinear system in strict feedback form

$$\begin{aligned}\dot{q}_r &= (m_1 + m_2)^{-1}p_r \\ \dot{p}_r &= -k_1 q_r + k_1 \gamma(q_2) \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\quad (5.13)$$

The reduced system with Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r) = \frac{1}{2}(m_1 + m_2)\dot{q}_r^2 - \frac{1}{2}k_1 q_r^2$$

satisfies the one-dimensional forced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = k_1 \gamma(q_2)$$

with a nonlinear input force  $F_r(q_2) = k_1 \gamma(q_2)$  which depends on the shape variable  $q_2$ . If  $q_2 \equiv 0$ , the reduced system is an oscillator with frequency  $\omega_0 = \sqrt{k_1/(m_1 + m_2)}$  (rad/sec). Using backstepping procedure, stabilization of the overall system reduces to the stabilization of the reduced undamped mass-spring system with reduced Lagrangian  $\mathcal{L}_r(q_r, \dot{q}_r)$ .

**Proposition 5.2.1.** *Consider the TORA system in cascade form (5.13). Then, the following statements hold:*

- i) *There exists a bounded state feedback*

$$q_2 = K_r(q_r, p_r) = -\sigma(c_1 q_r + c_2 p_r), \quad c_1, c_2 > 0 \quad (5.14)$$

*that globally asymptotically stabilizes the origin  $(q_r, p_r) = 0$  for reduced system of the TORA system in (5.13) where  $\sigma$  denotes a sigmoidal function satisfying  $|\sigma(\cdot)| \leq \pi/2$ .*

- ii) *The reduced system has a well-defined new Hamiltonian function  $\tilde{H}_r(q_r, p_r)$  that is decreasing by time, i.e.  $\dot{\tilde{H}}_r \leq 0$ . In addition,  $\tilde{H}_r(q_r, p_r)$  is a valid Lyapunov function for the reduced system. In other words, the closed-loop reduced system with controller  $K_r(q_r, p_r)$  is a passive Hamiltonian system.*
- iii) *There exists a state feedback in explicit form*

$$u = K(q_r, p_r, q_2, p_2)$$

*that globally asymptotically stabilizes the origin for the composite system in (5.13).*

**Proof.** We prove parts i) and ii) simultaneously and then part iii) follows from the first two parts. First, observe that if  $\sigma(x)$  is a sigmoidal function satisfying  $|\sigma(x)| \leq \pi/2$  for all  $x$ , then  $\sin(\sigma(x))$  and thus  $\gamma(\sigma(s))$  is a sigmoidal function as well. Define the *amended potential* function

$$\psi(q_r) = \int_0^{q_r} k_1 \gamma(\sigma(c_1 s)) ds = \int_0^{q_r} \frac{k_1 m_2 r \sin(\sigma(c_1 s))}{m_1 + m_2} ds$$

and notice that  $\psi(q_r)$  is positive definite function. Now, consider the following new Hamiltonian  $\tilde{H}_r$  as a candidate Lyapunov function

$$\tilde{H}_r(q_r, p_r) = H_r(q_r, p_r) + \psi(q_r)$$

where  $H_r$  is the (original) Hamiltonian of the reduced system given by

$$H_r(q_r, p_r) = \frac{1}{2}(m_1 + m_2)^{-1}p_r^2 + \frac{1}{2}k_1q_r^2$$

Since the input force is not identically zero, the Hamiltonian  $H_r$  is not a conserved quantity. Calculating  $\dot{\tilde{H}}_r$  along the solutions of the closed-loop reduced system with controller  $K_r(q_r, p_r)$  gives

$$\begin{aligned}\dot{\tilde{H}}_r &= \dot{H}_r + \dot{q}_r\psi'(q_r) \\ &= (m_1 + m_2)p_r[-\gamma(\sigma(c_1q_r + c_2p_r)) + \gamma(\sigma(c_1q_r))] \\ &= (m_1 + m_2)p_r[\bar{\sigma}(c_1q_r) - \bar{\sigma}(c_1q_r + c_2p_r)] \leq 0\end{aligned}$$

where  $\bar{\sigma}(x) = \gamma(\sigma(x))$  is a sigmoidal function. From the last equation,  $\dot{\tilde{H}}_r < 0$  for all  $p_r \neq 0$  and based on LaSalle's invariance principle, all the solutions of the closed-loop system converge to the largest invariant set  $\Omega$  with  $p_r = 0$ . In this case,  $\Omega = \{(0, 0)\}$  and the origin  $(q_r, p_r) = (0, 0)$  is globally asymptotically stable for the reduced system. Apparently,  $\tilde{H}(q_r, p_r)$  is a valid Hamiltonian for the closed-loop system with configuration  $q_r$ , a new potential energy

$$\tilde{V}(q_r) = \frac{1}{2}k_1q_r^2 + \psi(q_r)$$

and a new nonlinear damping force

$$F_d(q_r, p_r) = \gamma(\sigma(c_1q_r)) - \gamma(\sigma(c_1q_r + c_2p_r))$$

which satisfies  $p_r F_d(q_r, p_r) \leq 0$ . □

*Remark 5.2.1.* The controller obtained in [109] is equivalent to using the following state feedback

$$q_2 = K_r(p_r) = -\sigma(c_2p_r)$$

which is a globally asymptotically stabilizing feedback for the reduced system. This feedback is a special case of the state feedback in part i) of proposition 5.2.1 with  $c_1 = 0$ . In this special case, the amended potential  $\psi(q_r)$  is identically zero.

Figure 5-11 shows the simulation results for the TORA system with the choice of parameters  $m_1 = 10$ ,  $m_2 = 1$ ,  $k_1 = 5$ ,  $r = 1$  and  $I = 1$ . The controller  $K_r$  is in the form

$$K_r(q_r, p_r) = -a \tanh(c_1q_r + c_2p_r)$$

with  $a = \pi/2$  and  $c_1 = c_2 = 1$ . Figure 5-11 (c), clearly shows that the reduced system demonstrates a solution close to the trajectories of position and velocity corresponding to a damped mass-spring system with a linear damping force.

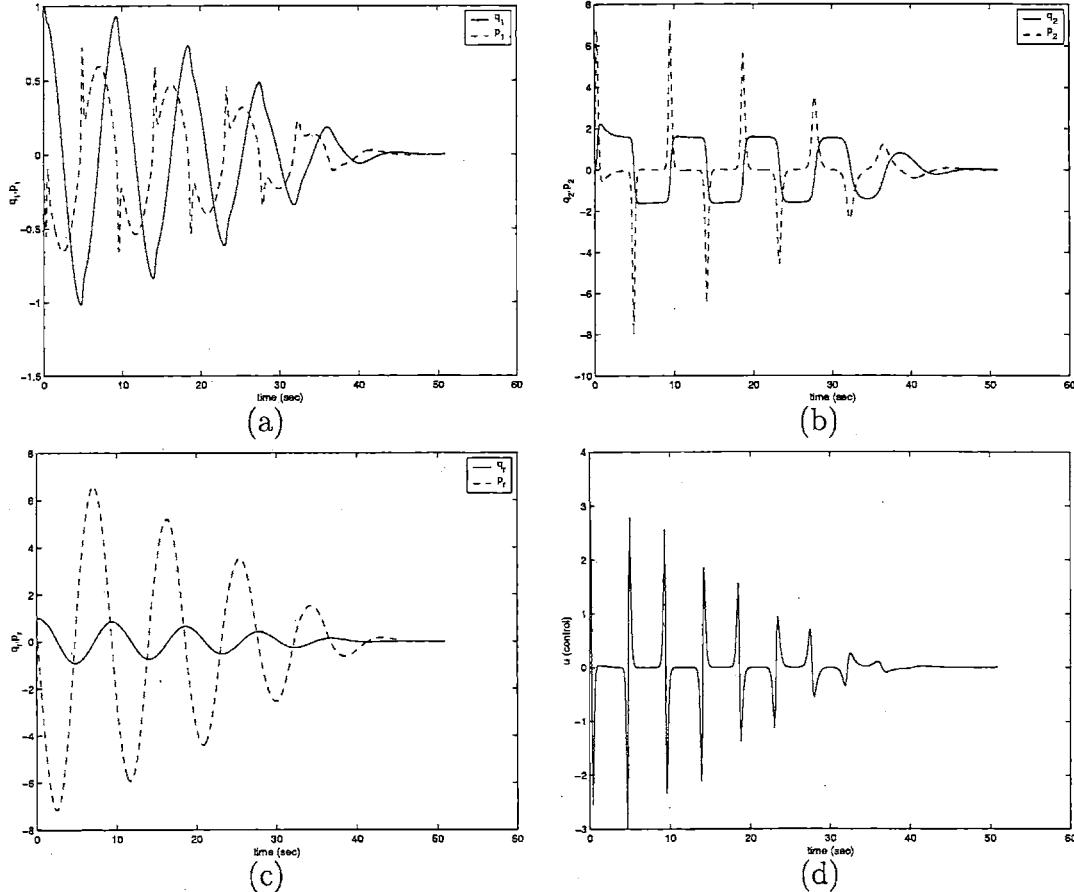


Figure 5-11: State trajectories and control input of the TORA system for the initial condition  $(q_1, q_2, p_1, p_2) = (1, 0, 0, 0)$ : (a)  $(q_1, p_1)$ , (b)  $(q_2, p_2)$ , (c)  $(q_r, p_r)$ , and (d)  $u$ .

### 5.3 The Inertia-Wheel Pendulum

The *Inertia-Wheel Pendulum* is a planar inverted pendulum with a revolving wheel at the end, as shown in Figure 5-12. The wheel is actuated and the joint of the pendulum at the base is unactuated. The inertia-wheel pendulum was first introduced by Spong *et al.* in [93] where a supervisory hybrid/switching control strategy is applied to asymptotic stabilization of the inertia-wheel pendulum around its upright equilibrium point. First, a passivity-based controller [28] swings up the pendulum. Then, a balancing controller that is obtained by Jacobian linearization or (local) exact feedback linearization stabilizes the pendulum around its upright position.

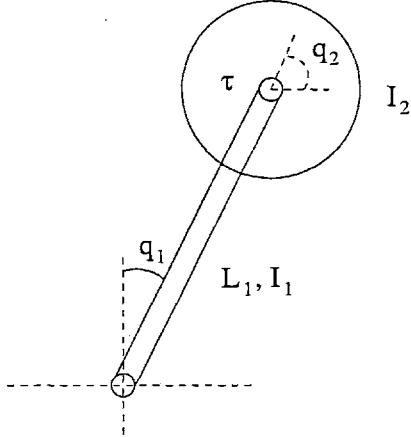


Figure 5-12: The Inertia-Wheel Pendulum

Here, we show that the inertia-wheel pendulum is an example of a Class-I underactuated system. Based on theorem 4.2.1, the dynamics of the inertia wheel pendulum can be transformed into a cascade nonlinear system in strict feedback form; using a global change of coordinates in an explicit form. Then, global asymptotic stabilization of the upright equilibrium point can be achieved using the standard backstepping procedure [36, 57, 80].

The Lagrangian of the inertia wheel pendulum is as the following

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} - V(q_1)$$

where  $q = (q_1, q_2)^T \in \mathbb{R}^2$  denotes the configuration vector of the IWP and  $M$  is a constant inertia matrix with elements

$$\begin{aligned} m_{11} &= m_1 l_1^2 + m_2 L_1^2 + I_1 + I_2 \\ m_{12} &= m_{21} = m_{22} = I_2 \end{aligned}$$

The potential energy of the IWP [93] is given by

$$V(q_1) = (m_1 l_1 + m_2 L_1) g_0 \cos(q_1) =: m_0 \cos(q_1)$$

The Euler-Lagrange equations of motion for the inertia-wheel pendulum are as the following

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right) - \frac{\partial \mathcal{L}}{\partial q_1} &= 0 \\ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) - \frac{\partial \mathcal{L}}{\partial q_2} &= \tau \end{aligned}$$

or

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 &= g_1(q_1) \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 &= \tau \end{aligned} \quad (5.15)$$

where  $g_1(q_1) = m_0 \sin(q_1)$ .

Due to the invariance of the kinetic energy of the inertia-wheel pendulum under the group action  $(q_1, q_2) \mapsto (q_1 + \alpha, q_2 + \beta)$  for  $(\alpha, \beta) \in \mathbb{R}^2$ , the inertia matrix is constant and all the Christoffel Symbols associated with  $M$  vanish. Thus, the inertia-wheel pendulum is a flat underactuated mechanical system with two degrees of freedom and kinetic symmetry w.r.t. both degrees of freedom. As a consequence, the inertia-wheel pendulum is a special case of underactuated systems with symmetry properties considered in [68] and can be globally asymptotically stabilized.

Based on theorem 4.2.1, the global change of coordinates

$$\begin{aligned} z_1 &= \partial \mathcal{L} / \partial \dot{q}_1 = m_{11}\dot{q}_1 + m_{12}\dot{q}_2 \\ z_2 &= q_1 \\ z_3 &= \dot{q}_2 \end{aligned}$$

transforms the dynamics of the inertia-wheel pendulum into a strict feedback system

$$\begin{aligned} \dot{z}_1 &= m_0 \sin(z_2) \\ \dot{z}_2 &= (z_1 - m_{12}z_3) / m_{11} \\ \dot{z}_3 &= u \end{aligned} \quad (5.16)$$

where the new control  $u$  and the torque  $\tau$  are related through the expression

$$\tau = (m_{22} - m_{21}m_{12}/m_{11})u + (m_{21}m_0/m_{11}) \sin(q_1)$$

Note that  $q_2$  does not play any important role in the dynamics of the IWP and is ignored as an state variable (also,  $q_2$  does not appear in  $\mathcal{L}(q, \dot{q})$ ).

*Remark 5.3.1.* Clearly, the nonlinear system in (5.16) is exact feedback linearizable with the choice of the output  $y = z_1$  over  $|z_2| < \pi/2$  (i.e.  $|q_1| < \pi/2$ ). This result has been previously obtained in [93] using a standard method from [36].

Here is our main result for the inertia wheel pendulum:

**Proposition 5.3.1.** *There exists a nonlinear state feedback law (in explicit form) that globally asymptotically stabilizes  $(z_1, z_2, z_3) = 0$  (i.e. the upright equilibrium point) for the Inertia Wheel Pendulum in (5.16).*

**Proof.** Consider the scalar nonlinear system

$$\dot{z}_1 = m_0 \sin(z_2)$$

with control input  $z_2$ . The following state feedback

$$z_2 = k_1(z_1) := c_0 \sigma_1(c_1 z_1); -\pi/2 < c_0 < 0, c_1 > 0$$

(where  $\sigma_1(s) = \tanh(s)$  is a sigmoidal function) globally asymptotically stabilizes  $z_1 = 0$  and  $V(z_1) = z_1^2$  is a valid Lyapunov function for the  $z_1$ -subsystem in (5.16). This is due to the fact that

$$\sigma(z_1) = -m_0 \sin(c_0 \sigma_1(c_1 z_1))$$

is a sigmoidal function that satisfies  $\sigma(0) = 0$ ,  $\sigma'(0) > 0$ , and  $s\sigma(s) > 0, \forall s \neq 0$ . Now, define the following change of coordinates and control

$$\mu_1 = z_2 - k_1(z_1), \mu_2 = \dot{\mu}_1, v = \dot{\mu}_2$$

which can be explicitly expressed as

$$\begin{aligned} \mu_1 &= z_2 - k_1(z_1) \\ \mu_2 &= (z_1 - m_{12}z_3)/m_{11} - \dot{k}_1 \\ v &= (m_0/m_{11}) \sin(z_2) - (m_{12}/m_{11})u - \ddot{k}_1 \end{aligned} \quad (5.17)$$

where by the assumption that  $\sigma_1(s) = \tanh(s)$ , we have

$$\begin{aligned} k_1 &= c_0 c_1 m_0 \sin(z_2)(1 - \sigma_1(c_1 z_1)^2) \\ \ddot{k}_1 &= c_0 c_1 m_0 (1 - \sigma_1(c_1 z_1)^2) (\cos(z_2)(z_1/m_{11} - m_{12}z_3/m_{11}) - 2c_1 m_0 \sigma_1(c_1 z_1) \sin(z_2)^2) \end{aligned}$$

In new coordinates, the dynamics of the inertia wheel pendulum in (5.16) can be written in the following cascade form

$$\begin{aligned} \dot{z}_1 &= m_0 \sin(k_1(z_1) + \mu_1) \\ \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= v \end{aligned} \quad (5.18)$$

The state feedback

$$v = -c_2 \mu_1 - c_3 \mu_2; \quad c_2, c_3 > 0 \quad (5.19)$$

globally exponentially stabilizes  $(\mu_1, \mu_2) = 0$  for the  $\mu$ -subsystem of (5.18). Due to the fact that for any exponentially vanishing function  $\mu_1(t)$ , the solution of the  $z_1$ -subsystem in (5.18) is uniformly bounded (the proof for this boundedness is rather elementary and is omitted), the origin  $(z_1, \mu_1, \mu_2) = 0$  for the closed loop cascade system in (5.18) is globally asymptotically and locally exponentially stable due to a theorem by Sontag [83]. The overall nonlinear state feedback  $u = k(z_1, z_2, z_3)$  can be calculated as the following

$$u = \frac{m_{11}}{m_{12}}(c_2 \mu_1 + c_3 \mu_2 + \frac{m_0}{m_{11}} \sin(z_2) - \ddot{k}_1)$$

which is an explicit function of the state  $(z_1, z_2, z_3)$ .  $\square$

*Remark 5.3.2.* Apparently, using standard backstepping procedure in [36] (chapter 9), a Lyapunov function  $V(z)$  and a globally stabilizing feedback law for (5.16) can

be obtained by explicit use of  $V_1(z_1) = z_1^2$  and propagation of the time derivative of  $V_1$  along the solutions of (5.16). However, this only complicates the calculation of the final stabilizing state feedback law and there is no need to do so in this special case.

Now, we present simulation results for global stabilization of the inertia-wheel pendulum to its upright equilibrium point. For comparison purposes, we used the same numeric values for the parameters of the dynamics of the IWP as in [93]. These parameters are as follows:  $m_{11} = 4.83 \times 10^{-3}$ ,  $m_{12} = m_{21} = m_{22} = 32 \times 10^{-6}$ ,  $\bar{m} = 38.7 \times 10^{-3}$ ,  $g_0 = 9.8$ , and  $m_0 = \bar{m}g_0$ .

The state trajectories and the control input of the inertia-wheel pendulum for the downward and vertical initial positions of the pendulum are shown in Figures 5-13 and 5-14, respectively. These simulation results demonstrate that the nonlinear controller aggressively stabilizes the pendulum to its up-right position. The values of the controller parameters for the simulation are  $c_0 = -\pi/10$ ,  $c_1 = .03$ ,  $c_2 = 16$ ,  $c_3 = 8$  and the maximal applied torque was  $\tau_{max} = 0.6$  Nm.

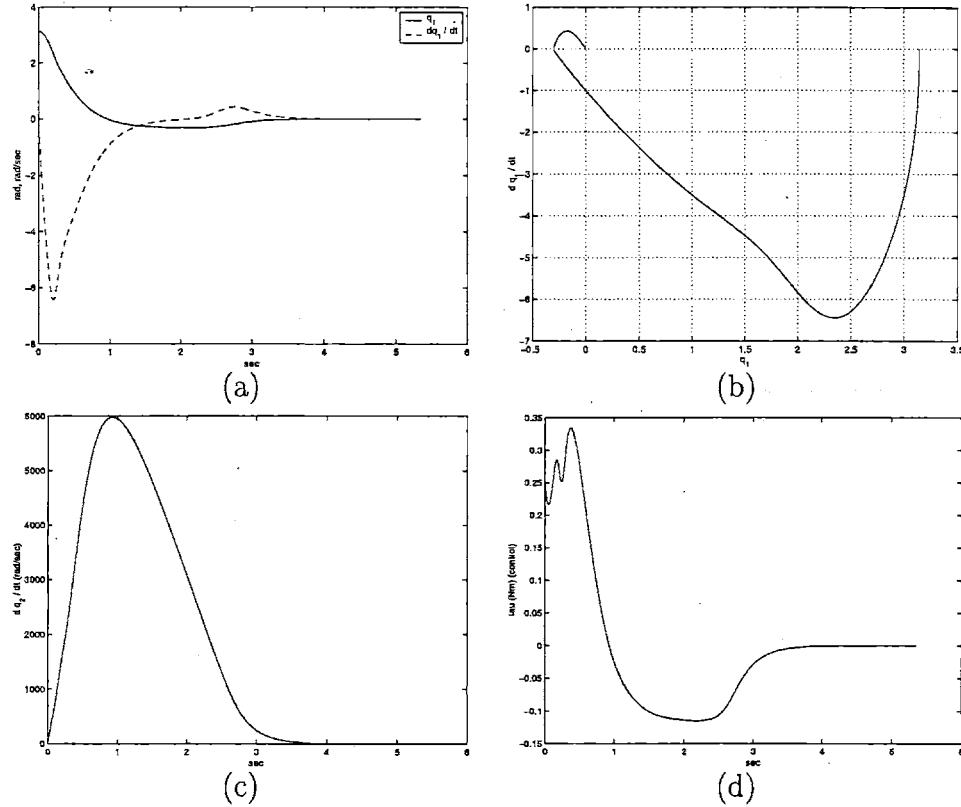


Figure 5-13: For the initial condition  $(\pi, 0, 0)$ , (a) shows the state trajectories for  $(q_1, \dot{q}_1)$ , (b) shows the state trajectory in  $(q_1, \dot{q}_1)$  plane, (c) shows  $\dot{q}_2$ , and (d) shows the control input  $\tau$ .

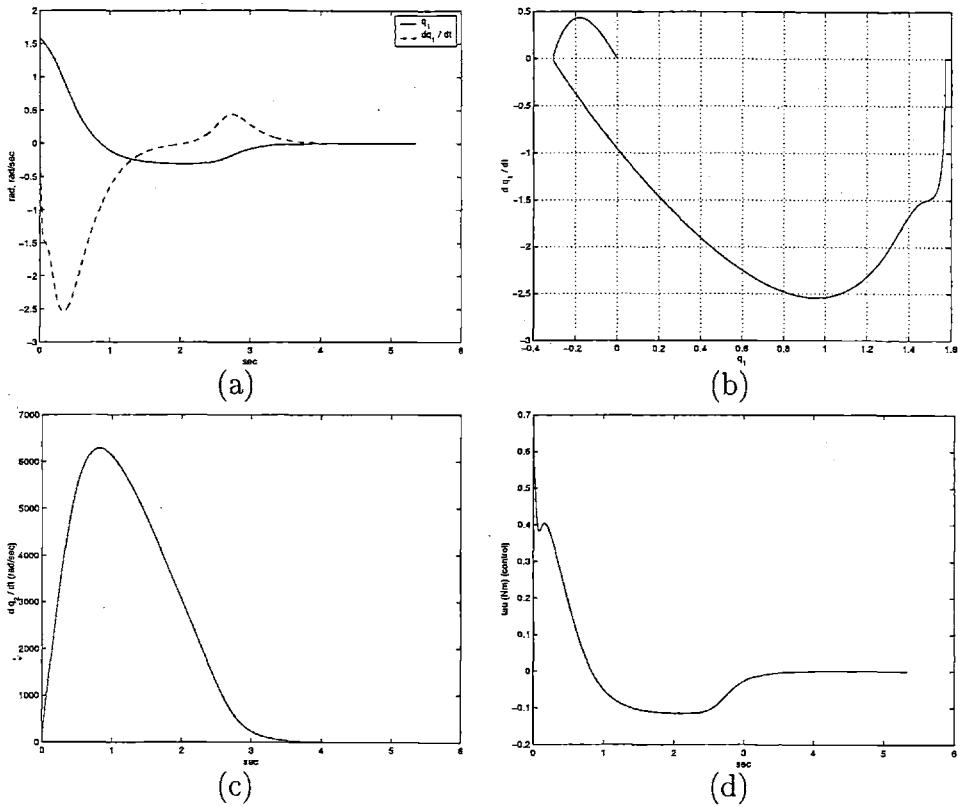


Figure 5-14: For the initial condition  $(\pi/2, 0, 0)$ , (a) shows the state trajectories for  $(q_1, \dot{q}_1)$ , (b) shows the state trajectory in  $(q_1, \dot{q}_1)$  plane, (c) shows  $\dot{q}_2$ , and (d) shows the control input  $\tau$ .

## 5.4 The Cart-Pole System

The *Cart-Pole* system consists of a cart and an inverted-pendulum on it, as shown in Figure 5-15. A horizontal force  $F$  as the control input is applied along  $q_1$  and the joint  $q_2$  is unactuated. The inertia matrix of the cart-pole system is given by (see section A.5)

$$\begin{aligned} m_{11} &= m_1 + m_2 \\ m_{12}(q_2) &= m_{21}(q_2) = m_2 l_2 \cos q_2 \\ m_{22} &= m_2 l_2^2 + I_2 \end{aligned}$$

where  $m_1$  is the mass of the cart and  $m_2, l_2, I_2$  are the mass, length of the center of mass, and the inertia of the pendulum, respectively. Also, the potential energy of the cart-pole system is in the form

$$V(q_2) = m_2 g l_2 \cos(q_2)$$

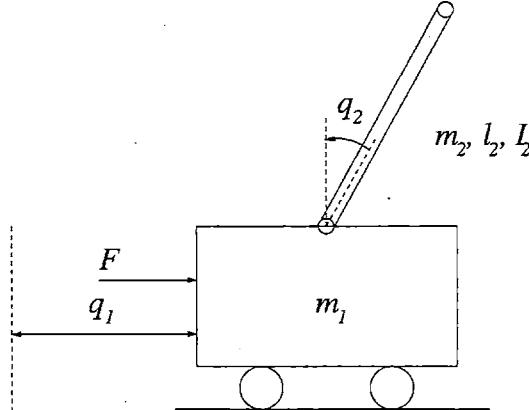


Figure 5-15: The Cart-Pole system.

The Euler-Lagrange equations of motion for the cart-pole system can be expressed as the following

$$\begin{aligned}(m_1 + m_2)\ddot{q}_1 + m_2l_2 \cos(q_2)\ddot{q}_2 - m_2l_2 \sin(q_2)\dot{q}_2^2 &= F \\ m_2l_2 \cos(q_2)\ddot{q}_1 + (m_2l_2^2 + I_2)\ddot{q}_2 - m_2gl_2 \sin(q_2) &= 0\end{aligned}$$

Apparently, the inertia matrix only depends on  $q_2$  and thus  $q_1$  is the external variable and  $q_2$  is the shape variable of the cart-pole system. Since the shape variable  $q_2$  is unactuated, the cart-pole system falls within the category of Class-II underactuated systems.

#### 5.4.1 Global Stabilization to an Equilibrium Point

In this section, we introduce a state feedback that globally asymptotically stabilizes the upright equilibrium point of the cart-pole system in the upper half-plane. First, we need to perform noncollocated feedback linearization on the dynamics of the cart-pole system. Notice that

$$(m_{12}(q_2) - \frac{m_{11}m_{22}}{m_{21}(q_2)})\ddot{q}_2 + (m_1 + m_2)g \tan(q_2) - m_2l_2 \sin(q_2)\dot{q}_2^2 = F$$

and the coefficient of  $\ddot{q}_2$  is negative thus applying the feedback law

$$F = (m_{12}(q_2) - \frac{m_{11}m_{22}}{m_{21}(q_2)})u + (m_1 + m_2)g \tan(q_2) - m_2l_2 \sin(q_2)\dot{q}_2^2 \quad (5.20)$$

reduces the dynamics of  $q_2$  to a double integrator as

$$\dot{q}_2 = p_2, \quad \dot{p}_2 = u$$

where  $u$  is the new control. Based on theorem 4.2.2, the following change of coordinates

$$\begin{aligned} q_r &= q_1 + \gamma(q_2) \\ p_r &= m_{21}(q_2)p_1 + m_{22}p_2 \end{aligned}$$

where

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{22}}{m_{21}(\theta)} d\theta = \frac{(m_2 l_2^2 + I_2)}{m_2 l_2} \log \left( \frac{1 + \tan(q_2/2)}{1 - \tan(q_2/2)} \right), \quad q_2 \in (-\pi/2, \pi/2)$$

transforms the dynamics of the cart-pole system into

$$\begin{aligned} \dot{q}_r &= \frac{1}{m_{21}(q_s)} p_r \\ \dot{p}_r &= g_r(q_s) + \frac{1}{m_{21}(q_s)} \frac{dm_{21}(q_s)}{dq_s} p_r p_s - \frac{m_{22}}{m_{21}(q_s)} \frac{dm_{21}(q_s)}{dq_s} p_s^2 \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned}$$

where  $q_s = q_2$  and  $g_r(q_s) = m_2 g l_2 \sin(q_s)$ . Clearly, the equation of  $\dot{p}_r$  is a quadratic form in  $(p_r, p_s)$  and the reason that the coefficient of  $p_r^2$  is identically zero is that  $m_{11}$  is constant for the cart-pole system. This property guarantees that the cart-pole system is a nonlinear system in strict feedforward form (see [102]). The (strict) feedforward property of the cart-pole system has been used for semiglobal stabilization of the system above the half-plane by Teel [101] as an application of nonlinear small-gain theorem. In [56], Mazenc and Praly constructed a Lyapunov function for the cart-pole system (after extremely lengthy and complicated algebraic calculations) that proves existence of a globally stabilizing state feedback law for the cart-pole system above the half-plane. Here is our main result for the cart-pole system:

**Proposition 5.4.1.** *There exists an state feedback law in the form of nested saturations that globally asymptotically stabilizes the origin  $(q_1, p_1, q_2, p_2) = 0$  for the cart-pole system over the upper half-plane.*

**Proof.** The proof relays on the aforementioned change of coordinates and further simplification of the reduced system and a famous result due to Teel in [101]. The equation of  $\dot{p}_r$  can be simplified as

$$\dot{p}_r = -\tan(q_s)p_r p_s + m_{22} \tan(q_s)p_s^2 + m_2 g l_2 \sin(q_s)$$

Applying the following change of coordinates

$$\begin{aligned} z_1 &= q_r \\ z_2 &= p_r/m_{21}(q_s) \end{aligned}$$

we get

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \tan(q_s)(g + \frac{m_2 l_2^2 + I_2}{m_2 l_2 \cos(q_s)} p_s^2)\end{aligned}$$

To map the upper half-plane to  $\mathbb{R}$ , we use another change of coordinates and control as the following

$$\begin{aligned}\xi_1 &= \tan(q_s) \\ \xi_2 &= (1 + \tan(q_s)^2)p_s \\ v &= (1 + \tan(q_s)^2)u + 2 \tan(q_s)(1 + \tan(q_s)^2)p_s\end{aligned}$$

This transforms the dynamics of the cart-pole system into

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1(k_1 + k_2 \frac{\xi_2^2}{(1 + \xi_1^2)^{\frac{3}{2}}}) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{5.21}$$

where  $k_1, k_2$  are the following positive constants

$$k_1 = g, \quad k_2 = \frac{m_2 l_2^2 + I_2}{m_2 l_2}$$

Observing that  $|\xi_1/(1 + \xi_1^2)^{\frac{3}{2}}| < 1$ , the nonlinear system in (5.21) is in strict feedforward form and satisfies all the conditions of Teel's theorem in [102]. Therefore, the origin  $(q, p) = 0$  for the cart-pole system can be globally asymptotically stabilized using small nested saturations in explicit form. The control law can be obtained as the following. First, based on [102], apply the change of coordinates and control

$$\begin{aligned}\eta_1 &= \frac{1}{k_1} z_1 + \frac{2}{k_1} z_2 + 2\xi_1 + \xi_2 \\ \eta_2 &= \frac{1}{k_1} z_2 + \xi_1 + \xi_2 \\ w &= \xi_1 + \xi_2 + v\end{aligned}$$

Then, the state feedback law

$$w = -\sigma_2(\eta_2 + \sigma_1(\eta_1))$$

or

$$v = -\xi_1 - \xi_2 - \sigma_2(\eta_2 + \sigma_1(\eta_1))$$

globally asymptotically stabilizes the origin for the cart-pole system where  $\sigma_i$ 's are saturation functions with appropriate small thresholds and magnitudes.  $\square$

Figure 5-16 shows the simulation results for the cart-pole system for the initial state  $(q_1, p_1, q_2, p_2) = (5, 0, \pi/3, 0)$ . Clearly, the controller with  $\sigma_i(x) = \text{sat}_1(x)$  stabilizes the inverted pendulum in its upright position at  $q_1 = 0$  after a rather short time. The parameters are chosen as  $m_1 = m_2 = 1, l_2 = 0.75, g = 9.8$  and  $\text{sat}_\epsilon(x) = \epsilon \cdot \text{sgn}(x) \cdot \min\{1, |x/\epsilon|\}$ . However, based on Teel's theorem the saturation functions must have sufficiently small thresholds and magnitudes. To see how the controller designed based on small nested saturations performs, we used  $\sigma_i = \text{sat}_{0.1}(\cdot)$ ,  $i = 1, 2$ . The thresholds and magnitudes of saturation functions are reduced by a factor of ten compared to the previous case, but the same initial condition  $(5, 0, \pi/3, 0)$  is chosen. The results are shown in Figure 5-17. Clearly, the controller has a poor performance in terms of the settling time and maximum peak of the position. The settling time increases by an approximate factor of 10 (which is expected) and the maximum peak of the position has increases by a factor of 4 (which is better than expected). In addition, the convergence rate of the position is almost linear. This reveals a large deterioration of the performance of the controller for nested saturations with relatively small magnitudes.

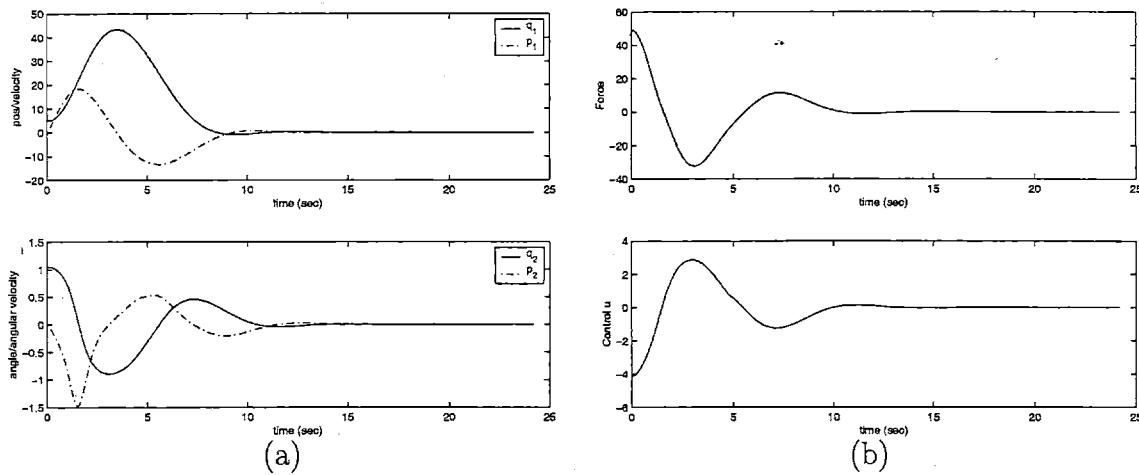


Figure 5-16: (a) State trajectory of the cart-pole system for the initial condition  $(5, 0, \pi/3, 0)$  and  $\sigma_i(\cdot) = \text{sat}_1(\cdot)$ , (b) The force  $F$  and control input  $u$ .

*Remark 5.4.1.* It is needless to mention that the controller obtained here for global asymptotic stabilization of the cart-pole system over the upper half-plane is much simpler than the one given by Mazenc and Praly [56].

In chapter 7, we introduce a different approach to stabilization of cart-pole system that is more general than the one presented here. This new method applies to nonlinear systems that are not necessarily in strict feedforward/feedback forms.

#### 5.4.2 Aggressive Swing-up of the Inverted Pendulum

In this section, we provide a state feedback law for aggressive swing-up of an inverted pendulum on a cart in a single action without swinging the pendulum. In the previous

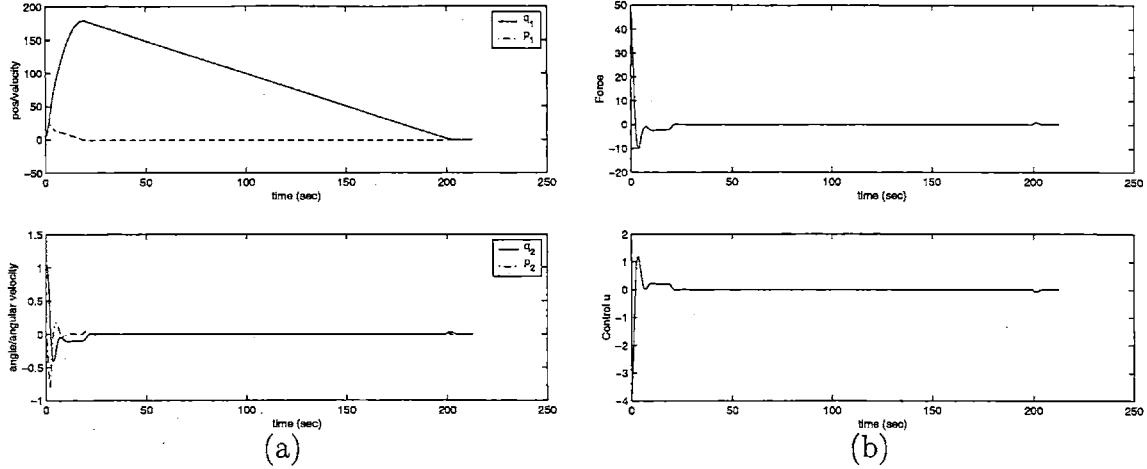


Figure 5-17: (a) State trajectory of the cart-pole system for the initial condition  $(5, 0, \pi/3, 0)$  and  $\sigma_i(\cdot) = \text{sat}_{0.1}(\cdot)$ , (b) The force  $F$  and control input  $u$ .

section, we showed that using noncollocated feedback linearization the dynamics of  $q_2$  can be reduced to a double integrator

$$\dot{q}_2 = p_2, \quad \dot{p}_2 = u$$

a globally exponentially stabilizing (i.e. aggressive) state feedback that asymptotically stabilizes  $(q_2, p_2) = 0$  is given by

$$u = -c_1 q_2 - c_2 p_2$$

where  $c_1, c_2 > 0$  are constants. The main difficulty in stabilizing the angle of the pendulum from initial angle of  $q_2 = \pi$  to final angle of  $q_2 = 0$  is that at some moment of time, the state trajectory of  $q_2$  passes through the point  $q_2 = \pi/2$ . Based on (5.20), this means  $F \rightarrow \infty$  due to the division by  $\cos(q_2)$ . To avoid this singularity, we saturate the control input  $F$  at a maximum level  $F_{\max}$  that is relatively large, i.e.  $F_{\max} \gg 1$ . In other words, we apply the following approximate control

$$F_{app}(q_2, p_2) = F_{\max} \text{sat}(F/F_{\max})$$

where

$$F(q_2, p_2) = \left( \frac{m_{11}m_{22}}{m_{21}(q_2)} - m_{12}(q_2) \right) (c_1 q_2 + c_2 p_2) + (m_1 + m_2)g \tan(q_2) - m_2 l_2 \sin(q_2) p_2^2$$

The difference between  $F$  and  $F_{app}$

$$F - F_{app} = \phi(F)$$

can be expressed in terms of the famous dead-zone nonlinearity  $\phi$  shown in Figure 5-18. The function  $\phi(F)$  is zero over the interval  $[-F_{\max}, F_{\max}]$  and linear with unit slope outside of this interval. The dynamics of the closed loop system with control

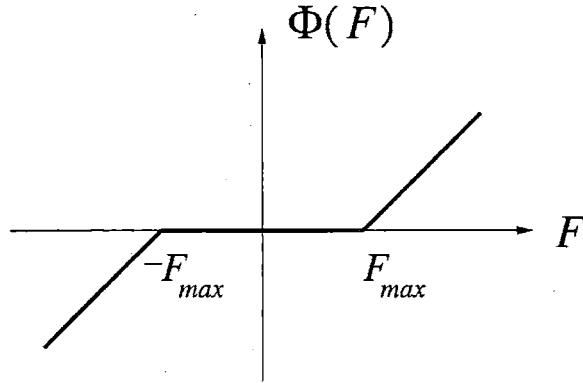


Figure 5-18: The dead-zone nonlinearity.

$F = F_{app}(q_2, p_2)$  can be expressed as

$$\begin{aligned}\dot{q}_2 &= p_2 \\ \dot{p}_2 &= -c_1 q_2 - c_2 p_2 + \frac{m_{21}(q_2)}{D(q_2)} \phi(F(q_2, p_2))\end{aligned}\quad (5.22)$$

where  $D(q_2) = \det(M(q_2)) > 0$ . Apparently, whenever  $|F| < F_{max}$  (i.e. the controller does not saturate)  $\phi(F) = 0$  and  $(q_2, p_2) = (0, 0)$  is exponentially stable as long as the solution passes through an area that the controller saturates, i.e.  $|F_{app}(q_2, p_2)| = F_{max}$ . The nonlinear feedback  $F(q_2, p_2)$  has a discontinuity along the lines  $q_2 = \pm\pi/2$  in the  $(q_2, p_2)$ -plane. This discontinuity property is closely related to the function  $p_2 = \alpha(q_2)$  defined by

$$\alpha(q_2) = -\frac{c_1}{c_2}q_2 - \frac{m_{11}m_{22}gl_2 \sin(q_2)}{c_2 D(q_2)}$$

In fact,  $F(q_2, p_2)$  is continuous at isolated points  $p_2 = \alpha(\pm\pi/2)$  on the lines  $q_2 = \pm\pi/2$ . The following result determines the area in which the controller  $F_{app}(q_2, p_2)$  saturates.

**Proposition 5.4.2.** *Consider the neighborhood*

$$N_\epsilon = \{(q_2, p_2) \in I \times \mathbb{R} \mid |q_2 - \pi/2| \leq \epsilon, |p_2| \leq n_0\pi, n_0 > 0\}$$

for some  $0 < \epsilon \leq \pi/2$  and  $I = [0, \pi]$ . Then, there exist  $F^*(\epsilon, n_0) > 0$  in the form

$$F^*(\epsilon, n_0) = \frac{a_1 + a_2 n_0}{\sin(\epsilon)} + a_3 n_0^2, \quad a_1, a_2, a_3 > 0$$

and a neighborhood  $U_\epsilon \subset N_\epsilon$  of  $q_2 = \pi/2$  such that for all  $F_{max} \geq F^*$ , the controller saturates (i.e.  $|F_{app}(q_2, p_2)| = F_{max}$ ) over  $(q_2, p_2) \in U_\epsilon$  and remains unsaturated over  $I \times [-n_0\pi, n_0\pi] \setminus N_\epsilon$ .

**Proof.**  $F(q_2, p_2)$  can be rewritten as

$$F = \frac{c_2 D(q_2)(p_2 - \alpha(q_2))}{m_2 l_2 \cos(q_2)} - m_2 l_2 \sin(q_2) p_2^2$$

which is discontinuous along the lines  $q_2 = \pm\pi/2$  and  $p_2 \neq \alpha(\pm\pi/2)$ . We have

$$\begin{aligned} |F| &\leq \frac{D_0 |c_1 q_2 + c_2 p_2| + m_{11} m_2 g l_2}{m_2 l_2} \frac{1}{|\cos(q_2)|} + m_2 l_2 p_2^2 \\ &\leq \frac{D_0 (c_1 + c_2 n_0) \pi + (m_1 + m_2) m_2 g l_2}{m_2 l_2} \frac{1}{|\cos(q_2)|} + m_2 l_2 n_0^2 \pi^2 \end{aligned}$$

where  $D_0 > 0$  is a constant satisfying  $0 < D(q_2) \leq D_0 := m_{11} \cdot m_{22}$  and the equality is achieved at  $q_2 = \pi/2$ . Since  $\cos(q_2)$  takes its maximum at the boundary of  $N_\epsilon$  where  $q_2 = \pi/2 \pm \epsilon$ , for all  $(q_2, p_2) \notin N_\epsilon$ ,  $D(q_2) < D_0$  and we get

$$|F| < \frac{D_0 (c_1 + c_2 n_0) \pi + (m_1 + m_2) m_2 g l_2}{m_2 l_2 \sin(\epsilon)} + m_2 l_2 n_0^2 \pi^2$$

Hence, defining

$$F^*(\epsilon, n_0) = \frac{D_0 (c_1 + c_2 n_0) \pi + (m_1 + m_2) m_2 g l_2}{m_2 l_2 \sin(\epsilon)} + m_2 l_2 n_0^2 \pi^2$$

the controller  $F_{app}$  does not saturate outside of the region  $N_\epsilon$ . On the other hand, for  $p_2 \neq \alpha(\pi/2)$ ,  $F(q_2, p_2) \rightarrow \infty$ . By definition, for any  $F_{max} \geq F^*$ , there exists a neighborhood  $U_\epsilon \subset N_\epsilon$  of the line  $q_2 = \pi/2$  where  $F(q_2, p_2) \geq F_{max}$  and therefore the controller saturates in the region  $U_\epsilon$ . Over the region  $N_\epsilon \setminus U_\epsilon$  the controller might or might not saturate.  $N_\epsilon$  is an upper bound (w.r.t.  $\subset$ ) for the maximal region  $U_\epsilon$  that the controller saturates.  $\square$

The values of  $F^*$  for some typical choices of  $\epsilon, n_0$  are given in Table 5.1. Using some standard phase plane arguments, one can show that  $F_{app}$  semiglobally asymptotically stabilizes  $(q_2, p_2) = 0$  for a sufficiently large  $F_{max} \geq F^*(\epsilon, n_0)$  and the solution exponentially converges to the origin over the region  $\{(q_2, p_2) \in \mathbb{R}^2 | q_2 \in [-\pi/2 + \epsilon, \pi/2 - \epsilon]\}$ . Due to  $\dot{q}_2 = p_2$ ,  $q_2$  is strictly decreasing in the region  $q_2 p_2 < 0$  and therefore the solution passes through the saturation region  $N_\epsilon$  once and exponentially converges to the origin under the condition that  $c_1, c_2 > 0$  are sufficiently large. The simulation

Table 5.1: The Values of  $F^*(\epsilon, n_0)$ .

$(n_0, \epsilon)$	$\pi/20$	$\pi/10$	$\pi/5$	$\pi/4$	$\pi/2$
1	454	234	127	107	78
2	637	337	192	164	125

results for this controller are shown in Figure 5-19. The parameters are chosen as  $F_{max} = 100$  and  $c_1 = c_2 = 4, g = 9.8$ . The pendulum starts from the downward position at  $q_2 = \pi$  and the angle of the pendulum (almost) exponentially converges to zero. The convergence is not truly exponential, due to the fact that  $F(q_2, p_2)$  is saturated for a relatively short period of time. A sample path in  $(q_2, p_2)$ -plane for the initial condition starting at  $(q_2, p_2) = (\pi, 0)$  is shown in Figure 5-19 (b).

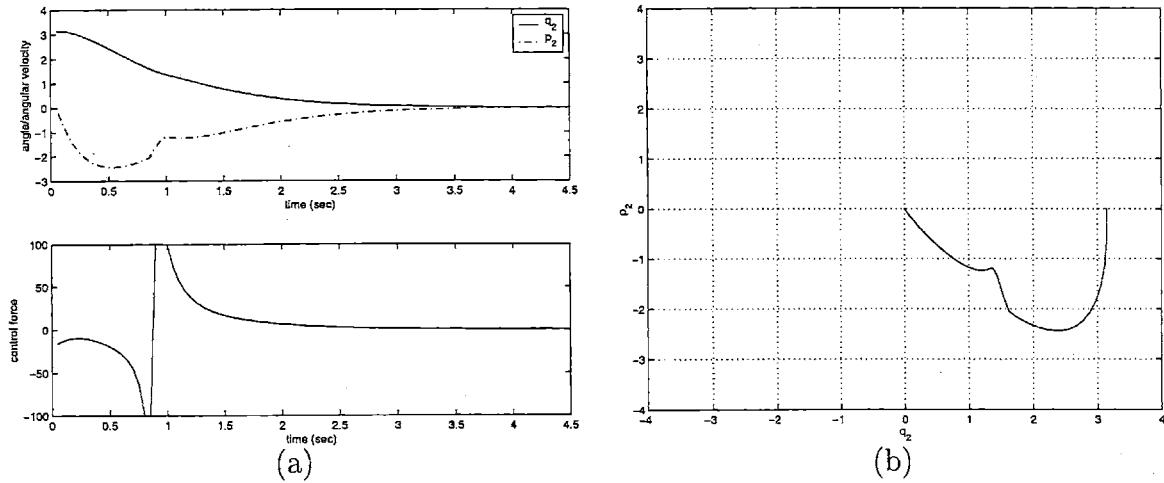


Figure 5-19: (a) Aggressive swing-up of an inverted pendulum on a cart from the downward position  $(q_2, p_2) = (\pi, 0)$ , (b) The path of the pendulum in  $(q_2, p_2)$ -plane.

### 5.4.3 Switching-based Controller

For the inverted initial condition  $(0, 0, \pi, 0)$  (i.e  $q_2 = \pi$ ), one can use the controller  $F_{app}$  to bring the pendulum to the upper half-plane area and switch (say whenever  $|q_2| \leq \pi/3$ ) to the globally stabilizing controller that makes use of nested saturations. This switching-based controller semiglobally asymptotically stabilizes the upright equilibrium point of the cart-pole system over the whole plane with  $q_2 \in [-\pi, \pi]$ . The semiglobal nature of this stabilization is due to bounded initial angular velocity and speed required for the former controller that brings the pendulum to the upper half plane. The simulation results for the overall stabilization of the cart-pole system for the initial condition  $(0, 0, \pi, 0)$  where the pendulum is in the downward position are shown in Figure 5-20. This figure demonstrates that the switching-based controller aggressively stabilizes the origin for the cart-pole system.

## 5.5 The Rotating Pendulum

The *Rotating Pendulum* consists of an inverted pendulum on a rotating arm, as shown in Figure 5-21. The elements of the inertia matrix for the rotating pendulum are given by (see section A.6)

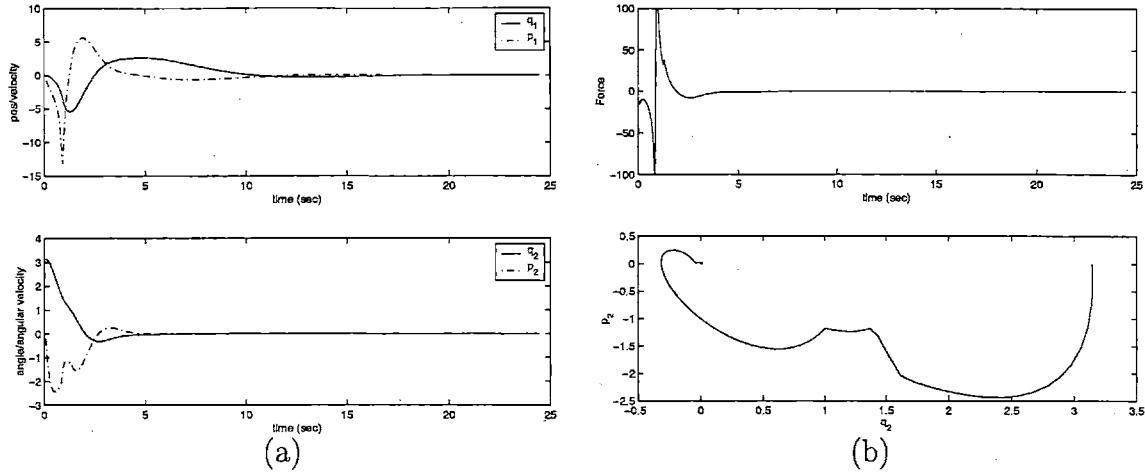


Figure 5-20: (a) State trajectory of the cart-pole system for the initial condition  $(0, 0, \pi, 0)$ , (b) The input force  $F$  and the trajectory in  $(q_2, p_2)$ -plane.

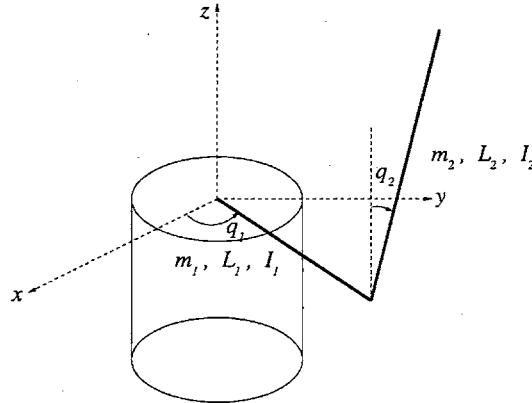


Figure 5-21: The Rotating Pendulum

$$\begin{aligned} m_{11}(q_2) &= I_1 + m_1 l_1^2 + m_2(L_1^2 + l_2^2 \sin^2(q_2)) \\ m_{12}(q_2) &= m_{21}(q_2) = m_2 L_1 l_2 \cos(q_2) \\ m_{22} &= I_2 + m_2 l_2^2 \end{aligned}$$

The potential energy for this system is

$$V(q_1, q_2) = m_2 g l_2 \cos(q_2)$$

Clearly, the inertia matrix of the rotating pendulum only depends on  $q_2$ . Therefore,  $q_x = q_1$  is the external variable and  $q_s = q_2$  is the shape variable of the rotating pendulum. Due to lack of actuation of the shape variable, *the rotating pendulum is a Class-II underactuated mechanical system*. The Euler-Lagrange equations of motion for the rotating pendulum are given by

$$\begin{aligned} m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + 2m_2 l_2^2 \sin(q_2) \cos(q_2) \dot{q}_1 \dot{q}_2 - m_2 l_2 L_1 \sin(q_2) \dot{q}_2^2 &= \tau \\ m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 - m_2 l_2^2 \sin(q_2) \cos(q_2) \dot{q}_1^2 - m_2 g l_2 \sin(q_2) &= 0 \end{aligned}$$

Noting that the second line of this equation can be simplified as

$$\ddot{q}_1 + \frac{m_{22}}{m_{21}(q_2)}\ddot{q}_2 - \frac{l_2}{L_1} \sin(q_2)\dot{q}_1^2 - \frac{g}{L_1} \tan(q_2) = 0$$

and applying the change of control

$$\begin{aligned}\tau &= (m_{12}(q_2) - \frac{m_{11}(q_2)m_{22}}{m_{21}(q_2)})u + 2m_2l_2^2 \sin(q_2) \cos(q_2)\dot{q}_1\dot{q}_2 - m_2l_2L_1 \sin(q_2)\dot{q}_2^2 \\ &+ m_{11}(q_2)\frac{l_2}{L_1} \sin(q_2) + m_{11}(q_2)\frac{g}{L_1} \tan(q_2)\end{aligned}$$

(which is well-defined for  $q_2 \neq \pi/2$ ) reduces the dynamics of the shape variable  $q_2$  to

$$\begin{aligned}\dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}$$

Based on theorem 4.2.2, we apply the following change of coordinates

$$\begin{aligned}q_r &= q_1 + \gamma(q_2) \\ p_r &= m_{21}(q_2)p_1 + m_{22}p_2\end{aligned}$$

or

$$\begin{aligned}z_1 &= q_r \\ z_2 &= p_r/m_{21}(q_2)\end{aligned}$$

where

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{22}}{m_{21}(\theta)} d\theta = \frac{(m_2l_2^2 + I_2)}{m_2l_2L_1} \log\left(\frac{1 + \tan(q_2/2)}{1 - \tan(q_2/2)}\right), \quad q_2 \in (-\pi/2, \pi/2)$$

that transforms the dynamics of the rotating pendulum into

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{g}{L_1} \tan(q_2) + \frac{l_2}{L_1} \sin(q_2)(z_2 - \frac{m_{22}}{m_{21}(q_2)}p_2)^2 + \frac{m_{22}}{m_{21}(q_2)} \tan(q_2)p_2^2 \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\tag{5.23}$$

The nonlinear system in (5.23) is neither in feedforward form (due to the term  $l_2 \sin(q_2)z_2^2/L_1$  in the equation of  $\dot{z}_2$  which depends on  $z_2$ ), nor in strict feedback form. Therefore, direct feedforwarding or backstepping procedures are not applicable to the rotating pendulum. Based on a recent work [10], global/semiglobal asymptotic stabilization of the rotating pendulum to its upright equilibrium point is considered an open problem. In chapter 7, we will address this problem based on a theory that is developed in this thesis for asymptotic stabilization of cascade nonlinear systems in nontriangular normal forms. To remove the constraint  $q_2 \in (-\pi/2, \pi/2)$ , we apply the

following invertible global change of coordinates and control (similar to the cart-pole system)

$$\begin{aligned}\xi_1 &= \tan(q_2) \\ \xi_2 &= (1 + \tan(q_2)^2)p_2 \\ v &= (1 + \tan(q_2)^2)(u + 2 \tan(q_2)p_2)\end{aligned}$$

This transforms the dynamics of the rotating pendulum in (5.23) into

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1\left(\frac{g}{L_1} + \frac{l_2}{L_1}\frac{1}{\sqrt{1+\xi_1^2}}(z_2 - \frac{m_{22}}{m_2 l_2 L_1}\frac{\xi_2}{\sqrt{1+\xi_1^2}})^2 + \frac{m_{22}}{m_2 l_2 L_1}\frac{\xi_2^2}{(1+\xi_1^2)^{\frac{3}{2}}}\right) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\quad (5.24)$$

Notice that in the equation of  $\dot{z}_2$  the quantity in the parentheses is strictly positive definite. This equation can be simplified further as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1(k_1 + k_2\frac{z_2^2}{(1+\xi_1^2)^{\frac{1}{2}}} - k_3\frac{z_2\xi_2}{(1+\xi_1^2)} + k_4\frac{\xi_2^2}{(1+\xi_1^2)^{\frac{3}{2}}}) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\quad (5.25)$$

where

$$k_1 = \frac{g}{L_1}, \quad k_2 = \frac{l_2}{L_1}, \quad k_3 = \frac{2m_{22}}{m_2 L_1^2}, \quad k_4 = \frac{m_{22}}{m_2 l_2 L_1}(1 + \frac{m_{22}}{m_2 L_1^2})$$

are all positive constants. Observe that  $k_i/k_1, i = 2, 3, 4$  all vanish as  $l_2/g \rightarrow 0$  (i.e. in relatively high gravity). We use this property later for semiglobal asymptotic stabilization of the rotating pendulum in its upright position.

### 5.5.1 Aggressive Swing-up of the Rotating Pendulum

In this section, we introduce a state feedback that aggressively swings up an inverted pendulum mounted on a rotating arm in a single action without swinging it back and forth (i.e. a passivity-based swing-up as in [6, 112]). The procedure is very similar to the one applied to the cart-pole system. Thus, we omit the details. Simply, one has to set

$$u = -c_1 q_2 - c_2 p_2, \quad c_1, c_2 > 0$$

which gives the following input torque

$$\begin{aligned}\tau &= \frac{D(q_2)}{m_2 l_2 L_1 \cos(q_2)} (c_1 q_2 + c_2 p_2) + 2m_2 l_2^2 \sin(q_2) \cos(q_2) \dot{q}_1 \dot{q}_2 - m_2 l_2 L_1 \sin(q_2) \dot{q}_2^2 \\ &+ m_{11}(q_2) \frac{l_2}{L_1} \sin(q_2) + m_{11}(q_2) \frac{g}{L_1} \tan(q_2)\end{aligned}$$

where  $D(q_2) = \det(M(q_2))$ . Due to division by  $\cos(q_2)$  in the last equation as  $q_2 \rightarrow \pi/2$ ,  $\tau(q_2, p_2, p_1) \rightarrow \infty$  for almost every  $p_2$ . To avoid this singularity, we apply the saturated torque as the following

$$\tau_{app}(q_2, p_2, p_1) = \tau_{max} \text{sat}(\tau/\tau_{max})$$

where  $\tau_{max} > 0$  is the maximum allowed torque (to be specified later). One obtains the following result which is very similar to proposition 5.4.2 for the cart-pole system.

**Proposition 5.5.1.** Consider a neighborhood  $N_\epsilon = \{(q_2, p_2, p_1) \in I \times \mathbb{R}^2 | |q_2 - \pi/2| \leq \epsilon, |p_2| \leq n_0 \pi, |p_1| < n_1 \pi, n_0 \geq n_1 > 0\}$  for some  $0 < \epsilon \leq \pi/2$  and  $I = [0, \pi]$ . Then, there exist  $\tau^*(\epsilon, n_0, n_1) > 0$  in the form

$$\tau^*(\epsilon, n_0, n_1) = \frac{a_1 + a_2 n_0}{\sin(\epsilon)} + a_3 n_0^2 + a_4 n_0 n_1 + a_5, \quad a_i > 0, i = 1, \dots, 5$$

and a neighborhood  $U_\epsilon \subset N_\epsilon$  of  $q_2 = \pi/2$  such that for all  $\tau_{max} \geq \tau^*$ ,  $\tau_{app}(q_2, p_2, p_1)$  saturates (i.e.  $|\tau_{app}(q_2, p_2)| = \tau_{max}$ ) over  $(q_2, p_2) \in U_\epsilon$  and it remains unsaturated over  $I \times [-n_0 \pi, n_0 \pi] \setminus N_\epsilon$ .

**Proof.** Following the line of proof of proposition 5.4.2, let us define

$$D_0 = D(\pi/2), \quad M_0 = m_{11}(\pi/2)$$

and note that for all  $q_2 \in [-\pi, \pi]$ ,  $D(q_2) \leq D_0$  and  $m_{11}(q_2) \leq M_0$ . For  $0 < \epsilon \leq \pi/2$  and  $q_2 = \pi/2 - \epsilon$ , we have

$$|\tau| < \frac{D_0(c_1 + n_0 c_2)\pi + M_0 m_2 g l_2}{m_2 l_2 L_1 \sin(\epsilon)} + 2m_2 l_2^2 n_0 n_1 \pi^2 + m_2 l_2 L_1 n_0^2 \pi^2 + \frac{M_0 l_2}{L_1}$$

Thus, we define  $\tau^*(\epsilon, n_0)$  as the right hand side of the last equation. In other words, we have

$$a_1 = \frac{D_0 c_1 + M_0 m_2 g l_2}{m_2 l_2 L_1}, \quad a_2 = \frac{D_0 c_2}{m_2 l_2 L_1}, \quad a_3 = m_2 l_2 L_1 \pi^2, \quad a_4 = 2m_2 l_2^2 \pi^2, \quad a_5 = \frac{M_0 l_2}{L_1}$$

□

For the choice of parameters  $m_1 = 1, l_1 = .5, m_2 = 1, l_2 = .75, g = 9.8, n_1 = 1$ , the values of  $\tau^*(\epsilon, n_0, 1)$  are shown in Table 5.2. The simulation results for almost exponential swing-up of the rotating pendulum from the downward position (i.e. initial condition  $(0, 0, \pi, 0)$ ) to  $(q_2, p_2) = (0, 0)$  are shown in Figure 5-22. Apparently,

Table 5.2: The Values of  $\tau^*(\epsilon, n_0, 1)$ .

$(n_0, \epsilon)$	$\pi/20$	$\pi/10$	$\pi/5$	$\pi/4$	$\pi/2$
1	1471	755	406	341	247
2	1885	981	541	459	340

the overall solution  $(q_2(t), p_2(t))$  is not exponentially vanishing but it has a relatively long exponentially vanishing tail—thus the name almost exponential. The maximal torque is chosen to be  $\tau_{max} = 200$ .

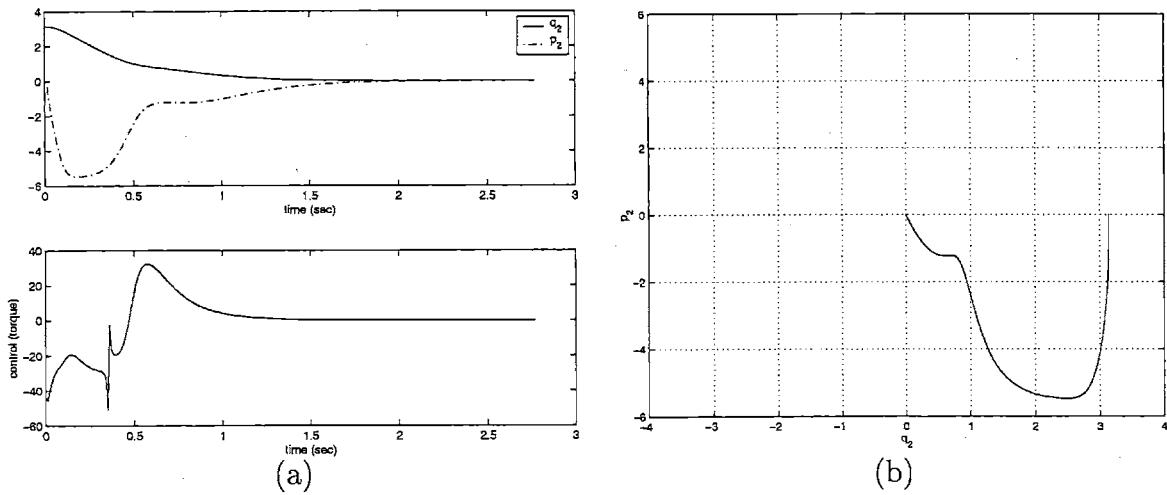


Figure 5-22: (a) Aggressive swing-up of an inverted pendulum on a rotating arm from the downward position  $(q_2, p_2) = (\pi, 0)$ , (b) Trajectory of the pendulum in  $(q_2, p_2)$ -plane.

## 5.6 The Pendubot

The *Pendubot* is a two-link revolute planar robot with one actuator at the shoulder, as shown in Figure 5-23. In this section, we provide an explicit global change of coordinates that transforms the dynamics of the Pendubot into a cascade nonlinear system in nontriangular quadratic normal form. Stabilization of the Pendubot to its upright equilibrium point using a single (i.e. without switching) static state feedback is addressed in chapter 7. The inertia matrix for the pendubot is given by (see section A.3)

$$\begin{aligned} m_{11}(q_2) &= m_1 l_1^2 + m_2(L_1^2 + l_2^2 + 2L_1l_2 \cos(q_2)) + I_1 + I_2 \\ m_{12}(q_2) &= m_{21}(q_2) = m_2(l_2^2 + L_1l_2 \cos(q_2)) + I_2 \\ m_{22}(q_2) &= m_2l_2^2 + I_2 \end{aligned}$$

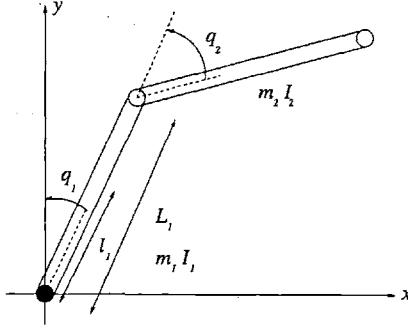


Figure 5-23: The Pendubot

where  $q_i$ ,  $m_i$ ,  $L_i$ ,  $l_i$ , and  $I_i$  denote angles, masses, lengths, lengths of center of masses, and inertia, respectively. For further use, it is simpler to use the following expressions for the inertia matrix

$$\begin{aligned} m_{11}(q_2) &= a + 2b \cos(q_2) \\ m_{12}(q_2) &= m_{21}(q_2) = c + b \cos(q_2) \\ m_{22}(q_2) &= c \end{aligned}$$

where  $a, b, c > 0$  are given by

$$\begin{aligned} a &= m_1 l_1^2 + m_2 (L_1^2 + l_2^2) + I_1 + I_2 \\ b &= m_2 l_2 L_1 \\ c &= m_2 l_2^2 + I_2 \end{aligned}$$

The potential energy of the pendubot is

$$V(q_1, q_2) = (m_1 l_1 + m_2 L_1)g \cos(q_1) + m_2 l_2 g \cos(q_1 + q_2).$$

The inertia matrix of the pendubot only depends on  $q_2$ . Thus,  $q_2$  is the shape variable of the pendubot. Since the shape variable  $q_2$  is unactuated, *the pendubot is a Class-II underactuated system*. The Euler-Lagrange equations of motion for the pendubot are as the following

$$\begin{aligned} m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 - 2b \sin(q_2)\dot{q}_1\dot{q}_2 - b \sin(q_2)\dot{q}_2^2 + g_1(q_1, q_2) &= \tau \\ m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 + b \sin(q_2)\dot{q}_1^2 + g_2(q_1, q_2) &= 0 \end{aligned} \quad (5.26)$$

where the gravity terms are

$$\begin{aligned} g_1(q_1, q_2) &= -(m_1 l_1 + m_2 L_1)g \sin(q_1) - m_2 l_2 g \sin(q_1 + q_2) \\ g_2(q_1, q_2) &= -m_2 l_2 g \sin(q_1 + q_2) \end{aligned}$$

Applying the feedback law

$$\tau = (m_{12} - \frac{m_{11}m_{22}}{m_{21}})u - b \sin(q_2)(\frac{m_{11}}{m_{21}}p_1^2 + 2p_1 p_2 + p_2^2) + g_1(q_1, q_2) - \frac{m_{11}}{m_{21}}g_2(q_1, q_2)$$

reduces the dynamics of  $q_2$  to a double integrator

$$\dot{q}_2 = p_2, \quad \dot{p}_2 = u$$

Applying the change of coordinates

$$\begin{aligned} q_r &= q_1 + \gamma(q_2) \\ p_r &= m_{11}(q_2)p_1 + m_{22}p_2 \end{aligned}$$

where

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{22}}{m_{21}(\theta)} d\theta = \frac{2c}{\sqrt{c^2 - b^2}} \arctan \left( \sqrt{\frac{c-b}{c+b}} \tan \left( \frac{q_2}{2} \right) \right); \quad c > b, \quad q_2 \in [-\pi, \pi]$$

and denoting  $q_s = q_2$ , we get

$$\begin{aligned} \dot{q}_r &= p_r/m_{21}(q_s) \\ \dot{p}_r &= m_2 l_2 g \sin(q_r - \gamma(q_s) + q_s) - b \sin(q_s) \frac{(p_r - m_{22}p_s)(p_r - (m_{11} + m_{22})p_s)}{m_{11}^2} \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned}$$

which apparently the equation of  $\dot{p}_r$  consists of a quadratic term in  $(p_r, p_s)$  and a reduced gravity term

$$g_r(q_r, q_s) = m_2 l_2 g \sin(q_r - \gamma(q_s) + q_s)$$

as in theorem 4.2.2. To further simplify the reduced dynamics of the pendubot, define the change of coordinates

$$\begin{aligned} z_1 &= q_1 + \gamma(q_2) \\ z_2 &= p_r/m_{21}(q_s) \end{aligned}$$

we get

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{m_2 l_2 g \sin(z_1 - \gamma(q_2) + q_2)}{m_{21}(q_2)} - \frac{b \sin(q_2)}{m_{21}(q_2)} [(z_2 - \frac{m_{22}}{m_{21}(q_2)} p_2)^2 + \frac{m_{22}}{m_{21}(q_2)} p_2^2] \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \tag{5.27}$$

Clearly, the dynamics of the pendubot in (5.27) is neither in strict feedback form, nor in strict feedforward form. Therefore, the overall dynamics of the pendubot cannot be stabilized using famous backstepping and forwarding approaches. Stabilization of nonlinear systems in nontriangular forms like Pendubot is addressed in chapter 7.

## 5.7 The Beam-and-Ball System

The beam-and-ball system consists of a beam and a ball with radius  $r$  on it (see Figure 5-24). The task is to bring the ball from any initial position (and speed) to the origin. Let  $d \geq 0$  denote the distance between the center of the mass of the ball and the beam ( $d = r$  in Figure 5-24). In the current literature [34, 102, 80, 79], what is known as the beam-and-ball system corresponds to the special case where  $d = 0$ . We refer to this case as the *conventional beam-and-ball system* and to the case where  $d > 0$  as the beam-and-ball system. The following treatment applies to the both cases of  $d = 0$  and  $d > 0$ . From section A.4, the elements of the inertia matrix of the

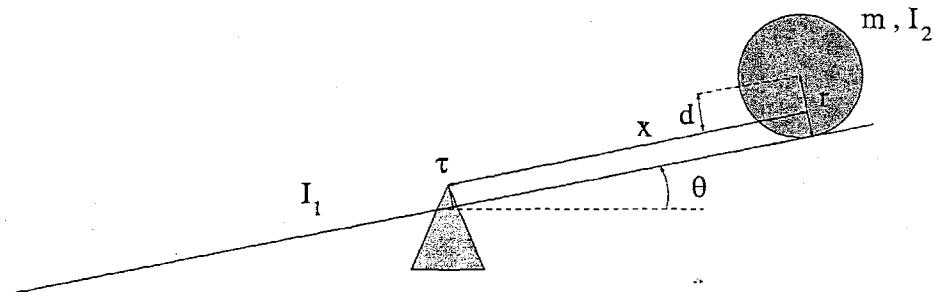


Figure 5-24: The Beam-and-Ball System.

beam-and-ball system are given by

$$\begin{aligned} m_{11}(x) &= I_1 + m(x^2 + d^2) \\ m_{12} &= m_{21} = -md \\ m_{22} &= m\lambda \end{aligned}$$

where  $I_1, m, I_2$  denote the inertia of the beam, the mass of the ball, and the inertia of the ball, respectively. Also,  $\lambda = 1 + I_2/mr^2 > 1$ . The inertia matrix of the beam-and-ball system only depends on the position of the ball  $x$ . This means that *the position of the ball is the shape variable of the beam-and-ball system*. Since the position is unactuated, *the beam-and-ball system is a Class-II underactuated system*. The equations of motion for the beam-and-ball system with  $d > 0$  is as the following

$$\begin{aligned} [I_1 + m(x^2 + d^2)]\ddot{\theta} - md\ddot{x} + 2mx\dot{x}\dot{\theta} + mg(x\cos(\theta) - d\sin(\theta)) &= \tau_d \\ -md\ddot{\theta} + m\lambda\ddot{x} - mx\dot{\theta}^2 + mg\sin(\theta) &= 0 \end{aligned} \quad (5.28)$$

While the dynamics of the conventional beam-and-ball system with  $d = 0$  is in the form

$$\begin{aligned} (I_1 + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} + mgx\cos(\theta) &= \tau_0 \\ m\lambda\ddot{x} - mx\dot{\theta}^2 + mg\sin(\theta) &= 0 \end{aligned} \quad (5.29)$$

Clearly,  $\ddot{\theta}$  does not appear in the second equation and after applying

$$\tau_0 = (I_1 + mx^2)u + 2mx\dot{x}\dot{\theta} + mgx\cos(\theta)$$

after normalizing the units of  $(x, v)$  by  $1/\lambda$ , the dynamics of the conventional beam-and-ball system can be written as

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= -g \sin(\theta) + x\omega^2 \\ \dot{\theta} &= \omega \\ \dot{\omega} &= u\end{aligned}\tag{5.30}$$

The last equation is exactly the same as the model of the beam-and-ball system originally obtained in [34] and used in [102, 80, 79]. In contrast, consider the beam-and-ball system with  $d > 0$  in (5.28). After applying the following global change of coordinates

$$\begin{aligned}z_1 &= -d\theta + \lambda x \\ z_2 &= -d\omega + \lambda v\end{aligned}\tag{5.31}$$

and collocated partially linearizing feedback law

$$\begin{aligned}\tau_d &= (I_1 + m(x^2 + d^2) - \frac{md^2}{\lambda})u - \frac{mdx}{\lambda}\omega^2 + \frac{mdg \sin(\theta)}{\lambda} \\ &\quad + 2mxv\omega + mg(x \cos(\theta) - d \sin(\theta))\end{aligned}\tag{5.32}$$

the dynamics of the beam-and-ball system with  $d > 0$  transforms into

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -g \sin(\theta) + (z_1 + d\theta)\omega^2 \\ \dot{\theta} &= \omega \\ \dot{\omega} &= u\end{aligned}\tag{5.33}$$

where the units of  $(z_1, z_2)$  are normalized by  $1/\lambda$ . Equation (5.33) has many similarities to the normal form of the conventional beam-and-ball system in (5.30).

## 5.8 Control of Multi-link Underactuated Planar Robot Arms

In this section, we discuss reduction of a planar multi-link planar robot arm. Let us start by a planar three-link robot. The inertia matrix of a three-link arm is as the following (see section A.2)

$$M(q_2, q_3) = \begin{bmatrix} m_{11}(q_2, q_3) & m_{12}(q_2, q_3) & m_{13}(q_2, q_3) \\ m_{21}(q_2, q_3) & m_{22}(q_3) & m_{23}(q_3) \\ m_{31}(q_2, q_3) & m_{32}(q_3) & m_{33} \end{bmatrix}$$

Notice that there is an interesting structure in the way the shape variables  $(q_2, q_3)$  appear in the inertia matrix of a three-link robot. The following result shows that such a nested structure in general exists for any  $n$ -link planar robot.

**Theorem 5.8.1.** Consider an  $n$ -link planar robot with revolute joints  $q_1, \dots, q_n$ . Define the following index sets

$$I_k := \{\alpha | \alpha = ij; (k, k) \leq (i, j) \leq (n, n)\}, \quad k = 1, \dots, n$$

$$J_k := I_k \setminus I_{k+1}, \quad k = 1, \dots, n; \quad I_{n+1} = \emptyset$$

Then  $\forall \alpha \in J_k : m_\alpha = m_\alpha(q_{k+1}, \dots, q_n), \forall k = 1, \dots, n-1$  and  $m_\alpha = \text{const.}$  for  $\alpha \in J_n$ .

**Proof.** The proof is by direct calculation using the rotation matrices  $R_i = R(q_i) \in SO(2)$  which represent the orientation of each link in  $\mathbb{R}^2$ . Notice that by definition  $J_k$ 's are disjoint sets.  $\square$

**Remark 5.8.1.** The inertia matrix of an  $n$ -link planar robot arm is independent of  $q_1$ . i.e.  $q_x = q_1$  and  $q_s = (q_2, \dots, q_n)$  are the vector of external variables and the vector of shape variables for a multi-link planar robot arm, respectively.

In examples 4.5.1 and 4.5.2, we showed that there are three possible actuation configurations for a three-link planar robot with two actuators. Furthermore, based on the method of momentum decomposition, we proved that stabilization of a three-link robot with two actuators reduces to stabilization of the Acrobot, or the Pendubot. In the following, we show that this fact can be generalized to a multi-link planar robot underactuated by one.

**Theorem 5.8.2.** There exists a change of coordinates in explicit form that transforms the dynamics of an  $n$ -link planar robot ( $n \geq 2$ ) with  $(n-1)$  actuators into the perturbed reduced normal forms of two-link planar robots (i.e. the Acrobot or the Pendubot) in cascade with  $(n-2)$  double-integrators.

**Proof.** First, assume  $q_1$  is not actuated and  $(n-1)$  of shape variables are actuated. Apparently, it is possible to linearize the dynamics of  $(n-2)$  shape variables using  $(n-2)$  control inputs. Only one actuated shape variable, say  $q_2$ , and  $q_1$  are left. Following the line of momentum decomposition procedure in chapter 4, set all other joint angles to a constant locked configuration (e.g. zero). The inertia matrix of the overall system only depends on  $q_2$  and the reduced form for this system is the perturbed normal form of the Acrobot. Now, assume  $q_1$  is actuated. Then,  $(n-2)$  controls are remained for  $(n-1)$  shape variables. Therefore, the dynamics of all shape variables except for one, say  $q_2$ , can be linearized using collocated partial feedback linearization. Only an unactuated shape variable  $q_2$  and an actuated external variable  $q_1$  are left. Thus, by applying momentum decomposition, one obtains the perturbed normal form of the Pendubot in cascade with  $(n-2)$  double-integrators (resulted from partial feedback linearization).  $\square$

**Remark 5.8.2.** The preceding proof applies to any underactuated (simple) mechanical system with  $n$  degrees of freedom, one external variable  $q_1$ ,  $(n-1)$  shape variables  $(q_2, \dots, q_n)$ , and  $(n-1)$  control inputs. The special structure of the inertia matrix of the  $n$ -link robot here plays no major role except for the fact that  $M$  is independent of  $q_1$ .

## 5.9 Trajectory Tracking for a Flexible One-Link Robot

In this paper, we introduce a method that provides a nonlinear noncollocated output for trajectory tracking of a flexible one-link robot. The link is modeled as a finite-order Lagrangian system obtained from truncated modal analysis. This noncollocated output is derived based on viewing a flexible-link robot arm as an underactuated mechanical system and then applying an appropriate change of coordinates that transforms the system into a cascade nonlinear system with a minimum-phase zero dynamics. The obtained output is the angle of rotation augmented with the (generalized) saturation of the weighted amplitudes of the deflection modes of the flexible link.

Flexible-link robots are finding their way in industrial and space robotics applications due to their lighter weight and faster response time compared to rigid robots. Control of flexible robots has been studied extensively for more than a decade by several researchers [12, 18, 25, 81, 110, 107] (see [24] for a recent survey). Despite their applications, control of flexible link robots has proven to be rather complicated. The simplest possible flexible arm is a robot with a single flexible link. The dynamic model for a flexible-link robot that has been used by almost all of the researchers is the Euler-Bernoulli model of a beam. This model is a fourth order equation that leads to an originally infinite-dimensional model of a flexible link. The common collocated output for trajectory tracking of flexible one-link robots is the angle of the rotation of the link at the base. The performance of this output measurement turns out to be not satisfactory due to a weak control of the vibrations of the link [20]. This initiated finding other noncollocated output measurements like the position of the end-point of the link [18]. Based on [85], this choice of the output leads to a non-minimum phase zero-dynamics. In [20], a noncollocated output is proposed as a linear combination of the angle of rotation and the slope of the beam at its tip. The model used in [20] is an infinite dimensional linear model of a flexible link robot. Then, an approximate finite order compensator is designed based on [107]. Here, we take a fundamentally different approach to find a noncollocated output. We use a finite-order state-space model of a flexible link derived by truncated modal analysis based on a Lagrangian formalism due to De Luca and Siciliano [25]. This model is naturally in the form of an underactuated mechanical system with  $(m + 1)$  degrees of freedom and a single actuator where  $m$  is the number of deformation modes of the flexible link. Based on our previous results on normal forms for underactuated mechanical systems [68, 66] and theorem 4.2.3, we propose a noncollocated output as the angle of the rotation augmented with a generalized saturated weighted linear combination of the deformation amplitudes of the link. We prove that the zero-dynamics corresponding to this output is minimum-phase.

### 5.9.1 Model of a Flexible Link

The flexible link depicted in Figure 5-25 is modeled as an Euler-Bernoulli beam [25] satisfying

$$EI \frac{\partial^4 w(\zeta, t)}{\partial \zeta^4} + \rho A L^4 \frac{\partial^2 w(\zeta, t)}{\partial t^2} = 0$$

where  $\zeta = x/L$  is the normalized position along the link of length  $L$  and  $A, I, E, \rho$  are physical parameters of the link. By the assumption of separability in time and space, the deformation of the beam can be expressed as  $W(\zeta, t) = \phi(\zeta)\delta(t)$ . After

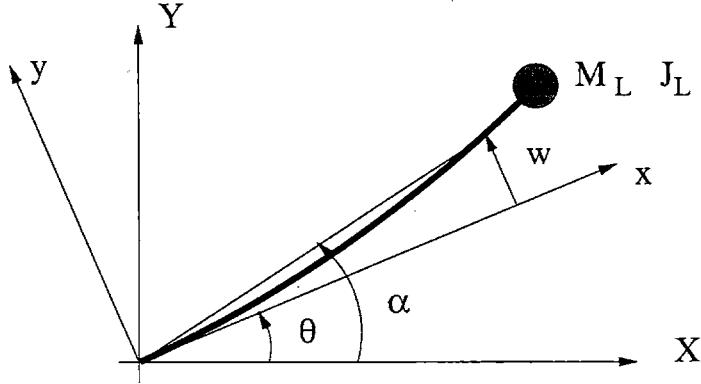


Figure 5-25: A flexible one-link robot arm.

performing a truncated modal analysis with  $m$  modes, the Lagrangian equations of the flexible link can be written as the following [25]

$$\begin{bmatrix} m_{11}(\delta) & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} c_1(\dot{\theta}, \delta, \dot{\delta}) \\ c_2(\dot{\theta}, \delta) \end{bmatrix} + \begin{bmatrix} 0 \\ K\delta + F\dot{\delta} \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \quad (5.34)$$

where  $\theta$  is the angle of rotation,  $\delta \in \mathbb{R}^m$  denotes the deformation amplitudes,  $M(\delta) = \{m_{ij}\}$  is the inertia matrix that is a positive definite symmetric matrix and  $K, F$  are positive definite matrices as well. The expressions for  $m_{ij}$ 's are explicitly given in [25]. Here, we only need an explicit form of  $m_{11}(\delta)$  (the rest of  $m_{ij}$ 's are constant) as

$$m_{11}(\delta) = k_1 + k_2(\Phi_e^T \delta)^2 \quad (5.35)$$

where  $k_1 > 0$ ,  $k_2 = M_L$  i.e. payload mass,  $\Phi_e \in \mathbb{R}^{m \times m}$  is a constant column vector that is determined by the  $m$  eigenfunctions obtained from the modal analysis. In addition, Coriolis and centrifugal terms can be explicitly given as

$$\begin{aligned} c_1(\dot{\theta}, \delta, \dot{\delta}) &= 2M_L \dot{\theta} (\Phi_e^T \delta)(\Phi_e^T \dot{\delta}) \\ c_2(\dot{\theta}, \delta) &= -M_L \dot{\theta}^2 (\Phi_e^T \delta) \Phi_e \end{aligned}$$

Apparently, from (5.34), the flexible one-link robot is an underactuated mechanical system with  $m + 1$  degrees of freedom and one control. The important feature of

(5.34) is that the system has kinetic symmetry w.r.t.  $\theta$ . In the next section, we show that this symmetry property has a crucial role in derivation of a minimum phase noncollocated output for the flexible one-link robot.

To apply theorem 4.2.3 to the case of the flexible one-link robot in (5.34), all the elements of

$$\omega_1 = m_{11}(\delta)^{-1}m_{12}d\delta$$

must be exact one-forms. If this holds, then a nonlinear noncollocated output in the form of  $y = \theta + \gamma(\delta)$  exists that transforms the system into a cascade nonlinear system. However, this is false and the one-form  $m_{11}(\delta)^{-1}m_{12}d\delta$  is not exact due to the fact that  $m_{12} \neq \lambda\Phi_e^T$  for any  $\lambda \in \mathbb{R}$ . This suggests an alternative way of constructing the desired output. By inspection of  $m_{11}(\delta)$  in (5.35), one can observe that the following *modified* one-form

$$\omega_2 = m_{11}(\delta)^{-1}\Phi_e^T d\delta$$

is exact and thus the expression

$$\dot{\theta} + m_{11}(\delta)^{-1}\Phi_e^T \dot{\delta}$$

is integrable as an output  $y = \theta + \gamma(\delta)$ . Next, we show that this is indeed the output we were seeking.

### 5.9.2 The Noncollocated Output

In this section, we derive a noncollocated output and prove that its corresponding zero-dynamics is minimum phase. But first we need to make some assumptions and give a lemma.

**Notation.** For any two vectors  $x, y \in \mathbb{R}^m$  and the positive definite  $m \times m$  matrix  $Q$ , define the following inner product

$$\langle x, y \rangle_Q := x^T Q y$$

**Assumption 5.9.1.** Suppose the following properties hold

- i)  $\Phi_e$  satisfies  $\langle m_{21}, m_{21} \rangle_{m_{22}^{-1}} > \langle \Phi_e, m_{21} \rangle_{m_{22}^{-1}}$ .
- ii) The matrix  $Q_0 = m_{21}m_{12} - m_{21}\Phi_e^T$  is positive definite.

*Remark 5.9.1.* By the specific structure of the inertia matrix  $M(\delta)$  for a flexible link robot in [25], assumptions (i) and (ii) are very reasonable and can be algebraically checked.

**Lemma 5.9.1.** Suppose Assumption 5.9.1 holds. Then the following matrices are positive definite and thus invertible

$$\begin{aligned} Q_1 &= m_{11}(\delta) - \Phi_e^T m_{22}^{-1} m_{21} \\ Q_2 &= m_{22} - m_{21} m_{11}^{-1} \Phi_e^T \end{aligned}$$

**Proof.** First, note that the following matrices are symmetric and positive definite

$$\begin{aligned} Q_3 &= m_{11}(\delta) - m_{12} m_{22}^{-1} m_{21} \\ Q_4 &= m_{22} - m_{21} m_{11}^{-1}(\delta) m_{12} \end{aligned}$$

The proof for this can be found in [90]. This is true regardless of how many deformation modes are chosen as long as  $m_{ij}$ 's are elements of a symmetric positive definite inertia matrix  $M$ . Noting that  $Q_1$  can be rewritten as

$$Q_1 = Q_3 + (m_{12} - \Phi_e^T) m_{22}^{-1} m_{21}$$

by i) in Assumption 5.9.1, the second term is positive and  $Q_3$  is positive as well. Thus,  $Q_1$  is positive definite. To prove  $Q_2$  is positive definite, notice that for any  $x \in \mathbb{R}^m$  we have

$$x^T Q_2 x = x^T Q_4 x + x^T m_{21} m_{11}(\delta)^{-1} (m_{12} - \Phi_e^T) x$$

or

$$x^T Q_2 x = x^T Q_4 x + m_{11}(\delta)^{-1} x^T Q_0 x$$

and because by Assumption 5.9.1  $Q_0$  is positive definite and also  $m_{11}(\delta)^{-1} > 0$ ,  $Q_2$  is a positive definite matrix.  $\square$

Now, we are ready to present our main result for output tracking of a flexible-link robot:

**Theorem 5.9.1.** The nonlinear noncollocated output

$$y = \theta + \gamma(\delta) \quad (5.36)$$

where  $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}$  is given by

$$\gamma(\delta) = \frac{1}{\sqrt{k_1 k_2}} \arctan\left(\sqrt{\frac{k_1}{k_2}} \Phi_e^T \delta\right) \quad (5.37)$$

has global uniform relative degree 3 w.r.t.  $u = \dot{\tau}$  and determines a global change of coordinates

$$z_1 = \delta, z_2 = \dot{\delta}, \xi_1 = y, \xi_2 = \dot{y}, \xi_3 = \ddot{y}$$

that transforms the dynamics of a flexible one-link robot augmented with an integrator

$u = \dot{\tau}$  into the following partially linear cascade nonlinear system

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z, \xi_1, \xi_2, \xi_3) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= v\end{aligned}\tag{5.38}$$

with a new control

$$v = \alpha(\delta)u + \beta(\delta, \dot{\delta}, \dot{\theta})$$

In addition, there exists an  $\epsilon > 0$  with the property  $\epsilon \propto \sqrt{\sigma_{\min}(F)}$  such that over a neighborhood  $U_\epsilon(0) = \{(z_1, z_2) : |z_1| < \epsilon, |z_2| < \epsilon\}$  of  $(z_1, z_2) = 0$ , for the zero-dynamics corresponding to this output the origin is locally uniformly asymptotically stable (i.e. the noncollocated output  $y$  is minimum phase).

**Proof.** By direct calculations, for  $y = \theta + \gamma(\delta)$  in (5.36), we have

$$\begin{aligned}\dot{y} &= \dot{\theta} + \frac{\Phi_e^T \dot{\delta} / k_2}{k_1/k_2 + (\Phi_e^T \delta)^2} \\ &= \dot{\theta} + \frac{\Phi_e^T \dot{\delta}}{k_1 + k_2(\Phi_e^T \delta)^2} \\ &= \dot{\theta} + m_{11}(\delta)^{-1}(\Phi_e^T \dot{\delta})\end{aligned}$$

also

$$\ddot{y} = \ddot{\theta} + m_{11}(\delta)^{-1}\Phi_e^T \ddot{\delta} - 2k_2 m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2$$

but from the second line of equation (5.34),  $\ddot{\delta}$  can be obtained as

$$\ddot{\delta} = -m_{22}^{-1}m_{21}\ddot{\theta} - m_{22}^{-1}(c_2 + K\delta + F\dot{\delta})$$

Thus, combining the last two equations, we get

$$\begin{aligned}\ddot{y} &= m_{11}(\delta)^{-1}(m_{11}(\delta) - \Phi_e^T m_{22}^{-1}m_{21})\ddot{\theta} \\ &- m_{11}(\delta)^{-1}\Phi_e^T m_{22}^{-1}(c_2 + K\delta + F\dot{\delta}) \\ &- 2k_2 m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2\end{aligned}$$

On the other hand, after cancelling  $\ddot{\delta}$  from the first and second lines of (5.34), one obtains

$$(m_{11}(\delta) - m_{12}m_{22}^{-1}m_{21})\ddot{\theta} + c_1 - m_{12}m_{22}^{-1}(c_2 + K\delta + F\dot{\delta}) = \tau$$

Hence, taking

$$\tilde{m}_{11}(\delta) = (m_{11}(\delta) - m_{12}m_{22}^{-1}m_{21}) > 0$$

we have

$$\begin{aligned}\ddot{y} &= m_{11}(\delta)(m_{11}(\delta) - \Phi_e^T m_{22}^{-1} m_{21}) \tilde{m}_{11}(\delta)^{-1} \cdot (\tau - c_1 + m_{12}m_{22}^{-1}(c_2 + K\delta + F\dot{\delta})) \\ &- m_{11}(\delta)^{-1} \Phi_e^T m_{22}^{-1} (c_2 + K\delta + F\dot{\delta}) - 2k_2 m_{11}(\delta)^{-2} (\Phi_e^T \delta) (\Phi_e^T \dot{\delta})^2\end{aligned}$$

This means that applying the change of control

$$w = (m_{11}(\delta) - m_{12}m_{22}^{-1}m_{21}) \cdot (m_{11}(\delta) - \Phi_e^T m_{22}^{-1} m_{21})^{-1} m_{11}(\delta) \tau + \beta(\delta, \dot{\delta}, \dot{\theta})$$

(where  $\beta$  can be explicitly calculated from the last equation of  $\ddot{y}$ ) partially linearizes the dynamics of the system as

$$\ddot{y} = w$$

and therefore the output  $y$  has global relative degree 2 w.r.t.  $\tau$  and global relative degree 3 w.r.t.  $v$ . To determine the stability of the zero-dynamics corresponding to  $y$ , let  $z_1 = \delta$ ,  $z_2 = \dot{\delta}$  and set  $y \equiv 0$  (thus,  $\dot{y}, \ddot{y} \equiv 0$ ). From the second equation in (5.34), the zero-dynamics system can be determined as

$$(m_{22} - m_{21}m_{11}(\delta)^{-1}\Phi_e^T)\ddot{\delta} + 2k_2 m_{21}m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2 + c_1 + K\delta + F\dot{\delta} = 0$$

but  $c_1 = -k_2\dot{\theta}^2(\Phi_e^T \delta)\Phi_e$  and  $\dot{\theta} = -m_{11}(\delta)^{-1}(\Phi_e^T \dot{\delta})$  (because  $\dot{y} = 0$ ), thus

$$c_1 = -k_2 m_{11}(\delta)^{-2} (\Phi_e^T \dot{\delta})^2 (\Phi_e^T \delta) \Phi_e$$

The equations of the zero-dynamics can be expressed as the following

$$(m_{22} - m_{21}m_{11}(\delta)^{-1}\Phi_e^T)\ddot{\delta} + k_2 m_{11}(\delta)^{-2}(\Phi_e^T \delta)(\Phi_e^T \dot{\delta})^2(2m_{21} - \Phi_e) + K\delta + F\dot{\delta} = 0 \quad (5.39)$$

Denoting

$$Q(\delta) = (m_{22} - m_{21}m_{11}(\delta)^{-1}\Phi_e^T)$$

and noting that  $Q(\delta)$  is a positive definite matrix, the following Lyapunov function for the zero-dynamics system can be proposed

$$V(\delta, \dot{\delta}) = \frac{1}{2} \dot{\delta}^T Q(\delta) \dot{\delta} + \frac{1}{2} \delta^T K \delta$$

Calculating  $\dot{V}$  along the solutions of the zero-dynamics in (5.39), we obtain

$$\dot{V} = \dot{\delta}^T Q(\delta) \ddot{\delta} + \dot{\delta}^T K \delta + \frac{1}{2} \dot{\delta}^T Q \dot{\delta}$$

where

$$\dot{Q} = \frac{\partial Q}{\partial \delta} \dot{\delta} = 2k_2 m_{21} m_{11}(\delta)^{-2} (\Phi_e^T \delta) (\Phi_e^T \dot{\delta}) \Phi_e^T$$

and

$$\frac{1}{2} \dot{\delta}^T \dot{Q} \dot{\delta} = k_2 m_{11}(\delta)^{-2} (m_{12} \dot{\delta}) (\Phi_e^T \delta) (\Phi_e^T \dot{\delta})^2$$

Hence

$$\dot{V} = -\dot{\delta}^T F \dot{\delta} - k_2 m_{11}(\delta)^{-2} (\Phi_e^T \dot{\delta})^2 (\Phi_e^T \delta) \cdot ((m_{12} \dot{\delta}) - (\Phi_e^T \dot{\delta}))$$

and because  $m_{11}(\delta)^{-2} \leq k_1^{-2}$ , we get

$$\dot{V} \leq -\sigma_{min}(F) |\dot{\delta}|^2 + \frac{k_2}{k_1^2} |\Phi_e|^3 (|m_{21} - \Phi_e|) |\dot{\delta}|^3 |\delta|$$

where  $\sigma_{min}(F)$  is the smallest singular value of  $F$ . Denoting

$$k_3 = \frac{k_2}{k_1^2} |\Phi_e|^3 |m_{21} - \Phi_e|$$

and defining

$$\epsilon = \sqrt{\frac{\sigma_{min}(F)}{k_3}}$$

for  $|\delta|, |\dot{\delta}| < \epsilon$  (i.e. over  $U_\epsilon(0)$ ),  $\dot{V} \leq 0$ . But the largest invariant set in  $z_2 = \dot{\delta} = 0$  (i.e.  $V = 0$ ) for (5.39) is  $(z_1, z_2) = (0, 0)$ . Therefore, from LaSalle's invariance principle the origin  $(\delta, \dot{\delta}) = (z_1, z_2) = 0$  is uniformly locally asymptotically stable for the zero-dynamics system, i.e. the output  $y$  is (locally) minimum phase.  $\square$

*Remark 5.9.2.* Viewing  $\sigma_{min}(F)$  as the *strength of damping* of the flexible link, the result of the preceding theorem can be interpreted as follows. The higher the strength of damping, the larger the region of attraction of the origin for the zero-dynamics.

*Remark 5.9.3.* For a sufficiently small deformation amplitudes  $\delta$ . The output  $y$  in (5.36) can be expressed as a linear combination of the angle of rotation and weighted deformation amplitudes as

$$y = \theta + \frac{1}{M_L} (\Phi_e^T \delta)$$

which depends on the payload mass  $M_L$ . This is in agreement with our intuition that the payload mass matters in trajectory tracking control design for a flexible arm.

### 5.9.3 Tracking Control

The control for trajectory tracking can be given as the following. Let  $y_d(t)$  denote the desired trajectory and define the error functions

$$e_1 = y - y_d(t), \quad e_2 = \dot{y} - \dot{y}_d, \quad e_3 = \ddot{y} - \ddot{y}_d$$

Then

$$\begin{aligned}\dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= u - y_d^{(3)}\end{aligned}$$

where  $y_d^{(k)} = \frac{d^k y_d(t)}{dt^k}$ . Thus, setting

$$u = y_d^{(3)} + c_p(y - y_d) + c_d(\dot{y} - \dot{y}_d) + c_a(\ddot{y} - \ddot{y}_d)$$

with  $c_p, c_d, c_a < 0$ , or applying the partial state feedback

$$u = y_d^{(3)} + c_p(\xi_1 - y_d) + c_d(\xi_2 - \dot{y}_d) + c_a(\xi_3 - \ddot{y}_d) \quad (5.40)$$

guarantees that  $(e_1, e_2, e_3) \rightarrow 0$  as  $t \rightarrow \infty$  and for sufficiently small initial conditions  $(\delta(0), \dot{\delta}(0))$  asymptotic output tracking can be achieved [36].

### 5.10 Configuration Stabilization for the VTOL Aircraft

In this section, we give a static state feedback law for global configuration stabilization of the VTOL (vertical take off and landing) aircraft. The simplified dynamics of the

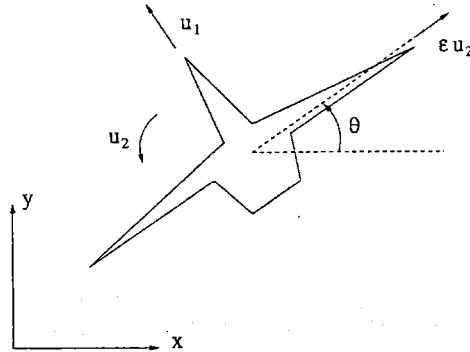


Figure 5-26: The VTOL aircraft.

VTOL aircraft, as shown in Figure 5-26, is given in [35, 55] as the following

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -u_1 \sin(\theta) + \epsilon u_2 \cos(\theta) \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= u_1 \cos(\theta) + \epsilon u_2 \sin(\theta) - g \\ \dot{\theta} &= \omega \\ \dot{\omega} &= u_2\end{aligned}\tag{5.41}$$

where  $\epsilon \neq 0$ . Here,  $\theta$  denotes the roll angle and the plane moves in a vertical  $(x_1, y_1)$ -plane. Clearly, the system has three degrees of freedom and only two actuators—thus it is underactuated. The effect of the body torque  $u_2$  appears in the translational dynamics of the VTOL aircraft by a factor of  $\epsilon$ . This captures a general property of real aircrafts with 6 DOF that exhibit a similar input coupling effect. The case with  $|\epsilon| \ll 1$  resulting in a weak input coupling has been mostly considered in the literature [35, 55, 80]. Here, we are interested in the strong input coupling case with an arbitrary  $\epsilon \neq 0$ . To decouple the effect of  $u_2$  in the translational dynamics of the VTOL aircraft, first let rewrite the dynamics of the VTOL aircraft as

$$\begin{aligned}m_{xx}\ddot{q}_x - g_x(q_x) &= F_r(q_s)\tau_r + F_x(q_s)\tau \\ m_{ss}\ddot{q}_s &= F_s(q_s)\tau\end{aligned}\tag{5.42}$$

where  $q_x = (x_1, y_1)^T$ ,  $q_s = \theta$ ,  $m_{xx} = I_{2 \times 2}$ ,  $m_{ss} = 1$ ,  $F_s = 1$ , and  $F_r, F_x, g_x$  are given by

$$F_r(q_s) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}, \quad F_x(q_s) = \begin{bmatrix} \epsilon \cos(\theta) \\ \epsilon \sin(\theta) \end{bmatrix}, \quad g_x = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

which clearly shows  $F_r(q_s)$  is a unit vector. Based on theorem 4.3.2, we need to check whether the vector of one-forms  $\omega$  defined as  $\omega = m_{xx}^{-1}F_x(q_s)F_s^{-1}(q_s)m_{ss}dq_s$  has exact elements or not. By direct calculation, we get

$$\omega = F_x(q_s)dq_s = \begin{bmatrix} \epsilon \cos(\theta)d\theta \\ \epsilon \sin(\theta)d\theta \end{bmatrix}$$

which has exact elements. Moreover,  $\omega$  has an exact differential  $\gamma(\theta) = [\epsilon \sin(\theta), -\epsilon(\cos(\theta) - 1)]^T$  that satisfies  $d\gamma(\theta) = \omega$ . Therefore, applying the following global change of co-ordinates

$$\begin{aligned}z_1 &= x_1 - \epsilon \sin(\theta) \\ z_2 &= x_2 - \epsilon \cos(\theta)\omega \\ w_1 &= y_1 + \epsilon(\cos(\theta) - 1) \\ w_2 &= y_2 - \epsilon \sin(\theta)\omega \\ \xi_1 &= \theta \\ \xi_2 &= \omega\end{aligned}\tag{5.43}$$

transforms the dynamics of the VTOL aircraft into

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\sin(\xi_1)\bar{u}_1 =: v_1 \\ \dot{w}_1 &= w_2 \\ \dot{w}_2 &= \cos(\xi_1)\bar{u}_1 - g =: v_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u_2\end{aligned}\tag{5.44}$$

and decouples the translational and attitude dynamics. Here,  $\bar{u}_1 = u_1 - \epsilon\xi_2^2$  is the new control. Due to [35], the system in (5.44) is dynamic feedback linearizable. In this work, we take a different approach that makes use of a static state feedback and backstepping procedure for global configuration stabilization of the VTOL aircraft. Clearly,  $(z_1, w_1)$  are two flat outputs of the VTOL aircraft which are obtained automatically as the by-products of the decoupling change of coordinates (see [29, 106, 105] for more details on differential flatness and flat outputs).

Applying the linear state feedback  $v_1 = c_1 z$  and bounded feedback  $v_2 = c_0 \sigma(c_2 w)$  where  $\sigma(s) = \tanh(s)$  and  $0 < c_0 < g$ , globally asymptotically and locally exponentially stabilizes  $(z, w) = (0, 0)$  for the  $(z, w)$ -subsystem in (5.44) where  $c_1, c_2 \in \mathbb{R}^2$  are coefficients of a Hurwitz polynomial (a rather similar choice is made for  $v_1$  in [80] but with  $\epsilon = 0$  which is not the case here). An important point is that, the unbounded choice of  $v_2 = c_2 w$  is not possible due to a singularity that appears later in control design. To stabilize the origin for the composite system, we use the backstepping procedure as follows. Solving for  $\bar{u}_1$  and  $\tan \xi_1$  in equation (5.44) gives

$$\begin{aligned}\bar{u}_1 &= k_1(z, w) := \sqrt{v_1^2 + (v_2 + g)^2} \\ \tan \xi_1 &= k_2(z, w) = \frac{-v_1}{v_2 + g}\end{aligned}$$

Then, defining the change of coordinates and control

$$\begin{aligned}\mu_1 &= \tan \xi_1 - k_2(z, w) \\ \mu_2 &= (1 + \tan^2 \xi_1)\xi_2 - \dot{k}_2 \\ \bar{u}_2 &= (1 + \tan^2 \xi_1)(u_2 + 2 \tan \xi_1 \xi_2^2) - \ddot{k}_2\end{aligned}$$

we get

$$\begin{aligned}\dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= \bar{u}_2\end{aligned}$$

Thus, applying  $\bar{u}_2 = -d_1\mu_1 - d_2\mu_2$  with  $d_1, d_2 > 0$  globally exponentially stabilizes  $(\mu_1, \mu_2) = (0, 0)$  for the  $\mu$ -subsystem. The dynamics of the closed-loop system is in the form

$$\begin{aligned}\dot{\eta} &= f(\eta, \mu) \\ \dot{\mu} &= A\mu\end{aligned}\tag{5.45}$$

where  $\eta = \text{col}(z, w)$ ,  $A$  is a Hurwitz matrix. Given  $\mu = 0$ , for  $\dot{\eta} = f(\eta, 0)$ ,  $\eta = 0$  is globally asymptotically and locally exponentially stable. It can be shown that for any solution of the  $\mu$ -subsystem the solution of the  $\eta$ -subsystem is uniformly bounded and the asymptotic stability of  $(\eta, \mu) = 0$  for the cascade system in (5.45) follows from a theorem due to Sontag in [82]. This guarantees global asymptotic stability and local exponential stability of the origin for the VTOL aircraft. Figure 5-27 (a) shows the trajectories of the VTOL aircraft from initial condition  $(2, 3, 4, 1, \pi/3, 1)$  with  $\epsilon = 0.1$ . The path of the center of mass of the aircraft is shown in Figure 5-27 (b). Based on the simulation result in Figure 5-27, the controller performs an aggressive maneuver to stabilize the position and attitude of the aircraft. In the sense that, after a rather short transient period, the states of the system almost exponentially converge to zero.

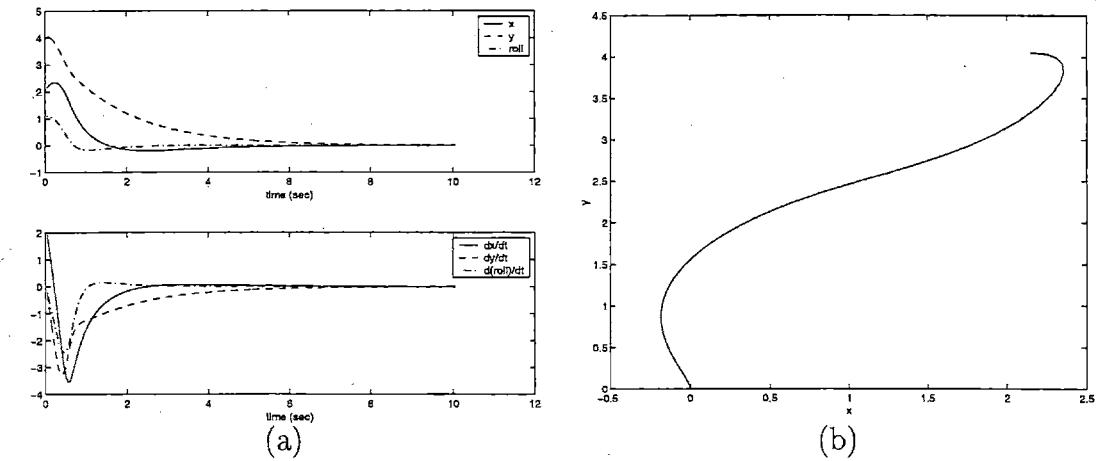


Figure 5-27: (a) The state trajectories of the VTOL aircraft, and (b) The path of the center of mass of the aircraft in  $(x_1, y_1)$ -plane.

## 5.11 Trajectory Tracking and Stabilization of An Autonomous Helicopter

This section is devoted to trajectory tracking and attitude stabilization nonlinear control design for an autonomous helicopter capable of performing aggressive maneuvers. A helicopter is an underactuated mechanical system with 6 DOF (i.e. the position and the attitude in  $\mathbb{R}^3 \times SO(3)$ ) and 4 control inputs which includes the thrust of the main rotor and 3 control moments in the body frame. As the dynamic model of the helicopter, we use a model introduced in [43]. The reason for this choice is that the helicopter model in [43] is rather accurate. In the sense that, both the effect of the body moments on the translational dynamics and the effect of the thrust of the main rotor on the attitude dynamics are taken into account. These two mutual effects complicate control design for the helicopter. To derive the differentially flat outputs [29, 106, 105] for this helicopter model, the effects of the body moment on the translational dynamics are ignored in [42, 43]. We present a method for decoupling of the translational and attitude dynamics of the helicopter. As a by-product, this decoupling method provides the differentially flat outputs for the accurate model of the helicopter. In addition, the problem of global exponential tracking of feasible trajectories of a helicopter is addressed.

We use a rotation matrix  $R \in SO(3)$  to represent the attitude of the helicopter. This way, the singularities of the Euler angle parameterization can be avoided. In the past, the use of a rotation matrix in  $SO(3)$  as the attitude, has proven to be efficient for attitude control and tracking of satellites [108, 16], spacecraft [17], and helicopters [43, 31].

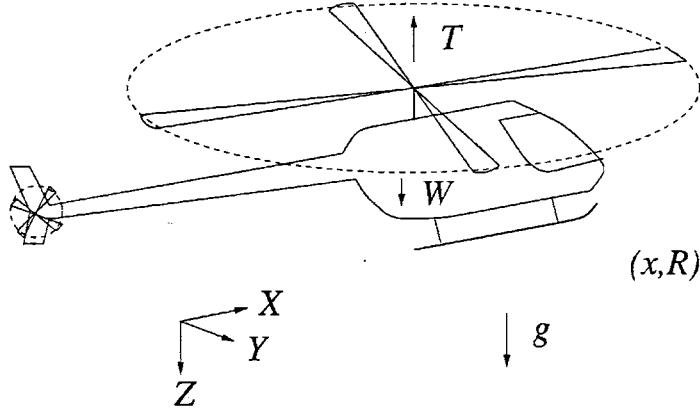


Figure 5-28: A helicopter.

### 5.11.1 Dynamic Model of A Helicopter

The dynamic model of the helicopter used in this work is based on a model of the helicopter given in [43]. Let  $q = (x, R) \in \mathbb{R}^3 \times SO(3)$  denote the configuration of the helicopter depicted in Figure 5-28. Here,  $R$  denotes that transformation from the

body frame to the reference frame and is defined as

$$R = \exp(\psi\hat{e}_3) \exp(\theta\hat{e}_2) \exp(\phi\hat{e}_1)$$

where  $e_i$ 's are standard basis in  $R^3$  and  $\eta = (\phi, \theta, \psi)$  denotes the vector of three Euler angles associated with rotations around  $x, y, z$  axes. The angles  $\phi, \theta, \psi$  are called roll, pitch, and yaw, respectively. The attitude kinematics is given by

$$\dot{R} = R\hat{\omega}$$

where  $\omega = (\omega_1, \omega_2, \omega_3)^T$  denotes the vector of angular velocities in the body frame and  $\hat{\omega}$  (i.e.  $\hat{\omega} = \omega \times \cdot$ ) is a skew-symmetric matrix

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The relation between  $\eta$  and  $\omega$  is as

$$\dot{\eta} = \Phi(\eta)\omega$$

where  $\Phi(\eta)$  is called the Euler matrix and is given by

$$\Phi(\eta) = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \quad (5.46)$$

The matrix  $\Phi(\eta)$  has a singularity at  $\theta = \pm\pi/2$ . For this reason, we use  $R$  for the tracking control design.

The equations of motion for the helicopter are as the following

$$\begin{cases} \dot{x} = v \\ m\dot{v} = RF_b \\ \dot{R} = R\hat{\omega} \\ J\dot{\omega} = -\omega \times J\omega + \tau_b \end{cases} \quad (5.47)$$

where  $F_b, \tau_b$  denote the forces and moments in the body frame and  $(x, v)$  denote the position and the velocity of the center of mass of the helicopter in the reference frame. The forces and moments in (5.47) are in the form

$$\begin{cases} F_b = mgR^T e_3 - Te_T + T_t e_2 \\ \tau_b = Q(T)e_T + D\tau \end{cases} \quad (5.48)$$

where  $T$  is the thrust of the main rotor,  $T_t$  is the thrust of the tail rotor,  $e_T$  is a unit

vector along the shaft of the main rotor defined by

$$e_T(a, b) = \begin{bmatrix} \sin(a) \\ -\cos(a) \sin(b) \\ \cos(a) \cos(b) \end{bmatrix} \quad (5.49)$$

with  $a, b$  denoting the pitch and roll tilt angles of the tip of the main rotor,  $\tau = (bT, aT, T_t)^T$ ,  $D = \text{diag}(l_b, l_a, l_t)$  is a constant matrix with positive diagonal elements, and

$$Q(T) = Q_0 + Q_1 T^{1.5}$$

with relatively small constants  $Q_0, Q_1 > 0$ .

**Assumption 5.11.1.** Notice that for  $(a, b) = (0, 0)$ ,  $e_T = e_3$ . In addition, for  $|a|, |b| \ll 1$ ,  $e_T$  can be approximated as

$$e_T \approx \begin{bmatrix} a \\ -b \\ 1 \end{bmatrix} = e_3 + \begin{bmatrix} a \\ -b \\ 0 \end{bmatrix}$$

Since,  $a, b$  are small in practice, we use this approximation of  $e_T$  instead of  $e_T$  itself. Moreover, we use the following approximation

$$Q(T)e_T \approx Q(T)e_3$$

This is justified by the fact that if  $Q_0, Q_1, a, b, \sim O(\epsilon)$ , then the first two elements of  $Q(T)e_T$  are of the order  $O(\epsilon^2)$  and can be ignored.

The following simplified forces and body moments are directly used in [31] as the original expressions of forces and moments. For the sake of clarity, we restate the approximate forces and moments in the body frame under the approximation assumption 5.11.1.

**Corollary 5.11.1.** Suppose assumption 5.11.1 holds and let  $u = D\tau \in \mathbb{R}^3$ . Then  $F_b, \tau_b$  take the following forms

$$\begin{aligned} F_b &= mgR^T e_3 - Te_3 + E \cdot u \\ \tau_b &= Q(T)e_3 + u \end{aligned} \quad (5.50)$$

where

$$E = \begin{bmatrix} 0 & \epsilon_2 & 0 \\ \epsilon_1 & 0 & \epsilon_3 \\ 0 & 0 & 0 \end{bmatrix}; \quad \epsilon_1 = 1/l_b, \epsilon_2 = -1/l_a, \epsilon_3 = 1/l_t$$

**Proof.** The proof is elementary and by direct substitution. □

Based on corollary 5.11.1, the *first-level approximate model of the helicopter* is in the form

$$\begin{aligned}\dot{x} &= v \\ m\dot{v} &= mge_3 - (Re_3)T + REu \\ \dot{R} &= R\hat{\omega} \\ J\dot{\omega} &= -\omega \times J\omega + Q(T)e_3 + u\end{aligned}\tag{5.51}$$

Under a further approximation, by setting  $E = 0_{3 \times 3}$  in 5.51, one obtains a *second-level approximate model of the helicopter* as

$$\begin{aligned}\dot{x} &= v \\ m\dot{v} &= mge_3 - (Re_3)T \\ \dot{R} &= R\hat{\omega} \\ J\dot{\omega} &= -\omega \times J\omega + Q(T)e_3 + u\end{aligned}\tag{5.52}$$

where the control input of the attitude dynamics  $u$  has no effect on the translational dynamics. This is the “approximate model” used in [42, 43, 42]. Here, our main goal is to address trajectory tracking for model (5.51) where  $E \neq 0_{3 \times 3}$ .

### 5.11.2 Input Moment Decoupling for A Helicopter

Consider the first-level approximate model of the helicopter in (5.51) with input moment  $u$ . In this section, we seek a differentially flat output [29, 106, 105] in the form  $y = x + \gamma(\eta)$  such that  $\dot{y}$  is independent of  $u$ . If  $\dot{y}$  is independent of  $u$  for all  $\eta$ , we call  $y = x + \gamma(\eta)$  an *exact flat output*. On the other hand, if  $y = x + \gamma(\eta)$  is independent of  $u$  at  $\eta = \eta_0$ , we call  $y = x + \gamma(\eta)$  a *weak flat output*. In the following, we show that there exists no exact flat output  $y = x + \gamma(\eta)$  for the helicopter model (5.51). Before doing so, let us present the dynamics of the helicopter using the Euler angle parameterization and identify the shape variables of the helicopter.

The first-level approximate model of the helicopter in equation (5.51) can be rewritten as

$$\begin{aligned}\dot{x} &= v \\ m\dot{v} &= mge_3 - n(\eta)T + R(\eta)Eu \\ \dot{\eta} &= \Phi(\eta)\omega \\ J\dot{\omega} &= -\omega \times J\omega + Q(T)e_3 + u\end{aligned}\tag{5.53}$$

where  $n(\eta) = R(\eta)e_3$  and we refer to it as the unit *normal vector*.  $R(\eta)$  is given by

$$R(\eta) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}$$

or equivalently

$$R(\eta) = \begin{bmatrix} \cos \theta \cos \psi & \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \\ \cos \theta \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi \\ -\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix}$$

Define  $\Psi(\eta) = \Phi^{-1}(\eta)$  and note that  $\Psi(\eta)$  can be calculated as

$$\Psi(\eta) = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \theta \cos \phi \end{bmatrix}$$

Denote the configuration vector of the helicopter by  $q = (x, \eta)$ . The Lagrangian of the helicopter is in the form

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}mv^T v + \frac{1}{2}\omega^T J\omega + mgx_3$$

which can be rewritten as

$$\mathcal{L}(x, \eta, \dot{x}, \dot{\eta}) = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix}^T \cdot \begin{bmatrix} mI_3 & 0 \\ 0 & \Psi(\eta)^T J \Psi(\eta) \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} + mgx_3 \quad (5.54)$$

The inertia matrix of the helicopter is a block diagonal matrix

$$M(\eta) = \text{diag}(mI_3, \Psi(\eta)^T J \Psi(\eta))$$

Therefore, the *Euler angles are the shape variables of a helicopter*, or  $q_s = (\phi, \theta, \psi)$  and the position  $q_x = x$  is the vector of external variables of a helicopter. We need the following result to check whether a flat output exists for a helicopter.

**Proposition 5.11.1.** *The attitude dynamics of the helicopter can be expressed in the form*

$$M_s(\eta)\ddot{\eta} + C_s(\eta, \dot{\eta})\dot{\eta} = \Psi(\eta)^T \tau_b \quad (5.55)$$

with a well-defined positive definite inertia matrix  $M_s(\eta) = \Psi(\eta)^T J \Psi(\eta)$  (for  $\theta \neq \pm\pi/2$ ) given by

$$M_s(\eta) = \begin{bmatrix} J_1 & 0 & -J_1 s_\theta \\ 0 & J_2 s_\phi^2 + J_3 c_\phi^2 & (J_2 - J_3) c_\theta c_\phi s_\phi \\ -J_1 s_\theta & (J_2 - J_3) c_\theta c_\phi s_\phi & J_1 s_\theta^2 + c_\theta^2 (J_2 s_\phi^2 + J_3 c_\phi^2) \end{bmatrix} \quad (5.56)$$

(“ $c_\theta$ ” and “ $s_\theta$ ” denote  $\cos(\theta)$  and  $\sin(\theta)$ , respectively) and the Coriolis and centrifugal matrix

$$C_s(\eta, \dot{\eta}) = \Psi(\eta)^T J \dot{\Psi}(\eta, \dot{\eta}) - \Psi^T(\eta)[(\Psi(\eta)\dot{\eta})^\wedge] J \Psi(\eta) \quad (5.57)$$

where  $J = (\text{diag}(J_1, J_2, J_3))$ ,  $\tau_b = Q(T)e_3 + u$  denotes the total body moment, and  $\wedge$

denotes the hat operation. In addition, for the case  $J_2 \neq J_3$  (i.e. the anti-symmetric case) the shape variables of the helicopter are  $\phi, \theta$ ; while given  $J_2 = J_3$  (i.e. the  $yz$ -symmetric case) the only shape variable of the helicopter is  $\theta$ .

**Proof.** Let us first rewrite the attitude dynamics as the following

$$\begin{aligned}\ddot{\eta} &= \dot{\Phi}(\eta, \dot{\eta})\omega + \Phi(\eta)\dot{\omega} \\ &= \dot{\Phi}(\eta, \dot{\eta})\Psi(\eta)\dot{\eta} - \Phi(\eta)J^{-1}\dot{\omega}J\omega + \Phi(\eta)J^{-1}\tau_b\end{aligned}$$

which multiplying both sides of the last equation by  $m_s(\eta) = \Psi(\eta)^T J \Psi(\eta)$  gives the attitude dynamics in the question with

$$C_s(\eta, \dot{\eta}) = \Psi(\eta)^T J \Psi(\eta) \dot{\Phi}(\eta, \dot{\eta}) \Psi(\eta) - \Psi(\eta)^T [(\Psi(\eta)\dot{\eta})^\wedge] J \Psi(\eta)$$

Noting that  $\dot{\Psi} = \Psi(\eta)\dot{\Phi}(\eta, \dot{\eta})\Psi(\eta)$  finishes the proof of the first part. The second part of the result for the anti-symmetric case follows from the fact that  $M_s(\eta)$  is independent of  $\psi$ . In addition, for the symmetric case where  $J_2 = J_3$ , the inertia matrix of the helicopter in (5.56) takes the form

$$M_s(\theta) = \begin{bmatrix} J_1 & 0 & -J_1 s_\theta \\ 0 & J_2 & 0 \\ -J_1 s_\theta & 0 & J_1 s_\theta^2 + J_2 c_\theta^2 \end{bmatrix} \quad (5.58)$$

which does not depend on  $\phi, \psi$ .  $\square$

Due to the fact that the attitude dynamics in (5.55) is a fully-actuated mechanical system for  $\theta \neq \pm\pi/2$ , it is exact feedback linearizable. For the sake of completeness, this is stated as the following corollary.

**Corollary 5.11.2.** *The collocated partially linearizing feedback law*

$$\tau_b = J\Psi(\eta)\tilde{u} + \Phi(\eta)^T C_s(\eta, \dot{\eta})\dot{\eta}$$

which is invertible for  $\theta \neq \pm\pi/2$ , transforms the attitude dynamics in (5.55) into a 3-dimensional double-integrator

$$\begin{aligned}\dot{\eta} &= \nu \\ \dot{\nu} &= \tilde{u}\end{aligned}$$

with a new control  $\tilde{u}$ .

Following the line of theorem 4.3.1 and based on proposition 5.11.1, the dynamics of the helicopter in (5.53) can be expressed as

$$\begin{aligned}m_{xx}\ddot{q}_x - g_x &= F_r(q_s)T + F_x(q_s)u \\ m_{ss}(q_s)\ddot{q}_s + h_s(q_s, \dot{q}_s) &= F_s(q_s)Q(T)e_3 + F_s(q_s)u\end{aligned} \quad (5.59)$$

where  $(q_x, q_s) = (x, \eta)$  and

$$\begin{aligned} m_{xx} &= mI_3 \\ m_{ss}(q_s) &= \Psi^T(\eta)J\Psi(\eta) \\ h_s(q_s, \dot{q}_s) &= C_s(\eta, \dot{\eta})\dot{\eta} \\ F_r(q_s) &= -n(\eta) \\ F_x(q_s) &= R(\eta)E \\ F_s(q_s) &= \Psi(\eta)^T \end{aligned}$$

Based on theorem 4.3.1, if the following vector of coupling one-forms

$$\omega_{xs} = m_{xx}^{-1}F_x(q_s)F_s^{-1}(q_s)m_{ss}(q_s)dq_s$$

which can be rewritten as

$$\omega_{xs} = m^{-1}R(\eta)EJ\Psi(\eta)d\eta \quad (5.60)$$

has exact elements. Then, there exists an exact flat output  $y = x + \gamma(\eta)$  such that  $\omega_{xs} = d\gamma(\eta)$ . Without loss of generality assume  $m = 1$ . The following result assures that such an exact flat output does not exist for a helicopter.

**Proposition 5.11.2.** *There exists no exact flat output  $y = x + \gamma(\eta)$  for the helicopter, i.e. there exists no local diffeomorphism  $\gamma(\eta) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $\omega_{xs} = d\gamma(\eta)$ .*

**Proof.** Let  $f_i(\eta)$  denote the  $i$ th column of  $R(\eta)EJ\Psi(\eta)$ . Then,  $\omega_{xs}$  has exact elements if and only if  $f_i$ 's satisfy the following property

$$\frac{\partial f_i(\eta)}{\partial \eta_j} = \frac{\partial f_j(\eta)}{\partial \eta_i}, \quad \forall i, j = 1, 2, 3, i \neq j \quad (5.61)$$

where  $(\eta_1, \eta_2, \eta_3) = (\phi, \theta, \psi)$ . By direct symbolic calculations in MATLAB, it has been shown that there exists no pair  $(i, j)$  with  $i \neq j$  that satisfies condition (5.61) (see Appendix B for the code that symbolically generates the difference of the left and right hand side of the equation (5.61)). Therefore, none of the elements of  $\omega_{xs}$  are exact one forms.  $\square$

The following proposition, provides the weak flat outputs for the first-level approximate model of the helicopter using momentum decomposition procedure in chapter 4.

**Proposition 5.11.3.** *Consider the dynamics of the helicopter as the following*

$$\begin{aligned} \dot{x} &= v \\ m\dot{v} &= mge_3 - (R(\eta)e_3)T + R(\eta)E\bar{u} \\ \dot{\eta} &= \nu \\ M_s(\eta)\dot{\nu} &= -C_s(\eta, \nu)\nu + \Psi(\eta)^T\bar{u} \end{aligned} \quad (5.62)$$

where  $\bar{u} = Q(T)e_3 + u$  and it is assumed that  $EQ(T)e_3 \approx 0$ . Define the coupling inertia  $\mu(\eta)$  as the following

$$\mu(\eta) = R(\eta)EJ\Psi(\eta) \quad (5.63)$$

and let  $\pi = \mu(\eta)\dot{\eta}$  denote the nonintegrable coupling momentum between the body moments and the translational dynamics of a helicopter. (The nonintegrability property of  $\pi$  is due to proposition 5.11.2). Then, using momentum decomposition procedure  $\pi$  can be represented as

$$\pi = \pi_l + \pi_e$$

where  $\pi_l = G_l(\eta)\dot{\eta}$  denotes the integrable locked momentum associated with locked configuration  $\bar{\eta} = 0$  which is given by

$$\pi_l = \begin{bmatrix} 0 & \epsilon_2 J_2 \cos \theta & -\epsilon_3 J_3 \sin \psi \\ \epsilon_1 J_1 \cos \phi & 0 & \epsilon_3 J_3 \cos \psi \\ \epsilon_1 J_1 \sin \phi & -\epsilon_2 J_2 \sin \theta & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (5.64)$$

and  $\pi_e$  denotes the non-integrable error momentum that vanishes identically at  $\eta = \bar{\eta}$ . In addition, there exists a diffeomorphism  $\gamma(\eta) = (\gamma_\phi(\eta), \gamma_\theta(\eta), \gamma_\psi(\eta))^T$  satisfying  $\gamma(0) = 0$  where

$$\begin{aligned} \gamma_\phi(\theta, \psi) &= \epsilon_2 J_2 \sin \theta + \epsilon_3 J_3 (\cos \psi - 1) \\ \gamma_\theta(\phi, \psi) &= \epsilon_1 J_1 \sin(\phi) + \epsilon_3 J_3 \sin \psi \\ \gamma_\psi(\phi, \theta) &= -\epsilon_1 J_1 (\cos \phi - 1) + \epsilon_2 J_2 (\cos \theta - 1) \end{aligned} \quad (5.65)$$

such that  $\dot{\gamma} = \pi_l$ . In addition, the following output

$$y_* = mx - \gamma(\eta) \quad (5.66)$$

is a weak flat output (i.e.  $\ddot{y}_*$  does not depend on the body moment  $\bar{u}$  at  $\eta = 0$ ) that satisfies

$$\ddot{y}_* = mge_3 - (Re_3)T + C_e(\eta, \dot{\eta})\dot{\eta} + \tilde{G}_e(\eta)u$$

where

$$\begin{aligned} C_e(\eta, \dot{\eta}) &= C_l(\eta, \dot{\eta}) - G_l(\eta)M_s^{-1}(\eta)C_s(\eta, \dot{\eta}) \\ C_l(\eta, \dot{\eta}) &= \dot{G}_l \\ G_e(\eta) &= G(\eta) - G_l(\eta) \\ \tilde{G}_e(\eta) &= G_e(\eta)\Psi^{-1}(\eta)J^{-1} \end{aligned}$$

and both  $G_e(\eta)$  and  $\tilde{G}_e(\eta)$  vanish at  $\eta = \bar{\eta}$ .

**Proof.** To obtain the locked momentum  $\pi_l = G_l(\eta)\dot{\eta}$  from the original non-integrable momentum  $\pi = G(\eta)\dot{\eta}$ , we use the momentum decomposition procedure as the fol-

lowing. Let

$$R_1 = \exp(\phi \hat{e}_1), R_2 = \exp(\theta \hat{e}_2), R_3 = \exp(\psi \hat{e}_3)$$

then,  $R = R_3 R_2 R_1$  and  $G(\eta)$  can be written as

$$G(\eta) = R_3 R_2 R_1 E J \Psi(\eta)$$

Consider the following matrices

$$G_1 = G(\eta)|_{\eta=(\phi,0,0)}, G_2 = G(\eta)|_{\eta=(0,\theta,0)}, G_3 = G(\eta)|_{\eta=(0,0,\psi)}$$

To find  $\pi_l$ , we need to calculate the  $i$ th column  $C_i$  of  $G_i$  for  $i = 1, 2, 3$ . This way each  $C_i$  is a function of a single variable. Note that  $\Psi(\eta) = \Psi(\phi, \theta)$ . By definition,  $G_i$ 's can be rewritten as

$$G_1(\phi) = R_1 E J \Psi(\phi, 0), G_2(\theta) = R_2 E J \Psi(0, \theta), G_3(\psi) = R_3 E J \Psi(0, 0)$$

Setting  $G_l(\eta) = [C_1(\phi), C_2(\theta), C_3(\psi)]$  gives the locked momentum  $\pi_l = G_l(\eta)\dot{\eta}$  in the question (see Appendix B for the symbolic calculation of  $G_l(\eta)$ ). Clearly,  $\pi_l$  is integrable and  $\ddot{\gamma} = G_l(\eta)\ddot{\eta}$ . Calculating  $\ddot{y}_*$ , we get

$$\ddot{y}_* = mge_3 - (Re_3)T + RE\bar{u} - \ddot{\gamma}$$

where

$$\begin{aligned} \ddot{\gamma} &= \dot{G}_l \dot{\eta} + G_l(\eta) \ddot{\eta} \\ &= C_l(\eta, \dot{\eta}) \dot{\eta} + (G(\eta) - G_e(\eta)) [M_s^{-1}(\eta) \Psi(\eta)^T \bar{u} - M_s^{-1}(\eta) C_s(\eta, \dot{\eta}) \dot{\eta}] \\ &= [C_l(\eta, \dot{\eta}) - G_l(\eta) M_s^{-1}(\eta) C_s(\eta, \dot{\eta})] \dot{\eta} + (G(\eta) - G_e(\eta)) \Psi^{-1}(\eta) J^{-1} \bar{u} \\ &= C_e(\eta, \dot{\eta}) \dot{\eta} + RE\bar{u} - G_e(\eta) \Psi^{-1}(\eta) J^{-1} \bar{u} \end{aligned}$$

where

$$C_e(\eta, \dot{\eta}) = C_l(\eta, \dot{\eta}) - G_l(\eta) M_s^{-1}(\eta) C_s(\eta, \dot{\eta})$$

Therefore,  $\ddot{y}_*$  takes the following form

$$\ddot{y}_* = mge_3 - (Re_3)T - C_e(\eta, \dot{\eta}) \dot{\eta} + G_e(\eta) \Psi^{-1}(\eta) J^{-1} \bar{u}$$

The weak flat property of the output  $y_*$  follows from the fact that  $G_e(\eta)$  vanishes at  $\eta = \bar{\eta} = 0$ .  $\square$

**Corollary 5.11.3.** *After applying the change of coordinates*

$$z_1 = y_*, z_2 = \dot{y}_*$$

the dynamics of the helicopter in (5.62) transforms into

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= mge_3 - (Re_3)T + \Delta(\eta, \nu, \bar{u}) \\ \dot{\eta} &= \nu \\ M_s(\eta)\dot{\nu} &= -C_s(\eta, \nu)\nu + \Psi(\eta)^T\bar{u}\end{aligned}\tag{5.67}$$

where the perturbation  $\Delta$  is given by

$$\Delta(\eta, \nu, \bar{u}) = -C_e(\eta, \nu)\nu + \tilde{G}_e(\eta)\bar{u}$$

In addition, assume  $\exists L_1, L_2 > 0 : \sigma_{max}(C_e(\eta, \nu)) \leq L_1\|\nu\|, \sigma_{max}(\tilde{G}_e(\eta)) \leq L_2\|\eta\|, \forall \eta, \nu$ . Let  $\bar{u} = K_a(z, \eta, \nu)$  be a state feedback such that  $(\eta(t), \nu(t), \bar{u}(t))$  is exponentially vanishing with a rate  $\sigma_1 > 0$ . Then,  $\Delta(t)$  is exponentially vanishing with a rate  $\sigma_2 = 2\sigma_1 > 0$ .

**Proof.** The proof is straightforward and is omitted.  $\square$

*Remark 5.11.1.* Based on the preceding corollary, one can solve the tracking problem for the unperturbed system with  $\Delta = 0$  and then perform a Lyapunov-based robust stability analysis for the overall system with a perturbation  $\Delta$ .

### 5.11.3 Nontriangular Normal Form of a Helicopter

The following proposition provides a non-triangular normal form for the first-level approximate model of a helicopter. This normal form will be used later in chapter 7 for stabilization of the helicopter without decoupling or ignoring the effect of the input moment on the translational dynamics.

**Proposition 5.11.4.** Define the state variables and control

$$z = \text{col}(x, v), \xi_1 = \eta, \xi_2 = \nu, w = \tilde{u}\tag{5.68}$$

and assume  $EQ(T)e_3 \approx 0_3$ . Then, the first-level approximate model of the helicopter can be expressed in the following nontriangular normal form quadratic in  $\xi_2$  and affine in  $w$

$$\begin{aligned}\dot{z} &= f(z) + g_1(\xi_1)T + g_2(\xi_1, \xi_2, \epsilon) + g_3(\xi_1, \epsilon)w \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= w\end{aligned}\tag{5.69}$$

where  $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^T$  and  $f, g_1, g_2, g_3$  are defined as the following

$$f(z) = \begin{bmatrix} v \\ ge_3 \end{bmatrix} \quad (5.70)$$

$$g_1(\xi_1) = \begin{bmatrix} 0_3 \\ -m^{-1}(R(\eta)e_3) \end{bmatrix} \quad (5.71)$$

$$g_2(\xi_1, \xi_2, \epsilon) = \begin{bmatrix} 0_3 \\ R(\eta)E(\epsilon)\Phi(\eta)^T C_s(\eta, \nu)\nu \end{bmatrix} \quad (5.72)$$

$$g_3(\xi_1, \epsilon) = \begin{bmatrix} 0_3 \\ R(\eta)E(\epsilon)J\Psi(\eta) \end{bmatrix} \quad (5.73)$$

with the property that  $g_2, g_3$  vanish at  $\epsilon = (0, 0, 0)^T$ .

**Proof.** The proof is by direct calculation and follows from proposition 5.11.1 and corollary 5.11.2.  $\square$

#### 5.11.4 Feedback Linearization of the Unperturbed Model of the Helicopter

The dynamics of the unperturbed model of the helicopter is in the following form

$$\begin{aligned} \dot{x} &= v \\ m\dot{v} &= mge_3 - (Re_3)T \\ \dot{R} &= R\hat{\omega} \\ J\dot{\omega} &= -\omega \times J\omega + \tau_b \end{aligned} \quad (5.74)$$

Both the translational and the attitude dynamics of the model of the helicopter in (5.74) can be linearized as

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= w \\ \dot{R} &= R\hat{\omega} \\ \dot{\omega} &= \tau \end{aligned} \quad (5.75)$$

where  $w \in \mathbb{R}^3$  is the control of the  $(x, v)$ -subsystem and  $\tau \in \mathbb{R}^3$  is the new control for the attitude dynamics. The relation between  $\tau_b$  and  $\tau$  is given by

$$\tau_b = J\tau + \omega \times J\omega \quad (5.76)$$

Based on equation (5.75), the relation between the new control  $w$ , the thrust  $T$ , and the *normal vector*  $n = Re_3$  is as the following

$$mge_3 - nT = mw$$

From the last equation, one can solve for  $T$  and  $n$  as

$$\begin{aligned} T &= m\|ge_3 - w\| \\ n &= \frac{ge_3 - w}{\|ge_3 - w\|} \end{aligned} \quad (5.77)$$

Assume that the controller for the translational dynamics is a state feedback

$$w = k(x, v)$$

Then, the feedback law  $w = k(x, v)$  defines both  $T$  and  $n$  as the following

$$\begin{aligned} T(x, v) &= m\|ge_3 - k(x, v)\| \\ n_c(x, v) &= \frac{ge_3 - k(x, v)}{\|ge_3 - k(x, v)\|} \end{aligned} \quad (5.78)$$

We refer to  $n_c = n_c(x, v)$  as the *control normal vector*.

*Remark 5.11.2.* The reader should notice that the normal vector  $n = Re_3$  is a function of the attitude  $R$ , while the control normal vector  $n_c(x, v)$  is a function of the position and the velocity (for a stabilization problem).

For the *stabilization problem*, the state feedback  $k(x, v)$  is in the form

$$w = k(x, v) = -c_p(x - x_0) - c_d v, \quad c_p, c_d > 0$$

where  $(x_0, 0)$  is a desired equilibrium point of the translational dynamics. In contrast, for the *tracking problem*, one is interested in tracking a trajectory  $x_d(\cdot)$  in  $\mathbb{R}^3$ . The control law for the tracking case takes the following form

$$w = k(\tilde{x}, \tilde{v}) = -c_p(x - x_d) - c_d(v - \dot{x}_d) + \ddot{x}_d, \quad c_p, c_d > 0$$

where  $\tilde{x} = x - x_d$  and  $\tilde{v} = v - \dot{x}_d$ . Thus, for tracking purposes, the control normal vector  $n_c = n_c(\tilde{x}, \tilde{v})$  is a function of the position and velocity errors. Notice that the stabilization problem is a special case of the tracking problem with  $(x_d, \dot{x}_d, \ddot{x}_d) = (x_0, 0, 0)$ . Therefore, from now on, by  $n_c$  we mean

$$n_c(\tilde{x}, \tilde{v}) = \frac{ge_3 - k(\tilde{x}, \tilde{v})}{\|ge_3 - k(\tilde{x}, \tilde{v})\|}$$

To achieve asymptotic tracking, the controller for the attitude dynamics must guarantee that the normal vector  $n = Re_3$  asymptotically tracks the control normal vector  $n_c = n_c(\tilde{x}, \tilde{v})$ . However, this does not uniquely determine the desired attitude  $R \in SO(3)$  required for such an asymptotic tracking. This problem is resolved in the next section.

### 5.11.5 Desired Control Attitude $R_d$ for Position Tracking

The desired control attitude is a rotation matrix  $R_d \in SO(3)$  satisfying  $R_d e_3 = n_c$ . However, this only determines one column of  $R_d$  and the other two columns need to be determined from the trajectory  $x_d$  and (possibly) the heading direction of the vehicle. For doing so, let us denote the three columns of  $R_d$  by

$$\begin{aligned} f_c &= R_d e_1 \\ b_c &= R_d e_2 \\ n_c &= R_d e_3 \end{aligned} \quad (5.79)$$

We refer to  $f_c, b_c$  as the control forward vector and the control binormal vector, respectively. This terminology is adopted from the names of the three unit vectors in the Frenet frame for a curve in  $\mathbb{R}^3$  [26]. Given a trajectory  $x_d(\cdot)$  in  $\mathbb{R}^3$ , the control law  $w = k(\tilde{x}, \tilde{v})$  determines  $n_c$ . To specify  $R_d$ , we need to determine  $f_d$  which is a unit vector pointing out towards the nose of the vehicle. Let  $t_d(t)$  denote a unit vector tangent to  $x_d(t)$ . Then, the tangent vector  $t_d$ , given by

$$t_d = \frac{v_d}{\|v_d\|}$$

is defined almost everywhere (a.e.) for  $v_d = \dot{x}_d \neq 0$ . Depending on whether  $n_c \parallel t_d$  and/or  $t_d \parallel e_3$ , or not three situations might arise.

- i)  $n_c \cdot t_d \neq \pm 1$ : In this case, there is well-defined  $(n_c, t_d)$ -plane. We define  $f_c$  as a unit vector in the  $(n_c, t_d)$ -plane orthogonal to  $n_c$  such that  $f_c \cdot t_d > 0$ . Therefore

$$f_c = \frac{t_d - (n_c \cdot t_d)n_c}{\sqrt{1 - (n_c \cdot t_d)^2}}, \quad n_c \cdot t_d \neq \pm 1$$

(where  $\cdot$  denotes the inner product of two vectors).

- ii)  $n_c \cdot t_d = \pm 1$  and  $e_3 \cdot t_d \neq \pm 1$ : These two conditions imply that  $n_c \not\parallel e_3$ , or  $|n_c \cdot e_3| < 1$ . For this case, we define  $f_c$  as a unit vector in the  $(n_c, e_3)$ -plane such that  $f_c \perp n_c$ . Thus

$$f_c = -\text{sgn}(n_c \cdot t_d) \cdot \text{sgn}(t_d \cdot e_3) \frac{e_3 - (n_c \cdot e_3)n_c}{\sqrt{1 - (n_c \cdot e_3)^2}}, \quad n_c \cdot t_d = \pm 1, e_3 \cdot t_d \neq \pm 1$$

- iii)  $n_c \cdot t_d = \pm 1$  and  $e_3 \cdot t_d = \pm 1$ : This is equivalent to the case where  $n_c \parallel t_d \parallel e_3$  and the vehicle moves in the vertical direction. For this case, we set

$$f_c = \text{sgn}(n_c \cdot t_d)e_1$$

( $e_1$  in the last equation can be replaced by any other pre-determined unit vector which defines a desired heading direction of the vehicle).

Given  $f_c$  and  $n_c$ , the control binormal vector is defined as

$$b_c = f_c \times n_c$$

Now, the three unit vectors  $f_d, b_d, n_d$  determine the desired control attitude  $R_d$  as  $R_d = [f_c, b_c, n_c]$ .

### 5.11.6 Asymptotic Tracking of a Desired Attitude $R_d \in SO(3)$

To do attitude tracking, we need to define a meaningful distance between two different attitudes in  $SO(3)$ . Let  $R = [f, b, n], R_d = [f_c, b_c, n_c]$  denote the current and desired attitudes, respectively. We define the *Frenet distance* between  $R, R_d \in SO(3)$  as

$$\delta_F(R, R_d) = \|f - f_c\|_2 + \|b - b_c\|_2 + \|n - n_c\|_2$$

On the other hand, there is a well-known function  $\varphi : SO(3) \rightarrow \mathbb{R}_{\geq 0}$  defined as

$$\varphi(R_0) := \frac{1}{2} \text{tr}(I - R_0)$$

which has similar properties to a norm [16] (“tr” denotes the trace operator over square matrices). In fact,  $\varphi$  satisfies the following triangular-type inequality:

$$\varphi(R_1 R_2^T) + \varphi(R_2 R_3^T) \geq \varphi(R_1 R_3^T), \quad \forall R_1, R_2, R_3 \in SO(3)$$

Defining the distance  $\delta$  as

$$\delta(R_1, R_2) := \varphi(R_1 R_2^T),$$

the triangular inequality over  $SO(3)$  takes a more familiar form

$$\delta(R_1, R_2) + \delta(R_2, R_3) \geq \delta(R_1, R_3)$$

In addition, based on Rodrigues's formula, the rotation by  $\theta$  around a unit vector  $\mathbf{k}$  is given by

$$R(\mathbf{k}, \theta) = I + \sin \theta \hat{\mathbf{k}} + (1 - \cos \theta) \hat{\mathbf{k}}^2 \tag{5.80}$$

Thus

$$\varphi(R(\mathbf{k}, \theta)) = 1 - \cos(\theta) \leq 2$$

In other words,  $\varphi : SO(3) \rightarrow [0, 2]$ . We prove that the Frenet distance  $\delta_F(R, R_d)$  and  $\varphi(R R_d^T)$  are equivalent upto a constant multiplicative factor. As a distance measure between two  $SO(3)$  matrices  $R$  and  $R_d$ ,  $\varphi(R R_d^T)$  has been effectively used for satellite attitude control [16] and helicopter motion planning [31].

**Lemma 5.11.1.** *Let*

$$\delta(R, R_d) := \varphi(RR_d^T) = \frac{1}{2} \text{tr}(I - RR_d^T)$$

*Then  $\delta$  and  $\delta_F$  satisfy the following identity*

$$\delta_F(R, R_d) = 4\delta(R, R_d)$$

**Proof.** We have

$$\begin{aligned} \delta_F(R, R_d) &= \sum_{i=1}^3 \| (R - R_d)e_i \|_2 \\ &= \sum_{i=1}^3 e_i^T (R - R_d)^T (R - R_d) e_i \\ &= \sum_{i=1}^3 e_i^T (2I - RR_d^T - R_d^T R) e_i \\ &= \text{tr}(I - RR_d^T) + \text{tr}(I - (RR_d^T)^T) \\ &= 2\text{tr}(I - RR_d^T) = 4\delta(R, R_d) \end{aligned}$$

□

From now on, we only use  $\delta(R, R_d)$  as a distance measure instead of the Frenet distance  $\delta_F$ . Note that  $\delta(R, R_d) = 0 \Leftrightarrow R = R_d$ . Since  $\delta \geq 0$ , the main idea in attitude tracking is to design  $\omega$ , as the control input for the attitude kinematics, such that  $\dot{\delta} < 0, \forall \delta > 0$ . Before presenting our control design for attitude tracking, we need to define some notations. Let  $\langle A, B \rangle$  denote the inner product of two  $3 \times 3$  matrices  $A, B$  defined as

$$\langle A, B \rangle := \frac{1}{2} \text{tr}(A^T B)$$

Then, for two skew-symmetric matrices  $\hat{\alpha}_1, \hat{\alpha}_2$ , we have

$$\langle \hat{\alpha}_1, \hat{\alpha}_2 \rangle = \alpha_1 \cdot \alpha_2, \quad \langle \hat{\alpha}, \hat{\alpha} \rangle = \|\alpha\|_2^2$$

where  $\cdot$  denotes the inner product of vectors in  $\mathbb{R}^3$ . Notice that any matrix  $A$  can be decomposed as

$$A = \text{skew}(A) + \text{sym}(A)$$

where

$$\text{skew}(A) = \frac{A - A^T}{2}, \quad \text{sym}(A) = \frac{A + A^T}{2}$$

Keeping this in mind, the main property of the distance  $\delta(R, R_d)$  is given by the following lemma [16].

**Lemma 5.11.2.** *Let  $\omega_d = (R_d^T \dot{R}_d)^\vee$ , then  $\delta(R, R_d) = \frac{1}{2} \text{tr}(I - RR_d^T)$  satisfies the following property*

$$\dot{\delta} = \text{skew}(RR_d^T)^\vee \cdot (\omega - \omega_d)$$

$\cdot^\vee$  over skew-symmetric matrices denotes the inverse of the hat operation  $\hat{\cdot}$ ).

**Proof.** By direct calculation, we have

$$\begin{aligned}
\dot{\delta} &= -\frac{1}{2}\text{tr}\left(\frac{d}{dt}(RR_d^T)\right) \\
&= -\frac{1}{2}\text{tr}(R\hat{\omega}R_d^T + R\hat{\omega}_d^TR_d^T) \\
&= -\frac{1}{2}\text{tr}(R(\hat{\omega} - \hat{\omega}_d)R_d^T) \\
&= -\frac{1}{2}\text{tr}(R_d^TR(\hat{\omega} - \hat{\omega}_d)) \\
&= -\frac{1}{2}\text{tr}((\text{skew}(R_d^TR) + \text{sym}(R_d^TR))(\hat{\omega} - \hat{\omega}_d)) \\
&= -\frac{1}{2}\text{tr}(\text{skew}(R_d^TR)(\hat{\omega} - \hat{\omega}_d)) \\
&= \frac{1}{2}\text{tr}(\text{skew}(RR_d^T)^T(\hat{\omega} - \hat{\omega}_d)) \\
&= \langle \text{skew}(R_d^TR), (\hat{\omega} - \hat{\omega}_d) \rangle \\
&= \text{skew}(R_d^TR)^\vee \cdot (\omega - \omega_d)
\end{aligned}$$

and the result follows.  $\square$

The following proposition is a direct result of lemma 5.11.2.

**Proposition 5.11.5.** *The state feedback law*

$$\omega = K_R(R, R_d, \omega_d) := \omega_d - c_0 \text{skew}(R_d^TR)^\vee, \quad c_0 > 0$$

(almost) globally asymptotically stabilizes the equilibrium  $\delta = 0$  of

$$\dot{\delta} = \text{skew}(R_d^TR)^\vee \cdot (\omega - \omega_d)$$

for all  $\delta(0) \neq 2$ . In other words,  $\omega = K_R$  renders  $R = R_d$  for

$$\dot{R} = R\hat{\omega}$$

(almost) globally asymptotically stable given that  $R(0) \in SO(3) \setminus \partial SO(3)$  where

$$\partial SO(3) = \{R \in SO(3) \mid \delta(R, R_d) = 2\}$$

**Proof.** For the closed-loop system

$$\dot{\delta} = -c_0 \|\text{skew}(R_d^TR)^\vee\|_2^2 < 0$$

for  $\delta > 0$ . Now, we need to show that  $\text{skew}(R_d^T R)^\vee = 0$  implies  $\delta = 0$ . For doing so, observe that  $\text{skew}(R_d^T R)^\vee = 0$  implies  $R_d^T R$  is a symmetric matrix, or

$$R_d^T R = (R_d^T R)^T \in SO(3)$$

The only symmetric matrices in  $SO(3)$  are  $I$  and  $I + 2\hat{\mathbf{k}}^2$  where  $\mathbf{k}$  is an arbitrary unit vector in  $\mathbb{R}^3$  (this follows from Rodrigues's formula). But  $R_d^T R = I + 2\hat{\mathbf{k}}^2$  means  $\delta = \delta(R, R_d) = 2$  and this contradicts the assumption that  $\delta(0) \neq 2$ . Hence,  $R_d^T R = I$  is the only acceptable solution for  $\text{skew}(R_d^T R)^\vee = 0$  which implies  $R = R_d$ , or  $\delta = \delta(R, R_d) = 0$ . In other words, the only invariant equilibrium is  $\delta = 0$  that proves (almost) global asymptotic stability of  $\delta = 0$  for  $\delta(0) < 2$ . This is equivalent to (almost) global asymptotic tracking of  $R = R_d$  for  $R(0) \notin \partial SO(3)$ .  $\square$

*Remark 5.11.3.* All the matrices in  $\partial SO(3)$  are obtained by rotation of  $R_d$  around a unit vector  $\mathbf{k}$  by  $\theta = \pi$  (see (5.80)). Therefore, to satisfy  $R(0) \notin \partial SO(3)$  in proposition 5.11.5, one can perturb the initial condition  $R(0)$  via multiplying  $R(0)$  by the rotation matrix  $R(\hat{\mathbf{k}}, \epsilon)$  defined in (5.80) that slightly rotates  $R(0)$ . In practice, due to measurement noise and round-off errors, it is unnecessary to apply such an initial perturbation by a small rotation angle around  $\mathbf{k}$ .

We restate the following lemma from [16] based on the notation and application of interest in this work.

**Lemma 5.11.3.** ([16]) For any  $0 < \epsilon \leq 2$ , there exist positive constants  $d_1, d_2 > 0$  such that  $\varphi(R_0) \leq 2 - \epsilon$  implies

$$d_1 \|\text{skew}(R_0)^\vee\|^2 \leq \varphi(R_0) \leq d_2 \|\text{skew}(R_0)^\vee\|^2$$

This helps us to prove that the state feedback law in proposition 5.11.5 exponentially stabilizes  $R = R_d$ .

*Remark 5.11.4.* This result in a more general form on Riemannian manifolds has been proven by Bullo [16], however it is crucial to emphasize the result for the special case of  $SO(3)$  matrices due to its important applications in attitude control of helicopters, aircrafts, satellites, spacecraft, and underwater vehicles. In addition, in [16] additional dissipation functions are used for attitude tracking/stabilization. Therefore, the control laws obtained here are slightly different and simpler than the ones given in [16].

**Proposition 5.11.6.** *The state feedback law*

$$\omega = K_R(R, R_d, \omega_d) := \omega_d - c_0 \text{skew}(R_d^T R)^\vee, \quad c_0 > 0$$

(almost) globally exponentially stabilizes  $R = R_d$  for

$$\dot{R} = R\hat{\omega}$$

for all the initial conditions  $R(0) \in SO(3) \setminus \partial SO(3)$  where

$$\partial SO(3) = \{R \in SO(3) : \delta(R, R_d) = 2\}$$

**Proof.** For all  $R(0) \in SO(3) \setminus \partial SO(3)$ ,  $\delta(0) = \varphi(RR_d^T) < 2$ . Setting  $\epsilon = (2 - \delta(0))/2$ , based on lemma 5.11.3, we have

$$\dot{\delta} = -c_0 \|skew(R_d^T R)^\vee\|_2^2 \leq -c_0 d_1 \delta = -\bar{c}_0 \delta$$

Therefore,  $\delta = 0$  is globally exponentially stable for all initial conditions  $\delta(0) \in [0, 2 - \epsilon_0]$ ,  $0 < \epsilon_0 \ll 1$ .  $\square$

*Remark 5.11.5.* In the following, we provide the interpretation of the (almost) global exponential attitude tracking control law in proposition 5.11.6 in terms of Rodrigues's rotation matrix  $R(\theta, \mathbf{k})$ . For doing so, let

$$R_0(\theta, \mathbf{k}) := R_d^T R$$

Then,  $\delta = \varphi(R_d^T R) = \varphi(R_0(\theta, \mathbf{k})) = 1 - \cos(\theta)$  and

$$\dot{\delta} = skew(R_d^T R)^\vee \cdot (\omega - \omega_d)$$

where

$$skew(R_d^T R)^\vee = skew(R_0(\theta, \mathbf{k}))^\vee = \sin(\theta)\mathbf{k}$$

Therefore, the control law in proposition 5.11.6 can be expressed as

$$\omega = \omega_d - c_0 \sin(\theta)\hat{\mathbf{k}}$$

which means  $\delta$  satisfies the following equation

$$\dot{\delta} = -c_0 \sin^2(\theta) = -c_0(1 + \cos(\theta))\delta$$

If for some  $0 < \epsilon \ll 1$ ,  $\varphi(R_0) \in [0, 2 - \epsilon]$ , then  $|\theta| < \pi$  and there exist  $d_1, d_2 \in (0, 2]$  such that

$$d_1 \leq 1 + \cos(\theta) \leq d_2 = 2$$

As a result, we get

$$\dot{\delta} \leq -c_0 d_1 \delta = \bar{c}_0 \delta$$

which guarantees (almost) global exponential stability of  $\delta = 0$ . Apparently, if initially  $\delta(0) = \varphi(R_0) = 2$ ,  $\theta = \pi$  and  $\dot{\delta} = 0$ . The stable solution in this case is  $\delta(t) = 2, \forall t > 0$  and thus  $\delta$  does not converge to zero. Furthermore, substituting  $\delta = 1 - \cos(\theta)$  in the differential equation of  $\delta$  gives

$$\dot{\theta} = -\sin(\theta), \quad \theta \in (-\pi, \pi)$$

which has an exponentially stable equilibrium point  $\theta = 0$ . Two possible Lyapunov functions for this system are  $V = \theta$  or  $V = 1 - \cos(\theta) = \delta$ . Here, we prefer the latter one due to its direct relation with the distance function  $\delta$ .

Having a stabilizing state feedback  $\omega = K_R$ , using the backstepping procedure, the controller for the composite system can be obtained in a straightforward way.

**Proposition 5.11.7.** *Consider the attitude dynamics of a helicopter as the following*

$$\begin{aligned}\dot{R} &= R\hat{\omega}, \\ \dot{\hat{\omega}} &= \tau\end{aligned}\tag{5.81}$$

*Then, the static state feedback law*

$$\tau = K_a(\omega, R, \omega_d, R_d) := -c_1(\omega - \omega_d) - c_1 c_0 \text{skew}(R_d^T R)^\vee + \dot{K}_R; \quad c_0, c_1 > 0\tag{5.82}$$

*where*

$$\begin{aligned}\dot{K}_R &= \dot{\omega}_d - c_0 \text{skew}(R_d^T R \hat{\omega} - \hat{\omega}_d R_d^T R)^\vee \\ \dot{\omega}_d &= (R_d^T \ddot{R}_d - \hat{\omega}_d^2)^\vee\end{aligned}$$

*achieves (almost) global asymptotic tracking for the solution  $(R_d, \omega_d)$  of the closed-loop system. In addition, the following positive definite function*

$$V = 2c_0 c_1 \delta(R, R_d) + \frac{1}{2} \|\omega - K_R\|^2$$

*is a valid Lyapunov function for the system.*

**Proof.** Calculating  $\dot{V}$ , we get

$$\begin{aligned}\dot{V} &= 2c_0 c_1 \dot{\delta} - c_1 (\omega - \omega_d + c_0 \text{skew}(R_d^T R)^\vee) \cdot (\omega - \omega_d + c_0 \text{skew}(R_d^T R)^\vee) \\ &= -c_1 c_0^2 \|\text{skew}(R_d^T R)^\vee\|_2^2 - c_1 \|\omega - \omega_d\|_2^2 < 0\end{aligned}$$

for  $(R, \omega) \neq (R_d, \omega_d)$ , because  $\|\text{skew}(R_d^T R)^\vee\| = 0 \Leftrightarrow \delta(R, R_d^T) = 0$ . Thus,  $\delta = 0$  (i.e.  $R = R_d$ ) and  $\omega = \omega_d$  are (almost) globally asymptotically stable for the closed-loop system, given that  $\delta(0) \neq 2$ .  $\square$

Here is our main result for the trajectory tracking/stabilization (in position) for an autonomous helicopter:

**Theorem 5.11.1.** *Consider a trajectory  $x_d(\cdot)$  as a curve in  $\mathbb{R}^3$  with an associated attitude  $R_d(\cdot)$ . Assume  $L_d := \|\ddot{x}_d\|_\infty < \infty$  and denote  $z = \text{col}(\tilde{x}, \tilde{v})$ . Then, there exists a closed ball  $B_r = \{z \in \mathbb{R}^6 \mid \|z\| \leq r\}$  around  $z = 0$  and a finite time  $t_0 > 0$  such that given the state feedback law*

$$\begin{aligned}\tau_b &= \omega \times J\omega - J[c_1(\omega - \omega_d) + c_1 \text{skew}(R_d^T R)^\vee - \dot{\omega}_d + c_0 \text{skew}(R_d^T R \hat{\omega} - \hat{\omega}_d R_d^T R)^\vee] \\ T(\tilde{x}, \tilde{v}) &= m\|g e_3 - k(x - x_d, v - v_d)\|\end{aligned}\tag{5.83}$$

the tracking error  $z(t)$  in position and velocity, for the closed-loop system, globally exponentially converges to  $B_r$ , i.e.

$$\exists t_0, \lambda_0, d_0 > 0 : \min_{p \in B_r} \|z(t) - p\| \leq d_0 \exp(-\lambda_0(t - t_0)), \quad \forall t \geq t_0$$

and remains in  $B_r$  thereafter. In addition, the total tracking error  $(\tilde{x}, \tilde{v}, \delta, \tilde{\omega})$  (almost) globally asymptotically converges to zero, given that  $\delta(0) \neq 2$ .

The following theorem, introduces a high-gain nonlinear controller for the approximate model of the helicopter with  $T, \omega$  as the input. This controller achieves an (almost) global exponential tracking of feasible trajectories for a helicopter.

**Theorem 5.11.2.** Consider a trajectory  $x_d(\cdot)$  as a curve in  $\mathbb{R}^3$  with an associated attitude  $R_d(\cdot)$ . Assume  $L_d := \|\ddot{x}_d\|_\infty < \infty$  and denote  $z = \text{col}(\tilde{x}, \tilde{v})$ . Then, there exist a finite time  $t_0 > 0$  and a positive constant  $c^* > 0$  such that given the state feedback law

$$\begin{aligned} \omega &= \omega_d - c_0 \text{skew}(R_d^T R)^\vee, \quad c_0 > 0 \\ T(\tilde{x}, \tilde{v}) &= m\|g e_3 - k(x - x_d, v - v_d)\| \end{aligned} \tag{5.84}$$

for all  $c_0 > c^*$ , after time  $t_0$ , the tracking error  $(z(t), \delta(t))$  for the closed-loop system, (almost) globally exponentially converges to zero, given that  $\delta(0) \neq 2$ .

We need the following lemma, before presenting the proof of the last two theorems.

**Lemma 5.11.4.** Assume  $L_d = \|\ddot{x}_d\|_\infty < \infty$  and denote  $z = \text{col}(\tilde{x}, \tilde{v})$ . Then, there exists a closed ball  $B_r = \{z \in \mathbb{R}^6 : \|z\| \leq r\}$ ,  $r > 0$  around  $z = 0$  in the form and a positive constant  $C_1 > 0$  such that the thrust state feedback  $T(z) = T(\tilde{x}, \tilde{v})$  in (5.83) satisfies the following properties:

- i)  $\|T(z)\| < C_1\|z\|$ ,  $\forall z \in (\mathbb{R}^6 \setminus B_r)$ ,
- ii)  $\|T(z)\| \leq C_1 r$ ,  $\forall z \in B_r$ .

**Proof.** Let  $c_m := \max\{c_p, c_d\} > 0$ . Since  $k(\tilde{x}, \tilde{v}) = -c_p \tilde{x} - c_d \tilde{v} + \ddot{x}_d$  and  $T(\tilde{x}, \tilde{v}) = m\|g e_3 - k(\tilde{x}, \tilde{v})\|$ , the thrust feedback  $T(z) = T(\tilde{x}, \tilde{v})$  satisfies the following inequality

$$\|T(z)\| \leq m(g + L_d) + \sqrt{2}c_m\|z\|$$

due to  $\|\tilde{x}\| + \|\tilde{v}\| \leq \sqrt{2}\|z\|$ . Setting  $r = r(c_m) := (g + L_d)/(c_m \sqrt{2})$ , we get

$$\|T(z)\| \leq \frac{1}{2}C_1(r + \|z\|)$$

where  $C_1 = 2\sqrt{2}mc_m$ . Thus, for all  $\|z\| > r$ ,  $\|T(z)\| < C_1\|z\|$  and for all  $\|z\| \leq r$ ,  $\|T(z)\| \leq C_1 r$ .  $\square$

**Proof.**(theorem 5.11.1) Denote  $\tilde{x} = x - x_d$  and  $\tilde{v} = v - v_d$ . The closed-loop dynamics of the helicopter in (5.74) can be written as the following

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{v} \\ \dot{\tilde{v}} &= -ge_3 + R_d e_3 T(\tilde{x}, \tilde{v}) + \Delta \\ \dot{R} &= R\hat{\omega} \\ \dot{\hat{\omega}} &= u\end{aligned}$$

where the perturbation  $\Delta \in \mathbb{R}^3$  is given by

$$\Delta = \frac{1}{m}(R - R_d)e_3 T(\tilde{x}, \tilde{v})$$

Calculating the upper bound of  $\Delta$ , we get

$$\|\Delta\| \leq \frac{1}{m}\|(R - R_d)e_3\| \|T(\tilde{x}, \tilde{v})\|$$

but  $\|(R - R_d)e_3\| \leq \delta_F(R, R_d) = 4\delta(R, R_d)$  and therefore

$$\|\Delta\| \leq C_2 \delta(R, R_d) \|T(z)\|$$

where  $z = \text{col}(\tilde{x}, \tilde{v})$  and  $C_2 := 4/m$ . Defining the new perturbation  $\bar{\Delta} = \text{col}[0, \Delta]$ , the closed-loop translational dynamics in  $z$ -coordinates can be written as

$$\dot{z} = Az + \bar{\Delta} \quad (5.85)$$

where

$$A = \begin{bmatrix} 0_3 & I_3 \\ -c_p I_3 & -c_d I_3 \end{bmatrix}$$

is a Hurwitz matrix and  $z \in \mathbb{R}^6$ . Consider the Lyapunov function  $V(z) = z^T P z$  where  $P$  is the symmetric and positive definite solution of the Lyapunov equation

$$A^T P + P A = -I$$

For all  $z \notin B_r$ , we have

$$\begin{aligned}\dot{V} &= -\|z\|^2 + 2z^T P \bar{\Delta} \\ &\leq -\|z\|^2 + 2\lambda_{\max}(P)C_2\|z\| \|T(z)\| \|\delta\| \\ &< -\|z\|^2(1 - 2\sqrt{2}\lambda_{\max}(P)C_1C_2\delta) \quad (\text{by lemma 5.11.4 and the property } \delta \geq 0)\end{aligned}$$

On the other hand,  $\delta = 0$  is exponentially stable for the attitude dynamics, thus  $\exists t_0 > 0 : \delta(t) \in [0, r_0], \forall t \geq t_0$  with

$$r_0 := \frac{1}{4\sqrt{2}\lambda_{\max}(P)C_1C_2} = \frac{1}{64c_m\lambda_{\max}(P)}, \quad c_m = \max\{c_p, c_d\} > 0$$

This implies that for  $t \geq t_0$ ,  $\dot{V} < -0.5V$  and therefore any solution  $z(t)$  outside the ball  $B_r$  exponentially converges to  $B_r$  and remains in it thereafter. Since with  $\delta = 0$ ,  $\bar{\Delta} = 0$  and  $z = 0$  is globally asymptotically stable for  $\dot{z} = Az$ , from Sontag's theorem on stability of cascade systems (see [83], or theorem 7.3.3 in chapter 7),  $(z, \delta, \tilde{\omega}) = 0$  is (almost) globally asymptotically stable for the closed-loop system given that  $\delta(0) \neq 2$ .  $\square$

**Proof.** (theorem 5.11.2) Based on the proof of theorem 5.11.1, after some finite time  $t_0 > 0$ , the position and velocity tracking error  $z(t)$  enters a closed ball  $B_r$  around  $z = 0$  exponentially fast and remains in it thereafter. From lemma 5.11.4, over  $B_r$  we have  $\|T(z)\| \leq C_3 := C_1r$ . From proposition 5.11.6,  $\dot{\delta} \leq -\bar{c}_0\delta$  where  $\bar{c}_0 = c_0d_1$  and  $d_1 > 0$  is a constant. Defining

$$W(z, \delta) = V(z) + \frac{1}{2}\delta^2$$

and keeping in mind that  $\delta \geq 0$ , we get

$$\begin{aligned} \dot{W} &= -\|z\|^2 + 2z^T P \bar{\Delta} + \delta \dot{\delta} \\ &\leq -\|z\|^2 + 2rC_1\lambda_{max}(P)\|z\|\delta - \bar{c}_0\delta^2 \\ &\leq -\frac{1}{2}\|z\|^2 - (\bar{c}_0^2 - 2r^2C_1^2\lambda_{max}^2(P))\delta^2 \end{aligned}$$

where the last inequality is obtained from

$$kab \leq la^2 + \frac{k^2}{4l}b^2$$

with  $a = \|z\|$ ,  $b = \delta$ ,  $k = 2rC_1\lambda_{max}(P)$ ,  $l = 1/2$ . Notice that  $C_4 := r(c_m)C_1(c_m)$  does not depend on  $c_m$  and is a constant because

$$C_4 = rC_1 = \frac{g + L_d}{\sqrt{2}c_m}(2\sqrt{2}mc_m) = 2m(g + L_d)$$

Taking

$$\bar{c}_0 \geq \bar{c}_0^* := (2C_4^2\lambda_{max}^2(P) + \frac{1}{4\lambda_{max}(P)})^{\frac{1}{2}}$$

or  $c_0 > c^* := \bar{c}_0^*/d_1$ , we get

$$\begin{aligned} \dot{W} &\leq -\frac{1}{2}\|z\|^2 - \frac{1}{4\lambda_{max}(P)}\delta^2 \\ &\leq -\frac{1}{2\lambda_{max}(P)}[\lambda_{max}(P)\|z\|^2 + \frac{1}{2}\delta^2] \\ &\leq -\frac{1}{2\lambda_{max}(P)}W \end{aligned}$$

and therefore  $W = 0$ , or  $(z, \delta) = 0$  is exponentially stable over  $B_r$ . This implies  $(z(t), \delta(t))$  (almost) globally exponentially converges to zero, after some finite time  $t_0 > 0$ , given that  $\delta(0) \neq 2$ .  $\square$

### 5.11.7 Thrust for Exponential Attitude Stabilization

For exponential attitude stabilization regardless of how the position in  $(e_1, e_2)$ -plane changes, one has to use a different control law for the thrust  $T$  than the one given before (for tracking/stabilization of the position). This situation arises during doing a maneuver from a (normal) hover mode to an (inverted) upside-down hover mode. This is equivalent to having no motions in the  $e_3$  direction, thus

$$e_3^T (R e_3) T = mg$$

or

$$T = T(R) = \frac{mg}{e_3^T R e_3}$$

However, the denominator in the above ratio vanishes as  $R e_3$  passes through a horizontal plane where the roll angle is  $\pm\pi/2$  and  $T \rightarrow \infty$ . This singularity is inherent in the dynamics of the helicopter and is impossible to avoid regardless of any parameterization of the attitude [31]. To avoid this problem, we suggest to apply a saturated thrust given by

$$\tilde{T} = T_{max} \text{sat}(T/T_{max})$$

where  $T_{max} \gg 1$  is a pre-specified maximum thrust. This results in a slight drop of the height of the position of the helicopter in the  $e_3$  direction during performing this particular maneuver (or in general, any maneuver passing through  $\theta = \pi/2$  or  $\phi = \pi/2$ ). Still, the main controller for stabilization of the attitude remains the same as in proposition 5.11.7 (with  $\omega_d = 0$ ). Notice that both  $T$  and  $\tau$  are independent of  $(x, v)$  in this case.

# Chapter 6

## Reduction and Control of Underactuated Nonholonomic Systems

This chapter is devoted to reduction and control of underactuated mechanical systems with nonholonomic first-order constraints and symmetry. A variety of real-life control systems including Car-type Vehicles, Mobile Robots, Surface Vessels, and Autonomous Underwater Vehicles (AUV) are examples of underactuated control systems with nonholonomic velocity constraints. All of these examples of nonholonomic systems possess certain symmetry properties, i.e. their kinetic energy, potential energy, or both are independent some configuration variables. This motivated us to focus mainly on reduction and control of nonholonomic systems with symmetry.

Our main theoretical result is that an underactuated system with first-order nonholonomic constraints and kinetic symmetry can be reduced to a well-defined reduced-order Lagrangian system in cascade with the constraint equation. Depending on whether the number of control inputs and constraint equations is less than or equal to the number of configuration variables, the reduced Lagrangian system is fully-actuated or underactuated, respectively. We present a number of examples with detailed reduction process. Namely, a rolling disk, a mobile robot [5], a dynamic model of a car, and the snakeboard [50]. Moreover, (almost) global exponential stabilization and tracking for a two-wheeled mobile robot is provided in great details.

Based on a result due to Brockett [15], there exists no smooth static state feedback that asymptotically stabilizes a mechanical system with first-order nonholonomic constraints to the origin. As a consequence, a two-wheeled nonholonomic mobile robot (Figure 6-2), cannot be asymptotically stabilized to the origin using a smooth feedback. Due to Astolfi [5], this mobile robot can be stabilized using a discontinuous change of coordinates and a discontinuous static state feedback. Our contribution is to introduce a smooth dynamic state feedback which achieves global exponential stabilization/tracking to a point/trajectory that is  $\epsilon$  far from the desired equilibrium/trajectory ( $\epsilon \ll 1$ ) for a two-wheeled mobile robot. This is based on the use of a new class of diffeomorphisms parameterized by  $\lambda$  that we call *near-identity diffeomorphism*. A near-identity diffeomorphism is equal to an identity function when-

ever  $\lambda$  vanishes. In addition, for all  $\lambda$ , a near-identity diffeomorphism remains within a distance  $\lambda$  of an identity function. The dynamic state feedback is designed such that  $\lambda(t) \rightarrow \epsilon$  as  $t \rightarrow \infty$  and  $\lambda(0) \approx O(1)$ .

## 6.1 Nonholonomic Systems with Symmetry

This section is devoted to reduction of underactuated mechanical systems with nonholonomic first-order constraints and symmetry. In the following, we precisely characterize a broad class of underactuated nonholonomic systems of interest by giving three assumptions.

To begin, consider a Lagrangian system with an  $n$ -dimensional configuration vector  $q$ , force matrix  $F(q)$ , and  $m$  nonholonomic first-order constraints as the following

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} &= W^T(q)\lambda + F(q)\tau \\ W(q)\dot{q} &= 0 \end{aligned} \quad (6.1)$$

where  $\lambda \in \mathbb{R}^m$  is the vector of Lagrange multipliers,  $\tau \in \mathbb{R}^l$  is the control input,  $l = \text{rank}(F(q))$ , and  $m + l \leq n$ . Due to  $m \geq 1$ ,  $l \leq (n - 1)$  and the nonholonomic system in (6.1) is an underactuated system. The term  $W^T(q)\lambda$  in (6.1) represents the effect of the constraint forces [64, 11, 8, 48]. This is based on the principle of virtual work which states “the constraint forces do not work on motions allowed by the constraints”. The principle of virtual work is an axiom of mechanics.

*Remark 6.1.1.* The last condition that  $m + l \leq n$  is one of the main conditions that distinguishes this work from the result of Bloch *et al.* in [11] on reduction of Caplygin control systems. In that work, it is assumed that  $m + l \geq n$ . Later, we show this assumption leads to over-actuated or fully-actuated reduced systems.

**Assumption 6.1.1.** Assume  $W(q)$  has full row rank. Then,  $q$  can be partitioned as  $(q_1, q_2)$  such that  $W(q) = (W_1(q), W_2(q))$  where  $W_1(q)$  is an invertible matrix. Therefore, the constraint equation in (6.1) can be rewritten as

$$W_1(q)\dot{q}_1 + W_2(q)\dot{q}_2 = 0 \quad (6.2)$$

**Assumption 6.1.2.** Assume  $M(q), F(q), W(q)$  are all independent of  $q_1$ .

**Assumption 6.1.3.** Assume the potential energy is in the form

$$V(q) = k^T q_1 + U(q_2) \quad (6.3)$$

where  $k \in \mathbb{R}^{n-m}$  is a constant.

*Remark 6.1.2.* Under Assumption 6.1.3 with  $k = 0$ , the notions of kinetic symmetry and classical symmetry coincide. Therefore, by saying “symmetry”, we refer to both of these notions.

*Remark 6.1.3.* Broad classes of mechanical systems with nonholonomic velocity constraints including mobile robots, car-type vehicles, surface vessels, rolling disk, Caplygin control systems [64, 11], and the snakeboard system [50, 9, 49, 73, 74] satisfy Assumptions 6.1.1 through 6.1.3. Moreover, all the aforementioned examples of nonholonomic systems are underactuated.

Under assumptions 6.1.1 through 6.1.3, the dynamics of the underactuated nonholonomic system in (6.1) can be expressed as

$$\begin{cases} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = W^T(q_2)\lambda + F(q_2)\tau \\ W_1(q_2)\dot{q}_1 + W_2(q_2)\dot{q}_2 = 0 \end{cases} \quad (6.4)$$

Nonholonomic systems in the form (6.4) (with  $F(q) \equiv 0$ ) are called *Caplygin systems* [64]. *Caplygin control systems* in the form (6.4) were first introduced by Bloch *et al.* in [11]. To eliminate  $\lambda$  from (6.4), one can multiply both sides of the forced Euler-Lagrange equation in (6.4) by a matrix  $A(q)$  that annihilates  $W^T(q)$  [11], i.e.  $A(q)W^T(q) = 0$ . For doing so, let us define

$$w_{12}(q_2) = -W_1^{-1}(q_2)W_2(q_2)$$

then  $\dot{q} = D(q_2)\dot{q}_2$  where

$$D(q_2) = \begin{bmatrix} w_{12}(q_2) \\ I_{n-m} \end{bmatrix}$$

By direct calculation, it can be readily shown that  $A(q_2) = D^T(q_2)$  annihilates  $W^T(q_2)$ . This eventually leads to the following reduction theorem for underactuated nonholonomic systems with symmetry.

**Theorem 6.1.1.** Consider the underactuated mechanical control system with nonholonomic constraints and symmetry in (6.4). Then, system (6.4) with  $(2n+m)$  first-order equations can be reduced to a system of  $(2n-m)$  first-order equations in the following cascade form

$$\begin{aligned} \dot{q}_x &= w_r(q_r)\dot{q}_r \\ M_r(q_r)\ddot{q}_r + C_r(q_r, \dot{q}_r)\dot{q}_r + G_r(q_r) &= F_r(q_r)\tau \end{aligned} \quad (6.5)$$

where  $(q_x, q_r) = (q_1, q_2)$ ,  $w_r(q_r) = w_{12}(q_2)$ , and

$$\begin{aligned} M_r(q_r) &= D(q_2)^T M(q_2) D(q_2) \\ G_r(q_r) &= w_r^T(q_r)k + \nabla_{q_r} U(q_r) \\ F_r(q_r) &= D^T(q_2)F(q_2) \end{aligned}$$

In addition, if  $V(q) = U(q_2)$  (or  $k = 0$ ), the reduced system is a well-defined La-

grangian system with configuration vector  $q_r = q_2$  and Lagrangian function

$$\mathcal{L}_r(q_r, \dot{q}_r) = \frac{1}{2} \dot{q}_r^T M_r(q_r) \dot{q}_r - U(q_r) \quad (6.6)$$

which satisfies a forced Euler-Lagrange equation in cascade with the constraint equation as the following

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = F_r(q_r) \tau \quad (6.7)$$

Moreover, if  $l + m < n$  (or  $l + m = n$ ), the reduced system with configuration vector  $q_r$  is an underactuated (or a fully-actuated) mechanical control system.

**Proof.** The forced-Euler-Lagrange equation in (6.4) can be rewritten as

$$M(q_2) \ddot{q} + C(q_2, \dot{q}) \dot{q} + G(q) = W^T(q_2) \lambda + F(q_2) \tau \quad (6.8)$$

where  $C(q_2, \dot{q})$  satisfies  $\dot{M} = C + C^T$  and  $G(q) = \nabla_q V(q)$ . Multiplying both sides of the last equation by  $A(q_2) = D^T(q_2)$  eliminates  $\lambda$  and gives

$$D^T(q_2) M(q_2) \ddot{q} + D^T(q_2) C(q_2, \dot{q}) \dot{q} + G_r(q) = F_r(q_2) \tau \quad (6.9)$$

where  $F_r(q_2) = D^T(q_2) F(q_2)$  and

$$G_r(q) = D^T(q_2) G(q) = [w_{12}^T(q_2) \ I] \begin{bmatrix} \nabla_{q_1} V(q) \\ \nabla_{q_2} V(q) \end{bmatrix} = w_{12}^T(q_2) k + \nabla_{q_2} U(q_2) =: G_r(q_2)$$

On the other hand,  $\dot{q} = D(q_2) \dot{q}_2$  implies that

$$\ddot{q} = D(q_2) \ddot{q}_2 + \dot{D}(q_2, \dot{q}_2) \dot{q}_2$$

Substituting  $\ddot{q}$  from the last equation in (6.9), we get

$$D^T(q_2) M(q_2) D(q_2) \ddot{q}_2 + [D^T(q_2) C(q_2, \dot{q}) D(q_2) + D^T(q_2) M(q_2) \dot{D}] \dot{q}_2 + G_r(q_2) = F_r(q_2) \tau$$

which after renaming  $q_r = q_2$  can be equivalently rewritten as

$$M_r(q_r) \ddot{q}_r + C_r(q_r, \dot{q}_r) \dot{q}_r + G_r(q_r) = F_r(q_r) \tau \quad (6.10)$$

where

$$M_r(q_r) := D^T(q_2) M(q_2) D(q_2) \quad (6.11)$$

$$C_r(q_r, \dot{q}_r) := D^T(q_2) C(q_2, D(q_2) \dot{q}_2) D(q_2) + D^T(q_2) M(q_2) \dot{D}(q_2, \dot{q}_2) \quad (6.12)$$

To establish that (6.10) is in fact equivalent to the forced Euler-Lagrange equation for the reduced system with the Lagrangian function  $\mathcal{L}_r(q_r, \dot{q}_r)$  and the force matrix  $F_r(q_r)$ , we need to prove  $C_r(q_r, \dot{q}_r)$  satisfies  $M_r = C_r + C_r^T$ . By direct calculation, we

have

$$\begin{aligned}
\dot{M}_r &= D^T \dot{M}D + \dot{D}^T M D + D^T M \dot{D} \\
&= D^T (C + C^T) D + \dot{D}^T M D + D^T M \dot{D} \\
&= (D^T C D + D^T M \dot{D}) + (D^T C^T D + \dot{D}^T M D) \\
&= C_r + C_r^T
\end{aligned}$$

and the result follows.  $\square$

*Remark 6.1.4.* Since the Lagrangian reduced system in (6.7) with the reduced configuration vector  $q_r$  and the reduced Lagrangian  $\mathcal{L}_r$  is itself an underactuated system for  $m + l < n$ , its reduction process can be addressed based on the reduction theory developed in Chapter 4 of this thesis. Also, the case of  $m + l = n$  is presented in corollary 6.1.1.

Many well-studied cases of nonholonomic systems including the rolling disk, a one-legged hopping robot in free-flying mode [51, 63], a two-wheeled mobile robot [5], and a planar three-link diver [22, 32] are special examples of underactuated nonholonomic systems considered in theorem 6.1.1 which all possess a fully-actuated reduced system. We formalize this special case in the following corollary (which has also been found independently in [11]).

**Corollary 6.1.1.** *Consider the underactuated nonholonomic system in (6.4). Assume the number of inputs and constraints add up to  $n$ , i.e.  $m + l = n$ . Suppose that  $q_2$  is actuated and the system has non-interacting inputs, i.e.  $F(q) = \text{col}(0, F_2(q_2))$  where  $F_2(q_2)$  is an invertible matrix. Then, there exists a change of control input that transforms the dynamics of (6.5) (or (6.4)) into the following nontriangular normal form with a vector double-integrator linear part*

$$\begin{aligned}
\dot{z} &= f(\xi_1, \xi_2) := w_r(\xi_1)\xi_2 \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= u
\end{aligned} \tag{6.13}$$

**Proof.** Notice that  $F_r(q_r) = D^T(q_2)F(q_2) = F_2(q_2)$  and thus  $F_r$  is invertible. After renaming the variables as  $z = q_x$ ,  $\xi_1 = q_r$ ,  $\xi_2 = \dot{q}_r$  and applying the change of control

$$\tau = F_r^{-1}(q_r)(M_r(q_r)u + C_r(q_r, \dot{q}_r)\dot{q}_r + G_r(q_r))$$

we get  $\ddot{q}_r = u$  and the result follows.  $\square$

## 6.2 Applications to Nonholonomic Robots

In this section, we provide applications of the theory developed in the preceding section for reduction and control of several examples of nonholonomic systems including Caplygin systems and mobile robots.

### 6.2.1 A Rolling Disk

A vertical rolling disk (see Figure 6-1) which is not allowed to slip is an example of mechanical system with nonholonomic velocity constraints. These constraints can be expressed as

$$\begin{aligned}\dot{x} &= r \cos(\theta)\dot{\varphi} \\ \dot{y} &= r \sin(\theta)\dot{\varphi}\end{aligned}$$

To view the rolling disk as a control system, assume the rotation angle  $\varphi$  and the

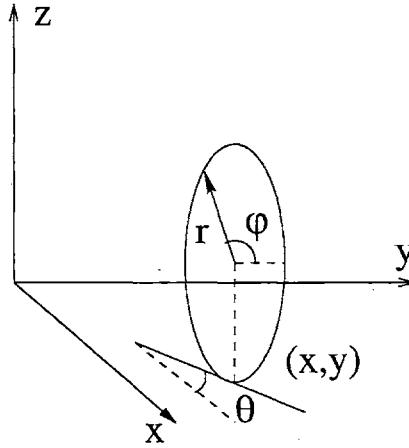


Figure 6-1: A vertical rolling disk

heading angle  $\theta$  of the disk can be controlled using torques  $\tau_1$  and  $\tau_2$ , respectively. The Lagrangian of the rolling disk with configuration vector  $q = (x, y, \varphi, \theta)$  is

$$\mathcal{L} = \frac{1}{2}\dot{q}^T M \dot{q}$$

with a constant inertia matrix

$$M = \text{diag}(m, m, J_1, J_2)$$

where  $m$  is the mass of the disk and  $J_1$  and  $J_2$  are the inertia of the disk. Thus, the rolling disk is a flat underactuated mechanical system with actuated variables  $(\varphi, \theta)$  and unactuated variables  $(x, y)$ . The constraint equation is in the form  $W(q)\dot{q} = 0$  with

$$W(q) = \begin{bmatrix} 1 & 0 & -r \cos(\theta) & 0 \\ 0 & 1 & -r \sin(\theta) & 0 \end{bmatrix}$$

Apparently  $W(q)$  has full row rank. Setting  $q_1 = (x, y)$  and  $q_2 = (\varphi, \theta)$ , one obtains  $W_1(q_2) = I_{2 \times 2}$  which is an invertible matrix. Since the number of controls and constraints add up to  $n = 4$ , based on theorem 6.1.1, the reduced system for this underactuated nonholonomic system is a fully actuated system. The configuration

of the reduced system is  $q_r = (\varphi, \theta)$  and the overall dynamics of the system can be expressed as

$$\begin{aligned}\dot{x} &= r \cos(\theta)\dot{\varphi} \\ \dot{y} &= r \sin(\theta)\dot{\varphi} \\ \ddot{\varphi} &= u_1 \\ \dot{\theta} &= u_2\end{aligned}\tag{6.14}$$

where  $u_i = \tau_i/J_i$ ,  $i = 1, 2$ . The reduced system of the rolling disk is a vector double-integrator  $\ddot{q}_r = u$  ( $u = (u_1, u_2)^T$ ). Controllability of the vertical rolling disk has been addressed in [11, 49].

Assume the input controls of the system are  $\dot{\varphi} = \omega_1$  and  $\dot{\theta} = \omega_2$ . Then, after normalization of  $(x, y)$  units by  $r$ , the kinematic model of the rolling disk can be written as

$$\begin{aligned}\dot{x} &= \cos(\theta)\omega_1 \\ \dot{y} &= \sin(\theta)\omega_1 \\ \dot{\theta} &= \omega_2\end{aligned}\tag{6.15}$$

Applying the following change of coordinates and control for  $\theta \in (-\pi/2, \pi/2)$

$$\begin{aligned}x_1 &= x \\ x_2 &= \tan(\theta) \\ x_3 &= y \\ v_1 &= \omega_1/\cos(\theta) \\ v_2 &= \omega_2/(1 + \tan^2(\theta))\end{aligned}$$

the kinematics of the rolling disk in (6.15) transforms into a first-order chained-type nonholonomic system as the following

$$\begin{aligned}\dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_3 &= x_2 v_1\end{aligned}\tag{6.16}$$

This is a special case of the general chained-type system in (2.8). Exponential stabilization of nonholonomic systems in the chained-form (6.16) using a discontinuous state feedback is addressed in [4]. Also, a homogeneous time-varying state feedback is used in [60, 59] for local exponential stabilization of such chained-form systems.

### 6.2.2 A Nonholonomic Mobile Robot

Consider the mobile robot depicted in Figure 6-2 [5, 4]. The robot has two rolling wheels that can be controlled independently using input torques. It is not difficult to see that the dynamics of this mobile robot is exactly the same as the vertical rolling disk in equation (6.14). Clearly, the same arguments hold for the mobile robot and using a change of coordinates and control the dynamics of this mobile robot

can be transformed into the chained-form system (6.16). Exponential stabilization

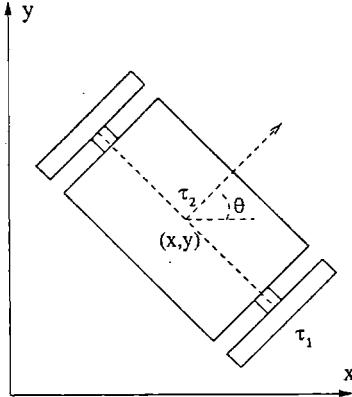


Figure 6-2: A mobile robot

of this mobile robot using a discontinuous change of coordinates and a discontinuous state feedback (both with singularities at the origin) has been addressed in [5]. Other researchers have tried time-varying and discontinuous state feedback or both to stabilize similar classes of nonholonomic systems [76, 19, 60]. Based on a theorem due to Brockett [15], systems with first-order nonholonomic constraints cannot be asymptotically stabilized using a  $C^1$  smooth state feedback.

We propose a new approach for tracking and stabilization of this mobile robot using a *smooth time-varying change of coordinates* and a *smooth dynamic state feedback law* that globally exponentially stabilizes the system to an equilibrium point arbitrary close to the origin (i.e. within a distance  $\epsilon$  of the desired equilibrium for  $\epsilon \ll 1$ ). To make this more precise, we need to define new notions of asymptotic stability and tracking. This is done in section 6.3. The details regarding stabilization and tracking for the two-wheeled nonholonomic robot are presented in section 6.4.

### 6.2.3 A Car

In this section, we address reduction of a dynamic model of a car as shown in Figure 6-3. The kinematic model of a car is a more common model of a car that has been studied in the literature [63]. This includes steering using sinusoids [63] and stabilization using discontinuous state feedback of the type used for a two-wheeled mobile robot [4]. Here, we take a different approach that later allows us to apply our reduction result for a car, to locomotion control design for the snakeboard [50]. The snakeboard is a complex nonholonomic system with certain similarities to a car and will be discussed in the next section. A dynamic model of a car is an example of an underactuated nonholonomic system with five degrees of freedom, two controls, and two velocity constraints corresponding to the rear and front wheels. Let  $q = (x, y, \theta, \psi, \phi) \in SE(2) \times S^1 \times S^1$  denote the configuration vector of a dynamic car. The position of the middle of the rear wheels of the car is denoted by  $(x, y)$ , the orientation of the body of the car is denoted by  $\theta$ , the angle of rotation of each wheel is denoted by  $\psi$ , and the orientation of the front wheels w.r.t. the body is denoted

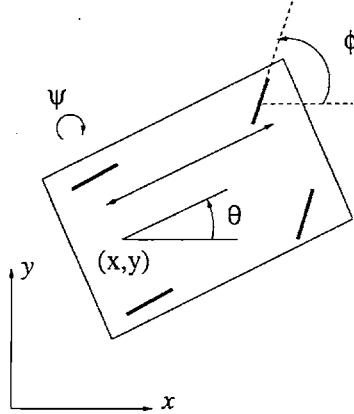


Figure 6-3: The model of a car.

by  $\phi$ . The velocity constraints for the front and rear wheels are as the following

$$\sin(\theta + \phi) \cdot \frac{d}{dt}(x + l \cos \theta) - \cos(\theta + \phi) \cdot \frac{d}{dt}(x + l \sin \theta) = 0 \quad (6.17)$$

$$\sin(\theta) \cdot \dot{x} - \cos(\theta) \cdot \dot{y} = 0 \quad (6.18)$$

or equivalently

$$\sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - l \cos \phi \dot{\theta} = 0 \quad (6.19)$$

$$\sin(\theta)\dot{x} - \cos(\theta)\dot{y} = 0 \quad (6.20)$$

These two constraints can be rewritten as  $W(q)\dot{q} = 0$  where  $W = (W_1, W_2)$  is partitioned according to  $q_x = (x, y)$  and  $q_r = (\theta, \psi, \phi)$ . The matrices  $W_1, W_2$  are given by

$$W_1(\theta, \phi) = \begin{bmatrix} \sin(\theta + \phi) & -\cos(\theta + \phi) \\ \sin \theta & -\cos \theta \end{bmatrix}, \quad W_2(\theta, \phi) = \begin{bmatrix} -l \cos \phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Notice that  $W_1(\theta, \phi)$  is not invertible at  $\phi = 0$ . We have

$$w_{12}(\theta, \phi) = -W_1^{-1}W_2 = \begin{bmatrix} \rho_1 & 0 & 0 \\ \rho_2 & 0 & 0 \end{bmatrix}$$

where  $\rho_1, \rho_2$  are defined as

$$\begin{aligned} \rho_1(\theta, \phi) &= \frac{l \cos \theta}{\tan \phi} \\ \rho_2(\theta, \phi) &= \frac{l \sin \theta}{\tan \phi} \end{aligned} \quad (6.21)$$

Thus, from  $w_{12}(q)$ ,  $D(q)$  can be obtained as the following

$$D(q) = D(\theta, \phi) = \begin{bmatrix} \rho_1 & 0 & 0 \\ \rho_2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Later, we use  $D(q)$  to calculate the reduced inertia matrix  $M_r(q_r)$ . The Lagrangian of a dynamic car is as the following

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}m[(\dot{x} - \frac{l}{2}\sin(\theta)\dot{\theta})^2 + (\dot{y} + \frac{l}{2}\cos(\theta)\dot{\theta})^2] \\ & + \frac{1}{2}(J_b + 2J_v)\dot{\theta}^2 + \frac{1}{2}[(2J_h)(1 + \frac{1}{\cos^2(\phi)})]\dot{\psi}^2 + \frac{1}{2}(2J_v)(\dot{\theta} + \dot{\phi})^2 \end{aligned}$$

where  $m$  is the mass of the car,  $J_b$  is the inertia of the body,  $J_h$  is the inertia of each wheel along a horizontal axes, and  $J_v$  is the inertia of each wheel along the vertical axes. After some simplification, the Lagrangian can be expressed as

$$\mathcal{L} = \frac{1}{2}\dot{q}^T \begin{bmatrix} m & 0 & -(ml/4)\sin\theta & 0 & 0 \\ 0 & m & (ml/4)\cos\theta & 0 & 0 \\ -(ml/4)\sin\theta & (ml/4)\cos\theta & J_\theta & 0 & 2J_v \\ 0 & 0 & 0 & 2J_h(2 + \tan^2(\phi)) & 0 \\ 0 & 0 & 2J_v & 0 & 2J_v \end{bmatrix} \dot{q} \quad (6.22)$$

where  $J_\theta = J_b + 4J_v + ml^2/4$ . The differential-algebraic equations of motion for the dynamic car are as the following

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} &= W^T(q) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0_{3 \times 2} \\ I_{2 \times 2} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \\ W(q)\dot{q} &= 0 \end{aligned} \quad (6.23)$$

where  $\lambda_1, \lambda_2 \in \mathbb{R}$  are Lagrange multipliers and  $\tau_1, \tau_2 \in \mathbb{R}$  are torques applied to the rear wheels and the steering wheel, respectively. Based on theorem 6.1.1, the dynamics of the car in (6.23) can be reduced to the cascade of the constraint equation and a reduced-order Lagrangian system with configuration vector  $q_r = (\theta, \psi, \phi)$  as the following

$$M_r(q_r)\ddot{q}_r + C_r(q_r, \dot{q}_r)\dot{q}_r = F_r(q_r)\tau$$

where

$$\begin{aligned} M_r(q_r) &= D^T(q)M(q)D(q) \\ C_r(q_r, \dot{q}_r) &= D^T C(q, \dot{q})D + D^T M(q)\dot{D} \\ F_r(q_r) &= D^T(q)F(q) \end{aligned}$$

By direct calculation, we have

$$F_r(q_r) = F_r = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$M_r(\phi) = \begin{bmatrix} \tilde{J}_\theta(\phi) & 0 & 2J_v \\ 0 & 2J_h(2 + \tan^2(\phi)) & 0 \\ 2J_v & 0 & 2J_v \end{bmatrix}$$

with

$$\tilde{J}_\theta(\phi) = J_\theta + m(\rho_1^2 + \rho_2^2) - \frac{\rho_1 ml}{2} \sin(\theta) + \frac{\rho_2 ml}{2} \cos(\theta) = J_\theta + \frac{ml^2}{\tan^2(\phi)}$$

Clearly, the reduced Lagrangian system is itself an underactuated system with three degrees of freedom  $(\theta, \psi, \phi)$  and two controls. In addition,  $(\theta, \psi)$  are the external variables and  $\phi$  is the shape variable of the car. The dynamics of the actuated variables  $(\phi, \psi)$  of the reduced system can be linearized as

$$\begin{aligned} \ddot{\psi} &= u_1 \\ \ddot{\phi} &= u_2 \end{aligned}$$

using an explicit collocated change of control in the form

$$\tau = \alpha(\phi)u + \beta(\phi, \dot{q}_r)$$

where

$$\alpha(\phi) = \begin{bmatrix} 2J_h(2 + \tan^2(\phi)) & 0 \\ 0 & 2J_v(1 - \frac{2J_v \tan^2(\phi)}{ml^2 + J_\theta \tan^2(\phi)}) \end{bmatrix}$$

is well-defined and positive definite for all  $\phi$ . From the first constraint equation, one can solve for  $\dot{\theta}$  to get

$$\dot{\theta} = \tan(\phi)(\cos(\theta)\dot{x} + \sin(\theta)\dot{y})$$

From this equation, the overall dynamics of the car can be written as

$$\begin{aligned}\dot{x} &= r \cos(\theta)\dot{\psi} \\ \dot{y} &= r \sin(\theta)\dot{\psi} \\ \dot{\theta} &= \frac{r}{l} \tan(\phi)\dot{\psi} \\ \ddot{\psi} &= u_1 \\ \ddot{\phi} &= u_2\end{aligned}\tag{6.24}$$

where  $r$  is the radius of the wheel. After normalization of the units of  $(x, y)$  by  $r$ , the kinematic model of the car can be expressed as

$$\begin{aligned}\dot{x} &= \cos(\theta)\omega_1 \\ \dot{y} &= \sin(\theta)\omega_1 \\ \dot{\theta} &= \frac{1}{l} \tan(\phi)\omega_1 \\ \dot{\phi} &= \omega_2\end{aligned}\tag{6.25}$$

where

$$\omega_1 = u_1, \omega_2 = u_2$$

Now, applying the following change of coordinates and control [63]

$$\begin{aligned}x_1 &= x \\ x_2 &= \frac{\tan(\phi)}{l \cos(\theta)^3} \\ x_3 &= \tan(\theta) \\ x_4 &= y \\ v_1 &= \cos(\theta)\omega_1 \\ v_2 &= \frac{(1 + \tan^2(\phi))}{l \cos(\theta)^3}\omega_2 + \frac{3 \tan(\theta) \tan^2(\phi)}{l^2 \cos(\theta)^3}\omega_1\end{aligned}\tag{6.26}$$

transforms the nonholonomic system in (6.25) into a chained-form system as the following

$$\begin{aligned}\dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_3 &= x_2 v_1 \\ \dot{x}_4 &= x_3 v_1\end{aligned}\tag{6.27}$$

The system in (6.27) can be stabilized using a discontinuous state feedback [4, 47], or sinusoid inputs [63].

### 6.2.4 The Snakeboard

The *snakeboard* system depicted in Figure 6-4 was first introduced in [50]. In that work, an experimental study was performed for locomotion of the snakeboard using different “gait” settings (i.e. a set of sinusoid inputs). The snakeboard example was one of the main motivations for developing a general framework for reduction of non-holonomic systems with symmetry on Riemannian manifolds in [9]. Controllability and motion planning for the snakeboard example as a control system was also considered in [49] and [73, 74], respectively. Here, we show that the snakeboard system can be reduced to an underactuated Lagrangian system in cascade with the constraint equation. Then, we use this representation to prove that locomotion control of the snakeboard is equivalent to stabilization of a kinematic car.

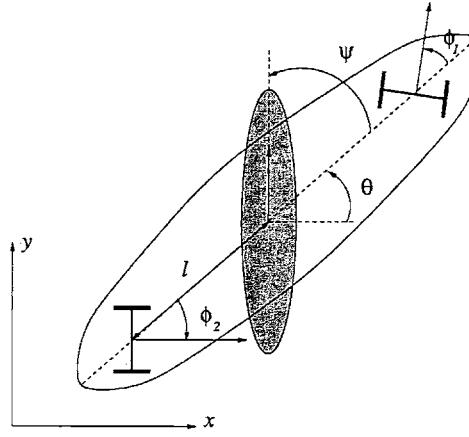


Figure 6-4: The Snakeboard.

### Dynamics and Reduction of the Snakeboard

The configuration vector for the snakeboard is  $q = (x, y, \theta, \psi, \phi_1, \phi_2) \in \mathbb{R}^2 \times S^4_1$ . The velocity constraints for the front and back wheels of the snakeboard are as the following

$$\begin{aligned}\frac{d}{dt}(x + l \cos \theta) \cdot \sin(\theta + \phi_1) - \frac{d}{dt}(x + l \sin \theta) \cdot \cos(\theta + \phi_1) &= 0 \\ \frac{d}{dt}(x - l \cos \theta) \cdot \sin(\theta - \phi_2) - \frac{d}{dt}(x - l \sin \theta) \cdot \cos(\theta - \phi_2) &= 0\end{aligned}$$

These constraints can be simplified as

$$\sin(\theta + \phi_1)\dot{x} - \cos(\theta + \phi_1)\dot{y} - l \cos(\phi_1)\dot{\theta} = 0 \quad (6.28)$$

$$\sin(\theta - \phi_2)\dot{x} - \cos(\theta - \phi_2)\dot{y} + l \cos(\phi_2)\dot{\theta} = 0 \quad (6.29)$$

Therefore, the constraints are in the form  $W(q)\dot{q} = 0$  with the constraint matrix  $W(q)$  given by

$$W(q) = \begin{bmatrix} \sin(\theta + \phi_1) & -\cos(\theta + \phi_1) & -l \cos(\phi_1) & 0 & 0 & 0 \\ \sin(\theta - \phi_2) & -\cos(\theta - \phi_2) & l \cos(\phi_2) & 0 & 0 & 0 \end{bmatrix}$$

Partitioning  $q$  as  $q_1 = (x, y)$  and  $q_2 = (\theta, \psi, \phi_1, \phi_2)$  and defining the vector  $\eta = (\theta, \phi_1, \phi_2)$  (i.e. all the variables that appear in  $W(q)$ ), the constraint matrix  $W(q)$  can be partitioned as  $(W_1, W_2)$  where  $W_1$  and  $W_2$  are defined as

$$W_1(\eta) = \begin{bmatrix} \sin(\theta + \phi_1) & -\cos(\theta + \phi_1) \\ \sin(\theta - \phi_2) & -\cos(\theta - \phi_2) \end{bmatrix}, \quad W_2(\eta) = \begin{bmatrix} -l \cos(\phi_1) & 0 & 0 & 0 \\ l \cos(\phi_2) & 0 & 0 & 0 \end{bmatrix}$$

and  $W_1(\eta)$  is invertible except for at isolated points  $\phi_1 + \phi_2 = k\pi, k \in \mathbb{Z}$ . Assume  $\phi_1 + \phi_2 \neq k\pi$ . By direct calculation, we have

$$w_{12}(\eta) = -W_1^{-1}W_2 = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ \rho_2 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\begin{aligned} \rho_1 &= \frac{l \cos \phi_1 \cos(\theta - \phi_2) + l \cos \phi_2 \cos(\theta + \phi_1)}{\sin(\phi_1 + \phi_2)} \\ \rho_2 &= \frac{l \cos \phi_1 \sin(\theta - \phi_2) + l \cos \phi_2 \sin(\theta + \phi_1)}{\sin(\phi_1 + \phi_2)} \end{aligned} \tag{6.30}$$

Hence

$$D(q_2) = D(\eta) = \begin{bmatrix} w_{12}(\eta) \\ I_{4 \times 4} \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ \rho_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is assumed that the torque  $\tau = (\tau_1, \tau_2, \tau_3)^T \in \mathbb{R}^3$  is applied to variables  $\psi, \phi_1, \phi_2$  and the rest of variables are unactuated. The Lagrangian of the snakeboard is in the form

$$\begin{aligned} \mathcal{L}(q, \dot{q}) &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J_b\dot{\theta}^2 + \frac{1}{2}J_r(\dot{\psi} + \dot{\theta})^2 \\ &+ \frac{1}{2}J_v(\dot{\phi}_1 + \dot{\theta})^2 + \frac{1}{2}J_v(\dot{\phi}_2 - \dot{\theta})^2 \end{aligned}$$

where  $J_b$ ,  $J_r$ , and  $J_v$  are the moment of inertia of the body, the rotor, and the wheel along the vertical axis, respectively [50]. This Lagrangian can be rewritten as

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}^T \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & J_\theta & J_r & J_v & -J_v \\ 0 & 0 & J_r & J_r & 0 & 0 \\ 0 & 0 & J_v & 0 & J_v & 0 \\ 0 & 0 & -J_v & 0 & 0 & J_v \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

with  $J_\theta = J_b + J_r + 2J_v$ . Clearly, the inertia matrix  $M$  is constant. Thus, the Coriolis and centrifugal matrix  $C(q, \dot{q})$  associated with  $M$  vanishes identically. The algebraic-differential equations of motion for this system is as the following

$$\begin{cases} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = W^T(q) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0_{3 \times 3} \\ I_{3 \times 3} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \\ W(q)\dot{q} = 0 \end{cases} \quad (6.31)$$

which can be expressed as

$$\begin{aligned} m\ddot{x} &= \lambda_1 \sin(\theta + \phi_1) + \lambda_2 \sin(\theta - \phi_2) \\ m\ddot{y} &= -\lambda_1 \cos(\theta + \phi_1) - \lambda_2 \cos(\theta - \phi_2) \\ J_\theta \ddot{\theta} + J_r \ddot{\psi} + J_v \ddot{\phi}_1 - J_v \ddot{\phi}_2 &= -\lambda_1 l \cos \phi_1 + \lambda_2 l \cos \phi_2 \\ J_r \ddot{\psi} + J_r \ddot{\theta} &= \tau_1 \\ J_v \ddot{\phi}_1 + J_v \ddot{\theta} &= \tau_2 \\ J_v \ddot{\phi}_2 - J_v \ddot{\theta} &= \tau_3 \end{aligned} \quad (6.32)$$

Based on theorem 6.1.1, the reduced Lagrangian system for the snakeboard system in (6.32) is an underactuated system with configuration vector  $q_r = (\theta, \psi, \phi_1, \phi_2)$  as the following

$$M_r(q_r)\ddot{q}_r + C_r(q_r, \dot{q}_r)\dot{q}_r = F_r(q_r)\tau \quad (6.33)$$

where

$$\begin{aligned} M_r(q_r) &= M_r(\eta) = D(\eta)^T M D(\eta) \\ C_r(q_r, \dot{q}_r) &= C_r(\eta, \dot{\eta}) = D^T(\eta) M \dot{D}(\eta, \dot{\eta}) \\ F_r(q_r) &= F_r = D^T(q) F(q) = \begin{bmatrix} 0_{1 \times 3} \\ I_{3 \times 3} \end{bmatrix} \end{aligned}$$

By direct calculation, we obtain the following inertia matrix

$$M_r(q_r) = \begin{bmatrix} \tilde{J}_\theta(\phi_1, \phi_2) & J_r & J_v & -J_v \\ J_r & J_r & 0 & 0 \\ J_v & 0 & J_v & 0 \\ -J_v & 0 & 0 & J_v \end{bmatrix}$$

where

$$\tilde{J}_\theta = J_\theta + m(\rho_1^2 + \rho_2^2)$$

Setting  $b := \rho_1^2 + \rho_2^2$ , we get

$$b(\phi_1, \phi_2) = l^2 \frac{\cos^2 \phi_1 + \cos^2 \phi_2 + 2 \cos \phi_1 \cos \phi_2 \cos(\phi_1 + \phi_2)}{\sin^2(\phi_1 + \phi_2)}$$

which means  $b \geq 0$  does not depend on  $\theta, \psi$ . Thus,  $M_r(q_r) = M_r(\phi_1, \phi_2)$ . In other words,  $(\phi_1, \phi_2)$  are the shape variables and  $(\theta, \psi)$  are the external variables of the reduced snakeboard. Notice that  $b(\phi_1, \phi_2)$  has an isolated singularity at  $(\phi_1, \phi_2) = 0$  (under the assumption that  $|\phi_i| < \pi/2; i = 1, 2$ ).

The reduced system in (6.33) has non-interacting control inputs (i.e.  $F_r = \text{col}(F_r^1, F_r^2)$  such that  $F_r^1 = 0$ ). Therefore, using collocated partial-feedback linearization, the dynamics of all three actuated configuration variables  $(\psi, \phi_1, \phi_2)$  can be linearized using an explicit change of control in the form

$$\tau = \alpha(\phi)u + \beta(\phi, \dot{q}_r)$$

where  $\phi = (\phi_1, \phi_2)$  and  $\alpha(\phi)$  is a positive definite matrix with a well-defined p.d. limit as  $\phi \rightarrow 0$ . After collocated partial feedback linearization, we get

$$\begin{aligned} \ddot{\psi} &= u_1 \\ \ddot{\phi}_1 &= u_2 \\ \ddot{\phi}_3 &= u_3 \end{aligned} \tag{6.34}$$

This gives the following asymptotic output tracking result for the snakeboard.

**Proposition 6.2.1.** *Denoting  $\xi_1 = (\psi, \phi_1, \phi_2)$  and  $\xi_2 = (\dot{\psi}, \dot{\phi}_1, \dot{\phi}_2)$ , we obtain*

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = u$$

with  $u \in \mathbb{R}^3$ . Let  $\xi_d(t) \in \mathbb{R}^3$  be a desired  $C^2$  smooth trajectory for the snakeboard. Then, the following partial state and output feedback

$$u = k_p(\xi_1 - \xi_d) + k_d(\xi_2 - \dot{\xi}_d) + \ddot{\xi}_d, \quad k_p, k_d < 0$$

achieves global exponential tracking of the trajectory  $\xi_d$  for the snakeboard.

**Proof.** Set  $e_1 = \xi_1 - \xi_d$  and  $e_2 = \xi_2 - \xi_d$ . Then

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = k_p e_1 + k_d e_2$$

and the  $e_i$ 's vanish exponentially.  $\square$

Proposition 6.2.1 allows exponential tracking of certain “gaits” or sinusoid inputs  $(\psi, \phi_1, \phi_2)$  for locomotion of the snakeboard [50]. Here, the contribution is that instead of algebraic manipulation of the equations of motion (6.32) in [50], we propose a systematic method for reduction and motion generation of a complex example like the snakeboard which is applicable to higher-order underactuated nonholonomic systems as well.

### Locomotion Control of the Snakeboard

The key to locomotion control of the snakeboard is the invariance of the generalized momentum conjugate to  $\theta$  in the Lagrangian reduced system of the snakeboard in equation (6.33). Since, the potential energy of the system is zero and  $\theta$  is an unactuated variable of the reduced system (6.33), the generalized momentum

$$\pi_\theta = \frac{\partial \mathcal{L}_r}{\partial \dot{\theta}} = \tilde{J}_\theta(\phi_1, \phi_2)\dot{\theta} + J_r\dot{\psi} + J_v\dot{\phi}_1 - J_v\dot{\phi}_1$$

is a conserved quantity. Assuming that the system was initially at rest (i.e. had zero initial velocity),  $\pi_\theta = 0, \forall t > 0$  and

$$\dot{\theta} = -\frac{J_r}{\tilde{J}_\theta(\phi_1, \phi_2)}\dot{\psi} - \frac{J_v}{\tilde{J}_\theta(\phi_1, \phi_2)}(\dot{\phi}_1 - \dot{\phi}_2) \quad (6.35)$$

Based on the following assumption, both the last equation and the expression for  $\rho_1, \rho_2$  in (6.30) can be highly simplified.

**Assumption 6.2.1.** Assume  $\phi_1 = \phi_2 = \phi$  and  $|\phi| < \pi/2$ .

**Lemma 6.2.1.** Under Assumption 6.2.1, the following equations hold:

$$\begin{aligned} \rho_1(\phi) &= \frac{l \cos(\theta)}{\tan(\phi)} \\ \rho_2(\phi) &= \frac{l \sin(\theta)}{\tan(\phi)} \\ \dot{\theta} &= -\frac{J_r \tan(\phi)^2}{ml^2 + J_\theta \tan(\phi)^2} \dot{\psi} \end{aligned} \quad (6.36)$$

In addition, the constraint equation for the snakeboard is equivalent to the following nonholonomic constraints

$$\begin{aligned} \sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - l \cos(\phi)\dot{\theta} &= 0 \\ \sin(\theta)\dot{x} - \cos(\theta)\dot{y} &= 0 \end{aligned} \quad (6.37)$$

**Proof.** Setting  $\phi_1 = \phi_2 = \phi$ , we get

$$\begin{aligned}\rho_1 &= l \frac{\cos(\phi_1) \cos(\theta - \phi_2) + \cos(\phi_2) \cos(\theta + \phi_1)}{\sin(\phi_1 + \phi_2)} \\ &= l \frac{\cos(\phi)[\cos(\theta - \phi) + \cos(\theta + \phi)]}{\sin(2\phi)} \\ &= l \frac{\cos(\phi)(2 \cos(\theta) \cos(\phi))}{2 \sin(\phi) \cos(\phi)} \\ &= \frac{l \cos(\theta)}{\tan(\phi)}\end{aligned}$$

Similarly, one obtains  $\rho_2 = l \sin(\theta) / \tan(\phi)$ . Thus,  $\tilde{J}_\theta$  can be rewritten as

$$\tilde{J}_\theta(\phi) = J_\theta + m(\rho_1^2 + \rho_2^2) = J_\theta + \frac{ml^2}{\tan^2(\phi)}$$

From this equation, we have

$$\dot{\theta} = -\frac{J_r}{\tilde{J}_\theta(\phi)} \dot{\psi} = -\frac{J_r \tan^2(\phi)}{ml^2 + J_\theta \tan^2(\phi)} \dot{\psi}$$

In addition, under Assumption 6.2.1, the velocity constraints for the snakeboard can be expressed as

$$\begin{aligned}\sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - l \cos(\phi)\dot{\theta} &= 0 \\ \sin(\theta - \phi)\dot{x} - \cos(\theta - \phi)\dot{y} + l \cos(\phi)\dot{\theta} &= 0\end{aligned}$$

By adding the last two equations, one gets

$$[\sin(\theta + \phi) + \sin(\theta - \phi)]\dot{x} - [\cos(\theta + \phi) + \cos(\theta - \phi)]\dot{y} = 0$$

which can be simplified as

$$\cos(\phi)[\sin(\theta)\dot{x} - \cos(\theta)\dot{y}] = 0$$

Due to  $|\phi| < \pi/2$ , the last constraint reduces to

$$\sin(\theta)\dot{x} - \cos(\theta)\dot{y} = 0$$

and the result follows.  $\square$

Notice that the velocity constraints in (6.37) are exactly the same as the velocity constraints of a car. Roughly speaking, a velocity constraint in the form

$$\sin(\theta)\dot{x} - \cos(\theta)\dot{y} = 0$$

for the snakeboard, implies that the back half of the snakeboard can be removed and two new virtual wheels can be added to the middle of the snakeboard. These virtual

wheels must be along the body of the snakeboard with orientation angles  $\theta$  w.r.t. the horizontal axis. This connection between the snakeboard and the car suggests the following formal equivalence between the two systems.

**Proposition 6.2.2.** *Let  $q = (x, y, \theta, \psi, \phi) \in SE(2) \times S^1 \times S^1$  denote the configuration vector of both the snakeboard and the car with one difference. For the snakeboard,  $\psi$  denotes the angle of the rotor, while  $\psi$  is the angle of the rotation of the rear wheels in a car. Then, the kinematic model of the snakeboard and the car are the same for  $\phi \neq 0$  (i.e. the diffeomorphism that maps the configuration of the snakeboard to the configuration of the car is an identity map). In addition, both the kinematic snakeboard and the kinematic car can be transformed into a chained-form system as the following*

$$\begin{aligned}\dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_3 &= x_2 v_1 \\ \dot{x}_4 &= x_3 v_1\end{aligned}\tag{6.38}$$

where the diffeomorphism

$$(x_1, x_2, x_3, x_4, v_1, v_2) = T(x, y, \theta, \phi, \omega_\psi, \omega_\phi)$$

for the snakeboard is given by

$$\begin{aligned}x_1 &= x \\ x_2 &= \frac{\tan(\phi)}{l \cos(\theta)^3} \\ x_3 &= \tan(\theta) \\ x_4 &= y \\ v_1 &= -\frac{l J_r \cos(\theta) \tan(\phi)}{ml^2 + J_\theta \tan^2(\phi)} \omega_\psi \\ v_2 &= \frac{(1 + \tan^2(\phi))}{l \cos^3(\theta)} \omega_\phi - \frac{3 \sin(\theta) \tan(\phi)^3 J_r}{l \cos(\theta)^3 (ml^2 + J_\theta \tan^2(\phi))} \omega_\psi\end{aligned}\tag{6.39}$$

and the dynamics of  $(\psi, \phi)$  is linear as the following

$$\begin{aligned}\dot{\psi} &= \omega_\psi \\ \dot{\phi} &= \omega_\phi \\ \dot{\omega}_\psi &= u_1 \\ \dot{\omega}_\phi &= u_2\end{aligned}\tag{6.40}$$

**Proof.** The dynamics of the snakeboard with  $\phi_1 = \phi_2 = \phi$  can be expressed as

$$\begin{aligned}\dot{x} &= \rho_1(\phi)\dot{\theta} \\ \dot{y} &= \rho_2(\phi)\dot{\theta} \\ \dot{\theta} &= -\frac{J_r \tan^2(\phi)}{ml^2 + J_\theta \tan^2(\phi)}\dot{\psi} \\ \ddot{\psi} &= u_1 \\ \ddot{\phi} &= u_2\end{aligned}\tag{6.41}$$

where

$$\rho_1(\phi) = \frac{l \cos(\theta)}{\tan(\phi)}, \quad \rho_2(\phi) = \frac{l \sin(\theta)}{\tan(\phi)}$$

Defining

$$w_1 = -\frac{l J_r \tan(\phi)}{ml^2 + J_\theta \tan^2(\phi)}\dot{\psi}$$

the kinematic model of the snakeboard takes the following form

$$\begin{aligned}\dot{x} &= \cos(\theta)w_1 \\ \dot{y} &= \sin(\theta)w_1 \\ \dot{\theta} &= \frac{1}{l} \tan(\phi)w_1 \\ \dot{\phi} &= w_2\end{aligned}\tag{6.42}$$

with control inputs  $(w_1, w_2) \in \mathbb{R}^2$ . Equation (6.42) is exactly the same as the kinematic model of a car with configuration  $(x, y, \theta, \phi)$  where  $w_1$  is the velocity of the rear wheels of the car and  $w_2$  is the velocity of the steering wheel. Now, applying the following change of coordinates and control (which is equivalent to the one in the question)

$$\begin{aligned}x_1 &= x \\ x_2 &= \frac{\tan(\phi)}{l \cos(\theta)^3} \\ x_3 &= \tan(\theta) \\ x_4 &= y \\ v_1 &= \cos(\theta)w_1 \\ v_2 &= \dot{x}_2\end{aligned}\tag{6.43}$$

transforms the kinematic model of the snakeboard into the chained-form nonholonomic system in (6.38).  $\square$

*Remark 6.2.1.* Proposition 6.2.2 reduces the locomotion control for the snakeboard to the stabilization problem for a car which is a rather well-studied problem [63, 4]. This is the first direct evidence between control of a locomotion system and a mobile system that are both transformable into chained-form nonholonomic systems. In

other words, this is an example of a case where the locomotion control for a system reduces to the stabilization of a simple system.

### Geometric Interpretation of $(\rho_1, \rho_2)$ and the Singularity in the Kinematics of the Snakeboard

The functions  $(\rho_1, \rho_2)$  in the kinematics and constraint equations of both the snakeboard and the car have a meaningful geometric interpretation that is stated in the following. Consider a car with a virtual rear wheel in the middle of the real rear wheels and a virtual front wheel. The path and geometry of the rear and front virtual wheels for a car are shown in Figure 6-5. For the snakeboard, the front wheels are the same but the rear wheel is a virtual wheel along the body of the snakeboard at the center of mass  $(x, y)$ . If one fixes the orientation of the front wheel at  $\phi$ . The

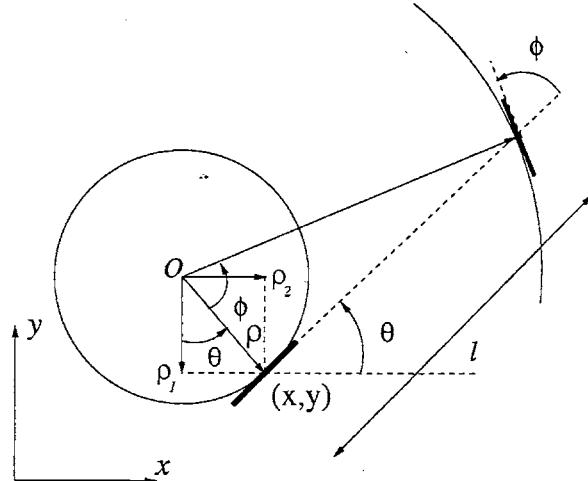


Figure 6-5: The geometry of the paths of the rear and front wheels of the car with a radius of rotation  $\rho$ .

car rotates around the center of rotation  $O$ . The path of both wheels are circles. Let  $\rho$  denote the radius of the rotation of the rear wheel located at point  $p = (x, y)$ . Then, the velocity  $v$  of point  $p$  is tangent to the circle  $(O, \rho)$  at  $p$  and is obtained by

$$v = \rho \dot{\theta} \quad (6.44)$$

Therefore, the vertical and horizontal components of  $v$  are

$$\begin{aligned} \dot{x} &= \rho \cos(\theta) \dot{\theta} \\ \dot{y} &= \rho \sin(\theta) \dot{\theta} \end{aligned} \quad (6.45)$$

In other words, setting  $\rho_1 = \rho \cos \theta$  and  $\rho_2 = \rho \sin \theta$ , the components of  $v$  satisfy the following equations

$$\begin{aligned} \dot{x} &= \rho_1 \dot{\theta} \\ \dot{y} &= \rho_2 \dot{\theta} \end{aligned} \quad (6.46)$$

From Figure 6-5, we have

$$\rho = \frac{l}{\tan(\phi)}$$

Thus,  $\rho_1$  and  $\rho_2$  are given by

$$\rho_1 = \frac{l \cos(\theta)}{\tan(\phi)}, \quad \rho_2 = \frac{l \sin(\theta)}{\tan(\phi)}$$

This is exactly the same as the definition of  $\rho_1, \rho_2$  for the car and the snakeboard with  $\phi_1 = \phi_2 = \phi$ . Notice that  $\rho$  uniquely determined by the orientation angle of the front wheel  $\phi$  and does not depend on  $\theta$ . Apparently, when  $\phi \rightarrow 0$ ,  $\rho \rightarrow \infty$  and  $\rho_1, \rho_2 \rightarrow \infty$ . This is equivalent to the fact that  $\theta$  is constant (or  $\dot{\theta} = 0$ ) and the radius  $\rho$  of a straight line is infinity.

Now, let  $\psi$  denote the angle of rotation of the rear wheel with unit radius in the car. Then,  $v = \rho\dot{\theta} = \dot{\psi}$ , which means

$$\dot{\theta} = \frac{1}{\rho}\dot{\psi} = \frac{\tan(\phi)}{l}\dot{\psi}$$

This gives the final equations of motion for the kinematic car as

$$\begin{aligned}\dot{x} &= \cos(\theta)v \\ \dot{y} &= \sin(\theta)v \\ \dot{\theta} &= \frac{\tan(\phi)}{l}v \\ \dot{\phi} &= w\end{aligned}$$

The last system is obtained without any explicit use of nonholonomic velocity constraints. However, the fact that the rear and front wheels always move on a circle with radius  $\rho$  is equivalent to the assumption that the wheels do not slip. The snakeboard does not satisfy the last equation since it does not have a real rotating rear wheel. Instead, it has a rotor with angle of rotation  $\psi$ . The invariance of the momentum conjugate to  $\theta$  in the reduced Lagrangian system of the snakeboard plays the role of the kinematic equation  $\dot{\theta} = \dot{\psi}/\rho$  in the car. One can observe that defining the curvature as  $\kappa = 1/\rho$ , we have

$$\dot{\theta} = \kappa(\phi)\dot{\psi}$$

where  $\kappa(\phi)$  is the curvature of the circle of rotation of the rear wheel of the car corresponding to the front wheel angle  $\phi$ . If the rear wheels of the car rotate with a constant non-zero velocity  $\dot{\psi}$ , the curvature  $\kappa(\phi)$  plays the role of the control input for the motion of the group element  $(x, y, \theta) \in SE(2)$ .

A rather similar property holds for the snakeboard with arbitrary front wheel and rear wheel angles  $(\phi_1, \phi_2)$ . The geometric interpretation of  $\rho$  is shown in Figure 6-6. In this figure,  $(r_1, r_2)$  denote the radii of rotation corresponding to the front and back

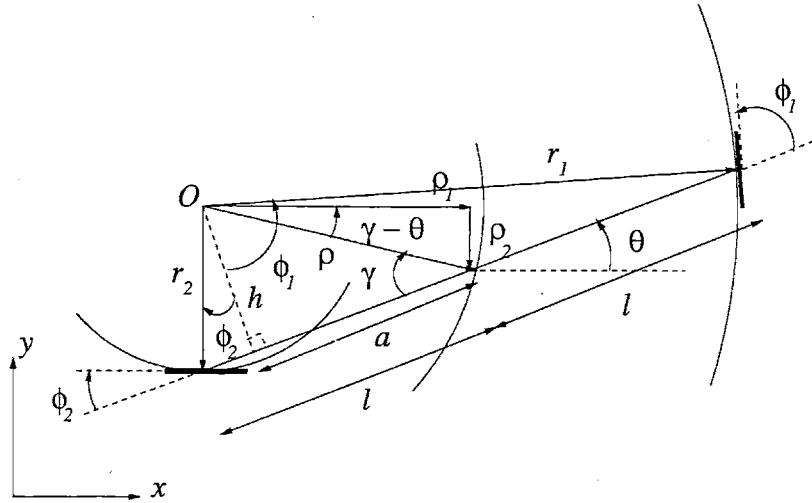


Figure 6-6: The geometry of the paths of the rear and front wheels and the center of mass of the snakeboard with a radius of rotation  $\rho$ .

wheels of the snakeboard. In addition,  $\rho$  is the radius of rotation of the center of mass of the snakeboard. This is due to the midsection theorem in planar geometry. Observe that

$$\rho^2 = a^2 + h^2$$

and the radii of rotation  $r_1, r_2$  satisfy

$$\begin{aligned} r_1^2 &= h^2 + (l + a)^2 \\ r_2^2 &= h^2 + (l - a)^2 \end{aligned}$$

Adding the last two equations gives

$$\rho^2 = \frac{r_1^2 + r_2^2}{2} - l^2$$

In other words, if  $r_1, r_2$  (or  $\phi_1, \phi_2$ ) are constant, then  $\rho$  is constant and it is uniquely determined by  $\phi_1, \phi_2, l$ . This means that the geometric path of  $(x, y)$  is a circle centered at  $O$  with radius  $\rho$ .

An elementary calculation shows that

$$h = \frac{2l}{\tan(\phi_1) + \tan(\phi_2)}$$

From  $h$ , the expressions of  $\rho, r_1, r_2, \gamma$  can be readily determined as functions of the shape variables  $(\phi_1, \phi_2)$ . Particularly, one obtains the following formula for  $\rho(\phi_1, \phi_2)$

$$\rho^2 = a^2 + h^2 = (l - h \tan(\phi_2))^2 + h^2 = l^2 \left[ \frac{(\tan(\phi_1) - \tan(\phi_2))^2 + 4}{(\tan(\phi_1) + \tan(\phi_2))^2} \right]$$

It is rather elementary to prove that the radius of rotation  $\rho$  from the last equation is exactly the same as  $\rho = \sqrt{\rho_1^2 + \rho_2^2}$  for  $\rho_1, \rho_2$  defined in equation (6.30).

### 6.3 Notions of $\epsilon$ -Stabilization and $\epsilon$ -Tracking

In this section, we define new stability/tracking notions for nonlinear systems called  $\epsilon$ -stability/tracking. Consider the following nonlinear system

$$\dot{x} = f(x) \quad (6.47)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$  smooth function.

**Definition 6.3.1.** (virtual equilibrium) We say  $x_0 \in \mathbb{R}^n$  is a *virtual equilibrium* of (6.47) within a distance  $\epsilon > 0$ , if there exists an equilibrium  $x_0^*$  of (6.47) that is within a distance  $\epsilon$  from  $x_0$ , i.e.  $\exists x_0^* \in \mathbb{R}^n : f(x_0^*) = 0, \|x_0^* - x_0\| \leq \epsilon$ . We call  $x_0^*$  the *adjacent equilibrium* to  $x_0$ .

**Definition 6.3.2.** ( $\epsilon$ -stable virtual equilibrium) We call  $x_0$  an (exponentially or asymptotically)  $\epsilon$ -stable virtual equilibrium of (6.47), if there exists an adjacent equilibrium  $x_0^*$  within a distance  $\epsilon > 0$  from  $x_0$  that is (exponentially or asymptotically) stable for (6.47) in the sense of Lyapunov.

We are interested in considering the solutions of (6.47) under specific classes of parameterized transformations where the notion of  $\epsilon$ -stability of solutions is useful. Assume there exists a global diffeomorphism  $\psi(x, \lambda) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  in  $x$  that is continuous in  $\lambda$  and satisfies  $\psi(x, \lambda) = x \implies \lambda = 0$ . By continuity at  $\lambda = 0$ , we have

$$\forall \epsilon > 0, \exists \lambda = \lambda(\epsilon) : \|\psi(x, \lambda) - x\| \leq \epsilon \quad (6.48)$$

We call  $\psi(x, \lambda)$  a *near identity diffeomorphism* for  $\|\lambda\| \ll 1$ . Denote the inverse of  $y = \psi(x, \lambda)$  by  $x = \phi(y, \lambda)$  so that

$$\phi(\psi(x, \lambda), \lambda) = x, \psi(\phi(y, \lambda), \lambda) = y$$

By definition  $\phi(x, 0) = x$  and  $\phi(x, \lambda) = x \implies \lambda = 0$ . Also,  $\phi(x, \lambda)$  is continuous w.r.t.  $\lambda$ . Applying the following change of coordinates

$$x_\lambda = \psi(x, \lambda)$$

one obtains a parameterized nonlinear system that we call  $\lambda$ -system

$$\dot{x}_\lambda = \tilde{f}(x_\lambda, \lambda) \quad (6.49)$$

where

$$\tilde{f}(x_\lambda, \lambda) = \left[ \frac{\partial \psi(x, \lambda)}{\partial x} f(x) \right]_{x=\phi(x_\lambda, \lambda)}$$

Apparently, any equilibrium point of (6.47) is an equilibrium point of (6.49) and vice versa.

**Definition 6.3.3.** ( $\epsilon$ -stability under transformation) For any  $\epsilon > 0$ , let  $\lambda = \lambda(\epsilon)$  be such that (6.48) holds. We say  $x = x^0$  is (exponentially/asymptotically)  $\epsilon$ -stable virtual equilibrium of (6.47) under the transformation  $\psi(x, \lambda)$ , iff  $x_\lambda = x^0$  is a (exponentially/asymptotically) stable equilibrium of the  $\lambda$ -system (6.49) in the sense of Lyapunov with  $\lambda = \lambda(\epsilon)$ , or equivalently iff  $x_\epsilon^0 = \phi(x^0, \lambda(\epsilon))$  is (an) a (exponentially/asymptotically) stable equilibrium of (6.47) in the sense of Lyapunov, i.e.  $x_\epsilon^0$  is an adjacent equilibrium to  $x^0$  for the  $\lambda$ -system.

The following result is a common sense corollary of the definition of  $\epsilon$ -stability which means the original system has an asymptotically stable equilibrium point near  $x_0$  in the usual sense of stability.

**Corollary 6.3.1.** If  $x^0$  is an (exponentially/asymptotically)  $\epsilon$ -stable equilibrium of (6.47) under the transformation  $\psi(x, \lambda)$ , then there exists an equilibrium point  $x_\epsilon^0$  of (6.47) that is (exponentially/asymptotically) stable equilibrium of (6.47) in the sense of Lyapunov within a distance  $\epsilon > 0$  from  $x^0$ , i.e.  $x_\epsilon^0$ .

**Proof.** Let  $x_\epsilon^0 = \phi(x^0, \lambda(\epsilon))$ , then

$$\|x_\epsilon^0 - x^0\| = \|x_\epsilon^0 - \psi(x^0, \lambda(\epsilon))\| < \epsilon$$

□

*Remark 6.3.1.* Two simple examples of  $\psi(x, \lambda)$  are the following

$$\psi(x, \lambda) = x + \lambda \vec{u}, \quad \lambda \in \mathbb{R}, \|\vec{u}\| = 1$$

and

$$\psi(x, \lambda) = x + A\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^p, A \text{ has full rank}$$

Now, consider a nonlinear control system with a dynamic state feedback

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{\lambda} &= g(x, \lambda) \end{aligned} \tag{6.50}$$

Let  $\psi(x, \lambda)$  be a global diffeomorphism as before. Applying the change of coordinates

$$x_\lambda = \psi(x, \lambda)$$

we obtain the following  $\lambda$ -system

$$\begin{aligned} \dot{x}_\lambda &= \tilde{f}(x_\lambda, \lambda, u) \\ \dot{\lambda} &= \tilde{g}(x_\lambda, \lambda) \end{aligned} \tag{6.51}$$

where

$$\begin{aligned}\tilde{f}(x, \lambda, u) &= \left[ \frac{\partial \psi(x, \lambda)}{\partial x} f(x, u) + \frac{\partial \psi(x, \lambda)}{\partial \lambda} g(x, \lambda) \right]_{x=\phi(x_\lambda, \lambda)} \\ \tilde{g}(x_\lambda, \lambda) &= g(\phi(x_\lambda, \lambda), \lambda)\end{aligned}$$

**Definition 6.3.4.** ( $\epsilon$ -stabilization) For a fix  $\epsilon > 0$ , let  $\lambda_f = \lambda(\epsilon)$  satisfy (6.48). Let  $x^0$  be a desired equilibrium point of the system with zero control, or  $f(x^0, 0) = 0$ . We say the dynamic state feedback

$$\begin{aligned}u &= k(x, \lambda) \\ \dot{\lambda} &= g(x, \lambda)\end{aligned}\tag{6.52}$$

achieves *globally asymptotic  $\epsilon$ -stabilization* of  $x^0$  for (6.50) under the transformation  $\psi(x, \lambda)$ , iff for the closed-loop  $\lambda$ -system

$$\begin{aligned}\dot{x}_\lambda &= \tilde{f}(x_\lambda, \lambda, \tilde{k}(x_\lambda, \lambda)) \\ \dot{\lambda} &= \tilde{g}(x_\lambda, \lambda)\end{aligned}$$

where  $\tilde{k}(x_\lambda, \lambda) = k(\phi(x_\lambda, \lambda), \lambda)$ ,  $(x^0, \lambda_f)$  is a globally asymptotically stable equilibrium in the sense of Lyapunov, or equivalently  $x_\epsilon^0 = \phi(x^0, \lambda_f)$  and  $\lambda_f$  are globally asymptotically stable equilibriums of (6.50).

The notion of  $\epsilon$ -tracking for a nonlinear control system  $\dot{x} = f(x, u)$  can be defined for an equivalent transformed control system with dynamic state feedback based on the notion of  $\epsilon$ -stabilization.

**Definition 6.3.5.** ( $\epsilon$ -tracking) Consider the following nonlinear system with input  $u$  and output  $y$

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x)\end{aligned}\tag{6.53}$$

We say a control law  $u$  achieves *asymptotic  $\epsilon$ -tracking* of a trajectory  $y_d(\cdot)$  for (6.53) under transformation  $\psi(x, \lambda)$ , if there exists a control law  $u$  that achieves asymptotic tracking of the desired trajectory  $y_\lambda = y_d$  for

$$\begin{aligned}\dot{x}_\lambda &= \tilde{f}(x_\lambda, \lambda, u) \\ \dot{\lambda} &= \tilde{g}(x_\lambda, \lambda) \\ y_\lambda &= h(\phi(x_\lambda, \lambda))\end{aligned}\tag{6.54}$$

## 6.4 Global Exponential $\epsilon$ -Stabilization/Tracking for a Mobile Robot

In this section, we design a nonlinear controller for global exponential  $\epsilon$ -stabilization and  $\epsilon$ -tracking of a nonholonomic mobile robot (Figure 6-2).

### 6.4.1 Dynamics of a Mobile Robot in $SE(2)$

In the majority of previous control design strategies for this nonholonomic mobile robot, the kinematic model of the robot is used and the input torque are then obtained using adding an integrator and high-gain control. Here, we directly use the dynamic model of the mobile robot as the following

$$\begin{aligned}\dot{x}_1 &= \cos(\theta)v_1 \\ \dot{x}_2 &= \sin(\theta)v_1 \\ \dot{\theta} &= v_2 \\ \dot{v}_1 &= \tau_1 \\ \dot{v}_2 &= \tau_2\end{aligned}\tag{6.55}$$

Since this mobile robot moves on a surface, a more appropriate form of representation of the dynamics of the mobile robot is in  $SE(2)$ .  $SE(2)$  is the Special Euclidean group in  $\mathbb{R}^2$  with elements  $g = (R, x)$  where  $R$  is the rotation matrix satisfying  $R^T R = I$ ,  $\det(R) = 1$  (i.e.  $R \in SO(2)$ ) and  $x \in \mathbb{R}^2$ . Equivalently  $g$  can be defined as

$$g = \begin{bmatrix} R & x \\ 0 & I \end{bmatrix}$$

where the group product  $g_1 g_2$  is matrix multiplication. From (6.55), the dynamics of the mobile robot in  $SE(2)$  can be expressed as the following

$$\begin{aligned}\dot{x} &= (R e_1) v_1 \\ \dot{R} &= R \hat{v}_2 \\ \dot{v} &= \tau\end{aligned}\tag{6.56}$$

where

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

and the skew-symmetric matrix  $\hat{v}_2$  is defined as follows

$$\hat{v}_2 = \begin{bmatrix} 0 & -v_2 \\ v_2 & 0 \end{bmatrix}$$

or equivalently the evolution of the group element  $g$  in (6.56) can be expressed as

$$\dot{g} = g \cdot \Omega, \dot{v} = \tau$$

where  $\Omega \in \mathfrak{se}(2)$  (i.e. the Lie algebra associated with  $SE(2)$ )

$$\Omega = \begin{bmatrix} \hat{v}_2 & v_1 \\ 0 & 0 \end{bmatrix}$$

We use the representation in (6.56) as the model of this nonholonomic mobile robot.

### 6.4.2 Near-Identity Diffeomorphism

Consider the following *near identity smooth change of coordinates*

$$z = \psi(x, \lambda) := x + \lambda R e_1 \quad (6.57)$$

where  $\lambda \in \mathbb{R}$ ,  $e_i$  is the  $i$ th standard basis in  $\mathbb{R}^2$ , and  $R = R(\theta)$  is the rotation matrix by  $\theta$  in  $\mathbb{R}^2$ . This transformation is schematically depicted in Figure 6-7. The inverse

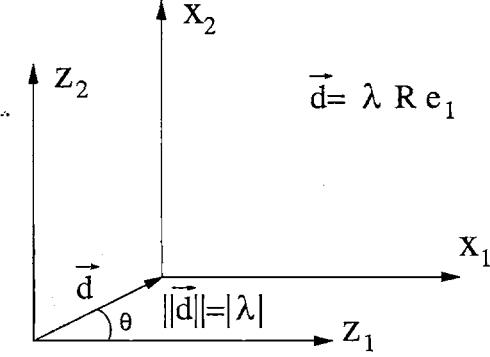


Figure 6-7: A near identity transformation with a parameter  $|\lambda| \ll 1$ .

of this transformation is a near identity diffeomorphism given by

$$x = \phi(z, \lambda) = z - \lambda R e_1$$

Apparently, for all  $x \in \mathbb{R}^2$  the following property holds

$$\|\psi(x, \lambda) - x\| = \|\lambda R e_1\| = |\lambda|$$

and for all  $\epsilon > 0$ ,  $\lambda = \epsilon$  achieves the property

$$\|\psi(x, \lambda) - x\| \leq \epsilon$$

Notice that these properties of the near identity diffeomorphism  $\psi(x, \lambda)$  are true for  $(R, x) \in SE(n)$  with  $n \geq 2$  and are not restricted to the special case of  $n = 2$ . The dynamics of the system in  $z$ -coordinates (i.e.  $\lambda$ -system) can be expressed as

$$\begin{aligned} \dot{z} &= R_\lambda v + (R_\lambda e_1)\dot{\lambda} \\ \dot{R}_\lambda &= R_\lambda \hat{v}_2(\lambda, \dot{\lambda}) \\ \dot{v} &= \tau \end{aligned} \quad (6.58)$$

where  $v = (v_1, v_2)^T$  and  $R_\lambda, \hat{v}_2$  are given by

$$R_\lambda = \begin{bmatrix} \cos(\theta) & -\lambda \sin(\theta) \\ \sin(\theta) & \lambda \cos(\theta) \end{bmatrix}, \quad \hat{v}_2(\lambda, \dot{\lambda}) = \begin{bmatrix} 0 & -\lambda v_2 \\ \frac{1}{\lambda} v_2 & \frac{\dot{\lambda}}{\lambda} \end{bmatrix}$$

Note that for  $\lambda \neq 1$ ,  $R_\lambda$  is not a rotation matrix and  $R_\lambda^T R_\lambda \neq I$ . In fact, we have

$$R_\lambda^T R_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 \end{bmatrix}, \quad \det R_\lambda = \lambda$$

Thus, for  $\lambda \neq 0$ , the matrix  $R_\lambda$  is invertible.

### 6.4.3 Control Design for $\epsilon$ -Stabilization/Tracking

In this section, we present our main  $\epsilon$ -stabilization/tracking results for a nonholonomic mobile robot.

**Proposition 6.4.1.** *Any desired position  $x_d \in \mathbb{R}^2$  for the nonholonomic mobile robot in (6.56) can be rendered globally exponentially  $\epsilon$ -stable using a smooth dynamic state feedback in an explicit form. Equivalently, this feedback law globally exponentially stabilizes (i.s.o. Lyapunov) an equilibrium  $x_d^\epsilon$  of (6.56) with a distance  $\epsilon$  from  $x_d$ , i.e.  $\|x_d^\epsilon - x_d\| = \epsilon$ .*

**Proof.** Fix an  $\epsilon > 0$ , and let the dynamics of  $\lambda$  be as

$$\dot{\lambda} = -c_\lambda(\lambda - \lambda_f), \quad \lambda(0) = \lambda_0 > \epsilon, \quad \lambda_f = \epsilon, \quad c_\lambda > 0$$

Clearly,  $\lambda = \lambda_f$  is exponentially asymptotically stable for this system and  $\lambda(t) \geq \epsilon > 0, \forall t \geq 0$ . The task in the question is to stabilize the position of the mobile robot to  $x = x_d$  (the orientation of the mobile robot can be trivially exponentially stabilized by setting  $v_1 = 0$  at  $x = x_d$ ). Thus, we set the desired equilibrium of the  $\lambda$ -system in (6.58) to  $z_d = x_d$ . The goal is to globally exponentially stabilize  $z = z_d$  for the  $\lambda$ -system. This renders the equilibrium  $x_d^\epsilon = \phi(x_d, \epsilon)$  of the original system is (6.56) globally exponentially stable (i.s.o. Lyapunov). But  $\|x_d^\epsilon - x_d\| = \epsilon$ , thus this achieves global exponential  $\epsilon$ -stabilization of  $x_d$  for the original system (6.56). To render  $z_d$  globally exponentially stable for the  $\lambda$ -system in (6.56) can be rendered globally exponentially stable. For doing so, let us calculate  $\ddot{z}$ . We have

$$\begin{aligned} \ddot{z} &= R_\lambda \dot{v} + \dot{R}_\lambda v + (\dot{R}_\lambda e_1) \dot{\lambda} + (R_\lambda e_1) \ddot{\lambda} \\ &= R_\lambda \tau + R_\lambda \hat{v}_2(\lambda, \dot{\lambda}) v + R_\lambda (\hat{v}_2(\lambda, \dot{\lambda}) - c_\lambda I) e_1 \dot{\lambda} \end{aligned}$$

Thus, defining the change of control

$$\tau = R_\lambda^{-1} u - \hat{v}_2(\lambda, \dot{\lambda}) v - (\hat{v}_2(\lambda, \dot{\lambda}) - c_\lambda I) \dot{\lambda} e_1$$

where  $\dot{\lambda} = -c_\lambda(\lambda - \epsilon)$ , we get

$$\ddot{z} = u$$

which is a two-dimensional double-integrator. Therefore, applying

$$u = -c_p(z - z_d) - c_d\dot{z}, \quad c_p, c_d > 0$$

or

$$u = -c_p(z - z_d) - c_d(R_\lambda v + R_\lambda e_1 \dot{\lambda})$$

renders the equilibrium point  $z = z_d$  of the  $\lambda$ -system globally exponentially stable. The overall smooth dynamic state feedback is in the form

$$\begin{aligned}\tau &= -c_p R_\lambda^{-1}(z - z_d) - c_d v - c_d \dot{\lambda} e_1 - \hat{v}_2(\lambda, \dot{\lambda})v - (\hat{v}_2(\lambda, \dot{\lambda}) - c_\lambda I)\dot{\lambda} e_1 \\ \dot{\lambda} &= -c_\lambda(\lambda - \epsilon)\end{aligned}\quad (6.59)$$

□

Figures 6-8 and 6-9 show the path of the mobile robot starting at position  $x = (2, 3)^T$  for the initial orientation angles  $\theta = k\pi/4, k = 0, \dots, 7$ . The trajectories for the position and input torques (controls) are shown in Figure 6-10

Similar results for trajectories starting at  $x = (0, 2)^T$  with the initial orientation angles  $\theta = 0, \pi/4, \pi/2, 3\pi/2$  are shown in Figure 6-11 (for the complement angles the trajectories are symmetric w.r.t. vertical axis). Also, trajectories of the position and input torques (controls) for initial position  $x = (0, 2)^T$  and orientation  $\theta = 0$  are shown in Figure 6-12. These results demonstrate that the controller efficiently stabilizes the origin for this nonholonomic mobile robot. Each trajectory exponentially converges to a point within a distance  $\epsilon = 0.01$  from the desired equilibrium  $x_d = 0$ . This is sufficiently close to the origin for all practical purposes. The values of the parameters in all simulations for the mobile robot were chosen as  $c_\lambda = 1, c_p = 1, c_d = 2, \lambda(0) = 0.5, \lambda_f = \epsilon = 0.01$ . The trace trajectories of this nonholonomic robot are shown in Figure 6-13. Based on this figure, graphically, the robot makes a turn to face the desired destination (i.e. the origin) and goes exponentially fast towards the destination.

Asymptotic  $\epsilon$ -tracking of a desired output  $z_d$  can be obtained based on the following result.

**Proposition 6.4.2.** *The following dynamic state feedback law achieves asymptotic  $\epsilon$ -tracking for the desired output  $y_d(\cdot)$  of the nonholonomic mobile robot (6.55)*

$$\begin{aligned}\tau &= -c_p R_\lambda^{-1}z - c_d v - c_d \dot{\lambda} e_1 - \hat{v}_2(\lambda, \dot{\lambda})v - (\hat{v}_2(\lambda, \dot{\lambda}) - c_\lambda I)\dot{\lambda} e_1 \\ &\quad + R_\lambda^{-1}(c_p y_d + c_d \dot{y}_d + \ddot{y}_d) \\ \dot{\lambda} &= -c_\lambda(\lambda - \epsilon)\end{aligned}\quad (6.60)$$

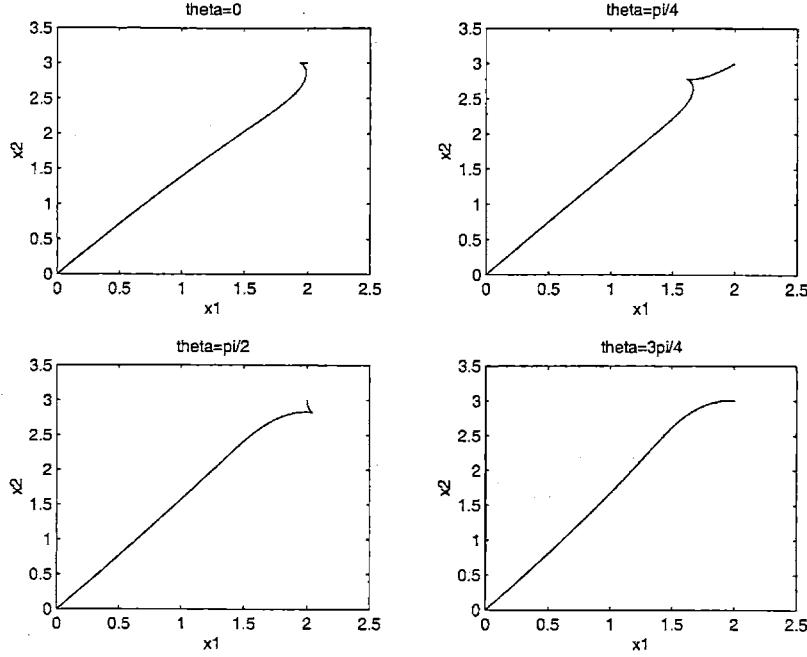


Figure 6-8: Trajectories of the nonholonomic mobile robot in  $(x_1, x_2)$ -plane for initial position  $x = (2, 3)^T$  ( $v = 0$ ) and orientation angles  $\theta = 0, \pi/4, \pi/2, 3\pi/4$ .

**Proof.** Set the desired trajectory in  $z$ -coordinates to  $z_d = y_d$ . Then, the following feedback

$$u = -c_p(z - z_d) - c_d(\dot{z} - \dot{z}_d) + \ddot{z}_d$$

guarantees that the error  $e = z - z_d$  globally asymptotically converges to zero and the result follows from the relation between  $\tau$  and  $u$ .  $\square$

Figure 6-14 demonstrates simulation results of the  $\epsilon$ -tracking for a nonholonomic robot starting at position  $(4, 4)$  with orientation angle  $\pi/2$ . The desired trajectory is an ellipse  $(x, y) = (3 \sin t, 4 \cos t)$ . Clearly, the robot very quickly converges to an  $\epsilon$ -neighborhood of the desired trajectory (e.g.  $\epsilon = 0.01$ ).

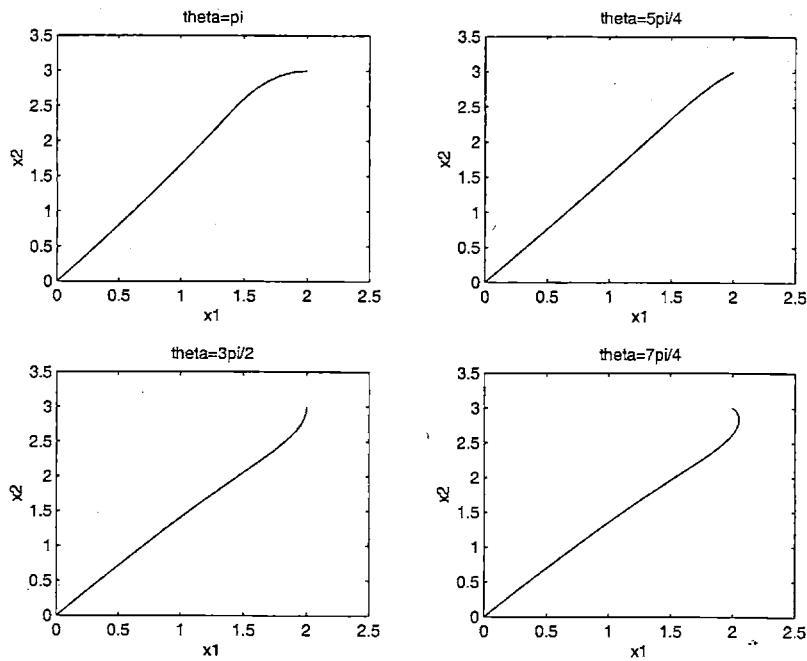


Figure 6-9: Trajectories of the nonholonomic mobile robot in  $(x_1, x_2)$ -plane for initial position  $x = (2, 3)^T$  ( $v = 0$ ) and orientation angles  $\theta = \pi, 5\pi/4, 3\pi/2, 7\pi/4$ .

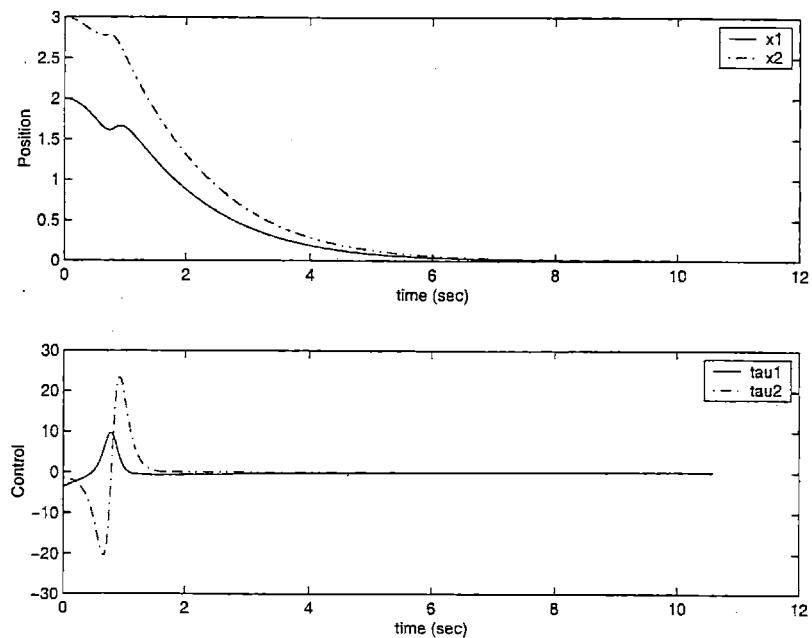


Figure 6-10: Trajectories of the nonholonomic mobile robot in  $(x_1, x_2)$ -plane starting at  $x = (2, 3)^T$  ( $v = 0$ ) with angle  $\theta = \pi/4$

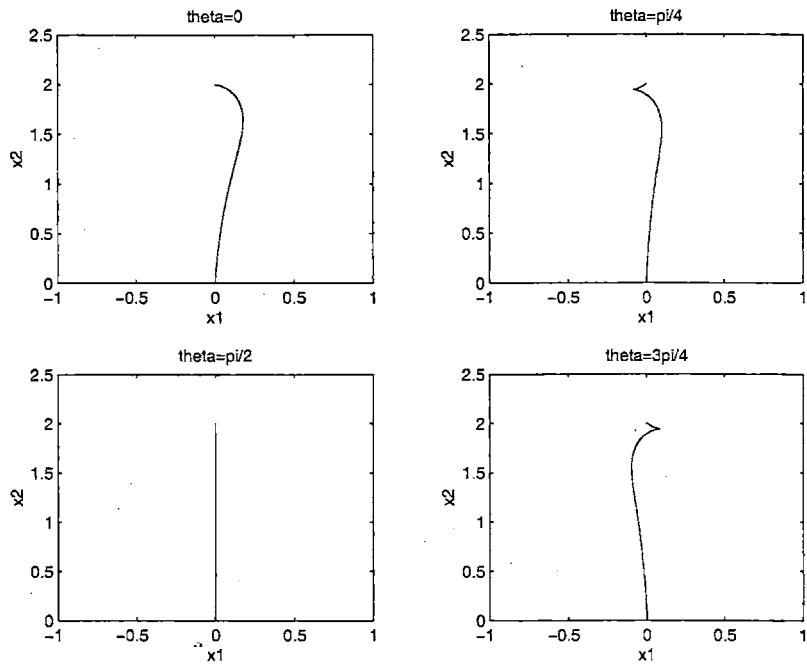


Figure 6-11: Trajectories of the nonholonomic mobile robot in  $(x_1, x_2)$ -plane for initial position  $x = (0, 2)^T$  ( $v = 0$ ) and initial orientation angles  $\theta = 0, \pi/4, \pi/2, 3\pi/4$ .

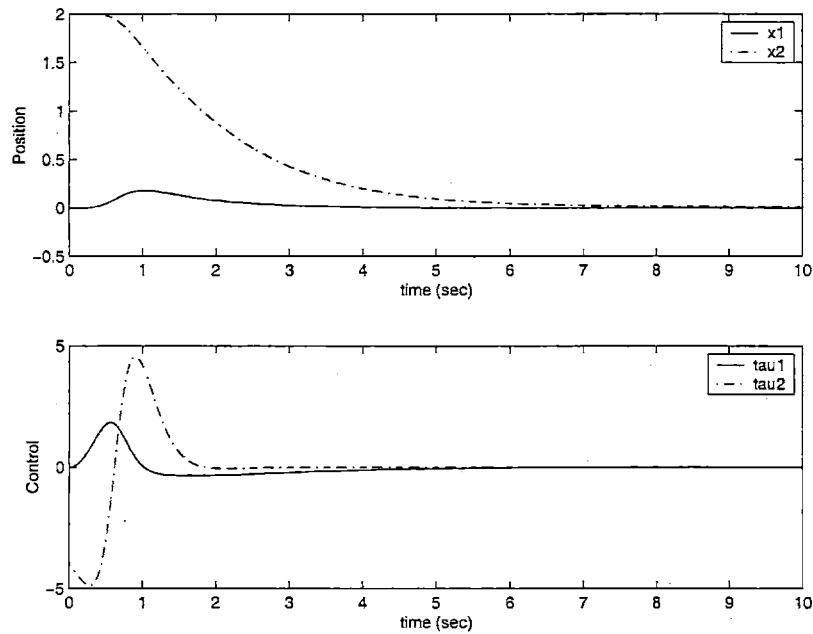


Figure 6-12: Evolution of position and controls for the nonholonomic mobile robot in  $(x_1, x_2)$ -plane starting at  $x = (0, 2)^T$  ( $v = 0$ ) with angle  $\theta = 0$

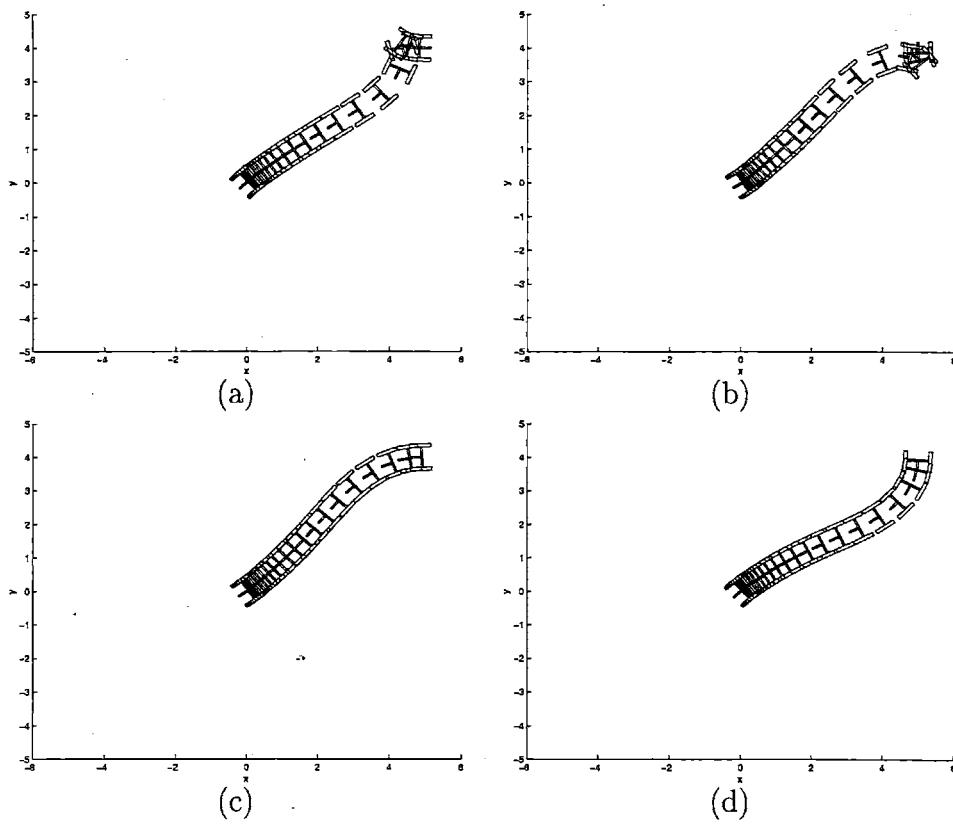


Figure 6-13: Trace trajectories of a two-wheel nonholonomic robot with initial position  $(5, 4)$  and heading angles (a)  $\theta = 0$ , (b)  $\theta = \pi/2$ , (c)  $\theta = \pi$ , and (d)  $\theta = 3\pi/2$ .

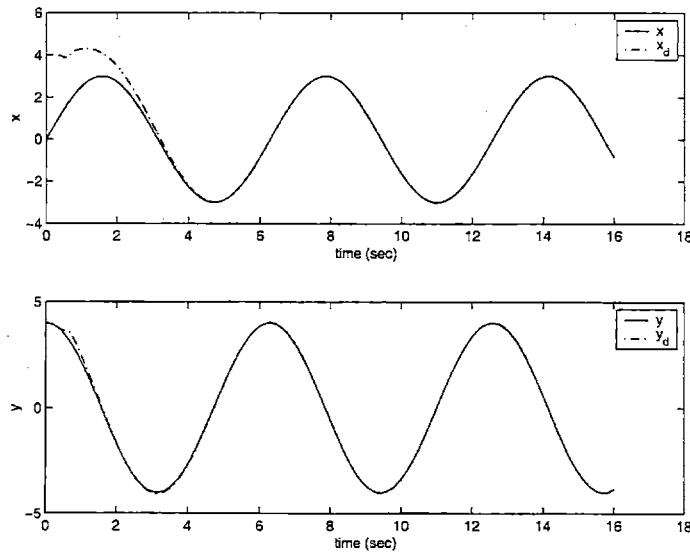


Figure 6-14: Position trajectory tracking for a mobile robot from the initial point  $(4, 4, \pi/2)$ .

# Chapter 7

## Control of Nonlinear Systems in Nontriangular Normal Forms

This chapter is devoted to control of nonlinear systems in normal forms that possess nontriangular structures. Control of nonlinear systems in *triangular forms*, i.e. (*strict*) *feedback and feedforward forms*, has been extensively studied in the past decade. However, few results are available that deal with nonlinear systems with *nontriangular structures*. later, we will be more precise on what we mean by nontriangular forms and present the challenging problems and stabilization limitations of nonlinear systems in nontriangular forms. The following notations are convenient in this chapter.

**Notation.** *GAS* stands for globally asymptotically stable.

**Notation.**  $\|\cdot\| = \langle x, x \rangle^{\frac{1}{2}}$  denotes the Euclidean norm and  $\|\cdot\|_p$  denotes the  $L_p$ -norm in  $\mathbb{R}^n$ .

**Notation.** (Lie derivative) Let  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be smooth functions. The *directional derivative* of  $f(x)$  with respect to  $g(x)$  is defined as any of the following equivalent forms

$$L_g f(x) := \frac{\partial f(x)}{\partial x} g(x) = \langle \nabla f(x), g(x) \rangle = Df(x) \cdot g(x) \quad (7.1)$$

where  $\langle x, y \rangle = x^T y$  denotes the inner product in  $\mathbb{R}^n$ . Moreover, the  $k$ th order Lie derivative of  $f$  w.r.t.  $g$  is denoted by  $L_g^{(k)} f$  and is defined as

$$L_g^{(k)} f(x) := L_g(L_g^{((k-1)} f(x)), \quad k \in \mathbb{Z}_+ \quad (7.2)$$

with  $L_g^{(0)} f(x) = f(x)$ .

The outline of this chapter is as follows: First, we provide some background on stabilization of nonlinear systems in triangular forms, namely, (*strict*) feedback and feedforward forms. Secondly, we focus on control of nonlinear systems with nontriangular structures.

## 7.1 Introduction

In the preceding chapters, we obtained three types of normal forms for underactuated mechanical systems. Namely, cascade nonlinear systems in strict feedback form, strict feedforward form, and different variations of nontriangular linear-quadratic forms. In the past, effective methods have been developed for stabilization of nonlinear systems in triangular forms (i.e. feedback/feedforward) [57, 36, 102, 103, 101, 56, 80]. Namely, the *backstepping* procedure [57, 36, 103] and *forwarding* methods [102, 101, 56, 80] are analytic tools for stabilization of nonlinear systems in feedback forms and feedforward forms, respectively.

The class of nonlinear systems affine in control

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^p \quad (7.3)$$

is one of the most well-studied types of nonlinear systems. Assume there exists an output  $y = h(x)$  that has uniform relative degree  $r \leq n$  with respect to the input  $u$  over an open neighborhood  $U$  of  $x = x_0 \in \mathbb{R}^n$ , i.e. there exist  $\alpha(x), \beta(x)$  such that the following properties hold over  $U$  for a constant  $r$

$$\begin{aligned} L_g^{(k)} h(x) &= 0, \quad \forall k \in \{0, 1, \dots, r-1\} \\ y^{(r)} &= \alpha(x)u + \beta(x) \end{aligned}$$

where  $\alpha(x)$  is an invertible matrix. According to [36], there exists a change of coordinates and control in the form

$$(z, \xi) = \Phi(x), \quad v = \alpha(x)u + \beta(x)$$

that transforms the nonlinear system in (7.3) into the *Byrnes-Isidori normal form* of degree  $r \leq n$

$$\begin{aligned} \dot{z} &= F(z, \xi_1, \xi_2, \dots, \xi_r) \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_r &= v \end{aligned} \quad (7.4)$$

Here,  $\Phi(x)$  is a diffeomorphism over  $U$ . If  $U = \mathbb{R}^n$ , then  $y = h(x)$  has global uniform relative degree  $r$  and the aforementioned change of coordinates is global. The  $z$ -subsystem

$$\dot{z} = F(z, \xi)$$

in (7.4) with  $\xi = (\xi_1, \dots, \xi_r)$  is called the *zero-dynamics* of (7.3) [36]. Given  $\xi_1 = \xi_2 = \dots = \xi_r = 0$ , if  $z = 0$  is asymptotically stable for the zero-dynamics, then both the output  $y = h(x)$  and the zero-dynamics are called *minimum phase*.

Under the condition that the vector field  $F$  in (7.4) does not depend on  $(\xi_2, \dots, \xi_r)$ ,

the nonlinear system in (7.4) reduces to

$$\begin{aligned}\dot{z} &= F(z, \xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ &\dots \\ \dot{\xi}_r &= v\end{aligned}\tag{7.5}$$

which is in *strict feedback form* [36, 57]. In general, if at most one of  $\xi_1, \dots, \xi_r$  appears in  $f$ , it is known how to stabilize the composite system in (7.4) under appropriate technical conditions (see [36], chapter 9). The importance of the strict feedback normal form in (7.5) is due to the existence of the following theorem called *standard backstepping* [36].

**Theorem 7.1.1.** (*standard backstepping*) Consider the nonlinear system in (7.5) and assume there exists a smooth state feedback  $\xi_1 = \alpha(z)$  with  $\alpha(0) = 0$  such that for the closed-loop zero-dynamics

$$\dot{z} = F(z, \alpha(z))\tag{7.6}$$

the origin  $z = 0$  is globally asymptotically stable (GAS). Define  $K_1(z) := \alpha'(z)$ . Then, there exists a state feedback

$$\xi_{j+1} = K_{j+1}(z, \xi_1, \dots, \xi_j), \quad j = 1, \dots, r\tag{7.7}$$

in an explicit form where  $\xi_{r+1} := v$  that renders the origin  $(z, \xi_1, \dots, \xi_j) = 0$  GAS for the closed-loop  $(z, \xi_1, \dots, \xi_j)$ -subsystem (i.e.  $j$ th subsystem) of (7.5). In addition, let  $V_0(z)$  be a smooth positive definite and proper Lyapunov function associated with the closed-loop zero-dynamics subsystem satisfying  $\dot{V}_0(z) := \nabla V_0(z) \cdot f(z, \alpha(z)) < 0, \forall z \neq 0$ . Then, the the Lyapunov function  $V_j(z, \xi_1, \dots, \xi_j), j = 1, \dots, r$  associated with the  $j$ th subsystem is given by

$$V_j(z, \xi_1, \dots, \xi_j) = V_0(z) + \frac{1}{2}(\xi_1 - K_1(z))^2 + 1_{\{j>1\}}(j) \frac{1}{2} \sum_{i=2}^j (\xi_i - K_i(z, \xi_1, \dots, \xi_{i-1}))^2$$

and satisfies  $\dot{V}_j < 0, \forall (z, \xi_1, \dots, \xi_j) \neq 0$  ( $1_A(\cdot)$  denotes the indicator function of the set  $A$ ).

**Proof.** See remark 7.3.2. □

In the next section, we provide a constructive method for obtaining both the state feedback  $K_j$  and the Lyapunov function  $V_j$  in a recursive manner (this can be found in [36], also see theorem 7.3.1 here).

In contrast to nonlinear systems in strict feedback forms, the following normal form

$$\begin{aligned}\dot{z} &= F(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{7.8}$$

has a *nontriangular structure*. Throughout this chapter, we assume  $F(z, \xi_1, \xi_2)$  depends on both  $\xi_1, \xi_2$  unless otherwise is stated. Our main focus in this chapter is on stabilization of nonlinear systems in the form (7.8). Our main motivation is that (almost) all nontriangular normal forms for underactuated systems are special classes of Byrnes-Isidori normal form with a vector double-integrator linear part as in (7.8). In addition, the following nontriangular normal form with a triple-integrator linear part

$$\begin{aligned}\dot{z} &= F(z, \xi_1, \xi_2, \xi_3) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ \dot{\xi}_3 &= v\end{aligned}\tag{7.9}$$

appeared in the study of flexible-link robots and a helicopter model with the effect of the input of the attitude dynamics in the translational dynamics. The actual input in both cases is  $\xi_3$  which is augmented with an integrator at the input.

*Remark 7.1.1.* Notice that by redefining the state variables and zero-dynamics vector field as

$$\zeta := \text{col}(z, \xi_1), \eta_1 := \xi_2, \eta_2 := \xi_3, F_0 := \text{col}(F(z, \xi_1, \xi_2, \xi_3), \xi_2)$$

The dynamics of system (7.9) can be rewritten as the following

$$\begin{aligned}\dot{\zeta} &= F_0(\zeta, \eta_1, \eta_2) \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= v\end{aligned}\tag{7.10}$$

which is a Byrnes-Isidori normal form of degree 2 (the same as (7.8)). Roughly speaking, this implies that certain stabilization problems for the class of nonlinear systems in (7.9) with a triple-integrator linear part reduce to equivalent problems for the class of nonlinear systems in (7.8) with a (vector) double-integrator linear part.

Stabilization of Byrnes-Isidori normal forms with a double-integrator linear part as in (7.8) is currently considered a major open problem [36]. Particular cases of interest include normal forms for mechanical systems where  $F(z, \xi_1, \xi_2)$  has quadratic dependence in the velocity  $\xi_2$  (e.g. the beam-and-ball system). Our main contribution in this chapter is to introduce an extension of the backstepping procedure for stabilization of important special classes of normal form (7.8). This new backstepping procedure is considerably different from standard backstepping procedure that relies on construction of appropriate Lyapunov functions.

Based on standard backstepping procedure, if a controller  $\xi_2 = K_1(z, \xi_1)$  exists that renders the origin GAS for

$$\begin{aligned}\dot{z} &= F(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2\end{aligned}\tag{7.11}$$

Then, the existence of a globally stabilizing state feedback  $v = K_2(z, \xi_1, \xi_2)$  for the

composite system in (7.8) is trivial. In other words, the main stabilization problem of interest is stabilization of nonlinear system (7.11) which is non-affine in control  $\xi_2$ .

Observe that it is nontrivial to formulate a stabilization problem for the nonlinear system (7.8) analogous to the one for a cascade system in strict feedback form, unless  $\alpha(z) \equiv 0$ . We formulate this problem for the case of  $\alpha(z) \equiv 0$  as the following:

**Problem 7.1.1.** *Consider the nonlinear system in (7.8). Assume that the zero-dynamics of (7.8) is globally minimum phase, i.e. given  $\xi_1 = \xi_2 = 0$ , the origin  $z = 0$  is GAS for the zero-dynamics*

$$\dot{z} = F(z, 0, 0)$$

*Find sufficient conditions such that there exists a state feedback  $v = k(z, \xi_1, \xi_2)$  that globally asymptotically stabilizes  $(z, \xi_1, \xi_2) = 0$  for (7.8).*

Due to a counter-example by Sussmann [98], there exists a third-order nonlinear system in the form (7.8) which is not semiglobally stabilizable. This major stabilization limitation of nonlinear systems in nontriangular forms as (7.8) is because of the *peaking phenomenon* that was introduced by Sussmann and Kokotovic [99]. The problem is that driving the state  $(\xi_1, \xi_2)$  exponentially fast to zero using a linear partial state feedback does not necessarily asymptotically stabilize the composite system in (7.8). Such a partial state feedback might cause the  $z$ -subsystem to have finite escape time. In [99], growth conditions are provided on  $F(z, \xi_1, \xi_2)$  for global stabilization of (7.8) using partial state feedback. The problem is that these growth conditions are rather restrictive. Nevertheless, the work in [99] is one possible solution to problem 7.1.1.

In many applications (e.g. tracking for flexible-link robots), the zero-dynamics is not globally minimum phase. In this thesis, we are mainly interested in stabilization problems for normal form (7.8) when the zero-dynamics is not minimum phase. A possible way to formulate a stabilization problem for normal form (7.8) analogous to the nature of the one addressed using backstepping procedure for systems in strict feedback form is the following problem.

**Problem 7.1.2.** *Consider the nonlinear system in (7.8). Assume there exists a smooth state feedback law  $\xi_1 = \alpha(z)$  with  $\alpha(0) = 0$  such that for the zero-dynamics locked at  $\xi_2 = 0$*

$$\dot{z} = F(z, \alpha(z), 0)$$

*$z = 0$  is globally asymptotically stable. Find sufficient conditions such that there exist state feedback laws  $\xi_2 = k_1(z, \xi_1)$  and  $u = k_2(z, \xi_1, \xi_2)$  which asymptotically stabilize the origin for the  $(z, \xi_1)$ -subsystem and the composite system (7.8), respectively.*

A third possible stabilization problem is as follows:

**Problem 7.1.3.** *Consider the nonlinear system in (7.8). Assume there exist smooth state feedback laws  $\xi_1 = \alpha(z)$  and  $\xi_2 = \beta(z)$  satisfying the following conditions:*

i)  $\alpha(0) = 0$  and  $\beta(0) = 0$ .

ii) For

$$\dot{z} = f(z, \alpha(z), 0)$$

$z = 0$  is GAS.

iii)  $(\xi_1, \xi_2) = (\alpha(z), \beta(z))$  is an invariant manifold for (7.8), i.e.  $\alpha(z), \beta(z)$  satisfy the equation

$$\beta(z) = \frac{\partial \alpha(z)}{\partial z} f(z, \alpha(z), \beta(z))$$

iv) For

$$\dot{z} = f(z, \alpha(z), \beta(z))$$

the origin  $z = 0$  is GAS.

Find sufficient conditions such that there exist state feedback laws  $\xi_2 = k_1(z, \xi_1)$  and  $u = k_2(z, \xi_1, \xi_2)$  which semiglobally/globally asymptotically stabilize the origin for the  $(z, \xi_1)$ -subsystem and the composite system (7.8), respectively.

*Remark 7.1.2.* Based on condition iii) in problem 7.1.3,  $\xi_2 = \beta(z)$  is a fixed point of equation

$$\xi_2 = \frac{\partial \alpha(z)}{\partial z} f(z, \alpha(z), \xi_2)$$

Thus,  $\alpha(z), \beta(z)$  are not arbitrary controllers. The state feedback  $\alpha(z)$  must be chosen such that it both stabilizes the zero-dynamics locked at  $\xi_2 = 0$  and the preceding fixed-point equation must be well-posed (i.e. have a solution). Both of these issues are addressed later in this chapter.

A common example of nonlinear systems with *feedforward structures* is a chain of integrators with higher-order perturbations as

$$\begin{aligned} \dot{x}_1 &= x_2 + p_1(x_2, x_3, \dots, x_n, u) \\ \dot{x}_2 &= x_3 + p_2(x_3, \dots, x_n, u) \\ &\dots \\ \dot{x}_n &= u + p_n(x_n, u) \end{aligned} \tag{7.12}$$

System (7.12) can be rewritten as

$$\dot{x} = Ax + Bu + p(x, u) \tag{7.13}$$

(with obvious definitions of  $A, B, p(x, u)$ ) where the perturbation terms  $p_i$  are at least quadratic with respect to their arguments [102]. Nonlinear systems in feedforward

forms can have more general forms as the following

$$\begin{aligned}\dot{x}_1 &= f_1(x_2, x_3, \dots, x_n, u) \\ \dot{x}_2 &= f_2(x_3, \dots, x_n, u) \\ &\dots \\ \dot{x}_n &= f_n(x_n, u)\end{aligned}\tag{7.14}$$

with an apparent upper triangular structure w.r.t. variables  $(x_1, x_2, \dots, x_n, u)$  [56]. Feedforward systems of types (7.12) and (7.14) are studied by Teel and Mazenc-Praly. Similar to systems in feedback forms recursive procedures called *feedforwarding* are developed for stabilization of feedforward systems [102, 101, 56]. The main draw back of all existing results on stabilization of feedforward systems is the low-gain nature of controllers that leads to relatively poor performance of control laws obtained from feedforwarding. Here, the performance measure is the settling time of the solutions of the closed-loop system. The controllers obtained using backstepping procedure do not suffer from this drawback.

## 7.2 Structure of Nontriangular Normal Forms of Underactuated Systems

The classification of underactuated mechanical systems presented in section 4.6 allows us to identify the structure of the vector field  $f$  in (7.8) (or (7.9)) for underactuated systems in nontriangular forms and feedforward forms.

**Proposition 7.2.1.** *All the normal forms for classes of underactuated systems in nontriangular/feedforward forms with a double-integrator linear part (minus the perturbation terms  $\varphi_i$ ) are special classes of the following normal form*

$$\begin{aligned}\dot{z}_1 &= z_2 + g_0(\xi_1)\xi_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) + z_2^T g_{11}(\xi_1)z_2 + z_2^T g_{12}(\xi_1)\xi_2 + \xi_2^T g_{22}(\xi_1)\xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{7.15}$$

where  $z_1, z_2 \in \mathbb{R}^n$ ,  $\xi_1, \xi_2, u \in \mathbb{R}^m$ . In addition, the functions  $g_0 : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$ ,  $f_0(z_1, \xi_1) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and cubic matrices  $g_{11} : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n \times n}$ ,  $g_{12} : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m \times n}$ , and  $g_{22} : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m \times n}$  are all smooth and  $g_0(0) = 0$ .

**Proof.** The proof is by the definition of each class in section 4.6. □

*Remark 7.2.1.* Equation (7.15) can be rewritten in the form

$$\begin{aligned}\dot{z} &= F(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{7.16}$$

where

$$F(z, \xi_1, \xi_2) = f(z_1, \xi_1) + L(\xi_1)\xi_2 + (z_2, \xi_2)Q(\xi_1)(z_2^T, \xi_2^T)^T \quad (7.17)$$

has a linear-quadratic structure with respect to  $\xi_2$  with a cubic matrix  $Q(\xi_1)$ .

*Remark 7.2.2.* Global stabilization of the core reduced system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) \end{aligned} \quad (7.18)$$

is addressed in section 4.7.

The following special classes of normal form (7.15) are important due to their application in control of underactuated systems:

- i)  $g_{11}, g_{22} = 0$ : nonlinear system (7.15) is in nontriangular form with a vector field affine in  $\xi_2$ .
- ii)  $f_0(z_1, \xi_1) = f_0(\xi_1), g_{11} = 0$ : nonlinear system (7.15) is in feedforward form with a vector field quadratic in  $\xi_2$ .
- iii)  $g_0 = 0$ : nonlinear system (7.15) is in nontriangular form with a vector field quadratic in  $\xi_2$ . Moreover,  $\dot{z}_1$  does not depend on  $(\xi_1, \xi_2)$ .

**Lemma 7.2.1.** *Normal form (7.15) can be represented as*

$$\begin{aligned} \dot{z} &= f(z, \xi_1) + g(z, \xi_1, \xi_2)\xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \quad (7.19)$$

where  $f, g$  are explicitly given by

$$f(z, \xi_1) = \begin{bmatrix} z_2 \\ f_0(z_1, \xi_1) + z_2^T g_{11}(\xi_1)z_2 \end{bmatrix}, \quad g(z, \xi_1, \xi_2) = \begin{bmatrix} g_0(\xi_1) \\ z_2^T g_{12}(\xi_1) + \xi_2^T g_{22}(\xi_1) \end{bmatrix}$$

**Proof.** The proof is by definition of  $f$  and  $g$ .  $\square$

*Remark 7.2.3.* The main condition in problems 7.1.2 and 7.1.3 is that there should exist a state feedback  $\xi_1 = \alpha(z)$  with  $\alpha(0) = 0$  such that for the closed-loop zero dynamics locked at  $\xi_2 = 0$

$$\dot{z} = F(z, \alpha(z), 0)$$

the equilibrium  $z = 0$  is globally asymptotically stable. Based on lemma 7.2.1, this is equivalent to the existence of a globally stabilizing state feedback for  $\dot{z} = f(z, \xi_1)$  that can be expressed as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f_0(z_1, \xi_1) + z_2^T g_{11}(\xi_1)z_2 \end{aligned} \quad (7.20)$$

Apparently, if  $g_{11} \equiv 0$ , then global stabilization of (7.20) is equivalent to global stabilization of the core reduced system which is addressed in section 4.7. In addition, for the special case where  $g_{11} = g_{11}(\xi_1, \epsilon)$  with the property  $g_{11}(\xi_1, 0) = 0$ , semiglobal stabilization of (7.20) can be achieved (under appropriate technical conditions).

**Definition 7.2.1.** (slightly nontriangular nonlinear systems) The following subclass of nontriangular nonlinear systems in (7.19)

$$\begin{aligned}\dot{z} &= f(z, \xi_1) + g(z, \xi_1, \xi_2, \epsilon) \xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{7.21}$$

with the property  $g(z, \xi_1, \xi_2, 0) = 0$  are called *slightly nontriangular nonlinear systems*.

Control design for slightly nontriangular systems is important due to their applications in aerospace vehicles. As an example, according to proposition 5.11.4, the normal form of the first-level approximate model of a helicopter is in the form

$$\begin{aligned}\dot{z} &= f(z) + g_1(\xi_1)T + g_2(\xi_1, \xi_2, \epsilon) + g_3(\xi_1, \epsilon)u \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{7.22}$$

where  $z = \text{col}(x, v) \in \mathbb{R}^6$ ,  $\xi_1 = \text{col}(\phi, \theta, \psi)$ , (i.e. the three Euler angles),  $\xi_2 \in \mathbb{R}^3$ , and  $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^T \in \mathbb{R}^3$ . Furthermore, both  $g_2$  and  $g_3$  vanish at  $\epsilon = (0, 0, 0)$ . Later, we provide more examples of slightly nontriangular systems in robotics applications.

## 7.3 Stabilization of Nonlinear Systems in Feedback Form

Backstepping procedure is an effective method for global stabilization of nonlinear systems in strict feedback form. There are two types of backstepping procedure. One that makes an explicit use of a Lyapunov function and is referred to as *standard backstepping* procedure. The other one is a type of backstepping procedure which does not require any explicit knowledge of a Lyapunov function and we call it *cascade backstepping* procedure. Here, we present both of them and clarify the differences between the two methods.

### 7.3.1 Standard Backstepping Procedure

We start by presenting the standard backstepping procedure for a nonlinear system augmented with a single integrator at the input. First, we need to state the following lemma [36, 40].

**Lemma 7.3.1.** Assume  $f(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  is a continuously differentiable function. Then, the following identity holds:

$$f(x, y) = f(x, 0) + g(x, y)y \quad (7.23)$$

where

$$g(x, y) = \int_0^1 D_y f(x, sy) ds$$

**Proof.** Fix  $(x, y) \in \mathbb{R}^{n+m}$  and let  $h(s) := f(x, sy)$  where  $s \in \mathbb{R}$ . We have  $h'(s) = D_y f(x, sy) \cdot y$ . Hence

$$h(1) - h(0) = \int_0^1 h'(s) ds = \left( \int_0^1 D_y f(x, sy) ds \right) \cdot y$$

and the result follows.  $\square$

*Remark 7.3.1.* An application of lemma 7.3.1 gives

$$f(x, y + \delta) = f(x, y) + g(x, y, \delta)\delta$$

The following version of backstepping theorem can be found in [36].

**Theorem 7.3.1. (standard backstepping)** Consider the following nonlinear system

$$\begin{aligned} \dot{z} &= f(z, \xi) \\ \dot{\xi} &= u \end{aligned} \quad (7.24)$$

where  $z \in \mathbb{R}^n$ ,  $\xi, u \in \mathbb{R}$ , and  $f$  is a smooth function satisfying  $f(0, 0) = 0$ . Assume there exists a smooth state feedback  $\xi = \alpha(z) : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying  $\alpha(0) = 0$  such that for

$$\dot{z} = f(z, \alpha(z))$$

$z = 0$  is GAS. Let  $V(z)$  be a smooth positive definite and proper Lyapunov function associated with the closed-loop  $z$ -subsystem satisfying

$$DV(z) \cdot f(z, \alpha(z)) < 0, \quad \forall z \neq 0$$

Then, the following smooth state feedback

$$u = -c(\xi - \alpha(z)) + \frac{\partial \alpha(z)}{\partial z} f(z, \xi_1, \xi_2) - \frac{\partial V(z)}{\partial z} \cdot h(z, \mu) \quad (7.25)$$

renders  $(z, \xi) = 0$  GAS for (7.24) where  $c > 0$  is a constant,  $\mu := \xi - \alpha(z)$ , and

$$h(z, \mu) := \int_0^1 D_\xi f(z, \alpha(z) + s\mu) ds$$

satisfies

$$f(z, \alpha(z) + \mu) = f(z, \alpha(z)) + h(z, \mu)\mu$$

**Proof.** Consider the following Lyapunov function candidate

$$W(z, \xi) = V(z) + \frac{1}{2}\mu^2$$

Calculating  $\dot{W}$  along the solutions of the closed-loop system in (7.24), we get

$$\begin{aligned} \dot{W} &= \dot{V} + \dot{\mu}\mu, \\ &= \frac{\partial V(z)}{\partial z} f(z, \alpha(z) + \mu) + \left(u - \frac{\partial \alpha(z)}{\partial z} f(z, \xi_1, \xi_2)\right)\mu, \\ &= \frac{\partial V(z)}{\partial z} f(z, \alpha(z)) + \left[u - \frac{\partial \alpha(z)}{\partial z} f(z, \xi_1, \xi_2) + \frac{\partial V(z)}{\partial z} h(z, \mu)\right]\mu, \quad (\text{lemma 7.3.1}) \\ &= \frac{\partial V(z)}{\partial z} f(z, \alpha(z)) - c\mu^2 < 0 \end{aligned}$$

for all  $(z, \mu) \neq 0$ . Therefore,  $(z, \xi) = 0$  is GAS for (7.24) and  $W(z, \xi)$  is a valid Lyapunov function for the composite system.  $\square$

*Remark 7.3.2.* Successive application of theorem 7.3.1 proves the statements in theorem 7.1.1.

In [57], the following theorem which is a special case of theorem 7.3.1 (with a  $z$ -subsystem affine in  $\xi$ ) is considered as standard backstepping procedure.

**Theorem 7.3.2.** ([57]) Consider the following cascade nonlinear system affine in  $\xi$

$$\begin{aligned} \dot{z} &= f(z) + g(z)\xi \\ \dot{\xi} &= u \end{aligned} \tag{7.26}$$

where  $z \in \mathbb{R}^n$ ,  $\xi, u \in \mathbb{R}$ , and  $f, g$  are smooth functions with  $f(0) = 0$ . Let the state feedback  $\xi = \alpha(z)$  with  $\alpha(0) = 0$  render  $z = 0$  GAS for the closed-loop  $z$ -subsystem

$$\dot{z} = f(z) + g(z)\alpha(z)$$

Then, the following state feedback

$$u = -c(\xi - \alpha(z)) + \frac{\partial \alpha(z)}{\partial z} f(z, \xi_1, \xi_2) - \frac{\partial V(z)}{\partial z} g(z) \tag{7.27}$$

renders  $(z, \xi) = 0$  GAS for the composite system (7.26).

**Proof.** Define  $\tilde{f}(z, \xi) := f(z) + g(z)\xi$ . Then

$$\tilde{f}(z, \xi + \mu) = \tilde{f}(z, \xi) + h(z, \mu)\mu$$

where  $h(z, \mu) := g(z)$ . Thus, state feedback (7.27) reduces to (7.25) and the result follows from theorem 7.3.1.  $\square$

The proof of standard backstepping procedure automatically provides a Lyapunov function  $W(z, \xi_1)$  for the closed-loop composite system. This is very valuable for (possible) robustness analysis of perturbed nonlinear systems in feedback forms (e.g. normal forms of Class-VI underactuated mechanical systems in chapter 4).

### 7.3.2 Cascade Backstepping Procedure

The knowledge about the Lyapunov function  $V(z)$  for the closed-loop  $z$ -subsystem is necessary in the computation of the control law (7.25) for the composite system. Such a knowledge about  $V(z)$  is not always available. This motivates developing a backstepping procedure which is not Lyapunov-based. We refer to this procedure that does not require explicit knowledge of any Lyapunov functions as *cascade backstepping procedure*.

The most important tool in developing a non-Lyapunov-based or cascade backstepping procedure in the following theorem due to Sontag.

**Theorem 7.3.3.** (*global stability of cascade systems [83]*) Consider the following cascade nonlinear system

$$\begin{aligned}\dot{z} &= f(z, \eta) \\ \dot{\eta} &= g(\eta)\end{aligned}\tag{7.28}$$

where  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$  are smooth functions satisfying  $f(0, 0) = 0$  and  $g(0) = 0$ . Assume the following conditions hold:

- i) For  $\dot{z} = f(z, 0)$ ,  $z = 0$  is globally asymptotically stable.
- ii) For  $\dot{\eta} = g(\eta)$ ,  $\eta = 0$  is globally asymptotically and locally exponentially stable.
- iii) For any solution  $\eta(t)$  with initial condition  $\eta_0 \in \mathbb{R}^m$ , the solution of the  $z$ -subsystem of (7.28) exists for all  $t \geq 0$  and is bounded.

Then,  $(z, \eta) = 0$  is globally asymptotically stable for (7.28).

The main use of theorem 7.3.3 is in stability analysis of the class of cascade systems

$$\begin{aligned}\dot{x} &= f(x, \eta) \\ \dot{\eta} &= g(\eta, u)\end{aligned}\tag{7.29}$$

where a control law  $u = k(\eta)$  is available that render  $\eta = 0$  GAS for  $\dot{\eta} = g(\eta, k(\eta))$ . In this chapter, we frequently use theorem 7.3.3 in the aforementioned context.

The only non-constructive aspect of theorem 7.3.3 is in its third condition where the boundedness of the solutions of the  $z$ -subsystem of (7.28) is assumed. One possible assumption to guarantee this boundedness assumption is to assume that the  $z$ -subsystem

$$\dot{z} = f(z, \eta)$$

is input-to-state stable, or simply ISS-stable [82]. Due to Sontag-Wang [84], a Lyapunov-type characterization of the ISS property reveals that the necessary and sufficient condition for a system to be ISS-stable is that there exist class-K functions  $\alpha_1, \alpha_2$  and a positive definite smooth ISS Lyapunov function  $V(z)$  such that the following property holds:

$$\frac{\partial V(z)}{\partial z} f(z, \eta) < -\alpha_1(\|z\|), \quad \text{for } \|z\| > \alpha_2(\|\eta\|) \quad (7.30)$$

Condition (7.30) is an extremely restrictive condition which might not hold for large classes of nonlinear systems of interest. Since, it essentially ignores the existence of the input  $\eta$ . In the sense that over the region  $\|z\| > \alpha_2(\|\eta\|)$ ,  $\dot{V} < 0$  and the solution of the  $z$ -subsystem converges towards the inner level surfaces around the origin  $z = 0$ , until the condition  $\|z\| > \alpha_2(\|\eta\|)$  is violated. However, to guarantee the boundedness of the solution of the  $z$ -subsystem it is sufficient that  $V(z(t))$  stays bounded. The time decay of  $V(z(t))$  is unnecessary. Keeping this in mind, here we provide a constructive condition that guarantees boundedness of the solutions of the  $z$ -subsystem. This condition is less restrictive than the ISS-stability property of the  $z$ -subsystem. This is influenced by the work of Sepulchre *et al.* [80] (theorem 4.7, p.129) and generalizes that result. First, we need to define a class of input disturbances.

**Definition 7.3.1.** (class- $\mathcal{C}_0$  functions) Define the following class of functions  $\delta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$

$$\mathcal{C}_0(l_0, l_1) := \{\delta(\cdot) \mid \|\delta\|_\infty \leq l_0, \|\delta\|_1 \leq l_1, \lim_{t \rightarrow \infty} \delta(t) = 0\}, \quad l_0, l_1 > 0$$

where  $\|\delta\|_\infty := \sup_{t \geq 0} |\delta(t)|$  and  $\|\delta\|_1 := \int_0^\infty |\delta(t)| dt$  denote  $L_\infty$  and  $L_1$  norms of  $\delta(\cdot)$ , respectively. We say a function  $\delta : t \rightarrow \delta(t)$  is class- $\mathcal{C}_0$  with bounds  $l_0, l_1 > 0$ , if  $\delta(\cdot) \in \mathcal{C}_0(l_0, l_1)$ .

**Theorem 7.3.4.** Consider the nonlinear system

$$\dot{z} = f(z, \eta) \quad (7.31)$$

with a class- $\mathcal{C}_0$  input disturbance  $\eta(\cdot) \in \mathcal{C}_0(l_0, l_1)$ . Assume for any  $l_0 > 0$ , there exist a smooth positive definite and proper function  $V(z)$ , positive constants  $\rho \geq 0$ ,  $k > 0$ , and  $\lambda \in (0, 1]$  such that the following conditions are satisfied:

- i)  $\nabla V(z) \cdot f(z, 0) \leq 0$
- ii)  $|\nabla V(z) \cdot h(z, \eta)| \leq kV(z)^\lambda$

for all  $\|z\| \geq \rho$  and  $\|\eta\| \leq l_0$  where

$$h(z, \eta) = \int_0^1 D_\eta f(z, s\eta) ds$$

Then, any solution of the system (7.31) is bounded.

**Proof.** Based on lemma 7.3.1, we have

$$f(z, \eta) = f(z, 0) + h(z, \eta)\eta$$

Calculating  $\dot{V}$  gives

$$\begin{aligned}\dot{V} &= \nabla V(z) \cdot f(z, \eta) \\ &= \nabla V(z) \cdot f(z, 0) + \nabla V(z) \cdot h(z, \eta)\eta \\ &\leq \nabla V(z) \cdot h(z, \eta)\eta \\ &\leq |\nabla V(z) \cdot h(z, \eta)| \cdot \|\eta\| \\ &\leq kV(z)^\lambda \|\eta\|\end{aligned}$$

We consider two cases: i)  $\lambda \in (0, 1)$  and ii)  $\lambda = 1$ . For  $\lambda \in (0, 1)$ , over the region  $\|z\| \geq \rho$ , the last inequality can be rewritten as

$$\frac{1}{k}V(z(t))^{-\lambda}\dot{V}(z(t)) \leq \|\eta(t)\|$$

Integrating both sides of the last inequality w.r.t.  $t$  gives

$$V(z(t))^{(1-\lambda)} - V(z(0))^{(1-\lambda)} \leq k(1-\lambda) \int_0^t \|\eta(t)\| dt \leq k(1-\lambda) \|\eta\|_1 \leq k(1-\lambda)l_1$$

which means  $z(t)$  is bounded uniformly in  $t$ , because

$$V(z(t)) \leq [V(z(0))^{(1-\lambda)} + k(1-\lambda)l_1]^{\frac{1}{1-\lambda}} =: L(k, \lambda, l_1, z(0)) \quad (7.32)$$

and  $V(z)$  is a proper function. Thus,  $\mathcal{K} := V^{-1}([0, L(k, \lambda, l_1, z(0))])$  is a compact set and  $z(t) \in \mathcal{K}, \forall t \geq 0$ . This implies  $\|z(t)\| < L_z := \max\{\rho, r(\mathcal{K})\}$ , for all  $t > 0$  where  $r(\mathcal{K}) = \max_{x \in \mathcal{K}} \|x\|$  denotes the radius of the compact set  $\mathcal{K}$ . The proof of the boundedness of  $z(t)$  for all  $t > 0$  for the case of  $\lambda = 1$  is very similar and is omitted. The only difference is that the inequality (7.32) is in the form

$$V(z(t)) \leq V(z(0)) \exp(kl_1) \quad (7.33)$$

□

Now, we are ready to introduce a non-Lyapunov-based backstepping procedure.

**Theorem 7.3.5.** (*cascade backstepping procedure*) Consider the following nonlinear system

$$\begin{aligned}\dot{z} &= f(z, \xi) \\ \dot{\xi} &= u\end{aligned} \quad (7.34)$$

Assume there exists a smooth state feedback  $\xi = \alpha(z)$  with  $\alpha(0) = 0$  such that for

$$\dot{z} = f(z, \alpha(z))$$

$z = 0$  is GAS. Let  $V(z)$  be a smooth positive definite and proper Lyapunov function associates with the closed-loop  $z$ -subsystem satisfying

$$\nabla V(z) \cdot f(z, \alpha(z)) \leq 0$$

In addition, suppose for any  $l_0 > 0$ , there exist  $\rho \geq 0$ ,  $k > 0$ , and  $\lambda \in (0, 1]$  such that

$$|\nabla V(z) \cdot h(z, \mu)| \leq kV(z)^\lambda, \quad \forall (z, \mu) : \|z\| \geq \rho, \|\mu\| \leq l_0 \quad (7.35)$$

where  $\mu = \xi - \alpha(z)$  and  $h(z, \mu) = \int_0^1 D_\xi f(z, \alpha(z) + s\mu) ds$ . Then, the following smooth state feedback

$$u = -c(\xi - \alpha(z)) + \frac{\partial \alpha(z)}{\partial z} f(z, \xi), \quad c > 0 \quad (7.36)$$

renders  $(z, \xi) = 0$  GAS for (7.34).

**Proof.** The closed-loop cascade system in (7.34) with controller (7.36) can be rewritten as

$$\begin{aligned} \dot{z} &= f(z, \alpha(z) + \mu) =: \tilde{f}(z, \mu) \\ \dot{\mu} &= -c\mu \end{aligned} \quad (7.37)$$

Notice that both  $\mu = 0$  is GAS for the  $\mu$ -subsystem and  $z = 0$  is GAS for  $\dot{z} = \tilde{f}(z, 0)$ . On the other hand, the solution of the  $z$ -subsystem is bounded according to theorem 7.3.4. To see this, note that  $\mu(t) = \mu(0) \exp(-ct)$  and thus  $\mu(t)$  is a class- $\mathcal{C}_0$  function satisfying  $\|\mu(t)\| \leq l_0 := \|\mu(0)\|, \forall t > 0$  and  $\|\mu(t)\|_1 = 1/c =: l_1$ . But

$$f(z, \alpha(z) + \mu) = f(z, \alpha(z)) + h(z, \mu)\mu$$

and by the assumption in the question we get

$$\dot{V} \leq kV(z)^\lambda \|\mu\|$$

which guarantees uniform boundedness of the solution  $z(t)$  for all  $t > 0$ . Based on theorem 7.3.3, the origin  $(z, \mu) = 0$  is GAS for the cascade closed-loop system. This implies  $(z, \xi) = 0$  is GAS for (7.34).  $\square$

**Example 7.3.1.** Consider the following nonlinear system in strict feedback form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \sigma(\xi) \\ \dot{\xi} &= u \end{aligned} \quad (7.38)$$

where  $\sigma(\xi)$  is a sigmoidal function that is odd and globally Lipschitz (e.g.  $\tanh(\xi)$ ). The linear state feedback

$$\xi = \alpha(z) := -k_1 z_1 - k_2 z_2, \quad k_1, k_2 > 0$$

renders  $z = 0$  GAS for the  $z$ -subsystem. To prove this, let us express the dynamics of the closed-loop  $z$ -subsystem as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\sigma(k_1 z_1 + k_2 z_2)\end{aligned}\tag{7.39}$$

Define  $\phi(z_1, z_2) := \sigma(k_1 z_1 + k_2 z_2)$  and notice that  $D_{z_2} \phi(z_1, z_2) > 0$  for all  $z \in \mathbb{R}^2$ . This means

$$z_2(\phi(z_1, 0) - \phi(z_1, z_2)) < 0, \quad \forall (z_1, z_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$$

Let us define

$$\psi(z_1) = \int_0^{z_1} \phi(s, 0) ds = \int_0^{z_1} \sigma(k_1 s) ds$$

and consider the following Lyapunov function candidate

$$V(z_1, z_2) = \psi(z_1) + \frac{1}{2} z_2^2$$

Calculating  $\dot{V}$  along the solutions of the closed-loop  $z$ -subsystem, we get

$$\dot{V} = z_2(\phi(z_1, 0) - \phi(z_1, z_2)) < 0, \quad \forall (z_1, z_2) \neq (0, 0)$$

and therefore  $(z_1, z_2) = 0$  is GAS for the  $z$ -subsystem. Taking  $f(z, \xi) := [z_2, \sigma(\xi)]^T$ , we have

$$f(z, \alpha(z) + \mu) = f(z, \alpha(z)) + h(z, \mu)\mu$$

where  $h(z, \mu) = [0, g(z, \mu)]^T$  and  $g(z, \mu) = \int_0^1 \sigma'(\alpha(z) + s\mu) ds$ . Since  $\sigma$  is globally Lipschitz, there exists an  $L > 0$  such that  $|g(z, \mu)| \leq L$  for all  $(z, \mu)$ . Now, we prove that based on theorem 7.3.5, the following nonlinear state feedback

$$u = -c(\xi - \alpha(z)) + \frac{\partial \alpha(z)}{\partial z} f(z, \xi), \quad c > 0$$

which can be written in an explicit form

$$u = -c(\xi + k_1 z_1 + k_2 z_2) - k_1 z_2 - k_2 \sigma(\xi)\tag{7.40}$$

renders  $(z, \xi) = 0$  GAS for (7.38). For doing so, we need to show that condition (7.35) holds. We have

$$|\nabla V(z) \cdot h(z, \mu)| = |z_2 g(z, \mu)| \leq L |z_2| \leq (2L)V(z)^{1/2}$$

Therefore, condition (7.35) holds with  $k = 2L$ ,  $\lambda = 1/2$  uniformly in  $\mu$ . This guarantees boundedness of the solution of the  $z$ -subsystem and the GAS property of  $(z, \xi) = 0$  for the composite system. Using standard backstepping procedure, one

obtains a different globally stabilizing control law

$$u = -c(\xi - \alpha(z)) + \frac{\partial \alpha(z)}{\partial z} f(z, \xi) - E(z, \xi), \quad c > 0$$

with an extra term  $E(z, \xi)$  compared to state feedback (7.40) which is given by

$$E(z, \xi) := \nabla V(z) \cdot h(z, \mu) = \int_0^1 z_2 \sigma'((1-s)\alpha(z) + s\xi) ds$$

*Remark 7.3.3.* If the sigmoidal function  $\sigma$  in (7.38) is neither bounded nor globally Lipschitz. Then, we choose the following control law

$$\xi = \alpha(z) := -\sigma_0(k_1 z_1 + k_2 z_2), \quad k_1, k_2 > 0$$

where  $\sigma_0$  is a bounded and globally Lipschitz sigmoidal function. Now,  $\tilde{\sigma}(\xi) = \sigma(\sigma_0(\xi))$  is both bounded and globally Lipschitz. The analysis in example 7.3.1 now continues in a similar way with replacing  $\sigma$  by  $\tilde{\sigma}$ .

## 7.4 Stabilization of Nonlinear Systems in Nontriangular Forms

The main class of nontriangular nonlinear systems considered in this chapter are in the form

$$\begin{aligned} \dot{z} &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{7.41}$$

where  $z \in \mathbb{R}^n$  and  $f$  is a smooth function satisfying  $f(0, 0, 0) = 0$ . Here, we consider the single input case with  $\xi_1, \xi_2, u \in \mathbb{R}$  for the sake of simplicity of notations. The generalization of the obtained results to the case of multiple inputs is trivial. Depending on whether  $f(z, \xi_1, \xi_2)$  is an affine function of  $\xi_1$  and/or  $\xi_2$  or none, we break the problem of stabilization of (7.41) into a number of cases and address each case separately. Throughout this chapter, we put especial emphasis on the structure of  $f$  as it appears in the normal forms of underactuated mechanical systems.

### 7.4.1 Nontriangular Nonlinear Systems Affine in $(\xi_1, \xi_2)$

In this section, our goal is to stabilize normal form (7.41) in its simplest possible form given as

$$\begin{aligned}\dot{z} &= f(z) + g_1(z)\xi_1 + g_2(z)\xi_2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}\tag{7.42}$$

where  $(z, \xi_1) \in \mathbb{R}^n \times \mathbb{R}$  and  $\xi_2 \in \mathbb{R}$  is the control input. The vector fields  $f(z), g_1(z), g_2(z)$  are all smooth and  $f(0) = 0$ . Due to the linearity property of the vector field of the  $z$  subsystem w.r.t.  $\xi_2$ , normal form (7.42) does not capture the effects of Coriolis and centrifugal terms in mechanical systems. However, it facilitates explaining the main necessary steps that are required to address the stabilization problem of systems in nontriangular forms non-affine in  $(\xi_1, \xi_2)$ , without excessive technical preliminaries. More importantly, it allows us to show that the method of standard backstepping and its particular way of construction of Lyapunov functions is not generalizable to nonlinear systems in normal form (7.42). In contrast, we provide a cascade backstepping procedure for stabilization of nonlinear systems in the form (7.42). Later, we present generalization of this non-Lyapunov-based backstepping procedure to classes of normal form (7.41) that are non-affine in  $(\xi_1, \xi_2)$ .

First, we demonstrate why the standard backstepping procedure is practically not applicable to (7.42). In the sense that, it requires conditions that are extremely restrictive.

**Proposition 7.4.1.** *Consider the nonlinear system in (7.42). Assume there exists a smooth state feedback  $\alpha(z) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\alpha(0) = 0$  such that for*

$$\dot{z} = f(z) + g_1(z)\alpha(z) =: f_1(z)\tag{7.43}$$

$z = 0$  is GAS. Let  $V(z)$  be the smooth positive definite and proper Lyapunov function associated with (7.43) satisfying  $L_{f_1}V(z) < 0, \forall z \neq 0$ . Assume  $\alpha(z)$  globally satisfies the condition  $L_{g_2}\alpha(z) < 1$ . Define  $\beta(z)$  as

$$\beta(z) := \frac{L_{f_1}\alpha(z)}{1 - L_{g_2}\alpha(z)}$$

so that  $\xi_2 = \beta$  is the fixed-point solution of the equation

$$\xi_2 = \nabla\alpha(z) \cdot [f(z) + g_1(z)\alpha(z) + g_2(z)\xi_2]$$

Suppose that  $L_{g_2}V(z) = 0, \forall z \in \mathbb{R}^n$ . Then, the following statements hold:

- i) For the closed-loop  $z$ -subsystem,  $\dot{z} = f(z) + g_1(z)\alpha(z) + g_2(z)\beta(z) =: f_2(z)$ , the origin  $z = 0$  is GAS.
- ii) there exist a smooth state feedback  $\xi_2 = K_1(z, \xi_1)$  that renders  $(z, \xi_1) = 0$  GAS for the  $(z, \xi_1)$ -subsystem of (7.42).

iii) there exists a smooth state feedback  $u = K_2(z, \xi_1, \xi_2)$  that renders  $(z, \xi_1, \xi_2) = 0$  GAS for the composite system (7.42).

**Proof.** The proof of i) is trivial due to the assumption  $L_{g_2}V(z) = 0$ . This implies

$$\dot{V} = L_{f_2}V(z) = L_{f_1}V(z) + \beta(z)L_{g_2}V(z) = L_{f_1}V(z) < 0$$

for all  $z \neq 0$  and therefore  $z = 0$  is GAS. To prove ii), first we need to apply a change of coordinates by setting  $\mu = \xi_1 - \alpha(z)$ . The dynamics of the  $(z, \xi_1)$ -subsystem of (7.42) in new coordinates can be expressed as

$$\begin{aligned}\dot{z} &= f_2(z) + [g_1(z) + A(z)g_2(z)]\mu_1 + B(z)g_2(z)\mu_2 \\ \dot{\mu}_1 &= \mu_2\end{aligned}\tag{7.44}$$

where  $\mu_2$  is the new control and the functions  $A(z), B(z) : \mathbb{R}^n \rightarrow \mathbb{R}$  are obtained as follows. By definition of  $\mu_2$ , we have

$$\begin{aligned}\mu_2 &= \xi_2 - \nabla\alpha(z) \cdot [f(z) + g_1(z)\xi_1 + g_2(z)\xi_2] \\ &= (1 - L_{g_2}\alpha(z))\xi_2 - L_{f_1}\alpha(z) - \mu_1 L_{g_1}\alpha(z)\end{aligned}$$

This change of variable is globally invertible due to  $1 - L_{g_2}\alpha(z) > 0$ . Solving the last equation for  $\xi_2$  gives

$$\xi_2 = \frac{L_{f_1}\alpha(z) + L_{g_1}\alpha(z)\mu_1 + \mu_2}{1 - L_{g_2}\alpha(z)} = \beta(z) + A(z)\mu_1 + B(z)\mu_2\tag{7.45}$$

where

$$A(z) = \frac{L_{g_1}\alpha(z)}{1 - L_{g_2}\alpha(z)}, \quad B(z) = \frac{1}{1 - L_{g_2}\alpha(z)}$$

Substituting  $\xi_2$  from (7.45) in equation (7.42) gives (7.44). Following the proof the standard backstepping theorem, let us consider the candidate Lyapunov function

$$W(z, \xi_1) := V(z) + \frac{1}{2}\mu_1^2$$

for the nonlinear system in (7.44). Calculating  $\dot{W}$ , we get

$$\begin{aligned}\dot{W} &= \dot{V} + \mu_1\mu_2 \\ &= \nabla V(z) \cdot [f(z) + g_1(z)(\alpha(z) + \mu_1) + g_2(z)(\beta(z) + A(z)\mu_1 + B(z)\mu_2)] \\ &= L_{f_2}V(z) + [L_{g_1}V(z) + A(z)L_{g_2}V(z) + \mu_2]\mu_1 + B(z)(L_{g_2}V(z))\mu_2\end{aligned}$$

Unlike the case of systems in strict feedback form, for the case of nontriangular nonlinear systems, an extra term  $\delta = B(z)(L_{g_2}V(z))\mu_2$  appears in the expression of  $\dot{W}$  which is sign indefinite. Unless  $\delta = 0$ , or  $L_{g_2}V(z) = 0$  for all  $z$ ,  $\dot{W}$  cannot be made

negative definite. But by assumption,  $L_{g_2}V(z) = 0$  and applying the state feedback

$$\mu_2 = -c_1\mu_1 - L_{g_1}V(z), \quad c_1 > 0$$

renders  $\dot{W}$  negative definite for all  $(z, \mu_1) \neq 0$  because

$$\dot{W} = L_{f_2}V(z) - c_1\mu_1^2$$

This means that

$$\xi_2 = K_1(z, \xi_1) := \beta(z) + [A(z) - B(z)c_1](\xi_1 - \alpha(z)) - B(z)L_{g_1}V(z) \quad (7.46)$$

renders  $(z, \xi_1) = 0$  GAS for the  $(z, \xi_1)$ -subsystem of (7.42) and  $W(z, \xi_1)$  is a valid Lyapunov function for the  $(z, \xi_1)$ -subsystem. Finally, part *iii*) follows from part *ii*) according to the standard backstepping theorem.  $\square$

*Remark 7.4.1.* The most restrictive condition in theorem is the assumption that  $L_{g_2}V(z) = 0$  for all  $z \in \mathbb{R}^n$ . However, without this assumption, in the opinion of the author, there does not seem to exist any ways to render  $\dot{W}$  negative definite for all  $(z, \xi_1) \neq 0$ . This demonstrates the weakness of the particular choice of the Lyapunov function used in the standard backstepping procedure, in dealing with nonlinear systems in nontriangular forms.

The following result is the non-Lyapunov-based version of proposition 7.4.1.

**Proposition 7.4.2.** *Assume all the conditions in proposition 7.4.1 hold. In addition, suppose there exist  $\rho \geq 0$ ,  $k > 0$ , and  $\lambda \in (0, 1]$  such that*

$$|L_{g_1}V(z)| \leq kV(z)^\lambda, \quad \|z\| \geq \rho$$

*Then, the smooth state feedback*

$$\xi_2 = K(z, \xi_1) = \beta(z) + G(z)(\xi_1 - \alpha(z)) \quad (7.47)$$

*(which does not depend on the Lyapunov function  $V(z)$ ) renders  $(z, \xi_1) = 0$  GAS for the  $(z, \xi_1)$ -subsystem of (7.42) where the nonlinear gain  $G(z)$  is defined as*

$$G(z) = A(z) - c_1B(z), \quad c_1 > 0$$

**Proof.** Setting  $\mu_2 = -c_1\mu_1$ , from equation (7.44), we have  $\dot{\mu}_1 = -c_1\mu_1$  (thus  $\mu_1 = 0$  is globally exponentially stable) and

$$\begin{aligned} \dot{V} &= L_{f_2}V + (L_{g_1}V)\mu_1 + G(z)L_{g_2}V \\ &= L_{f_2}V + (L_{g_1}V)\mu_1 \\ &\leq |L_{g_1}V(z)||\mu_1| \\ &\leq kV(z)^\lambda|\mu_1| \end{aligned}$$

Based on theorem 7.3.4, the solutions of the  $z$ -subsystem are bounded. Now, according to theorem 7.3.3, the origin  $(z, \mu_1) = 0$  is GAS for the cascade nonlinear system

(7.44) which proves the statement in the question.  $\square$

Now, we would like to present a non-Lyapunov-based backstepping procedure for stabilization of nontriangular nonlinear system (7.42) which does not require the restrictive condition  $L_{g_2}V(z) = 0, \forall z \in \mathbb{R}^n$ . First, we need to make some assumptions.

**Assumption 7.4.1.** There exists a state feedback  $\alpha(z) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\alpha(0) = 0$  satisfying the following conditions:

- i) for  $\dot{z} = f(z) + g_1(z)\alpha(z) =: f_1(z)$ , the origin  $z = 0$  is globally asymptotically stable,
- ii) there exists  $L_1 > 0$  such that  $|L_{g_1}\alpha(z)| \leq L_1$  for all  $z \in \mathbb{R}^n$ ,
- iii) there exists  $L_2 > 0$  such that  $L_{g_2}\alpha(z) \leq 1 - 1/L_2$  for all  $z \in \mathbb{R}^n$ .

The following proposition is the main analytical tool that justifies the use of the cascade backstepping procedure for stabilization of a nontriangular system as (7.42).

**Proposition 7.4.3.** Consider the nonlinear system in (7.42) and suppose assumption 7.4.1 holds. Then, after applying the global change of coordinates  $\mu = \Phi(z, \xi)$  and change of control defined by

$$\begin{aligned}\mu_1 &= \xi_1 - \alpha_1(z) \\ \mu_2 &= M(z)(\xi_2 - \alpha_2(z, \xi_1)) \\ w &= M(z)(u - \alpha_3(z, \xi_1, \xi_2))\end{aligned}\tag{7.48}$$

with  $M(z) = 1 - L_{g_2}\alpha(z) > 1/L_2 > 0$ , the dynamics of the nonlinear system in (7.42) transforms into

$$\begin{aligned}\dot{z} &= f_0(z) + [g_1(z) + A(z)g_2(z)]\mu_1 + B(z)g_2(z)\mu_2 \\ \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= w\end{aligned}\tag{7.49}$$

where  $f_0(z) := f(z) + g_1(z)\alpha_1(z) + g_2(z)\alpha_2(z, \alpha_1(z))$  and  $\alpha_i$ 's are defined recursively as the following

$$\begin{aligned}\alpha_1(z) &:= \alpha(z) \\ \alpha_2(z, \xi_1) &:= [L_f\alpha_1(z) + \xi_1 L_{g_1}\alpha_1(z)]/M(z) \\ \alpha_3(z, \xi_1, \xi_2) &:= [\frac{\partial \alpha_2}{\partial z}(f(z) + g_1(z)\xi_1 + g_2(z)\xi_2) + \frac{\partial \alpha_2}{\partial \xi_1}\xi_2] - \frac{\dot{M}}{M}[\xi_2 - \alpha_2(z, \xi_1)]\end{aligned}$$

with

$$\dot{M}(z, \xi_1, \xi_2) := L_f M(z) + \xi_1 L_{g_1} M(z) + \xi_2 L_{g_2} M(z)$$

In addition,  $\xi = \Psi(z, \mu)$ , the inverse of  $\Phi$ , is given in closed-form by

$$\begin{aligned}\xi_1 &= \alpha(z) + \mu_1 \\ \xi_2 &= \beta(z) + A(z)\mu_1 + B(z)\mu_2\end{aligned}\tag{7.50}$$

where

$$\beta(z) = \alpha_2(z, \alpha_1(z)), A(z) = \frac{L_{g_1}\alpha(z)}{M(z)}, B(z) = \frac{1}{M(z)} \quad (7.51)$$

Moreover,  $A(z), B(z)$  satisfy the following bounds

$$|A(z)| \leq L_1 L_2, 0 < B(z) \leq L_2$$

for all  $z \in \mathbb{R}^n$  (the constants  $L_1, L_2 > 0$  are defined in assumption 7.4.1).

**Proof.** Let us calculate the first two time derivatives of  $\mu_1 = \xi_1 - \alpha(z)$ . We get

$$\begin{aligned} \mu_2 &:= \dot{\mu}_1 = \dot{\xi}_2 - [L_f\alpha(z) + \xi_1 L_{g_1}\alpha(z) + \xi_2 L_{g_2}\alpha(z)] \\ &= (1 - L_{g_2}\alpha(z))\dot{\xi}_2 - [L_f\alpha(z) + \xi_1 L_{g_1}\alpha(z)] \\ &= M(z)(\dot{\xi}_2 - \frac{L_f\alpha(z) + \xi_1 L_{g_1}\alpha(z)}{M(z)}) \end{aligned}$$

Now, according to the definition of  $\alpha_2(z, \xi_1)$

$$\mu_2 = M(z)(\dot{\xi}_2 - \alpha_2(z, \xi_1))$$

Denoting  $w := \dot{\mu}_2$ ,  $w$  can be calculated as

$$\begin{aligned} w &= M(z)(u - \dot{\alpha}_2) + \dot{M}(\dot{\xi}_2 - \alpha_2(z, \xi_1)) \\ &= M(z)(u - \alpha_3(z, \xi_1, \xi_2)) \end{aligned}$$

where

$$\alpha_3(z, \xi_1, \xi_2) := \dot{\alpha}_2(z, \xi_1, \xi_2) - \frac{\dot{M}}{M}(\dot{\xi}_2 - \alpha_2(z, \xi_1))$$

So far, we proved that the new variables  $\mu_1, \mu_2, w$  satisfy the chain of integrators

$$\dot{\mu}_1 = \mu_2, \dot{\mu}_2 = w$$

To calculate the inverse of  $\mu = \Phi(z, \xi)$ , we need to solve  $\mu = \Phi(z, \xi)$  in  $(\xi_1, \xi_2)$ . One can solve for  $\xi_1$  as  $\xi_1 = \alpha(z) + \mu_1$ . Now, solving for  $\xi_2$  from the definition of  $\mu_2$  gives

$$\begin{aligned} \xi_2 &= \alpha_2(z, \xi_1) + \frac{1}{M(z)}\mu_2 \\ &= \alpha_2(z, \alpha(z) + \mu_1) + B(z)\mu_2 \\ &= \alpha_2(z, \alpha_1(z)) + A(z)\mu_1 + B(z)\mu_2 \end{aligned}$$

which after denoting  $\beta(z) = \alpha_2(z, \alpha_1(z))$ , it agrees with the definition of  $\beta(z)$  in the

question. Substituting  $\xi_1, \xi_2$  in the first equation of (7.42) shows that

$$\begin{aligned}\dot{z} &= f(z) + g_1(z)(\alpha(z) + \mu_1) + g_2(z)(\beta(z) + A(z)\mu_1 + B(z)\mu_2) \\ &= f_0(z) + [g_1(z) + A(z)g_2(z)]\mu_1 + B(z)g_2(z)\mu_2\end{aligned}$$

and this completes the proof of equation (7.49). The bounds for  $A(z)$  and  $B(z)$  follow from parts ii) and iii) of assumption 7.4.1.  $\square$

The following assumption guarantees that the zero-dynamics of the cascade nonlinear system in (7.49) with the output  $y = \mu_1$  is globally minimum phase.

**Assumption 7.4.2.** Assume for

$$\dot{z} = f(z) + g_1(z)\alpha(z) + g_2(z)\beta(z) =: f_0(z) \quad (7.52)$$

the origin  $z = 0$  is globally asymptotically stable.

**Assumption 7.4.3.** Let  $V(z)$  be a smooth positive definite and proper Lyapunov function satisfying  $L_{f_0}V(z) \leq 0$ . Suppose there exists  $\rho \geq 0$ ,  $k > 0$ , and  $\lambda \in (0, 1]$  such that the following conditions are satisfied

$$|L_{g_i}V(z)| \leq kV(z)^\lambda, \quad \forall \|z\| \geq \rho, \quad i = 1, 2 \quad (7.53)$$

Here is our main result on global asymptotic stabilization of nontriangular nonlinear systems affine in  $(\xi_1, \xi_2)$ :

**Theorem 7.4.1.** Consider the nonlinear system in (7.42). Suppose assumptions 7.4.1, 7.4.2, and 7.4.3 hold. Then, the following smooth state feedback in the form of a PD controller with nonlinear gains

$$u = \alpha_3(z, \xi_1, \xi_2) - \frac{c_1}{M(z)}(\xi_1 - \alpha_1(z)) - \frac{c_2}{M(z)}(\xi_2 - \alpha_2(z, \xi_1)), \quad c_1, c_2 > 0 \quad (7.54)$$

that globally asymptotically stabilizes  $(z, \xi_1, \xi_2) = 0$  for the nontriangular nonlinear system (7.42) with  $M(z) = 1 - L_{g_2}\alpha(z)$ .

**Proof.** Based on proposition 7.4.3, nonlinear system (7.42) can be transformed into the cascade system

$$\begin{aligned}\dot{z} &= f_0(z) + [g_1(z) + A(z)g_2(z)]\mu_1 + B(z)g_2(z)\mu_2 =: F(z, \mu_1, \mu_2) \\ \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= w\end{aligned} \quad (7.55)$$

such that for  $\dot{z} = F(z, 0, 0) = f_0(z)$ ,  $z = 0$  is GAS. Let us stabilize the  $\mu$ -subsystem with a PD controller

$$w = -c_1\mu_1 - c_2\mu_2, \quad c_1, c_2 > 0$$

Then,  $\mu = 0$  is globally exponentially stable. Based on theorem 7.3.3, to show that  $(z, \mu) = 0$  is globally asymptotically stable for the closed-loop cascade system in (7.55), we need to prove that any solution of the  $z$ -subsystem is bounded. For doing so, let us calculate  $\dot{V}$  along the solutions of (7.55). One obtains

$$\begin{aligned}\dot{V} &= L_{f_0}V(z) + [L_{g_1}V(z) + A(z)L_{g_2}V(z)]\mu_1 + B(z)(L_{g_2}V(z))\mu_2 \\ &\leq (|L_{g_1}V(z)| + |A(z)||L_{g_2}V(z)|)|\mu_1| + B(z)|L_{g_2}V(z)||\mu_2| \\ &\leq (1 + L_1L_2 + L_2)kV(z)^\lambda\|\mu\| \quad (\text{because } |\mu_i| \leq \|\mu\|, i = 1, 2) \\ &= \tilde{k}V(z)^\lambda\|\mu\|\end{aligned}$$

where  $\tilde{k} := 1 + L_1L_2 + L_2$ . On the other hand,  $\mu(t)$  is exponentially vanishing and  $\|\mu(\cdot)\|_1 < \infty$ . Thus, from the proof of theorem 7.3.4, it follows that any solution of the  $z$ -subsystem is bounded. The PD feedback  $w = -c_1\mu_1 - c_2\mu_2$  in the original coordinate can be written as

$$u = \alpha_3(z, \xi_1, \xi_2) - \frac{c_1}{M(z)}\mu_1 - \frac{c_2}{M(z)}\mu_2$$

which is a PD controller with nonlinear gains  $-c_i/M(z)$ ,  $i = 1, 2$ .  $\square$

*Remark 7.4.2.* The equation of  $\alpha_3$  in the state feedback (7.54) can be expressed in a slightly more explicit way as follows. Define  $p(z), q(z)$  as

$$p(z) := \frac{L_f\alpha(z)}{M(z)}, \quad q(z) := \frac{L_{g_1}\alpha(z)}{M(z)}$$

and observe that they satisfy  $\alpha_2(z, \xi_1) = p(z) + \xi_1 q(z)$ . Then, the right hand side of

$$\dot{\alpha}_3(z, \xi_1, \xi_2) = \dot{\alpha}_2 - \frac{\dot{M}}{M^2}\mu_2$$

can be explicitly determined as the following

$$\dot{\alpha}_2 = L_f p(z) + \xi_1 L_{g_1} q(z) + \xi_2 (q(z) + L_{g_2} q(z)) \quad (7.56)$$

$$\dot{M} = L_f M(z) + \xi_1 L_{g_1} M(z) + \xi_2 L_{g_2} M(z) \quad (7.57)$$

A crucial assumption in theorem 7.4.1 is assumption 7.4.2. In the following, we provide sufficient conditions on  $g_2(z)$  such that assumption 7.4.2 holds.

**Proposition 7.4.4.** *Consider the following nonlinear system*

$$\dot{z} = f(z) + g_1(z)\xi_1 + g_2(z)\xi_2$$

*and assume the following conditions hold:*

- i) *there exists a smooth state feedback  $\xi_1 = \alpha(z)$  such that for*

$$\dot{z} = f(z) + g_1(z)\alpha(z) =: f_1(z)$$

$z = 0$  is GAS,

ii)  $L_{g_2}\alpha(z) < 1$  for all  $z \in \mathbb{R}^n$ ,

iii) there exists a function  $h(z) : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $g_2(z) = h(z)f_1(z)$ .

Let

$$\beta(z) := \frac{L_{f_1}\alpha(z)}{1 - L_{g_2}\alpha(z)}$$

Then, for the closed-loop system

$$\dot{z} = f(z) + g_1(z)\alpha(z) + g_2(z)\beta(z) \quad (7.58)$$

$z = 0$  is GAS.

**Proof.** The closed-loop system with control  $(\xi_1, \xi_2) = (\alpha(z), \beta(z))$  can be written as

$$\dot{z} = \frac{f_1(z) + \Delta(z)}{1 - L_{g_2}\alpha(z)} \quad (7.59)$$

where

$$\Delta(z) := (L_{f_1}\alpha(z))g_2(z) - (L_{g_2}\alpha(z))f_1(z)$$

But  $g_2(z) \parallel f_1(z)$ , or  $g_2(z) = h(z)f_1(z)$  implies

$$(L_{f_1}\alpha(z))g_2(z) = (L_{g_2}\alpha(z))f_1(z), \quad \forall z \in \mathbb{R}^n$$

Thus,  $\Delta(z) \equiv 0$ . This means the closed-loop nonlinear system in (7.59) reduces to

$$\dot{z} = S(z)f_1(z) \quad (7.60)$$

with a scaling factor  $S(z) := (1 - L_{g_2}\alpha(z))^{-1} > 0$ . Since  $S(z)$  is a positive definite scalar,  $z = 0$  is GAS for  $\dot{z} = S(z)f_1(z)$  and the result follows.  $\square$

In the following result, we slightly remedy the restrictive condition  $g_2(z) \parallel f_1(z)$  in proposition 7.4.4.

**Proposition 7.4.5.** Consider the following nonlinear system

$$\dot{z} = f(z) + g_1(z)\xi_1 + g_2(z)\xi_2$$

and assume the following conditions hold:

i) there exists a smooth state feedback  $\xi_1 = \alpha(z)$  such that for

$$\dot{z} = f(z) + g_1(z)\alpha(z) =: f_1(z)$$

$z = 0$  is GAS.

- ii) Let  $V(z)$  be a smooth positive definite and proper Lyapunov function associated with  $\dot{z} = f_1(z)$  such that  $\nabla V(z)f_1(z) \leq 0$ . Assume the largest invariant set in  $\Omega = \{z \mid \nabla V(z)f_1(z) = 0\}$  is  $\{0\}$ .
- iii)  $L_{g_2}\alpha(z) < 1$  for all  $z \in \mathbb{R}^n$ ,
- iv) Assume  $V(z)$  satisfies the inequality

$$(L_{f_1}\alpha(z))(L_{g_2}V(z)) \leq (L_{g_2}\alpha(z))(L_{f_1}V(z)), \quad \forall z \in \mathbb{R}^n$$

where the equality is achieved for  $g_2(z) = h(z)f_1(z)$  with  $h(z) : \mathbb{R}^n \rightarrow \mathbb{R}$ .

Define

$$\beta(z) := \frac{L_{f_1}\alpha(z)}{1 - L_{g_2}\alpha(z)}$$

Then, for the closed-loop system

$$\dot{z} = f(z) + g_1(z)\alpha(z) + g_2(z)\beta(z) \quad (7.61)$$

$z = 0$  is GAS.

**Proof.** The closed-loop system with control  $(\xi_1, \xi_2) = (\alpha(z), \beta(z))$  is in the form

$$\dot{z} = S(z)(f_1(z) + \Delta(z)) \quad (7.62)$$

where  $S(z) = (1 - L_{g_2}\alpha(z))^{-1}$  and

$$\Delta(z) = (L_{f_1}\alpha(z))g_2(z) - (L_{g_2}\alpha(z))f_1(z)$$

Thus

$$L_\Delta V(z) = (L_{f_1}\alpha(z))(L_{g_2}V(z)) - (L_{g_2}\alpha(z))(L_{f_1}V(z)) \leq 0$$

which implies

$$\dot{V} = S(z)(L_{f_1}V(z) + L_\Delta V(z)) \leq L_{f_1}V(z) \leq 0$$

but the largest invariant set in  $\Omega$  is  $\{0\}$ , therefore  $z = 0$  is GAS for the closed-loop nonlinear system.  $\square$

**Example 7.4.1.** Consider the following cascade nonlinear system which is in a non-

triangular form affine in  $(\xi_1, \xi_2)$

$$\begin{aligned}\dot{z}_1 &= z_2 - \frac{z_2 \xi_2}{2(1 + \sqrt{z_1^2 + z_2^2})} \\ \dot{z}_1 &= \xi_1 + \frac{\tanh(z_1 + z_2) \xi_2}{2(1 + \sqrt{z_1^2 + z_2^2})} \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u\end{aligned}$$

We show that there exists a state feedback  $\xi_1 = \alpha(z)$  which satisfies all the conditions in theorem 7.4.1. and thus the last nonlinear system can be globally asymptotically stabilized to the origin. Take  $\alpha(z) = -\tanh(z_1 + z_2)$ . Then, for

$$\dot{z} = f_1(z) := \begin{bmatrix} z_2 \\ -\tanh(z_1 + z_2) \end{bmatrix}$$

$z = 0$  is GAS. This is due to the fact that

$$V(z) = \int_0^{z_1} \tanh(s) ds + \frac{1}{2} z_2^2 \quad (7.63)$$

is a valid positive definite Lyapunov function for  $\dot{z} = f_1(z)$ . Since

$$L_{g_2} \alpha(z) = \frac{\partial \alpha(z)}{\partial z} g_2(z) = (1 - \tanh^2(z_1 + z_2)) \frac{(z_2 - \tanh(z_1 + z_2))}{2(1 + \|z\|)},$$

we get

$$|L_{g_2} \alpha(z)| \leq |1 - \tanh^2(z_1 + z_2)| \frac{1 + |z_2|}{2(1 + \|z\|)} \leq \frac{1}{2}$$

and thus  $1/2 \leq 1 - L_{g_2} \alpha(z) \leq 3/2$  (i.e.  $L_2 = 2$  and  $1/2 \leq M(z) \leq 3/2$ ). In addition, due to

$$|L_{g_1} \alpha(z)| = \left| \frac{\partial \alpha(z)}{\partial z_2} \right| = 1 - \tanh^2(z_1 + z_2) \leq 1$$

we obtain  $L_1 = 1$ . So far, we have proved that assumption 7.4.1 holds. Using proposition 7.4.4, we can show that assumption 7.4.2 holds as well. This is a consequence of the fact that  $g_2(z)$  satisfies the relation  $g_2(z) = h(z)f_1(z)$  with

$$h(z) := \frac{-1}{2(1 + \|z\|)}$$

Finally, assumption 7.4.3 can be verified by direct calculation as follows. The Ly-

punov function  $V(z)$  in equation (7.63) satisfies

$$|L_{g_1}V(z)| = \left| \frac{\partial V(z)}{\partial z_2} \right| = |z_2| \leq \sqrt{2}V(z)^{\frac{1}{2}}, \quad \forall z \in \mathbb{R}^2$$

and

$$|L_{g_2}V(z)| = \frac{|z_2[\tanh(z_1 + z_2) - \tanh(z_1)]|}{2(1 + \|z\|)} \leq |z_2| \leq \sqrt{2}V(z)^{\frac{1}{2}}, \quad \forall z \in \mathbb{R}^2$$

The last inequality is due to the global Lipschitz property of  $\tanh(x)$  which implies

$$|\tanh(z_1 + z_2) - \tanh(z_1)| \leq |z_2|$$

Hence for  $k = \sqrt{2}$ ,  $\lambda = 1/2$ , the following inequalities hold

$$|L_{g_i}V(z)| \leq kV(z)^{\lambda}, \quad \forall z \in \mathbb{R}^2, i = 1, 2$$

Therefore, based on theorem 7.4.1, the following PD controller with nonlinear gains

$$u = \alpha_3(z, \xi_1, \xi_2) - \frac{c_1}{M(z)}(\xi_1 - \alpha_1(z)) - \frac{c_2}{M(z)}(\xi_2 - \alpha_2(z, \xi_1)), \quad c_1, c_2 > 0$$

globally asymptotically stabilizes the origin  $(z, \xi_1, \xi_2) = 0$  for the nontriangular nonlinear system in the question.

#### 7.4.2 Nontriangular Nonlinear Systems Non-affine in $(\xi_1, \xi_2)$

In this section, we generalize our results on cascade backstepping procedure for stabilization of nontriangular nonlinear systems affine in  $(\xi_1, \xi_2)$  to cascade nonlinear systems that their vector fields are non-affine in  $(\xi_1, \xi_2)$ . We consider the following class of partially-linear cascade nonlinear systems

$$\begin{aligned} \dot{z} &= F(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \tag{7.64}$$

where  $F(z, \xi_1, \xi_2) : \mathbb{R}^{n_z} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n_z}$  is assumed to be a smooth function satisfying  $F(0) = 0$  and non-affine in  $(\xi_1, \xi_2)$ . The function  $F(z, \xi_1, \xi_2)$  can be rewritten as

$$F(z, \xi_1, \xi_2) = f(z) + g_1(z, \xi_1)\xi_1 + g_2(z, \xi_1, \xi_2)\xi_2 \tag{7.65}$$

where

$$f(z) := F(z, 0, 0) \quad (7.66)$$

$$g_1(z, \xi_1) := \int_0^1 (D_{\xi_1} F)(z, s\xi_1, 0) ds \quad (7.67)$$

$$g_2(z, \xi_1, \xi_2) := \int_0^1 (D_{\xi_2} F)(z, \xi_1, s\xi_2) ds \quad (7.68)$$

In this section, we provide sufficient conditions for global stabilization of nonlinear systems in normal form (7.64). Depending on the particular structure of  $f, g_1, g_2$  a number of important cases arise that will be discussed later in this chapter. The importance of each case is due to its application as the normal form of special classes of mechanical systems of interest. The main emphasis will be on control design for nontriangular normal forms of underactuated systems. Before presenting our main result, we need to make some assumptions.

**Assumption 7.4.4.** Assume there exists a smooth state feedback  $\xi_1 = \alpha(z)$  with  $\alpha(0) = 0$  that globally asymptotically stabilizes  $z = 0$  for

$$\dot{z} = F(z, \xi_1, 0) =: f_1(z, \xi_1) \quad (7.69)$$

Let  $V(z)$  be the smooth positive definite and proper Lyapunov function associated with the closed-loop system such that  $\nabla V(z) \cdot f_1(z, \alpha(z)) < 0, \forall z \neq 0$ .

**Assumption 7.4.5.** The exists a smooth function  $\beta(z) : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\beta(0) = 0$  such that the following conditions hold:

i)  $\xi_2^* = \beta(z)$  is the unique solution of the following fixed-point equation in  $\xi_2$

$$\xi_2 = \frac{\partial \alpha(z)}{\partial z} F(z, \alpha(z), \xi_2)$$

ii)  $\nabla V(z) \cdot F(z, \alpha(z), \beta(z)) < 0, \forall z \neq 0$ .

**Assumption 7.4.6.** Define

$$\psi(z, \xi_1, \xi_2) := -c_1(\xi_1 - \alpha(z)) + \frac{\partial \alpha(z)}{\partial z} F(z, \xi_1, \xi_2) \quad (7.70)$$

and assume there exists an  $a > 0$  such that

$$\frac{\partial \psi}{\partial \xi_2} = \frac{\partial \alpha(z)}{\partial z} \frac{\partial F(z, \xi_1, \xi_2)}{\partial \xi_2} \leq 1 - a, \quad \forall (z, \xi_1, \xi_2)$$

This implies

$$M(z, \xi_1, \xi_2) := 1 - \frac{\partial \psi}{\partial \xi_2} \geq a > 0, \quad \forall (z, \xi_1, \xi_2) \quad (7.71)$$

**Assumption 7.4.7.** Assume the change of coordinates  $\mu = T(z, \xi)$  given by

$$\begin{aligned}\mu_1 &= \xi_1 - \alpha(z) \\ \mu_2 &= \xi_2 - \psi(z, \xi_1, \xi_2)\end{aligned}$$

is a global diffeomorphism for each  $z \in \mathbb{R}^n$  with an inverse  $\xi = T^{-1}(z, \mu)$  as the following

$$\begin{aligned}\xi_1 &= \mu_1 + \alpha(z) \\ \xi_2 &= \phi(z, \mu_1, \mu_2)\end{aligned}$$

where  $\phi(z, 0, 0) = \beta(z)$ .

Now, the following result clarifies our purpose of making the last four assumptions.

**Proposition 7.4.6.** Consider the nonlinear system in (7.64) and suppose assumptions 7.4.4 through 7.4.7 hold. Then, the change of coordinates  $\mu = T(z, \xi)$  and the change of control given by

$$\begin{aligned}\mu_1 &= \xi_1 - \alpha(z) \\ \mu_2 &= \xi_2 - \psi(z) \\ w &= M(z, \xi_1, \xi_2)(u - \gamma(z, \xi_1, \xi_2))\end{aligned}\tag{7.72}$$

is globally invertible where

$$\gamma(z, \xi_1, \xi_2) = \frac{D_z \psi \cdot F(z, \xi_1, \xi_2) + D_{\xi_1} \psi \cdot \xi_2}{1 - D_{\xi_2} \psi}\tag{7.73}$$

In addition, it transforms the dynamics of (7.64) into the following cascade form

$$\begin{aligned}\dot{z} &= F_0(z, \mu_1, \mu_2) \\ \dot{\mu}_1 &= -c_1 \mu_1 + \mu_2 \\ \dot{\mu}_2 &= w\end{aligned}\tag{7.74}$$

where

$$F_0(z, \mu_1, \mu_2) = F(z, \alpha(z) + \mu_1, \phi(z, \mu_1, \mu_2))$$

Moreover, for

$$\dot{z} = F_0(z, 0, 0) = F(z, \alpha(z), \beta(z))$$

$z = 0$  is GAS.

**Proof.** Based on assumptions 7.4.6 and 7.4.7,  $\mu = T(z, \xi)$  is a global diffeomorphism and  $M(z, \xi_1, \xi_2) \geq a > 0$ . Thus, the change of coordinates and control in (7.74) is globally invertible. By definitions of  $\mu_1$  and  $\psi$ , we have

$$\dot{\mu}_1 = \xi_2 - \frac{\partial \alpha(z)}{\partial z} F(z, \xi_1, \xi_2) = -c_1 \mu_1 + \mu_2$$

and because  $\mu_2 = \xi_2 - \psi(z, \xi_1, \xi_2)$ , one gets

$$\begin{aligned}\dot{\mu}_2 &= (1 - D_{\xi_2}\psi)u - (D_z\psi \cdot F(z, \xi_1, \xi_2) + D_{\xi_1}\psi) \\ &= M(z, \xi_1, \xi_2)(u - \gamma(z, \xi_1, \xi_2)) =: w\end{aligned}$$

which means the  $\mu$ -subsystem has the following linear dynamics

$$\begin{aligned}\dot{\mu}_1 &= -c_1\mu_1 + \mu_2 \\ \dot{\mu}_2 &= w\end{aligned}$$

global asymptotic stability of  $z = 0$  for

$$\dot{z} = F_0(z, 0, 0)$$

follows from part ii) of assumption 7.4.5.  $\square$

**Assumption 7.4.8.** Rewrite  $F_0(z, \mu_1, \mu_2)$  as

$$F_0(z, \mu_1, \mu_2) = F_0(z, 0, 0) + h_1(z, \mu_1)\mu_1 + h_2(z, \mu_1, \mu_2)\mu_2$$

and assume for any  $l_1, l_2 > 0$ , there exist  $\rho_i \geq 0$ ,  $k_i > 0$ ,  $\lambda_i \in (0, 1]$  for  $i = 1, 2$  such that  $V(z)$  satisfies

$$\begin{aligned}|\nabla V(z) \cdot h_1(z, \mu_1)| &\leq k_1 V(z)^{\lambda_1}, \quad \|z\| \geq \rho_1, |\mu_1| \leq l_1 \\ |\nabla V(z) \cdot h_2(z, \mu_1, \mu_2)| &\leq k_2 V(z)^{\lambda_2}, \quad \|z\| \geq \rho_2, |\mu_1| \leq l_1, |\mu_2| \leq l_2\end{aligned} \quad (7.75)$$

**Lemma 7.4.1.** For any  $l > 0$ , take  $l_1, l_2 \geq l$  and suppose assumption 7.4.8 holds. Then, the following condition is satisfied

$$|\nabla V(z)h_i| \leq kV(z)^\lambda, \quad \|z\| \geq \rho, \|\mu\| < l, i = 1, 2 \quad (7.76)$$

with  $\rho = \max\{\rho_1, \rho_2\} \geq 0$ ,  $k = \max\{k_1, k_2\} > 0$ , and  $\lambda = \max\{\lambda_1, \lambda_2\} \in (0, 1]$ .

**Proof.** Since  $|\mu_i| \leq \|\mu\|$  for  $i = 1, 2$  and the function  $\varphi(k, l) = kx^\lambda$  for  $x > 0$  is an increasing function of  $\lambda$  (or  $k$ ) uniformly in  $k$  (or  $\lambda$ ) the result holds.  $\square$

We are now ready to present our main result on global stabilization of nontriangular nonlinear systems non-affine in  $(\xi_1, \xi_2)$ :

**Theorem 7.4.2.** Consider the nonlinear system in (7.64). Suppose that assumptions 7.4.4 through 7.4.8 hold. Then, the following state feedback in explicit form

$$u = K(z, \xi_1, \xi_2) := \gamma(z, \xi_1, \xi_2) - \frac{c_2}{M(z, \xi_1, \xi_2)}(\xi_2 - \psi(z, \xi_1, \xi_2)), \quad c_2 > 0 \quad (7.77)$$

globally asymptotically stabilizes the origin  $(z, \xi_1, \xi_2) = 0$  for (7.64).

**Proof.** According to proposition 7.4.6, after applying the change of coordinates and control in equation (7.72), one obtains cascade system (7.74). The linear  $\mu$ -subsystem

of (7.74) can be globally exponentially stabilized using  $w = -c_2\mu_2$ ,  $c_2 > 0$ . The closed-loop  $\mu$ -subsystem is in the form  $\dot{\mu} = A\mu$  where

$$A = \begin{bmatrix} -c_1 & 1 \\ 0 & -c_2 \end{bmatrix}$$

is a Hurwitz matrix with eigenvalues  $-c_1, -c_2 < 0$ . In new coordinates, the control  $u$  can be expressed as the following

$$u = \gamma(x) - \frac{c_2}{M(x)}(\xi_2 - \psi(x))$$

with  $x = \text{col}(z, \xi_1, \xi_2)$  as given in (7.77). Based on assumption 7.4.8 and lemma 7.4.1, any solution of the closed-loop  $z$ -subsystem in (7.74) with state feedback  $w = -c_2\mu_2$  (or (7.77)) is bounded. Thus, based on theorem 7.3.3,  $(z, \mu) = 0$  is GAS for the cascade nonlinear system in (7.74). This implies the origin  $(z, \xi_1, \xi_2) = 0$  is GAS for the closed-loop nonlinear system (7.64).  $\square$

### 7.4.3 Global Existence of Fixed-Point Control Laws

A crucial assumption in the global version of the cascade backstepping procedure in theorem 7.77 is that the fixed point equation in  $\xi_2$  of the form

$$\xi_2 = \frac{\partial \alpha(z)}{\partial z} F(z, \alpha(z), \xi_2) =: \Psi(z, \xi_2) \quad (7.78)$$

(see assumption 7.4.5) globally (w.r.t  $z$ ) has solutions in  $\xi_2$ . More precisely, there exists  $\beta : \mathbb{R}^{n_0} \rightarrow \mathbb{R}$  such that

$$\beta(z) = \Psi(z, \beta(z)), \quad \forall z \in \mathbb{R}^{n_0}$$

In this section, we provide sufficient conditions for global existence of a solution  $\xi_2^* = \beta(z)$  for the fixed point equation

$$\xi_2 = \Psi(z, \xi_2) \quad (7.79)$$

We refer to the solution  $\beta(z)$  of this fixed point equation as a *fixed point control law*. Due to the structure of  $F(z, \xi_1, \xi_2)$  in equation (7.15), we focus on global existence of fixed-point control laws for nontriangular systems with the following dynamics

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z, \xi_1, \xi_2) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned} \quad (7.80)$$

where

$$f(z, \xi_1, \xi_2) = f_0(z, \xi_1) + g_0(z, \xi_1, \xi_2) \xi_2^2 \quad (7.81)$$

Before presenting our main result, we need to make some definitions and give some lemmas. Let us focus our attention to the following class of state feedback laws  $\alpha(z)$ .

**Definition 7.4.1.** (bounded functions with compact domain of attention) We say a function  $\alpha(z) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  with  $\alpha(0) = 0$  and  $z = \text{col}(z_1, z_2)$  is *bounded with compact domain of attention around the origin w.r.t.  $z_2$*  if there exist a compact neighborhood  $\mathcal{K}_0 \subset \mathbb{R}^n$  of  $z_2 = 0$  and constants  $L_i > 0, i = 0, 1, 2$  such that

- i)  $|\alpha(z)| \leq L_0, \forall z \in \mathbb{R}^{2n}$
- ii)  $D_{z_2}\alpha(z) = 0, \forall z \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \mathcal{K}_0)$
- iii)  $\|D_{z_2}\alpha(z)\| \leq L_1, \forall z \in \mathbb{R}^{2n}$
- iv)  $\|D_{z_1}\alpha(z)\| \leq L_2, \forall z \in \mathbb{R}^{2n}$

**Definition 7.4.2.** (bounded sigmoidal functions with flat tails) We say  $\sigma(x) : \mathbb{R} \rightarrow \mathbb{R}$  is a *bounded sigmoidal function with a flat tail* if the following conditions hold:

- i)  $\exists L > 0 : |\sigma(x)| \leq L, \forall x$ .
- ii)  $x\sigma(x) > 0, \forall x \neq 0$  and  $\sigma(0) = 0$ .
- iii)  $\exists I \subset \mathbb{R}, 0 \in I : \sigma'(x) = 0, \forall x \notin I$ .

Clearly, over the region  $\mathbb{R} \setminus I$ ,  $\sigma(x)$  is constant (which explains the name “flat tail”).

**Example 7.4.2.** (sigmoidal functions) A nonsmooth example of a sigmoidal function with a flat tail is the saturation function  $\text{sat}(x) = \text{sgn}(x) \min\{|x|, 1\}$ . A  $C^1$  smooth bounded sigmoidal function with flat tails is

$$\sigma(x) = \begin{cases} \sin(\frac{\pi}{2}x) & x \in [-1, 1] \\ 1 & x > 1 \\ -1 & x < -1 \end{cases} \quad (7.82)$$

or in a more compact form

$$\sigma(x) = \sin(\frac{\pi}{2}x) \cdot 1_I(x) + (1 - 1_I(x)) \cdot \text{sgn}(x), \quad I = [-1, 1] \quad (7.83)$$

where  $1_I(x)$ ,  $\text{sgn}(x)$  denote the indicator function and the sign function, respectively. In general, it is possible to construct a  $C^r$  smooth bounded sigmoidal function with flat tails and (relatively) large linear region as

$$\sigma_f(x; a) = \begin{cases} x & |x| \leq a \\ \text{sgn}(x) \cdot P(|x|) & a < |x| \leq 1 \\ \text{sgn}(x) & |x| > 1 \end{cases} \quad (7.84)$$

where  $a \in (0, 1)$  is a constant threshold and  $P(x)$  is a polynomial of an appropriate order. For example, with  $a = 0.75$  and  $I = [-1, 1]$  the following 5th order polynomial gives a  $C^2$  smooth bounded sigmoidal function with flat tails

$$P(x) = 768x^5 - 3328x^4 + 5728x^3 - 4896x^2 + 2080x - 351 \quad (7.85)$$

The function  $\sigma_f(x; 0.75)$  with its first two derivatives is depicted in Figure 7-1. In this case,  $\sigma_f(x; 0.75)$  is strictly increasing over the interval  $I = [-1, 1]$ .

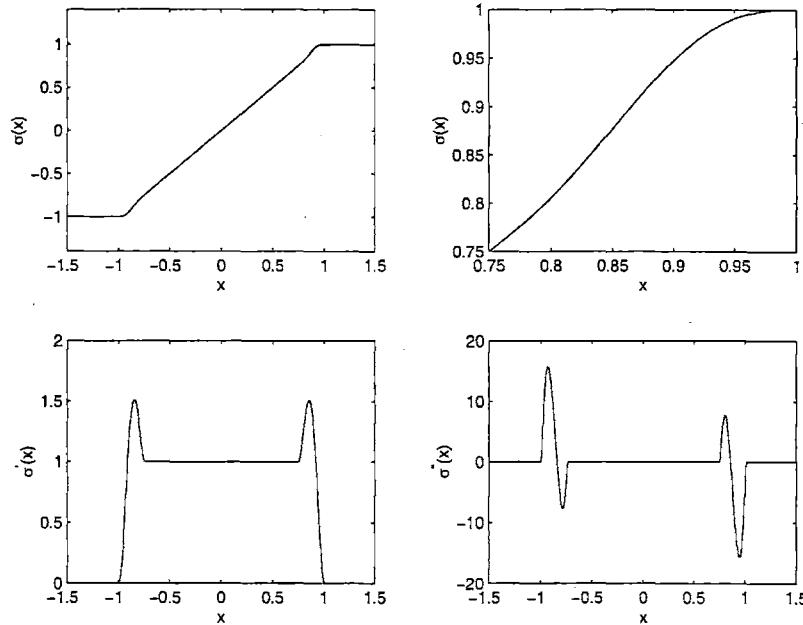


Figure 7-1: A  $C^2$  smooth sigmoidal function with flat tails with its derivatives.

**Example 7.4.3.** Consider the following double-integrator with a saturation control

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \text{sat}(\xi_1) \end{aligned} \quad (7.86)$$

where  $\text{sat}(x) = \text{sgn}(x) \min\{|x|, 1\}$ . System (7.86) is a special case of the  $z$ -subsystem of (7.80) under the assumption that  $f(z, \xi_1, 0) = \text{sat}(\xi_1)$ . The following state feedback

$$\alpha(z) = -\sigma_1(z_1 + \sigma_2(z_2)) \quad (7.87)$$

globally asymptotically stabilize  $(z_1, z_2) = 0$  for (7.86) where  $\sigma_1$  is a strictly increasing bounded sigmoidal function and  $\sigma_2$  is a bounded sigmoidal function with flat tails outside  $I = [-1, 1]$ . By definition,  $\alpha(z)$  in (7.87) has compact domain of attention w.r.t.  $z_2$  with  $\mathcal{K}_0 = I$ .

**Lemma 7.4.2.** Consider the nonlinear system in (7.86). Let  $\alpha(z)$  be the state feedback in (7.87) with  $\sigma_1$  and  $\sigma_2$  which are bounded sigmoidal functions with flat tails outside  $I = [-1, 1]$ . Then, the following statements hold:

i)  $z = 0$  is globally asymptotically stable for (7.86) with state feedback  $\alpha(z)$ .

ii) there exists a compact neighborhood  $\mathcal{K} \subset \mathbb{R}^2$  of  $z = 0$  as the following

$$\mathcal{K} = \{(z_1, z_2) \in \mathbb{R}^2 \mid |z_1 + \sigma_2(z_2)| \leq 1, |z_2| \leq 1\} \subset [-2, 2] \times [-1, 1] \quad (7.88)$$

such that  $D_{z_2}\alpha(z) = 0$  for  $z \notin \mathcal{K}$ .

**Proof.** Let us define the following global change of coordinates

$$\begin{aligned} y_1 &= z_1 + \sigma_2(z_2) \\ y_2 &= z_2 \end{aligned} \quad (7.89)$$

In new coordinates, the dynamics of the closed-loop system (7.86) takes the form

$$\begin{aligned} \dot{y}_1 &= y_2 - \sigma'_2(y_2)\sigma_1(y_1) \\ \dot{y}_2 &= -\sigma_1(y_1) \end{aligned} \quad (7.90)$$

Define

$$\phi(y_1) = \int_0^{y_1} \sigma_1(s)ds$$

and consider the following positive definite and smooth Lyapunov function candidate

$$V(y_1, y_2) = \phi(y_1) + \frac{1}{2}y_2^2$$

which satisfies

$$\dot{V} = -\sigma'_2(y_2)\sigma_1(y_1)^2 \leq 0$$

We need to find the largest invariant set in  $\Omega = \{y \in \mathbb{R}^2 \mid \dot{V} = 0\}$ , or

$$\Omega = \{y \in \mathbb{R}^2 \mid |y_2| \geq 1 \text{ or } y_1 = 0\}$$

A simple analysis shows that there is no invariant set in the region  $|y_2| > 1$  and the only invariant set for  $y_1 = 0$  is  $\{(0, 0)\}$ . Thus, from LaSalle's invariance principle,  $y = 0$  (or  $z = 0$ ) is GAS. To prove part ii), notice that  $D_{z_2}\alpha(z) = \sigma'_2(y_2)\sigma'_1(y_1) = 0$  for all  $y \in S = \{y \mid |y_1| > 1, |y_2| > 1\}$ . Clearly,  $\mathcal{K}$  is the complement of  $S$  in  $\mathbb{R}^2$ . Also, assuming  $|z_1| > 2$ , we get

$$|y_1| \geq |z_1| - |\sigma_2(z_2)| > 1$$

and therefore  $D_{z_2}\alpha(z) = 0$ . This means outside  $\tilde{\mathcal{K}} = [-2, 2] \times [-1, 1] \supset \mathcal{K}$ ,  $D_{z_2}\alpha(z)$  vanishes.  $\square$

The main benefit of lemma 7.4.2 is that outside a compact set  $\mathcal{K}$ , one can explicitly determine the fixed point control law  $\xi_2^* = \beta(z)$  of equation (7.78). Since  $D_{z_2}\alpha(z) = 0$

over  $\mathbb{R}^2 \setminus \mathcal{K}$ , we obtain

$$\beta(z) = \frac{\partial \alpha(z)}{\partial z_1} z_2$$

and the stability analysis for the closed-loop  $z$ -subsystem

$$\dot{z} = F(z, \alpha(z), \beta(z))$$

reduces to a stability analysis over the sets  $(\mathbb{R}^2 \setminus \mathcal{S})$  and  $(\mathcal{S} \setminus \mathcal{K})$  where the set  $\mathcal{S}$  is called a *sigmoidal ribbon* defined as the following

$$\mathcal{S} := \{z \in \mathbb{R}^2 \mid |z_1 + \sigma_2(z_2)| \leq 1\}$$

By definition,  $\mathcal{K} \subset \mathcal{S}$ . We have

$$\beta(z) = \begin{cases} D_{z_1} \alpha(z) \cdot z_2 & z \in (\mathcal{S} \setminus \mathcal{K}) \\ 0 & z \in (\mathbb{R}^2 \setminus \mathcal{S}) \end{cases} \quad (7.91)$$

Figure 7-2 shows different regions including the sigmoidal ribbon  $\mathcal{S}$  (light shaded) and the compact set  $\mathcal{K}$  (dark shaded). The dashed regions in Figure 7-2 show where the control law  $\alpha(z)$  is constant (i.e.  $D\alpha(z) = 0$ ). It remains to find  $\beta(z)$  over the

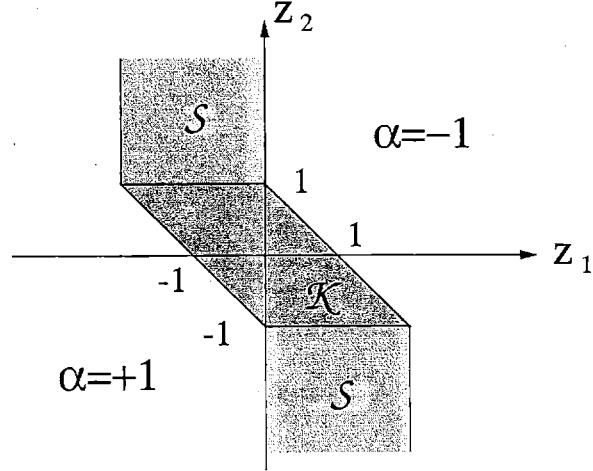


Figure 7-2: Sigmoidal ribbon  $\mathcal{S}$  and compact set  $\mathcal{K}$ .

compact set  $\mathcal{K}$ . This is done in the following for the general case where  $z_1, z_2 \in \mathbb{R}^n$ .

**Assumption 7.4.9.** Suppose that for any compact set  $0 \in \mathcal{K} := \mathcal{K}_{z_2} \times \mathcal{K}_{\xi_1} \times \mathcal{K}_{\xi_2} \subset \mathbb{R}^{n+2}$ , there exist constants  $a_i > 0, b_i \geq 0, i = 1, 2, 3$  such that  $f_0, g_0, D_{\xi_2} f$  have incremental linear growth in  $z_1$  for all  $(z_2, \xi_1, \xi_2) \in \mathcal{K}$ , i.e.

- i)  $\|f_0(z, \xi_1)\| \leq a_1 + b_1 \|z_1\|$
- ii)  $\|g_0(z, \xi_1, \xi_2)\| \leq a_2 + b_2 \|z_1\|$

$$\text{iii}) \quad \|D_{\xi_2} f(z, \xi_1, \xi_2)\| \leq a_3 + b_3 \|z_1\|$$

The following scaling of the function  $\alpha(z)$  is influenced by the work of Sepulchre [79].

**Lemma 7.4.3.** *Let the smooth function  $\alpha(z) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  with  $\alpha(0) = 0$  be a bounded function that has a compact domain of attention  $\mathcal{K}_0$  around  $z_2 = 0$  with constants  $L_i > 0, i = 0, 1, 2$  satisfying all the conditions in definition 7.4.1. Define*

$$\alpha_\kappa(z) = \kappa(z_1) \alpha(z) \quad (7.92)$$

where

$$\kappa(z_1) := \begin{cases} 1 & , \text{if } b_i = 0, i = 1, 2, 3 \\ \frac{1}{\sqrt{1 + \|z_1\|^2}} & , \text{otherwise} \end{cases}$$

Then,  $\alpha_\kappa(z)$  is also a bounded function with compact domain of attention  $\mathcal{K}_0$  around  $z_2 = 0$  and associated constants  $L_0^\kappa = L_0$ ,  $L_1^\kappa = L_1$ , and  $L_2^\kappa = L_0 + L_2$ .

**Proof.** Noting that  $0 < \kappa(z_1) \leq 1$ , part i) of definition 7.4.1 immediately follows. To prove properties ii) and iii) of  $\alpha_\kappa(z)$  (in the same definition), notice that

$$D_{z_2} \alpha_\kappa(z) = \kappa(z_1) D_{z_2} \alpha(z)$$

Thus,  $D_{z_2} \alpha_\kappa(z)$  vanishes for all  $z_2 \notin \mathcal{K}_0$ . In addition,  $\|D_{z_2} \alpha_\kappa(z)\| \leq \|D_{z_2} \alpha(z)\| \leq L_1$ ,  $\forall z \in \mathbb{R}^2$ . To prove the part iv), let us calculate  $D_{z_1} \alpha_\kappa(z)$  as

$$D_{z_1} \alpha_\kappa(z) = (D_{z_1} \kappa(z_1)) \alpha(z) + \kappa(z_1) D_{z_1} \alpha(z)$$

but  $D_{z_1} \kappa(z_1) = z_1 \kappa(z_1)^3$  and  $\|D_{z_1} \kappa(z_1)\| \leq \|z_1 \kappa(z_1)\| \kappa(z_1)^2 \leq 1$ . Hence, we get

$$\|D_{z_1} \alpha_\kappa(z)\| \leq L_0 + L_2$$

□

**Theorem 7.4.3.** *Suppose that Assumption (7.4.9) holds. Let the smooth function  $\alpha(z) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  with  $\alpha(0) = 0$  be a bounded function that has a compact domain of attention  $\mathcal{K}_0$  around  $z_2 = 0$  with constants  $L_i > 0, i = 0, 1, 2$ . In addition, let  $\alpha_\kappa(z)$  denote the scaled  $\alpha(z)$  defined in equation (7.92). Then, there exist  $L_i^*(\mathcal{K}_0) > 0, i = 0, 1, 2$  such that for all  $\alpha(z)$  with associated constants  $0 < L_i < L_i^*, i = 0, 1, 2$ , the following fixed-point equation*

$$\xi_2 = D_z \alpha_\kappa(z) \cdot F(z, \alpha_\kappa(z), \xi_2) \quad (7.93)$$

globally has a unique smooth solution

$$\xi_2^* = \beta(z)$$

with  $\beta(0) = 0$ .

**Proof.** See section C.1 in Appendix C. □

*Remark 7.4.3.* Based on the contraction mapping theorem, the fixed point  $\xi_2^* = \beta(z)$  in theorem 7.4.3 can be obtained using the following iteration in  $k$

$$\xi_2^{(k+1)} = \Psi(z, \xi_2^{(k)}), \quad \xi_2^{(0)} = 0 \quad (7.94)$$

and satisfies  $\beta(z) = \lim_{k \rightarrow \infty} \xi_2^{(k)} = \xi_2^*$ .

**Proposition 7.4.7.** Assume  $F$  has the following structure

$$F(z, \xi_1, \xi_2) = \begin{bmatrix} z_2 \\ f_0(z, \xi_1) + g_0(z, \xi_1) \xi_2^2 \end{bmatrix}$$

Then, the fixed point equation in theorem 7.4.3 reduces to the following quadratic equation in  $\xi_2$

$$\xi_2 = D_{z_1} \alpha_\kappa(z) \cdot z_2 + D_{z_2} \alpha_\kappa(z) \cdot [f_0(z, \alpha_\kappa(z)) + g_0(z, \alpha_\kappa(z))] \xi_2^2 \quad (7.95)$$

so that its solutions can be given in explicit form.

**Proof.** The fixed point equation in (7.95) can be rewritten as a standard quadratic equation in  $\xi_2$  in the form

$$a(z) \xi_2^2 + b \xi_2 + c(z) = 0 \quad (7.96)$$

where

$$\begin{aligned} a(z) &= D_{z_2} \alpha_\kappa(z) g_0(z, \alpha_\kappa(z)) \\ b &= -1 \\ c(z) &= D_{z_1} \alpha_\kappa(z) \cdot z_2 + D_{z_2} \alpha_\kappa(z) \cdot f_0(z, \alpha_\kappa(z)) \end{aligned}$$

The condition for global existence of a fixed point solution  $\xi_2^* = \beta(z)$  can be readily expressed as  $\Delta(z) = 1 - 4a(z)c(z) > 0$ . Then the solution of the quadratic equation (7.96) in  $\xi_2$  with the smallest absolute value [72] is given by

$$\beta(z) = \frac{2c(z)}{1 + \sqrt{1 - 4a(z)c(z)}} \quad (7.97)$$

The solution in (7.97) is the same as the following standard solution of (7.96)

$$\beta(z) = \begin{cases} \frac{1 - \sqrt{1 - 4a(z)c(z)}}{2a(z)} & a(z) \neq 0 \\ c(z) & a(z) = 0 \end{cases}$$

□

*Remark 7.4.4.* Following the line of proposition 7.4.7, let  $\mathcal{K}$  be a compact set such that  $D_{z_2}\alpha(z) = 0$  for all  $z \notin \mathcal{K}$  (see part (ii) of lemma 7.4.2). Set  $\kappa(z) = 1$ , or let  $\alpha_\kappa(z) = \alpha(z)$ . Then, the existence condition of the fixed point  $\xi_2^*$

$$a(z)c(z) < \frac{1}{4}$$

has to hold over the compact set  $\mathcal{K} \subset \mathbb{R}^{n_0}$  instead of the whole  $\mathbb{R}^{n_0}$ . This tremendously reduces the complexity of checking the global existence of a fixed point control law. We demonstrate this in the following. Since  $\mathcal{K}$  is a compact set, there exists constants  $L_{f_0}, L_{g_0}, L_{z_2} > 0$  such that

$$\|f_0(z, \alpha(z))\| \leq L_{f_0}, \|g_0(z, \alpha(z))\| \leq L_{g_0}, \|z_2\| \leq L_{z_2}$$

for all  $z \in \mathcal{K}$ . Hence, the following condition

$$|a(z)c(z)| \leq L_1 L_{g_0} (L_2 L_{z_2} + L_1 L_{f_0}) < \frac{1}{4}$$

is satisfied by choosing the constants  $L_1, L_2$  associated with  $\alpha(z)$  according to

$$0 < L_1, L_2 < L^* := \frac{1}{2\sqrt{L_{g_0}(L_{f_0} + L_{z_2})}} \quad (7.98)$$

where  $L^*$  can be readily calculated.

**Example 7.4.4.** (beam-and-ball system) Consider the dynamics of the beam-and-ball system given in equation (5.33). The third-order subsystem of (5.33) is in the form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -g \sin(\xi_1) + (z_1 + d\xi_1)\xi_2^2 \\ \dot{\xi}_1 &= \xi_2 \end{aligned} \quad (7.99)$$

where  $(\xi_1, \xi_2) = (\theta, \omega)$ . Define

$$\begin{aligned} f_0(z, \xi_1) &= -g \sin(\xi_1) \\ g_0(z, \xi_1) &= z_1 + d\xi_1 \end{aligned} \quad (7.100)$$

Clearly,  $f_0, g_0$  satisfy assumption 7.4.9. Let  $\alpha(z) = -c_0 \sigma_2(z_1 + \sigma_1(z_1))$  where  $\sigma_i$ 's are sigmoidal functions with flat tails outside  $[-1, 1]$ . In addition,  $|\sigma_2(x)| \leq 1, \forall x$  and  $0 < c_0 < \pi/2$  is a constant (to be determined). One can take  $c_0$  to be sufficiently small such that condition  $a(z)c(z) < 1/4$  in remark 7.4.4 is satisfied over the compact set  $\mathcal{K}$  (shown in Figure 7-2). Based on proposition 7.4.7, the state feedback  $\beta(z)$  can be explicitly and globally obtained from (7.97).

#### 7.4.4 Slightly Nontriangular Nonlinear Systems

In this section, we consider stabilization of a perturbed version of a nonlinear system in strict feedback form such that the overall system is non-triangular. We call this class of systems *slightly nontriangular systems*.

**Theorem 7.4.4.** *Consider the following nonlinear system*

$$\begin{aligned}\dot{z} &= f_0(z, \xi_1) + g_0(z, \xi_1, \xi_2, \epsilon), \\ \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= u,\end{aligned}\tag{7.101}$$

where  $f_0, g_0$  are smooth functions with  $f_0(0) = 0, g_0(0) = 0, g_0(z, \xi_1, 0, \epsilon) = 0, g_0(z, \xi_1, \xi_2, 0) = 0$ , and  $\partial g_0 / \partial \xi_2$  vanishes at  $\xi_2 = 0$  (i.e.  $\exists p_0 \in C^1 : g_0 = p_0(z, \xi_1, \xi_2, \epsilon) \xi_2^2 \epsilon$ ). Suppose there exists a state feedback law  $\xi_1 = \alpha_1(z)$  with  $\alpha_1(0) = 0$  such that for

$$\dot{z} = f_0(z, \alpha_1(z))$$

$z = 0$  is globally asymptotically and locally exponentially stable. Let  $V(z)$  be the smooth positive definite proper Lyapunov function associated with this system satisfying the property  $\nabla V(z) \cdot f_0(z, \alpha_1(z)) < 0, \forall z \neq 0$ . Define

$$\begin{aligned}\psi_2(z, \xi_1, \xi_2, \epsilon) &= -\sigma(\xi_1 - \alpha_1(z)) + \frac{\partial \alpha_1(z)}{\partial z} f(z, \xi_1, \xi_2, \epsilon) \\ \psi_3(z, \xi_1, \xi_2, u, \epsilon) &= -\sigma(\xi_2 - \psi_2) + \frac{\partial \psi_2}{\partial z} f(z, \xi_1, \xi_2, \epsilon) + \frac{\partial \psi_2}{\partial \xi_1} \xi_2 + \frac{\partial \psi_2}{\partial \xi_2} u\end{aligned}\tag{7.102}$$

where  $f = f_0(z, \xi_1) + g_0(z, \xi_1, \xi_2, \epsilon)$  and  $\sigma(\cdot)$  is a sigmoidal function. In addition, assume for any  $l_0 > 0$ , there exist  $k > 0, r_0 \geq 0$  and  $\lambda \in (0, 1]$  such that

$$\left| \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \delta) \right| \leq k V(z)^\lambda, \quad \forall |z| > r_0, \forall |\delta| \leq l_0\tag{7.103}$$

where  $h_0$  is given by

$$h_0(z, \xi_1, \delta) = \int_0^1 \frac{\partial f_0}{\partial \xi_1}(z, \xi_1 + s\delta) ds$$

and satisfies

$$f_0(z, \xi_1 + \delta) = f_0(z, \xi_1) + h_0(z, \xi_1, \delta)\delta$$

Then, there exists an  $\epsilon^* > 0$  such that for all  $0 < \epsilon < \epsilon^*$  the following hold:

- i) The change of coordinates  $\mu = T(z, \xi)$  as

$$\begin{aligned}\mu_1 &= \xi_1 - \alpha_1(z) \\ \mu_2 &= \xi_2 - \psi_2(z, \xi_1, \xi_2, \epsilon)\end{aligned}$$

is invertible over a compact domain  $\mathcal{K}_z \times \mathcal{K}_\xi$  with a smooth inverse  $\xi = T^{-1}(z, \mu)$  given by

$$\begin{aligned}\xi_1 &= \beta_1(z, \mu_1) := \mu_1 + \alpha_1(z) \\ \xi_2 &= \beta_2(z, \mu_1, \mu_2, \epsilon)\end{aligned}$$

with  $\beta_1(0, 0) = 0$  and  $\beta_2(0, 0, 0, \epsilon) = 0$ .

ii) The fixed point equations

$$\begin{aligned}\xi_2 &= \psi_2(z, \xi_1, \xi_2, \epsilon) \\ u &= \psi_3(z, \xi_1, \xi_2, u, \epsilon)\end{aligned}$$

have unique smooth solutions

$$\begin{aligned}\xi_2^* &= \alpha_2(z, \xi_1, \epsilon) \\ u^* &= \alpha_3(z, \xi_1, \xi_2, \epsilon)\end{aligned}$$

with  $\alpha_2(0, 0, \epsilon) = 0$  and  $\alpha_3(0, 0, 0, \epsilon) = 0$  over a compact domain  $\mathcal{K}_z \times \mathcal{K}_\xi$ .

- iii) The state feedback  $\xi_2 = \alpha_2(z, \xi_1, \epsilon)$  semiglobally asymptotically and locally exponentially stabilizes  $(z, \xi_1) = 0$  for the  $(z, \xi_1)$ -subsystem of (7.101).
- iv) The state feedback  $u = \alpha_3(z, \xi_1, \xi_2, \epsilon)$  semiglobally asymptotically and locally exponentially stabilizes  $(z, \xi_1, \xi_2) = 0$  for the composite system in (7.101).

**Proof.** See section C.4 in Appendix C. □

The proof of the existence of unique smooth fixed points in theorem 7.4.4 relies on the following lemma.

**Lemma 7.4.4.** Consider the following fixed point equation in  $y$

$$y = \phi_0(x) + \phi_1(x, y, \epsilon)\epsilon =: \phi(x, y, \epsilon) \quad (7.104)$$

where  $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^m$  is continuously differentiable with  $\phi(0, 0, \epsilon) = 0$ . Then, for any compact set  $\mathcal{K}_x$  and the interval  $I_\epsilon = [0, \epsilon_1]$ , there exists a compact set  $\mathcal{K}_y$  and an  $0 < \epsilon^* \in I_\epsilon$  such that for all  $0 < \epsilon < \epsilon^*$  the fixed point equation (7.104) has a unique smooth solution

$$y^* = \alpha(x, \epsilon)$$

that belongs to  $\mathcal{K}_y$  and satisfies  $\alpha(0, \epsilon) = 0$  uniformly in  $\epsilon$ .

**Proof.** See section C.3 in Appendix C. □

The next theorem is a Lyapunov-based fixed point backstepping theorem for stabilization of slightly nontriangular nonlinear systems. First, we need to make a definition.

**Definition 7.4.3.** (residual of a function w.r.t. its  $i$ th argument) Consider a continuously differentiable function  $f(x_1, \dots, x_m) : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \rightarrow \mathbb{R}^p$ . Define

$$h(x_1, \dots, x_m, a, \delta) = \int_0^1 D_{x_i} f(x_1, \dots, x_m)|_{x_i=a+s\delta} ds$$

and denote

$$h(x_1, \dots, x_m, a, \delta) = \text{res}_{(i,a,\delta)}(f(x_1, \dots, x_m))$$

Read  $\text{res}_{(i,a,\delta)}(f(x_1, \dots, x_m))$ , the residual of  $f$  w.r.t. its  $i$ th argument with perturbation  $\delta$  around  $x_i = a$ . Then, the following decomposition of  $f$  is always possible

$$f(x_1, \dots, x_i + \delta, \dots, x_m) = f(x_1, \dots, x_m) + h(x_1, \dots, x_m, \delta)\delta$$

Based on this notation, we have  $f(x_1, x_2) = f(x_1, 0) + h(x_1, x_2)x_2$  with  $h(x_1, x_2) = \text{res}_{(2,0,x_2)}(f(x_1, x_2))$ .

**Theorem 7.4.5.** Consider the perturbed nonlinear system in (7.101) where  $f_0, g_0$  are smooth functions satisfying  $f_0(0) = 0, g_0(0) = 0, g_0(z, \xi_1, 0, \epsilon) = 0$ , and  $g_0(z, \xi_1, \xi_2, 0) = 0$  (i.e.  $\exists p_0 \in C^1 : g_0 = p_0(z, \xi_1, \xi_2, \epsilon)\xi_2\epsilon$ ). Suppose there exists a state feedback  $\xi_1 = \alpha_1(z)$  with  $\alpha_1(0) = 0$  such that for

$$\dot{z} = f_0(z, \alpha_1(z))$$

$z = 0$  is globally asymptotically and locally exponentially stable. Let  $V(z)$  be the Lyapunov function for this system with

$$\frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) < 0, \forall z \neq 0$$

Decompose  $f_0$  as the following

$$f_0(z, \xi_1 + \delta) = f_0(z, \xi_1) + h_0(z, \xi_1, \delta)\delta$$

and define

$$\begin{aligned} \psi_2(z, \xi_1, \xi_2, \epsilon) &= -\sigma_1(\xi_1 - \alpha_1(z)) + \frac{\partial \alpha_1(z)}{\partial z}(f_0(z, \xi_1) + g_0(z, \xi_1, \xi_2, \epsilon)) \\ &\quad - \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \xi_1 - \alpha_1(z)) \end{aligned} \tag{7.105}$$

Then, there exists an  $\epsilon^* > 0$  such that for all  $0 < \epsilon < \epsilon^*$  the following hold

i) The fixed point equation

$$\xi_2 = \psi_2(z, \xi_1, \xi_2, \epsilon) \tag{7.106}$$

for all  $(z, \xi_1) \in \mathcal{K}_z \times \mathcal{K}_{\xi_1}$  has a unique smooth solution  $\xi_2^* = \alpha_2(z, \xi_1, \epsilon)$  with  $\alpha_2(0, 0, \epsilon) = 0$ .

ii) For a sufficiently large  $c_1 = \sigma'_1(0)$ , the state feedback  $\xi_2 = \alpha_2(z, \xi_1, \epsilon)$  semiglobally asymptotically stabilizes the origin for the  $(z, \xi_1)$ -subsystem of (7.101).

iii) Fix an  $\epsilon > 0$  valid for (ii). Then, there exists a static state feedback  $u = \alpha_3(z, \xi_1, \xi_2, \epsilon)$  in the following form

$$u = -\sigma_2(\xi_2 - \alpha_2(\bar{z}, \epsilon)) + \frac{\partial \alpha_2(\bar{z}, \epsilon)}{\partial \bar{z}} f_2(\bar{z}, \xi_2, \epsilon) - \frac{\partial W(\bar{z})}{\partial \bar{z}} h_2(\bar{z}, \alpha_2(z, \xi_1, \epsilon), \epsilon, \xi_2 - \alpha_2) \quad (7.107)$$

that semiglobally asymptotically stabilizes the origin for the composite system in (7.101) for a sufficiently large  $c_2 = \sigma'_2(0) > 0$  ( $\sigma_2$  is a sigmoidal function) where

$$\begin{aligned} \bar{z} &= \text{col}(z, \xi_1) \\ W(\bar{z}) &= V(z) + \frac{1}{2}(\xi_1 - \alpha_1(z))^2 \\ f_2(z, \xi_1, \xi_2, \epsilon) &= \text{col}(f, \xi_2) \\ h_2(\bar{z}, \xi_2, \epsilon, \delta) &= \text{res}_{(3, \xi_2, \delta)}(f_2(z, \xi_1, \xi_2, \epsilon)) \end{aligned}$$

**Proof.** See section C.5 in Appendix C.  $\square$

Now, we present some examples of nontriangular nonlinear systems that can be stabilized using theorems 7.4.4 and 7.4.5.

**Example 7.4.5.** (perturbed chain of integrators) Consider the following perturbed chain of integrators

$$\begin{aligned} \dot{x} &= Ax + Bu + p(x, u, v) \\ \dot{u} &= v \end{aligned} \quad (7.108)$$

where  $p(x, u, v)$  is smooth and  $p(x, u, 0) = 0$ . In addition,  $A = \{a_{ij}\}$  with  $a_{ij} = 1$  for  $j = i + 1$  and zero otherwise, and  $B = (0, \dots, 0, 1)^T$ . Define the following change of scale and time-scale

$$\begin{aligned} z_i &= \epsilon^{(n+1-i)} x_i, \quad i = 1, \dots, n \\ \xi_1 &= u \\ \xi_2 &= v/\epsilon \end{aligned}$$

and denote

$$\zeta_\epsilon := (\epsilon^{-n} z_1, \dots, \epsilon^{-1} z_n, \xi_1, \epsilon \xi_2)$$

Define a new perturbation  $\bar{p} = (\bar{p}_1, \dots, \bar{p}_n)^T$  by its elements as the following

$$\bar{p}_i(z, \xi_1, \xi_2, \epsilon) := \epsilon^{n-i} p_i(\zeta_\epsilon).$$

In proposition 7.4.8, we show that if the perturbation  $p$  satisfies the following growth condition, then the nonlinear system in (7.108) can be semiglobally asymptotically stabilized.

**Assumption 7.4.10.** The perturbation  $p$  satisfies the following condition

$$\epsilon^{n-i} p_i(\zeta_\epsilon)|_{\epsilon=0} = 0, \quad \forall (z, \xi_1, \xi_2) \in \mathbb{R}^{n+2}, i = 1, \dots, n$$

i.e.  $\bar{p}(z, \xi_1, \xi_2, 0) = 0$ .

**Proposition 7.4.8.** Suppose Assumption 7.4.10 holds. Then, there exists a static state feedback law in the form of a fixed point controller that semiglobally asymptotically stabilizes  $(x, u) = 0$  for (7.108).

**Proof.** Apply the aforementioned change of scale and time-scale  $\tau = \epsilon t$ . From now on, (with a little abuse of notation) let  $(\cdot)$  denote  $d/d\tau$  in new scale and time-scale. The dynamics of the system in new coordinates can be expressed as

$$\begin{aligned} \dot{z} &= Az + B\xi_1 + \bar{p}(z, \xi_1, \xi_2, \epsilon) \\ \dot{\xi}_1 &= \xi_2 \end{aligned} \tag{7.109}$$

Clearly,  $\bar{p}(z, \xi_1, 0, \epsilon) = 0$  and  $\bar{p}(z, \xi_1, \xi_2, 0) = 0$ . Thus, system (7.109) is in the form (7.101). Noting that a linear state feedback  $\xi_1 = Kz$  globally asymptotically stabilizes  $z = 0$  with a quadratic Lyapunov function  $V(z) = z^T P z$  where  $P$  is the solution of the Lyapunov equation

$$(A + BK)^T P + P(A + BK) = -I$$

a semiglobally asymptotically stabilizing feedback for (7.108) can be obtained from part (ii) of theorem 7.4.5.  $\square$

*Remark 7.4.5.* The problem of global asymptotic stabilization of a perturbed chain of integrators with nontriangular nonlinear perturbation  $p$  in (7.108) that satisfies the condition

$$\frac{\partial p}{\partial v}(x, u, 0) = 0$$

is addressed in [79] using a state feedback in the form of sum of saturations with relatively small magnitudes. The work in [79] is an extension of the method of nested saturations due to Teel [102]. The main difference between the result of proposition 7.4.8 and the work in [79] is that the condition that  $p(x, u, v) = \tilde{p}(x, u, v)v^2$  is not needed in here. The next example clarifies this difference.

**Example 7.4.6.** Consider the following nonlinear system non-affine in control

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1 v^2 + uv + x_2 v \\ \dot{x}_2 &= u + x_2 v^{\frac{4}{3}} + x_1 v^3 + uv + x_1 x_2 u v^4 \\ \dot{u} &= v\end{aligned}\tag{7.110}$$

the system is in the form of a chain of two integrators with a perturbation  $p = (p_1, p_2)^T$  where  $p_1(x, u, v) = x_1 v^2 + uv + x_2 v$  and  $p_2(x, u, v) = x_2 v^{\frac{4}{3}} + x_1 v^3 + uv + x_1 x_2 u v^4$ . This system is not in feedforward form and the perturbation term does not satisfy  $\frac{\partial p}{\partial v}(x, u, 0) = 0$  for all  $(x, u)$ . Thus, it can neither be controlled using Teel's nested saturations [102], nor the method in [79]. Applying the change of scale and time-scale

$$z_1 = \epsilon^2 x_1, z_2 = \epsilon x_2, \xi_1 = u, \xi_2 = v/\epsilon, \tau = \epsilon t$$

we get

$$\begin{aligned}\dot{z}_1 &= z_2 + \epsilon z_1 \xi_2^2 + \epsilon^2 \xi_1 \xi_2 + \epsilon z_2 \xi_2 \\ \dot{z}_2 &= \xi_1 + \epsilon^{\frac{1}{3}} z_2 \xi_2^{\frac{4}{3}} + \epsilon z_1 \xi_2^3 + \epsilon \xi_1 \xi_2 + \epsilon z_1 z_2 \xi_1 \xi_2^4 \\ \dot{\xi}_1 &= \xi_2\end{aligned}$$

Apparently, this system satisfies all the conditions of proposition 7.4.8 and can be semiglobally stabilized using a static state feedback law in the form of a fixed point controller.

#### 7.4.5 Notion of Partial Semiglobal Asymptotic Stabilization

In this section, we introduce a new notion of stabilizability that is in a sense close to semiglobal asymptotic stabilizability but rather weaker than that. A reason behind defining this notion is that it is not always possible to transform any perturbed chain of integrators with a nontriangular structure into the form (7.101) using a change of scale and time-scale as described in proposition 7.4.8. The following example clarifies this issue.

**Example 7.4.7.** Consider the following nonlinear system in non-triangular form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \eta_1 + \eta_1 x_2^3 \eta_2^2 \\ \dot{\eta}_1 &= \eta_2\end{aligned}\tag{7.111}$$

applying a change of scale and time-scale as

$$z_1 = \epsilon^2 x_1, z_2 = \epsilon x_2, \xi_1 = \eta_1, \xi_2 = \eta_2/\epsilon, \tau = \epsilon t$$

one obtains

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1 + \frac{1}{\epsilon} \xi_1 z_2^3 \xi_2^2 \\ \dot{\xi}_1 &= \xi_2\end{aligned}$$

Apparently, as  $\epsilon \rightarrow 0$  the perturbation term  $\frac{1}{\epsilon} \xi_1 z_2^3 \xi_2^2$  goes unbounded and does not vanish. A modification of the change of scale and time-scale as the following

$$z_1 = \epsilon x_1, z_2 = x_2, \xi_1 = \eta_1/\epsilon, \xi_2 = \eta_2/\epsilon^2, \tau = \epsilon t$$

resolves this problem and we get

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1 + \epsilon^2 \xi_1 z_2^3 \xi_2^2 \\ \dot{\xi}_1 &= \xi_2\end{aligned}\tag{7.112}$$

which is clearly in the form (7.101). Now, the problem is that semiglobal stabilization of (7.112) does not guarantee semiglobal stabilization of the original system in (7.111). The obtained region of attraction for (7.111) is in the form  $\mathcal{K}_x \times \mathcal{K}_{\eta_1}$  where  $\mathcal{K}_x$  is larger than the set of initial conditions  $x(0)$  and  $\mathcal{K}_{\eta_1}$  is smaller than the set of initial conditions  $\eta_1(0)$  due to the fact that  $\eta_1 = \epsilon \xi_1$  for a relatively small  $\epsilon > 0$ .

This motivates us to define a new stabilizability notion that is weaker than semiglobal stabilizability but still is rather similar to it.

**Definition 7.4.4.** (partial semiglobal stabilizability) Consider the following system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, u)\end{aligned}\tag{7.113}$$

We say the origin  $(x_1, x_2) = 0$  for (7.113) is *partially semiglobally asymptotically stabilizable with respect to  $x_1$*  if for any compact set of (partial) initial conditions  $\mathcal{K}_1 \ni x_1(0)$ , there exist a neighborhood  $\mathcal{A}_2$  of  $x_2 = 0$  and a control law  $u$  such that the closed-loop system in (7.113) has a region of attraction  $\mathcal{A}_1 \times \mathcal{A}_2$  with  $\mathcal{A}_1 \supset \mathcal{K}_1$ .

*Remark 7.4.6.* Apparently, partial semiglobal asymptotic stabilization is stronger than local asymptotic stabilization.

Now, based on theorem 7.4.5,  $(x, \eta_1) = 0$  is partially semiglobally stabilizable w.r.t.  $x$  for (7.111) using a static state feedback.

## 7.5 Applications

In this section, we present applications of fixed point control laws to stabilization of a number of challenging benchmark systems in robotics.

### 7.5.1 The Cart-Pole System with Small Length/Strong Gravity Effects

In this section, we analyze the effects of relatively small lengths or large gravity constants in stabilization of the Cart-Pole system depicted in Figure 7-3. In both

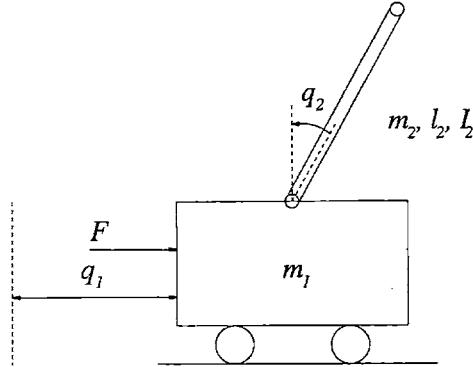


Figure 7-3: The Cart-Pole System

cases, we provide semiglobally stabilizing fixed point state feedback laws for the Cart-Pole system. The cascade normal form for the Cart-Pole system is given in equation (5.21) as the following

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1(k_1 + k_2 \frac{\xi_2^2}{(1 + \xi_1^2)^{\frac{3}{2}}}) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{7.114}$$

where  $k_1, k_2$  are the following positive constants

$$k_1 = g \geq 1, \quad k_2 = \frac{m_2 l_2^2 + I_2}{m_2 l_2}$$

Assume the link  $l_2$  has uniform mass density. Thus,  $I_2 = m_2 l_2^2 / 3$  and  $k_2 = \frac{4}{3} l_2$ . Denote  $\epsilon = k_2/k_1 > 0$  and apply the following change of scale

$$y_1 = \frac{1}{k_1} z_1, \quad y_2 = \frac{1}{k_1} z_2$$

Since  $k_1 \geq 1$ , under this change of scale, semiglobal stabilization in new coordinates and original coordinates are equivalent (i.e. one implies the other). The dynamics of

the Cart-Pole system in new scale can be rewritten as

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= \xi_1(1 + \epsilon \frac{\xi_2^2}{(1 + \xi_1^2)^{\frac{3}{2}}}) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{7.115}$$

Under either of the following conditions:

- i)  $g \gg 1$ , i.e. relatively high gravity.
- ii)  $l_2 \ll 1$ , i.e. relatively small lengths.

we obtain an  $\epsilon$  that is relatively small, i.e.  $\epsilon = k_2/k_1 \ll 1$ . Clearly, this is in the form (7.101) and the origin for this system can be semiglobally asymptotically stabilized according to theorem 7.4.5. Here, we show that theorem 7.4.4 can be also used to obtain a stabilizing state feedback for this Cart-Pole system. Setting  $\xi_2 = 0$  in the  $y$ -subsystem, the remaining system is a double-integrator that can be globally asymptotically stabilized using a bounded control  $\xi_1 = -\sigma(y_1 + y_2)$  ( $\sigma$  is a sigmoidal function) with a positive definite Lyapunov function

$$V(y) = \phi(y_1) + \phi(y_1 + y_2) + y_2^2$$

where  $\phi(x) = \int_0^x \sigma(s)ds$  and  $\dot{V} < 0$  for all  $y \neq 0$ . To prove the boundedness of the solutions of the  $y$ -subsystem, one can use a simpler Lyapunov function

$$V_1(y) = \phi(y_1) + \frac{1}{2}y_2^2$$

Because  $f_0(y, \xi_1) = (y_2, \xi_1)^T$ ,  $f_0(y, \xi_1 + \delta) = f_0(y, \xi_1) + h_0\delta$  with  $h_0 = (0, 1)^T$ . Thus, we have

$$\frac{\partial V_1(y)}{\partial y} h_0 = \frac{\partial V_1(y)}{\partial y_2} = y_2$$

or (independent of  $\delta$ )

$$|\frac{\partial V_1(y)}{\partial y} h_0| \leq |y_2| < \sqrt{2}V_1(y)^{\frac{1}{2}}, \forall y \neq 0$$

and condition (7.103) of theorem 7.4.4 is satisfied with  $k = \sqrt{2}$ , any  $r_0 > 0$ , and  $\lambda = \frac{1}{2}$ . Thus, a semiglobally stabilizing state feedback that makes no explicit use of any Lyapunov functions can be obtained using the cascade backstepping procedure in theorem 7.4.4). Figure 7-4, shows the simulation results for the cart-pole system starting at the initial state  $(1, 1, \pi/3, 0)$  for the choice of  $\xi_1 = -\tanh(y_1 + y_2)$ .

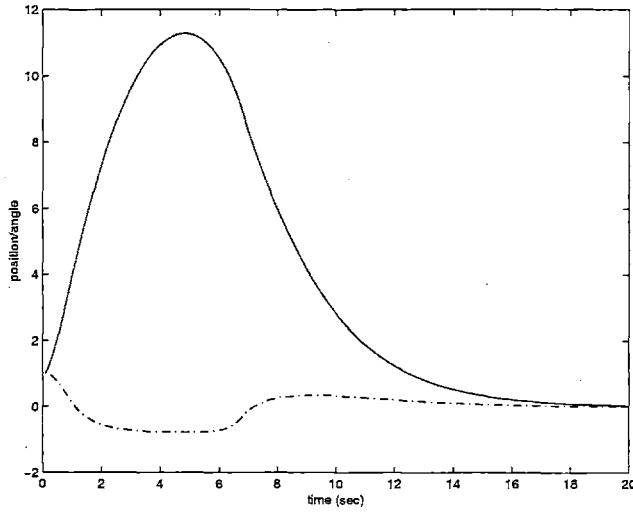


Figure 7-4: The state trajectories of the Cart-Pole system.

### 7.5.2 The Cart-Pole System with Large Length/Weak Gravity Effects

Consider the normal form of the Cart-Pole system in (7.114) and apply the following change of scale and time-scale given in [67]

$$y_1 = \epsilon^2 z_1, y_2 = \epsilon z_2, \eta_1 = \xi_1, \eta_2 = \xi_2/\epsilon, w = u/\epsilon^2, \tau = \epsilon t$$

The dynamics of the system in new coordinates and time-scale is in the form

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= \eta_1(k_1 + k_2\epsilon^2 \frac{\eta_2^2}{(1+\eta_1^2)^{\frac{3}{2}}}), \\ \dot{\eta}_1 &= \eta_2, \\ \dot{\eta}_2 &= w \end{aligned} \tag{7.116}$$

Semiglobal stabilization for (7.116) is very similar to the one presented for system (7.115) and will be repeated here.

Now, without loss of generality assume  $I_2 = 0$ . Thus,  $k_2 = l_2$ . This case has an interesting physical interpretation. Normalizing the units of  $y_1, y_2$  by  $k_1 = g$  as

$$x_1 = \frac{1}{g}y_1, x_2 = \frac{1}{g}y_2$$

and setting

$$\epsilon = \sqrt{\frac{k_1}{k_2}} = \sqrt{\frac{g}{l_2}} = \omega_0$$

(where  $\omega_0$  is the natural frequency of a pendulum of length  $l_2$ ) we obtain the normal

form of the Cart-Pole system with normalized units as

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \eta_1 \left( 1 + \frac{\eta_2^2}{(1 + \eta_1^2)^{\frac{3}{2}}} \right), \\ \dot{\eta}_1 &= \eta_2, \\ \dot{\eta}_2 &= w\end{aligned}\tag{7.117}$$

In either of the following cases:

- i)  $g \ll 1$ , i.e. relatively weak gravity,
- ii)  $l_2 \gg 1$ , i.e. relatively large length;

the obtained  $\epsilon$  (or the natural frequency  $\omega_0$ ) is relatively small, i.e.  $\omega_0 = \epsilon \ll 1$ . This means the solutions in scaled coordinates and time scale are relatively slow. Observe that the conditions of large length/weak gravity are in complete opposition with the case considered in section 7.5.1 with small length/large gravity effects.

### 7.5.3 The Rotating Pendulum

Consider the Rotating Pendulum as shown in Figure 7-5. The nontriangular normal form for the Rotating Pendulum is given in (5.24) as the following

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1 \left( \frac{g}{L_1} + \frac{l_2}{L_1} \frac{1}{\sqrt{1 + \xi_1^2}} \left( z_2 - \frac{m_{22}}{m_2 l_2 L_1} \frac{\xi_2}{\sqrt{1 + \xi_1^2}} \right)^2 + \frac{m_{22}}{m_2 l_2 L_1} \frac{\xi_2^2}{(1 + \xi_1^2)^{\frac{3}{2}}} \right) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{7.118}$$

where  $m_{22} = m_2 l_2^2 + I_2$ . Without loss of generality assume  $I_2 = 0$ , or  $m_{22} = m_2 l_2^2$

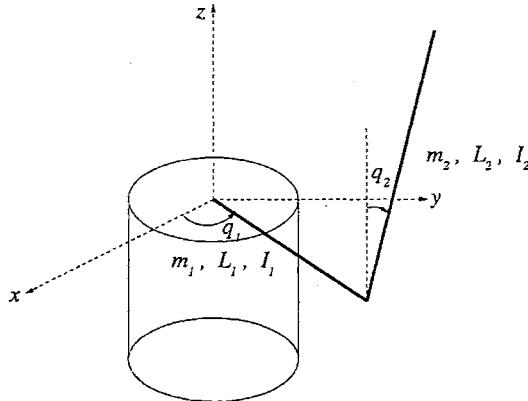


Figure 7-5: The Rotating Pendulum

(otherwise,  $I_2 = k_0 m_2 l_2^2$  with some  $k_0 > 0$  and the line of analysis is very similar).

Thus, equation (7.118) can be rewritten as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1[k_1 + k_2 \frac{1}{(1 + \xi_1^2)^{\frac{1}{2}}} (z_2 - k_2 \frac{\xi_2}{(1 + \xi_1^2)^{\frac{1}{2}}})^2 + k_2 \frac{\xi_2^2}{(1 + \xi_1^2)^{\frac{3}{2}}}] \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{7.119}$$

where

$$k_1 = \frac{g}{L_1}, k_2 = \frac{l_2}{L_1}$$

*Remark 7.5.1.* Let us define

$$y_1 = \frac{1}{k_1} z_1, \quad y_2 = \frac{1}{k_1} z_2, \quad \epsilon = \frac{k_2}{k_1}$$

where  $k_2$  is constant and  $k_1 \gg 1$ . This change of scale transforms (7.119) into the following nonlinear system

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= \xi_1[1 + \frac{k_2}{\epsilon} \frac{1}{(1 + \xi_1^2)^{\frac{1}{2}}} z_2^2 - 2\epsilon^2 \frac{z_2 \xi_2}{(1 + \xi_1^2)} + \epsilon(1 + k_2^2) \frac{\xi_2^2}{(1 + \xi_1^2)^{\frac{3}{2}}}] \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}$$

Apparently, the last equation is not a well-defined slightly nontriangular system. Since as  $\epsilon \rightarrow 0$ , the coefficient of the term containing  $z_2^2$  goes unbounded, i.e.  $k_2/\epsilon \rightarrow \infty$ .

Now, consider the following change of scale and time scale

$$y_1 = \epsilon z_1, \quad y_2 = z_2, \quad \eta_1 = \xi_1/\epsilon, \quad \eta_2 = \xi_2/\epsilon, \quad w = v/\epsilon^3, \quad \tau = \epsilon t\tag{7.120}$$

From (7.119), we obtain

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= \eta_1[k_1 + \frac{k_2}{(1 + \epsilon^2 \eta_1^2)^{\frac{1}{2}}} (y_2 - \frac{\epsilon^2 k_2 \eta_2}{(1 + \epsilon^2 \eta_1^2)^{\frac{1}{2}}})^2 + \frac{\epsilon^4 k_2 \eta_2^2}{(1 + \epsilon^2 \eta_1^2)^{\frac{3}{2}}}] \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= w\end{aligned}\tag{7.121}$$

but

$$\frac{k_2}{(1 + \epsilon^2 \eta_1^2)^{\frac{1}{2}}} = k_2 \left(1 + \frac{1}{(1 + \epsilon^2 \eta_1^2)^{\frac{1}{2}}} - 1\right) = k_2 - \frac{k_2 \epsilon^2}{(1 + \epsilon^2 \eta_1^2)^{\frac{1}{2}} [1 + (1 + \epsilon^2 \eta_1^2)^{\frac{1}{2}}]}$$

Hence

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= \eta_1(k_1 + k_2 y_2^2) + g(y_2, \eta_1, \eta_2, \epsilon) + \Delta(y_2, \eta_1, \epsilon) \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= w\end{aligned}\tag{7.122}$$

where

$$\begin{aligned}g(y_2, \eta_1, \eta_2, \epsilon) &= -\frac{2\epsilon^2 k_2^2 y_2 \eta_1 \eta_2}{(1 + \epsilon^2 \eta_1^2)} + \frac{\epsilon^4 k_2 (1 + k_2^2) \eta_2^2}{(1 + \epsilon^2 \eta_1^2)^{\frac{3}{2}}} \\ \Delta(y_2, \eta_1, \epsilon) &= -\frac{\epsilon^2 \eta_1 y_2^2}{(1 + \epsilon^2 \eta_1^2)^{\frac{1}{2}}}\end{aligned}$$

Notice that  $g(y_2, \eta_1, \eta_2, \epsilon)$  and  $\Delta(y_2, \eta_1, \epsilon)$  vanish at  $\epsilon = 0$ . In addition, the perturbation  $\Delta$  has an upper bound that vanishes uniformly in  $\eta_1$  due to

$$|\Delta(y_2, \eta_1, \epsilon)| \leq \epsilon k_2 y_2^2$$

The nonlinear system in (7.122) without the perturbation  $\Delta$  is in the form (7.101) and satisfies the conditions of theorem 7.4.5. Thus, it can be semiglobally asymptotically stabilized using the choice of  $\eta_1 = \alpha(y) = -\sigma(y_1 + y_2)$  ( $\sigma$  is a sigmoidal function). This state feedback renders  $y = 0$  globally asymptotically stable for

$$\begin{aligned}\dot{y}_1 &= y_2 \\ \dot{y}_2 &= \eta_1(k_1 + k_2 y_2^2)\end{aligned}\tag{7.123}$$

The following Lyapunov function

$$V(y) = \phi(y_1) + \phi(y_1 + y_2) + \frac{1}{k_2} \log(1 + \frac{k_2}{k_1} y_2^2)\tag{7.124}$$

with  $\phi(x) = \int_0^x \sigma(s)ds$  is a valid Lyapunov function for the closed-loop system in (7.123) satisfying  $\dot{V} < 0$  for all  $y \neq 0$ . Now, semiglobal asymptotic stabilization of (7.122) with the high-order perturbation  $\Delta$  can be shown using an elementary argument. It is important to notice that semiglobal asymptotic stabilization of the origin for (7.122) implies *partial semiglobal asymptotic stabilization of the origin for the dynamics of the Rotating Pendulum* in the original coordinates. Figure 7-6 shows the simulation results for the rotating pendulum with initial condition  $(\pi/3, 0, \pi/4, 0)$ .

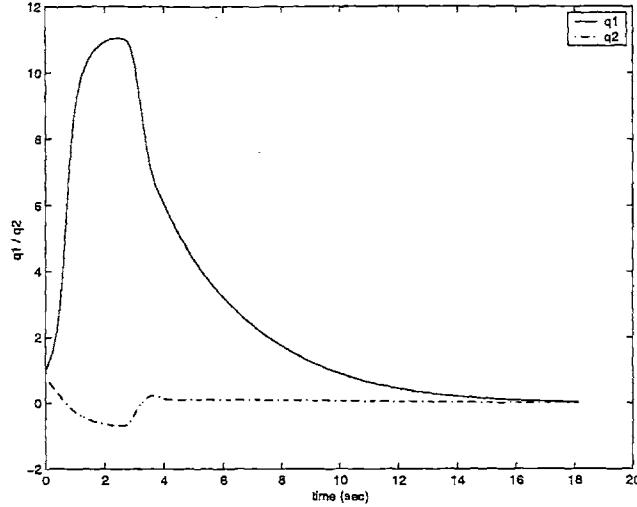


Figure 7-6: The state trajectories for the Rotating Pendulum.

#### 7.5.4 The Generalized Beam-and-Ball System

The *generalized beam-and-ball* system is a partially-linear nontriangular cascade nonlinear system in the following form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= g \sin(\eta_1) + x_1^\nu \eta_2^2, \quad \nu \geq 1 \\ \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= u\end{aligned}\tag{7.125}$$

for  $\nu = 1$ , this system is the same as the famous beam-and-ball system that has been extensively studied and used as an example by many researchers in the past decade [34, 102, 80, 79]. Here, we consider two cases: i)  $1 \leq \nu < 3$ , and ii)  $3 \leq \nu < \infty$ . For the sake of simplicity assume  $g = 1$ .

Case (i): Applying the change of scale and time scale

$$z_1 = \epsilon x_1, z_2 = x_2, \xi_1 = \eta_1/\epsilon, \xi_2 = \eta_2/\epsilon^2, w = v/\epsilon^3, \tau = \epsilon t$$

In the new scale, we obtain

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1 + \epsilon^2 O(|\xi_1|^3) + \epsilon^{(3-\nu)} z_1^\nu \xi_2^2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{7.126}$$

where

$$\xi_1 + \epsilon^2 O(|\xi_1|^3) = \frac{\sin(\epsilon \xi_1)}{\epsilon}, \quad \epsilon > 0$$

Clearly, the origin for system (7.126) can be semiglobally asymptotically stabilized using a state feedback in the form of a fixed point controller. This way, the origin for system (7.125) can be only partially semiglobally stabilized w.r.t.  $(x_1, x_2)$ .

Case (ii): It is easy to see that using the change of scale and time-scale

$$z_1 = x_1, z_2 = x_2/\epsilon, \xi_1 = \eta_1/\epsilon^2, \xi_2 = \eta_2/\epsilon^3, w = v/\epsilon^4, \tau = \epsilon t$$

the dynamics of the system transforms into

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \xi_1 + \epsilon^5 O(|\xi_1|^3) + \epsilon^5 z_1^\nu \xi_2^2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v\end{aligned}\tag{7.127}$$

and the origin for the system (7.125) can be partially semiglobally stabilized w.r.t.  $x_1$ . The simulation results for the generalized beam-and-ball system are shown in Figure 7-7.

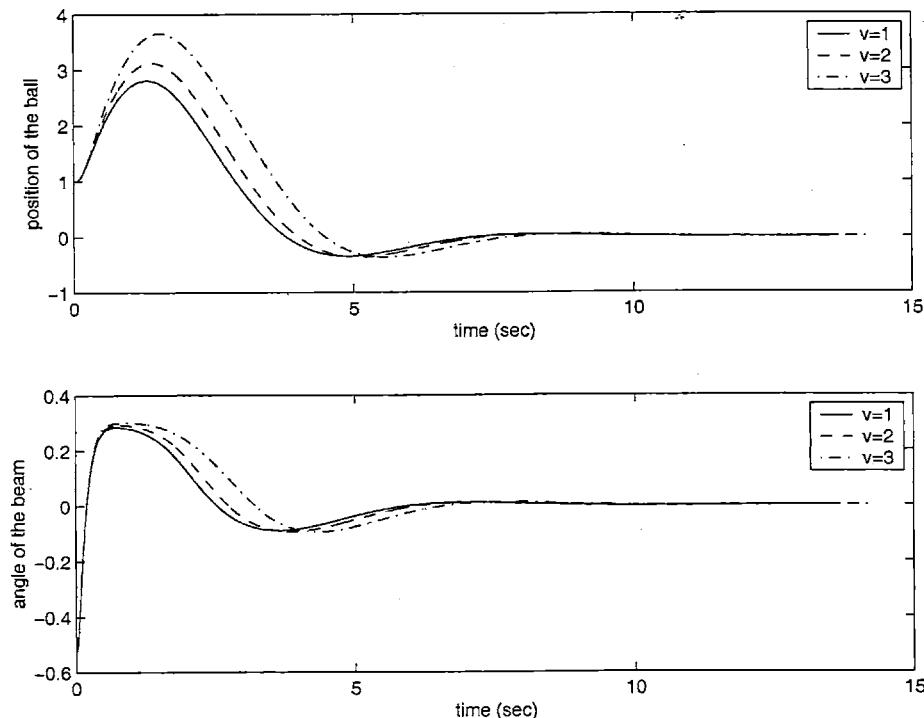


Figure 7-7: The state trajectories of the generalized beam-and-ball system with initial condition  $(1, 0, -\pi/6, 0)$  for  $g = -9.8$  and  $\nu = 1, 2, 3$ .

# Chapter 8

## Conclusions

In this chapter, we summarize the results of this thesis and make concluding remarks. This thesis creates a bridge between control of mechanical systems and nonlinear control theory by providing transformations in explicit forms that take an underactuated mechanical system to a cascade nonlinear system with structural properties. Due to the special partially-linear cascade structure of the obtained normal forms for underactuated systems, control design for the original (possibly) high-order underactuated system reduces to control of a low-order nonlinear part of the transformed system. The main focus of this thesis was on identifying broad classes of underactuated systems with kinetic symmetry where such a transformation for them could be found in closed-form. It turned out that the majority of obtained normal forms for underactuated systems are in nontriangular forms. Since control of nonlinear systems with nontriangular structures was an important open problem, a great deal of effort in this thesis was devoted to the development of control design tools for stabilization of nontriangular nonlinear systems in chapter 7.

Reduction of underactuated systems with nonholonomic velocity constraints and symmetry is addressed in chapter 6. As a consequence, the corresponding class of an underactuated system with nonholonomic velocity constraints is determined by the class of its reduced system which is a well-defined underactuated (or fully-actuated) mechanical system itself.

The theoretical results of this thesis are applied to challenging control problems for a variety of robotics and aerospace systems. Almost for all examples, detailed control design and simulation results are provided. Among these examples are trajectory tracking for a flexible-link arm, automatic calculation of the differentially flat outputs of the VTOL aircraft, (almost) global asymptotic/exponential tracking of feasible trajectories of an autonomous helicopter, global exponential  $\epsilon$ -tracking and stabilization for a two-wheeled mobile robot, global stabilization of the Acrobot, global asymptotic stabilization of the 2D and 3D Cart-Pole systems to an equilibrium point, and semiglobal asymptotic stabilization of the Rotating Pendulum to an equilibrium point.

## 8.1 Summary

The summary of this thesis is as follows.

In chapter 1, an introduction was provided to control of mechanical systems, nonlinear control theory, and underactuated mechanical systems with their applications.

In chapter 2, some background on different types of mechanical systems and basic notions in mechanics are given. These mechanical notions include classical symmetry, kinetic symmetry, shape variables, conservation laws, nonholonomicity, and flatness.

In chapter 3, the dynamics of underactuated mechanical system is defined as the forced Euler-Lagrange equations of motion. Several examples of underactuated systems were presented in chapter 3. For each example, only the inertia matrix and the potential energy of the system are given. This allows the reader to clearly observe the kinetic symmetry property in each example and identify its corresponding shape variables. In addition, different methods for partial feedback linearization of underactuated mechanical systems are introduced. This includes collocated and noncollocated feedback linearization methods as special cases. Finally, the problem of decoupling of the dynamics of the actuated and unactuated subsystems of a class of underactuated systems after partial feedback linearization is addressed. The decoupling is achieved using a global change of coordinates so that after transformation, the obtained cascade system was in Byrnes-Isidori normal form with a set of double-integrator linear part. This result then was applied to complete classification of underactuated mechanical systems with two degrees of freedom and kinetic symmetry to three classes. By “complete”, we mean “all” such underactuated systems. It was proven that these three classes of underactuated systems could be transformed into three types of cascade normal forms. Namely, nonlinear systems in strict feedback form, strict feedforward form, and nontriangular quadratic form. All the transformations are performed using change of coordinates in closed-form.

In chapter 4, some of the main contributions of this thesis are presented. Reduction and classification of high-order underactuated systems is certainly the most important result of this thesis. The number of obtained classes are eight with certain classes which contain two subclasses. In addition to classification of underactuated systems a correspondence is established between classes of underactuated systems and cascade systems with triangular and nontriangular structural properties. This correspondence allows control design for the original underactuated system. The main tools in stabilization of underactuated systems with triangular normal forms are backstepping and forwarding methods. In contrast, the main approach in control of nontriangular nonlinear systems is a fixed-point based backstepping procedure that is introduced in chapter 7. To deal with reduction of underactuated systems with non-integrable momentums, we introduced a procedure call momentum decomposition. This procedure represents a non-integrable momentum as the sum of an integrable momentum and an error momentum term which is non-integrable. Momentum decomposition was applied to reduction of multi-link underactuated robots and calculation of weakly flat outputs for an accurate model of an autonomous helicopter in chapter 5. For Class-I,II,III underactuated systems both the transformations and the obtained normal forms are physically meaningful. This allows construction

of physically meaningful Lyapunov functions for stability and robustness analysis of these normal forms. The fact that “the core reduced system for (almost) all classes of underactuated systems is itself a Lagrangian mechanical system with shape variables as the control”, creates a unity among different classes of underactuated systems. Global asymptotic stabilization of this core reduced system is addressed in chapter 4.

In chapter 5, several examples of underactuated systems are provided with detailed control design and simulation results. A method is introduced for aggressive swing-up control using bounded input for special classes of underactuated systems that involve inverted pendulums. Furthermore, trajectory tracking for a flexible one-link robot and an autonomous helicopter is addressed in chapter 5. The connection between differential flatness and a decoupling change of coordinates for underactuated systems with input coupling are demonstrated for the case of the VTOL aircraft and an accurate model of an autonomous helicopter.

In chapter 6, it was shown that underactuated systems with nonholonomic velocity constraints and symmetry can be represented as the cascade of the constraint equation and a lower-order underactuated (or fully-actuated) mechanical system. Depending on the specific class of the obtained reduced subsystem, the associated class of the original nonholonomic system is determined. This framework allowed us to present a systematic reduction of a snake-type robot called the snake-board. It was demonstrated that the kinematic model of the snake-board with anti-symmetric heading-angles of the front and back wheels is diffeomorphic to the kinematic model of a car. Moreover, the kinematics of the snake-board can be transformed into a first-order chained-form nonholonomic system. The notations of  $\epsilon$ -stabilization and  $\epsilon$ -tracking are formally defined in chapter 6. Furthermore, global exponential  $\epsilon$ -tracking and  $\epsilon$ -stabilization is achieved for a two-wheeled mobile robot.

In chapter 7, the problem of stabilization of nonlinear systems in nontriangular normal forms is addressed for several important special cases. This includes vector fields that are affine, quadratic, linear-quadratic, and non-affine in variables  $(\xi_1, \xi_2)$ . In addition, stabilization of slightly nontriangular systems that are small nonlinear perturbations of systems in strict feedback forms is addressed. This has applications in stabilization/tracking for the accurate model of a helicopter, flexible link-robots, and in general underactuated systems with unactuated shape variables and/or input coupling.

## 8.2 Future Work

This section provides potential future directions of research in continuation of this work.

### 8.2.1 Hybrid Lagrangian Systems

From normal form of different classes of underactuated systems, it is clear that the shape variables have a fundamental role as the main control inputs of the reduced subsystems. For example, in the case of class-I underactuated systems, the reduced

system is itself a Lagrangian system where its Lagrangian function is parameterized with the shape variables. One potential way to control the system is by switching between a finite set of the values of the shape vector  $q_s$ . Let  $\Theta := \{\theta_1, \dots, \theta_k\}$  denote a finite set of shape values  $\theta_j \in Q_s, j = 1, \dots, k$  where  $Q_s$  denotes the shape manifold. Now, consider the following nonlinear switching system

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s)\end{aligned}\tag{8.1}$$

where  $q_s = v(t)$  and  $v(t) : \mathbb{R}_{\geq 0} \rightarrow \Theta$  is a piece-wise constant control input. A stabilizing control  $v(t)$  is a switching signal that appropriately puts one of the following nonlinear dynamics in effect at any time  $t \geq 0$

$$\begin{aligned}\dot{q}_r &= m_r^{-1}(\theta_j)p_r \\ \dot{p}_r &= g_r(q_r, \theta_j)\end{aligned}\tag{8.2}$$

such that  $(q_r, p_r) = 0$  is asymptotically stable for (8.1). Setting  $z = \text{col}(q_r, p_r)$ , this nonlinear switching system can be rewritten as

$$\dot{z} = f_j(z), \quad z = \text{col}(q_r, p_r)\tag{8.3}$$

According to [52], if there exists a *common Lyapunov function*  $V(z)$  such that

$$\nabla V(z) \cdot f_j(z) < 0, \quad \forall z \neq 0, j \in \{1, \dots, k\}$$

Then, given any arbitrary switching signal  $v(t) : \mathbb{R}_{\geq 0} \rightarrow \Theta$ ,  $z = 0$  is globally asymptotically stable for the closed-loop system in (8.1). The existence of such a common Lyapunov function is an extremely restrictive condition.

Consider a second scenario and assume  $g_r(q_r, q_s) = g_r(q_s)$  and  $m_r(q_s) = m_r$  is a constant. In addition,  $g_r(-q_s) = -g_r(q_s)$  (i.e.  $g_r(0) = 0$ ) and  $\exists \theta_0 : g_r(\theta_0) > 0$ . Setting  $\Theta = \{0, \pm \theta_0\}$ , one possible stabilizing controller for the double-integrator

$$\begin{aligned}\dot{q}_r &= m_r^{-1}p_r \\ \dot{p}_r &= u\end{aligned}\tag{8.4}$$

with  $u = g_r(v)$  is a time-optimal controller that takes values in the set  $\mathcal{U} = \{0, \pm \bar{u}\}$  where  $\bar{u} = g_r(\theta_0)$  is a positive constant. Roughly speaking, stabilization using such a piece-wise constant control law is equivalent to *stabilization of the reduced system by switching between a finite set of possible shapes which the system could take*. We call this approach *control via changing the shape*. The method of “control via changing the shape” is in complete agreement with natural locomotion in a swimming fish, a flying bird, and a walking human being. In the sense that in all the aforementioned cases, locomotion is achieved via changing the physical shape of the system. A formal and systematic analysis of the method of “changing the shape” for locomotion and control purposes is needed.

### **8.2.2 Uncertainty and Robustness**

Control of underactuated systems is rather challenging even in lack of any parametric uncertainties. However, it is important to formulate and address robust stabilization/tracking problems for classes of underactuated systems where a satisfactory nonlinear control design is available for them in lack of uncertainties (e.g. Class-I systems). Normal forms for underactuated systems offer a set of structured nonlinear systems with a common core reduced system that can be exploited for construction of Lyapunov functions. The obtained Lyapunov functions could be later employed in robustness analysis of underactuated systems with parametric uncertainties.

### **8.2.3 Output Feedback Stabilization and Tracking**

The majority of control laws presented in this work are in the form of full state feedback. The exceptions to this statement include trajectory tracking for an autonomous helicopter, a mobile robot, and a flexible-link robot. From theoretical point of view, it is important to design nonlinear observers and develop output feedback stabilization laws for control of underactuated systems (wherever possible). For example, in the normal form of Class-I underactuated systems, the equation of the zero-dynamics is globally linear in the velocity (or momentum)  $z_2$ . This promises the existence of an output stabilization feedback for Class-I underactuated systems (under appropriate technical conditions). The same argument applies to Class-III systems and Class-II systems without quadratic terms in  $z_2$ .

### **8.2.4 The Effects of Bounded Control Inputs**

In real-life most of actuators are subject to limitations in their actuation power. A formal analysis of nonlinear systems with *saturated nonlinear state feedback* similar to the ones presented in sections 5.4.2 and 5.5.1 has yet to be developed.

### **8.2.5 Small Underactuated Mechanical Systems (SUMS)**

Due to the availability of Micro-electro-mechanical systems (MEMS), it is possible to build underactuated micro-robots and physically test the effects of the relatively small scale of the components of a robot in control design for underactuated systems. This leads to control design for nonlinear systems with slightly nontriangular normal forms. Control of SUMS is very challenging because of both the limitation of the actuation power and the presence of parametric uncertainties that are inevitable in dealing with semiconductor devices.

### **8.2.6 Underactuated Systems in Strong/Weak Fields**

The behavior of the zero-dynamics of underactuated systems in a strong potential field is dominant with the gravity terms, i.e. any nontriangular normal forms for such underactuated systems are slightly nontriangular. In contrast, in zero-gravity the

behavior of a Class-I underactuated system is the same as a mechanical system with first-order nonholonomic constraints. This suggests a future analysis for possible extensions of some of the existing nonlinear control design approaches for nonholonomic systems to control methods applicable to underactuated/nonholonomic mechanical systems in low-gravity fields. This is of great importance in space exploration applications.

# Appendix A

## Dynamics of Mechanical Systems

### A.1 Underactuated Mechanical Systems with Two DOF and Kinetic Symmetry

In this section, we present the closed-form Euler-Lagrange equations of motion for a mechanical system with two degrees of freedom ( $q_1, q_2$ ) and kinetic symmetry with respect to  $q_1$ . This class of simple Lagrangian systems can be characterized with the following Lagrangian function

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - V(q_1, q_2) \quad (\text{A.1})$$

where  $V(q_1, q_2)$  is the potential energy of the system. From forced Euler-Lagrange equations of motion

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \begin{bmatrix} F_1(q) \\ F_2(q) \end{bmatrix} \tau \quad (\text{A.2})$$

where  $\tau \in \mathbb{R}$  is the control input and for all  $q \in Q$  either  $F_1(q) \neq 0$ , or  $F_2(q) \neq 0$ . For the special cases where  $F(q) = (1, 0)^T$  or  $F(q) = (0, 1)$ , we say the system has non-interacting inputs. Otherwise, we say the system has input-coupling. The equations of motion in (A.2) can be explicitly written as

$$\begin{aligned} m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + m'_{11}(q_2)\dot{q}_1\dot{q}_2 + m'_{12}(q_2)\dot{q}_2^2 + g_1(q_1, q_2) &= F_1(q)\tau \\ m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 - \frac{1}{2}m'_{11}(q_2)\dot{q}_1^2 + \frac{1}{2}m'_{22}(q_2)\dot{q}_2^2 + g_2(q_1, q_2) &= F_2(q)\tau \end{aligned} \quad (\text{A.3})$$

where ' denotes  $d/dq_2$  and the gravity terms are given by

$$g_i(q_1, q_2) = \frac{\partial V(q_1, q_2)}{\partial q_i}, \quad i = 1, 2$$

## A.2 A Three-link Planar Robot

Consider a planar three-link robot arm with revolute joints depicted in Figure A-1. The mass, inertia, length, and length of the center of mass of the  $i$ th link is denoted by  $m_i, I_i, L_i, l_i$ , respectively. To calculate the kinetic energy of this system we follow a general method used in [95] which decomposes the kinetic energy as the sum of translational and rotational kinetic energies, i.e.  $K = K_t + K_r$ . The translational kinetic energy is given by

$$K_t = \frac{1}{2}v^T m v$$

where  $v = (v_1, v_2, v_3)^T$  is the vector of linear velocities of the center of mass of the links and  $m = \text{diag}(m_1, m_2, m_3)$ . The coordinates of the center of mass of three links

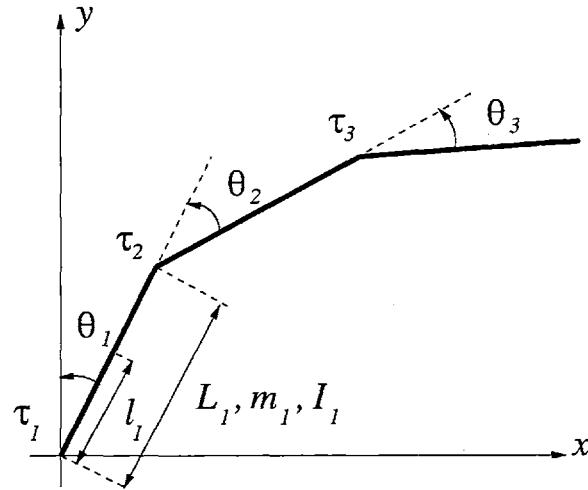


Figure A-1: A planar three-link robot arm.

are as follows

$$\begin{aligned} x_1 &= \begin{bmatrix} l_1 \sin \theta_1 \\ l_1 \cos \theta_1 \end{bmatrix} \\ x_2 &= \begin{bmatrix} L_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \\ x_3 &= \begin{bmatrix} L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \end{bmatrix} \end{aligned}$$

Thus, denoting  $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T$ , we have

$$\begin{aligned} v_1 &= J_1 \dot{\theta} = \begin{bmatrix} l_1 \cos \theta_1 & 0 & 0 \\ -l_1 \sin \theta_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\ v_2 &= J_2 \dot{\theta} = \begin{bmatrix} L_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) & 0 \\ -L_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\ v_3 &= J_3 \dot{\theta} = \begin{bmatrix} L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) & l_3 \cos(\theta_1 + \theta_2 + \theta_3) & 0 \\ -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) - l_3 \sin(\theta_1 + \theta_2 + \theta_3) & -l_3 \sin(\theta_1 + \theta_2 + \theta_3) & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \end{aligned}$$

This gives

$$K_t = \frac{1}{2} \dot{\theta}^T (J_1^T m_1 J_1 + J_2^T m_2 J_2 + J_3^T m_3 J_3) \dot{\theta}$$

Now, let us calculate the rotational kinetic energy  $K_r$ . Denoting the unit vector orthogonal to the  $(x, y)$ -frame by  $e_3 = (0, 0, 1)^T$ , the angular velocities of the links in reference frame are as follows

$$\begin{aligned} \omega_1 &= \dot{\theta}_1 e_3 \\ \omega_2 &= (\dot{\theta}_1 + \dot{\theta}_2) e_3 \\ \omega_3 &= (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) e_3 \end{aligned}$$

Hence

$$K_r = \frac{1}{2} \omega_1^T I_1 \omega_1 + \frac{1}{2} \omega_2^T I_2 \omega_2 + \frac{1}{2} \omega_3^T I_3 \omega_3 = \frac{1}{2} \dot{\theta}^T I \dot{\theta}$$

where

$$I = \begin{bmatrix} I_1 + I_2 + I_3 & I_2 + I_3 & I_3 \\ I_2 + I_3 & I_2 + I_3 & I_3 \\ I_3 & I_3 & I_3 \end{bmatrix}$$

Therefore, the total kinetic energy is given by

$$K = \frac{1}{2} \dot{\theta}^T (I + J_1^T m_1 J_1 + J_2^T m_2 J_2 + J_3^T m_3 J_3) \dot{\theta}$$

The inertia matrix of the system takes the following form

$$M = M(\theta_2, \theta_3) = \begin{bmatrix} m_{11}(\theta_2, \theta_3) & m_{12}(\theta_2, \theta_3) & m_{13}(\theta_2, \theta_3) \\ m_{21}(\theta_2, \theta_3) & m_{22}(\theta_3) & m_{23}(\theta_3) \\ m_{31}(\theta_2, \theta_3) & m_{32}(\theta_3) & m_{33} \end{bmatrix}$$

with elements

$$\begin{aligned} m_{11}(\theta_2, \theta_3) &= I_1 + I_2 + I_3 + m_1 l_1^2 + m_2 (L_1^2 + l_2^2) + m_3 (L_1^2 + L_2^2 + l_3^2) \\ &+ 2(m_2 L_1 l_2 + m_3 L_1 L_2) \cos \theta_2 \\ &+ 2m_3 L_1 l_3 \cos(\theta_2 + \theta_3) + 2m_3 L_2 l_3 \cos \theta_3 \\ m_{12}(\theta_2, \theta_3) &= m_{21} = I_2 + I_3 + m_2 l_2^2 + m_3 (L_2^2 + l_3^2) + (m_2 L_1 l_2 + m_3 L_1 L_2) \cos \theta_2 \\ &+ 2m_3 L_2 l_3 \cos \theta_3 + m_3 L_1 l_3 \cos(\theta_2 + \theta_3) \\ m_{13}(\theta_2, \theta_3) &= m_{31} = I_3 + m_3 l_3^2 + m_3 L_1 l_3 \cos(\theta_2 + \theta_3) + m_3 L_2 l_3 \cos \theta_3 \\ m_{22}(\theta_3) &= I_2 + I_3 + m_2 l_2^2 + m_3 (L_2^2 + l_3^2) + 2m_3 L_2 l_3 \cos \theta_3 \\ m_{23}(\theta_3) &= m_{32} = I_3 + m_3 l_3^2 + m_3 L_2 l_3 \cos \theta_3 \\ m_{33} &= I_3 + m_3 l_3^2 \end{aligned} \tag{A.4}$$

The potential energy of this three-link robot is given by

$$\begin{aligned} V(\theta) &= (m_1 l_1 + m_2 L_1 + m_3 L_1) g \cos \theta_1 + (m_2 l_2 + m_3 L_2) g \cos(\theta_2 + \theta_3) \\ &+ m_3 l_3 g \cos(\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

Notice that the potential energy  $V(\theta)$  is identically zero for a three-link robot in a horizontal plane. The Euler-Lagrange equations of motion for this system are as the following

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \tau$$

where  $\tau = (\tau_1, \tau_2, \tau_3)^T$  is the input torque (control) and the Lagrangian  $\mathcal{L}$  is given by

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^T M(\theta_2, \theta_3) \dot{\theta} - V(\theta)$$

Clearly,  $\theta_1$  is the external variable and  $\theta_2, \theta_3$  are the shape variables of this robot. The following lemma is useful and shows that an three-link robot with one or two actuators is an example of an underactuated system with non-integrable normalized momentums. It is important to notice that this property does not hold for a planar two-link robot and both normalized momentums of the Acrobot (or the Pendubot) are integrable.

**Lemma A.2.1.** *All three normalized momentums*

$$\pi_i = m_{ii}^{-1}(\theta) \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}, \quad i = 1, 2, 3$$

*of a planar three-link robot are non-integrable, i.e.  $\exists h : h = \pi_1$ .*

**Proof.** First, we prove

$$\pi_1 = \dot{\theta}_1 + \frac{m_{12}(\theta_2, \theta_3)}{m_{11}(\theta_2, \theta_3)} \dot{\theta}_2 + \frac{m_{13}(\theta_2, \theta_3)}{m_{11}(\theta_2, \theta_3)} \dot{\theta}_3$$

is non-integrable. For the purpose of contradiction, assume  $\pi_1$  is integrable. Then,  $\pi_1 - \dot{\theta}_1$  is trivially integrable and the following one-form must be exact

$$\omega_1 = \frac{m_{12}(\theta_2, \theta_3)}{m_{11}(\theta_2, \theta_3)} d\theta_2 + \frac{m_{13}(\theta_2, \theta_3)}{m_{11}(\theta_2, \theta_3)} d\theta_3$$

A sufficient and necessary condition that  $\omega_1$  is exact is

$$\frac{\partial}{\partial \theta_3} \frac{m_{12}(\theta_2, \theta_3)}{m_{11}(\theta_2, \theta_3)} = \frac{\partial}{\partial \theta_2} \frac{m_{13}(\theta_2, \theta_3)}{m_{11}(\theta_2, \theta_3)}$$

for all  $(\theta_2, \theta_3) \in S_1 \times S_1$ . A rather lengthy but straightforward calculation shows that the last equality does not hold for all  $\theta_2, \theta_3$  and therefore  $\pi_1$  is non-integrable. In a similar way but after a short calculation, it can be shown that  $\pi_2, \pi_3$  are non-integrable normalized momentums as well.  $\square$

**Proposition A.2.1.** *A three-link robot in a horizontal plane or zero gravity (i.e.  $V(\theta) = 0$ ) with two actuators at  $(\theta_2, \theta_3)$ , a passive joint  $\theta_1$ , and zero initial velocity is an underactuated mechanical system with a nonholonomic velocity constraint  $\pi_1 = 0$ . In addition, there is an invertible change of control*

$$v = \alpha(\theta)(\tau_2, \tau_3)^T + \beta(\theta, \dot{\theta})$$

that transforms the dynamics of the system into a driftless nonlinear system augmented with an integrator

$$\begin{aligned} \dot{\theta}_1 &= g_2(\theta_2, \theta_3)u_2 + g_3(\theta_2, \theta_3)u_3 \\ \dot{\theta}_2 &= u_2 \\ \dot{\theta}_3 &= u_3 \\ \dot{u} &= v \end{aligned} \tag{A.5}$$

where  $u = (u_2, u_3)^T$  and

$$g_i = -\frac{m_{1i}(\theta_2, \theta_3)}{m_{11}(\theta_2, \theta_3)}, \quad i = 2, 3$$

**Proof.** Assume  $\tau_1 = 0$ , then from Euler-Lagrange equations of motion

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{\partial \mathcal{L}}{\partial \theta_1} = 0$$

Thus  $m_{11}\pi_1$  is a conserved quantity, i.e.  $m_{11}\pi_1 = 0, \forall t \geq 0$ . Since  $m_{11}(\theta_2, \theta_3) > 0$ , based on lemma A.2.1,  $\pi_1 = 0$  is a nonholonomic (i.e. non-integrable) velocity

constraint. The normal form in (A.5) is a direct result of collocated partial feedback linearization and definition of  $\pi_1$ .  $\square$

**Theorem A.2.1.** *The inertia matrix of an n-link planar robot with n revolute joints  $(\theta_1, \dots, \theta_n)$  (as shown in Figure A-1 for  $n = 3$ ) has the following structure*

$$M(\theta) = \begin{bmatrix} m_{11}(\theta_2, \dots, \theta_n) & m_{12}(\theta_2, \dots, \theta_n) & \dots & m_{1n}(\theta_2, \dots, \theta_n) \\ m_{21}(\theta_2, \dots, \theta_n) & m_{22}(\theta_3, \dots, \theta_n) & \dots & m_{2n}(\theta_3, \dots, \theta_n) \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1}(\theta_2, \dots, \theta_n) & m_{n2}(\theta_3, \dots, \theta_n) & \dots & m_{nn} \end{bmatrix}$$

In other words,  $M(\theta) = \{m_{ij}(\theta_s, \dots, \theta_n)\}$  where  $s = \min\{i, j\} + 1$  for  $i, j < n$  and  $m_{nn}$  is constant.

**Proof.** The proof is by direct calculation similar to the one applied to a three-link robot. However, it is easier to prove this by induction. One assumes that the theorem is true for  $k = 2$  and  $k = n > 2$  links and adds an  $(n+1)$ th link to the end point of the  $n$ th link. It is rather straightforward to prove that the theorem holds for  $k = n + 1$ . This proves that the theorem holds for and  $n \geq 2$ .  $\square$

### A.3 Two-link Planar Robots: the Acrobot and the Pendubot

A two-link planar robot with joint angles  $\theta_1, \theta_2$  can be viewed as a special case of a three-link planar robot with  $l_3 = 0$  (and thus  $m_3 = I_3 = 0$ ) (see Figure A-1). In other words, the inertia matrix for a two-link planar robot with revolute joints is given by

$$\begin{aligned} m_{11}(\theta_2) &= I_1 + I_2 + m_1 l_1^2 + m_2 (L_1^2 + l_2^2) + 2m_2 L_1 l_2 \cos(\theta_2) \\ m_{12}(\theta_2) &= I_2 + m_2 l_2^2 + m_2 L_1 l_2 \cos(\theta_2) \\ m_{22} &= I_2 + m_2 l_2^2 \end{aligned} \quad (\text{A.6})$$

and  $m_{21}(\theta_2) = m_{12}(\theta_2)$ . The potential energy of a two-link robot in a vertical plane is in the form

$$V(\theta_1, \theta_2) = (m_1 l_1 + m_2 L_1) g \cos(\theta_1) + m_2 l_2 g \cos(\theta_1 + \theta_2) \quad (\text{A.7})$$

Based on equation (A.3), the Euler-Lagrange equations of motion for this two-link robot can be expressed as

$$\begin{aligned} m_{11}(\theta_2) \ddot{\theta}_1 + m_{12}(\theta_2) \dot{\theta}_2 + h_1(\theta, \dot{\theta}) &= F_1 \tau \\ m_{21}(\theta_2) \ddot{\theta}_1 + m_{22}(\theta_2) \dot{\theta}_2 + h_2(\theta, \dot{\theta}) &= F_2 \tau \end{aligned} \quad (\text{A.8})$$

where

$$\begin{aligned} h_1(\theta, \dot{\theta}) &= -m_2 L_1 l_2 \sin(\theta_2)(2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) + g_1(\theta_1, \theta_2) \\ h_2(\theta, \dot{\theta}) &= m_2 L_1 l_2 \sin(\theta_2)\dot{\theta}_1^2 + g_2(\theta_1, \theta_2) \\ g_1(\theta_1, \theta_2) &= -(m_1 l_1 + m_2 L_1)g \sin(\theta_1) - m_2 g l_2 \sin(\theta_1 + \theta_2) \\ g_2(\theta_1, \theta_2) &= -m_2 g l_2 \sin(\theta_1 + \theta_2) \end{aligned}$$

Depending on whether  $\theta_1$  is actuated and  $\theta_2$  is unactuated, or vice versa; two important examples of underactuated systems with two degrees of freedom arise.

**Definition A.3.1.** (Acrobot and Pendubot) The underactuated system in (A.8) is called *Acrobot* if  $(F_1, F_2) = (0, 1)$ . It is called *Pendubot* if  $(F_1, F_2) = (1, 0)$ .

Defining the state vector  $x = (q_1, \dot{q}_1, q_2, \dot{q}_2)$ , the dynamics of the the system in (A.8) can be expressed as a nonlinear system affine in control

$$\dot{x} = f(x) + g_i(x)u \quad (\text{A.9})$$

where  $u = \tau$  and  $i = a, p$  correspond to the Acrobot and the Pendubot, respectively. Apparently, the drift vector field  $f(x)$  is the same for the Acrobot and the Pendubot. However, the control vector fields  $g_1, g_2$  are different and can be expressed as

$$g_a(q_2) = \frac{1}{D(q_2)} \begin{bmatrix} 0 \\ -m_{12}(q_2) \\ 0 \\ m_{11}(q_2) \end{bmatrix}, \quad g_p(q_2) = \frac{1}{D(q_2)} \begin{bmatrix} 0 \\ m_{22}(q_2) \\ 0 \\ -m_{21}(q_2) \end{bmatrix}$$

where  $D(q_2) = \det(M(q_2)) = m_{11}m_{22} - m_{12}m_{21} > 0$ .

*Remark A.3.1.* The distribution generated by nested Lie brackets of  $f(x), g_a(x)$  is not the same as the distribution generated by  $f(x), g_p(x)$ ; unless  $g_p(x) = \lambda g_a(x)$  for a constant  $\lambda \in \mathbb{R}$ . But this means  $m_{21} = -\lambda m_{11}$  and  $m_{22} = -\lambda m_{12}$ , or equivalently  $D(q_2) = 0$  which contradicts the property  $D(q_2) > 0$ . Therefore, the controllability properties of the Acrobot and the Pendubot are different.

## A.4 The Beam-and-Ball System

The beam-and-ball system consists of a beam and a ball with radius  $r$  on it (see Figure A-2). Let  $d \geq 0$  denote the distance between the center of the mass of the ball and the beam ( $d = r$  in Figure 5-24). In the current literature [34, 102, 80, 79], what is known as the beam-and-ball system corresponds to the special case where  $d = 0$ . We refer to this case as the *conventional beam-and-ball system* and to the case where  $d > 0$  as the beam-and-ball system. The following treatment applies to the both cases of  $d = 0$  and  $d > 0$ . Let  $I_1, m, I_2$  denote the inertia of the beam, the mass of the ball, and the inertia of the ball, respectively. The position and velocity of the

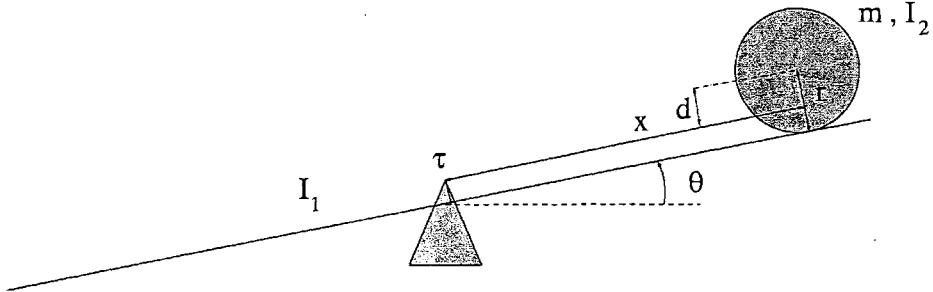


Figure A-2: The Beam-and-Ball System.

center of mass of the ball is

$$\begin{aligned}x_c &= (x \cos(\theta) - d \sin(\theta), x \sin(\theta) + d \cos(\theta)) \\v_c &= (v \cos(\theta) - x \sin(\theta)\omega - d \cos(\theta)\omega, v \sin(\theta) + x \cos(\theta)\omega - d \sin(\theta)\omega)\end{aligned}$$

where  $\dot{x} = v$  and  $\dot{\theta} = \omega$ . Thus, the translational part of the kinetic energy is given by

$$K_{trans} = \frac{1}{2}mv_c^2 = \frac{1}{2}m(v^2 + (x^2 + d^2)\omega^2 - 2d\omega v)$$

The rotational kinetic energy of the beam-and-ball system is

$$K_{rot} = \frac{1}{2}I_1\omega^2 + \frac{1}{2}I_2\left(\frac{v}{r}\right)^2$$

Hence, the total kinetic energy  $K = K_{trans} + K_{rot}$  equals

$$K = \frac{1}{2}(I_1 + m(x^2 + d^2))\omega^2 - md\omega v + \frac{1}{2}\lambda mv^2$$

where

$$\lambda = 1 + \frac{I_2}{mr^2} > 1$$

is a constant. The potential energy of the beam-and-ball system is given by

$$V(x, \theta) = mg(x \sin \theta + d \cos \theta)$$

Therefore, the Lagrangian of the beam-and-ball system is as the following

$$\mathcal{L} = \frac{1}{2} \begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix}^T \begin{bmatrix} I_1 + m(x^2 + d^2) & -md \\ -md & m\lambda \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x} \end{bmatrix} - mg(x \sin \theta + d \cos \theta) \quad (\text{A.10})$$

Based on (A.10), the elements of the inertia matrix of the beam-and-ball system are

$$\begin{aligned} m_{11}(x) &= I_1 + m(x^2 + d^2) \\ m_{12} &= m_{21} = -md \\ m_{22} &= m\lambda \end{aligned}$$

inertia matrix of the beam-and-ball system only depends on the position of the ball  $x$ . This means that *the position of the ball is the shape variable of the beam-and-ball system* which is unactuated. Therefore, *the beam-and-ball system is a Class-II underactuated system*. The Euler-Lagrange equations of motion for the beam-and-ball system with  $d > 0$  is given by

$$\begin{aligned} [I_1 + m(x^2 + d^2)]\ddot{\theta} - md\ddot{x} + 2mx\dot{x}\dot{\theta} + mg(x\cos(\theta) - d\sin(\theta)) &= \tau_d \\ -md\ddot{\theta} + m\lambda\ddot{x} - mx\dot{\theta}^2 + mg\sin(\theta) &= 0 \end{aligned} \quad (\text{A.11})$$

While for the conventional beam-and-ball system with  $d = 0$ , the Euler-Lagrange equations take the following form

$$\begin{aligned} (I_1 + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} + mgx\cos(\theta) &= \tau_0 \\ m\lambda\ddot{x} - mx\dot{\theta}^2 + mg\sin(\theta) &= 0 \end{aligned} \quad (\text{A.12})$$

## A.5 The Cart-Pole System

The Cart-Pole system depicted in Figure A-3 consists of a pendulum on a cart. Let  $(q_1, q_2)$  denote the configuration vector of the Cart-Pole system. The coordinates of

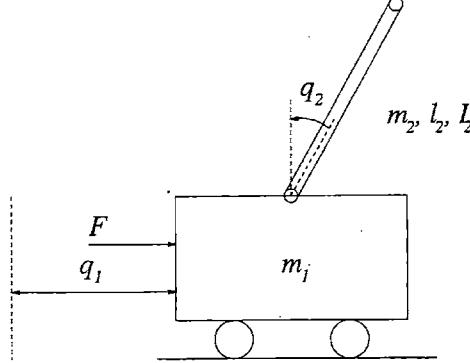


Figure A-3: The Cart-Pole system.

the center of mass of the cart and pendulum are given by

$$x_1 = \begin{bmatrix} q_1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} q_1 + l_2 \sin(q_2) \\ l_2 \cos(q_2) \end{bmatrix}$$

with linear velocities

$$\dot{x}_1 = \begin{bmatrix} \dot{q}_1 \\ 0 \end{bmatrix}, \quad \dot{x}_2 = \begin{bmatrix} \dot{q}_1 + l_2 \cos(q_2)\dot{q}_2 \\ -l_2 \sin(q_2)\dot{q}_2 \end{bmatrix}$$

Thus, the total kinetic energy of the system is

$$K = \frac{1}{2}m_1|\dot{x}_1|^2 + \frac{1}{2}m_2|\dot{x}_2|^2 + \frac{1}{2}I_2\dot{q}_2^2$$

After simplification, we obtain

$$K = \frac{1}{2}(m_1 + m_2)\dot{q}_1^2 + m_2l_2 \cos(q_2)\dot{q}_1\dot{q}_2 + \frac{1}{2}(m_2l_2^2 + I_2)\dot{q}_2^2$$

The potential energy of the Cart-Pole system is in the form

$$V(q) = V(q_2) = m_2gl_2 \cos(q_2)$$

Therefore, the Lagrangian of the Cart-Pole system can be expressed as

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} m_1 + m_2 & m_2l_2 \cos(q_2) \\ m_2l_2 \cos(q_2) & m_2l_2^2 + I_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - m_2gl_2 \cos(q_2) \quad (\text{A.13})$$

and the elements of the inertia matrix for the Cart-Pole system are given by

$$\begin{aligned} m_{11} &= m_1 + m_2 \\ m_{12}(q_2) &= m_{21}(q_2) = m_2l_2 \cos(q_2) \\ m_{22} &= m_2l_2^2 + I_2 \end{aligned} \quad (\text{A.14})$$

The Euler-Lagrange equations of motion for the Cart-Pole system is in the form

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

In other words, *the Cart-Pole system is an underactuated system with two DOF and an unactuated shape variable  $q_2$ .*

## A.6 The Rotating Pendulum

The Rotating Pendulum consists of an inverted pendulum on a rotating arm as shown in Figure A-4. This system was first introduced in [6]. The coordinates of the center of mass of both links are given by

$$x_1 = \begin{bmatrix} l_1 \cos(q_1) \\ l_1 \sin(q_1) \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} L_1 \cos(q_1) - l_2 \sin(q_1) \sin(q_2) \\ L_1 \sin(q_1) + l_2 \cos(q_1) \sin(q_2) \\ l_2 \cos(q_2) \end{bmatrix}$$

with linear velocities

$$\dot{x}_1 = \begin{bmatrix} -l_1 \sin(q_1)\dot{q}_1 \\ l_1 \cos(q_1)\dot{q}_1 \\ 0 \end{bmatrix}, \quad \dot{x}_2 = \begin{bmatrix} -L_1 \sin(q_1)\dot{q}_1 - l_2 \cos(q_1) \sin(q_2)\dot{q}_1 - l_2 \sin(q_1) \cos(q_2)\dot{q}_2 \\ L_1 \cos(q_1)\dot{q}_1 - l_2 \sin(q_1) \sin(q_2)\dot{q}_1 + l_2 \cos(q_1) \cos(q_2)\dot{q}_2 \\ -l_2 \sin(q_2)\dot{q}_2 \end{bmatrix}$$

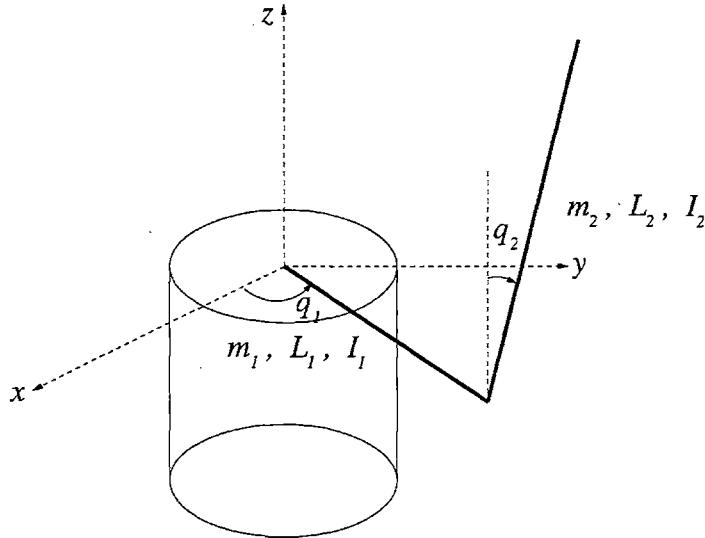


Figure A-4: The Rotating Pendulum.

The kinetic energy of the Rotating Pendulum is

$$K = \frac{1}{2}m_1|\dot{x}_1|^2 + \frac{1}{2}m_2|\dot{x}_2|^2 + \frac{1}{2}I_1\dot{q}_1^2 + \frac{1}{2}I_2\dot{q}_2^2$$

which can be simplified as

$$K = \frac{1}{2}(m_1l_1^2 + I_1 + m_2L_1^2 + m_2l_2^2 \sin^2(q_2))\dot{q}_1^2 + m_2l_2 \cos(q_2)\dot{q}_1\dot{q}_2 + \frac{1}{2}(m_2l_2^2 + I_2)\dot{q}_2^2$$

The potential energy of the Rotating Pendulum is given by

$$V(q) = V(q_2) = m_2gl_2 \cos(q_2)$$

Thus, the Lagrangian of the Rotating Pendulum can be written as

$$\mathcal{L} = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} m_1l_1^2 + I_1 + m_2L_1^2 + m_2l_2^2 \sin^2(q_2) & m_2l_2 \cos(q_2) \\ m_2l_2 \cos(q_2) & m_2l_2^2 + I_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - m_2gl_2 \cos(q_2) \quad (\text{A.15})$$

Therefore, the elements of the inertia matrix for the Rotating Pendulum are given by

$$\begin{aligned} m_{11}(q_2) &= m_1l_1^2 + I_1 + m_2L_1^2 + m_2l_2^2 \sin^2(q_2) \\ m_{12}(q_2) &= m_{21}(q_2) = m_2l_2 \cos(q_2) \\ m_{22} &= m_2l_2^2 + I_2 \end{aligned} \quad (\text{A.16})$$

The Euler-Lagrange equations of motion for the Rotating Pendulum is as the following

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}$$

which means *the Rotating Pendulum is an underactuated system with two DOF and one unactuated shape variable  $q_2$* .

## A.7 The 3D Cart-Pole System with Unactuated Roll and Pitch Angles

Consider an inverted pendulum mounted on a moving platform via an unactuated 2 DOF joint ( $\theta, \phi$ ) (i.e. the roll and pitch angles) with two external forces as shown in Figure A-5. The angles  $\theta, \phi$  denote rotation around  $x_2$ -axis and  $x_1$ -axis, respectively. Denote the positions of the centers of mass of  $M$  and  $m$  as the following by  $x_{c_1}$  and

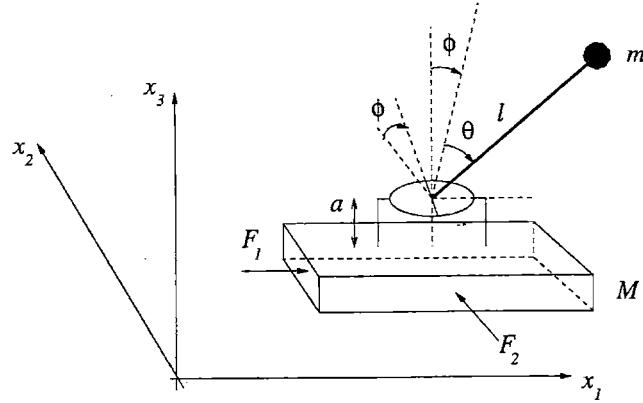


Figure A-5: The 3D Cart-Pole system with unactuated roll and pitch angles.

$x_{c_2}$ , respectively.

$$x_{c_1} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \quad x_{c_2} = \begin{bmatrix} x_1 + l \sin(\theta) \\ x_2 + l \cos(\theta) \sin(\phi) \\ a + l \cos(\theta) \cos(\phi) \end{bmatrix}$$

The linear velocities associated with  $(x_{c_1}, x_{c_2})$  are given by

$$\dot{x}_{c_1} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ 0 \end{bmatrix}, \quad \dot{x}_{c_2} = \begin{bmatrix} \dot{x}_1 + l \cos(\theta) \dot{\theta} \\ \dot{x}_2 - l \sin(\theta) \sin(\phi) \dot{\theta} + l \cos(\theta) \cos(\phi) \dot{\phi} \\ -l \sin(\theta) \cos(\phi) \dot{\theta} - l \cos(\theta) \sin(\phi) \dot{\phi} \end{bmatrix}$$

The kinetic and potential energy of the system are

$$K = \frac{1}{2} M \dot{x}_{c_1}^T \dot{x}_{c_1} + \frac{1}{2} m \dot{x}_{c_2}^T \dot{x}_{c_2}, \quad V(q) = mgl \cos \theta \cos \phi$$

The expression for the kinetic energy can be simplified as

$$K = \frac{1}{2} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}^T \begin{bmatrix} M+m & 0 & ml \cos \theta & 0 \\ 0 & M+m & -ml \sin \theta \sin \phi & ml \cos \theta \cos \phi \\ ml \cos \theta & -ml \sin \theta \sin \phi & ml^2 & 0 \\ 0 & ml \cos \theta \cos \phi & 0 & ml^2 \cos^2 \theta \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \quad (\text{A.17})$$

Clearly, the inertia matrix for this system only depends on  $(\theta, \phi)$ . Therefore,  $(\theta, \phi)$  are the shape variables and  $(x_1, x_2)$  are the external variables of the 3D Cart-Pole system. The Euler-Lagrange equations of motion for the 3D Cart-Pole system with  $q = (x_1, x_2, \theta, \phi)^T$  is as the following

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \end{bmatrix}$$

In other words, *the 3D Cart-Pole System is an underactuated system with four DOF and two unactuated shape variables.* The equations of motion for the 3D Cart-Pole system can be explicitly written as

$$\begin{aligned} (M+m)\ddot{x}_1 + ml \cos \theta \ddot{\theta} &= ml \sin \theta \dot{\theta}^2 + F_1 \\ (M+m)\ddot{x}_2 - ml \sin \theta \sin \phi \ddot{\theta} + ml \cos \theta \cos \phi \ddot{\phi} &= ml \cos \theta \sin \phi \dot{\theta}^2 + 2ml \sin \theta \cos \phi \dot{\theta} \dot{\phi} \\ &\quad + ml \cos \theta \sin \phi \dot{\phi}^2 + F_2 \\ ml \cos \theta \ddot{x}_1 - ml \sin \theta \sin \phi \ddot{x}_2 + ml^2 \ddot{\theta} &= mgl \sin \theta \cos \phi - ml^2 \sin \theta \cos \theta \dot{\phi}^2 \\ ml \cos \theta \cos \phi \ddot{x}_2 + ml^2 \cos^2 \theta \ddot{\phi} &= mgl \cos \theta \sin \phi + 2ml^2 \sin \theta \cos \theta \dot{\phi} \end{aligned} \quad (\text{A.18})$$

## A.8 An Underactuated Surface Vessel

A surface vessel with no side thruster is illustrated in Figure A-6. Thus, it is an underactuated system with three degrees of freedom and two actuators. The configuration vector for this vessel is  $\eta = (x, y, \theta)$  where  $(x, y)$  denotes the position and  $\theta$  determines the orientation of the vessel (i.e. the yaw angle). The velocity vector in the body frame is  $\nu = (v_1, v_2, \omega)$  where  $(v_1, v_2)$  are the linear velocities and  $\omega$  is the angular velocity. The complete dynamics of an underactuated surface vessel can be found in [30, 111]. Defining the following rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

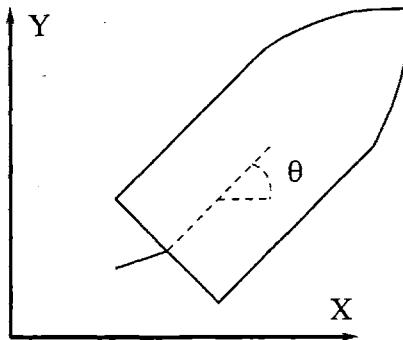


Figure A-6: An underactuated surface vessel.

we have

$$\dot{\eta} = R(\theta)\nu$$

The kinetic energy of this system is in the form

$$K = \frac{1}{2}\nu^T J \nu$$

with  $J = \text{diag}(J_1, J_2, J_3)$ . Thus, the kinetic energy in the reference frame is

$$K = \frac{1}{2}\dot{\eta}^T M \dot{\eta}$$

which means the inertia matrix of a surface vessel is

$$M = R(\theta)JR^T(\theta) = J$$

or *the surface vessel is a flat underactuated mechanical system*. The potential energy of a surface vessel is identically zero. The rotation matrix  $R$  satisfies

$$\dot{R} = R\hat{\omega}$$

where  $\hat{\omega}$  is the following skew symmetric matrix

$$\hat{\omega} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The dynamics of this surface vessel can be expressed as

$$\begin{aligned} \dot{\eta} &= R(\theta)\nu \\ J\dot{\nu} &= -\hat{\omega}J\nu - D\nu + \begin{bmatrix} \tau_1 \\ 0 \\ \tau_3 \end{bmatrix} \end{aligned} \tag{A.19}$$

where  $D\nu$  is a damping term with  $D = \text{diag}(d_1, d_2, d_3)$  and  $d_i > 0$ . The term  $\hat{\omega}J\nu$  has the following explicit form

$$\hat{\omega}J\nu = \begin{bmatrix} 0 & -J_2v_2\omega & 0 \\ J_1v_1\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = C(\nu)\nu$$

where

$$C(\nu) = \begin{bmatrix} 0 & 0 & J_2v_2 \\ 0 & 0 & -J_1v_1 \\ -J_2v_2 & J_1v_1 & 0 \end{bmatrix}$$

The overall dynamics of the surface vessel can be rewritten as

$$\begin{aligned} \dot{\eta} &= R(\theta)\nu \\ J\dot{\nu} + C(\nu)\nu + D\nu &= \begin{bmatrix} \tau_1 \\ 0 \\ \tau_3 \end{bmatrix} \end{aligned} \tag{A.20}$$

From the second line of the dynamic equation in (A.20), we have

$$J_2\dot{v}_2 + J_1v_1\omega + d_2v_2 = 0$$

which is a *nonholonomic second-order constraint* (i.e. a constraint that involves second-order time derivatives of the configuration variables) that holds regardless of the control inputs  $\tau_1, \tau_3$ .

## Appendix B

# Symbolic Calculation of the Flat Outputs of the Helicopter

This appendix provides a MATLAB code for symbolic calculations required to prove that the vector of one-forms  $\omega_{xs} = R(\eta)EJ\Psi(\eta)d\eta$  in proposition 5.11.2 has no exact elements. In addition, using momentum decomposition, an exact part of  $\omega_{xs}$  is obtained that is denoted by  $C(\eta)d\eta$  in the following.

```
% (a,b,c)=(phi,theta,psi)=(roll,pitch,yaw) are three Euler angles.  
syms a b c;  
J1=sym('J1'); J2=sym('J2'); J3=sym('J3'); J=diag([J1,J2,J3]);  
e1=sym('e1'); e2=sym('e2'); e3=sym('e3'); E=[0 e2 0; e1 0 e3; 0 0 0];  
% A=Psi(eta) is the inverse of the Euler matrix.  
A=[1 0 -sin(b); 0 cos(a) cos(b)*sin(a); 0 -sin(a) cos(b)*cos(a)];  
R1=[1 0 0; 0 cos(a) -sin(a); 0 sin(a) cos(a)];  
R2=[cos(b) 0 sin(b); 0 1 0 ; -sin(b) 0 cos(b)];  
R3=[cos(c) -sin(c) 0; sin(c) cos(c) 0; 0 0 1];  
R=simplify(R3*R2*R1); G=simplify(R*E*J*A);  
% if d1,d2,d3 are not identically zero, G(eta)deta is not exact.  
d1=simplify(diff(G(:,2), 'c')-diff(G(:,3), 'b'));  
d2=simplify(diff(G(:,1), 'c')-diff(G(:,3), 'a'));  
d3=simplify(diff(G(:,1), 'b')-diff(G(:,2), 'a'));  
% Observation: none of d1,d2,d3 are identically zero  
% A=A(a,b); A1=A(a,0); A2=A(0,b); A3=A(0,0);  
A1=[1 0 0; 0 cos(a) sin(a); 0 -sin(a) cos(a)];  
A2=[1 0 -sin(b); 0 1 0; 0 0 cos(b)];  
A3=eye(3);  
G1=simplify(R1*E*J*A1); C1=G1(:,1); G2=simplify(R2*E*J*A2); C2=G2(:,2);  
G3=simplify(R3*E*J*A3); C3=G3(:,3);  
% The obtained locked vector of one-forms omega_l = C(eta)deta has exact elements  
C=[C1 C2 C3];  
C = [ 0, cos(b)*e2*J2, -sin(c)*e3*J3] [ cos(a)*e1*J1, 0, cos(c)*e3*J3] [  
sin(a)*e1*J1, -sin(b)*e2*J2, 0]
```

## Appendix C

# Proofs of Theorems in Chapter 7

### C.1 Proof of Theorem 7.4.3:

Equation (7.93) can be rewritten as

$$\xi_2 = D_{z_1} \alpha_\kappa(z) \cdot z_2 + D_{z_2} \alpha_\kappa(z) \cdot f(z, \alpha_\kappa(z), \xi_2) =: \psi(z, \xi_2)$$

Since  $D_{z_2} \alpha_\kappa(z) = 0, \forall z_2 \notin \mathcal{K}_0$ , the fixed point equation (7.93) has a (trivial) smooth unique solution as the following

$$\beta(z) = D_{z_1} \alpha_\kappa(z) \cdot z_2$$

for all  $z_2 \notin \mathcal{K}_0$ . For the rest of the proof, assume  $z_2 \in \mathcal{K}_0$  and set  $r_0 = \max_{z_2 \in \mathcal{K}_0} \|z_2\|$ . Denote

$$\begin{aligned} \mathcal{K}_{\xi_1} &= B(0, L_0) \subset \mathbb{R} \\ \mathcal{K}_{\xi_2} &= B(0, 1) \subset \mathbb{R} \end{aligned}$$

(where  $B(x_0, r) = \{x \in \mathbb{R}^{n_x} : |x - x_0| \leq r\}$ ) and set  $S := \mathbb{R}^n \times \mathcal{K}_0 \times \mathcal{K}_{\xi_2}$ . Based on Assumption (7.4.9), over  $S$  we have

$$\begin{aligned} |\psi(z, \xi_2)| &\leq \|D_{z_1} \alpha_\kappa(z)\| \cdot \|z_2\| + \|D_{z_2} \alpha_\kappa(z)\| \cdot (\|f_0(z, \xi_1)\| + \|g_0(z, \xi_1, \xi_2)\| \cdot \|\xi_2\|^2) \\ &\leq (L_0 + L_2)r_0 + L_1((a_1 + a_2)\kappa(z_1) + (b_1 + b_2)|z_1\kappa(z_1)|) \\ &\leq (L_0 + L_2)r_0 + L_1(a_1 + a_2 + b_1 + b_2) \leq 1 \end{aligned}$$

where the last inequality can be achieved by taking

$$L_0, L_2 \leq \frac{1}{3}r_0^{-1}, L_1 \leq \frac{1}{3}(a_1 + a_2 + b_1 + b_2)^{-1}$$

This guarantees that  $\psi(S) \subseteq \mathcal{K}_{\xi_2}$ . Calculating  $D_{\xi_2}\psi(z, \xi_2)$  for all  $(z, \xi_2) \in S$ , we get

$$\begin{aligned} |D_{\xi_2}\psi(z, \xi_2)| &\leq \|D_{z_2}\alpha_\kappa(z)\| \cdot |D_{\xi_2}f(z, \alpha_\kappa(z), \xi_2)| \\ &\leq L_1(a_3\kappa(z_1) + b_3\|z_1\kappa(z_1)\|) \\ &\leq L_1(a_3 + b_3) < 1 \end{aligned}$$

where the last inequality can be satisfied for  $L_1 < (a_3 + b_3)^{-1}$  that guarantees the mapping  $\psi(z, \xi_2)$  is contractive. Thus, after taking

$$L_0^* = L_2^* = \frac{1}{3}r_0^{-1}, \quad L_1^* = \min\left\{\frac{1}{3}(a_1 + a_2 + b_1 + b_2)^{-1}, (a_3 + b_3)^{-1}\right\}$$

the result follows from the contraction mapping theorem. The property  $\beta(0) = 0$  is due to the uniqueness of the solution  $\xi_2^* = 0$  of (7.93) at  $z = 0$ .

## C.2 Proof of the Stability of $\mu$ -Subsystem in Theorem 7.4.4

**Lemma C.2.1.** *The origin  $(\mu_1, \mu_2) = 0$  for*

$$\begin{aligned} \dot{\mu}_1 &= -\sigma(\mu_1) + \mu_2 \\ \dot{\mu}_2 &= -\sigma(\mu_2) \end{aligned} \tag{C.1}$$

*is globally asymptotically and locally exponentially stable.*

**Proof.** First, in the linearization of this system  $\dot{\mu} = A\mu$ ,  $A$  is a Hurwitz matrix and  $\mu = 0$  is locally exponentially stable. On the other hand,  $\mu_2 = 0$  is globally asymptotically stable for  $\dot{\mu}_2 = -\sigma(\mu_2)$  (because  $V_2(\mu_2) = \mu_2^2$  is a valid Lyapunov function that satisfies  $\dot{V}_2 < 0$  for  $\mu_2 \neq 0$ ). Thus, for any solution  $\mu_2(t)$  of  $\dot{\mu}_2 = -\sigma(\mu_2)$  with  $\mu_2(0) \in \mathbb{R}$ ,  $|\mu_2(t)| \leq |\mu_2(0)|$  for all  $t \geq 0$ . Taking  $V_1(\mu_1) = \frac{1}{2}\mu_1^2$ , we have

$$\begin{aligned} \dot{V}_1 &= -\mu_1\sigma(\mu_1) + \mu_1\mu_2 \\ &\leq -|\mu_1|\sigma(|\mu_1|) + |\mu_1| \cdot |\mu_2| \\ &\leq -|\mu_1|(\sigma(|\mu_1|) - |\mu_2(0)|) \end{aligned}$$

Let  $r_{\mu_1} := \sigma^{-1}(|\mu_2(0)|)$  and observe that for  $|\mu_1| > r_{\mu_1}$ ,  $\dot{V}_1 < 0$  and any solution starting out of the closed ball  $B(0, r_{\mu_1})$ , enters  $B(0, r_{\mu_1})$  after some finite time  $T_0 > 0$  and remains in it thereafter. Meaning that the solutions of system (C.1) stay uniformly bounded for all  $t > 0$  (with a bound that depends on  $\mu(0)$ ). But given  $\mu_2 = 0$  for  $\dot{\mu}_1 = -\sigma(\mu_1)$ ,  $\mu_1 = 0$  is GAS. Therefore, based on Sontag's theorem on stability of cascade nonlinear systems [83],  $(\mu_1, \mu_2) = 0$  is GAS for the cascade (C.1).  $\square$

### C.3 Proof of Lemma 7.4.4:

First, we construct  $K_y$ . Take  $r_0 = \max_{x \in K_x} |\phi_0(x)|$ ,  $\tilde{K}_y = B(0, 2r_0)$ , and  $r_1 = \max |\phi_1(x, y, \epsilon)|$  over  $K_x \times \tilde{K}_y \times I_\epsilon$ . We have

$$|\phi(x, y, \epsilon)| \leq |\phi_0(x)| + \epsilon |\phi_1(x, y, \epsilon)| \leq r_0 + \epsilon r_1 \leq 2r_0$$

for all  $0 < \epsilon \leq \epsilon_2 := \min\{r_0/r_1, \epsilon_1\}$ . Thus, setting  $K_y = \tilde{K}_y$ ,  $\phi$  maps  $K_x \times K_y \times \tilde{I}_\epsilon$  into  $K_y$  where  $\tilde{I}_\epsilon = [0, \epsilon_2]$ . Define

$$L_y = \max\{|\frac{\partial \phi_1}{\partial y}(x, y, \epsilon)|, 1\} \geq 1$$

over  $K_x \times K_y \times \tilde{I}_\epsilon$  and set  $\epsilon^* = \min\{\epsilon_2, 1/L_y\}$ . Then, for all for all  $0 < \epsilon < \epsilon^*$  the following holds

$$|\frac{\partial \phi}{\partial y}| = \epsilon |\frac{\partial \phi_1}{\partial y}| \leq \epsilon L_y < 1$$

over  $K_x \times K_y \times (0, \epsilon^*]$  and the result follows from contraction mapping theorem (or Banach fixed point theorem). In addition, due to uniqueness of the fixed point,  $\alpha(0, \epsilon) = 0$  independent of  $\epsilon$ .

### C.4 Proof of Theorem 7.4.4:

Noting that  $\partial \psi_3 / \partial u$  and  $\partial \psi_2 / \partial \xi_2$  are equal and both vanish at  $\epsilon = 0$ , the proof of (i) and (ii) follows directly from lemma 7.4.4 with appropriate choices of  $x, y$ . To prove (iii), first we show that  $\alpha_2$  locally asymptotically stabilizes the origin  $(z, \xi_1) = 0$  for the  $(z, \xi_1)$ -subsystem in (7.101). Then, we prove any solution of the system starting in a compact set of initial conditions for a sufficiently small  $\epsilon > 0$  stays uniformly bounded and asymptotically converges to the origin. For doing so, take  $\mu_1 = \xi_1 - \alpha_1(z)$ , we have  $\dot{\mu}_1 = -\sigma(\mu_1)$  and immediately it follows that  $\mu_1 = 0$  is globally asymptotically and locally exponentially stable for this system (with a Lyapunov function  $V_1(\mu_1) = \mu_1^2$ ). The dynamics of the system in new coordinates is a cascade nonlinear system as the following

$$\begin{aligned} \dot{z} &= f_0(z, \alpha_1(z) + \mu_1) + g_0(z, \alpha_1(z) + \mu_1, \alpha_2(z, \alpha_1(z) + \mu_1, \epsilon), \epsilon) \\ \dot{\mu}_1 &= -\sigma(\mu_1) \end{aligned} \tag{C.2}$$

or

$$\begin{aligned} \dot{z} &= f_0(z, \alpha_1(z)) + h_0(z, \alpha_1(z), \mu_1)\mu_1 + \tilde{g}_0(z, \mu_1, \epsilon) \\ \dot{\mu}_1 &= -\sigma(\mu_1) \end{aligned} \tag{C.3}$$

where

$$\tilde{g}_0(z, \mu_1, \epsilon) = g_0(z, \alpha_1(z) + \mu_1, \alpha_2(z, \alpha_1(z) + \mu_1, \epsilon), \epsilon)$$

Note that  $g_0$  can be rewritten as the following

$$g_0(z, \xi_1, \xi_2, \epsilon) = p_0(z, \xi_1, \xi_2, \epsilon) \xi_2^2 \epsilon$$

where  $p_0$  is a smooth function. Thus

$$\frac{\partial \tilde{g}_0}{\partial z} \Big|_{(z, \mu_1)=0} = \left[ \frac{\partial p_0}{\partial z} \alpha_2^2 \epsilon + \frac{\partial p_0}{\partial \xi_1} \frac{\partial \alpha_1(z)}{\partial z} \alpha_2^2 + 2\alpha_2 \frac{\partial \alpha_2}{\partial z} p_0 \right]_{(z, \mu_1)=0} = 0$$

where in the last equation  $\xi_1 = \alpha_1(z) + \mu_1$ ,  $\xi_2 = \alpha_2(z, \xi_1, \epsilon)$ , and  $\alpha_2(0, 0, \epsilon) = 0$ . Calculating the Jacobian matrix of (C.3) at the origin  $(z, \mu_1) = 0$ , we obtain

$$A_{z, \mu_1} = \begin{bmatrix} \frac{\partial f_0}{\partial z}(z, \alpha_1(z)) + \frac{\partial f_0}{\partial \xi_1}(z, \alpha_1(z)) \frac{\partial \alpha_1}{\partial z} + \left( \frac{\partial h_0}{\partial z} + \frac{\partial h_0}{\partial \xi_1} \frac{\partial \alpha_1}{\partial z} \right) \mu_1 + \frac{\partial \tilde{g}_0}{\partial z} & * \\ 0 & -\frac{d\sigma(\mu_1)}{d\mu_1} \end{bmatrix}_{(z, \mu_1)=(0,0)}$$

or

$$A_{z, \mu_1} = \begin{bmatrix} A_z & * \\ 0 & -\sigma'(0) \end{bmatrix}$$

where

$$A_z = \frac{\partial f_0}{\partial z}(z, \alpha_1(z)) + \frac{\partial f_0}{\partial \xi_1}(z, \alpha_1(z)) \frac{\partial \alpha_1(z)}{\partial z} \Big|_{z=0}$$

is a Hurwitz matrix (by assumption,  $\xi_1 = \alpha_1(z)$  locally exponentially stabilizes  $z = 0$  for  $\dot{z} = f_0(z, \xi_1)$ ) and  $-\sigma'(0) < 0$ . Therefore,  $(z, \mu_1) = (0, 0)$  is locally exponentially stable for (C.3) and there exists a local region of attraction  $U_0$  as a small open neighborhood of the origin  $(z, \mu_1) = 0$  that is forward invariant in time. Now, we show the solutions of (C.3) starting in a compact set stay uniformly bounded. To do so, consider the following Lyapunov function candidate

$$W(z, \mu_1) = V(z) + \frac{1}{2} \mu_1^2$$

Calculating  $\dot{W}$  along the solutions of the closed-loop system, we get

$$\dot{W} = \frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) + \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \mu_1 + \frac{\partial V(z)}{\partial z} \tilde{g}_0(z, \mu_1, \epsilon) - \mu_1 \sigma(\mu_1)$$

Take  $r_1 \geq r_0$  to be sufficiently large such that  $B(0, r_1) \supset U_0$ . Consider the compact set  $\Omega(c_1) = \{(z, \mu_1) | W(z, \mu_1) \leq c_1\}$  (where  $c_1 > 0$  to be determined later) with  $c_1 > c_2 > 0$  such that  $\Omega(c_2) \supset (\mathcal{K}_0 \cup B(0, r_1))$  ( $\Omega(c_2)$  is fixed). Then, there exists  $\epsilon_1(c_1) > 0$  such that for all  $0 < \epsilon < \epsilon_1(c_1)$ , we have

$$\left| \frac{\partial V(z)}{\partial z} \tilde{g}_0(z, \mu_1, \epsilon) \right| < -\frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) + \mu_1 \sigma(\mu_1)$$

over  $\Omega(c_1) \setminus U_0$ . Thus, we have

$$\begin{aligned}\dot{W} &< \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \mu_1(t) \\ &\leq \left| \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \right| \cdot |\mu_1(t)| \\ &\leq kV(z)^\lambda |\mu_1(t)|, \quad \forall |z| > r_1, \forall t \geq 0\end{aligned}$$

First, assume  $\lambda = 1$ . It follows that

$$\dot{W} < kV(z)|\mu_1(t)| \leq kW|\mu_1(t)|$$

but  $\mu_1(t)$  is an  $L_1$  function and  $\exists l_1 > 0 : \int_0^\infty |\mu_1(t)| dt \leq l_1$ . This means that

$$W(z(t), \mu_1(t)) < W(z(0), \mu_1(0)) \exp(kl_1) =: c_3$$

where  $c_3 > 0$  is a constant that only depends on  $k, l_1, W(z(0), \mu_1(0))$ . In other words,  $c_3$  is independent of  $\epsilon > 0$ . Thus, taking a fixed  $c_1 > \max\{c_2, c_3\}$  determines a compact set  $\mathcal{K}_1 := \Omega(c_1) \supset \mathcal{K}_0$  such that the solutions  $(z(t), \mu_1(t)) \in \mathcal{K}_1$  for all  $t \geq 0$ . The case  $0 < \lambda < 1$  is rather similar, we get

$$\dot{W} < kV(z)^\lambda \leq kW(z, \mu_1)^\lambda$$

Thus,  $c_3$  can be defined in the following way

$$\frac{d}{dt}(W^{(1-\lambda)}) < (1-\lambda)k\mu_1(t)$$

or

$$W(z, \mu_1) < (W(z(0), \mu_1(0))^{(1-\lambda)} + (1-\lambda)kl_1)^{\frac{1}{(1-\lambda)}} =: c_3(\lambda), \quad \forall t > 0$$

and the boundedness result follows (also, observe that  $\lim_{\lambda \rightarrow 1^-} c_3(\lambda) = c_3$  for  $\lambda = 1$ ). It remains to prove that the solutions starting in  $\mathcal{K}_0$  asymptotically converge to the origin. Over the compact set  $\mathcal{K}_1$ , take

$$l_2 = \max_{(z, \mu_1) \in \mathcal{K}_1} \left| \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \right|$$

and

$$\epsilon_0 = \max_{(z, \mu_1) \in \mathcal{K}_1 \setminus U_0} -\frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) + \mu_1 \sigma(\mu_1) > 0$$

Then, there exists  $\epsilon_2 > 0$  such that for all  $0 < \epsilon < \epsilon_2$

$$\left| \frac{\partial V(z)}{\partial z} \tilde{g}_0(z, \mu_1, \epsilon) \right| < \frac{\epsilon_0}{3}$$

Also, because  $\mu_1(t) \rightarrow 0$  as  $t \rightarrow \infty$ , there exists  $T_0 > 0$  such that for all  $t > T_0$

$$\left| \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \mu_1 \right| \leq l_2 |\mu_1(t)| < \frac{\epsilon_0}{3}$$

Therefore, over  $\mathcal{K}_1 \setminus U_0$ , we have

$$\begin{aligned} \dot{W} &\leq \frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) - \mu_1 \sigma(\mu_1) + \left| \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \mu_1 \right| + \left| \frac{\partial V(z)}{\partial z} \tilde{g}_0(z, \mu_1, \epsilon) \right| \\ &< -\epsilon_0 + \frac{\epsilon_0}{3} + \frac{\epsilon_0}{3} \\ &= -\frac{\epsilon_0}{3} < 0 \end{aligned}$$

This means that any solution of the system starting in  $\mathcal{K}_0$  stays in  $\mathcal{K}_1$  for all  $t > 0$  and enters  $U_0$  after some finite time  $T_1 \geq T_0 > 0$  and then asymptotically converges to  $(z, \mu_1) = 0$ , or  $(z, \xi_1) = 0$ . This finishes the proof of (iii). The proof of (iv) is very similar to (iii) and is omitted.

## C.5 Proof of Theorem 7.4.5:

The proof of (i) is based on contraction mapping theorem (see lemma 7.4.4). To prove (ii), first we establish local asymptotic stability of the origin  $(z, \xi_1) = 0$  for the closed loop  $(z, \xi_1)$ -subsystem. Over a small open neighborhood of the origin  $U$ , fix  $\epsilon_1 > 0$  such that for all  $0 < \epsilon \leq \epsilon_1$  (i) holds. Let  $\xi_2 = \alpha_2(z, \xi_1, \epsilon)$  be the unique fixed point of (7.106) over  $U$ . Around  $(z, \xi_1) = 0$ ,  $f_0$  and  $g_0$  can be expressed as

$$\begin{aligned} f_0(z, \alpha_1(z)) &= A_0 z + \tilde{f}_0(z) \\ g_0(z, \xi_1, \xi_2, \epsilon) &= \tilde{g}_0(z, \xi_1, \xi_2, \epsilon) \xi_2 \epsilon \end{aligned}$$

where

$$A_0 = \left[ \frac{\partial f_0}{\partial z}(z, \alpha_1(z)) + \frac{\partial f_0}{\partial \xi_1}(z, \alpha_1(z)) \frac{\partial \alpha_1(z)}{\partial z} \right]_{z=0}$$

is a Hurwitz matrix by assumption and  $\tilde{f}_0$  with all its first order partial derivatives vanish at  $z = 0$ . Consider the quadratic Lyapunov function

$$V_0(z) = \frac{1}{2} z^T P z$$

where  $P$  is the solution of the following Lyapunov equation

$$A_0^T P + PA_0 = -I$$

Applying the change of coordinates  $\mu_1 = \xi_1 - \alpha_1(z)$ , we show that the candidate Lyapunov function

$$W_0(z, \mu_1) = V_0(z) + \frac{1}{2}\mu_1^2$$

locally over  $U$  is a valid Lyapunov function for a sufficiently large  $c_1 = \sigma'_1(0) > 0$ . Note that locally  $\sigma_1(\mu_1) = c_1\mu_1 + O(|\mu_1|^2)$ . Calculating  $\dot{W}_0$ , we have

$$\begin{aligned} \dot{W}_0 &= -|z|^2 + z^T P \tilde{f}_0(z) + \mu_1 z^T P h_0(z, \alpha_1(z), \mu_1) \\ &\quad + z^T P \tilde{g}_0(z, \alpha_1(z) + \mu_1, \alpha_2) \cdot \alpha_2(z, \alpha_1(z) + \mu_1, \epsilon) \epsilon \\ &\quad - c_1 \mu_1^2 + O(|\mu_1|^3) + \mu_1 \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \end{aligned}$$

We need to determine a local approximation of each term over  $U$  in the preceding expression. The second term  $z^T P \tilde{f}_0(z) \sim O(|z|^3)$ , and the third term satisfies

$$|\mu_1 z^T P h_0(z, \alpha_1(z), \mu_1)| < l_1 |\mu_1| \cdot |z|$$

for some  $l_1 > 0$ . Also,  $\alpha_2$  vanishes at the origin, thus  $|\alpha_2(z, \alpha_1(z) + \mu_1, \epsilon)| < \tilde{l}_2 |z| + \tilde{l}_3 |\mu_1|$ . This means that

$$|z^T P \tilde{g}_0(z, \alpha_1(z) + \mu_1, \alpha_2) \cdot \alpha_2(z, \alpha_1(z) + \mu_1, \epsilon)| < l_2 |z|^2 + l_3 |\mu_1| \cdot |z|$$

for some  $l_2, l_3 > 0$ . Also, because  $\partial V(z)/\partial z$  vanishes at  $z = 0$ , there exists  $\tilde{l}_4 > 0$  such that

$$|\frac{\partial V(z)}{\partial z}| \leq \tilde{l}_4 |z|$$

Thus

$$|\mu_1 \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1)| < l_4 |\mu_1| \cdot |z|$$

for some  $l_4 > 0$ . We obtain

$$\dot{W}_0 \leq -|z|^2(1 - \epsilon l_2) + (\epsilon l_3 + l_4) |\mu_1| \cdot |z| - c_1 |\mu_1|^2 + O(|z|^3) + O(|\mu_1|^3)$$

Take  $\epsilon_2 = \min\{1/3l_2, \epsilon_1\}$  and a sufficiently small open neighborhood  $U_0 \subset U$  of the origin in the form  $U_0 = U_0(c_0) = \{(z, \mu_1) | W_0(z, \mu_1) < c_0\}$  for a sufficiently small

$c_0 > 0$  such that

$$O(|z|^3) < \frac{1}{3}|z|^2, O(|\mu_1|^3) < b_1|\mu_1|^2$$

for some  $b_1 > 0$  (e.g.  $b_1 = 0.1$ ). Then, for all  $0 < \epsilon \leq \epsilon_2$ , we have

$$\dot{W}_0 \leq -\frac{1}{3}|z|^2 + \left(\frac{l_3}{3l_2} + l_4\right)|\mu_1| \cdot |z| - (c_1 - b_1)|\mu_1|^2$$

by completing the squares

$$\dot{W}_0 \leq -\frac{1}{3}\left(|z| + \frac{3}{2}b_2|\mu_1|\right)^2 - (c_1 - b_1 - \frac{3}{4}b_2^2)|\mu_1|^2$$

where  $b_2 = (l_3/3l_2) + l_4$ . Therefore, by taking

$$c_1 > c_1^* := b_1 + \frac{3}{4}b_2^2 > 0$$

there exists a positive definite matrix  $Q$  such that

$$\dot{W}_0 < -\bar{z}^T Q \bar{z} < 0, \forall \bar{z} \neq 0$$

where  $\bar{z} = \text{col}(z, \mu_1)$  and thus the origin  $(z, \mu_1) = 0$  is locally exponentially stable. Moreover,  $U_0$  is forward invariant in time. To establish semiglobal asymptotic stabilization of  $(z, \xi_1) = 0$ , consider the solutions starting in a compact set  $\mathcal{K}_0 := \mathcal{K}_z^0 \times \mathcal{K}_{\mu_1}^0 \ni (z(0), \mu_1(0))$  that includes  $U_0$ . Take the Lyapunov function  $W(z, \mu_1) = V(z) + \frac{1}{2}\mu_1^2$ . Let  $\Omega_0 = \{(z, \mu_1) | W(z, \mu_1) \leq c_0\}$  for a sufficiently large  $c_0 > 0$  such that  $\Omega_0 \supset \mathcal{K}_0$ . Fix  $\epsilon_3 \leq \epsilon_2$  such that for all  $0 < \epsilon < \epsilon_3$  (i) holds over  $\Omega_0$  (note that  $W$  and thus  $\Omega_0$  are independent of  $\epsilon$ ). Calculating  $\dot{W}$  along the solutions of the closed loop system, we get

$$\begin{aligned} \dot{W} &= \frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) + \frac{\partial V(z)}{\partial z} g_0(z, \xi_1, \xi_2, \epsilon) \\ &+ \mu_1(\xi_2 - \frac{\partial \alpha_1(z)}{\partial z}) f_0(z, \xi_1) - \frac{\partial \alpha_1(z)}{\partial z} g_0(z, \xi_1, \xi_2, \epsilon) \\ &+ \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1) \\ &= \frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) - \mu_1 \sigma(\mu_1) + \frac{\partial V(z)}{\partial z} g_0(z, \xi_1, \xi_2, \epsilon) \\ &= \frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) - \mu_1 \sigma(\mu_1) + \frac{\partial V(z)}{\partial z} g_0(z, \alpha_1(z) + \mu_1, \alpha_2(z, \alpha_1(z) + \mu_1, \epsilon), \epsilon) \end{aligned}$$

Noting that  $\Omega_0^c := \Omega_0 \setminus U_0$  is a compact set and defining

$$\epsilon_0 = \min_{z \in \Omega_0^c} -\frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) + \mu_1 \sigma(\mu_1) > 0$$

there exists an  $\epsilon^* > 0$  such that for all  $0 < \epsilon < \epsilon^*$ ,  $|\frac{\partial V(z)}{\partial z} g_0| < \epsilon_0$  and  $\dot{W} < 0$  over  $\Omega_0^\epsilon$ . Thus, any solution of the system enters  $U_0$  after some finite time and exponentially converges to the origin. This guarantees semiglobal asymptotic stabilization of  $(z, \mu_1) = 0$  and therefore semiglobal asymptotic stabilization of  $(z, \xi_1) = 0$ . It remains to prove (iii). This follows from standard backstepping theorem applied to the composite system (7.101) rewritten in the following form

$$\begin{aligned}\dot{\bar{z}} &= f_2(\bar{z}, \xi_2, \epsilon) \\ \dot{\xi}_2 &= u\end{aligned}$$

where  $z = \text{col}(z, \xi_1)$  and  $f_2 = \text{col}(f, \xi_2)$ . Noting that based on (ii), there exists a semiglobally asymptotically stabilizing feedback law  $\xi_2 = \alpha_2(\bar{z}, \epsilon)$  for the  $\bar{z}$ -subsystem. Then, given the control law (7.107), similar to the argument in the proof of (ii), it can be shown that there exists a small neighborhood of the origin  $\tilde{U}_0$  such that the solution of the closed loop system is exponentially stable and the positive definite proper function

$$Q(\bar{z}, \xi_2, \epsilon) = W(\bar{z}) + \frac{1}{2}(\xi_2 - \alpha_2(\bar{z}, \epsilon))^2$$

is a valid Lyapunov function for the whole system with the property that  $\dot{Q} < 0$  over  $\tilde{\Omega} \setminus \tilde{U}_0$  where  $\tilde{\Omega}$  is an appropriate invariant set of the Lyapunov function  $Q$ . This finishes the proof.

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