FIRST-ORDER PERTURBATION ANALYSIS OF SECSI WITH GENERALIZED UNFOLDINGS

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ABSTRACT

Tensor decompositions are regarded as a powerful tool for multi-dimensional signal processing. In this contribution, we focus on the well-known Canonical Polyadic (CP) decomposition and present a first-order perturbation analysis of the **SE**mi-algebraic framework for approximate **C**P decompositions via **SI**multaneous matrix diagonalization with Generalized Unfoldings (SECSI-GU), which is advantageous for tensors of an order higher than three. Numerical results indicate that the analytical relative Mean Square Factor Error (rMSFE) of the estimated factor matrices resulting from each generalized unfolding considered in SECSI-GU matches the empirical rMSFE very well. As SECSI-GU considers all possible partitionings of the tensor modes resulting in a large number of candidate factor matrix estimates, an exhaustive search-based criterion to select the final factor matrix estimates leads to a prohibitive computational complexity. The accurate performance prediction achieved by the first-order perturbation analysis conducted in this paper will significantly facilitate the selection of the final factor matrix estimates in an efficient manner and will therefore contribute to a low-complexity enhancement of SECSI-GU.

Index Terms— Canonical Polyadic decomposition, generalized unfoldings, first-order perturbation analysis

1. INTRODUCTION

The R-way Canonical Polyadic (CP) decomposition, also known as Parallel Factor (PARAFAC) analysis [1] or Canonical Decomposition (CANDECOMP) [2], has found applications in a variety of research fields, including array signal processing, wireless communications, and image processing [3], [4], [5]. To accomplish the challenging task of computing an approximate CP decomposition of observed signals of interest in additive noise, a **SE**mi-algebraic framework for approximate **C**P decompositions based on **SI**multaneous matrix diagonalizations (SECSI) was proposed in [6], [7], [8]. SECSI algebraically rephrases the CP decomposition into a set of less complex Simultaneous Matrix Diagonalization (SMD) problems. Combining generalized unfoldings with the idea of considering all possible generalized unfoldings to obtain multiple candidate CP models as in SECSI leads to SECSI with Generalized Unfoldings (SECSI-GU) [9]. For tensors with $R > 3$ dimensions, SECSI-GU enhances the identifiability and outperforms SECSI in terms of estimation accuracy, and it is very flexible in controlling the complexity-accuracy trade-off [9]. Such semi-algebraic approaches exhibit superiority over the Alternating Least Squares (ALS) procedure [1], [2] which may require a large number of iterations and is sensitive to ill-conditioned data. In addition, the non-iterative nature of the semi-algebraic methods enables a parallelized implementation, which is not possible with iterative methods such as ALS.

Recently, the performance analysis of the truncated Higher-Order SVD (HOSVD) [10] and the approximate CP decomposition via SECSI [11] for 3-D tensors have been carried out. These new advances in perturbation analysis of tensor decompositions further spark the interest in developing a performance analysis framework for SECSI-GU. Taking into account all possible generalized unfoldings, SECSI-GU accordingly forms and solves a large number of SMDs (up to $3^R - 3 \cdot 2^R + 3$ for an R-D tensor [9] and twice this amount if the considerations at the end of Section 3 are taken into account). Several heuristic selection criteria have been proposed to determine which SMDs to solve and how to select the final estimates of the factor matrices [9]. A performance analysis of SECSI-GU will enable us to predict the performance of SECSI-GU with respect to each of the possible generalized unfoldings and consequently to select the generalized unfolding leading to the "best" solutions in terms of, e.g., the minimum relative Mean Square Factor Error (rMSFE). Hence, conducting an analytical performance evaluation of SECSI-GU is not only of theoretical but also of practical interest.

In this paper, we present a first-order perturbation analysis of SECSI-GU. Note that SECSI-GU constructs the matrices for each SMD following the concept of the "Semi-Algebraic Tensor Decomposition" (SALT) algorithm [12], later named DIAG (DIrect AlGorithm for canonical polyadic decomposition) [13], which is essentially different from that of SECSI. The performance analysis of SECSI-GU presented here has thus fundamental differences from that of SECSI in [11]. Moreover, the analytical performance evaluation of SECSI [11] was derived only for 3-D tensors, while the results shown in this contribution are applicable also to tensors with more than three dimensions. On the other hand, DIAG constructs only a single SMD from a single appropriately selected generalized unfolding, whereas all possible generalized unfoldings are considered in SECSI-GU, giving rise to multiple candidates of the factor matrix estimates. Owing to this fact, a performance evaluation of the DIAG algorithm is inherent in the proposed performance analysis framework for SECSI-GU. As the least-squares Khatri-Rao factorization [14] is employed in the final steps of SECSI-GU to obtain the estimates of the factor matrices [9], we are able to utilize our previous results of its performance analysis [15] to get the closed-form expression of the rMSFE. As we show via numerical simulations, it leads to a very accurate prediction for the rMSFE of SECSI-GU.

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Throughout this paper, the r-mode product between an R-way tensor with size I_r along mode $r = 1, 2, \ldots, R$ represented as $\boldsymbol{\mathcal{A}} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_R}$ and a matrix $\boldsymbol{U} \in \mathbb{C}^{P_r \times I_r}$ is written as $\boldsymbol{\mathcal{A}} \times_r \boldsymbol{U}$. It is computed by multiplying all r-mode vectors of $\boldsymbol{\mathcal{A}}$ with $\boldsymbol{U}$, whereas the r-mode vectors of $\boldsymbol{\mathcal{A}}$ are obtained by varying the r-th index from 1 to I_r and keeping all other indices fixed. Aligning all r-mode vectors as the columns of a matrix yields the r-mode unfolding of $\boldsymbol{\mathcal{A}}$ which is denoted by $[\boldsymbol{\mathcal{A}}]_{(r)} \in \mathbb{C}^{I_r \times I_{r+1} \cdot \ldots \cdot I_R \cdot I_1 \cdot \ldots \cdot I_{r-1}}$. In other words, $[\boldsymbol{\mathcal{A}} \times_r \boldsymbol{U}]_{(r)} = \boldsymbol{U} \cdot [\boldsymbol{\mathcal{A}}]_{(r)}$. Here the reverse cyclical ordering of the columns, as proposed in [16], is used for the r-mode unfoldings. The tensor $\boldsymbol{\mathcal{I}}_{R,d}$ is an R-dimensional identity tensor of size $d \times d \times \ldots \times d$, which is equal to one if all R indices are equal and zero otherwise. A $d \times d$ identity matrix, on the other hand, is denoted by $\boldsymbol{I}_d$. Moreover, the Kronecker product between two matrices is expressed by $\boldsymbol{A} \otimes \boldsymbol{B}$ and the Khatri-Rao (column-wise Kronecker) product by $\boldsymbol{A} \diamond \boldsymbol{B}$. The vectorization operation of a matrix is symbolized by $\operatorname{vec}\{\cdot\}$. We use the superscript $^+$ for the Moore-Penrose pseudo inverse of a matrix.

2. CP DECOMPOSITION VIA SECSI-GU

The CP decomposition of an R-way rank-d tensor $\boldsymbol{\mathcal{X}}_0$ is written as

$$\boldsymbol{\mathcal{X}}_0 = \boldsymbol{\mathcal{I}}_{R,d} \times_1 \boldsymbol{F}_1 \times_2 \cdots \times_R \boldsymbol{F}_R \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R}, \qquad (1)$$

where $\boldsymbol{F}_r \in \mathbb{C}^{M_r \times d}$ $(r = 1, 2, \ldots, R)$ represent the factor matrices. Dividing the set of indices $(1, 2, \ldots, R)$ into a P-dimensional subset $\boldsymbol{\alpha}^{(1)} = [\alpha_1, \alpha_2, \ldots, \alpha_P]$ and an $(R-P)$-dimensional subset $\boldsymbol{\alpha}^{(2)} = [\alpha_{P+1}, \alpha_{P+2}, \ldots, \alpha_R]$ with $1 \leq P < R$, SECSI-GU considers generalized unfoldings

$$[\boldsymbol{\mathcal{X}}_0]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)} = (\boldsymbol{F}_{\alpha_1} \diamond \cdots \diamond \boldsymbol{F}_{\alpha_P}) \cdot \left(\boldsymbol{F}_{\alpha_{P+1}} \diamond \cdots \diamond \boldsymbol{F}_{\alpha_R}\right)^{\mathrm{T}}, \qquad (2)$$

where the first P indices are arranged into the rows and the remaining $R-P$ indices into the columns. Assigning the different modes into three non-empty groups yields

$$\begin{aligned} \boldsymbol{F}_{\mathrm{A}} &= \boldsymbol{F}_{\alpha_1} \diamond \cdots \diamond \boldsymbol{F}_{\alpha_t} \in \mathbb{C}^{M_{\mathrm{A}} \times d} \\ \boldsymbol{F}_{\mathrm{B}} &= \boldsymbol{F}_{\alpha_{t+1}} \diamond \cdots \diamond \boldsymbol{F}_{\alpha_P} \in \mathbb{C}^{M_{\mathrm{B}} \times d} \\ \boldsymbol{F}_{\mathrm{C}} &= \boldsymbol{F}_{\alpha_{P+1}} \diamond \cdots \diamond \boldsymbol{F}_{\alpha_R} \in \mathbb{C}^{M_{\mathrm{C}} \times d}, \end{aligned}$$

where $M_{\mathrm{A}} = \prod_{r=1}^{t} M_{\alpha_r}$, $M_{\mathrm{B}} = \prod_{r=t+1}^{P} M_{\alpha_r}$, and $M_{\mathrm{C}} = \prod_{r=P+1}^{R} M_{\alpha_r}$ with $1 \leq t < P < R$. In addition, $\boldsymbol{\alpha}_{\mathrm{A}}$, $\boldsymbol{\alpha}_{\mathrm{B}}$, and $\boldsymbol{\alpha}_{\mathrm{C}}$ contain the indices assigned to each of the three groups, respectively, which appear later in the legend of the figures in Section 4. Consequently, the generalized unfolding in (2) can be written as

$$[\boldsymbol{\mathcal{X}}_0]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)} = (\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}) \cdot \boldsymbol{F}_{\mathrm{C}}^{\mathrm{T}}. \qquad (3)$$

For a certain generalized unfolding of the perturbed version of $\boldsymbol{\mathcal{X}}_0$ given by $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}_0 + \boldsymbol{\mathcal{N}}$, we summarize SECSI-GU as follows:

- Compute the truncated SVD of $[\boldsymbol{\mathcal{X}}]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)} \in \mathbb{C}^{M_{\mathrm{A}} \cdot M_{\mathrm{B}} \times M_{\mathrm{C}}}$ and obtain

$$[\boldsymbol{\mathcal{X}}]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)} \approx \hat{\boldsymbol{U}}^{[\mathrm{s}]} \cdot \hat{\boldsymbol{\Sigma}}^{[\mathrm{s}]} \cdot \hat{\boldsymbol{V}}^{[\mathrm{s}]^{\mathrm{H}}}, \qquad (4)$$

  where $\hat{\boldsymbol{U}}^{[\mathrm{s}]} \in \mathbb{C}^{M_{\mathrm{A}} \cdot M_{\mathrm{B}} \times d}$, $\hat{\boldsymbol{V}}^{[\mathrm{s}]} \in \mathbb{C}^{M_{\mathrm{C}} \times d}$, and $\hat{\boldsymbol{\Sigma}}^{[\mathrm{s}]} \in \mathbb{C}^{d \times d}$. The column space of $\hat{\boldsymbol{U}}^{[\mathrm{s}]}$ is an estimate of the column space of $[\boldsymbol{\mathcal{X}}]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)}$. Also define $\hat{\boldsymbol{Z}}^{[\mathrm{s}]} = \hat{\boldsymbol{\Sigma}}^{[\mathrm{s}]} \cdot \hat{\boldsymbol{V}}^{[\mathrm{s}]^{\mathrm{H}}} \in \mathbb{C}^{d \times M_{\mathrm{C}}}$.

- Partition $\hat{\boldsymbol{U}}^{[\mathrm{s}]} \in \mathbb{C}^{M_{\mathrm{A}} \cdot M_{\mathrm{B}} \times d}$ into M_{A} blocks of size $M_{\mathrm{B}} \times d$ denoted by $\hat{\boldsymbol{U}}_m^{[\mathrm{s}]}$, $m = 1, 2, \ldots, M_{\mathrm{A}}$, such that

$$\hat{\boldsymbol{U}}^{[\mathrm{s}]} = \begin{bmatrix} \hat{\boldsymbol{U}}_1^{[\mathrm{s}]} \\ \vdots \\ \hat{\boldsymbol{U}}_{M_{\mathrm{A}}}^{[\mathrm{s}]} \end{bmatrix}. \qquad (5)$$

- Construct a set of matrices $\hat{\boldsymbol{\Gamma}}_{p,m} = \hat{\boldsymbol{U}}_p^{[\mathrm{s}]^+} \cdot \hat{\boldsymbol{U}}_m^{[\mathrm{s}]} \in \mathbb{C}^{d \times d}$, where $m = 1, 2, \ldots, M_{\mathrm{A}}$, and p is chosen according to

$$p = \underset{n=1,2,\ldots,M_{\mathrm{A}}}{\arg\min} \ \operatorname{cond}\left\{\hat{\boldsymbol{U}}_n^{[\mathrm{s}]}\right\}, \qquad (6)$$

  with $\operatorname{cond}\{\cdot\}$ representing the condition number.

- Compute an (approximate) SMD of $\hat{\boldsymbol{\Gamma}}_{p,m} \in \mathbb{C}^{d \times d}$ $(m = 1, 2, \ldots, M_{\mathrm{A}})$

$$\hat{\boldsymbol{\Gamma}}_{p,m} \approx \hat{\boldsymbol{T}} \cdot \hat{\boldsymbol{D}}_m \cdot \hat{\boldsymbol{T}}^{-1}, \qquad (7)$$

  where $\hat{\boldsymbol{D}}_m$ are diagonal matrices, thereby obtaining $\hat{\boldsymbol{T}} \in \mathbb{C}^{d \times d}$, which approximates the two Khatri-Rao products $\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}$ and $\boldsymbol{F}_{\mathrm{C}}$ as follows:

$$\hat{\boldsymbol{U}}^{[\mathrm{s}]} \cdot \hat{\boldsymbol{T}} \approx \boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}} = \boldsymbol{F}_{\alpha_1} \diamond \cdots \diamond \boldsymbol{F}_{\alpha_P} \qquad (8)$$

$$\hat{\boldsymbol{Z}}^{[\mathrm{s}]^{\mathrm{T}}} \cdot \hat{\boldsymbol{T}}^{-\mathrm{T}} \approx \boldsymbol{F}_{\mathrm{C}} = \boldsymbol{F}_{\alpha_{P+1}} \diamond \cdots \diamond \boldsymbol{F}_{\alpha_R}. \qquad (9)$$

- Perform the least-squares Khatri-Rao factorization of $\hat{\boldsymbol{U}}^{[\mathrm{s}]} \cdot \hat{\boldsymbol{T}}$ and of $\hat{\boldsymbol{Z}}^{[\mathrm{s}]^{\mathrm{T}}} \cdot \hat{\boldsymbol{T}}^{-\mathrm{T}}$, respectively, to obtain estimates of the factor matrices $\hat{\boldsymbol{F}}_r$ $(r = 1, 2, \ldots, R)$.

3. PERFORMANCE ANALYSIS OF SECSI-GU

Let us denote the perturbations of $\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}$ and $\boldsymbol{F}_{\mathrm{C}}$ caused by additive noise as $\Delta(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}})$ and $\Delta \boldsymbol{F}_{\mathrm{C}}$, respectively. In this section, we derive the closed-form expression of $\operatorname{vec}\{\Delta(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}})\}$ and $\operatorname{vec}\{\Delta \boldsymbol{F}_{\mathrm{C}}\}$. They can be regarded as the input of the already established first-order perturbation analysis of the least-squares Khatri-Rao factorization [15] to finally obtain the closed-form expression of the rMSFE of each factor matrix.

With $\Delta \boldsymbol{U}^{[\mathrm{s}]}$ and $\Delta \boldsymbol{T}$ representing the perturbations of $\boldsymbol{U}^{[\mathrm{s}]}$ and $\boldsymbol{T}$, respectively, as in

$$\hat{\boldsymbol{U}}^{[\mathrm{s}]} = \boldsymbol{U}^{[\mathrm{s}]} + \Delta \boldsymbol{U}^{[\mathrm{s}]} \ \text{ and } \ \hat{\boldsymbol{T}} = \boldsymbol{T} + \Delta \boldsymbol{T}, \qquad (10)$$

$\Delta(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}})$ can be expressed as

$$\Delta(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}) = \Delta \boldsymbol{U}^{[\mathrm{s}]} \cdot \boldsymbol{T} + \boldsymbol{U}^{[\mathrm{s}]} \cdot \Delta \boldsymbol{T} + \mathcal{O}(\Delta^2), \qquad (11)$$

where $\mathcal{O}(\Delta^2)$ includes all terms with an order higher than one. The SVD of $[\boldsymbol{\mathcal{X}}_0]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)}$ is given by

$$[\boldsymbol{\mathcal{X}}_0]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)} = \left[\boldsymbol{U}^{[\mathrm{s}]} \ \ \boldsymbol{U}^{[\mathrm{n}]}\right] \cdot \begin{bmatrix} \boldsymbol{\Sigma}^{[\mathrm{s}]} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \cdot \left[\boldsymbol{V}^{[\mathrm{s}]} \ \ \boldsymbol{V}^{[\mathrm{n}]}\right]^{\mathrm{H}},$$

where the columns of $\boldsymbol{U}^{[\mathrm{s}]} \in \mathbb{C}^{M_{\mathrm{A}} \cdot M_{\mathrm{B}} \times d}$, $\boldsymbol{V}^{[\mathrm{s}]} \in \mathbb{C}^{M_{\mathrm{C}} \times d}$, and $\boldsymbol{U}^{[\mathrm{n}]} \in \mathbb{C}^{M_{\mathrm{A}} \cdot M_{\mathrm{B}} \times (M_{\mathrm{A}} \cdot M_{\mathrm{B}} - d)}$ span the column space, row space, and null space of $[\boldsymbol{\mathcal{X}}_0]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)}$, respectively. In addition, $\boldsymbol{\Sigma}^{[\mathrm{s}]} \in \mathbb{C}^{d \times d}$ is a diagonal matrix whose diagonal elements are the d non-zero singular values of $[\boldsymbol{\mathcal{X}}_0]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)}$. Based on the first-order perturbation analysis of the SVD [17], we have

$$\Delta \boldsymbol{U}^{[\mathrm{s}]} = \boldsymbol{\Upsilon}^{[\mathrm{n}]} \cdot [\boldsymbol{\mathcal{N}}]_{\left(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\right)} \cdot \boldsymbol{V}^{[\mathrm{s}]} \cdot \boldsymbol{\Sigma}^{[\mathrm{s}]^{-1}} + \mathcal{O}(\Delta^2), \qquad (12)$$

where $\boldsymbol{\Upsilon}^{[\mathrm{n}]} = \boldsymbol{U}^{[\mathrm{n}]} \cdot \boldsymbol{U}^{[\mathrm{n}]^{\mathrm{H}}} \in \mathbb{C}^{M_{\mathrm{A}} \cdot M_{\mathrm{B}} \times M_{\mathrm{A}} \cdot M_{\mathrm{B}}}$ is the projection matrix into the noise subspace of $[\boldsymbol{\mathcal{X}}_0]_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}$, and $[\boldsymbol{\mathcal{N}}]_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}$ is the generalized unfolding of $\boldsymbol{\mathcal{N}}$. The vectorization of $\Delta \boldsymbol{U}^{[\mathrm{s}]}$ is obtained as

$$\operatorname{vec}\left\{\Delta \boldsymbol{U}^{[\mathrm{s}]}\right\} = \left(\boldsymbol{\Sigma}^{[\mathrm{s}]^{-1}} \cdot \boldsymbol{V}^{[\mathrm{s}]^{\mathrm{T}}} \otimes \boldsymbol{\Upsilon}^{[\mathrm{n}]}\right) \cdot \operatorname{vec}\left\{[\boldsymbol{\mathcal{N}}]_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}\right\} + \mathcal{O}(\Delta^2). \tag{13}$$

Assume that the covariance matrix $\boldsymbol{R}_{\mathrm{nn}}$ of the vectorization of the 1-mode unfolding of the noise tensor $\boldsymbol{\mathcal{N}}$, given by $\boldsymbol{n}_1 = \operatorname{vec}\left\{[\boldsymbol{\mathcal{N}}]_{(1)}\right\}$, is known. We now define a permutation matrix $\boldsymbol{P} \in \{1,0\}^{M \times M}$ that satisfies $\operatorname{vec}\left\{[\boldsymbol{\mathcal{N}}]_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}\right\} = \boldsymbol{P} \cdot \boldsymbol{n}_1$ and consequently write the vectorization of $\Delta \boldsymbol{U}^{[\mathrm{s}]}$ in the form of

$$\operatorname{vec}\left\{\Delta \boldsymbol{U}^{[\mathrm{s}]}\right\} = \boldsymbol{K}_0 \cdot \boldsymbol{n}_1 + \mathcal{O}(\Delta^2), \tag{14}$$

where $\boldsymbol{K}_0 = \left(\boldsymbol{\Sigma}^{[\mathrm{s}]^{-1}} \cdot \boldsymbol{V}^{[\mathrm{s}]^{\mathrm{T}}} \otimes \boldsymbol{\Upsilon}^{[\mathrm{n}]}\right) \cdot \boldsymbol{P} \in \mathbb{C}^{d \cdot M_{\mathrm{A}} \cdot M_{\mathrm{B}} \times M}$.

To compute $\hat{\boldsymbol{T}}$ in SECSI-GU, we employ the JDTM (Joint Diagonalization based on Targeting hyperbolic Matrices) algorithm [18] for the SMD of $\hat{\boldsymbol{\Gamma}}_{p,m} \in \mathbb{C}^{d \times d}$ ($m = 1, 2, \ldots, M_{\mathrm{A}}$). According to the performance analysis of JDTM in [19], we have

$$\operatorname{vec}\{\Delta \boldsymbol{T}\} = -\boldsymbol{A}^{+} \cdot \boldsymbol{B} \cdot \boldsymbol{\gamma} + \mathcal{O}(\Delta^2), \tag{15}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_{M_{\mathrm{A}}} \end{bmatrix}, \quad \boldsymbol{B} = \boldsymbol{I}_{M_{\mathrm{A}}} \otimes \boldsymbol{B}_0, \quad \boldsymbol{\gamma} = \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \vdots \\ \boldsymbol{\gamma}_{M_{\mathrm{A}}} \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{B}_0 &= \boldsymbol{J}_{(d)} \cdot \left(\boldsymbol{T}^{\mathrm{T}} \otimes \boldsymbol{T}^{-1}\right) \in \mathbb{C}^{d^2 \times d^2}, \\ \boldsymbol{A}_m &= \boldsymbol{J}_{(d)} \cdot \left(\boldsymbol{I}_d \otimes \left(\boldsymbol{T}^{-1} \cdot \boldsymbol{\Gamma}_{p,m}\right) - \boldsymbol{D}_m \otimes \boldsymbol{T}^{-1}\right) \in \mathbb{C}^{d^2 \times d^2}, \\ \boldsymbol{\gamma}_m &= \operatorname{vec}\{\Delta \boldsymbol{\Gamma}_{p,m}\} \in \mathbb{C}^{d^2}. \end{aligned} \tag{16}$$

Here a selection matrix is defined as $\boldsymbol{J}_{(d)} \in \{0,1\}^{d^2 \times d^2}$ such that $\operatorname{vec}\{\operatorname{Off}(\boldsymbol{X})\} = \boldsymbol{J}_{(d)} \cdot \operatorname{vec}\{\boldsymbol{X}\}$, where the $\operatorname{Off}(\cdot)$ operator sets the diagonal elements of its input matrix to zeros. Note that $\boldsymbol{\Gamma}_{p,m}$ and $\boldsymbol{D}_m$ are constructed from the noiseless tensor $\boldsymbol{\mathcal{X}}_0$ and therefore can be regarded as the "true" version of $\hat{\boldsymbol{\Gamma}}_{p,m}$ and $\hat{\boldsymbol{D}}_m$, respectively. To derive the perturbation of $\boldsymbol{\Gamma}_{p,m}$ denoted by $\Delta \boldsymbol{\Gamma}_{p,m}$ in (16), let us write $\hat{\boldsymbol{\Gamma}}_{p,m}$ in the following form

$$\boldsymbol{\Gamma}_{p,m} + \Delta \boldsymbol{\Gamma}_{p,m} = \left(\boldsymbol{U}_p^{[\mathrm{s}]} + \Delta \boldsymbol{U}_p^{[\mathrm{s}]}\right)^{+} \cdot \left(\boldsymbol{U}_m^{[\mathrm{s}]} + \Delta \boldsymbol{U}_m^{[\mathrm{s}]}\right). \tag{17}$$

Based on [20], the matrix pseudo inversion in (17) is expressed as

$$\left(\boldsymbol{U}_p^{[\mathrm{s}]} + \Delta \boldsymbol{U}_p^{[\mathrm{s}]}\right)^{+} = \boldsymbol{U}_p^{[\mathrm{s}]^{+}} - \boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \Delta \boldsymbol{U}_p^{[\mathrm{s}]} \cdot \boldsymbol{U}_p^{[\mathrm{s}]^{+}} + \mathcal{O}(\Delta^2). \tag{18}$$

Inserting (18) into (17), we obtain the perturbation $\Delta \boldsymbol{\Gamma}_{p,m}$ as

$$\Delta \boldsymbol{\Gamma}_{p,m} = \boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \Delta \boldsymbol{U}_m^{[\mathrm{s}]} - \boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \Delta \boldsymbol{U}_p^{[\mathrm{s}]} \cdot \boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \boldsymbol{U}_m^{[\mathrm{s}]} + \mathcal{O}(\Delta^2).$$

Define two block selection matrices $\boldsymbol{J}_m$ and $\boldsymbol{J}_p$ such that $\Delta \boldsymbol{U}_m^{[\mathrm{s}]} = \boldsymbol{J}_m \cdot \Delta \boldsymbol{U}^{[\mathrm{s}]}$ and $\Delta \boldsymbol{U}_p^{[\mathrm{s}]} = \boldsymbol{J}_p \cdot \Delta \boldsymbol{U}^{[\mathrm{s}]}$, respectively. Consequently, $\boldsymbol{\gamma}_m = \operatorname{vec}\{\Delta \boldsymbol{\Gamma}_{p,m}\}$ first given in (16) is now written as

$$\begin{aligned} \boldsymbol{\gamma}_m = & \left(\boldsymbol{I}_d \otimes \left(\boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \boldsymbol{J}_m\right)\right) \cdot \operatorname{vec}\left\{\Delta \boldsymbol{U}^{[\mathrm{s}]}\right\} + \mathcal{O}(\Delta^2) \\ & - \left(\left(\boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \boldsymbol{U}_m^{[\mathrm{s}]}\right)^{\mathrm{T}} \otimes \left(\boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \boldsymbol{J}_p\right)\right) \cdot \operatorname{vec}\left\{\Delta \boldsymbol{U}^{[\mathrm{s}]}\right\}. \end{aligned} \tag{19}$$

Substituting $\operatorname{vec}\left\{\Delta \boldsymbol{U}^{[\mathrm{s}]}\right\}$ in (20) by (14) allows us to express $\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}_m$ ($m = 1, 2, \ldots, M_{\mathrm{A}}$) are stacked, as

$$\boldsymbol{\gamma} = \boldsymbol{K}_2 \cdot \boldsymbol{n}_1 + \mathcal{O}(\Delta^2), \tag{20}$$

with $\boldsymbol{K}_2 = \begin{bmatrix} \boldsymbol{K}_1^{(1)^{\mathrm{T}}} & \ldots & \boldsymbol{K}_1^{(M_{\mathrm{A}})^{\mathrm{T}}} \end{bmatrix}^{\mathrm{T}}$ containing the stacking of $\boldsymbol{K}_1^{(m)} \in \mathbb{C}^{d^2 \times M}$ ($m = 1, 2, \ldots, M_{\mathrm{A}}$) defined via

$$\begin{aligned} \boldsymbol{K}_1^{(m)} = & \Big(\boldsymbol{I}_d \otimes \left(\boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \boldsymbol{J}_m\right) \\ & - \left(\boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \boldsymbol{U}_m^{[\mathrm{s}]}\right)^{\mathrm{T}} \otimes \left(\boldsymbol{U}_p^{[\mathrm{s}]^{+}} \cdot \boldsymbol{J}_p\right)\Big) \cdot \boldsymbol{K}_0. \end{aligned} \tag{21}$$

Accordingly, we obtain $\operatorname{vec}\{\Delta \boldsymbol{T}\}$ first given in (15) as

$$\operatorname{vec}\{\Delta \boldsymbol{T}\} = \boldsymbol{K}_3 \cdot \boldsymbol{n}_1 + \mathcal{O}(\Delta^2), \tag{22}$$

where $\boldsymbol{K}_3 = -\boldsymbol{A}^{+} \cdot \boldsymbol{B} \cdot \boldsymbol{K}_2 \in \mathbb{C}^{d^2 \times M}$.

The vectorization of $\Delta\left(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}\right)$ takes the form

$$\begin{aligned} \operatorname{vec}\{\Delta\left(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}\right)\} = & \left(\boldsymbol{T}^{\mathrm{T}} \otimes \boldsymbol{I}_{M_{\mathrm{A}} \cdot M_{\mathrm{B}}}\right) \cdot \operatorname{vec}\left\{\Delta \boldsymbol{U}^{[\mathrm{s}]}\right\} \\ & + \left(\boldsymbol{I}_d \otimes \boldsymbol{U}^{[\mathrm{s}]}\right) \cdot \operatorname{vec}\{\Delta \boldsymbol{T}\} + \mathcal{O}(\Delta^2). \end{aligned} \tag{23}$$

By inserting (14) and (22) into (23), we have

$$\operatorname{vec}\{\Delta\left(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}\right)\} = \boldsymbol{K}_4 \cdot \boldsymbol{n}_1 + \mathcal{O}(\Delta^2), \tag{24}$$

where $\boldsymbol{K}_4 = \left(\boldsymbol{T}^{\mathrm{T}} \otimes \boldsymbol{I}_{M_{\mathrm{A}} \cdot M_{\mathrm{B}}}\right) \cdot \boldsymbol{K}_0 + \left(\boldsymbol{I}_d \otimes \boldsymbol{U}^{[\mathrm{s}]}\right) \cdot \boldsymbol{K}_3$.

In the following, we proceed to derive $\operatorname{vec}\{\Delta \boldsymbol{F}_{\mathrm{C}}\}$. To this end, let us express $\hat{\boldsymbol{F}}_{\mathrm{C}}$ as

$$\boldsymbol{F}_{\mathrm{C}} + \Delta \boldsymbol{F}_{\mathrm{C}} = \left(\boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}} + \Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}}\right) \cdot (\boldsymbol{T} + \Delta \boldsymbol{T})^{-\mathrm{T}}, \tag{25}$$

where $\Delta \boldsymbol{Z}^{[\mathrm{s}]}$ denotes the perturbation of $\boldsymbol{Z}^{[\mathrm{s}]}$ such that $\hat{\boldsymbol{Z}}^{[\mathrm{s}]} = \boldsymbol{Z}^{[\mathrm{s}]} + \Delta \boldsymbol{Z}^{[\mathrm{s}]}$. Owing to the fact that [20]

$$(\boldsymbol{T} + \Delta \boldsymbol{T})^{-1} = \boldsymbol{T}^{-1} - \boldsymbol{T}^{-1} \cdot \Delta \boldsymbol{T} \cdot \boldsymbol{T}^{-1} + \mathcal{O}(\Delta^2), \tag{26}$$

$\Delta \boldsymbol{F}_{\mathrm{C}}$ can be further written as

$$\Delta \boldsymbol{F}_{\mathrm{C}} = \Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}} \cdot \boldsymbol{T}^{-\mathrm{T}} - \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}} \cdot \boldsymbol{T}^{-\mathrm{T}} \cdot \Delta \boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{T}^{-\mathrm{T}} + \mathcal{O}(\Delta^2).$$

Vectorizing $\Delta \boldsymbol{F}_{\mathrm{C}}$ leads to

$$\begin{aligned} \operatorname{vec}\{\Delta \boldsymbol{F}_{\mathrm{C}}\} = & \left(\boldsymbol{T}^{-1} \otimes \boldsymbol{I}_{M_{\mathrm{C}}}\right) \cdot \operatorname{vec}\left\{\Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}}\right\} + \mathcal{O}(\Delta^2) \\ & - \left(\boldsymbol{T}^{-1} \otimes \left(\boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}} \cdot \boldsymbol{T}^{-\mathrm{T}}\right)\right) \cdot \operatorname{vec}\left\{\Delta \boldsymbol{T}^{\mathrm{T}}\right\}. \end{aligned} \tag{27}$$

Defining a permutation matrix $\boldsymbol{P}_{(d,d)} \in \{0,1\}^{d^2 \times d^2}$ satisfying $\operatorname{vec}\left\{\Delta \boldsymbol{T}^{\mathrm{T}}\right\} = \boldsymbol{P}_{(d,d)} \cdot \operatorname{vec}\{\Delta \boldsymbol{T}\}$ enables us to use $\operatorname{vec}\{\Delta \boldsymbol{T}\}$ already obtained in (22). To derive $\operatorname{vec}\left\{\Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}}\right\}$ for the final expression of $\operatorname{vec}\{\Delta \boldsymbol{F}_{\mathrm{C}}\}$ given in (27), we rewrite $\hat{\boldsymbol{Z}}^{[\mathrm{s}]}$ originally defined as $\hat{\boldsymbol{Z}}^{[\mathrm{s}]} = \hat{\boldsymbol{\Sigma}}^{[\mathrm{s}]} \cdot \hat{\boldsymbol{V}}^{[\mathrm{s}]^{\mathrm{H}}}$ into

$$\hat{\boldsymbol{Z}}^{[\mathrm{s}]} = \boldsymbol{Z}^{[\mathrm{s}]} + \Delta \boldsymbol{Z}^{[\mathrm{s}]} = \hat{\boldsymbol{U}}^{[\mathrm{s}]^{\mathrm{H}}} \cdot [\boldsymbol{\mathcal{X}}]_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}. \tag{28}$$

The resulting expression of $\Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}}$

$$\Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}} = [\boldsymbol{\mathcal{X}}_0]_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}^{\mathrm{T}} \cdot \Delta \boldsymbol{U}^{[\mathrm{s}]^{*}} + [\boldsymbol{\mathcal{N}}]_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}^{\mathrm{T}} \cdot \boldsymbol{U}^{[\mathrm{s}]^{*}} + \mathcal{O}(\Delta^2)$$

is further simplified into

$$\Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}} = [\boldsymbol{\mathcal{N}}]^{\mathrm{T}}_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})} \cdot \boldsymbol{U}^{[\mathrm{s}]^*} + \mathcal{O}(\Delta^2)$$

due to the observation that the first term $[\boldsymbol{\mathcal{X}}_0]^{\mathrm{T}}_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})} \cdot \Delta \boldsymbol{U}^{[\mathrm{s}]^*}$ is a zero matrix according to the definition of $\Delta \boldsymbol{U}^{[\mathrm{s}]}$ in (12). Taking the vectorization of $\Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}}$ gives

$$\operatorname{vec}\left\{\Delta \boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}}\right\} = \boldsymbol{K}_5 \cdot \boldsymbol{n}_1 + \mathcal{O}(\Delta^2), \tag{29}$$

where $\boldsymbol{K}_5 = \left(\boldsymbol{U}^{[\mathrm{s}]^{\mathrm{H}}} \otimes \boldsymbol{I}_{M_{\mathrm{C}}}\right) \cdot \widetilde{\boldsymbol{P}} \in \mathbb{C}^{d \cdot M_{\mathrm{C}} \times M}$ and $\widetilde{\boldsymbol{P}} \in \{1,0\}^{M \times M}$ is a permutation matrix leading to $\operatorname{vec}\left\{[\boldsymbol{\mathcal{N}}]^{\mathrm{T}}_{(\boldsymbol{\alpha}^{(1)},\boldsymbol{\alpha}^{(2)})}\right\} = \widetilde{\boldsymbol{P}} \cdot \boldsymbol{n}_1$.

Finally, $\operatorname{vec}\{\Delta \boldsymbol{F}_{\mathrm{C}}\}$ takes the form

$$\operatorname{vec}\{\Delta \boldsymbol{F}_{\mathrm{C}}\} = \boldsymbol{K}_6 \cdot \boldsymbol{n}_1 + \mathcal{O}(\Delta^2), \tag{30}$$

where

$$\begin{aligned}\boldsymbol{K}_6 = &\left(\boldsymbol{T}^{-1} \otimes \boldsymbol{I}_{M_{\mathrm{C}}}\right) \cdot \boldsymbol{K}_5 \\ &- \left(\boldsymbol{T}^{-1} \otimes \left(\boldsymbol{Z}^{[\mathrm{s}]^{\mathrm{T}}} \cdot \boldsymbol{T}^{-\mathrm{T}}\right)\right) \cdot \boldsymbol{P}_{(d,d)} \cdot \boldsymbol{K}_3 \in \mathbb{C}^{d \cdot M_{\mathrm{C}} \times M}.\end{aligned}$$

Taking $\operatorname{vec}\{\Delta\left(\boldsymbol{F}_{\mathrm{A}} \diamond \boldsymbol{F}_{\mathrm{B}}\right)\}$ and $\operatorname{vec}\{\Delta \boldsymbol{F}_{\mathrm{C}}\}$ as the input of the performance analysis of the least-squares Khatri-Rao factorization [15], we are able to obtain the first-order perturbation of $\boldsymbol{F}_r$ denoted by $\Delta \boldsymbol{F}_r$. Its vectorization $\operatorname{vec}\{\Delta \boldsymbol{F}_r\}$ can be expressed as $\operatorname{vec}\{\Delta \boldsymbol{F}_r\} = \boldsymbol{K} \cdot \boldsymbol{n}_1 + \mathcal{O}\left(\Delta^2\right)$ similar to the vectorization of the perturbation terms derived above. Subsequently, we get the closed-form expression of the rMSFE for each factor matrix in terms of the second-order moments of the noise, i.e., the covariance matrix $\boldsymbol{R}_{\mathrm{nn}}$ with respect to $\boldsymbol{n}_1$. For detailed derivations and the resulting explicit expression of $\boldsymbol{K}$ as well as the rMSFE, the reader is referred to [15].

It is worth noting that $\hat{\boldsymbol{T}}$ can be alternatively obtained via the joint diagonalization of another set matrices given by $\boldsymbol{\Omega}_{p,m} = \hat{\boldsymbol{Z}}^{[\mathrm{s}]}_m \cdot \hat{\boldsymbol{Z}}^{[\mathrm{s}]^+}_p \in \mathbb{C}^{d \times d}$, where $\hat{\boldsymbol{Z}}^{[\mathrm{s}]}_m \in \mathbb{C}^{d \times M_{\mathrm{C}}^{(2)}}$ $(m = 1, 2, \ldots, M_{\mathrm{C}}^{(1)})$ are obtained by partitioning $\hat{\boldsymbol{Z}}^{[\mathrm{s}]}$ as

$$\hat{\boldsymbol{Z}}^{[\mathrm{s}]} = \begin{bmatrix} \hat{\boldsymbol{Z}}^{[\mathrm{s}]}_1 & \cdots & \hat{\boldsymbol{Z}}^{[\mathrm{s}]}_{M_{\mathrm{C}}^{(1)}} \end{bmatrix} \tag{31}$$

with $M_{\mathrm{C}}^{(1)} = \prod_{r=P+1}^{q} M_{\alpha_r}$ and $M_{\mathrm{C}}^{(2)} = \prod_{r=q+1}^{R} M_{\alpha_r}$, i.e., $M_{\mathrm{C}} = M_{\mathrm{C}}^{(1)} \cdot M_{\mathrm{C}}^{(2)}$. The index p corresponds to the block $\hat{\boldsymbol{Z}}^{[\mathrm{s}]}_p$ that has the minimum condition number. Due to space limitations, the first-order perturbation analysis of SECSI-GU where the SMDs are constructed as mentioned above is not included in this paper.

4. SIMULATION RESULTS

To demonstrate the validity of our first-order performance analysis of SECSI-GU, we present comparisons between the analytical total rMSFE and empirical ones obtained via Monte Carlo simulations. The factor matrices $\boldsymbol{F}_r$ $(r = 1, 2, \ldots, R)$ contain elements drawn independently from a zero-mean Gaussian distribution with unit variance, while elements of the noise tensor $\boldsymbol{\mathcal{N}}$ were drawn similarly with variance σ_n^2. Accordingly, we define $\mathrm{SNR} = 1/\sigma_n^2$. Fig. 1 depicts the results for a real-valued case where $R = 4$, $M_1 = 4$, $M_2 = 7$, $M_3 = 6$, $M_4 = 4$, and $d = 4$. A complex-valued case is considered in Fig. 2, where $R = 5$, $M_r = 4$ $(r = 1, \ldots, R)$, and $d = 4$. In the $R = 4$ and $R = 5$ scenarios, the total number

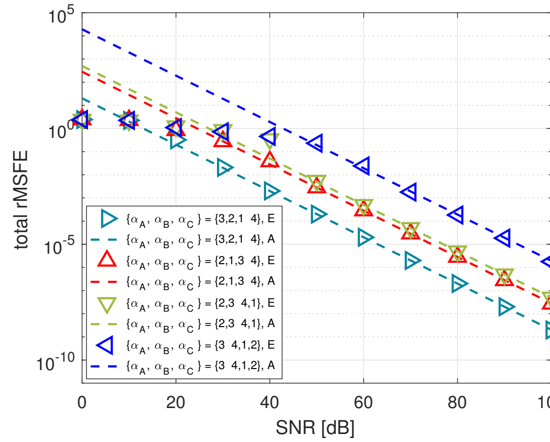

Fig. 1. Empirical and analytical total rMSFE versus SNR for a real-valued scenario where $R = 4$, $M_1 = 4$, $M_2 = 7$, $M_3 = 6$, $M_4 = 4$, and $d = 4$; "E" in the legend is short for "Empirical", whereas "A" for "Analytical"

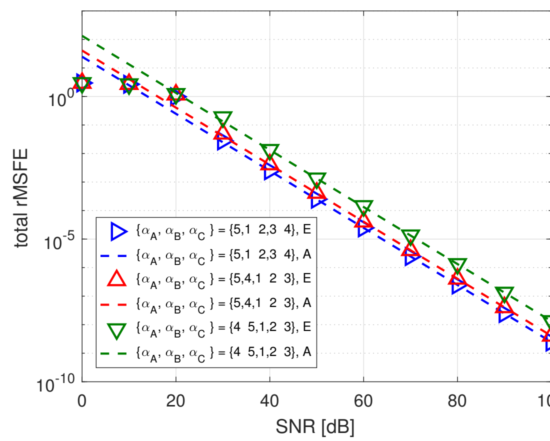

Fig. 2. Empirical and analytical total rMSFE versus SNR for a complex-valued scenario where $R = 5$, $M_r = 4$ $(r = 1, \ldots, R)$, and $d = 4$; "E" in the legend is short for "Empirical", whereas "A" for "Analytical"

of generalized unfoldings to be considered reaches 36 and 150, respectively. For clarity of the figures, only the results with respect to a few generalized unfoldings are shown as representative examples. In both cases, a good match between the analytical and empirical results is evident, especially in the higher SNR regime.

5. CONCLUSION

We have presented a first-order perturbation analysis of SECSI-GU for the approximate CP decomposition of noise-corrupted tensors of an order higher than three. To obtain the closed-form expression of the rMSFE for each factor matrix, we derived the first-order perturbations of all intermediate outcomes at every step of the SECSI-GU framework, ranging from the formulation of target matrices for the SMDs, to the estimation of the Khatri-Rao products of the factor matrices. Simulation results show that our performance analysis of SECSI-GU is able to predict the rMSFEs for each possible generalized unfolding very accurately, especially in the higher SNR regime. For future work, an efficient selection scheme for SECSI-GU will be designed, which determines the generalized unfolding to be considered based on the performance prediction provided by this first-order perturbation analysis. It will avoid computing all possible factor matrix estimates corresponding to all possible generalized unfoldings.

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