

# Perturbation Analysis of Joint Eigenvalue Decomposition Algorithms

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**Abstract**—Joint EigenValue Decomposition (JEVD) algorithms are widely used in many application scenarios. These algorithms can be divided into different categories based on the cost function that needs to be minimized. Most of the frequently used algorithms in the literature use indirect least square (LS) criteria as a cost function. In this work, we perform a first order perturbation analysis for the JEVD algorithms based on the indirect LS criterion. We also present closed-form expressions for the eigenvector and eigenvalue matrices. The obtained expressions are asymptotic in the signal-to-noise ratio (SNR). Additionally, we use these results to obtain a statistical analysis, where we only assume that the noise has finite second order moments. The simulation results show that the proposed analytical expressions match well to the empirical results of JEVD algorithms which are based on the LS cost function.

**Index Terms**—Perturbation analysis, joint eigenvalue decomposition.

## I. INTRODUCTION

The problem of Joint EigenValue Decomposition (JEVD) of a set of jointly diagonalizable matrices is often encountered in many different applications such as independent component analysis [1], the Canonical Polyadic (CP) decomposition [2], [3], and in achieving automatic pairing in multidimensional harmonic retrieval problems [4]. A perturbation analysis that describes the reconstruction error is of major importance when analyzing the performance of JEVD (or JEVD based) algorithms. Many JEVD algorithms such as Sh-Rt [5], JDTM [6], coupled JEVD [7], JET-O and JET-U [8] have been investigated in the literature and have been compared in [7], [8]. Moreover, a perturbation analysis for the classical eigenvalue decomposition has already been performed in [9], but to the best of our knowledge, an analytical perturbation analysis for the aforementioned JEVD algorithms is not present in the literature. In this work we provide a first order perturbation analysis of the above mentioned algorithms. The results show that the analytical expressions match the performance of empirical simulations in the high signal-to-noise ratio (SNR) regime.

Computing the JEVD of a given set of  $K$  jointly diagonalizable matrices  $\mathbf{S}_k \in \mathbb{C}^{M \times M}$ ,  $\forall k = 1, 2, \dots, K$  consists of finding the matrix  $\mathbf{T}$  such that

$$\mathbf{S}_k = \mathbf{T} \cdot \mathbf{D}_k \cdot \mathbf{T}^{-1}, \quad \forall k = 1, 2, \dots, K, \quad (1)$$

where  $\mathbf{T} \in \mathbb{C}^{M \times M}$  is an invertible matrix and  $\mathbf{D}_k \in \mathbb{C}^{M \times M}$  are diagonal matrices for  $k = 1, 2, \dots, K$ . This problem should not be confused with the classical problem of Joint Diagonalization by Congruence (JDC), for which  $\mathbf{T}^{-1}$ , in (1), is replaced by  $\mathbf{T}^H$ . Furthermore, in the presence of noise, let  $\hat{\mathbf{S}}_k = \mathbf{S}_k + \Delta \mathbf{S}_k \in \mathbb{C}^{M \times M}$  be a noisy observation of  $\mathbf{S}_k$ , where  $\Delta \mathbf{S}_k \in \mathbb{C}^{M \times M}$  can be modeled as a random perturbation matrix. Note that the set of noisy observations is not fully diagonalizable, and therefore, a JEVD algorithm obtains estimates  $\hat{\mathbf{T}} = \mathbf{T} + \Delta \mathbf{T} \in \mathbb{C}^{M \times M}$  and  $\hat{\mathbf{D}}_k = \mathbf{D}_k + \Delta \mathbf{D}_k \in \mathbb{C}^{M \times M}$  (with  $\Delta \mathbf{D}_k$  being a diagonal perturbation matrix) that approximately, but not fully, diagonalize the set of matrices  $\hat{\mathbf{S}}_k$ ,  $\forall k = 1, 2, \dots, K$  as

$$\hat{\mathbf{S}}_k \approx \hat{\mathbf{T}} \cdot \hat{\mathbf{D}}_k \cdot \hat{\mathbf{T}}^{-1}, \quad \forall k = 1, 2, \dots, K.$$

Most of these JEVD algorithms are based on Jacobi-like updates (sweeping procedure) and look for a factorized form of the updating matrix. First, an algorithm based on the polar decomposition, referred as Sh-Rt, was introduced in [5]. The same factorization is also at the heart of the JUST and the JDTM algorithms [6] while JET-U and JET-O resort to the LU factorization [8]. Therefore, the algorithms such as JDTM, Sh-Rt, and coupled JEVD aim to find the  $\hat{\mathbf{T}}$  that minimizes the indirect LS cost function

$$J = \sum_{k=1}^K \left\| \text{Off} \left( \hat{\mathbf{T}}^{-1} \cdot \hat{\mathbf{S}}_k \cdot \hat{\mathbf{T}} \right) \right\|_{\text{F}}^2, \quad (2)$$

where  $\|\cdot\|_{\text{F}}$  denotes the Frobenious norm of a matrix, and the  $\text{Off}(\cdot)$  operator is defined as  $\text{Off}(\mathbf{X}) = \mathbf{X} - \text{diag}(\text{diag}(\mathbf{X}))$ , with the  $\text{diag}(\cdot)$  operator defined as in Matlab. This implies that, ideally, these algorithms estimate  $\mathbf{T}$  iteratively by solving

$$\hat{\mathbf{T}} = \underset{\mathbf{T}}{\text{argmin}} \left( \sum_{k=1}^K \left\| \text{Off} \left( \hat{\mathbf{T}}^{-1} \cdot \hat{\mathbf{S}}_k \cdot \hat{\mathbf{T}} \right) \right\|_{\text{F}}^2 \right), \quad (3)$$

and the estimates  $\hat{\mathbf{D}}_k$  are computed as

$$\hat{\mathbf{D}}_k = \text{Ddiag} \left( \hat{\mathbf{T}}^{-1} \cdot \hat{\mathbf{S}}_k \cdot \hat{\mathbf{T}} \right), \quad \forall k = 1, 2, \dots, K, \quad (4)$$

where the  $\text{Ddiag}(\cdot)$  operator is defined as  $\text{Ddiag}(\mathbf{X}) = \text{diag}(\text{diag}(\mathbf{X}))$ . Therefore, we refer to these algorithms which are based on the indirect LS cost function as indirect LS. In this work, we focus on the perturbation analysis for these indirect LS algorithms. Since other algorithms such as JET-O and JET-U [8] are not based on the indirect LS cost function, a separate perturbation analysis would be required for these algorithms. However, we also compare our results with these algorithms.

## II. PERTURBATION ANALYSIS FOR INDIRECT LEAST SQUARES ALGORITHMS

Let  $\Delta \mathbf{T}$  be the perturbation present in  $\hat{\mathbf{T}}$ . Therefore, we can write  $\hat{\mathbf{T}} = \mathbf{T} + \Delta \mathbf{T}$ . With this setup we can expand the term  $\hat{\mathbf{T}}^{-1} \cdot \hat{\mathbf{S}}_k \cdot \hat{\mathbf{T}}$ , inside equation (2), to

$$\begin{aligned} \hat{\mathbf{T}}^{-1} \cdot \hat{\mathbf{S}}_k \cdot \hat{\mathbf{T}} &= (\mathbf{T} + \Delta \mathbf{T})^{-1} \cdot (\mathbf{S}_k + \Delta \mathbf{S}_k) \cdot (\mathbf{T} + \Delta \mathbf{T}) \\ &= (\mathbf{T}^{-1} - \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{T}^{-1}) \cdot (\mathbf{S}_k + \Delta \mathbf{S}_k) \cdot (\mathbf{T} + \Delta \mathbf{T}) + \mathcal{O}(\Delta^2) \\ &= \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \mathbf{T} + \mathbf{T}^{-1} \cdot \Delta \mathbf{S}_k \cdot \mathbf{T} \\ &\quad + \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \Delta \mathbf{T} - \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \mathbf{T} + \mathcal{O}(\Delta^2) \\ &= \mathbf{D}_k + \mathbf{T}^{-1} \cdot \Delta \mathbf{S}_k \cdot \mathbf{T} + \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \Delta \mathbf{T} - \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{D}_k + \mathcal{O}(\Delta^2). \end{aligned} \quad (5)$$

Note that, we use the assumption that all the error terms (i.e.  $\Delta \mathbf{T}$ ,  $\Delta \mathbf{S}_k$ ,  $\Delta \mathbf{D}_k$ ) are small enough, such that the second order terms (and higher) inside  $\mathcal{O}(\Delta^2)$  are negligible. Furthermore, since  $\text{Off}(\mathbf{D}_k) = 0$ , we use equation (5) to compute

$$\begin{aligned} \text{Off} \left( \hat{\mathbf{T}}^{-1} \cdot \hat{\mathbf{S}}_k \cdot \hat{\mathbf{T}} \right) &= \text{Off} \left( \mathbf{T}^{-1} \cdot \Delta \mathbf{S}_k \cdot \mathbf{T} \right) + \text{Off} \left( \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \Delta \mathbf{T} \right) \\ &\quad - \text{Off} \left( \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{D}_k \right) + \mathcal{O}(\Delta^2). \end{aligned} \quad (6)$$

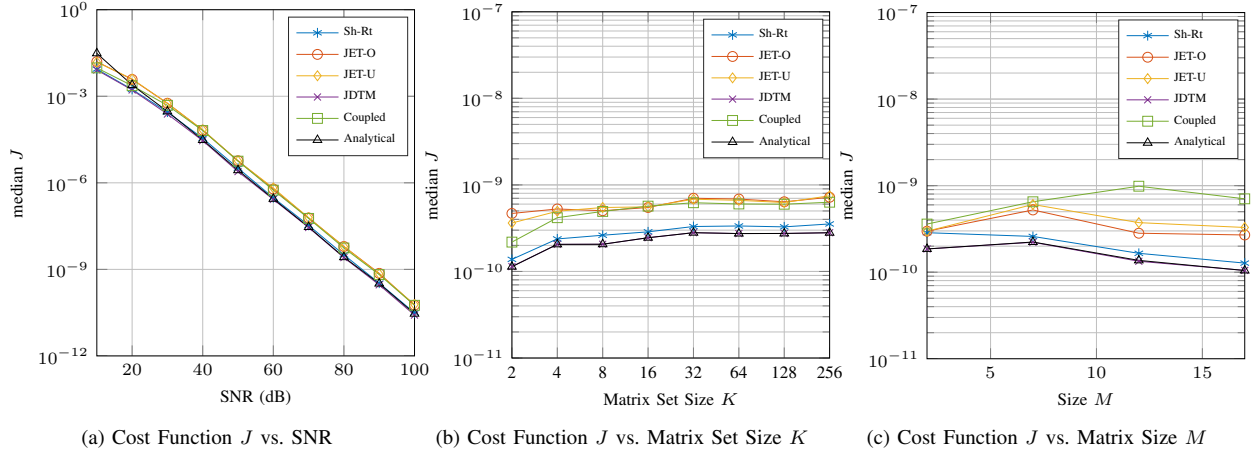


Fig. 1: (1a): Median  $J$  in equation (2) vs. SNR, for  $M = 4$  and  $K = 20$ . (1b): Median  $J$  in equation (2) vs. matrix set size  $K$ , for  $M = 4$  and SNR = 90 dB. (1c): Median  $J$  in equation (2) vs. matrix size  $M$ , for  $K = 20$  and SNR = 90 dB.

Now we can vectorise the result from equation (6) to find

$$\begin{aligned}
 & \text{vec} \left\{ \left( \text{Off} \left( \hat{T}^{-1} \cdot \hat{S}_k \cdot \hat{T} \right) \right) \right\} = \\
 & = \mathbf{Q} \cdot \text{vec} \left\{ \mathbf{T}^{-1} \cdot \Delta \mathbf{S}_k \cdot \mathbf{T} \right\} + \mathbf{Q} \cdot \text{vec} \left\{ \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \Delta \mathbf{T} \right\} \\
 & \quad - \mathbf{Q} \cdot \text{vec} \left\{ \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{D}_k \right\} + \mathcal{O}(\Delta^2) \\
 & = \mathbf{Q} \cdot \left( \mathbf{T}^T \otimes \mathbf{T}^{-1} \right) \cdot \text{vec} \left\{ \Delta \mathbf{S}_k \right\} + \mathbf{Q} \cdot \left( \mathbf{I} \otimes \mathbf{T}^{-1} \cdot \mathbf{S}_k \right) \cdot \text{vec} \left\{ \Delta \mathbf{T} \right\} \\
 & \quad - \mathbf{Q} \cdot \left( \mathbf{D}_k \otimes \mathbf{T}^{-1} \right) \cdot \text{vec} \left\{ \Delta \mathbf{T} \right\} + \mathcal{O}(\Delta^2), \quad (7)
 \end{aligned}$$

where  $\text{vec} \{ \cdot \}$  is the vectorization operator,  $\otimes$  denotes the Kronecker product, and  $\mathbf{Q} \in \{0, 1\}^{M^2 \times M^2}$  is a selection matrix that satisfies the relation  $\text{vec} \{ (\text{Off}(\mathbf{X})) \} = \mathbf{Q} \cdot \text{vec} \{ \mathbf{X} \}$  and can be constructed as  $\mathbf{Q} = \mathbf{I}_{M^2} - \text{diag}(\text{vec} \{ \mathbf{I}_M \})$ .

Let us define  $\mathbf{B}_0 \in \mathbb{C}^{M^2 \times M^2}$ ,  $\mathbf{n}_k \in \mathbb{C}^{M^2 \times 1}$ ,  $\mathbf{A}_k \in \mathbb{C}^{M^2 \times M^2}$ , and  $\mathbf{w} \in \mathbb{C}^{M^2 \times 1}$  as

$$\mathbf{B}_0 = \mathbf{Q} \cdot \left( \mathbf{T}^T \otimes \mathbf{T}^{-1} \right) \quad (8)$$

$$\mathbf{n}_k = \text{vec} \{ \Delta \mathbf{S}_k \} \quad (9)$$

$$\mathbf{A}_k = \mathbf{Q} \cdot \left[ \left( \mathbf{I}_M \otimes \mathbf{T}^{-1} \cdot \mathbf{S}_k \right) - \left( \mathbf{D}_k \otimes \mathbf{T}^{-1} \right) \right] \quad (10)$$

$$\mathbf{w} = \text{vec} \{ \Delta \mathbf{T} \}. \quad (11)$$

This notation allows us to reformulate equation (7) as

$$\text{vec} \left\{ \text{Off} \left( \hat{T}^{-1} \cdot \hat{S}_k \cdot \hat{T} \right) \right\} = \mathbf{B}_0 \cdot \mathbf{n}_k + \mathbf{A}_k \cdot \mathbf{w} + \mathcal{O}(\Delta^2). \quad (12)$$

Now, by substituting the values of equation (12) in equation (2) and neglecting the terms that contain  $\mathcal{O}(\Delta^2)$ , we approximate the indirect LS cost function to

$$\begin{aligned}
 J &= \sum_{k=1}^K \left\| \text{Off} \left( \hat{T}^{-1} \cdot \hat{S}_k \cdot \hat{T} \right) \right\|_F^2 \approx \sum_{k=1}^K \left\| \mathbf{B}_0 \cdot \mathbf{n}_k + \mathbf{A}_k \cdot \mathbf{w} \right\|^2 \\
 &= \left\| \mathbf{B} \cdot \mathbf{n} + \mathbf{A} \cdot \mathbf{w} \right\|^2, \quad (13)
 \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_K \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_0 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \vdots \\ \mathbf{n}_K \end{bmatrix}. \quad (14)$$

Since we have approximated equation (2) by a conventional LS problem in equation (13), the  $\mathbf{w} = \mathbf{w}_{\text{opt}}$  that approximately minimizes  $J$  is given by

$$\mathbf{w}_{\text{opt}} = -\mathbf{A}^+ \cdot \mathbf{B} \cdot \mathbf{n}, \quad (15)$$

where the operator  $^+$  denotes the Moore-Penrose pseudo-inverse of a matrix. Moreover, let  $\mathbf{R}_k$  be defined as  $\mathbf{R}_k \triangleq (\mathbf{I}_M \otimes \mathbf{D}_k \cdot \mathbf{T}^{-1}) -$

$(\mathbf{D}_k \otimes \mathbf{T}^{-1})$ . Note that  $\mathbf{R}_k$  is a block-diagonal matrix with  $M$  blocks, each of size  $M \times M$ , where, due to the diagonal structure of  $\mathbf{D}_k$ , the  $m$ -th row in the  $m$ -th block is zero (for all  $m = 1, 2, \dots, M$ ). Therefore  $\mathbf{R}_k$  has  $M$  all-zero rows at  $1, M+2, 2 \cdot M+3, \dots, M^2$ . Consequently,  $\mathbf{A} \in \mathbb{C}^{M^2 \cdot K \times M^2}$  is rank-deficient, having a column-rank of  $M \cdot (M-1)$  rather than  $M^2$ . This is due to the fact that each  $\mathbf{A}_k = \mathbf{Q} \cdot \mathbf{R}_k$  has the same  $M$ -dimensional null-space (which depends on  $\mathbf{T}$ ), so that the concatenation of these matrices does not improve the rank of  $\mathbf{A}$ , which shares the same null-space as all  $\mathbf{A}_k$ 's. Consequently,  $\mathbf{w}_{\text{opt}}$  in equation (15) is not a unique minimizer of the LS criterion, since any vector  $\mathbf{v}$  in the null space of  $\mathbf{A}$  can be added to  $\mathbf{w}_{\text{opt}}$  in forming a new  $\mathbf{w} = \mathbf{w}_{\text{opt}} + \mathbf{v}$  with the same resulting off-diagonal terms  $\hat{\mathbf{B}} \cdot \mathbf{n} + \hat{\mathbf{A}} \cdot \mathbf{w}$ . Nevertheless, we arbitrarily select  $\mathbf{w}_{\text{opt}}$  of (15) as our solution, which is also known to be the minimum-norm solution (i.e., the vector  $\mathbf{w}$  with the minimum 2-norm among all vectors minimizing the LS criterion), thereby reflecting the minimum possible perturbation of  $\mathbf{T}$  (under the high SNR assumption). Furthermore, we can analytically approximate the  $\Delta \mathbf{T}$  that minimizes the indirect LS cost function  $J$  by solving equation (3), with the value of  $\mathbf{w}_{\text{opt}}$  obtained in equation (15), to obtain

$$\Delta \mathbf{T} \approx \text{unvec}_{M \times M} \{ \mathbf{w}_{\text{opt}} \}, \quad (16)$$

where  $\text{unvec}_{M \times M} \{ \cdot \}$  denotes the inverse of the  $\text{vec} \{ \cdot \}$  operator<sup>1</sup>. Moreover, substituting equation (5) into (4) leads to

$$\begin{aligned}
 \mathbf{D}_k + \Delta \mathbf{D}_k &= \text{Ddiag} \left( \hat{T}^{-1} \cdot \hat{S}_k \cdot \hat{T} \right) \approx \\
 &\approx \mathbf{D}_k + \text{Ddiag} \left( \mathbf{T}^{-1} \cdot \Delta \mathbf{S}_k \cdot \mathbf{T} \right) \\
 &\quad + \text{Ddiag} \left( \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \Delta \mathbf{T} - \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{D}_k \right), \quad (17)
 \end{aligned}$$

We now rewrite the last term of equation (17) as

$$\begin{aligned}
 & \text{Ddiag} \left( \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \Delta \mathbf{T} - \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{D}_k \right) = \\
 &= \text{unvec}_{M \times M} \left\{ \mathbf{P} \cdot \text{vec} \left( \mathbf{T}^{-1} \cdot \mathbf{S}_k \cdot \Delta \mathbf{T} - \mathbf{T}^{-1} \cdot \Delta \mathbf{T} \cdot \mathbf{D}_k \right) \right\} \\
 &= \text{unvec}_{M \times M} \left\{ \mathbf{P} \cdot \left[ \left( \mathbf{I}_M \otimes \mathbf{T}^{-1} \cdot \mathbf{S}_k \right) - \left( \mathbf{D}_k \otimes \mathbf{T}^{-1} \right) \right] \cdot \text{vec}(\Delta \mathbf{T}) \right\} \\
 &= \text{unvec}_{M \times M} \left\{ \mathbf{P} \cdot \left[ \left( \mathbf{I}_M \otimes \mathbf{D}_k \cdot \mathbf{T}^{-1} \right) - \left( \mathbf{D}_k \otimes \mathbf{T}^{-1} \right) \right] \cdot \text{vec}(\Delta \mathbf{T}) \right\} \\
 &= \text{unvec}_{M \times M} \left\{ \mathbf{P} \cdot \mathbf{R}_k \cdot \text{vec}(\Delta \mathbf{T}) \right\}, \quad (18)
 \end{aligned}$$

where  $\mathbf{P}$  is a selection matrix defined as  $\mathbf{P} = \text{diag}(\text{vec} \{ \mathbf{I}_M \})$ . Furthermore, the structure present in  $\mathbf{R}_k$  as discussed above ensures that  $\mathbf{P} \cdot \mathbf{R}_k$  is always equal to zero. This leads to equation (18) being equal to zero and, therefore, equation (17) leads to

$$\Delta \mathbf{D}_k \approx \text{Ddiag} \left( \mathbf{T}^{-1} \cdot \Delta \mathbf{S}_k \cdot \mathbf{T} \right). \quad (19)$$

<sup>1</sup>If  $\mathbf{x} = \text{vec} \{ \mathbf{X} \}$ , then  $\mathbf{X} = \text{unvec}_{M \times M} \{ \mathbf{x} \} \in \mathbb{C}^{M \times M}$ .

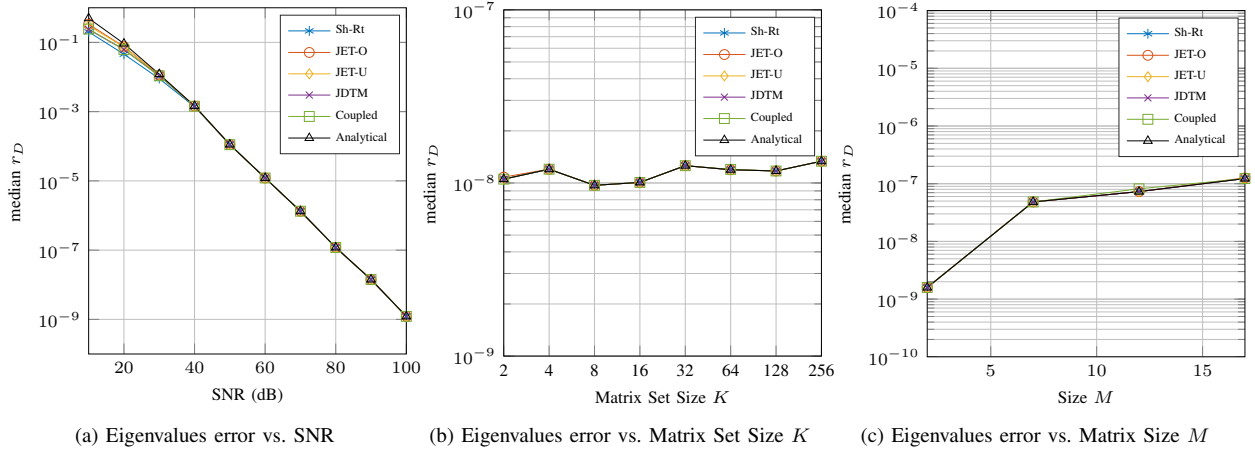


Fig. 2: (2a): Median  $r_D$  in equation (25) vs. SNR, for  $M = 4$  and  $K = 20$ . (2b): Median  $r_D$  in equation (25) vs. matrix set size  $K$ , for  $M = 4$  and SNR = 90 dB. (2c): Median  $r_D$  in equation (25) vs. matrix size  $M$ , for  $K = 20$  and SNR = 90 dB.

### III. PERTURBATION ANALYSIS USING NOISE STATISTICS

#### A. General Expressions

Let  $\mathbf{R}_{nn} \in \mathbb{C}^{K \cdot M^2 \times K \cdot M^2}$  and  $\mathbf{R}_{nn}^{(k)} \in \mathbb{C}^{M^2 \times M^2}$  be the correlation matrices of  $\mathbf{n}$  and  $\mathbf{n}_k$  (in Eqs. (14) and (9)), given by  $\mathbf{R}_{nn} \triangleq \mathbb{E}\{\mathbf{n} \cdot \mathbf{n}^H\}$  and  $\mathbf{R}_{nn}^{(k)} \triangleq \mathbb{E}\{\mathbf{n}_k \cdot \mathbf{n}_k^H\}$ , where  $\mathbb{E}\{\cdot\}$  denotes the expected value operator. Using this definition, and equations (15) (16), we find

$$\begin{aligned} \mathbb{E}\{\|\Delta \mathbf{T}\|_F^2\} &= \mathbb{E}\{\mathbf{w}_{\text{opt}}^H \cdot \mathbf{w}_{\text{opt}}\} = \mathbb{E}\{\text{Tr}(\mathbf{w}_{\text{opt}} \cdot \mathbf{w}_{\text{opt}}^H)\} \\ &= \mathbb{E}\{\text{Tr}(\mathbf{A}^+ \cdot \mathbf{B} \cdot \mathbf{n} \cdot \mathbf{n}^H \cdot \mathbf{B}^H \cdot \mathbf{A}^{+H})\} \\ &= \text{Tr}(\mathbf{A}^+ \cdot \mathbf{B} \cdot \mathbf{R}_{nn} \cdot \mathbf{B}^H \cdot \mathbf{A}^{+H}), \end{aligned} \quad (20)$$

where  $\text{Tr}(\cdot)$  denotes the trace operator. In the same way, we derive a closed-form expression for  $\mathbb{E}\{\|\Delta \mathbf{D}_k\|_F^2\}$  by using the vectorization operation on  $\Delta \mathbf{D}_k$  in equation (19), leading to  $\text{vec}\{\Delta \mathbf{D}_k\} = \mathbf{P} \cdot (\mathbf{T}^T \otimes \mathbf{T}^{-1}) \cdot \mathbf{n}_k = \mathbf{C}_k \cdot \mathbf{n}_k$ , where  $\mathbf{C}_k \triangleq \mathbf{P} \cdot (\mathbf{T}^T \otimes \mathbf{T}^{-1})$ . Therefore, we get

$$\begin{aligned} \mathbb{E}\{\|\Delta \mathbf{D}_k\|_F^2\} &= \mathbb{E}\{\text{vec}(\Delta \mathbf{D}_k)^H \cdot \text{vec}(\Delta \mathbf{D}_k)\} \\ &= \text{Tr}(\mathbf{C}_k \cdot \mathbf{R}_{nn}^{(k)} \cdot \mathbf{C}_k^H). \end{aligned} \quad (21)$$

#### B. Special Case of Uncorrelated Noise

In this section we simplify the expressions, from equations (20) and (21) for the special case of uncorrelated noise with equal variance  $\sigma^2$ . In this simplified scenario (which may not be realistic in practice, but is convenient to check our results in the simulation), the noise correlation matrix  $\mathbf{R}_{nn}$  is given by  $\mathbf{R}_{nn} = \sigma^2 \cdot \mathbf{I}_{M^2 \cdot K}$ . Therefore,  $\mathbb{E}\{\|\Delta \mathbf{T}\|_F^2\}$  in equation (20) is simplified to

$$\mathbb{E}\{\|\Delta \mathbf{T}\|_F^2\} = \sigma^2 \cdot \|\mathbf{A}^+ \cdot \mathbf{B}\|_F^2. \quad (22)$$

In the same manner,  $\mathbb{E}\{\|\Delta \mathbf{D}_k\|_F^2\}$  in equation (21) is simplified to

$$\mathbb{E}\{\|\Delta \mathbf{D}_k\|_F^2\} = \sigma^2 \cdot \|\mathbf{C}_k\|_F^2. \quad (23)$$

### IV. SIMULATION RESULTS

In this section, we corroborate the analytical results obtained in the previous sections using empirical results. We evaluate the performance in different scenarios, e.g., by varying the SNR, the matrix size  $M$ , and the size  $K$  of the matrix set. Furthermore, these

scenarios are simulated over 1000 independent realizations using these parameters:

- 1) *SNR*: During all these realizations the matrix size is fixed to  $M = 4$  and the matrix set size is fixed to  $K = 20$ , while the SNR varies.
- 2) *Matrix Size  $M$* : During all these realizations the SNR is fixed to 90 dB and the matrix set size is fixed to  $K = 20$ , while the matrix size  $M$  varies.
- 3) *Matrix Set Size  $K$* : During all these realizations the SNR is fixed to 90 dB and the matrix size is fixed to  $M = 4$ , while the matrix set size  $K$  varies.

#### A. Perturbation Analysis in terms of a known Noise Tensor

For these simulations we generate, at every realization, random matrix sets according to

$$\hat{\mathbf{S}}_k = \frac{\mathbf{T} \cdot \mathbf{D}_k \cdot \mathbf{T}^{-1}}{\|\mathbf{T} \cdot \mathbf{D}_k \cdot \mathbf{T}^{-1}\|_F} + \sigma \cdot \frac{\Delta \mathbf{S}_k}{\|\Delta \mathbf{S}_k\|_F} \quad \forall k = 1, 2, \dots, K,$$

where  $\mathbf{T}$ ,  $\Delta \mathbf{S}_k$ , and  $\mathbf{D}_k$  are randomly generated using a zero-mean Gaussian distribution (independently in each trial), and the SNR is defined as  $\text{SNR} = -20 \cdot \log(\sigma)$  [8]. All the tested algorithms stop iterating when the deviation (in terms of their corresponding criterion) between two consecutive iterations is less than  $10^{-6}$ , or when the maximum number of 50 iterations is reached. Similarly to [6], we define  $r_T$  to be the relative squared error between the eigenvector matrices as

$$r_T = \frac{\|\mathbf{T} - \hat{\mathbf{T}}\|_F^2}{\|\mathbf{T}\|_F^2}, \quad (24)$$

where the permutation and scaling ambiguities, between  $\mathbf{T}$  and  $\hat{\mathbf{T}}$ , have been removed. Likewise,  $r_D$  is the relative squared eigenvalue error, defined as

$$r_D = \frac{1}{K} \sum_{k=1}^K \frac{\|\mathbf{D}_k - \hat{\mathbf{D}}_k\|_F^2}{\|\mathbf{D}_k\|_F^2}. \quad (25)$$

Furthermore, the algorithms are compared in terms of three criteria: 1)  $J$  in equation (2), 2)  $r_T$  in equation (24), and 3)  $r_D$  in equation (25).

In Figure 1, we evaluate the cost function  $J$ , in equation (2), for the three test scenarios discussed above. We observe that the JET-O and JET-U algorithms do not approach the analytically predicted values, since they are not based on the indirect LS cost function. Moreover, the JDTM algorithm reaches the predicted values, while the Sh-Rt algorithm is slightly worse. Nevertheless, the coupled JEVD, despite being an indirect LS algorithm, does not approach the predicted values for  $J$ . This is more evident in Figures 1b and 1c, where

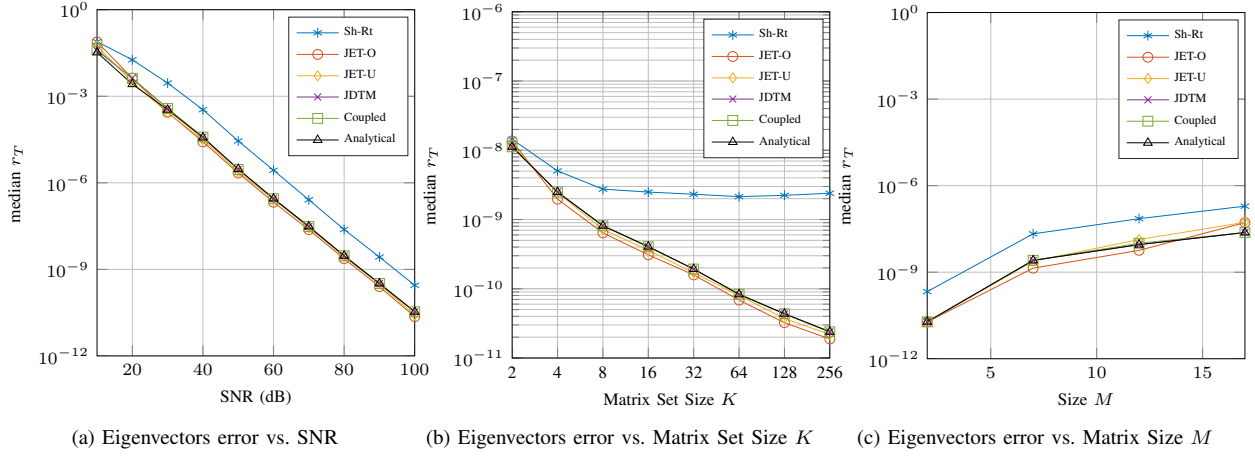


Fig. 3: (3a): Median  $r_T$  in equation (24) vs. SNR, for  $M = 4$  and  $K = 20$ . (3b): Median  $r_T$  in equation (24) vs. matrix set size  $K$ , for  $M = 4$  and SNR = 90 dB. (3c): Median  $r_T$  in equation (24) vs. matrix size  $M$ , for  $K = 20$  and SNR = 90 dB.

the test scenarios 2 and 3 are simulated. Yet, when we evaluate the performance in terms of  $r_D$ , in Figure 2, we observe that all JEVD algorithms achieve the same performance in the high SNR regime. But when we compare the algorithms in terms of  $r_T$ , as shown in Figure 3, we observe that the analytically predicted values match the performance of two indirect LS algorithms (JD TM and coupled JEVD), while the Sh-Rt algorithm does not achieve the analytical performance. As expected, we observe that this does not hold true for algorithms that are not based on the indirect LS (JET-O and JET-U), where another performance analysis specific for these algorithms is needed.

#### B. Closed-Form Expressions in terms of the Noise Statistics

In this section we show results for the special case of uncorrelated noise with equal variance. For these simulations we generate, at every realization, random matrix sets according to

$$\hat{S}_k = \frac{\mathbf{T} \cdot \mathbf{D}_k \cdot \mathbf{T}^{-1}}{\|\mathbf{T} \cdot \mathbf{D}_k \cdot \mathbf{T}^{-1}\|_F} + \sigma \cdot \Delta \mathbf{S}_k \quad \forall k = 1, 2, \dots, K,$$

where the entries of  $\Delta \mathbf{S}_k$  are randomly generated using a zero-mean Gaussian distribution with variance  $\sigma^2$ . Moreover,  $\mathbf{T}$  and  $\mathbf{D}_k$  are randomly generated at the beginning of the simulation but fixed through the 1000 realizations. Here, we use  $E\{r_T\} = \frac{E\{\|\Delta \mathbf{T}\|_F^2\}}{\|\mathbf{T}\|_F^2}$  and

$E\{r_D\} = \frac{1}{K} \sum_{k=1}^K \frac{E\{\|\Delta \mathbf{D}_k\|_F^2\}}{\|\mathbf{D}_k\|_F^2}$  for the performance analysis, where  $E\{\|\Delta \mathbf{T}\|_F^2\}$  and  $E\{\|\Delta \mathbf{D}_k\|_F^2\}$  are computed analytically using equations (22) and (23), respectively. The results are shown in Figure 4 and Figure 5 for the eigenvalues and eigenvectors, respectively. They show that the analytical expression presents an excellent match for both the eigenvalue and eigenvector perturbations ( $r_D$  and  $r_T$ , respectively) for the JD TM and Coupled JEVD algorithms. As shown in the previous results, the Sh-Rt algorithm does not achieve the analytical performance analysis in the eigenvector estimation sense.

#### V. CONCLUSION

In this work, we have performed a first order perturbation analysis for several state-of-the-art JEVD algorithms. We have presented closed-form expressions for the eigenvector and the eigenvalue matrices, as well as the indirect LS cost function. The obtained expressions are asymptotic in the SNR. Moreover, we have used empirical simulations to illustrate the excellent match between the closed-form expressions and the empirical results.

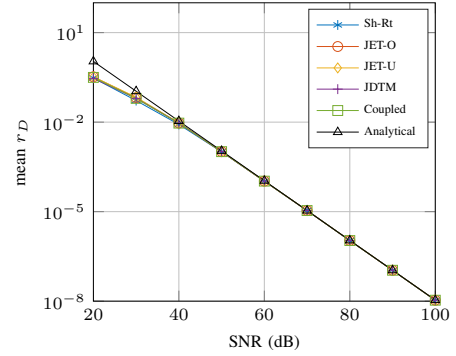


Fig. 4: Average  $r_D$  (Eq. (25)) against SNR, for  $M = 4$  and  $K = 20$ .

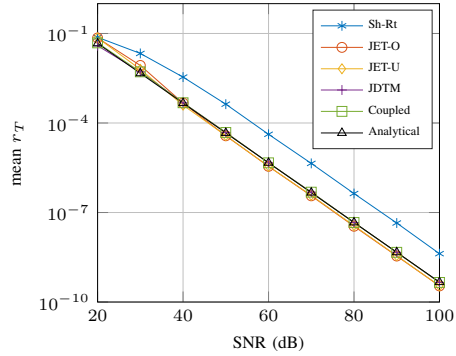


Fig. 5: Average  $r_T$  (Eq. (24)) against SNR, for  $M = 4$  and  $K = 20$ .

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