BOUNDS ON PASSIVE TDOA ESTIMATION IN MIXTURES

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ABSTRACT

We consider the problem of Time Difference of Arrival (TDOA) estimation in mixtures, namely when several sources are received by several receivers, possibly with different delays and attenuations. Under the assumption that the sources are stationary Gaussian with known spectra (a semi-blind scenario), we derive the Cramér-Rao Lower Bound on the Mean Squared Error (MSE) in unbiased joint estimation of the delays and of the mixing coefficients. We then analyze the results, drawing conclusions on the effects of the different model parameters (mixing coefficients, delay differences, signal to noise ratio) on the resulting bound, pointing out essential differences from the classical cases of static mixtures (with no delays) on one hand, and of single-source TDOA estimation on the other hand.

Index Terms— TDOA, source separation, Cramér-Rao lower bound, maximum likelihood.

1. INTRODUCTION

Passive Time Difference of Arrival (TDOA) estimation is widely used in applications such as direction finding and emitter localization (e.g., [1–5]). Traditionally, it assumes that a single source is received by two or more spatially separated sensors. The received signals are differently time-shifted (possibly noisy) versions of the source, where their TDOA, which is related to the spatial layout of the source and the sensors, can be used for subsequent position estimation.

In reality, more than one source (e.g., a number of emitters spatially separated in the reception range of the sensors) may be simultaneously active within the same frequency band. In this scenario, each sensor measures a linear combination of all the sources, each possibly being differently attenuated and delayed. This extended model implies a more intricate, joint TDOA estimation-Blind Source Separation (BSS) problem, where the delays and the sources are to be jointly estimated.

While TDOA estimation in the single source scenario has been extensively studied and analyzed in the literature over the past four decades (e.g., [1, 6–8] as only a few examples), mixtures of multiple sources have been more sparsely addressed. Porat and Friedlander [9] and Nehorai *et al.* [10] proposed methods based on modeling the sources, as well as the received signals, as Auto-Regressive Moving Average (ARMA) processes. Comon and Emile proposed an approach based on second- and/or fourth-order statistics in [11]. In their subsequent work [12], the case of more sources than sensors was presented, yet like in [11], it does not accommodate unknown mixing coefficients in addition to the delays (in fact, it assumes all mixing coefficients are known and are equal to 1). In their other work [13], the model allows unknown mixing parameters and the sources are assumed to be non-Gaussian Moving-Average (MA) processes. In different work by Yeredor [14], the sources are assumed to be general Wide-Sense Stationary (WSS) processes (allowing, but not exploiting, Gaussianity), and estimation based on "extended" approximate joint diagonalization of frequency-domain matrices is proposed. In [15] Chabriel and Barrére propose to describe the delayed signals in terms of their Taylor series expansion, thereby turning the problem into an approximate static mixture of dependent sources, consisting of the sources together with their derivatives.

In some earlier, different work (focused on localization of Gaussian, WSS sources), Wax and Kailath [16] derived the Maximum Likelihood Estimate (MLE) of the sources' locations, as well as the Cramér-Rao Lower Bound (CRLB) on the Mean Squared Error (MSE) matrix, which implicitly encompasses a bound on the TDOA estimates. However, the mixing coefficients are assumed in [16] to be weakly related to the sources' positions, and the respective derivatives are therefore neglected in obtaining the CRLB, rendering the implied CRLB on the TDOA estimates partial, as it essentially corresponds only to the case of known mixing coefficients (similarly to [11, 12]).

Thus, to the best of our knowledge, some aspects of the attainable performance remained unstudied, particularly the full CRLB (accounting for unknown mixing coefficients and TDOAs) and its analysis w.r.t. principal parameters of the problem. Our goal in this paper is therefore to further examine the theoretical bounds on TDOA estimation in mixtures. We shall consider a semi-blind scenario, assuming all signals involved to be WSS and Gaussian - a framework which conveniently lends itself to derivation of the CRLB and of the MLE (we shall not present the MLE in here due to space limitations, however we did use the MLE to verify that the derived bounds are attained asymptotically).

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The rest of the paper is organized as follows. In Section 2 we present the problem formulation. In Section 3 we present and analyze the CRLB on the MSE of unbiased TDOA estimates in this model. In section 4 the predicted dependence of the bound on the model parameters is visualized in several plots. Section 5 concludes the paper.

2. PROBLEM FORMULATION

Consider the following (continuous-time) M sources - L sensors model, where for each sensor $\ell \in \{1, \dots, L\}$

$$x_\ell(t) = \sum_{m=1}^{M} a_{\ell m} s_m(t - \tau_{\ell m}) + v_\ell(t). \tag{1}$$

Here $s_m(t)$, $x_\ell(t)$ and $v_\ell(t)$ are the sources, measured mixtures and noises, resp., $\{a_{\ell m}\}$ are the unknown (deterministic) mixing coefficients and $\{\tau_{\ell m}\}$ are the unknown (deterministic) time delays from source m to sensor ℓ. Our goal is to estimate the unknown delays and (possibly) separate the sources. To simplify the exposition, we assume the following:

1. All the signals and parameters involved are real-valued;
2. The sources $s_m(t)$ are all band-limited WSS, Gaussian, mutually independent random processes with Power Spectral Density (PSD) functions $P_m^s(\omega)$, resp.;
3. The noise signals $v_\ell(t)$ are mutually independent zero-mean white Gaussian processes, independent of the sources, all with the same spectral density level N_0;
4. The received signals are low-pass filtered (LPF) (the LPF bandwidth exceeds the bandwidth of the widest source) and sampled at (at least) the Nyquist rate.
5. Without loss of generality (wlog) we assume that the sampling rate is 1 (hence, the LPF bandwidth is π), and denote the sampled signals $x_\ell[n] \triangleq x_\ell(n \cdot 1), \quad n = 1, \dots, N$, where N is assumed even;

Fourier transforming (1), we obtain the equivalent frequency-domain representation (for all $\ell \in \{1, \dots, L\}$)

$$\tilde{x}_\ell(\omega) = \sum_{m=1}^{M} a_{\ell m} \tilde{s}_m(\omega) e^{-j\omega\tau_{\ell m}} + \tilde{v}_\ell(\omega). \tag{2}$$

Note that strictly speaking, sample functions of a WSS process do not admit a Fourier Transform (FT), but we assume that the received signals vanish outside our finite observation interval. However, this assumption induces inaccuracies in (2) due to edge-effects, which we assume to be negligible if the observation interval is sufficiently long.

Rewriting (2) in vector-matrix form,

$$\widetilde{\boldsymbol{x}}(\omega) = \widetilde{\boldsymbol{A}}(\omega)\widetilde{\boldsymbol{s}}(\omega) + \widetilde{\boldsymbol{v}}(\omega) \in \mathbb{C}^{L\times 1}, \tag{3}$$

where we defined the (frequency dependent) matrices

$$\widetilde{\boldsymbol{A}}(\omega) \triangleq \boldsymbol{A} \odot \boldsymbol{D}(\omega) \in \mathbb{C}^{L\times M}, \tag{4}$$

in which $\boldsymbol{A} \in \mathbb{R}^{L\times M}$ denotes the constant mixing matrix whose elements are $\{a_{\ell m}\}$, $\boldsymbol{D}(\omega) \in \mathbb{C}^{L\times M}$ is a matrix whose elements are $\{e^{-j\omega\tau_{\ell m}}\}$, ω is the (angular) frequency and $\odot$ denotes the Hadamard (element-wise) product. Similarly, we shall denote by $\boldsymbol{T} \in \mathbb{R}^{L\times M}$ the matrix containing the delays $\{\tau_{\ell m}\}$ as its elements.

Now, recall that the (normalized) Discrete-Fourier Transforms (DFTs) of the available samples $\{x_\ell[n], 1 \le n \le N\}_{\ell=1}^{L}$ are equal to the respective FTs of the continuous-time signals, at the discrete Fourier frequencies. More precisely, if $\boldsymbol{F} \in \mathbb{C}^{N\times N}$ is the (normalized) DFT matrix with elements $F_{kn} = \frac{1}{\sqrt{N}} \exp\left(-j2\pi(k-1)(n-1)/N\right)$, then

$$\boldsymbol{F}\boldsymbol{x}_\ell = \widetilde{\boldsymbol{x}}_\ell \Rightarrow \tilde{x}_\ell[k] = \tilde{x}_\ell(\omega_k), \tag{5}$$

where $\boldsymbol{x}_\ell^{\mathrm{T}} \triangleq [x_\ell[1] \dots x_\ell[N]] \in \mathbb{R}^{1\times N}$, $\omega_k \triangleq \frac{2\pi(k-1)}{N}$ and we distinguish the DFT from the FT by using brackets instead of parentheses. Since the DFT is merely a linear invertible transformation, the available data may be written (without loss of information) in the frequency domain as

$$\widetilde{\boldsymbol{x}}[k] = \widetilde{\boldsymbol{A}}[k]\widetilde{\boldsymbol{s}}[k] + \widetilde{\boldsymbol{v}}[k] \in \mathbb{C}^{L\times 1}, \forall k \in \{1, \dots, N\}. \tag{6}$$

At this point, recall that the sources, as well as the noises, are all stationary Gaussian and statistically independent. Hence, all the mixtures are stationary Gaussian as well. In addition, since the DFT is a linear complex-valued transformation, the k-th frequency source vector $\widetilde{\boldsymbol{s}}[k]$ and noise vector $\widetilde{\boldsymbol{v}}[k]$ are (circular) Complex Normal (CN) [17] (except for $k = 1, \frac{N+2}{2}$, for which they are real-valued Normal),

$$\widetilde{\boldsymbol{s}}[k] \sim \mathcal{CN}\left(\boldsymbol{0}_M, \boldsymbol{P}_k^s\right), \forall k \in \{1, \dots, N\}\backslash\{1, \tfrac{N+2}{2}\}, \tag{7}$$

$$\widetilde{\boldsymbol{v}}[k] \sim \mathcal{CN}\left(\boldsymbol{0}_L, \sigma_v^2 \boldsymbol{I}_L\right), \forall k \in \{1, \dots, N\}\backslash\{1, \tfrac{N+2}{2}\}, \tag{8}$$

where $\boldsymbol{0}_L \in \mathbb{R}^{L\times 1}$ is the all-zeros vector, $\boldsymbol{I}_L$ is the $L \times L$ identity matrix, $\sigma_v^2 = 2N_0$ is the sampled noise variance and asymptotically (namely, for N large enough) $\boldsymbol{P}_k^s$ is a diagonal matrix containing the PSDs of the sources at the k-th frequency, i.e., its (m,m)-th element equals $P_m^s(\omega_k)$ (asymptotically, namely for N sufficiently large, due to the stationarity of the sources). Consequently, since $\widetilde{\boldsymbol{A}}[k]$ is also a linear complex-valued transformation and using the independence between the sources and the noises, from (6) we have that

$$\widetilde{\boldsymbol{x}}[k] \sim \mathcal{CN}\left(\boldsymbol{0}_L, \boldsymbol{C}_k^x\left(\boldsymbol{A}, \boldsymbol{T}\right)\right), \forall k \in \{1, \dots, N\}\backslash\{1, \tfrac{N+2}{2}\}, \tag{9}$$

where $\boldsymbol{C}_k^x\left(\boldsymbol{A}, \boldsymbol{T}\right) \triangleq \widetilde{\boldsymbol{A}}[k]\boldsymbol{P}_k^s\widetilde{\boldsymbol{A}}[k]^\dagger + \sigma_v^2\boldsymbol{I}_L$ and where $(\cdot)^\dagger$ denotes the Hermitian (conjugate transpose) operator. Notice that since all the considered time-domain signals are real-valued, the sufficient statistics are actually the first $N/2$ frequency components[1]. Note further, that due to the stationar-

[1]From now on, we shall assume that the sampling rate is strictly higher than the Nyquist rate so that $\widetilde{\boldsymbol{x}}\left[\frac{N+2}{2}\right] = \boldsymbol{0}$. In addition, we shall not pay special attention to the DC component ($k = 1$) since it follows the same derivation like all other frequency components with slight immaterial modifications.

ity of the sources and noise, these frequency components are (asymptotically) mutually statistically independent.

The problem at hand can now be formulated compactly in the frequency domain as follows. Given the statistically independent measurements $\{\widetilde{\boldsymbol{x}}[k]\}_{k=1}^{N/2}$ whose distributions are prescribed by (9), estimate the unknown (deterministic) parameters $\boldsymbol{T}$ and $\boldsymbol{A}$.

From now on, to further simplify the exposition, we shall assume that the number of sources equals the number of sensors, i.e., $M = L$. In addition, in order to circumvent the inherent time-origin ambiguity in this problem, we shall assume wlog that $\tau_{\ell\ell} = 0$ for all $\ell \in \{1, \ldots, L\}$.

In the next section we present and analyze the CRLB on the MSE of any unbiased time delays estimates for this model.

3. CRLB - DERIVATION AND ANALYSIS

We start by recalling a general result (e.g., [18]): for $\boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{C}(\boldsymbol{\theta}))$, where $\boldsymbol{C}(\boldsymbol{\theta})$ is parametrized by a deterministic unknown vector $\boldsymbol{\theta} \in \mathbb{R}^{Q\times 1}$, the (q, r)-th element of the Fisher Information Matrix (FIM) $\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}) \in \mathbb{R}^{Q\times Q}$ is given by

$$\mathcal{I}_{qr}(\boldsymbol{\theta}) = \text{Tr}\left(\boldsymbol{C}^{-1}(\boldsymbol{\theta})\frac{\partial \boldsymbol{C}(\boldsymbol{\theta})}{\partial\theta_q}\boldsymbol{C}^{-1}(\boldsymbol{\theta})\frac{\partial \boldsymbol{C}(\boldsymbol{\theta})}{\partial\theta_r}\right), \quad (10)$$

where $\text{Tr}(\cdot)$ denotes the trace operator.

Returning to our problem, since $\{\widetilde{\boldsymbol{x}}[k]\}_{k=1}^{N/2}$ are statistically independent, the overall FIM is given by

$$\boldsymbol{\mathcal{I}}(\boldsymbol{A}, \boldsymbol{T}) = \sum_{k=1}^{N/2} \boldsymbol{\mathcal{I}}^{(k)}(\boldsymbol{A}, \boldsymbol{T}) \in \mathbb{R}^{M(2M-1)\times M(2M-1)}, \quad (11)$$

where $\boldsymbol{\mathcal{I}}^{(k)}(\boldsymbol{A}, \boldsymbol{T})$ is the FIM of the k-th frequency component. Following (10) and (9) we have for all $\ell, m \in \{1, \ldots, M\}$

$$\frac{\partial \boldsymbol{C}_k^x(\boldsymbol{A}, \boldsymbol{T})}{\partial a_{\ell m}} = \frac{\partial\left(\widetilde{\boldsymbol{A}}[k]\boldsymbol{P}_k^s\widetilde{\boldsymbol{A}}[k]^\dagger + \sigma_v^2\boldsymbol{I}_K\right)}{\partial a_{\ell m}} = \left(\boldsymbol{E}_{\ell m}\odot\boldsymbol{D}(\omega_k)\right)\boldsymbol{P}_k^s\widetilde{\boldsymbol{A}}[k]^\dagger + \widetilde{\boldsymbol{A}}[k]\boldsymbol{P}_k^s\left(\boldsymbol{E}_{\ell m}\odot\boldsymbol{D}(\omega_k)\right)^\dagger, \quad (12)$$

where $\boldsymbol{E}_{\ell m}$ denotes an all-zeros matrix, except for a 1 as its (ℓ, m)-th element. Likewise, for all $m \neq \ell \in \{1, \ldots M\}$

$$\frac{\partial \boldsymbol{C}_k^x(\boldsymbol{A}, \boldsymbol{T})}{\partial \tau_{\ell m}} = j\omega_k\left[e^{j\omega_k\tau_{\ell m}}\widetilde{\boldsymbol{A}}[k]\boldsymbol{P}_k^s\left(\boldsymbol{A}\odot\boldsymbol{E}_{\ell m}\right)^\dagger - e^{-j\omega_k\tau_{\ell m}}\left(\boldsymbol{A}\odot\boldsymbol{E}_{\ell m}\right)\boldsymbol{P}_k^s\widetilde{\boldsymbol{A}}[k]^\dagger\right]. \quad (13)$$

By the Matrix Inversion Lemma (e.g., [19]), we also have

$$\left(\boldsymbol{C}_k^x(\boldsymbol{A}, \boldsymbol{T})\right)^{-1} = \frac{1}{\sigma_v^2}\left[\boldsymbol{I}_M - \widetilde{\boldsymbol{A}}[k]\left(\sigma_v^2\left(\boldsymbol{P}_k^s\right)^{-1} + \widetilde{\boldsymbol{A}}[k]\widetilde{\boldsymbol{A}}[k]^\dagger\right)^{-1}\widetilde{\boldsymbol{A}}[k]^\dagger\right], \quad (14)$$

so substituting (12)-(14) in (10), we obtain all the required closed-form expressions for obtaining the FIM $\boldsymbol{\mathcal{I}}(\boldsymbol{A}, \boldsymbol{T})$.

An interesting particular case is the noiseless case, i.e., $\sigma_v^2 = 0$, for which (12), (13) remain unchanged, and (14) becomes

$$\left(\boldsymbol{C}_k^x(\boldsymbol{A}, \boldsymbol{T})\right)^{-1} = \widetilde{\boldsymbol{B}}[k]^\dagger\left(\boldsymbol{P}_k^s\right)^{-1}\widetilde{\boldsymbol{B}}[k], \quad (15)$$

where we have defined $\widetilde{\boldsymbol{B}}[k] \triangleq \left(\widetilde{\boldsymbol{A}}[k]\right)^{-1}$. Note that unlike the case of a single-source TDOA estimation, where under noiseless conditions the CRLB vanishes, here even in the noiseless case a positive lower bound on the attainable estimation accuracy exists.

We note further that the CRLB for this problem may be studied w.r.t. TDOA estimation and source separation (e.g., differences from the classical static noiseless case and identifiability conditions). However, due to space limitation, in what follows we focus on the CRLB for TDOA estimation only. To this end, define the following block matrices

$$\boldsymbol{\mathcal{I}}(\boldsymbol{A}, \boldsymbol{T}) \triangleq \begin{bmatrix}\boldsymbol{\mathcal{I}}_{A} & \boldsymbol{\mathcal{I}}_{AT}\\ \boldsymbol{\mathcal{I}}_{AT}^{\mathrm{T}} & \boldsymbol{\mathcal{I}}_{T}\end{bmatrix} \in \mathbb{R}^{2M(M-1)\times 2M(M-1)}, \quad (16)$$

where $\boldsymbol{\mathcal{I}}_A$ and $\boldsymbol{\mathcal{I}}_T$ are the individual FIM of $\boldsymbol{A}$ and $\boldsymbol{T}$, resp., and $\boldsymbol{\mathcal{I}}_{AT}$ is the "cross-terms" related block matrix between $\boldsymbol{A}$ and $\boldsymbol{T}$. Using the Schur complement ([20]) of (16), the CRLB on the MSE of any unbiased time delay estimate reads

$$E\left[\left(\hat{\tau}_{\ell m} - \tau_{\ell m}\right)^2\right] \geq \boldsymbol{e}_{i(\ell,m)}^{\mathrm{T}}\boldsymbol{\mathcal{I}}_T^{-1}\left[\boldsymbol{I}_K - \boldsymbol{\mathcal{I}}_{AT}^{\mathrm{T}}\cdot\right. \quad (17)$$

$$\left.\left(\boldsymbol{\mathcal{I}}_A - \boldsymbol{\mathcal{I}}_{AT}\boldsymbol{\mathcal{I}}_T^{-1}\boldsymbol{\mathcal{I}}_{AT}^{\mathrm{T}}\right)^{-1}\boldsymbol{\mathcal{I}}_{AT}\boldsymbol{\mathcal{I}}_T^{-1}\right]\boldsymbol{e}_{i(\ell,m)}, \quad (18)$$

(requiring only that $\boldsymbol{\mathcal{I}}_T$ be non-singular), where $i(\ell, m)$ is the index corresponding to the (ℓ, m)-th element of $\boldsymbol{T}$ in the individual FIM $\boldsymbol{\mathcal{I}}_T$ and where the "pinning vector" $\boldsymbol{e}_i$ denotes the i-th column of $\boldsymbol{I}_{M(M-1)}$.

3.1. Analysis of the Resulting Bound

We now turn to explore the effects of three factors on the bound: the mixing coefficients, the time delays and the Signal-to-Noise-Ratio (SNR). For simplicity, we shall consider the case $M = 2$ in here.

We start by studying the impact of the mixing coefficients and the delays on the bound. To this end, consider the ideal noiseless case ($\sigma_v^2 = 0$). Using (10), substituting (12), (13) and (15) into (11) and simplifying the expressions (using straightforward algebra) we obtain the k-th individual FIM elements of $\boldsymbol{T}$

$$\mathcal{I}^{(k)}[\tau_{12}, \tau_{12}] = 2\omega_k^2 a_{12}^2 b_{21}^2 \cdot\left[\left(1 - \cos(2\omega_k(\tau_{12} + \tau_{21}))\right) + \frac{b_{11}^2 P_2^s(\omega_k)}{b_{21}^2 P_1^s(\omega_k)}\right], \quad (19)$$

$$\mathcal{I}^{(k)}[\tau_{12}, \tau_{21}] = -2\omega_k^2 a_{12}a_{21}b_{11}b_{22}\cos(2\omega_k(\tau_{12} + \tau_{21})), \quad (20)$$

where $\{b_{m\ell}\}$ are the elements of $\boldsymbol{B} \triangleq \boldsymbol{A}^{-1}$. Obviously, $\mathcal{I}^{(k)}[\tau_{12}, \tau_{21}] = \mathcal{I}^{(k)}[\tau_{21}, \tau_{12}]$ and $\mathcal{I}^{(k)}[\tau_{21}, \tau_{21}]$ is similarly obtained by switching indices in (19). We shall refer to these terms in the sequel.

Consider an orthonormal (rotation) mixing matrix $\boldsymbol{A}_\alpha \triangleq \left[\begin{smallmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{smallmatrix}\right]$. Obviously, when $\alpha = 0, \pm\frac{\pi}{2}, \pi$, each sensor receives only one (different) source, so the delays are non-identifiable, and we expect the bound to be infinite. Notice that these results are in full agreement with (19) and (20). This simple example suggests that there should exist "optimal" rotation angles α for which the bound is (locally) minimized. Indeed, as we show in Fig. 1 (section 4), for this example the optimal rotation angle (in $(0, \frac{\pi}{2})$) is $\frac{\pi}{4}$, regardless of the sources' spectra or the delays. This result implies that the most informative scenario w.r.t. the delays is when both signals are received in each sensor at equal gains.

We proceed to examine the bound w.r.t. the values of the delays. From (19), we see that when $\tau_{12} + \tau_{21} = 0$ (namely, when both sources have the same TDOA), $\mathcal{I}^{(k)}[\tau_{12}, \tau_{12}]$, as well as $\mathcal{I}^{(k)}[\tau_{21}, \tau_{21}]$ are minimized, whereas the absolute value of $\mathcal{I}^{(k)}[\tau_{11}, \tau_{21}]$ is maximized - thereby minimizing both the trace and the determinant of $\boldsymbol{\mathcal{I}}_{\boldsymbol{\tau}}$. This directly implies maximization of the bound on the TDOAs estimates when $\boldsymbol{A}$ is known, and, as we show in the next section, also when $\boldsymbol{A}$ is unknown.

Lastly, we examine the bound w.r.t. the SNR. Recalling (9), when σ_v^2 is sufficiently large (at the low SNR regime), $\boldsymbol{C}_k^x(\boldsymbol{A}, \boldsymbol{T})$ is dominated by $\sigma_v^2 \boldsymbol{I}_L$. Since its derivatives (cf. (12), (13)) do not depend on σ_v^2, in this case $\boldsymbol{\mathcal{I}}(\boldsymbol{A}, \boldsymbol{T})$ is inversely proportional to σ_v^4, so the bound is linearly proportional to σ_v^4. At the high-SNR regime, when σ_v^2 is negligibly small, the bound tends (asymptotically in $-\log(\sigma_v^2)$) to a fixed positive value (unlike the case of a single source, where the bound vanishes at infinite SNR (e.g., [1])).

We note in passing (without a detailed proof), that according to the FIM, the time delays and the mixing coefficients may still be estimated to within a finite precision (decreasing with N) as long as $\tau_{12} \neq -\tau_{21}$, even when the spectra of the Gaussian sources are identical. Thus, unlike the classical case of static mixtures, Gaussian sources with identical spectra are asymptotically separable when $\tau_{12} \neq -\tau_{21}$.

4. BOUND VISUALIZATION

We demonstrate the behavior of the CRLB in three scenarios, each focusing on one out of the three factors analyzed above: mixing coefficient, time delays and SNR. In all three experiments we used $L = M = 2$ sources and sensors, and assumed an observation length of $N = 1000$ samples. The two spectra of the sources were arbitrarily set to represent MA processes of order 10 with arbitrary coefficients (when using the same spectrum for both sources we used the spectrum of source 1). When using fixed TDOAs, their values were $\tau_{12} = 2.6$ and $\tau_{21} = -4.37$. In the noiseless case we used $\sigma_v^2 = 0$ and a "noisy case" is defined as 0[dB] SNR ($\sigma_v^2 = 1$).

In the first experiment the mixing matrix $\boldsymbol{A}_\alpha$ was a rotation matrix by α (see Section 3 above), and the CRLB (for both TDOAs) is presented as a function of α with fixed delays, both for the noiseless and noisy cases, and both for the case of different spectra and identical spectra. In all cases, the minimum-bound point is seen to occur at $\alpha = \frac{\pi}{4}$, as expected. In the second experiment we fixed the mixing matrix at $A_{\pi/8}$, fixed $\tau_{12} = 0$ and varied τ_{21}. Again, both the noiseless and noisy cases, for both distinct and identical cases are shown. In the last experiment we varied the SNR while fixing $A_{\pi/8}$ with fixed delays and both distinct and identical spectra. In all three cases the bound is seen to behave as analyzed above.

5. CONCLUSION

In the context of passive TDOA estimation and semi-blind separation of stationary Gaussian sources, we presented the CRLB on the MSE for unbiased estimates of the TDOA and the mixing coefficients. We briefly analyzed the behavior of the bound w.r.t. the mixing coefficient, time delays and SNR, depicted numerically in several related scenarios.

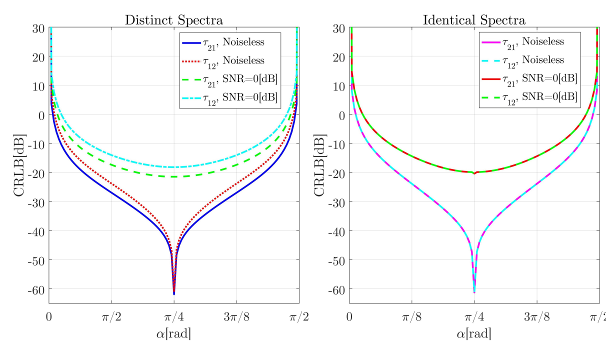

Fig. 1: Experiment 1: CRLB vs. α

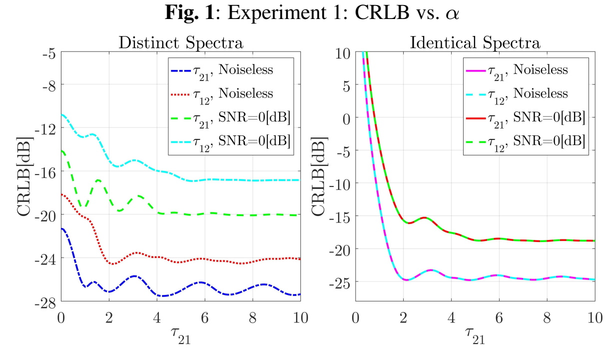

Fig. 2: Experiment 3: CRLB vs. τ_{21}

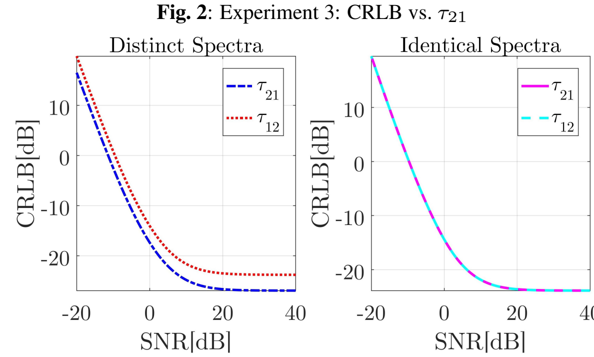

Fig. 3: Experiment 3: CRLB vs. SNR

6. REFERENCES

[1] Charles H. Knapp and G. Clifford Carter, "The generalized correlation method for estimation of time delay," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 24, no. 4, pp. 320–327, 1976.

[2] Benjamin Friedlander and Anthony J. Weiss, "Eigenstructure methods for direction finding with sensor gain and phase uncertainties," in *Proc. of ICASSP*, 1988, pp. 2681–2684.

[3] Xunxue Cui, Kegen Yu, and Songsheng Lu, "Direction finding for transient acoustic source based on biased TDOA measurement," *IEEE Trans. on Instrumentation and Measurement*, vol. 65, no. 11, pp. 2442–2453, 2016.

[4] Arie Yeredor and Eyal Angel, "Joint TDOA and FDOA estimation: A conditional bound and its use for optimally weighted localization," *IEEE Trans. on Signal Processing*, vol. 59, no. 4, pp. 1612–1623, 2011.

[5] Oren Jean and Anthony J. Weiss, "Passive localization and synchronization using arbitrary signals," *IEEE Trans. on Signal Processing*, vol. 62, no. 8, pp. 2143–2150, 2014.

[6] M.-A. Pallas and Genevieve Jourdain, "Active high resolution time delay estimation for large BT signals," *IEEE Trans. on Signal Processing*, vol. 39, no. 4, pp. 781–788, 1991.

[7] Jitendra K. Tugnait, "On time delay estimation with unknown spatially correlated Gaussian noise using fourth-order cumulants and cross cumulants," *IEEE Trans. on Signal Processing*, vol. 39, no. 6, pp. 1258–1267, 1991.

[8] K.C. Ho, Xiaoning Lu, and La-or Kovavisaruch, "Source localization using TDOA and FDOA measurements in the presence of receiver location errors: Analysis and solution," *IEEE Trans. on Signal Processing*, vol. 55, no. 2, pp. 684–696, 2007.

[9] Boaz Porat and Benjamin Friedlander, "Estimation of spatial and spectral parameters of multiple sources," *IEEE Trans. on Information Theory*, vol. 29, no. 3, pp. 412–425, 1983.

[10] Arye Nehorai, Guaning Su, and Martin Morf, "Estimation of time differences of arrival by pole decomposition," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 31, no. 6, pp. 1478–1492, 1983.

[11] Pierre Comon and Bruno Emile, "Estimation of time delays in the blind mixture problem," in *Proc. EUSIPCO, Edinburgh, Scotland*, 1994, pp. 482–485.

[12] Bruno Emile, Pierre Comon, and Joël Le Roux, "Estimation of time delays with fewer sensors than sources," *IEEE Trans. on Signal Processing*, vol. 46, no. 7, pp. 2012–2015, 1998.

[13] Bruno Emile and Pierre Comon, "Estimation of time delays between unknown colored signals," *Signal Processing*, vol. 69, no. 1, pp. 93–100, 1998.

[14] Arie Yeredor, "Blind source separation with pure delay mixtures," in *Int Workshop on Independent Component Analysis and Blind Source Separation Conference (ICA), San Diego, CA*, 2001.

[15] Gilles Chabriel and Jean Barrére, "An instantaneous formulation of mixtures for blind separation of propagating waves," *IEEE Trans. on Signal Processing*, vol. 54, no. 1, pp. 49–58, 2006.

[16] Mati Wax and Thomas Kailath, "Optimum localization of multiple sources by passive arrays," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 31, no. 5, pp. 1210–1217, 1983.

[17] Benedikt Loesch and Bin Yang, "Cramér-Rao bound for circular and noncircular complex independent component analysis," *IEEE Trans. on Signal Processing*, vol. 61, no. 2, pp. 365–379, 2013.

[18] Sandra L. Collier, "Fisher information for a complex Gaussian random variable: Beamforming applications for wave propagation in a random medium," *IEEE Trans. on Signal Processing*, vol. 53, no. 11, pp. 4236–4248, 2005.

[19] Max A. Woodbury, "Inverting modified matrices," *Memorandum report*, vol. 42, no. 106, pp. 336, 1950.

[20] Emilie V. Haynsworth, "On the Schur complement," Tech. Rep., Basel Univ. (Switzerland) Mathematics Inst., 1968.