Analytical Performance Analysis of the SEmi-algebraic framework for approximate CP decompositions via SImultaneous matrix diagonalizations (SECSI)

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***Abstract*—The Canonical Polyadic (CP) decomposition of R-way arrays is a powerful tool in multi-dimensional signal processing. There exists many methods to compute the CP decomposition. In particular, the Semi-Algebraic framework for the approximate Canonical Polyadic (CP) decomposition via SImultaneaous matrix diagonalization (SECSI) is an efficient and flexible framework for the computation of the CP decomposition. In this work, we perform a first-order performance analysis of the SECSI framework for the computation of the approximate CP decomposition of a noise corrupted low-rank 3-way tensor. We provide closed-form expressions of the relative Mean Square Factor Error (rMSFE) for each of the estimated factor matrices. The derived expressions are formulated in terms of the second-order moments of the noise, such that apart from a zero mean, no assumptions on the noise statistics are required. The numerical results depict the excellent match between the closed-form expressions and the empirical results.**

***Index Terms*—Perturbation analysis, Canonical Polyadic (CP), tensor signal processing.**

I. Introduction

One of the widely used tensor decomposition methods is the Canonical Polyadic (CP) decomposition, also referred as canonical decomposition (CANDECOMP) or parallel factor (PARAFAC) analysis. It allows to decompose a tensor into a sum of rank-one components. The biggest advantage of the CP decomposition comes from the fact that the factor matrices are essentially unique under mild conditions, which makes it very useful, also in cases when no or only very limited a priori information is available. It has been recognized as a basic tool for signal separation and data analysis, with many concrete applications in telecommunication, array processing, and machine learning [1]–[3]. The CP decomposition can be viewed as one extension of the matrix Singular Value Decomposition (SVD) to higher orders, with the difference that the factor matrices are generally non-orthogonal and the core tensor is an identity tensor.

To solve CP decomposition problems, iterative Alternating Least Squares (ALS) methods are widely employed [4], [5]. However, these methods are not efficient since they may require a large number of iterations to converge and are not guaranteed to reach the global optimum. Therefore, these methods are computationally expensive. Alternatively, semi-algebraic solutions such as the Semi-Algebraic canonicaL decompositTion (SALT) [6] and the SEmi-algebraic Cp decomposition via SImultaneous matrix diagonalizationn (SECSI) [7] have been proposed in the literature. By exploiting the structure of the tensor, the SECSI framework identifies the whole set of joint eigenvalue decompositions (JEVDs) (also called simultaneous matrix diagonalization (SMD)) [8]. Solving all of the JEVD problems yields several estimates of the factor matrices. In the final step, an appropriate solution is selected that results into a more robust algorithm with an improved performance. Moreover, in contrast to ALS, the SECSI framework facilitates the implementation on a parallel hardware architecture.

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In this work, we perform a first-order perturbation analysis of the SECSI framework. A perturbation analysis allows us to assess the behavior of the algorithm in different scenarios without the need of computationally expensive Monte-Carlo trials. To the best of our knowledge, there exists no such analytical assessment in the literature. The SECSI framework performs three distinct steps to compute the approximate CP decomposition of a noisy tensor. In the first step, the truncated HOSVD is used to suppress the noise. In the second step, the whole set of JEVDs is constructed from the truncated core tensor that results in several estimates of the factor matrices. Finally, the best factor matrices are selected from these estimates by using an appropriate selection strategy as presented in [7]. Depending on the chosen strategy, a performance complexity trade-off can be obtained. To compute a first-order perturbation analysis of the SECSI framework, we need a perturbation analysis for each of the steps. We have already presented a first-order perturbation analysis of low-rank tensor approximations based on the truncated HOSVD in [9]. We have also performed a first-order perturbation analysis of JEVD algorithms that are based on the indirect least squares (LS) cost function in [10]. These indirect least squares (LS) cost function based JEVD algorithms are used in the SECSI framework. Using theses results, we carry out a first-order perturbation analysis of the SECSI framework in this work, where apart from zero-mean and finite second order moments, no assumptions about the noise are required.

The organization of the remainder of the paper is as follows. We describe the data model in Section II. We also provide a brief overview of the SECSI framework. Then in Section III, we present a first-order perturbation analysis of the SECSI framework as a function of known noise tensor. The closed-form relative Mean Square Factor Error (rMSFE) expressions for each of the factor matrices are presented in Section IV. The simulation results are discussed in Section V. Finally, the work is concluded in Section VI.

Notation: For the sake of notation, we use a, $\mathbf{a}$, $\boldsymbol{A}$, and $\boldsymbol{\mathcal{A}}$ for a scalar, column vector, matrix, and tensor, respectively, where, $\boldsymbol{\mathcal{A}}(i,j,k)$ defines the element (i,j,k) of a tensor $\boldsymbol{\mathcal{A}}$. The same applies to a matrix $\boldsymbol{A}(i,j)$ and a vector $\mathbf{a}(i)$. The superscripts ${}^{-1}$, ${}^{+}$, *, ${}^{\mathrm{T}}$, and ${}^{\mathrm{H}}$ denote the matrix inverse, Moore-Penrose pseudo inverse, conjugate, transposition, and conjugate transposition, respectively. We also use the notation $\mathbb{E}\{\cdot\}$, $\mathrm{tr}\{\cdot\}$, $\otimes$, $\diamond$, $\|\cdot\|_{\mathrm{F}}$, and $\|\cdot\|_2$ for the expectation, trace, Kronecker product, the Khatri-Rao (column-wise Kronecker) product, Frobenius norm, and 2-norm operators, respectively. Moreover, the $\mathrm{diag}(\cdot)$ is applied to a matrix and result in a column vector while $\mathrm{Diag}(\cdot)$ is applied to a vector that gives a diagonal matrix. We also use operators $\mathrm{Ddiag}(\cdot)$ and $\mathrm{Off}(\cdot)$ where $\mathrm{Ddiag}(\cdot)$ sets all the off-diagonal elements of $\boldsymbol{X}$ to zero, while the $\mathrm{Off}(\cdot)$ operator sets the diagonal elements of $\boldsymbol{X}$ to zero. Finally, for the sake of notational simplicity, we define the following notation for r-mode products

$$\boldsymbol{\mathcal{A}} \bigtimes_{r=1}^{R} {}_r \boldsymbol{X}^{(r)} = \boldsymbol{\mathcal{A}} \times_1 \boldsymbol{X}^{(1)} \times_2 \boldsymbol{X}^{(2)} \times_3 \cdots \times_R \boldsymbol{X}^{(R)}.$$

$[\boldsymbol{\mathcal{A}}]_{(r)}$ denotes the r-mode unfolding of $\boldsymbol{\mathcal{A}}$ which is performed according to the reverse cyclical order [7].

II. Overview of the SECSI Framework

Let us assume a 3-way noiseless tensor $\boldsymbol{\mathcal{X}}_0 \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ of tensor rank d. The CP decomposition of such a low-rank noiseless tensor is given by

$$\boldsymbol{\mathcal{X}}_0 = \boldsymbol{\mathcal{I}}_{3,d} \times_1 \boldsymbol{F}^{(1)} \times_2 \boldsymbol{F}^{(2)} \times_3 \boldsymbol{F}^{(3)}, \tag{1}$$

where $\boldsymbol{F}^{(r)} \in \mathbb{C}^{M_r \times d}, \forall r = 1,2,3$ is the factor matrix in the r-th mode and $\boldsymbol{\mathcal{I}}_{3,d}$ is the 3-way identity tensor of size $d \times d \times d$. In real data-driven applications, we observe a noisy version of this noiseless tensor that is given as

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{X}}_0 + \boldsymbol{\mathcal{N}} \in \mathbb{C}^{M_1 \times M_2 \times M_3}, \tag{2}$$

where $\boldsymbol{\mathcal{N}} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ is a zero-mean additive noise tensor. In the following, we summarize the approximate CP decomposition of the noise corrupted low-rank tensor using the SECSI framework. The details can be found in [7].

1) First the noise is suppressed by truncating the HOSVD of the noisy tensor $\boldsymbol{\mathcal{X}}$ as

$$\hat{\boldsymbol{\mathcal{X}}} = \hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} \times_1 \hat{\boldsymbol{U}}_1^{[\mathrm{s}]} \times_2 \hat{\boldsymbol{U}}_2^{[\mathrm{s}]} \times_3 \hat{\boldsymbol{U}}_3^{[\mathrm{s}]}, \tag{3}$$

where $\hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} \in \mathbb{C}^{d \times d \times d}$ is the truncated core tensor and $\hat{\boldsymbol{U}}_r^{[\mathrm{s}]} \in \mathbb{C}^{M_r \times d}, \forall r = 1,2,3$ are the signal subspace matrices. The signal subspace matrices are obtained from the SVD of the r-mode unfolding of the observed noisy tensor $\boldsymbol{\mathcal{X}}$. The truncated core tensor is obtained by the expression $\hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} = \boldsymbol{\mathcal{X}} \bigtimes_{r=1}^{3} {}_r \hat{\boldsymbol{U}}_r^{[\mathrm{s}]^{\mathrm{H}}}$. Moreover, r-mode tensors for the JEVDs are defined as

$$\hat{\boldsymbol{\mathcal{S}}}_r^{[\mathrm{s}]} = \hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} \times_r \hat{\boldsymbol{U}}_r^{[\mathrm{s}]} \in \mathbb{C}^{M_1 \times d \times d} \quad \forall r = 1,2,3 \tag{4}$$

2) In the 3-way case, we can construct up to 6 JEVD problems in the SECSI framework from eq. (4) [7]. As an example, we consider the two JEVD problems constructed from the 3-mode, i.e., $\hat{\boldsymbol{\mathcal{S}}}_3^{[\mathrm{s}]} = \hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} \times_3 \hat{\boldsymbol{U}}_3^{[\mathrm{s}]}$. We define the 3-mode slices of $\hat{\boldsymbol{S}}_3$ as

$$\hat{\boldsymbol{S}}_{3,k} \triangleq \hat{\boldsymbol{\mathcal{S}}}_3^{[\mathrm{s}]} \times_3 \boldsymbol{e}_{M_3,k}^{\mathrm{T}} \in \mathbb{C}^{d \times d}, \quad k = 1,2,...M_3, \tag{5}$$

where $\boldsymbol{e}_{M_3,k} \in \{0,1\}^{M_3 \times 1}$ denotes the k-th standard basis vector and $\hat{\boldsymbol{S}}_{3,k}$ represents the k-th slice (along the third dimension) of $\hat{\boldsymbol{\mathcal{S}}}_3^{[\mathrm{s}]}$. Moreover, these slices satisfy

$$\mathrm{Diag}\left\{\hat{\boldsymbol{F}}^{(3)}(k,:)\right\} \approx \hat{\boldsymbol{T}}_1^{-1} \cdot \hat{\boldsymbol{S}}_{3,k} \cdot \hat{\boldsymbol{T}}_2^{-1}, \tag{6}$$

where $\hat{\boldsymbol{T}}_1$ and $\hat{\boldsymbol{T}}_2$ are transformation matrices obtained by solving the associated JEVD problems.

3) Next, we select the slice of $\hat{\boldsymbol{\mathcal{S}}}_3^{[\mathrm{s}]}$ with the lowest condition number, i.e., $\hat{\boldsymbol{S}}_{3,p}$ where $p = \mathrm{argmin}_k \left\{\mathrm{cond}\left(\hat{\boldsymbol{S}}_{3,k}\right)\right\}$ and $\mathrm{cond}(\cdot)$ denotes the condition number operator. This leads to two sets of matrices for the 3-mode, namely the right-hand-side (rhs) set and the left-hand-side (lhs) set that are defined as

$$\hat{\boldsymbol{S}}_{3,k}^{\mathrm{rhs}} \triangleq \hat{\boldsymbol{S}}_{3,k} \cdot \hat{\boldsymbol{S}}_{3,p}^{-1}, \quad \forall k = 1,2,\ldots,M_3 \tag{7}$$

$$\hat{\boldsymbol{S}}_{3,k}^{\mathrm{lhs}} \triangleq \left(\hat{\boldsymbol{S}}_{3,p}^{-1} \cdot \hat{\boldsymbol{S}}_{3,k}\right)^{\mathrm{T}}, \quad \forall k = 1,2,\ldots,M_3. \tag{8}$$

Using the results in eq. (6), it is easy to show that the two sets of matrices in eq. (7) and eq. (8) correspond to the following JEVD problems

$$\hat{\boldsymbol{S}}_{3,k}^{\mathrm{rhs}} \approx \hat{\boldsymbol{T}}_1 \cdot \hat{\boldsymbol{D}}_{3,k} \cdot \boldsymbol{T}_1^{-1} \quad \forall k = 1,2,\ldots,M_3, \tag{9}$$

$$\hat{\boldsymbol{S}}_{3,k}^{\mathrm{lhs}} \approx \hat{\boldsymbol{T}}_2 \cdot \hat{\boldsymbol{D}}_{3,k} \cdot \hat{\boldsymbol{T}}_2^{-1}, \quad \forall k = 1,2,\ldots,M_3, \tag{10}$$

respectively, where the diagonal matrices $\hat{\boldsymbol{D}}_{3,k}$ are defined as

$$\hat{\boldsymbol{D}}_{3,k} \triangleq \mathrm{Diag}\left\{\hat{\boldsymbol{F}}^{(3)}(k,:)\right\} \cdot \mathrm{Diag}\left\{\hat{\boldsymbol{F}}^{(3)}(p,:)\right\}^{-1}. \tag{11}$$

The matrices $\hat{\boldsymbol{T}}_1$, $\hat{\boldsymbol{T}}_2$, and $\hat{\boldsymbol{D}}_{3,k}$ can be computed by an approximate joint diagonalization of the matrix slices $\hat{\boldsymbol{S}}_{3,k}^{\mathrm{rhs}}$ and $\hat{\boldsymbol{S}}_{3,k}^{\mathrm{lhs}}$. This can be achieved via joint diagonalization algorithms such as Sh-Rt [11] or JDTM [12].

4) Next we estimate the factor matrices from the JEVD results obtained for each mode. As as example, we discuss the factor matrices obtained from the 3-mode rhs. The factor matrix $\boldsymbol{F}^{(1)}$ can be estimated from the transform matrix $\hat{\boldsymbol{T}}_1$ via $\hat{\boldsymbol{F}}^{(1)} = \hat{\boldsymbol{U}}_1^{[\mathrm{s}]} \cdot \hat{\boldsymbol{T}}_1$. Then the factor matrix $\boldsymbol{F}^{(3)}$ is estimated from the diagonals of $\hat{\boldsymbol{D}}_{3,k}$ as $\hat{\hat{\boldsymbol{F}}}^{(3)}(k,:) = \mathrm{diag}\left(\hat{\boldsymbol{D}}_{3,k}\right)^{\mathrm{T}}, \forall k = 1,2,\ldots,M_3$. Finally, the factor matrix $\boldsymbol{F}^{(2)}$ is estimated via a linear LS fit as $\hat{\boldsymbol{F}}^{(2)} = [\boldsymbol{\mathcal{X}}]_{(2)} \cdot \left(\hat{\boldsymbol{F}}^{(3)} \diamond \hat{\boldsymbol{F}}^{(1)}\right)^{+\mathrm{T}}$.

Note that these three factor matrix estimates are obtained from 3-mode rhs JEVD problem. Similarly, another set of factor matrix estimates can be obtained by solving the lhs JEVD problem for $\hat{\boldsymbol{T}}_2$ and $\hat{\boldsymbol{D}}_{3,k}, \forall k = 1,2\ldots,M_3$ in eq. (10). Therefore, we obtain a total of six estimates for each factor matrix (two from each mode in eq. (4)). The final estimates can be selected by using the best matching scheme (exhaustive search) or one of the low-complexity heuristic alternatives that are discussed in [7].

III. First-Order Perturbation Analysis

In the following, we perform a first order perturbation analysis of a noise corrupted version of a low-rank tensor $\boldsymbol{\mathcal{X}}_0$. Due to space limitations, we do not provide the details of the derivations. But these derivations can be found in [13]. Throughout the work, the matrices with the $(\hat{\cdot})$ denote estimates computed from noise corrupted data, e.g., we use $\hat{\boldsymbol{F}}^{(1)}$, $\hat{\boldsymbol{F}}^{(2)}$, and $\hat{\boldsymbol{F}}^{(3)}$ to denote the factor matrix estimates. Moreover, we write all the noisy estimates as a function of the true matrices and the perturbations that are denoted by a "Δ term". For example, the noisy estimates in eq. (3) can be expressed as

$$\hat{\boldsymbol{U}}_r^{[\mathrm{s}]} \triangleq \boldsymbol{U}_r^{[\mathrm{s}]} + \Delta \boldsymbol{U}_r^{[\mathrm{s}]}, \quad \forall r = 1,2,3 \tag{12}$$

$$\hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} \triangleq \boldsymbol{\mathcal{S}}^{[\mathrm{s}]} + \Delta \boldsymbol{\mathcal{S}}^{[\mathrm{s}]}. \tag{13}$$

To calculate the perturbations in the final factor estimates, i.e., $\Delta \boldsymbol{F}^{(1)}$, $\Delta \boldsymbol{F}^{(2)}$, and $\Delta \boldsymbol{F}^{(3)}$, we need to calculate the perturbations in each of step involved in the SECSI framework, as discussed in Section II. In the first step, the perturbation in the r-mode signal subspace estimate $\hat{\boldsymbol{U}}_r^{[\mathrm{s}]}$ is given by [14]

$$\Delta \boldsymbol{U}_r^{[\mathrm{s}]} = \boldsymbol{U}_r^{[\mathrm{n}]} \cdot \boldsymbol{U}_r^{[\mathrm{n}]^{\mathrm{H}}} \cdot [\boldsymbol{\mathcal{N}}]_{(r)} \cdot \boldsymbol{V}_r^{[\mathrm{s}]} \cdot \boldsymbol{\Sigma}_r^{[\mathrm{s}]^{-1}} + \mathcal{O}(\Delta^2), \tag{14}$$

where $\boldsymbol{U}_r^{[\mathrm{n}]}$ contains a basis for left null space of $[\boldsymbol{\mathcal{X}}_0]_{(r)}$ and $\boldsymbol{\Sigma}_r^{[\mathrm{s}]}$ contains the singular values of $[\boldsymbol{\mathcal{X}}_0]_{(r)}$. The perturbation in the truncated core estimates can be obtained by expanding the expression $\hat{\boldsymbol{\mathcal{S}}}^{[\mathrm{s}]} = \boldsymbol{\mathcal{X}} \bigtimes_{r=1}^{3}{}_r \hat{\boldsymbol{U}}_r^{[\mathrm{s}]^{\mathrm{H}}}$ and using the relations in eq. (2) and eq. (12). This results in

$$\Delta\boldsymbol{\mathcal{S}}^{[\mathrm{s}]} = \boldsymbol{\mathcal{N}} \bigtimes_{r=1}^{3}{}_r \boldsymbol{U}_r^{[\mathrm{s}]^{\mathrm{H}}} + \mathcal{O}(\Delta^2)\,. \tag{15}$$

Note that all terms that include products of more than one "Δ term" are included in $\mathcal{O}(\Delta^2)$. Moreover, in our first-order perturbation analysis, $\boldsymbol{\mathcal{N}}$ is also considered as a "Δ term". The SECSI framework takes advantage of multiple JEVDs, i.e., up to 6 JEVDs can be constructed for a 3-way array, resulting in multiple estimates for each factor matrices (one factor matrix estimate from each JEVD). Due to space limitations, we only carry out the perturbation analysis for the factor matrices originating from the 3-mode rhs explicitly. However, a similar procedure can be adopted for the remaining factor matrix estimates. The noisy estimate for the 3-mode tensor $\hat{\boldsymbol{\mathcal{S}}}_3^{[\mathrm{s}]}$ can be expressed as

$$\hat{\boldsymbol{\mathcal{S}}}_3^{[\mathrm{s}]} \triangleq \boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]} + \Delta\boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]}. \tag{16}$$

Using eq. (4), eq. (12), and eq. (13), we get

$$\boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]} + \Delta\boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]} = \left(\boldsymbol{\mathcal{S}}^{[\mathrm{s}]} + \Delta\boldsymbol{\mathcal{S}}^{[\mathrm{s}]}\right) \times_3 \left(\boldsymbol{U}_3^{[\mathrm{s}]} + \Delta\boldsymbol{U}_3^{[\mathrm{s}]}\right).$$

This leads to

$$\Delta\boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]} = \Delta\boldsymbol{\mathcal{S}}^{[\mathrm{s}]} \times_3 \boldsymbol{U}_3^{[\mathrm{s}]} + \boldsymbol{\mathcal{S}}^{[\mathrm{s}]} \times_3 \Delta\boldsymbol{U}_3^{[\mathrm{s}]} + \mathcal{O}(\Delta^2). \tag{17}$$

Similarly, the perturbations in the k-th slice are defined as

$$\hat{\boldsymbol{S}}_{3,k} \triangleq \boldsymbol{S}_{3,k} + \Delta\boldsymbol{S}_{3,k}, \quad \forall k = 1,2,\ldots,M_3, \tag{18}$$

where

$$\Delta\boldsymbol{S}_{3,k} = \Delta\boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]} \times_3 \boldsymbol{e}_{M_3,k}^{\mathrm{T}} + \mathcal{O}(\Delta^2). \tag{19}$$

Next, we consider the perturbation in the 3-mode rhs set. To this end, the perturbation in $\hat{\boldsymbol{S}}_{3,k}^{\mathrm{rhs}}$ is defined as

$$\hat{\boldsymbol{S}}_{3,k}^{\mathrm{rhs}} \triangleq \boldsymbol{S}_{3,k}^{\mathrm{rhs}} + \Delta\boldsymbol{S}_{3,k}^{\mathrm{rhs}}, \quad \forall k = 1,2,\ldots,M_3. \tag{20}$$

Then we expand eq. (7), as

$$\boldsymbol{S}_{3,k}^{\mathrm{rhs}} + \Delta\boldsymbol{S}_{3,k}^{\mathrm{rhs}} = \left(\boldsymbol{S}_{3,k} + \Delta\boldsymbol{S}_{3,k}\right)\left(\boldsymbol{S}_{3,p} + \Delta\boldsymbol{S}_{3,p}\right)^{-1} + \mathcal{O}(\Delta^2).$$

By using Taylor's expansion of the matrix inverse, we get

$$\Delta\boldsymbol{S}_{3,k}^{\mathrm{rhs}} = \Delta\boldsymbol{S}_{3,k} \cdot \boldsymbol{S}_{3,p}^{-1} - \boldsymbol{S}_{3,k} \cdot \boldsymbol{S}_{3,p}^{-1} \cdot \Delta\boldsymbol{S}_{3,p} \cdot \boldsymbol{S}_{3,p}^{-1} + \mathcal{O}(\Delta^2), \tag{21}$$

for all $k = 1,2,\ldots,M_3$. The matrix $\hat{\boldsymbol{S}}_{3,k}^{\mathrm{rhs}}$ is defined by eq. (9) where $\hat{\boldsymbol{T}}_1$ and $\hat{\boldsymbol{D}}_{3,k}$ are computed by an approximate joint diagonalization algorithm such as Sh-Rt or JDTM. These algorithms are based on the indirect least squares (LS) cost function. In [10], we have already presented a first order perturbation analysis of such JEVD algorithms. We can use those results to obtain analytical expressions for the perturbation in the estimates $\hat{\boldsymbol{T}}_1$ and $\hat{\boldsymbol{D}}_{3,k}, \forall k = 1,2,\ldots,M_3$. In the same fashion as in [10], we define the following matrices

$$\boldsymbol{B}_0 = \boldsymbol{J}_{(d)} \cdot \left(\boldsymbol{T}_1^{\mathrm{T}} \otimes \boldsymbol{T}_1^{-1}\right) \tag{22}$$

$$\boldsymbol{s}_k = \mathrm{vec}\left\{\Delta\boldsymbol{S}_{3,k}^{\mathrm{rhs}}\right\} \tag{23}$$

$$\boldsymbol{A}_k = \boldsymbol{J}_{(d)} \cdot \left[\left(\boldsymbol{I}_M \otimes \boldsymbol{T}_1^{-1} \cdot \boldsymbol{S}_{3,k}^{\mathrm{rhs}}\right) - \left(\boldsymbol{D}_{3,k} \otimes \boldsymbol{T}_1^{-1}\right)\right], \tag{24}$$

where $\boldsymbol{J}_{(d)} \in \{0,1\}^{d^2 \times d^2}$ is a selection matrix that satisfies the relation $\mathrm{vec}\{\mathrm{Off}(\boldsymbol{X})\} = \boldsymbol{J}_{(d)} \cdot \mathrm{vec}\{\boldsymbol{X}\}$, for any given $\boldsymbol{X} \in \mathbb{C}^{d \times d}$. We further arrange these quantities as in [10]

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_{M_3} \end{bmatrix}, \quad \boldsymbol{B} = \boldsymbol{I}_{M_3} \otimes \boldsymbol{B}_0, \quad \boldsymbol{s} = \begin{bmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \vdots \\ \boldsymbol{s}_{M_3} \end{bmatrix}, \tag{25}$$

to get

$$\mathrm{vec}\{\Delta\boldsymbol{T}_1\} = -\boldsymbol{A}^+ \cdot \boldsymbol{B} \cdot \boldsymbol{s} + \mathcal{O}(\Delta^2) \tag{26}$$

$$\Delta\boldsymbol{D}_{3,k} = \mathrm{Ddiag}\left(\boldsymbol{T}_1^{-1} \cdot \Delta\boldsymbol{S}_{3,k}^{\mathrm{rhs}} \cdot \boldsymbol{T}_1\right) + \mathcal{O}(\Delta^2), \tag{27}$$

for the rhs JEVD problem. Finally, we calculate the perturbations in the factor matrix estimates. The factor matrix $\boldsymbol{F}^{(1)}$ is estimated from the transform matrix $\hat{\boldsymbol{T}}_1$ via $\hat{\boldsymbol{F}}^{(1)} = \hat{\boldsymbol{U}}_1^{[\mathrm{s}]} \cdot \hat{\boldsymbol{T}}_1$. We expand this equation as

$$\boldsymbol{F}^{(1)} + \Delta\boldsymbol{F}^{(1)} = \left(\boldsymbol{U}_1^{[\mathrm{s}]} + \Delta\boldsymbol{U}_1^{[\mathrm{s}]}\right) \cdot \left(\boldsymbol{T}_1 + \Delta\boldsymbol{T}_1\right) + \mathcal{O}(\Delta^2).$$

This results in

$$\Delta\boldsymbol{F}^{(1)} = \Delta\boldsymbol{U}_1^{[\mathrm{s}]} \cdot \boldsymbol{T}_1 + \boldsymbol{U}_1^{[\mathrm{s}]} \cdot \Delta\boldsymbol{T}_1 + \mathcal{O}(\Delta^2). \tag{28}$$

The kth row of the factor matrix $\boldsymbol{F}^{(3)}$ is estimated from the diagonal of $\hat{\boldsymbol{D}}_{3,k}, \forall k = 1,2,\ldots,M_3$, that leads to

$$\hat{\tilde{\boldsymbol{F}}}^{(3)}(k,:) = \tilde{\boldsymbol{F}}^{(3)}(k,:) + \mathrm{diag}\left(\Delta\boldsymbol{D}_{3,k}\right)^{\mathrm{T}} + \mathcal{O}(\Delta^2),$$

where the perturbation in $\hat{\tilde{\boldsymbol{F}}}^{(3)}(k,:)$ is defined as

$$\hat{\tilde{\boldsymbol{F}}}^{(3)}(k,:) = \tilde{\boldsymbol{F}}^{(3)}(k,:) + \Delta\tilde{\boldsymbol{F}}^{(3)}(k,:).$$

This results in

$$\Delta\tilde{\boldsymbol{F}}^{(3)}(k,:) = \mathrm{diag}\left(\Delta\boldsymbol{D}_{3,k}\right)^{\mathrm{T}} + \mathcal{O}(\Delta^2).$$

To get an expression of the corresponding factor matrix estimate $\hat{\boldsymbol{F}}^{(3)}$, we take into account eq. (11) via $\hat{\boldsymbol{F}}^{(3)} \triangleq \hat{\tilde{\boldsymbol{F}}}^{(3)} \cdot \mathrm{Diag}\left(\boldsymbol{F}^{(3)}(p,:)\right)$. This leads to

$$\Delta\boldsymbol{F}^{(3)}(k,:) = \mathrm{diag}\left(\Delta\boldsymbol{D}_{3,k}\right)^{\mathrm{T}} \cdot \mathrm{Diag}\left(\boldsymbol{F}^{(3)}(p,:)\right) + \mathcal{O}(\Delta^2). \tag{29}$$

Lastly, the factor matrix $\boldsymbol{F}^{(2)}$ can be estimated via a LS fit

$$\hat{\boldsymbol{F}}^{(2)} = \left[\boldsymbol{\mathcal{X}}_0 + \boldsymbol{\mathcal{N}}\right]_{(2)} \cdot \left[\left(\boldsymbol{F}^{(3)} + \Delta\boldsymbol{F}^{(3)}\right) \diamond \left(\boldsymbol{F}^{(1)} + \Delta\boldsymbol{F}^{(1)}\right)\right]^{+\mathrm{T}}.$$

Since $\hat{\boldsymbol{F}}^{(2)} = \boldsymbol{F}^{(2)} + \Delta\boldsymbol{F}^{(2)}$, we solve above equation by using this relation and utilizing Taylor's expansion to get

$$\begin{aligned}\Delta\boldsymbol{F}^{(2)} = {}& -\boldsymbol{F}^{(2)} \cdot \left[\left(\Delta\boldsymbol{F}^{(3)} \diamond \boldsymbol{F}^{(1)}\right) + \left(\boldsymbol{F}^{(3)} \diamond \Delta\boldsymbol{F}^{(1)}\right)\right]^{\mathrm{T}} \\ & \cdot \left(\boldsymbol{F}^{(3)} \diamond \boldsymbol{F}^{(1)}\right)^{+\mathrm{T}} + [\boldsymbol{\mathcal{N}}]_{(2)} \cdot \left[\boldsymbol{F}^{(3)} \diamond \boldsymbol{F}^{(1)}\right]^{+\mathrm{T}} + \mathcal{O}(\Delta^2).\end{aligned} \tag{30}$$

Similar expressions for the lhs can be obtained via the procedure shown in [13]. The obtained expressions can also be used for the first and second mode estimates, since such estimates can be derived by applying the SECSI framework to a permuted version of $\boldsymbol{\mathcal{X}}$. For example, we can obtain the first mode estimates by applying the SECSI framework on the third mode of $\mathrm{permute}(\boldsymbol{\mathcal{X}}, [2,3,1])$, where $\mathrm{permute}\{\cdot\}$ operator rearranges the dimensions, as defined in Matlab. Similarly, the second mode estimates are obtained by using the SECSI framework on the third mode of $\mathrm{permute}(\boldsymbol{\mathcal{X}}, [1,3,2])$.

$\mathrm{vec}\{\cdot\}$	compute
$[\Delta\boldsymbol{\mathcal{S}}^{[\mathrm{s}]}]_{(1)}$	$\boldsymbol{L}_0 = \left(\boldsymbol{U}_2^{[\mathrm{s}]^{\mathrm{H}}} \otimes \boldsymbol{U}_3^{[\mathrm{s}]^{\mathrm{H}}} \otimes \boldsymbol{U}_1^{[\mathrm{s}]^{\mathrm{H}}}\right)$
$[\Delta\boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]}]_{(1)}$	$\boldsymbol{L}_1 = \boldsymbol{P}_{(d,d,M_3)}^{(3)^{\mathrm{T}}} \cdot \left(\boldsymbol{I}_{d^2} \otimes \boldsymbol{U}_3^{[\mathrm{s}]}\right) \cdot \boldsymbol{P}_{(d,d,d)}^{(3)} \cdot \boldsymbol{L}_0 + \boldsymbol{P}_{(d,d,M_3)}^{(3)^{\mathrm{T}}} \cdot \left([\boldsymbol{\mathcal{S}}^{[\mathrm{s}]}]_{(3)}^{\mathrm{T}} \cdot \boldsymbol{\Sigma}_3^{[\mathrm{s}]^{-1}} \boldsymbol{V}_3^{[\mathrm{s}]^{\mathrm{T}}} \otimes \boldsymbol{\Gamma}_3^{[\mathrm{n}]}\right) \cdot \boldsymbol{P}_{(M_1,M_2,M_3)}^{(3)}$
$\Delta\boldsymbol{S}_{3,k}$	$\boldsymbol{L}_2^{(k)} = (\boldsymbol{I}_d \otimes \boldsymbol{e}_{M_3,k}^{\mathrm{T}} \otimes \boldsymbol{I}_d) \cdot \boldsymbol{L}_1$
$\Delta\boldsymbol{S}_{3,k}^{\mathrm{rhs}}$	$\boldsymbol{L}_3^{(k)} = (\boldsymbol{S}_{3,p}^{-\mathrm{T}} \otimes \boldsymbol{I}_d) \cdot \boldsymbol{L}_2^{(k)} - (\boldsymbol{S}_{3,p}^{-\mathrm{T}} \otimes \boldsymbol{S}_{3,k} \cdot \boldsymbol{S}_{3,p}^{-1}) \cdot \boldsymbol{L}_2^{(p)}$
Stack $\boldsymbol{s}_k$	$\boldsymbol{L}_3 = \left[\boldsymbol{L}_3^{(1)} \quad \boldsymbol{L}_3^{(2)} \cdots \boldsymbol{L}_3^{(M_3)}\right]^T$
$\Delta\boldsymbol{T}_1$	$\boldsymbol{L}_4 = -\boldsymbol{A}^+ \cdot \boldsymbol{B} \cdot \boldsymbol{L}_3$
$\Delta\boldsymbol{F}^{(3)}(k,:)$	$\boldsymbol{L}_4^{(k)} = \boldsymbol{W}_{(d)}^{\mathrm{red}} \cdot \left(\mathrm{diag}\left(\boldsymbol{F}^{(3)}(p,:)\right) \cdot \boldsymbol{T}_1^{\mathrm{T}} \otimes \boldsymbol{T}_1^{-1}\right) \cdot \boldsymbol{L}_3^{(k)}$
$\Delta\boldsymbol{F}^{(3)}$	$\boldsymbol{L}_5 = \boldsymbol{Q}_{(M_3,d)}^{\mathrm{T}} \cdot \begin{bmatrix} \boldsymbol{L}_4^{(1)} \\ \vdots \\ \boldsymbol{L}_4^{(M_3)} \end{bmatrix} \quad \forall k = 1, 2, \cdots, M_3$
$\Delta\boldsymbol{F}^{(1)}$	$\boldsymbol{L}_6 = \left(\boldsymbol{I}_d \otimes \boldsymbol{U}_1^{[\mathrm{s}]}\right) \cdot \boldsymbol{L}_4 + \left(\boldsymbol{T}_1^{\mathrm{T}} \cdot \boldsymbol{\Sigma}_1^{[\mathrm{s}]^{-1}} \boldsymbol{V}_1^{[\mathrm{s}]^{\mathrm{T}}} \otimes \boldsymbol{\Gamma}_1^{[\mathrm{n}]}\right)$
$\Delta\boldsymbol{F}^{(2)}$	$\boldsymbol{L}_7 = \left(\left(\boldsymbol{F}^{(3)} \diamond \boldsymbol{F}^{(1)}\right)^+ \otimes \boldsymbol{I}_{M_2}\right) \cdot \boldsymbol{P}_{(M_1,M_2,M_3)}^{(2)} - \left(\left(\boldsymbol{F}^{(3)} \diamond \boldsymbol{F}^{(1)}\right)^+ \otimes \boldsymbol{F}^{(2)}\right) \cdot \boldsymbol{Q}_{(M_1 \cdot M_3,d)} \cdot \left(\boldsymbol{H}(\boldsymbol{F}^{(1)}, M_3) \cdot \boldsymbol{L}_5 + \boldsymbol{G}(\boldsymbol{F}^{(3)}, M_1) \cdot \boldsymbol{L}_6\right)$

Table I: Definitions of perturbations and matrices required in eq. (34).

IV. Closed-Form rMSFE expressions

In this section, we derive closed-form rMSFE expressions for the three factor matrices that are obtained from the 3-mode rhs. However, the extension to the lhs and other modes is straightforward and further details can be found in [13]. The rMSFE in the r-th factor matrix is defined as

$$\mathrm{rMSFE}^{(r)} = \mathbb{E}\left\{\frac{\left\|\boldsymbol{F}^{(r)} - (\boldsymbol{F}^{(r)} + \Delta\boldsymbol{F}^{(r)}) \cdot \tilde{\boldsymbol{P}}^{(r)} \cdot \boldsymbol{P}_{\mathrm{opt}}^{(r)}\right\|_{\mathrm{F}}^2}{\left\|\boldsymbol{F}^{(r)}\right\|_{\mathrm{F}}^2}\right\}, \tag{31}$$

where $\tilde{\boldsymbol{P}}^{(r)}$ is a diagonal matrix modeling the scaling ambiguity since the scaling ambiguity is only relevant for the perturbation analysis. The scaling ambiguity is inherent in the estimation of the loading matrices, since the CP decomposition is unique up to scaling and permutation. Moreover, $\boldsymbol{P}_{\mathrm{opt}}^{(r)}$ is the optimal column scaling matrix that resolve this ambiguity. Next, we vectorize the term $\boldsymbol{F}^{(r)} - (\boldsymbol{F}^{(r)} + \Delta\boldsymbol{F}^{(r)}) \cdot \tilde{\boldsymbol{P}}^{(r)} \cdot \boldsymbol{P}_{\mathrm{opt}}^{(r)}$. This results in

$$\begin{aligned} &\mathrm{vec}\left\{\boldsymbol{F}^{(r)} - (\boldsymbol{F}^{(r)} + \Delta\boldsymbol{F}^{(r)}) \cdot \tilde{\boldsymbol{P}}^{(r)} \cdot \boldsymbol{P}_{\mathrm{opt}}^{(r)}\right\} \\ &\approx \mathrm{vec}\left\{\boldsymbol{F}^{(r)} \cdot \mathrm{Ddiag}\left(\boldsymbol{F}^{(r)^{\mathrm{H}}} \cdot \Delta\boldsymbol{F}^{(r)}\right) \cdot \boldsymbol{K}_r^{-1} - \Delta\boldsymbol{F}^{(r)}\right\}, \end{aligned} \tag{32}$$

where $\boldsymbol{K}_r = \mathrm{Ddiag}\left(\boldsymbol{F}^{(r)^{\mathrm{H}}} \cdot \boldsymbol{F}^{(r)}\right)$. The above equation can be further simplified to

$$\begin{aligned} &\mathrm{vec}\left\{\boldsymbol{F}^{(r)} - (\boldsymbol{F}^{(r)} + \Delta\boldsymbol{F}^{(r)}) \cdot \tilde{\boldsymbol{P}}^{(r)} \cdot \boldsymbol{P}_{\mathrm{opt}}^{(r)}\right\} \\ &= \left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(r)}\right) \cdot \left(\boldsymbol{K}_r^{-1} \otimes \boldsymbol{I}_d\right) \cdot \boldsymbol{W}_{(d)} \cdot \\ &\quad \left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(r)^{\mathrm{H}}}\right) \cdot \mathrm{vec}\left\{\Delta\boldsymbol{F}^{(r)}\right\} - \mathrm{vec}\left\{\Delta\boldsymbol{F}^{(r)}\right\}, \end{aligned} \tag{33}$$

where $\boldsymbol{W}_{(d)} \in \{0,1\}^{d^2 \times d^2}$ is a selection matrix that selects the diagonal elements such that $\mathrm{vec}\{\mathrm{Ddiag}(\boldsymbol{Z})\} = \boldsymbol{W}_{(d)} \cdot \mathrm{vec}\{\boldsymbol{Z}\} \in \mathbb{C}^{d^2 \times 1}$ for any square matrix $\boldsymbol{Z} \in \mathbb{C}^{d \times d}$. Note that the resulting expression contains the vectorization of the perturbation in the respective factor matrix estimates. The perturbations in the three factor matrix estimates depend upon different perturbations, i.e., $\Delta\boldsymbol{U}_r^{[\mathrm{s}]}$, $\Delta\boldsymbol{\mathcal{S}}^{[\mathrm{s}]}$, $\Delta\boldsymbol{\mathcal{S}}_3^{[\mathrm{s}]}$, $\Delta\boldsymbol{S}_{3,k}$, $\Delta\boldsymbol{S}_{3,k}^{\mathrm{rhs}}$, and $\Delta\boldsymbol{D}_{3,k}$. Therefore, we vectorize all of these quantities in the following linear fashion

$$\mathrm{vec}\{\cdot\} = \boldsymbol{L}_{\{\cdot\}} \cdot \boldsymbol{n}_1 + \mathcal{O}(\Delta^2), \tag{34}$$

where $\boldsymbol{n}_1 = \mathrm{vec}\left\{[\boldsymbol{\mathcal{N}}]_{(1)}\right\}$ is the 1-mode noise vector. The results of all the vectorization are summarized in Table I. We have used the following definitions for the results presented in Table I,

- $\boldsymbol{W}_{(d)}^{\mathrm{red}} \in \{0,1\}^{d \times d^2}$ is a reduced dimensional diagonal elements selection matrix that selects only the diagonal elements, i.e., $\mathrm{vec}\{\mathrm{Ddiag}(\boldsymbol{Z})\} = \boldsymbol{W}_{(d)} \cdot \mathrm{vec}\{\boldsymbol{Z}\} \in \mathbb{C}^{d^2 \times 1}$ for any square matrix $\boldsymbol{Z} \in \mathbb{C}^{d \times d}$.
- $\boldsymbol{Q}_{(M_1,M_2)} \in \{0,1\}^{(M_1 \cdot M_2) \times (M_1 \cdot M_2)}$ is a permutation matrix that satisfies the relation $\mathrm{vec}\left\{\boldsymbol{Z}^{\mathrm{T}}\right\} = \boldsymbol{Q}_{(M_1,M_2)} \cdot \mathrm{vec}\{\boldsymbol{Z}\}$ for any $\boldsymbol{Z} \in \mathbb{C}^{M_1 \times M_2}$.
- The following relation for two matrices $\boldsymbol{X} = [\boldsymbol{x}_1, \cdots, \boldsymbol{x}_d] \in \mathbb{C}^{M_1 \times d}$ and $\boldsymbol{Y} = [\boldsymbol{y}_1, \cdots, \boldsymbol{y}_d] \in \mathbb{C}^{M_2 \times d}$ holds for the vectorization of Khatri-Rao products.

  $$\mathrm{vec}\{\boldsymbol{X} \diamond \boldsymbol{Y}\} = \boldsymbol{G}(\boldsymbol{X}, M_2) \cdot \mathrm{vec}\{\boldsymbol{Y}\} = \boldsymbol{H}(\boldsymbol{Y}, M_1) \cdot \mathrm{vec}\{\boldsymbol{X}\},$$

  where

  $$\boldsymbol{G}(\boldsymbol{X}, M_2) \triangleq \begin{bmatrix} (\boldsymbol{x}_1 \otimes \boldsymbol{I}_{M_2}) \cdot \left(\boldsymbol{e}_{d,1}^{\mathrm{T}} \otimes \boldsymbol{I}_{M_2}\right) \\ \vdots \\ (\boldsymbol{x}_d \otimes \boldsymbol{I}_{M_2}) \cdot \left(\boldsymbol{e}_{d,d}^{\mathrm{T}} \otimes \boldsymbol{I}_{M_2}\right) \end{bmatrix}$$
  $$\boldsymbol{H}(\boldsymbol{Y}, M_1) \triangleq \begin{bmatrix} (\boldsymbol{I}_{M_1} \otimes \boldsymbol{y}_1) \cdot \left(\boldsymbol{e}_{d,1}^{\mathrm{T}} \otimes \boldsymbol{I}_{M_1}\right) \\ \vdots \\ (\boldsymbol{I}_{M_1} \otimes \boldsymbol{y}_d) \cdot \left(\boldsymbol{e}_{d,d}^{\mathrm{T}} \otimes \boldsymbol{I}_{M_1}\right) \end{bmatrix}.$$

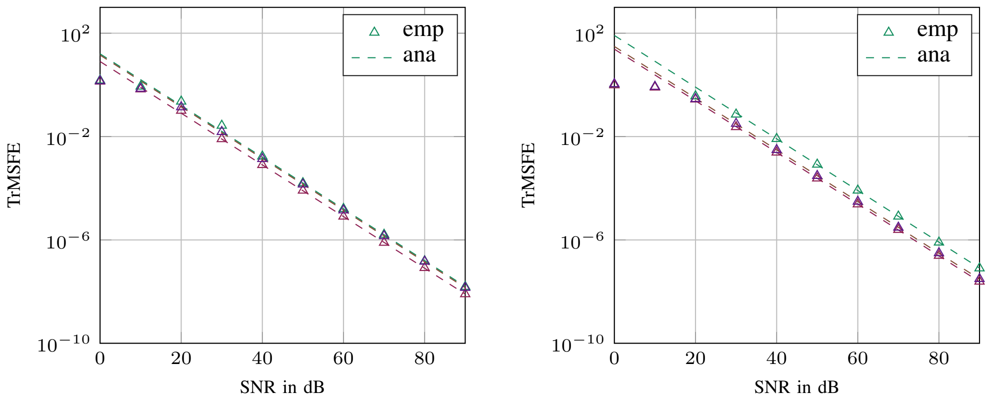
Once we have derived the expressions for the vectorization of the perturbation in the respective factor matrix estimates, we can further simplify eq. (33) in the following fashion

$$\mathrm{vec}\left\{\boldsymbol{F}^{(r)} - (\boldsymbol{F}^{(r)} + \Delta\boldsymbol{F}^{(r)}) \cdot \tilde{\boldsymbol{P}}^{(r)} \cdot \boldsymbol{P}_{\mathrm{opt}}^{(r)}\right\} \approx \boldsymbol{L}_{\boldsymbol{F}_r} \cdot \boldsymbol{n}_1, \tag{35}$$

where the $\boldsymbol{L}_{\boldsymbol{F}_r}$ matrices for $r = 1, 2, 3$ are given by

$$\begin{aligned} \boldsymbol{L}_{\boldsymbol{F}_1} &= \left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(1)}\right) \left(\boldsymbol{K}_1^{-1} \otimes \boldsymbol{I}_d\right) \boldsymbol{W}_{(d)} \left[\left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(1)^{\mathrm{H}}}\right) \boldsymbol{L}_6\right] - \boldsymbol{L}_6, \\ \boldsymbol{L}_{\boldsymbol{F}_2} &= \left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(2)}\right) \left(\boldsymbol{K}_2^{-1} \otimes \boldsymbol{I}_d\right) \boldsymbol{W}_{(d)} \left[\left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(2)^{\mathrm{H}}}\right) \boldsymbol{L}_7\right] - \boldsymbol{L}_7, \\ \boldsymbol{L}_{\boldsymbol{F}_3} &= \left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(3)}\right) \left(\boldsymbol{K}_3^{-1} \otimes \boldsymbol{I}_d\right) \boldsymbol{W}_{(d)} \left[\left(\boldsymbol{I}_d \otimes \boldsymbol{F}^{(1)^{\mathrm{H}}}\right) \boldsymbol{L}_5\right] - \boldsymbol{L}_5. \end{aligned}$$

Finally, the closed-form expressions for the three factor matrices $\mathrm{rMSFE}_{\boldsymbol{F}_r}$ are approximated by using eq. (35) in eq. (31) that leads



(a) Scenario I: a real valued tensor of size $5 \times 8 \times 7$ and $d = 3$

(b) Scenario II: a complex valued tensor of size $3 \times 17 \times 70$ and $d = 3$

Figure 1: TrMSFE of the 3-mode rhs estimates using the SECSI framework

to the following general expression

$$\text{rMSFE}_{\boldsymbol{F}_r} = \frac{\text{tr}\left(\boldsymbol{L}_{\boldsymbol{F}_r} \cdot \boldsymbol{R}_{\text{nn}}^{(1)} \cdot \boldsymbol{L}_{\boldsymbol{F}_r}^{\text{H}}\right)}{\left\|\boldsymbol{F}^{(r)}\right\|_{\text{F}}^{2}}, \tag{36}$$

where $\boldsymbol{R}_{\text{nn}}^{(1)} \triangleq \mathbb{E}\{\boldsymbol{n}_1 \cdot \boldsymbol{n}_1^{\text{H}}\}$ is the 1-mode noise covariance matrix.

V. Simulation Results

In this section, we evaluate the performance using the derived analytical results and compared them to empirical simulations. For this purpose, we show results for two scenarios. In the first scenario, a real valued tensor of size $5 \times 8 \times 7$ and $d = 3$ is used, while a complexed valued tensor of size $3 \times 15 \times 70$ and $d = 3$ is used in scenario 2. However, we have used real and complex valued JDTM algorithms for scenario 1 and scenario 2, respectively. The simulations are carried for 5000 trials. The noise tensor $\boldsymbol{\mathcal{N}}$ is randomly generated and has uncorrelated zero-mean Gaussian entries with variance $\sigma_N^2 = \|\boldsymbol{\mathcal{X}}_0\|_{\text{H}}^2/(\text{SNR} \cdot M)$. We plot three realizations of the experiments (therefore different $\boldsymbol{\mathcal{X}}_0$) on top of each other to provide a better insight of the algorithm. The results are shown in the form of the Total rMSFE (TrMSFE) since it reflects the total factor matrix estimation accuracy of the tested algorithms. The TrMSFE is defined as $\text{TrMSFE} = \sum_{r=1}^{3} \text{rMSFE}^{(r)}$. The results in Fig. 1 show an excellent match between the empirical results obtained using Monte-Carlo simulations and the analytical results obtained from the proposed first-order perturbation analysis.

VI. Conclusions

In this work, we have performed a first-order perturbation analysis of the SECSI framework for the approximate CP decomposition of 3-way noise-corrupted low-rank tensors. We provide closed-form expressions for the rMSFE for each of the estimated factor matrices. To obtain the closed-form expression of the rMSFE for each factor matrix, we derive the first-order perturbations of all intermediate outcomes at every step involved in the SECSI framework. The simulation results depict the excellent match between the closed-form expressions and the empirical results for both real and complex valued data.

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