Learn Algebra

MathopediaForAI Tutorial 1.1

This is a tutorial that develops a firm foundation of algebra starting from basics such as equations and techniques for solving simultaneous topics to other intermediate level topics such as conics, complex numbers, series, mathematical proofs, types of mean, types of inequalities, and other interesting problems solving strategies.

1. Solving Equations - Recap of Basic Techniques

Typically, linear equations can be solved using 2 basic methods, substitution and elimination. Substitution is when one variable is expressed in terms of another and then substituted to find out both the variables. Elimination is when given a set of simultaneous equations, one (or more) variables are eliminated by performing basic operations (addition, subtraction, multiplication or division) on different equations. Ultimately, both these techniques work in sync with one another - sometimes substitution may lead to the elimination of some variables and elimination might result in simpler substitution forms.

However, not all simultaneous equations have solutions. Based on this, simultaneous equations can be divided into multiple categories with regards to the equations they contain. The two main ways of classifying simultaneous equations are:

1. **Consistency**

a. Consistent system of equations have at least one set of solutions for the defined system

b. Inconsistent system of equations do not have any common solutions

2. **Dependency**

a. An equation that is a part of a system is dependent on, and can be obtained from, another equation in the same system. A dependent equation can also be removed from the system as it is simply a different expression of an already existing equation.

b. Independent equations are unique in nature and cannot be obtained by performing any basic mathematical operations on the existing equations

Apart from the basic mathematical operations, if any other operator is applied for solving a set of equations (for example, mod, log, exponents, etc.), it is important to ensure that no extraneous solutions are obtained and that all obtained solutions satisfy the given equation(s).

Other Strategies:

- For solving non-linear sets of simultaneous equations, elimination is an effective technique as it would allow the simplification of equations and remove several exponential terms that are otherwise hard to deal with.
- Certain systems of equations can be simplified by multiplying/dividing all equations by a common term, especially if some of them are equated to 0.
- Look out for expressions which seem to be common in multiple equations and try to eliminate these directly instead of focusing on each individual variable separately.
- For equations with polynomials, factorization can always help!
- For problems that require values of a certain expression, it may not be necessary to obtain values of each individual variable.

- While implementing elimination / substitution, look for the variable which has the easiest substitution available or can be eliminated most easily.
- You can also assign a variable to certain repetitive expressions.

2. Functions

Functions are mathematical algorithms which map every input x to a defined output $f(x)$ which is always mathematically dependent on the input. Every function has a definition which consists of a defining expression, domain, and range. The expression corresponds to the mathematical link between x and $f(x)$, domain corresponds to the set of all values of x for which f is defined and range is the set of all values of $f(x)$ which can be obtained with the given domain and definition. Thus, any general function can be defined in the following format.

$$f(x) := [\text{expression}], x \in D, f(x) \in R$$

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The Vertical Line Test

Any defined relation must clear the vertical line test to be considered a function. This means f is a function if for any values of a and b, $a = b \iff f(a) = f(b)$ where $a, b \in \text{Domain of } f$. Geometrically, this means that for any graph that corresponds to a function, every x value can be mapped to at most 1 y value. Thus, if a vertical line was to be drawn at any $x \in D$ on the graph of f, it will intersect the graph at maximum one point.

Graph Transformations:

Simpler functions plotted on the cartesian plane can often be transformed to other complex versions of themselves. For instance, a function $f(x) = x^2$ can be changed and instead be defined as $y = f(x + k)$. This would also lead to certain characteristics changes on the graph of f. In general, the rules of graph transformations are defined for all $k > 0$. Assuming this, we have that $y = f(x) + k$ will be a vertically upward shift of the graph by k units, $y = f(x) - k$ will be a vertically downward shift by k units, $y = f(x + k)$ will be a horizontal shift towards left by k units and $y = f(x - k)$ will be a horizontal shift towards right by k units.

We also have rules related to reflection factor multiplication in graph transformations. In general, for $k > 0$, $y = -f(x)$ is reflection of $y = f(x)$ across the x-axis and $y = f(-x)$ is a reflection of $y = f(x)$ across the y-axis.

Composite Functions:

Composite functions are those that are defined by merging two existing functions with one another. They are represented as $(f \circ g)(x)$ which is equivalent to $f(g(x))$. To make notations simpler, if a function is to be composed with itself, we represent the number of times composition with an exponent. This means that $f(f(x))$ can be written as $f^2(x)$ and $f^5(x) = f(f(f(f(f(x)))))$.

Inverse Functions:

Inverse functions are those that invert the effect of one another. Mathematically, f and g are considered inverses of one another if the condition below is met.

$$g(f(x)) = x \text{ for all values of } x \text{ in the domain of } f, \text{ and}$$
$$f(g(x)) = x \text{ for all values of } x \text{ in the domain of } g.$$

Conventionally, if g is an inverse of f, it is represented as f^{-1}. Graphically, inverses also share an interesting relation graphically. If f has an inverse denoted as f^{-1}, then the graph of $y = f^{-1}(x)$ is a reflection of $y = f(x)$ over the line $y = x$.

3. Complex Numbers

Complex numbers are numbers that can include a combination of real and imaginary numbers in their definition (though a completely real number or a completely imaginary number is also part of the complex set!). They can be represented in the form $a + bi$ where a and b are real number and $i^2 = -1$. This form of expression complex numbers is also called rectangular notation since it can be used to plot complex numbers on the Argand plane. Thus, for a complex number defined as $z = a + bi$, $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$ where $\mathrm{Re}(z)$ represents the real part of z and $\mathrm{Im}(z)$ denotes the imaginary part of z.

Further, the conjugate of $z = a + ib$ is denoted as $\overline{a + bi}$ and is equal to $a - ib$ while its magnitude is denoted as $|a + bi|$ and is equal to $\sqrt{a^2 + b^2}$.

Operations on complex Numbers:

Addition operations for complex numbers can be defined as shown below.

$$(a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i.$$

Likewise, multiplication is defined as

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i.$$

And division is defined as

$$\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}} = \frac{w\overline{z}}{|z|^2}.$$

Other complex numbers identities:

Similarly, these operations can also be implemented amongst conjugates of a single or multiple complex numbers. While they can all be derived from the simple operations, it's always better to keep them ready in your complex number identities toolbox.

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z} \cdot \overline{w}$
3. $z\overline{z} = a^2 + b^2$
4. $\overline{\overline{z}} = z$
5. $z\overline{z} = |z|^2$
6. $|zw| = |z||w|$

We can also determine the real and imaginary parts of any complex using the following relationships.

1. $\mathrm{Re}(z) = \dfrac{z+\overline{z}}{2}$
2. $\mathrm{Im}(z) = \dfrac{z-\overline{z}}{2i}$

While these identities can be very useful in solving complex number equations, sometimes basic algebra techniques such as factoring, grouping, equating like coefficients, etc. can make problems much simpler as well. Also, whenever dealing with $|z|$, see if you can square both sides of the given equation and substitute $a^2 + b^2$ for $|z|$ and eliminate the square root sign completely.

4. Quadratics

Quadratic functions are degree-2 polynomials which can be written in the form $ax^2 + bx + c$ where $a \neq 0$. The roots of a quadratic function defined as $ax^2 + bx + c = 0$ can be derived by using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When analysing quadratic functions, the quadratic discriminant can be a great place to begin. In a quadratic polynomial defined as $ax^2 + bx + c$, the discriminant is the value of $b^2 - 4ac$. There could be 3 possibilities for this value:

1. $b^2 - 4ac \geq 0$: implies that the quadratic function $ax^2 + bx + c = 0$ would have at 2 real roots (roots can be repeated or distinct)
2. $b^2 - 4ac < 0$: implies that the quadratic function $ax^2 + bx + c = 0$ does not have any real roots
3. $b^2 - 4ac = 0$ implies that the quadratic function $ax^2 + bx + c = 0$ has a pair of repeated roots

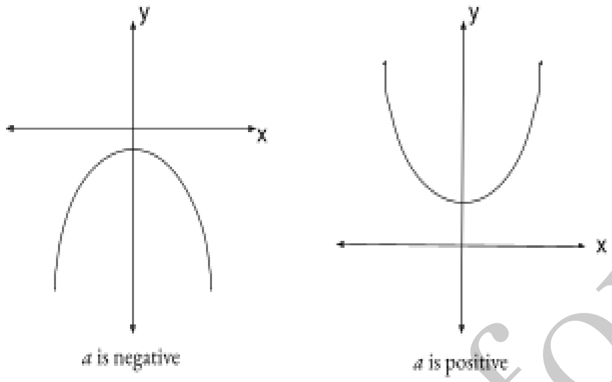
Another useful concept is Vieta's formulas. While the general formula proposed by Vieta is not limited to quadratic equations, a specific case for which $n = 2$ gives us the following results.

$$r + s = -\frac{b}{a} \qquad \text{and} \qquad rs = \frac{c}{a}$$

Where r and s are the roots of the quadratic equation $ax^2 + bx + c = 0$.

With these formulae, solving problems with asymmetric information about a given quadratic equation should become much simpler. For instance, if you are given one or two of the coefficients of the quadratic equation along with a root, Vieta's formulas can come in handy for determining the rest of the information about the quadratic as well.

Another useful tip is to remember that quadratic functions which do not have any real roots are typically those that do not intersect the x-axis at all.



These are the shapes of the graphs which will be obtained when a quadratic function with no real roots is plotted.

Completing the square is another way of solving a quadratic equation if the formula seems more complicated at times. In fact, completing the square is a technique which could be used to obtain the quadratic formula as well!

5. Conics

Conics can come in many different forms. The more popular classes of conics that we will be looking at in this section include parabolas, circles, ellipses and hyperbola.

1. Parabolas:

The general equation of a parabola is given as $y = a(x-h)^2 + k$. This is essentially an alternate form of a standard quadratic equation which is obtained upon completing the square. Parabola is a locus (a collection of points) which comprises all the points equidistant from the focus and the directrix. The directrix is a line drawn such that the line which is perpendicular to it and also passes through the focus will be the line of symmetry of the parabola.

2. Circle:

The general equation of a circle is given as $(x-h)^2 + (y-k)^2 = r^2$. Quite simply, a circle is a locus which comprises all the points that are equidistant from a single point, i.e. the center. This distance is the radius of the circle and is denoted by r in the equation of the circle.

3. Ellipse:

The general equation of an ellipse is given as $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. This locus has two foci, F_1 and F_2 and it comprises all the points for which the sum of their distances from the two foci remain the same. Also, ellipses typically have a major axis and a minor axis, both of which are essentially chords in the ellipse. While the major axis is the chord that connects the two foci, the minor axis is the line that goes through the center and is perpendicular to the major axis.

4. Hyperbola

Hyperbolas can be broadly of two types - vertically opening hyperbola and horizontally opening hyperbola. The general equation for a horizontally opening parabola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ while the general equation for a vertically opening parabola is $\frac{(y-h)^2}{a^2} - \frac{(x-k)^2}{b^2} = 1$. The center for both such hyperbolas with these general equations can be given by (h, k). Hyperbola is a locus which comprises all the points for which the difference between the distances from F_1 and F_2 stays constant.

These 4 loci are also known as conic sections because they represent the different sections that are obtained when a cone is 'sliced' at different angles. For instance, a circle would represent the section obtained when a horizontal plane cuts a right-angled cone and a hyperbola would represent the section obtained when a vertical plane cuts a right-angled cone.

Another clue to be mindful of lies in the quadratic formula itself - for problems involving quadratic optimisation, remember that the minimum or maximum point of a parabola *always* occurs at its vertex. Since the vertex is also the only point of the parabola which lies on the line of symmetry, the x-coordinate of this point will always be $\frac{-b}{2a}$.

6. Polynomial Division

To understand polynomial division, we must define a polynomial mathematically first. Conventionally, a polynomial is defined as $a_n x^n + a_{n-1}x^{n-1} + a_{n-1}x^{n+2} + \ldots + a_1 x + a_0$. Polynomial division can be done in two distinct methods: long division and synthetic division. Long division is slightly more time taking than synthetic division but it can always be used regardless of the degree of the dividend and divisor. On the other hand, synthetic division can only be implemented when the divisor is of degree 1 (i.e. the denominator of the polynomial is linear).

Polynomial division also introduces the Remainder Factor theorem. The remainder factor theorem states that the remainder obtained through polynomial division (using either of the methods mentioned above) will always be equal to the negative of the number in the linear divisor. Simply put, if $(x - a)$ is dividing some polynomial given by $y = f(x)$, then the remainder of the division will be equal to $f(a)$. Reflecting upon this, we can also see how this may work out for factors. If $(x - b)$ is factor of a polynomial given by $y = g(x)$, then the remainder obtained by dividing

$g(x)$ by $(x-b)$ will be $f(b)$. We also know that $f(b)=0$ since we have assumed $(x-b)$ is a factor (which implies that b is a root of $g(x)=0$). Thus, all conditions are met and the theorem is validated for factors.

<u>Key tip</u>: Synthetic division (or even long division for that matter) has a major pothole which usually tricks students - missing terms. Make sure you consider the missing terms appropriately and draw the synthetic division table accordingly!

7. Polynomial Roots

Polynomial roots are essentially the roots of the equation $f(x)=0$ given that $f(x)$ is a polynomial. There are several theorems associated with this idea which can help us identify these roots with minimal information from the question.

1. **Factor Theorem:** Defines the behavior of a function when there exists a number a (real or complex) that is a root, it can be said that $(x-a)$ is a factor.
2. **Fundamental Theorem of Algebra:** every one-variable polynomial of degree n has exactly n complex roots
3. **Rational root theorem:** For a polynomial defined as $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ with roots $r_1,r_2,r_3,\ldots,r_n$, the polynomial can be re-written as $f(x)=a_n(x-r_1)(x-r_2)\ldots(x-r_n)$.

<u>Key tip</u>: Remember that if asked to find the roots of a particular polynomial, mathematical inspection can be used to find the first few roots proceeding which, the factors obtained from these roots can be used as divisors for the original polynomial. At the end, the quotient obtained after long division (or synthetic division) can then be tackled more easily to find the remaining roots. For example: To find the roots of a cubic polynomial, use inspection to determine the first root, then use synthetic division to obtain a quadratic quotient. Finally, use the quadratic formula to obtain the last two roots.

Another interesting theorem about polynomials is the **Identity Theorem**. According to this theorem, for a set of polynomials with degree n (let's call them $p(x)$ and $q(x)$), if there are $n+1$ number of x for which $p(x) = q(x)$, then $p(x) = q(x)$ for all x.

This seems simple enough - let's delve into some of its implications.

We can use the identity theorem to identify if a given polynomial can be equated to a single constant term. Essentially, this would contract the entire polynomial's expansion.

For example, if we figure out that $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-1}x^{x-2} + \ldots + a_1 x + a_0$ is such that there exist at least n_1 values of x for which $p(x) = 0$, then $p(x) = 0$, i.e. $p(x)$ is a constant polynomial.

Fun fact: Adrien Marie Legendre proved that no constant polynomial can ever give a prime number as an output for every integer input.
Keytip: Remember that to prove statements about polynomials proved by contradiction (otherwise known as indirect proofs) are often helpful as they prove the contrary of statements false.

Vieta's Formulas are also a popular method of analyzing the roots obtained for an N degree polynomial. The general statement states that for a polynomial defined as $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ which has roots listed as $r_1, r_2, r_3, \ldots, r_n$ we can generalize that

$$s_k = (-1)^k \frac{a_{n-k}}{a_n}$$