Cubing - Magic or math?

Cubing is a hobby many teens embrace. Most of us see it as a sequence of algorithms that magically solve the cube by sorting every color in its appropriate phase. But you know what's cooler than magic? Math (yep, Peter Parker said that). So, what makes cubing algorithms work so perfectly in sync with one another? Let's find out!

To start, we'll redefine cubing notation in terms of mathematical functions. Consider the following functions:

1. $R :=$ R up in 3x3 $\Rightarrow$ $R^{-1} :=$ R down in 3x3
2. $L :=$ L down in 3x3 $\Rightarrow$ $L^{-1} :=$ L up in 3x3
3. $U :=$ U clockwise in 3x3 $\Rightarrow$ $U^{-1} :=$ U anti-clockwise in 3x3
4. $D :=$ D clockwise in 3x3 $\Rightarrow$ $D^{-1} :=$ D anti-clockwise in 3x3
5. $F :=$ F clockwise in 3x3 $\Rightarrow$ $F^{-1} :=$ F anti-clockwise in 3x3
6. $B :=$ B clockwise in 3x3 $\Rightarrow$ $B^{-1} :=$ B anti-clockwise in 3x3

Let's look at some of the properties of a rubik's cube:

1. **<u>The Commutative Property:</u>**

   A commutative operation is an operation in which the order of the individual terms does not matter. For example, take addition: writing 2+3=5 is equivalent to writing 3+2=5. Now, in a solved Rubik's cube, first move the top layer of the cube (U) and then move the right side (R) of the same face. You can either take a snap of the current cube or use your memory to remember the current position of the cubicles (I personally recommend taking a snap to avoid confusion). Next, resolve the cube and do the same thing again. But this time, do the right move (R) before the top layer (U). Again, take a snapshot and after doing so, compare the 2 images and note that they are not the same.

2. **<u>Conjugations:</u>**

   Just like how complex numbers have conjugates, even Rubik's cube notations have conjugate pairs. Essentially, a conjugate notation in Rubik's cube undoes what the original notation had done. For example, if I do the moves R U in the same order, I can undo the same operation by performing U' R' and bring my cube back to its original state. The main question though is "How can we find the conjugate of a certain Rubik notation?". To answer that we can use a simple logical explanation :

3. **Disjoint Permutations:**

   In the little investigation that we just did above, we saw that Rubik's cubes are not always commutative. But like there are exceptions to every rule made in chemistry (for eg. All metals have high melting and boiling points except mercury) there are a few exceptions to this property of Rubik's cubes as well. For example if you move the front face (F) and then the back face (B) and then do the same thing while doing (B) before (F), you'd notice that the resulting pattern is the same in both the cases. Why? Because both the moves that we made did not have any of the cubicles in common.

4. **The Associative Property:**

   Rubik's cube notations in general are not associative but when we come to the depth of it we will realize that when we use the Grout Theory to explain Rubik's cubes, multiplication becomes Associative. In general according to the associative property, a*(b*c) is the same as b*(a*c) and c*(a*b). To put this more concretely, every permutation of a, b and c would give the same result when multiplied.

Introduction to Group Theory:

Group Theory is a concept that allows us to combine various objects to form a set known as a "Group". Essentially, it is quite similar to how arrays in Java work. The array would have several digit places for us to enter value in and then all these values are carried together whenever we call that array for use. Likewise, a Group contains several numbers (integers and non-integers) or in other words, a set of objects and a binary operator (*) which is used to differentiate between terms and also acts as a mathematical operator (we will shortly see what this operator does). In general, we can also perform logical and mathematical operations on the elements of the group that we are working with but there are certain rules and properties that we must follow to do so.

The Rubik Group:

Rubik's cube patterns fall under a specific type of group known as that Non-Singular Matrices group (The name sounds kind of scary but it's quite easy to understand!). We already saw in the previous post that Rubik cubes are not commutative and therefore we can also call this group a non-commutative or a Non-Abelian group.
Properties:

1. Generally, we use a capital letter to denote the name of a group and small letters to show the elements of a group.
2. When multiplied using the binary operator, two elements from the same group always give an output that is an element in the group.
3. The * operator is associative.
4. In a group, every element has an inverse that is relative to the operation * in a way such that $g*g^{-1} = e$. (Note that g^{-1} is the inverse of the element g in the group G).
5. The * operator essentially acts as a concatenating operator that denotes combinations of different Rubik's cube moves. For instance, if I say that I will do the moves fg from the Group, I could also write the same moves as f*g.

Inverses:

Like we saw before, the inverse of an element in a group is denoted as the element with -1 as an exponent. From the logic obtained in previous section, we also realize that $(f*g)^{-1} = f^{-1} * g^{-1}$. In the previous blog, we also noticed that for any set of moves on the cube that need to be reversed, we would have to repeat those steps in the reverse order (in the opposite direction). If that's a little hard to visualize, you can also think of it as the process of packing a box. While packing you first put in the items and then seal the box with tape. But while unpacking a box, you first remove the tape and then take out the items.
Permutations:

The different permutations of a cube can be represented in various formats some of which are the Cycle notation and the General Rubik's notation. Usually we prefer writing it in terms of cycle notations though because it provides more precise information in less space as it tells us which cubicle or color exactly is being moved and by how much.

Subgroups:

A Group in general consists of several elements and so if we take a few of these elements and bunch them up together, we get what is called a Subgroup. But why do we need subgroups for Rubik's cubes again?? Let me ask you a question before I answer that. Have you ever tried a self solving trick on a Rubik's cube? (A self solve is basically a sequence of moves that when performed repeatedly on a solved cube would make the cube look jumbled at certain point but would eventually solve it back) If you haven't tried it then I recommend you to take a solved cube and do the moves M U repeatedly and see what happens. But how does a cube get magically solved on repeating these moves? The secret is that if we repeat any subgroup sequence on a cube, it will restore its original state after some time (the original state need not always be the solved cube; if you start from a jumbled cube, you would get back to that exact jumble after a few cycles).

The Lagrange's Theorem:

But, how do we know the number of times that we must do the sequence to finish the self solve? For all we know it could be as large as 781621836 (Don't worry, that's just a random number; Though we could have very large numbers!) moves…. The answer was given by the very famous Lagrange's Theorem: For every self-solving algorithm, the number of cycles must be divisible by the total number of permutations of the cube and can be obtained by a mathematical formula involving co-sets.

In all, Lagrange's theorem, and Group Theory allow us to analyze the 'algorithms' which are designed to solve cubes. While these algorithms are based entirely on the individual permutation of the cube at each stage, the fact that they can be deconstructed by being represented as mathematical sequences is pretty cool! Once we know that each of them can be represented in so many different ways, we can also derive their conjugate pairs and inverses to unsolve the cube in specific patterns like a checkerboard, donut and so much more!

So, it turns out that the magic algorithm that helped you make the American flag on the cube is not so much magic, after all!