Learn Calculus

MathopediaForAI Tutorial 3.1

This is a tutorial that develops a firm foundation of calculus starting from a thorough revision of the pre-requisites such as set notations and functions and then covering advanced topics including limits, differential calculus and its applications, integral calculus and its applications and finally differential equations.

1. Revisiting Set notation

The common definition of a set is to think of it as a collection of numbers encapsulated by curly brackets. They can either be defined through a specific correlation / pattern, or they can be pre-defined sets that are used within a problem.

$$\text{Pre-defined set:} \mathbb{R}, \mathbb{Z}, \ldots$$
$$\text{Self-defined set:} \{1, 3, 5, 7, 9, \ldots, 1001\}, \{4, 8, 12, 16, 20, \ldots\}, \ldots$$

Note that sets can have a finite or infinite number of elements based on how they are defined. For instance, the real number set $\mathbb{R}$ is an infinite set as there are infinitely many infinite numbers but the first self-defined set shown in the example is a finite set because it has a countable number of elements within it.

Here are some examples of set notation being used to define different kinds of sets.

- $\{x|x \text{ is an integer}\}$
- $\{x|x \text{ is a real number and } x^2 < 0\} = \emptyset$ where $\emptyset$ is a set that contains no elements.

Set theory also has several key terms that are used to refer to the different characteristics of sets.

- A **subset** is a set that contains all the elements of its parent set. Essentially, if A is a subset of B, then A can be thought of as a part of B. This is mathematically represented as $A \subset B$.
- Intuitively, this also helps defining a **superset** as this is exactly the opposite of what a subset is. If A is a subset of B, then B must be the superset of A meaning that all the elements of A must be present in B.
- If it is given that A is a subset of B, and $A \neq B$, then A is also called a proper subset of B. This is represented as $A \subseteq B$.
- The **cardinality** of a set is defined as the number of elements in that set and is written as $\#(A)$.
- The **union** of sets stands for elements of A and B combined. So, if $C = A \cup B$, then C contains *all* the elements of A and B.
- The **intersection** of sets represents the set of elements common between the two sets which are intersecting. So, $\{1, 4, 7, 24, 6, 3, 23\} \cap \{2, 4, 7, 23\} = \{4, 7, 23\}$. Sets that have no common elements (i.e., their intersection gives a null set) are called relatively **disjoint**.

Bounds exist for some sets and they define the upper and lower most values within that set. This brings us to the idea of intervals and how they can be represented using interval notation. This notation consists of two main components: the brackets, which represent the nature of the bounds and the union of multiple intervals together to form a bigger, unique interval.

Moving on to the revision of functions. Here are some of the key terms you might want to revisit: domain, range, composite functions, inverse functions, function notations, and other basic terminology from Algebra.

2. Functions

Functions form the fundamental base for all of calculus. In this section, we'll take a look at some of the types of functions (although they are infinite in total) and explore their unique characteristics. Note that this part of the calculus course is linked to Mathopedia For AI's Algebra tutorials and requires basic understanding of function notation and terms (domain, range, inverse, etc.).

Trigonometric functions come up often in any calculus or geometry problem. As a quick review, the basic functions that we need to know about include $\sin, \cos, \text{ and } \tan$. Their inverses are $\sin^{-1}, \cos^{-1}, \text{ and } \tan^{-1}$ respectively. Although these might seem familiar, what's more interesting is that trigonometric functions give surprisingly neat relationships with other types of exponential functions. For instance,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is also commonly known as the Euler's form of complex numbers and beautifully links two seemingly distinct concepts of trigonometry and complex numbers to Euler's constant.

Another important thing to note is that trigonometric functions, by convention, use radians. Of course they are used with degrees, but when applying any of the formulae or identities, make sure your units are consistent with those in the formula. Here are some common and useful identities:

1. $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$
2. $\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$
3. $\tan x = \frac{\text{opposite}}{\text{adjacent}}$
4. $\sin^2 x + \cos^2 x = 1$
5. $\sin 2x = 2 \sin x \cos x$
6. $\cos 2x = 1 - 2 \sin x$
7. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
8. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Another useful concept related to functions is that of periodicity. In simple words, if a function oscillates in a particular way within the defined domain, then the function is called periodic. The simplest visualization for this can be the sine and cosine curves. By definition, in a periodic function, $f(x) = f(x + k)$ where k is the period of the function. Can you identify the value of k for a sine and cosine graph? What about the graph of $y = \tan x$?

3. Limits and continuity

Limits are typically calculated to understand what value a particular function is likely to generate as x approaches the specified value (which may be a number or infinity). Mathematically, the limit of $f(x)$ as x approaches a can be written as $L = \lim_{x \to a} f(x)$.

Also, limits can be evaluated in terms of which 'direction' we approach a particular point on a graph. More specifically, if we approach a particular point on a graph from the positive direction, then the limit can be represented as $\lim_{x \to a^+} f(x)$. Likewise, if the same point is approached from the negative direction then the limit is written as $\lim_{x \to a^-} f(x)$. In general, we say that the limit $\lim_{x \to a} f(x)$ exists at a particular point iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

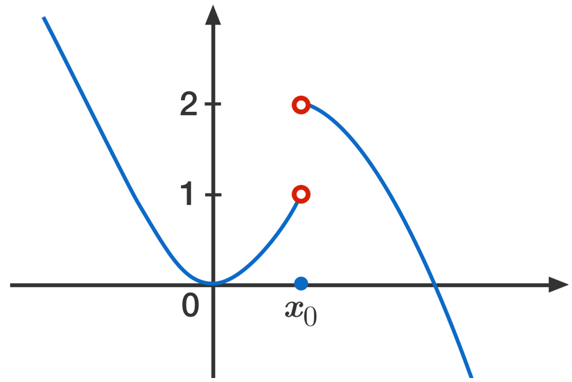


Figure 1: Linking the ideas of continuity and limits

The sketch above shows an example of when a general limit may not exist at x_0 exist although the function is defined at $x = x_0$. This happens in cases when the function is discontinuous or has a 'sharp turn' at the point for which the limit needs to be determined.

Limits can also be used to define derivatives in a unique method by considering the fact that a derivative is simply the instantaneous rate of change of a function, which can be calculated as the average rate of change between two of its points as the distance between these two points approaches 0.

This definition of a limit can also be used to define the continuous nature of certain graphs. A function $f(x)$ is considered to be continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$
Note that for this to happen, the limit from positive direction must be the same as that from the left too.

The concept of continuity gives rise to multiple ideas such as the Intermediate value theorem. This theorem states that given that $f(x)$ is continuous over $[a, b]$ and any random number M between $f(a)$ and $f(b)$, then there must exist a number c such that $a < c < b$, and $f(c) = M$. Here's how this looks graphically.

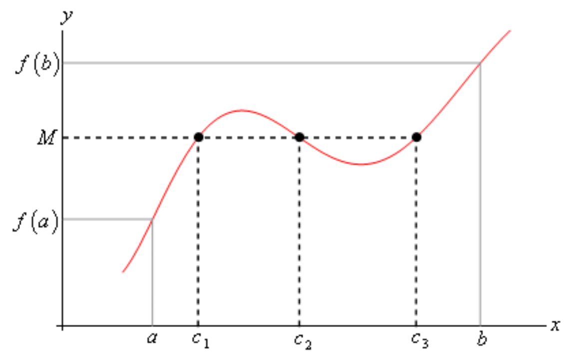


Figure 2: Graphical representation of Intermediate value theorem

Limits don't always need to be finite - in fact, limits involving infinity give us a special feature of graphs called asymptotes. In general, it can be said that:

$$\lim_{x\to a} f(x) = \infty \implies x = a \text{ is a vertical asymptote}$$

$$\lim_{x\to\infty} f(x) = b \implies y = b \text{ is a horizontal asymptote}$$

Like most calculus, limits have their own unique rules of algebra. Here are some identities related to limits that can come in handy whenever you need to do any manipulations in equations involving limits. Given that $lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$:

1. $\lim_{x\to a}(f+g)(x) = L + M$
2. If $c \in \mathbb{R}$, then $\lim_{x\to a}(cf)(x) = cL$
3. If $M \neq 0$, then $\lim_{x\to a}(f/g)(x) = L/M$

Related Theorems:

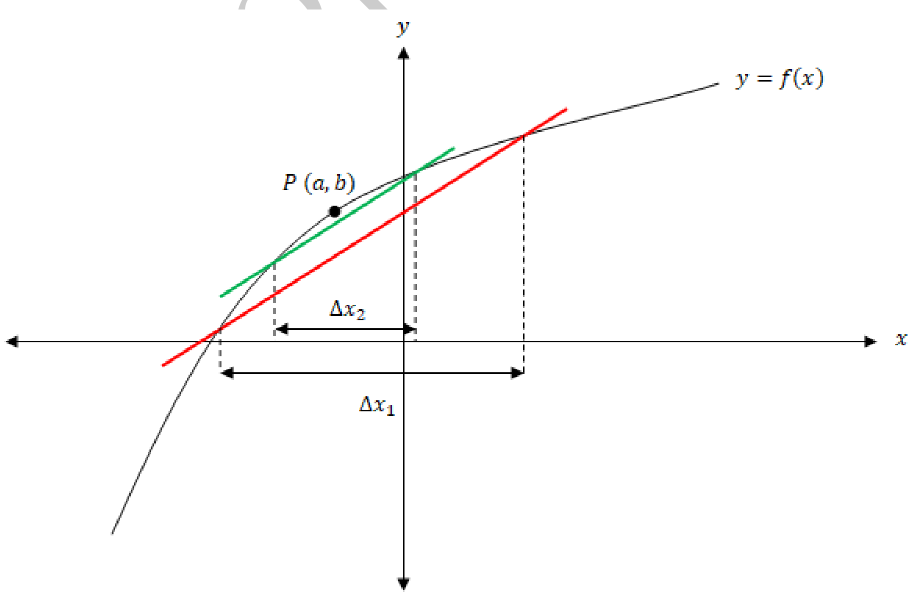
Squeeze Theorem: If f, g, h are real-valued functions such that $f(x) \leq g(x) \leq h(x)$ for x over an open interval that includes a and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Intermediate Value Theorem: Given that $f(x)$ is a continuous function and that $f(x)$ attains values a and b for some values of x, then $f(x)$ must also attain the value c where $a \leq c \leq b$.

Boundedness Theorem: If a function f is defined over a closed interval $[a, b]$, then the function must have a value $M \in \mathbb{R}$ which marks the maximum value attained by that function within that interval. The same applies for a minimum bound too. Note that the boundedness theorem only applies for functions defined over closed intervals. If a function is defined over an open interval, then it need not attain a maximum value over the domain since it might just exponentially rise near the open end. Example: if we define $g(x) = \frac{1}{x}; \{x \in (0, 1)\}$ then there is no maximum attained as g becomes infinitely large as $x \to 0$.

4. Derivatives and differentiation

Simply put, derivatives are a special kind of limit which generate a function that corresponds to the gradient function of a graph. To understand this better, we can consider a secant which connects two points on a graph as shown below.



Derivatives representation through secants

Notice that from the figure above, as the secants come closer and closer to $P(a, b)$, the value of $\triangle x$ will also reduce. Also, when $\triangle x$ becomes equal to 0, the secant line will no longer be cutting the graph at 2 points - it will only touch the graph at one point (i.e. P). The definition of a tangent is the line which is obtained as $\triangle x$ approaches 0. Note that the equation for such a line can be written as

$$g(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}.$$

The conventional notation is to represent the derivative of a function $y = f(x)$ as $f'(x)$ or $\frac{dy}{dx}$ and to write $\triangle x$ as h. Thus,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Reflecting back on this definition of a derivative, note that a function must be continuous over the given domain for it to have a derivative. Recall from before that if a function is not continuous, then it will not have a limit at the point of discontinuity as the left approaching limit will not be equal to the right approaching limit. Consequently, we can conclude that if a function is differentiable, it must be continuous. But is the converse also true?

No. Take the absolute function as an example: $y = f(x) = |x|$. At $(0, 0)$, this graph has a sharp turning point. To the left of this point the gradient is constant and -1 while to the right of this point the gradient is constant and +1. This means that the left hand limit is not equal to the right hand limit and therefore f is not differentiable. So, we can conclude that differentiability implies continuity but continuity alone does not always imply differentiability.

To summarize the link between differentiability and continuity, here are the different cases in which a function may not be differentiable:

1. Function is not continuous at that point

2. Function has a 'sharp corner' at that point
3. Function has a vertical tangent (i.e. gradient of the line is ∞)

Techniques for differentiation:

Differentiation is essentially an operation much like addition or subtraction - only more complex. To "differentiate" a function means to convert a function into a different form which gives the generalized equation for that function's gradient. This process involves several rules such as those listed below.

1. **Rule of linearity:** $(cf)' = c(f')$
2. **Exponents rule:**

$$\frac{d}{dx}x^n = nx^{n-1}$$

   This holds true for all real integer values of n except -1.

3. **Product rule:**

$$(fg)' = f'g + fg'$$

4. **Quotient rule:**

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

5. **Chain rule:** Given that $F(x) = f(g(x))$,

$$F'(x) = F'(g(x)) \cdot g'(x)$$

6. Inverse function rule:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

Knowing these rules along with the standard derivatives for trigonometric and logarithmic functions should give you a basic idea of differentiation. Here's a list of some of the standard derivatives that fit well in the differentiation toolkit.

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
6. $\frac{d}{dx}(\cot x) = -\csc^2 x$

Additional theorems:

1. **Rolle's theorem:**

   Given that f is continuous over $[a, b]$ and differentiable over (a, b), and that $f(a) = f(b)$ for some a and b then there must exist some c for which $f'(c) = 0$.

2. **Mean value theorem:**

   Given that f is continuous over $[a, b]$ and differentiable over (a, b) where $a < b$, there must exist some c for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

5. Implicit differentiation

The idea of implicit differentiation is based on multivariate equations which might have a complex form that does not allow for the isolation of a single variable or results in an even more complex form if isolation is done. In such cases, implicit differentiation allows us to differentiate both sides of the equation in which the desired derivative can be isolated and expressed in terms of one or more of the variables present in the original equation. The rules for implicit differentiation are similar to those of simple differentiation as both sides of the equation are to be differentiated with respect to the same variable (most commonly x).

Here's a simple example of how implicit differentiation might work.

Our aim is to differentiate the following equation and obtain $\frac{dy}{dx}$.

$$2x^2 = 4y^3 + x$$

Step 1: Differentiate both sides with respect to x.

$$\frac{d}{dx}(2x^2) = \frac{d}{dx}(4y^3 + x)$$

Step 2: Apply the differentiation formulas

$$4x = 12y^2\frac{dy}{dx} + 1$$

Step 3: Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{4x - 1}{12y^2}$$

The same 3 steps can be applied on any equation to differentiate implicitly. Note that in most cases implicit differentiation is not a need but rather an alternative which is used for simplification and convenience.