Unveiling the Math Behind Juggling - an orchestra of balls

Juggling is actually quite similar to a lot of our day to day experiences. Think about a simple tune like *Coffin dance* - doesn't it sound so much different when it's played with beats versus when it's played on a piano? Most songs have multiple instruments all playing in a synchronized manner to create a musical tone. In such performances, even if a single guitarist or pianist messes up, it can impact the whole performance ruining that perfectly mesmerizing tune.

Similarly, in juggling, every throw is an individual instrument and if even a single one of them is thrown at the wrong time or to the wrong height, chances are high that the entire sequence would become messy eventually leading to a drop.

Beats:

While throwing the balls (or any other object that one may juggle with), there is a set rhythm that one has to follow – much like that in a music piece. This 'tempo' is measured in terms of 'Beats'. Every beat is like a drum banging to signal that you must throw the next ball. You can also think of it as a march-past in which all throws are released at a regular interval of time. The more prompt your throws are, the better it looks and the lesser the chances of dropping.

Siteswap notations:

Throughout this series, we will be using some numbers and letters to denote particular Juggling patterns. Generally, these numbers represent the type of throw and also give a clue as to how high the throws will be. Further details about the same have been provided in the next section of this blog. As for now, all we need to know is that every throw has its own Siteswap notation.

Why are Notations and Beats Needed?

Apart from creating an 'Orchestra' or a synchronized 'March-past', in Juggling, the beats and Notations are also very helpful creating new patterns and also to analyze the pre-existing ones. The big question we ask ourselves at this point is "What does it mean to analyze a juggling pattern?". Well, the 'analysis' generally covers certain basic parameters such as:

- The total number of objects (for eg. balls or clubs) required to complete a pattern

- The period of the pattern (Or in other words, the number of different throws that the pattern accommodates)
- Will there ever be a collision while juggling?

Note that by the word 'collision', we are not referring to the balls 'colliding' with each other. 'Collision' in this context refers to the time when we are required to throw more than 1 ball at a time. If a pattern ever encounters a collision, it is not considered as a valid Juggling pattern. That being said, there are cases in which we have to throw multiple balls at a time (for eg: The fountain pattern using 4 balls), but in this series we will mainly be focusing on "Vanilla Siteswap patterns" which are basically patterns that will never have a 'collision'.

Expectations Versus Reality

When we see a juggler juggling objects, it almost seems as if they are 'throwing' the balls up in the air and although I have used the word 'throw' to describe the motion of the ball in most of the write-ups, that is theoretically wrong. On watching a live performance, or even while juggling objects yourself, it seems like we are throwing the ball up in the air and catching it after some time only to keep repeating the pattern. But actually, when analyzing juggling mathematically, we introduce another rule that contradicts most of our expectations:

For mathematical conventions, we assume that the objects that we seem to be 'throwing' while juggling are actually just being 'pushed' (against gravity) so that they don't fall onto the ground.

In simple terms, this means that the objects (for example- balls) are actually being 'caught' and 'thrown' at the same time, telling us that in every beat, we must be able to catch and throw the same ball or else the pattern will fail.

So far so good. But if throws don't take time in a pattern, then what does? And how do we define a pattern in terms of the time taken? For this, we introduce what is one of the most-spoken topics from classical physics - period.

The Period

A common misunderstanding

The term 'Period' in itself indicates that it has something to do with the length of the pattern. But that's wrong. Most people confuse the term "Period" with the period used in describing a pendulum. In a pendulum, the period is the amount of time taken to complete

one oscillation but in Juggling, a period has got absolutely nothing to do with the time or the length of the pattern.

What does a period tell us?

A period is basically just the number of digits that make up the Siteswap notation of a particular pattern. This means, a pattern with Siteswap notation 5 3 1 would have a period of 3 (because there are 3 digits in the notation). Likewise, a pattern with notation 3 would have a period of 1. But the question remains: What does a period tell us? Well, a period in itself doesn't actually tell us much about the "analysis" of the Juggling pattern but when used with other attributes such as the Beats and the Siteswap notations, it can help identify the validity of a pattern, the total number of objects required in the pattern and a lot more!

Tables for representing Siteswap Notation:

A table is just a visual representation of any data that we collect / observe related to a particular pattern. Much like how we draw tables with each column representing a certain variable in physics or any other science to compare the results and study the relation between the independent and dependent variables respectively. What are these variables? Here's a list:
1. Independent variable : No. of Beats, Siteswap Number
2. Dependent variable : The Object being thrown

At first sight, it seems rather counterintuitive to have the 'Object being thrown' as the Dependent variable; After all, at the end of the day we are just alternating between the 2 balls that are to be thrown right? Well, yes that is true but while considering it, we make an important assumption that leads to further confusion when moving on. Here, we are assuming that we precisely have 2 balls. What if I just have 1? Or 10? That's when the math changes things. A Lot.

Before we move on to learn how to draw a table, I am just going to introduce a quick concept. Imagine, as a Juggler, you throw up a ball in the air. How will you know when to throw it next? While some people prefer to call it their "muscle memory", we (Math Lovers) can also calculate this time by simple addition. For now, just remember that the time at which we throw a particular ball next is equal to the current Siteswap number of the ball added with the current beat at which it was thrown.

Next beat for throwing ball = Current beat + Current Siteswap Number

Current Beat	0	1	2	3	4	5	6	7	8	9
Current Siteswap No.	5	3	1	5	3	1	5	3	1	5
Object being thrown	A	B	C	C	B	A	A	B	C	C
Next beat for throwing	5+0=5	3+1=4	2+1=3	5+3=8	3+4=7	1+5=6	5+6=11	3+7=10	1+8=9	5+9=14

The Table above shows the sequential flow of the 3 balls – A, B and C – from Beat 0 to Beat 9 in the Siteswap sequence 5 3 1. Note that every beat has exactly 1 ball that is being thrown on it. If any beat was empty (i.e, no ball to throw on the beat is available), then we would introduce a new object D to fill in that space to make the pattern valid. If we had a situation in which we would have to throw 2 or more balls at the same beat, then the particular Siteswap sequence would not be considered a valid Vanilla Siteswap pattern.

A little bit of Math:

In this section, we will discuss a few of the mathematical formulae that can help us identify certain attributes of Siteswap sequences much more effectively and faster as they do not require one to make long tables (of course, one can also use Excel and other similar softwares but then again, nothing is as customisable as a pen and paper!).

The Average Theorem:

This one is quite simple. All you need to do is take the average of all the numbers present in the Siteswap notation and Boom! You will precisely get the total number of objects to satisfy the entire pattern. Cool right? But what if you get the average as 3.769? How can we juggle 3.769 objects? The answer is simple. We can't juggle a siteswap sequence if its average turns out to be decimal. This means that if a Siteswap sequence has a decimal average, then it is invalid.

However, we must keep in mind that the Average theorem is not a very reliable method of identifying valid Siteswap sequences because it only eliminates the sequences with a decimal average. So, as long as a sequence has an Integer average it can pass this test but remember, a sequence with the same numbers can also have different permutations! This means, if we take the sequence 5 3 1, we can rearrange the numbers like 1 3 5 or 3 5 1 etc. and although all of them have the same average, we will soon find out that not all of them are valid Siteswap sequences because some of them involve collisions.

Permutation and Reverse permutation:

The Permutation test is the solution to having a proof that a particular pattern is valid. This test works because unlike the average test (in which you just add all numbers in any random order and take average), it involves calculations that are affected by the position of the particular digit (or number) in the Siteswap sequence. A reverse permutation test is basically just an inverse function of the Permutation test because it follows the exact same steps in the reverse order. In simple words, if you plug in the output of the Permutation test as input in the Reverse Permutation test, you would get the input of the Permutation test as the output. The table below shows the steps of Permutation and Reverse Permutation test for better comparison:

Permutation Test	Reverse Permutation Test
Add the Siteswap number to the corresponding Beat number (Beat number starting with 0)	Take the result from the Permutation test and add or subtract any multiple of the sequence's period (P) to them
Take the Mod of the number obtained in Step 1 with respect to the period of the sequence	Take the average of all the numbers obtained after Step 1
Repeat the same for all the digits in the Siteswap sequence	If the average value is a multiple of 5, then the sequence is valid. If not, the sequence is invalid
If All numbers from 0 to the period of the sequence are present, then it is a valid sequence. Else, the sequence is invalid.	

Inventing Patterns

Once we have validated a pattern by using the Permutation test, we can create our own patterns using the validated pattern and be sure that those will be valid as well. For instance, if we take the Siteswap sequence 2 3 1, we can add any multiple of the period of the pattern to any of the digits and obtain a pattern that is also valid. Since the period (Let's call it P) is 3, we can add any multiple of 3 to any of the digits above to obtain valid Patterns. For example, 2 6 1, 5 3 1 and 2 3 7 are all valid Siteswap patterns because one of their

numbers has been added with a multiple of 3 (which is what the period of the sequence is). Of course, we can also add different (or even the same) multiples of P to all the digits in the sequence and that would still generate a valid Siteswap sequence.
That tells us how to create new patterns from existing ones and we have figured out the exact physical significance of individual siteswap numbers. We also know that Sideswap numbers are always non – negative, so we can safely say that we have covered all the possible cases for a sideswap number. Or can we?

A Special Case:

If a sideswap number is "Non – Negative", doesn't that mean it can be any number that's not negative? So, technically speaking, a sideswap number can also be 0. But what does a sideswap number 0 mean? Does it mean we throw the object with the height of 0? Yes. If a siteswap sequence contains the value 0, it means that the object does not get displaced at all; In other words, no ball is thrown when Siteswap number is 0. That is a logical explanation but since we have also learnt such wonderful theorems and methods of analyzing patterns, let's bring in a mathematical proof!

Let's start by making a table for a sideswap sequence that has a 0 in it. Take 4 5 0 for an example:

Current Beat	0	1	2	3	4	5	6	7	8
Current Sideswap No.	4	5	0	4	5	0	4	5	0
Object being thrown	A	B	C	D	A	E	B	D	G
Next beat for throwing	4+0=4	5+1=6	0+2=2	4+3=7	5+4=9	0+5=5	4+6=10	5+7=12	0+8=8

Do you see a problem there? If we look closely we would realise that every time the Sideswap number is 0, a new object has to be introduced. Why? Because every time the sideswap number is 0, the beat at which the object should be thrown is the current beat number + 0 which is always equal to the current beat number. So according to that explanation, we would need infinitely many objects to complete the sequence but is it possible to juggle infinitely many objects? So far, with the technology and knowledge that we have, the answer is no. At the same time, if we use the Average theorem to obtain the

total number of objects required to ‘fill’ the pattern, we would get (4+5+0)/3 which is equal to 3. That means that the total number of objects used in this pattern are 3 and we have already introduced 3 objects that do not lie on the Siteswap number 0 (these 3 objects are A, B and D).

Therefore, to fill in all the blocks with Siteswap number 0, we must consider them to be an Empty Hand since it is not possible to throw an object infinitely many times in 1 beat. This is one of the exceptions in Vanilla Siteswaps wherein one of the beats is left empty as no object is being thrown at that beat.

It might seem intuitively unobvious that juggling can be described using math - but that’s exactly what we just did! We used Siteswap notations to map patterns through numbers and utilizes several characteristics like period, collision, hand siteswap, etc. to define patterns. This just goes to show that math extends way beyond textbooks and classrooms - it's integrated in every step we take and ball we throw - literally!