Learn Precalculus

MathopediaForAI Tutorial 2.1

This is a tutorial that develops a firm foundation of pre-calculus starting from basics such as different categories of functions and techniques for visualizing them to other intermediate level topics such as functions, polar coordinates, cylindrical coordinates, linear algebra, trigonometry, etc..

1. Functions

A function is defined based on its domain, range, and how it maps a set of values of x to a different set of values, y. A domain defines the set of x values that can be inserted into a function $f(x)$ for it to be defined, while the range is the set of values of y that can possibly be 'given out' by a function as an input. Seems simple enough. Although, not every expression with a domain and a range can be called a function. For a specific relationship between x and y to be called a function $f(x)$, it must pass the vertical line test. Essentially, if a vertical line is draw anywhere in the graph of $y = f(x)$, it can interest a maximum of one point in the curve of the graph. Mathematically, this means that every x value is mapped onto *maximum* 1 point on the graph. As such,

Functions form the fundamental base for all of calculus. In this section, we'll take a look at some of the types of functions (although they are infinite in total) and explore their unique characteristics. Note that this part of the calculus course is linked to Mathopedia For AI's Algebra tutorials and requires basic understanding of function notation and terms (domain, range, inverse, etc.).

Trigonometric functions come up often in any calculus or geometry problem. As a quick review, the basic functions that we need to know about include $\sin, \cos, \text{ and } \tan$. Their inverses are $\sin^{-1}, \cos^{-1}, \text{ and } \tan^{-1}$ respectively. Although these might seem familiar, what's more interesting is that trigonometric functions give surprisingly neat relationships with other types of exponential functions. For instance,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is also commonly known as the Euler's form of complex numbers and beautifully links two seemingly distinct concepts of trigonometry and complex numbers to Euler's constant.

Another important thing to note is that trigonometric functions, by convention, use radians. Of course they are used with degrees, but when applying any of the formulae or identities, make sure your units are consistent with those in the formula. Here are some common and useful identities:

1. $\sin x = \frac{\text{opposite}}{\text{hypotenuse}}$
2. $\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$
3. $\tan x = \frac{\text{opposite}}{\text{adjacent}}$
4. $\sin^2 x + \cos^2 x = 1$
5. $\sin 2x = 2\sin x\cos x$
6. $\cos 2x = 1 - 2\sin x$
7. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
8. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Another useful concept related to functions is that of periodicity. In simple words, if a function oscillates in a particular way within the defined domain, then the function is called periodic. The simplest visualization for this can be the sine and cosine curves. By definition, in a periodic function,

$f(x) = f(x+k)$ where k is the period of the function. Can you identify the value of k for a sine and cosine graph? What about the graph of $y = \tan x$?

A) Exponential functions

The standard form of an exponential function is $f(x) = a^x$. This is a monotonically increasing function because as the value of x increases, the value of $f(x)$ also increases. Exponential models, such as those for exponential decay, can be modeled using this function and a common use of this is also in determining half life for radioactive substances.

B) Logarithmic functions

The standard form of a logarithmic function is $x = \log_a b$ and logarithms is the inverse function of exponential functions. Another link between the two is $\log_a b = x \implies b = a^x$. Some of the useful logarithmic identities are as listed below.

a) $\log_a a^p = p$

b) $\log_a b^c = c \log_a b$

c) $\log_{a^n} b^n = \log_a b$

d) $\log_a b + \log_a c = \log_a bc$

e) $(\log_a b)(\log_b c) = \log_a c$

f) $\dfrac{\log_a b}{\log_a c} = \log_c b$

g) $a^{\log_a x} = x$

Remember that to perform any of the addition, subtraction, or most of the other mathematical operations, the base of the logarithms must be constant. Typically, when the base is left unstated, it stands for base 10 and logarithms written as $\ln$ use a base of

e. A key tip is to convert complicated logarithmic expressions into factors so they can be easily evaluated.

C) Radicals

Functions involving radicals typically are often constructed to introduce extraneous solutions so watch out for them, and recheck your final answers by plugging them back in the equation. It is usually convenient to remove radicals from the denominator of fractions as this allows manipulations and simplification in the numerator. Rationalizing the denominator can be done by multiplying both parts of the fraction by the conjugate of the denominator. In general if the denominator is $\sqrt{a}+\sqrt{b}$ then the conjugate can be given as $\sqrt{a}-\sqrt{b}$.

<u>Properties of functions</u>:

- There are several properties based on which functions can be categorized such as even/odd, rational/irrational,monotonically increasing/monotonically decreasing .
- A function that can be expressed as a quotient of two polynomials is called a rational function.This definition must be true for all the values within the domain for the function to be called a rational function.
- When rational functions need to be eliminated an effective strategy to implement is by isolating all expressions and equating them to 0 so that extra terms can be eliminated and simplified.
- An even function is one that satisfies $f(x) = f(-x)$. The graph of such a function is a reflection about the y-axis. Similarly, an odd function is one for which $f(x) = -f(-x)$ and its graph is a reflection about the origin.
- A monotonically increasing function is a function for which $f(a) > f(b)$ implies $a > b$ and vice versa . Similarly a monotonically decreasing function is a function for which

$f(a) < f(b)$ implies $a < b$ and vice versa.To identify whether a function is monotonically increasing or decreasing check the sign of $f(a) - f(b)$ for $a > b$. If this is positive then the function must be increasing and if it is negative the function must be decreasing.

Key Tips:

- Substitution is usually an effective strategy to convert complicated equations into simpler ones. Try to obtain variables in a specific form that helps in simplification (for example eliminate the radical sign or removes the denominator)
- The substitution $x + \frac{1}{x}$ is commonly used for simplifying polynomials with symmetric coefficients.
- It is always helpful to consider ratios and writing variables one in terms of the other especially in homogenous equations as this can help in isolating the variable that needs to be solved for.

2. Complex numbers

Complex numbers are numbers that can include a combination of real and imaginary numbers in their definition (though a completely real number or a completely imaginary number is also part of the complex set!). They can be represented in the form $a + bi$ where a and b are real number and $i^2 = -1$. This form of expression complex numbers is also called rectangular notation since it can be used to plot complex numbers on the Argand plane. Thus, for a complex number defined as $z = a + bi$, $\text{Re}(z) = a$ and $\text{Im}(z) = b$ where $\text{Re}(z)$ represents the real part of z and $\text{Im}(z)$ denotes the imaginary part of z.

Further, the conjugate of $z = a + ib$ is denoted as $\overline{a + bi}$ and is equal to $a - ib$ while its magnitude is denoted as $|a + bi|$ and is equal to $\sqrt{a^2 + b^2}$.

Operations on complex Numbers:

Additional operations for complex numbers can be defined as shown below.

$$(a+bi)+(c+di) = a+c+bi+di = (a+c)+(b+d)i.$$

Likewise, multiplication is defined as

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$

And division is defined as

$$\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}} = \frac{w\overline{z}}{|z|^2}.$$

Other complex numbers identities:

Similarly, these operations can also be implemented amongst conjugates of a single or multiple complex numbers. While they can all be derived from the simple operations, it's always better to keep them ready in your complex number identities toolbox.

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z} \cdot \overline{w}$
3. $z\overline{z} = a^2 + b^2$
4. $\overline{\overline{z}} = z$
5. $z\overline{z} = |z|^2$
6. $|zw| = |z||w|$

We can also determine the real and imaginary parts of any complex using the following relationships.

1. $\text{Re}(z) = \dfrac{z + \overline{z}}{2}$
2. $\text{Im}(z) = \dfrac{z - \overline{z}}{2i}$

While these identities can be very useful in solving complex number equations, sometimes basic algebra techniques such as factoring, grouping, equating like coefficients, etc. can make problems much simpler as well. Also, whenever dealing with $|z|$, see if you can square both sides of the given equation and substitute $a^2 + b^2$ for $|z|$ and eliminate the square root sign completely.

Complex numbers and trigonometry:

The standard complex numbers can also be represented in a trigonometric form, otherwise known as the Euler's form. This is given as:

$$z = \cos\theta + i\sin\theta$$

This definition of complex numbers combined with the idea of roots of unity allows us to find complex roots of equations with much greater ease. For example, the roots of unity for $z^4 = 1$ would be $1, -1, i, -i$.

3. Vectors

To begin, let's observe the differences between simple line segments and vectors in order to understand the significance of vectors in solving geometry related problems.

1. Vectors are free to move - unlike line segments which connect two specific points on the graph, vectors can move anywhere on a given plane as long as their length and angle (with respect to the positive x-axis) remains the same.
2. Vectors have a direction and this is important in defining them unlike line segments. For example, a line which goes from (0,0) to (1,0) is considered the same as a line segment which goes from (1,0) to (0,0) but the same won't hold true for vectors.
3. Vector planes can be redefined or transformed, unlike the standard cartesian plane which never changes.

<u>**Important notations / conventions**</u>:

When it comes to representing vectors, here are some key conventions that you must be able to recognize. $\overrightarrow{C}$ represents a vector from the origin (0,0) to a point C. $\overrightarrow{AB}$ represents a vector from point A to point B.

Magnitude: For a vector defined as $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, the magnitude is given as $||\vec{v}|| = \sqrt{a^2 + b^2}$. This represents the length of the vector. When the magnitude of a vector is exactly 1, we call that vector a unit vector.

Dot product: The dot product is defined as $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v_1 w_1 + v_2 w_2$. When looking at dot products, the following properties are important:

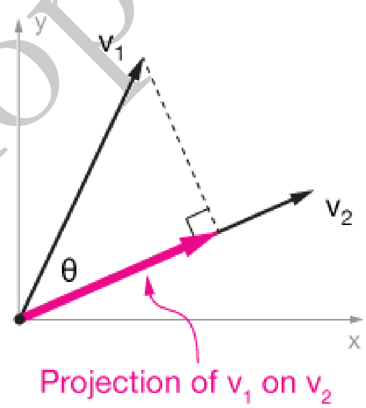
- ➔ $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- ➔ $\vec{u} \cdot (a\vec{v}) = a(\vec{v} \cdot \vec{u})$
- ➔ $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- ➔ $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$
- ➔ $\vec{v} \cdot \vec{w} = ||\vec{v}||\ ||\vec{w}||$

Linear dependence: If two or more vectors are the same as each other except a scalar factor, they are called linearly dependent (i.e. they have the same direction but a different magnitude). Otherwise they are linearly independent.

Projections: The easiest way to think about projections is to think of them as a shadow, assuming that the sun is exactly perpendicular to the ground surface. This can be given as:

$$\mathrm{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w}$$

Here's how you can visualize projections in vectors to look like.



Vectors and parametrization:

Parametrization is an important aspect in precalculus as it helps us redefine vectors in terms of a singular variable namely t thereby offering flexibility in the operations that we do on vectors.

An example of a parametrized vector would be as follows:

$$\mathbf{v} = \begin{pmatrix} 2t + 1 \\ 4t - 1 \end{pmatrix}$$

For now, note that parametrization offers to trace motion with respect to time and is used frequently with vectors especially because of their ability to transform through matrix transformation. We will look at this again in the next section after learning about 2D and 3D matrices and the polar coordinate system.