Learn Algebra

MathopediaForAI Tutorial 1.2

This is a tutorial that develops a firm foundation of algebra starting from basics such as equations and techniques for solving simultaneous topics to other intermediate level topics such as conics, complex numbers, series, mathematical proofs, types of mean, types of inequalities, and other interesting problems solving strategies.

1. Recap of Polynomials and Polynomial division

So far we have seen n degree polynomials and analyze them using different concepts such as the fundamental theorem of algebra, factor theorem, rational root theorem and the identity theorem. We also observed how the Vieta's formulas (along with their single general form) can be implemented to establish a link between the coefficients of polynomials ($a_1, a_2, a_3, \ldots, a_n$) and the roots of the given polynomial ($r_1, r_2, r_3, \ldots, r_n$)

Now let's look at a few problem solving strategies which may come in handy:

- Factor out complicated polynomials through inspection and synthetic division to obtain a simpler form. This typically gives a quadratic polynomial in the end whose roots can be much easily determined by using the quadratic formula.
- When given simultaneous equations with n degree polynomials either use Vieta's formulas to obtain expressions for the coefficients in terms of the roots or isolate a single variable in all the equations given and solved henceforth.
- Ensure that all steps (for example squaring both sides of an equation) are reversible or you may end up with extraneous solutions to the said polynomial.

2. Factoring Multivariable Polynomials

When given a multivariable polynomial the most convenient method of factoring involves grouping of like and unlike terms. The pattern of factors obtained can never be generalized as most polynomials have fractional or repeated roots. Consider the example below

$$ap + aq + bp + bq = a(p+q) + b(p+q) = (a+b)(p+q).$$

In the example above the given multivariable polynomial was factored based on symmetry. Simon's favorite factoring trick utilizes this approach of multivariable factoring to factor out polynomials that may not be as symmetrical. The idea here is to take any polynomial and bring it to a form which can be symmetrically factorized by adding or subtracting a constant from the left and right hand sides of the equation.

For example , to factor the equation $6xy + 4x + 9y = 247$, the number 6 can be added to both sides following which the equation becomes $6xy + 4x + 9y = 253$. This can then be rewritten as $(2x+3)(3y+2) = 253$.

In general the following hold true

1. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$
2. $x^{2n+1} + y^{2n+1} = (x+y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \ldots - xy^{2n-1} + y^{2n})$

The factors of a polynomial can also be obtained by using the factor theorem as an extension to multivariable polynomials. For a polynomial which has 2 variables , this can be generalized as follows:

Define $f(a,b)$ and $h(b)$ such that both polynomial satisfy the equation $f(h(b),b) = 0$ for $b \in \mathbb{R}$. Then , it can be said that $f(a,b) = (a - h(b))g(a,b)$ where $g(a,b)$ is a polynomial. The same can be applied for multivariable polynomials with more than two variables as well .

- When given a complicated set of products as the definition for a multivariable polynomial, factorize each term separately until a linear factor is obtained in order to reduce the degree of the polynomial to be factored by 1 .
- Mathematical inspection is usually effective in identifying the first few roots and therefore simplifying the polynomial by several degrees.

Fun fact: An equation which has all integer roots is known as a Diophantine equation which is named after the famous Mathematician who is known for his unique method of solving multivariable polynomial equations by expressing all unknowns in terms of a single unknown.

3. Sequence & Series

There are often misconceptions about series and sequences representing the same ideas. This is wrong. While a sequence simply lists out the elements within a set , a series sums up these terms and is often concatenated in terms of sigma notation. Note that series can also be defined recursively as the sum of every term within a sequence .

Series can typically be of infinite types but the most commonly known types of series include arithmetic and geometric series . Some of the key features for each of these are listed below.

- **Arithmetic Series:**

  The nth term is given as $T_n = a + (n-1)d$ where a is the first term in the series and d is the common difference between each set of consecutive terms within the series.

  This definition gives us multiple implications :

  - ➔ The average of all the terms in a series is always equal to the average of the first and the last term of the arithmetic series. Consequently a, b, c is an arithmetic series if and only if the average of a and c is equal to b.

- The sum of all the terms in an arithmetic series can be given as the average of the first and the last term (that is the median of the series) multiplied with the number of terms in the series.

Following this the sum of the first n terms of an arithmetic series (S_n) can be given as :

$$S_n = \frac{n(2a+(n-1)d)}{2}$$

- **Geometric Series:**

- The nth term is given as $T_n = ar^{n-1}$ where a is the first term in the series and r is the common ratio between the consecutive terms. As a result the sum of the first n terms is given as $\frac{a(1-r^n)}{1-r}$.
- Geometric Series have a specific characteristic which makes them unique:they can converge to a finite sum even if the series has an infinite number of terms under specific circumstances.
- If $|r| < 1$ then $S_\infty = \frac{a}{1-r}$.

Key tips:

- Summation Notation can often be confusing especially for series with alternating science . Thus for any geometric series with a negative common ratio it's always better to write the first and last few terms separately.
- If needed check the sum to infinity by using limits on the summation which generally defines the series.

Another Special type of series is known as the telescoping series . These series often use terms which are alternatingly positive and negative in a way such that certain manipulations can help in expressing the series in a closed form. A common method of doing this is by using partial fraction decomposition which allows us to express the rational function in terms of a rational term only with a simpler denominator .

When an arithmetic series is combined with a geometric one an arithmetico-geometric series is formed. The general term and the sum of first n terms of such a series can be obtained by individually summing up the linear and exponential part of the series.

4. Identities, manipulations, and induction

An identity is an equation which holds true for a specific range of values of variables. Identities can be proven either by using brute force (ie expanding both sides of the identity to czech if one equals the other. This however may not be useful for identities that are defined recursively. In such cases mathematical induction is used to utilize the domino theory for proving recursively defined sequences . This method is classically into three parts:

A. Base step - show that the identity holds true for the lowest value of n.
B. Inductive hypothesis - state what needs to be proven in terms of n=k+1.
C. Inductive Step - show that the assumption implies the inductive hypothesis.

Mathematical Induction can be used for proving all identities which are based on series .A popular use of this principle is the proof of binomial theorem which states that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{n-i} x^{n-i} y^i$$
.

<u>Key tips</u>:

- By definition an identity must hold true for all possible values of the included variables. Therefore in order to prove any identity it might be useful to plugin random values and observe patterns related to the identity.
- The first clue that a problem requires the implementation of the principle of mathematical induction is if it states or otherwise implies that the identity holds true for all positive integers.
- Direct proof for any identity does not always have to flow forward . If you are able to bring the LHS to a certain intermediate value ,check if you can bring the RHS to the same value too.

5. Inequalities

There are several standard inequalities which can be used as tools to prove others by using certain manipulations. Some of these are as follows:

A. **Trivial Inequality**- states that for any real x and a, $(x+a)^2 \geq 0$

B. **AM-GM Inequality**- states that if the sequence $a_1, a_2, \ldots, a_n$ are all non negative numbers then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

C. **Cauchy-Schwarz Inequality** - Given any two sequences defined as $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. It can be said that $(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$ where if $a_i = 0$ or if $b_i = ta_i$ for all i, then equality occurs.

Key tips:

- For proving complicated inequalities try breaking them down into simpler terms or factors that appear easier to deal with. When merging these back together be cautious of the signs as they might change because of the intermediate step.
- Factoring , expanding and regrouping terms is the most effective way of manipulating inequalities to bring them in the AM-GM form.
- For inequalities involving non negative expressions AM-GM is the most likely way to go whereas for inequalities involving squares or product of two sums with the same number of terms the Cauchy-Schwarz inequality might be more helpful.

Types of Mean:

To analyze inequalities we can use several types of mean as demonstrated below.

A. Quadratic Mean(QM)

$$\mathrm{GM} = \sqrt{\frac{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}{n}}$$

B. Harmonic Mean (HM)

$$\mathrm{HM} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}}$$

C. Arithmetic Mean (AM)

$$\mathrm{AM} = \frac{1}{n}\sum_{i=0}^{n} a_i$$

D. Geometric Mean (GM)

$$\mathrm{GM} = \sqrt[n]{\Pi_{i=1}^{n} x_i}$$

These means can be compared through the following inequality.

$$QM \geq AM \geq GM \geq HM.$$

They can be used to compare other series in terms of their individual terms. In general the condition for equality for all of them is when all the x_i terms are equal (i.e. the series has one single constant term only).

Key Tips:

- Note that although inequalities are usually defined for all real numbers, if a question specifies the definition of an inequality for all positive integers, mathematical induction might be the way to go!
- The QM-AM-GM-HM inequality chain can be used to implement any manipulations and obtain one of the means by using the others.
- Try looking for the upper and lower bounds of a complicated inequality to map out its behavior over a range of values - this might give clues on what can be done to solve the problem.

6. Functions

Functions can be broadly divided into multiple categories such as monotonically increasing or monotonically decreasing. A monotonically increasing function There can be many different types of functions, some of which are as listed below.

A) Exponential functions

The standard form of an exponential function is $f(x) = a^x$. This is a monotonically increasing function because as the value of x increases, the value of $f(x)$ also increases. Exponential models, such as those for exponential decay, can be modeled using this function and a common use of this is also in determining half life for radioactive substances.

B) Logarithmic functions

The standard form of a logarithmic function is $x = \log_a b$ and logarithms is the inverse function of exponential functions. Another link between the two is $\log_a b = x \implies b = a^x$. Some of the useful logarithmic identities are as listed below.

a) $\log_a a^p = p$

b) $\log_a b^c = c \log_a b$

c) $\log_{a^n} b^n = \log_a b$

d) $\log_a b + \log_a c = \log_a bc$

e) $(\log_a b)(\log_b c) = \log_a c$

f) $\dfrac{\log_a b}{\log_a c} = \log_c b$

g) $a^{\log_a x} = x$

Remember that to perform any of the addition, subtraction, or most of the other mathematical operations, the base of the logarithms must be constant. Typically, when the base is left unstated, it stands for base 10 and logarithms written as $\ln$ use a base of e. A key tip is to convert complicated logarithmic expressions into factors so they can be easily evaluated.

C) Radicals

Functions involving radicals typically are often constructed to introduce extraneous solutions so watch out for them, and recheck your final answers by plugging them back in the equation. It is usually convenient to remove radicals from the denominator of fractions as this allows manipulations and simplification in the numerator. Rationalizing the denominator can be done by multiplying both parts of the fraction by the conjugate of the denominator. In general if the denominator is $\sqrt{a}+\sqrt{b}$ then the conjugate can be given as $\sqrt{a}-\sqrt{b}$.

<u>Properties of functions</u>:

- There are several properties based on which functions can be categorized such as even/odd, rational/irrational,monotonically increasing/monotonically decreasing .
- A function that can be expressed as a quotient of two polynomials is called a rational function.This definition must be true for all the values within the domain for the function to be called a rational function.
- When rational functions need to be eliminated an effective strategy to implement is by isolating all expressions and equating them to 0 so that extra terms can be eliminated and simplified.
- An even function is one that satisfies $f(x)=f(-x)$. The graph of such a function is a reflection about the y-axis. Similarly, an odd function is one for which $f(x)=-f(-x)$ and its graph is a reflection about the origin.
- A monotonically increasing function is a function for which $f(a)>f(b)$ implies $a>b$ and vice versa . Similarly a monotonically decreasing function is a function for which $f(a)<f(b)$ implies $a<b$ and vice versa.To identify whether a function is monotonically

increasing or decreasing check the sign of $f(a) - f(b)$ for $a > b$. If this is positive then the function must be increasing and if it is negative the function must be decreasing.

Key Tips:

- Substitution is usually an effective strategy to convert complicated equations into simpler ones. Try to obtain variables in a specific form that helps in simplification (for example eliminate the radical sign or removes the denominator)
- The substitution $x + \frac{1}{x}$ is commonly used for simplifying polynomials with symmetric coefficients.
- It is always helpful to consider ratios and writing variables one in terms of the other especially in homogenous equations as this can help in isolating the variable that needs to be solved for.

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