Learn Precalculus

MathopediaForAI Tutorial 2.2

This is a tutorial that develops a firm foundation of pre-calculus starting from basics such as different categories of functions and techniques for visualizing them to other intermediate level topics such as functions, polar coordinates, cylindrical coordinates, linear algebra, trigonometry, etc..

1. Matrices

Matrices which represent the 2 dimensional plane are structured in a 2×2 grid. They are typically used to transform vectors from one form to another. Here are some of the basic notations and properties of 2D matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The matrix given above is an example of a 2D matrix which can be used upon 2D vectors.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Also note that conventionally, matrices are represented using capital letters and vectors in small letters. We can also have operations between two or more matrices themselves.

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$$

Remember that matrix multiplication is associative and distributive but not commutative.

Not all matrices transform vectors into a different form. For every set of dimensions there is always a matrix such that its application to a vector does not change it at all. For a two dimensional cartesian plane this matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Does this look familiar? This is known as the identity matrix which is represented as $\mathbf{I}$.

Thus we have that $\mathbf{Iv} = \mathbf{v}$ and that $\mathbf{IA} = \mathbf{A}$. Also observe that the individual columns of the identity matrix represent the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

The determinant is an important property of every matrix and is denoted as $(\mathbf{A})$. This can be calculated as

$$\mathrm{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

A matrix is called singular if its determinant is equal to 0. If a matrix (say $\mathbf{A}$) is not singular, then for any vector $\mathbf{v}$, there must exist some vector $\mathbf{c}$ such that $\mathbf{v} = \mathbf{Ac}$. But what's the graphical significance of a determinant? Determinants of matrices represent the areas of parallelograms which have sides represented by the matrix (in a 3D matrix, the determinant would give the volume of the parallelepiped).

Just like functions, matrices also have inverses. In functions, an inverse is a function whose graph is a reflection of the original across the line $y = x$. In matrices, an inverse matrix is the one which when multiplied with the original matrix gives the identity matrix of that dimension. So,

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$

An easy to find the inverse matrix is to follow the formula below.

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

2. 3-dimensional Linear Algebra

Three dimensional vectors are represented as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. The basic operations are quite similar to those in 2D vectors and can be shown as follows.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a + x \\ b + y \\ c + z \end{pmatrix}$$

$$k\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3$$

Similar to 2D vectors, 3D vectors also have a magnitude which represents its length in a scalar form.

$$\left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| = \sqrt{x^2 + y^2 + z^2}$$

Note that the base vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are all unit vectors as their magnitude, or length, is 1 unit.

We can also have dot products within 3D vectors. These are given as:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \cos\theta$$

But 3D vectors cannot be transformed with 2D matrices! So, we also introduce 3D matrix transformation.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}.$$

Again, 3D matrix operations are commutative and associative just like 2D matrix operations meaning that:

$$\mathbf{A}(k\mathbf{v}) = k\mathbf{A}\mathbf{v}$$

and

$$\mathbf{A}(\mathbf{v} + \mathbf{w}) = \mathbf{A}\mathbf{v} + \mathbf{A}\mathbf{w}$$

3D matrices can also be multiplied amongst themselves. Although this might seem daunting at first sight, remember that regardless of which dimension matrices you work with, the multiplication rule and pattern remains the same.

Further,

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$
$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$
$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

And what about the identity matrix of 3-dimensions? Well, since the identity matrix ideally consists of all the base vectors of that dimension, the identity matrix $\mathbf{I}$ of 3D matrices is given as:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Extra: Another interesting idea is that of permutation matrices, denoted by $\mathbf{P}$. A permutation matrix is one whose rows are a rearrangement of the original matrix. So, for the identity matrix, the following could be the permutation matrices.

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. Parametrization

The essential idea of parametrization is to rewrite an equation involving x and y (and perhaps z for 3D graphs) in terms of a single variable (most commonly taken as t). Under this topic, there are two things which you must know:

1. Converting general equations into parametric ones
2. Converting parametric equations into general ones

Example of converting simple equation into a parametric pair:

Parametrize the equation $x^2 - 3y^2 = 1$.

One of the best tricks to use when in doubt is to look for trigonometric identities which look similar to the equation you need to parametrize. In this case, we need an identity which involves squares and subtracts one trigonometric term from another to give 1. Sounds familiar? Yep, we can use any one of the following:

$$\sec^2\theta - \tan^2\theta = 1$$

and

$$\csc^2\theta - \cot^2\theta = 1$$

For now, let's use the first one and see what happens. Let $x^2 = \sec^2\theta$ and $3y^2 = \tan^2\theta$. We know that the equation would still hold true because of the sec-tan identity. Now, we can isolate x and y to obtain the parametric equations as:

$$\boxed{x = \sec\theta}$$

and

$$\boxed{y = \frac{1}{\sqrt{3}} \tan \theta}$$

Similarly, we can also do the opposite - i.e. converting parametric equations to a 'normal' equation. Let's look at an example.

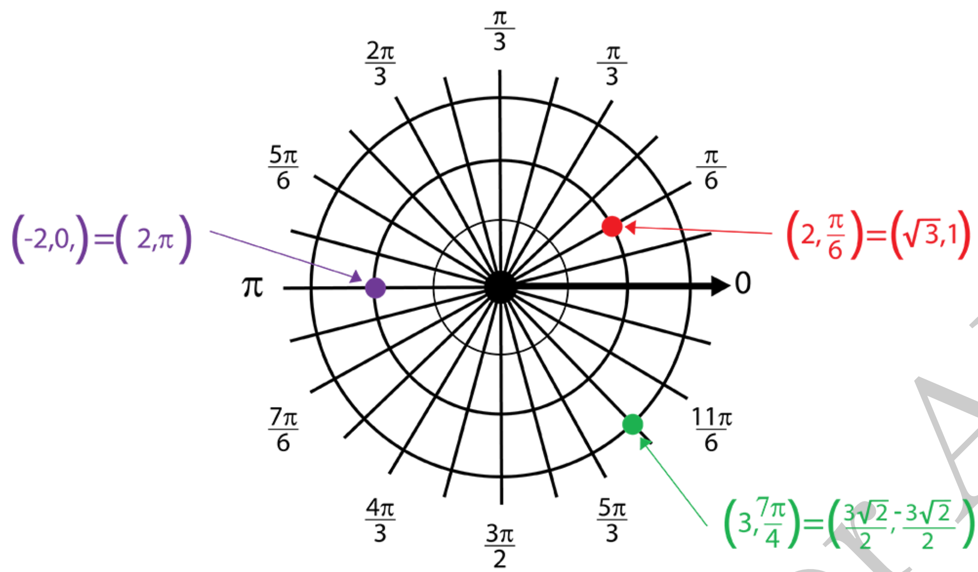
Given the parametric equations $x = \sin \theta$ and $y = 3 - 2\cos 2\theta$, find the equation which can be used to sketch a graph of this model on the x-y cartesian plane.

This isn't too difficult - we only need to express y in terms of x (or vice versa and then isolate y) to get our answer. We know that $\cos 2\theta = \cos^2\theta - \sin^2\theta$. Also, $\cos^2\theta = 1 - \sin^2\theta$. Putting it all together we have that $y = 3 - 2(1 - 2\sin^2\theta) = 1 + 4\sin^2\theta$. We can now express y in terms of x by saying that $y = 1 + 4x^2$.

But it doesn't end there. Remember that we need to specify the domain and range of any function in order to complete its definition. So, once we specify that the un-parametrized equation holds valid for $-1 \leq x \leq 1$ and $1 \leq y \leq 5$, we are done!

4. Polar coordinates

Polar coordinates offer another method for denoting the location of a point on a plane. As with rectangular coordinates, we start with a point we call the origin and we identify the location of each point with an ordered pair, (r, θ). The r-coordinate is the distance from the point to the origin; we call this coordinate the radial coordinate. The θ-coordinate is the angular coordinate, which we define in the same way we related angles to points on the unit circle.



If (x, y) in rectangular coordinates is the same point as (r, θ) in polar coordinates, then

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x},$$

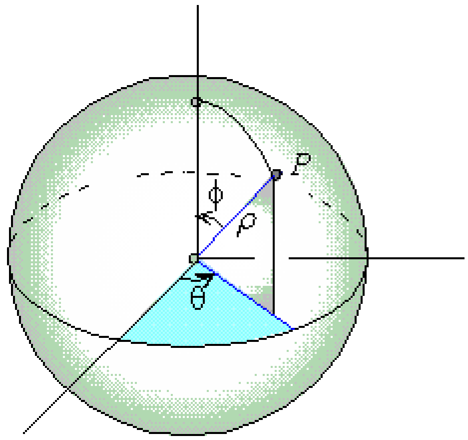
where $\tan\theta = \frac{y}{x}$ holds only if $x \neq 0$. If $x = 0$ and $y \neq 0$, then $\cos\theta = 0$ and $r^2 = y^2$. If $x = y = 0$, then $r = 0$ and θ can be anything.

The image of the point (x, y) upon a rotation θ counterclockwise about the origin is

$$(x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta).$$

5. 3D coordinate systems

When talking about the 3D coordinate system, there can be several different methods of representation. While the most commonly known is the x-y-z cartesian coordinate system, in this section, we'll specifically focus on other unique 3D representations including the spherical coordinate system. Have a look at the diagram below.



As labeled on the figure above, the spherical coordinate notation is classically written as (ρ, θ, ϕ) where:

$$x = \rho \sin \phi \cos \theta,$$
$$y = \rho \sin \phi \sin \theta,$$
$$z = \rho \cos \phi.$$

Key Tips:

- If parametric equations $x = f(t)$, $y = g(t)$ satisfy an equation in terms of x and y, we only know that the graph of the parametric equations is part of the graph of the non-parametric equation.
- To show that the graph of the parametric equations is the same as the graph of the non-parametric equation, we must show that every point in the graph of the non-parametric equation is in the graph of the parametric equations.
- We can usually do so by showing that for each point (x_1, y_1) on the graph of the non-parametric equation, there is a value of t such that $x_1 = f(t)$, $y_1 = g(t)$.

- The graph of parametric equations is not necessarily the same as that of a non-parametric equation satisfied by the parametric equations. Sometimes the parametric equations have restrictions that limit the graph of the parametric equations to only a portion of the graph of the non-parametric equation.
- When reading about spherical coordinates in another source, make sure you know what each variable stands for and what the restrictions on each are. While most sources will define the two angles in spherical coordinates in essentially the same way we have here, some will use different variables than we have.
- Even more confusing, some will use the same variables, but reverse their meanings. Furthermore, some will restrict ρ to be nonnegative, and some will use r rather than ρ.