Learn Calculus

MathopediaForAI Tutorial 3.2

This is a tutorial that develops a firm foundation of calculus starting from a thorough revision of the pre-requisites such as set notations and functions and then covering advanced topics including limits, differential calculus and its applications, integral calculus and its applications and finally differential equations.

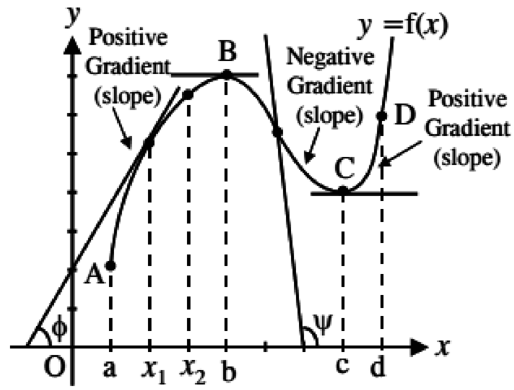
6. Applications of differentiation

Differentiation can be used in distinct applications within pure math as well as real life scenarios to model situations or optimize outcomes. Some of these applications are listed below.

- Identifying the increasing and decreasing parts of a function
- Points of inflection
- Rate of change
- Kinematics
- Error and approximation
- Tangents and normal
- Optimization through global minimum and maximum

We'll start by discussing the graphical significance of a derivative. We know that derivatives give us the slope of a curve at any point. So, if a derivative is positive, then the slope of the tangent must be going upwards as we move left to right and if the derivative is negative then the tangent must be going downwards as we move right to left.

The diagram below shows how the slope varies across a function as it increases and decreases in different parts.



Here are conditions which define how the function is behaving at any given point.

- **Increasing:** $f'(x) \geq 0$
- **Strictly increasing:** $f'(x) > 0$
- **Decreasing:** $f'(x) \leq 0$
- **Strictly decreasing:** $f'(x) < 0$

If a function is either strictly increasing or strictly decreasing over a certain interval, then we can also say that the function is **strictly monotonic** over that interval.

Another key characteristic which can be observed to analyze graphs is inflection points. This is also used to determine the concavity of a graph over a given interval. The general rule is as follows:

- If $f''(x) > 0$ over $[a, b]$ then the graph is concave up over $[a, b]$
- If $f''(x) < 0$ over $[a, b]$ then the graph is concave down over $[a, b]$

Rate of change is another application of differentiation which comes up most often in calculus. From problems related to changes in volume to problems related to optimization, rate of change is a critical application which is important for the thorough understanding of differentiation. The basic idea behind rate of change is that all derivatives are to be evaluated with respect to t, or a common variable present in all equations.

For example, we've been given that the rate of change of the radius of a sphere is 4 cm/second when the radius is 7 cm and are asked to find the rate at which the volume changes. The only real "trick" other than the actual differentiation is to interpret the question in terms of mathematical notation. In this case, the rate of change of radius can be written as $\frac{dr}{dt}$ while the rate of change of volume can be written as $\frac{dV}{dt}$. We also know that $V = \frac{4}{3}\pi r^3$. Thus, by differentiating both sides of this volume equation, we can say that $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. We can now substitute the values of r and $\frac{dr}{dt}$ respectively to find out the value of $\frac{dV}{dt}$.

Kinematics is another interesting application of differentiation (although it often requires integration too for complete understanding). Let displacement be represented by the function $s(t)$, velocity be represented as the function $v(t)$ and acceleration be represented as $a(t)$. Then, we can say that:

$$a(t) = \frac{d}{dx}(v(t)) = \frac{d^2}{dy^2}(s(t))$$

This is also relatively intuitive since acceleration is defined as the rate of change of velocity and velocity is defined as the rate of change of distance.

Tangents are commonly obtained by differentiating the function and finding the gradient at any particular point. The only thing to remember here is that:

$$m_{\text{normal}} = \frac{-1}{m_{\text{tangent}}}$$

where m_{normal} is the gradient of the normal and m_{tangent} is the gradient of the tangent.

7. Integration

Integration can be thought of as the inverse operation of differentiation - meaning that it undoes whatever changes differentiation causes to a function. As per the fundamental theorem of calculus, we have that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

and

$$\int_a^b f(x)dx = F(b) - F(a)$$

From these statements, we can define integrals as anti-derivatives – essentially the opposite of derivatives. For instance,

$$\int \cos x dx = \sin x$$

because

$$\frac{d}{dx} \sin x = \cos x$$

Seems simple enough. But this isn't entirely right. Let's think about the derivative of $\sin x + 5$. Well, by the addition rule, $\frac{d}{dx}(\sin x + 5) = \cos x$. But we said that $\int \cos x dx = \sin x$. So what's wrong? Actually, integration alone gives us an entire family of functions which are written as the actual integrated part plus a constant (usually represented as C). So, although $\frac{d}{dx} \sin x = \cos x$, integration of both sides gives us $\int \cos x dx = \sin x + C$.

This leads us to exploring how each of the rules that we previously saw for derivatives work for integrals. For instance, the chain rule for differentiation can be written as follows for integration.

$$\int f(x)g(x)dx = f(x) \int g(x)dx - \int \left(\frac{d}{dx} f(x) \int g(x)dx \right) dx.$$

Conventionally, $f(x)$ is written as u and $g(x)$ is written as v. This is also why the formula above can be rewritten as $\int uvdx = u \int vdx - \int (u' \cdot \int vdx)dx$. This method of integration is also commonly known as integration by parts.

The main challenge in using integration by parts is to identify which function can be considered as the u and which one will be v (i.e. which function should we integrate and which one to differentiate. While in some cases quick mental math can help in identifying this selection, many times this may not be enough as simplification of expression within the negative integral can be more complicated. There's no fixed rule as to how one might allot these functions but here's a general tip which you might find useful while solving problems:

- **L** - Logarithms
- **I** - Inverses
- **A** - Algebraic
- **T** - Trigonometric
- **E** - Exponential

You can also reverse the chain rule for differentiation to translate it to an integration-friendly formula. In differentiation, we use the chain rule when we have one function embedded into another, i.e. a form similar to $f(g(g))$. For integration, the chain rule can be applied when we need to calculate an integral of the form $\int g(f(x))f'(x)dx$. In general, we can say that:

1. Let $u = f(x) \implies du = f'(x)$
2. Substituting this into $\int g(f(x))f'(x)dx$, we have that $\int g(u)du$.

Try evaluating the following integrals using integration by parts and / or the chain rule of integration.

a) $\displaystyle\int \frac{x-3}{2x^2-12x}dx$

b) $\displaystyle\int (e^x+1)^3 dx$

c) $\displaystyle\int x\sqrt{3x^2-1}dx$

Refer to the Mathopedia For AI problem sets for more practice questions on integration.

8. Definite and indefinite integrals

Definite integrals are those integrals which have specified bounds so that they can be calculated over a certain specified interval. Calculating definite integrals just has one extra step, i.e. evaluating the integral over the specified bounds. For demonstration, let's calculate $\int_1^2 xe^{x^2}dx$. To do this, we will substitute with $u = x^2 \implies du = 2xdx$. Here the main thing to watch out for is to ensure that the bounds are altered according to the substitution before rewriting the integral in terms of u. As per how we have defined u when $x = 1$, u must also be 1 and when $x = 2$, then u must be 4. Hence the integral can be written as

$$\int_1^2 xe^{x^2}dx = \int_1^4 \frac{1}{2}e^u du$$

We can use the standard antiderivative technique to further evaluate this integral to obtain

$$\int_1^4 \frac{1}{2}e^u du = \frac{1}{2}e^u\bigg|_1^4 = \boxed{\frac{1}{2}(e^4 - e)}$$

The same can also be done to evaluate definite integrals involving other techniques such as integration by parts or partial fraction decomposition. Note that most often you may require more than one of these strategies to evaluate the desired integral. Here is an example of how substitution and IBP work together with definite integrals.

Let's evaluate $\int_2^1 e^x \sin x dx$. Let $u = e^x \implies du = e^x dx$ and $v = \implies dv = \cos x dx$. Using integration by parts we have that

$$\begin{aligned} \int e^x \sin x dx &= e^x \cos x + \int e^x \cos x dx \\ &= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x dx\right) \\ &= e^x(\sin x - \cos x) - \int e^x \sin x dx \end{aligned}$$

Shifting the integral to the left hand side, we have that:

$$2\int e^x \sin x dx = e^x(\sin x - \cos x)$$

Therefore,

$$\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2} + C$$

9. Applications of Integration

Applications of integration range from graphical analysis to real world uses . The most obvious implementation of integration is in determining areas under specific curves between defined bounds .

This is exactly the same as evaluating definite integrals. The extension of the application lies in the concept of obtaining volumes which can be obtained for prism-like structures as well as solids of revolution. Let's derive the formula for the volume of a sphere to understand how the process looks like.

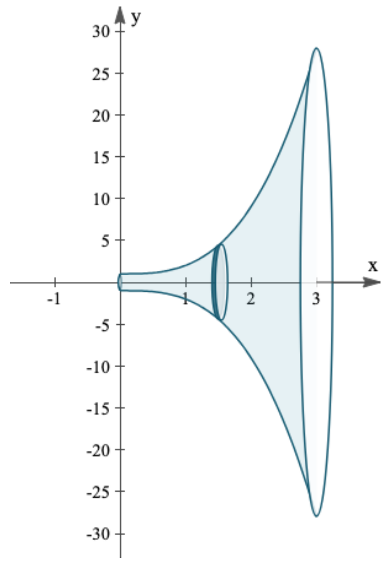
The idea here is to identify the area of individual 'disks' within a sphere and integrate this with respect to the bounds to get volume. Let's call the radius of the sphere r whose vertical distance from the center of the sphere is x and the horizontal distance from the center is y. From this, we can tell that $y = \sqrt{r^2 - x^2}$ and this implies that the are of the disk which is y units above the center is given as $\pi(r^2 - x^2)$. Now, we simply need to integrate the expression for the area of one disc over the bounds $-r$ to r. Doing this we get

$$\begin{aligned}\int_{-r}^{r} \pi(r^2 - x^2)dx &= \pi\left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} \\ &= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right)\end{aligned}$$

After simplifying we find that

$$\text{Volume of Sphere } = \boxed{\frac{4}{3}\pi r^3}$$

This is the standard technique used for finding volumes in objects with a standard cross section . A larger part of this application relies on objects of revolution or objects which may or may not have standard and consistent cross sections always. For instance imagine the revolution of the graph of $y = e^x$ around the x axis .



As shown by the graph above the revolution of the exponential function around the x-axis gives a non standard object. Note that the cross sectional area at any point in the figure will always be circular as the graph is being rotated for a complete revolution. As a result, this cross sectional area can be given as $\pi(f(x))^2$. Now all we need to do to obtain the volume is integrate this expression over the defined bounds . Reflecting back we can see that we haven't used the definition of $f(x)$ anywhere yet and therefore we can use our solution to generalize the volume for solids of revolution regardless of the nature of the initial 2D functions. Thus ,

$$\text{Volume for revolution around x-axis } = \pi \int_a^b (f(x))^2 dx$$

A similar process can be used to analyze the volume for solids of revolution about the y-axis. Doing so we see that the surface area of each layer will be given as $2\pi x f(x)$. Since we are dealing with the volumes, this will be integrated again over the bounds a to b .

$$\text{Volume for revolution around y-axis } = 2\pi \int_a^b x f(x) dx$$

Another interesting application of integration lies in its use for determining the length of particular curves over a certain domain. The main logic behind this is to "break" an inconsistent or non-standard curve into infinitely many small straight lines which can then be used to find the overall length of the curve. This can be done by using the formula below.

$$\text{Length of curve} = \int_a^b \sqrt{(f'(x))^2 + 1}dx$$

.

By glancing at this integral, it can be said that this quantity won't be the easiest to obtain - which is why arc lengths are not as commonly asked for as areas under curves. However, remember that although integration would have given us the exact length, we can still approximate arc lengths by assuming finitely many straight lines making up a curve and summing these up. A similar strategy is also used for the approximation of areas and this is known as the Reimann sum.

10. Series and calculus

We've already seen series in the Algebra part of this course while analyzing arithmetic and geometric series. In this section, we'll be looking specifically at infinite series and how to determine if a series converges or diverges over ∞. The standard notation for representing series is to call them a_n. When a series approaches infinity, we can represent this as:

Here's a list of the key terms associated with sequences (with the general form $\{a_n\}_{n=1}^{\infty}$:

1. Monotonically increasing: $a_k \leq a_{k+1}$ for all $f \geq 1$.
2. Monotonically decreasing: $a_k \geq a_{k+1}$ for all $f \geq 1$.
3. Strictly monotonically increasing: $a_k < a_{k+1}$ for all $f \geq 1$.
4. Strictly monotonically decreasing: $a_k > a_{k+1}$ for all $f \geq 1$.

Infinite sequences can also be analyzed by theorems such as Lagrange's theorem which allow us to determine properties such as convergence of the sum to infinity (represented with sigma notation).