COMPLEX NUMBER

Part I

Complex Numbers

A number $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$; x = Real part or Re(z); y = Imaginary part or Im(z)

Magnitude

$|z| = \sqrt{x^2 + y^2}$

$|z| = |\bar{z}|$

Argument

amp (z) = arg (z) = $\theta = \tan^{-1}\dfrac{y}{x}$

General Argument : $2n\pi + \theta,\ n \in \mathbb{N}$

Principal Argument : $-\pi < \theta \leq \pi$

Least Positive Argument : $0 < \theta \leq 2\pi$

Complex Conjugate

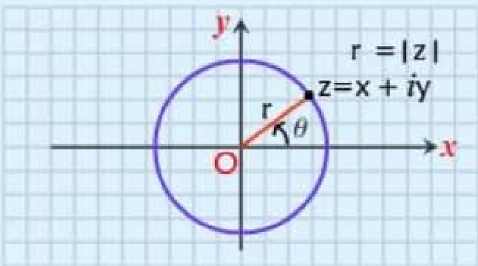
If $z = x + iy$

then the conjugate of 'z' is

$\bar{z} = x - iy$

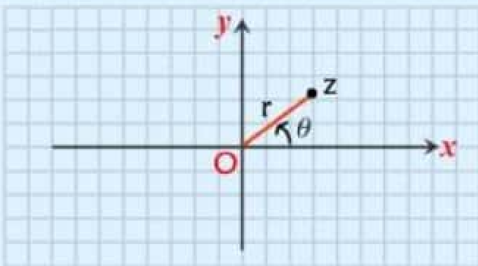
Representation

Polar Representation



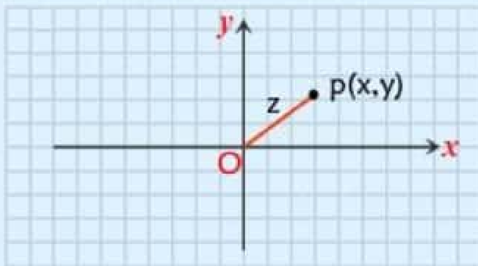
$x = r\cos\theta,\ y = r\sin\theta$

Exponential Form



$z = re^{i\theta}$ (where $e^{i\theta} = \cos\theta + i\sin\theta$)

Vector Representation



$z = x + iy$ may be considered as a position vector of point P.

Properties of argument of a Complex Number

If z, z_1 and z_2 are complex numbers, then

1. arg(any real positive number) = 0
2. arg(any real negative number) = π
3. $\arg(z - \bar{z}) = \pm\dfrac{\pi}{2}$
4. $\arg(z_1 . z_2) = \arg(z_1) + \arg(z_2)$
5. $\arg(z_1 . \bar{z}_2) = \arg(z_1) - \arg(z_2)$
6. $\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2)$
7. $\arg(\bar{z}) = -\arg(z) = \arg(1/z)$
8. $\arg(-z) = \arg(z) \pm \pi$
9. $\arg(z^n) = n\arg(z)$
10. $\arg(z) + \arg(\bar{z}) = 0$
11. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$
12. $|z_1 - z_2| = |z_1 + z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \dfrac{\pi}{2}$
13. $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$
14. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.
15. $|z_1 - z_2|^2 \leq (|z_1| - |z_2|)^2 + (\arg(z_1) - \arg(z_2))^2$
16. $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 + (\arg(z_1) - \arg(z_2))^2$
17. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2)$
18. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ or $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\bar{z}_2)$

COMPLEX NUMBER

Part II

Properties of Complex Conjugate

If $z = a + ib \Rightarrow \bar{z} = a - ib$

- $(\bar{\bar{z}}) = z$
- $z + \bar{z} = 2a = 2\,\mathrm{Re}(z) =$ purely real
- $z - \bar{z} = 2ib = 2i\,\mathrm{Im}(z) =$ purely imaginary
- $z\bar{z} = a^2 + b^2 = |z|^2 = \{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2$
- $z + \bar{z} = 0$ or $z = -\bar{z} \Rightarrow z = 0$ or z is purely imaginary
- $z = \bar{z} \Rightarrow z$ is purely real

Properties of Modulus

- $z\bar{z} = |z|^2$
- $z^{-1} = \dfrac{\bar{z}}{|z|^2}$
- $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}(z_1\bar{z}_2)$
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

Square roots of a Complex Number

The square root of $z = a + ib$ is $\sqrt{a+ib} = \pm\left[\sqrt{\dfrac{|z|+a}{2}} + i\sqrt{\dfrac{|z|-a}{2}}\right]$ for $b > 0$ and $\pm\left[\sqrt{\dfrac{|z|+a}{2}} - i\sqrt{\dfrac{|z|-a}{2}}\right]$ for $b < 0$

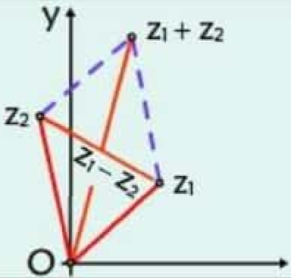
Inequalities

Triangle Inequalities

(1) $|z_1 \pm z_2| \le |z_1| + |z_2|$ (2) $|z_1 \pm z_2| \ge |z_1| - |z_2|$

Parallelogram Identity

(1) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

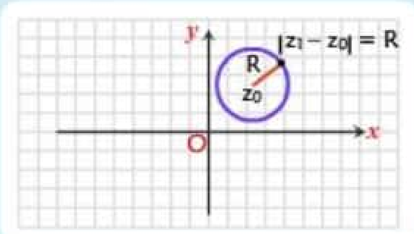


Points to Remember

- If ABC is an equilateral triangle having vertices z_1, z_2, z_3 then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ or $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$
- If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$
- If z_1, z_2, z_3 are affixes of the Points A, B and C in the Argand plane, then

  (a) $\angle BAC = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$ (b) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$

- The equation of a circle whose centre is at point having affix z_0 and radius R is $|z - z_0| = R$



- If a, b are positive real numbers then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$

Integral powers of iota

$i = \sqrt{-1}$ so $i^2 = -1$; $i^3 = -i$ and $i^4 = 1$ $i^{4n+3} = -i$; i^{4n} or $i^{4n+4} = 1$

Hence $i^{4n+1} = i$; $i^{4n+2} = -1$

COMPLEX THEOREM

Statement

(i) if $n \epsilon Z$ (the set of integers), then $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

(ii) if $n \epsilon$ Q (the set of rational number), then $\cos(n\theta) + i\sin(n\theta)$ one of the values of $(\cos\theta + i\sin\theta)^n$.

Roots of Unity

Let $z = a + ib$ be a complex number, and let $r(\cos\theta + i\sin\theta)$ be the polar form of z.

Then by De Moivre's theorem $r^{1/n}\left\{\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right\}$ is one of the values of $z^{1/n}$.

Cube Roots of unity

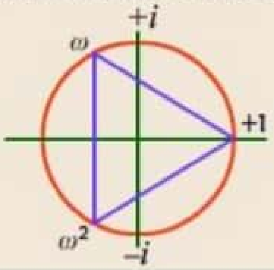
$z = (1)^{1/3}$

Roots : $1, \omega, \omega^2$, where $\omega = e^{i\frac{2\pi}{3}}$

Properties of Cube Roots of Unity

- $1 + \omega^r + \omega^{2r} = 0 \qquad r \neq 3n$
- $\omega = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

  $\omega^2 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
- The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.



n^{th} Roots of unity

$z = (1)^{1/n}$

Roots : $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$

$\alpha_r = e^{i\frac{2\pi r}{n}} = \cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n}$

Properties of n^{th} Roots of Unity

- They are in G.P. with common ratio $e^{i\frac{2\pi}{n}}$
- $1^P + \alpha_1^P + \alpha_2^P + \ldots + \alpha_{n-1}^P = 0$ if $p \neq kn$
- $1^P + (\alpha_1)^P + (\alpha_2)^P + \ldots + (\alpha_{n-1})^P = n$ if $p = kn$
- $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1}) = n$
- $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd
- $(1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots \alpha_{n-1}) = (-1)^{n-1}$
- $(\omega-\alpha_1)(\omega-\alpha_2)\ldots(\omega-\alpha_{n-1}) = \begin{cases} 0 & \text{if } n = 3k \\ 1 & \text{if } n = 3k+1 \\ 1+\omega & \text{if } n = 3k+2 \end{cases}$

Point to Remember

Centroid, Incentre, Orthocentre & Circumcentre of a triangle on a complex plane

(a) Centroid 'G' $= \dfrac{z_1 + z_2 + z_3}{3}$

(b) Incentre 'I' $= \dfrac{a z_1 + b z_2 + c z_3}{a + b + c}$

(c) Orthocentre 'Z_H' $= \dfrac{Z_1 \tan A + Z_2 \tan B + Z_3 \tan C}{\tan A + \tan B + \tan C}$

(D) Circumcentre 'Z_S' $= \dfrac{Z_1(\sin 2A) + Z_2(\sin 2B) + Z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$