

2.a

Sunday, April 17, 2022 4:18 PM

$$TF = \frac{\text{Forward Path}}{1 + \text{loop}}$$

$$T_{yr} = \frac{(I + GK_1)^{-1}}{(I + GK_2)}$$

Deal w/ the pieces

$$(I + GK_1) = (I + G) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^2 + a^2}{s^2 + a^2} & 0 \\ 0 & \frac{s^2 + a^2}{s^2 + a^2} \end{bmatrix} + \begin{bmatrix} \frac{s - a^2}{s^2 + a^2} & \frac{a(s+1)}{s^2 + a^2} \\ \frac{-a(s+1)}{s^2 + a^2} & \frac{s - a^2}{s^2 + a^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^2 + s}{s^2 + a^2} & \frac{a(s+1)}{s^2 + a^2} \\ \frac{-a(s+1)}{s^2 + a^2} & \frac{s^2 + s}{s^2 + a^2} \end{bmatrix} = \frac{s+1}{(s^2 + a^2)} \begin{bmatrix} s & a \\ -a & s \end{bmatrix}$$

$$\Rightarrow (I + GK_1)^{-1} = \left(\frac{s+1}{s^2 + a^2} \begin{bmatrix} s & a \\ -a & s \end{bmatrix} \right)^{-1}$$

Remember the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(I + GK_1)^{-1} = \frac{s^2 + a^2}{s+1} \left(\frac{1}{s^2 + a^2} \right) \begin{bmatrix} s & -a \\ a & s \end{bmatrix}$$

$$(I + GK_1)^{-1} = \frac{1}{s+1} \begin{bmatrix} s & -a \\ a & s \end{bmatrix}$$

$$GK_2 = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix} \frac{1}{1+a^2} \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}$$

$$\begin{aligned}
 GK_2 &= \frac{1}{s^2 + a^2} \begin{bmatrix} s & -a \\ -a(s+1) & s-a^2 \end{bmatrix} \frac{1}{1+a^2} \begin{bmatrix} a & 1 \\ 1 & -a \end{bmatrix} \\
 &= \frac{1}{(s^2 + a^2)(1+a^2)} \begin{bmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{bmatrix} \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} \\
 &= \frac{1}{(s^2 + a^2)(1+a^2)} \begin{bmatrix} (s-a^2) + a^2(s+1) & (as+a^3) + as + a \\ -a(s+1) + (as - a^3) & a^2(s+1) + (s-a^2) \end{bmatrix} \\
 &= \frac{1}{(s^2 + a^2)(1+a^2)} = \begin{bmatrix} s(1+a^2) & a(1+a^2) \\ -a(1+a^2) & s(1+a^2) \end{bmatrix} \\
 \boxed{GK_2 = \frac{1}{(s^2 + a^2)} \begin{bmatrix} s & a \\ -a & s \end{bmatrix}}
 \end{aligned}$$

Put it all together

$$\begin{aligned}
 (1+GK_1)^{-1}(GK_2) &= \frac{1}{(s+1)(s^2 + a^2)} \begin{bmatrix} s & -a \\ a & s \end{bmatrix} \begin{bmatrix} s & a \\ -a & s \end{bmatrix} \\
 &= \frac{1}{(s+1)(s^2 + a^2)} \begin{bmatrix} s^2 + a^2 & as - a^2s \\ as - a^2s & s^2 + a^2 \end{bmatrix} \\
 \boxed{T_{yr} = \frac{1}{(s+1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}
 \end{aligned}$$