Polygon Triangulation

Simple Polygons

- Definition

1. A polygon is the region of a plane bounded by a finite collection of line segments forming a **simple closed curve**.

2. “Simple closed curve” means a certain deformation of a circle.

Simple Polygons

- Definition

1. Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be n points in the plane in the cyclic ordering.
2. Let $e_0 = v_0v_1$, $e_1 = v_1v_2$,…, $e_i = v_iv_{i+1}$,…, $e_{n-1} = v_{n-1}v_0$ be n segments connecting the points.
3. These segments bound a polygon *iff*
    1. The intersection of each pair of segments adjacent in the cyclic ordering is the single point shared between them.
    2. Nonadjacent segments do not intersect.

Simple Polygons

1. The reason these segments define a **curve** is that they are connected end to end
2. The reason the curve is **closed** is that they form a cycle
3. The reason the closed curve is **simple** is that nonadjacent segments do not intersect.

Jordan Curve Theorem

Every simple closed plane curve divides the plane into two components.

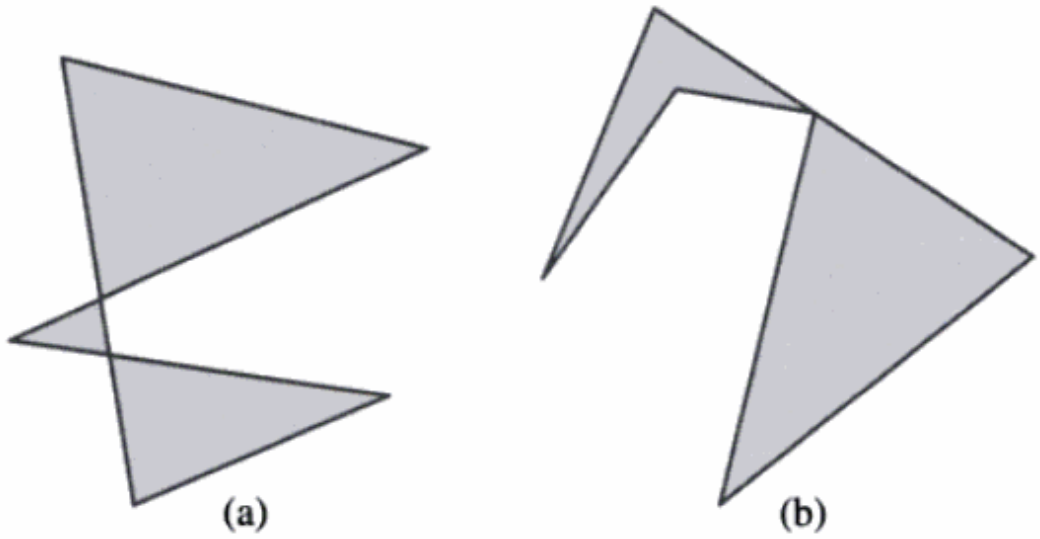


FIGURE 1.1 Nonsimple polygons.

- Nonsimple polygons
  1. For both objects in the figure, the segments satisfy condition (1) (adjacent segments share a common point)
  2. but not condition (2): nonadjacent segments intersect.

Visibility

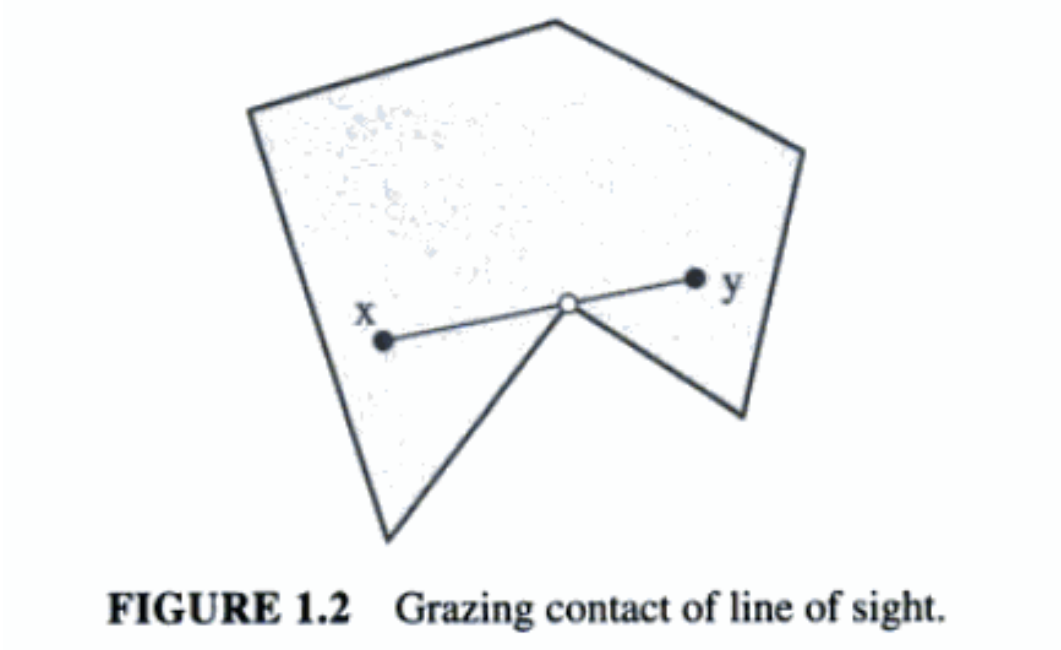


FIGURE 1.2 Grazing contact of line of sight.

- Visibility
  1. Point x can see point y (or y is visible to x) *iff* the closed segment xy is nowhere exterior to the polygon P ($xy \subseteq P$)

Visibility

- Visibility
  1. x has clear visibility to y if $xy \subseteq P$ and $xy \cap \partial P \subseteq \{x,y\}$
  2. ∂P means the boundary of a polygon P
  3. By definition, $\partial P \subseteq P$
  4. A guard is a point.
  5. A set of guards is said to cover a polygon if every point in the polygon is visible to some guard.

Triangulation

- Diagonals and Triangulation
  1. A diagonal of a polygon P is a line segment between two of its vertices a and b that are clearly visible to one another
  2. The open segment from a to b does not intersect ∂P; thus a diagonal cannot make grazing contact with the boundary.
  3. Two diagonals are noncrossing if their intersection is a subset of their endpoints. They share no interior points.
  4. A triangulation of a polygon P
     - Add as many noncrossing diagonals to a polygon as possible so that the interior can be partitioned into triangles.

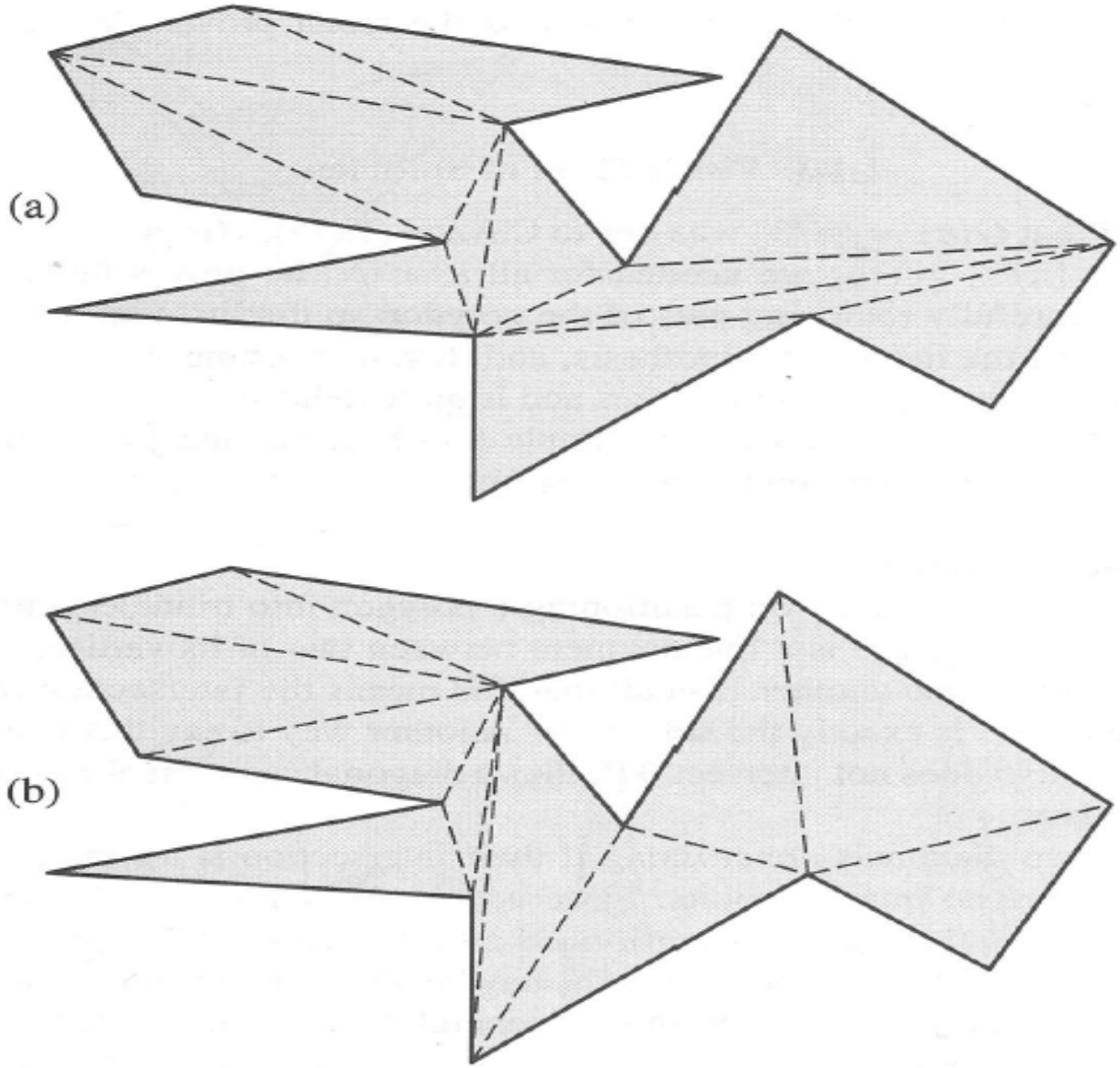
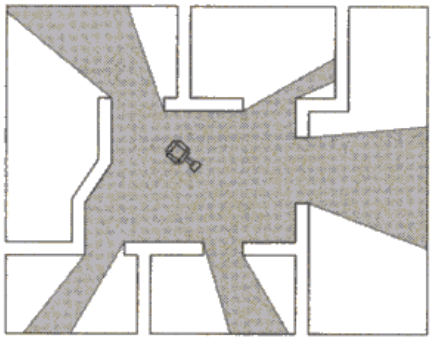
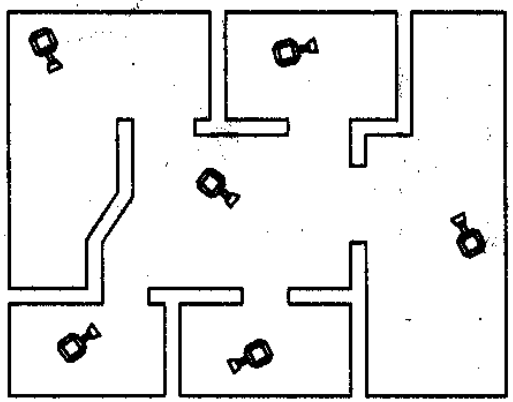


FIGURE 1.6 Two triangulations of a polygon of $n = 14$ vertices.

The Art Gallery Theorem

- Problem Definition
  1. A problem posed by **Klee** [1973]
  2. Imagine an art gallery room whose floor plan can be modeled by a polygon of n vertices. How many stationary guards are needed to guard the room?
  3. Each guard is considered a **fixed** point that can see in every direction
  4. Of course a guard cannot see through a wall of the room.





The Art Gallery Theorem

- Max over Min Formulation

1. For any given fixed polygon
   - there is some minimum number of guards that are necessary for complete coverage

2. Consider all polygons of n vertices
   - Find the maximum (worst case) of the fewest guards needed to cover the polygon

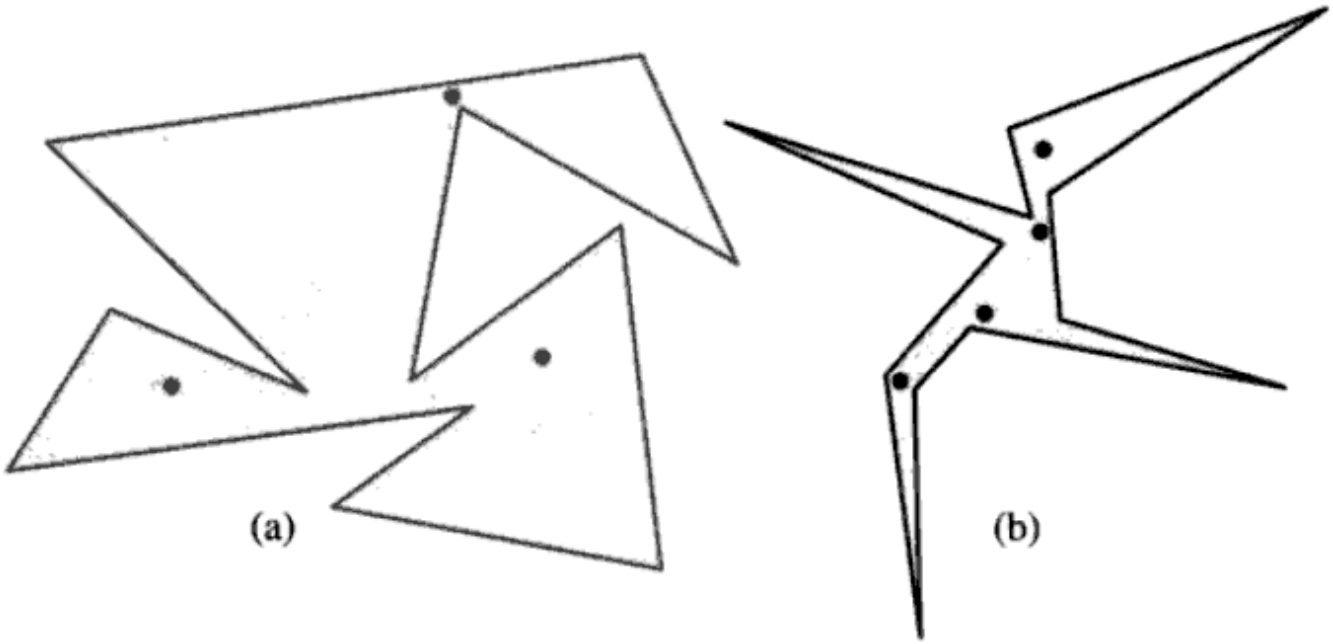


FIGURE 1.3 Two polygons of $n = 12$ vertices: (a) requires 3 guards; (b) requires 4 guards.

1. What is the largest number of guards that any polygon of 12 vertices needs?
2. Express as a function of n, the smallest number of guards that suffice to cover any polygon of n vertices.

The Art Gallery Theorem

- Max over Min Formulation

1. For any given fixed polygon
   - Let g(P) be the smallest number of guards needed to cover polygon P

2. Consider all polygons of n vertices
   - G(n) is the maximum of g(P_n) over all polygons of n vertices: G(n) = $\max_{Pn}$ g(P_n)

The Art Gallery Theorem

- Empirical Exploration
  1. Certainly at least one guard is always n necessary. A lower bound $1 \leq G(n)$
     - example?
  2. n guards suffice for any polygon. An upper bound $G(n) \leq n$
     - why?
  3. For small n, we can guess $G(n)$
     - For example, $G(3) = 1$. Clearly every triangle requires just one guard.

The Art Gallery Theorem

- Empirical Exploration

1. n=4 (convex and reflex)
   - Intuitively a polygon is convex if it has no dents.
   - A vertex is called reflex if its internal angle is strictly greater than π. Otherwise convex.
   - A quadrilateral can have at most one reflex vertex
   - Even quadrilaterals with a reflex vertex can be covered by a single guard placed near that vertex (Figure 1.4(a))
2. Thus G(4) = 1

The Art Gallery Theorem

- Empirical Exploration

1. $n=5$ (convex, 1 reflex, and 2 reflex)
   - A convex pentagon needs just one
   - A pentagon with one reflex vertex needs only one guard for the same reason as in a quadrilateral
   - Two reflex vertices may be either adjacent or separated by a convex vertex; in each case one guard suffices. (Figure 1.4(c) and (d))
2. Thus G(5) = 1
3. G(6) = 2 (Figure 1.4(e) and (f))

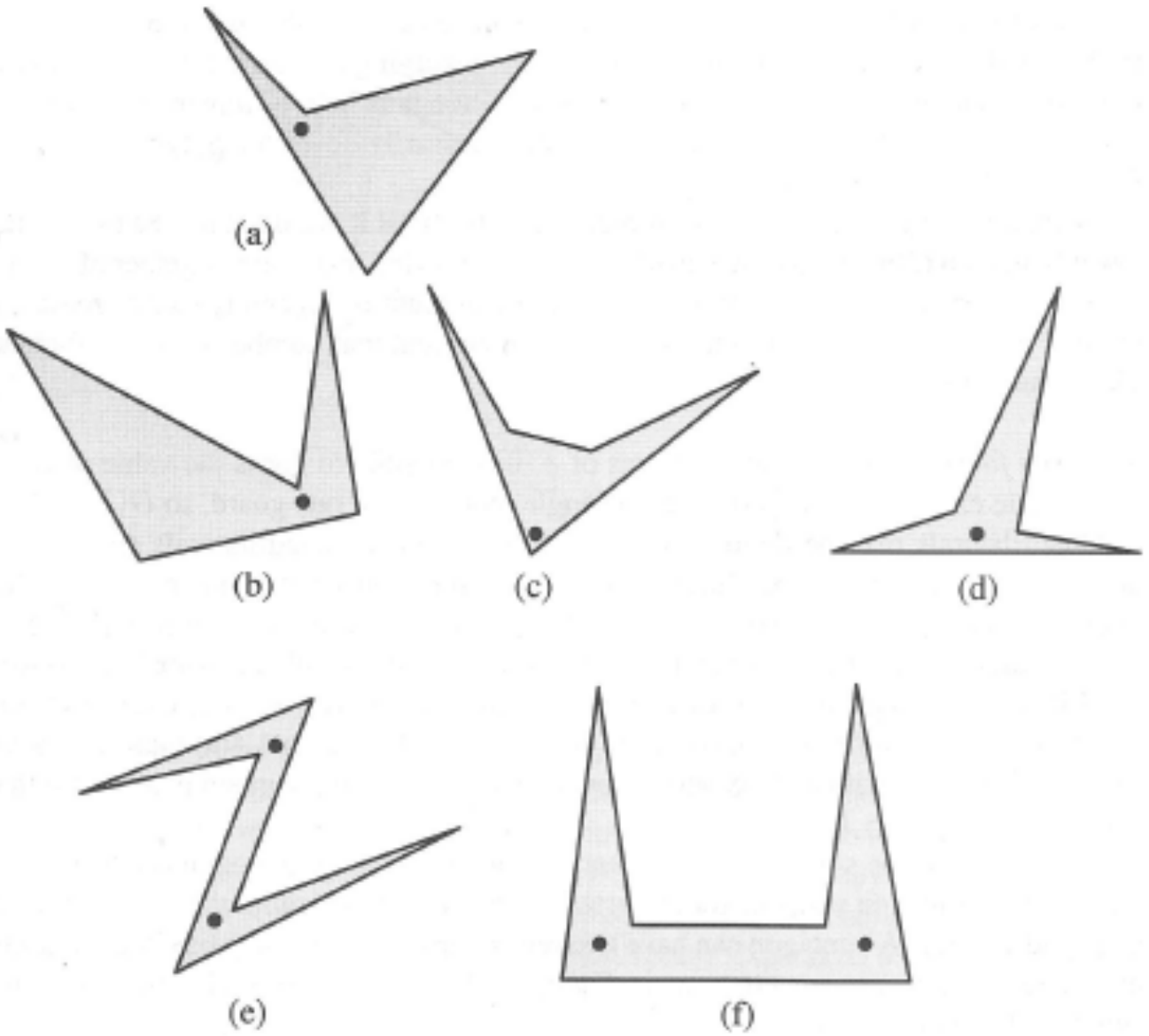


FIGURE 1.4 Polygons of $n = 4, 5, 6$ vertices.

The Art Gallery Theorem

- Necessity $O(\lfloor n/3 \rfloor)$

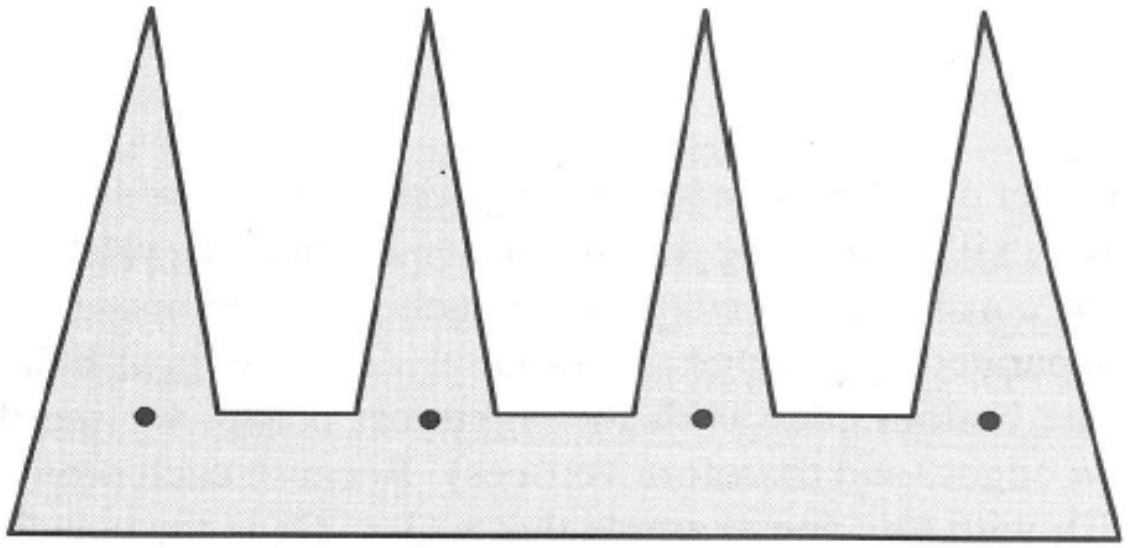


FIGURE 1.5 Chvátal's comb for $n = 12$.

- This "comb" shape consists of k teeth, with each tooth composed of two edges, and adjacent teeth separated by an edge.

The Art Gallery Theorem

- Necessity O⌊ n/3 ⌋

1. A comb of k teeth has $n = 3k$ edges
2. Because each tooth requires its own guard, $n/3 \leq G(n)$ for $n = 3k$
3. Notice G(3) = G(4) = G(5) might lead one to conjecture that $G(n) = \lfloor n/3 \rfloor$

Fisk's Proof

- Three Coloring
  1. To prove sufficiency of $\lfloor n/3 \rfloor$ guards, three coloring is used.
  2. A k-coloring of a graph is an assignment of k colors to the nodes of the graph, such that no two nodes connected by an arc are assigned the same color.
  3. Three-colorings of the triangulations in Figure 1.6 are shown in Figure 1.7
  4. Starting at the vertex indicated by the arrow, and coloring its triangle arbitrarily with three colors, the remainder of the coloring is completely forced.

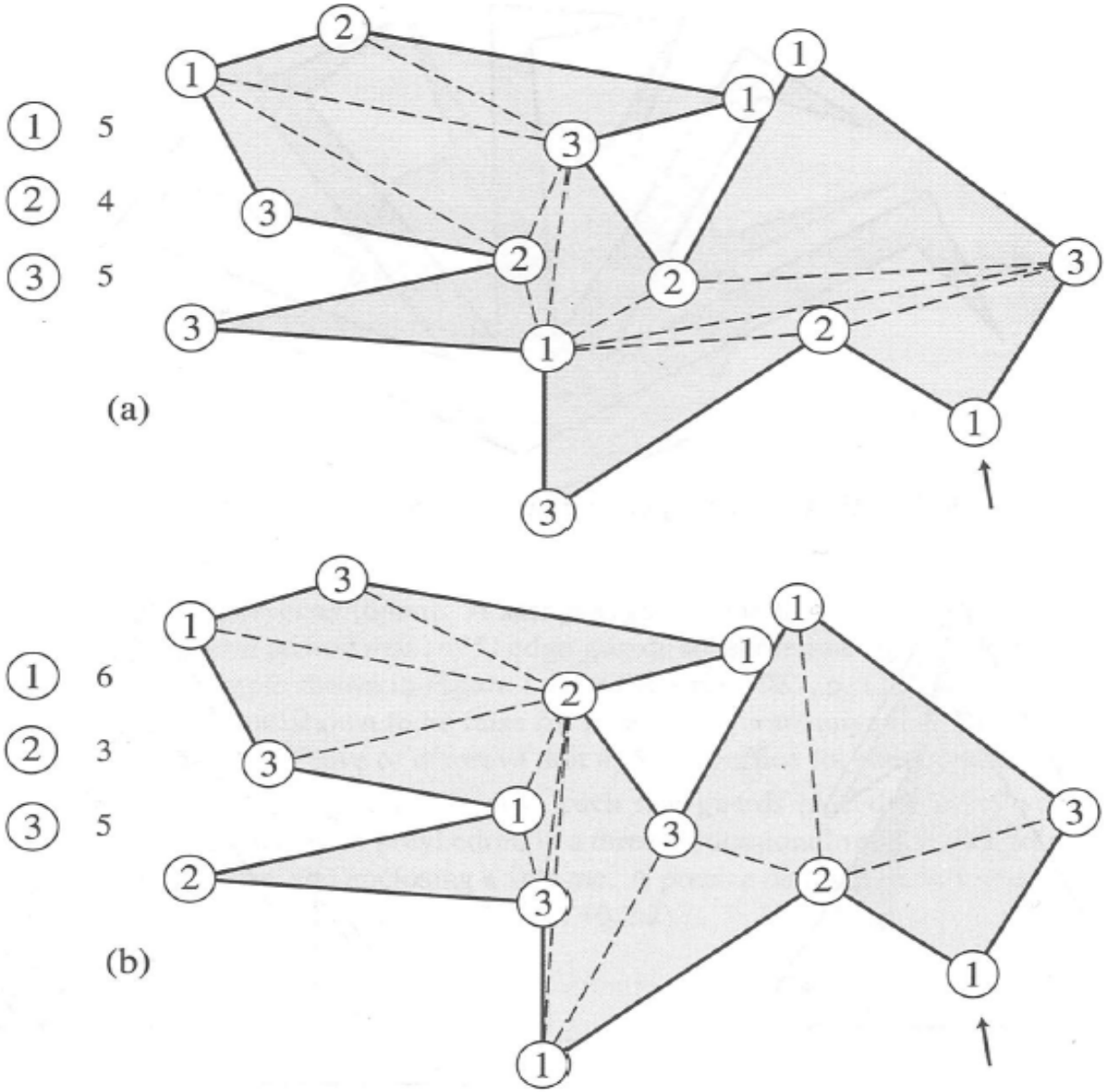


FIGURE 1.7 Two 3-colorings of a polygon of $n = 14$ vertices, based on the triangulations shown in Figure 1.6.

Fisk's Proof

- Three Coloring
  1. Each triangle must have each of the three colors(red, green, blue) at its three corners.
  2. Thus every triangle has a red(green or blue) node at one corner.
  3. Suppose guards are placed at every red (green or blue) node. Then every triangle has a guard in one corner
  4. Thus, the entire polygon is covered.

Polygon Triangulation

Triangulation Theory

Existence of a Diagonal

- Lemma 1.2.1
  1. Every polygon must have at least one strictly convex vertex.
  2. Proof

Existence of a Diagonal

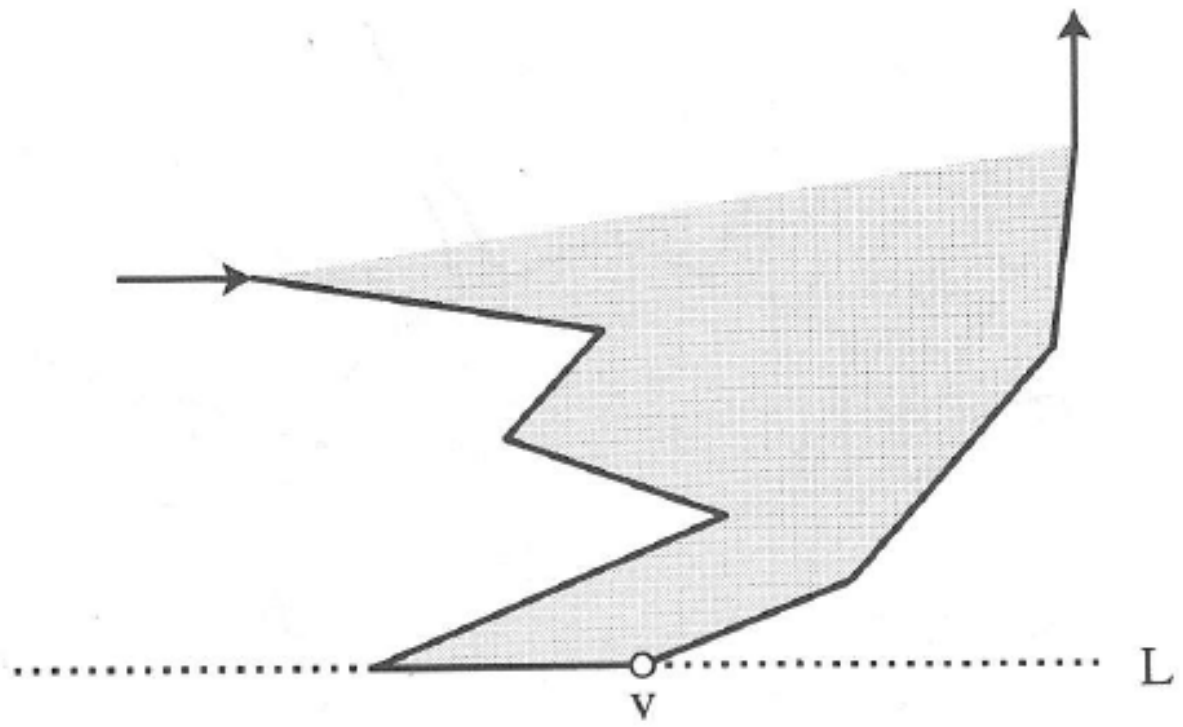


FIGURE 1.11 The rightmost lowest vertex must be strictly convex.

Existence of a Diagonal

- Lemma 1.2.2 (Meisters)
  1. Every polygon of $n \geq 4$ vertices has a diagonal.
  2. Proof

Existence of a Diagonal

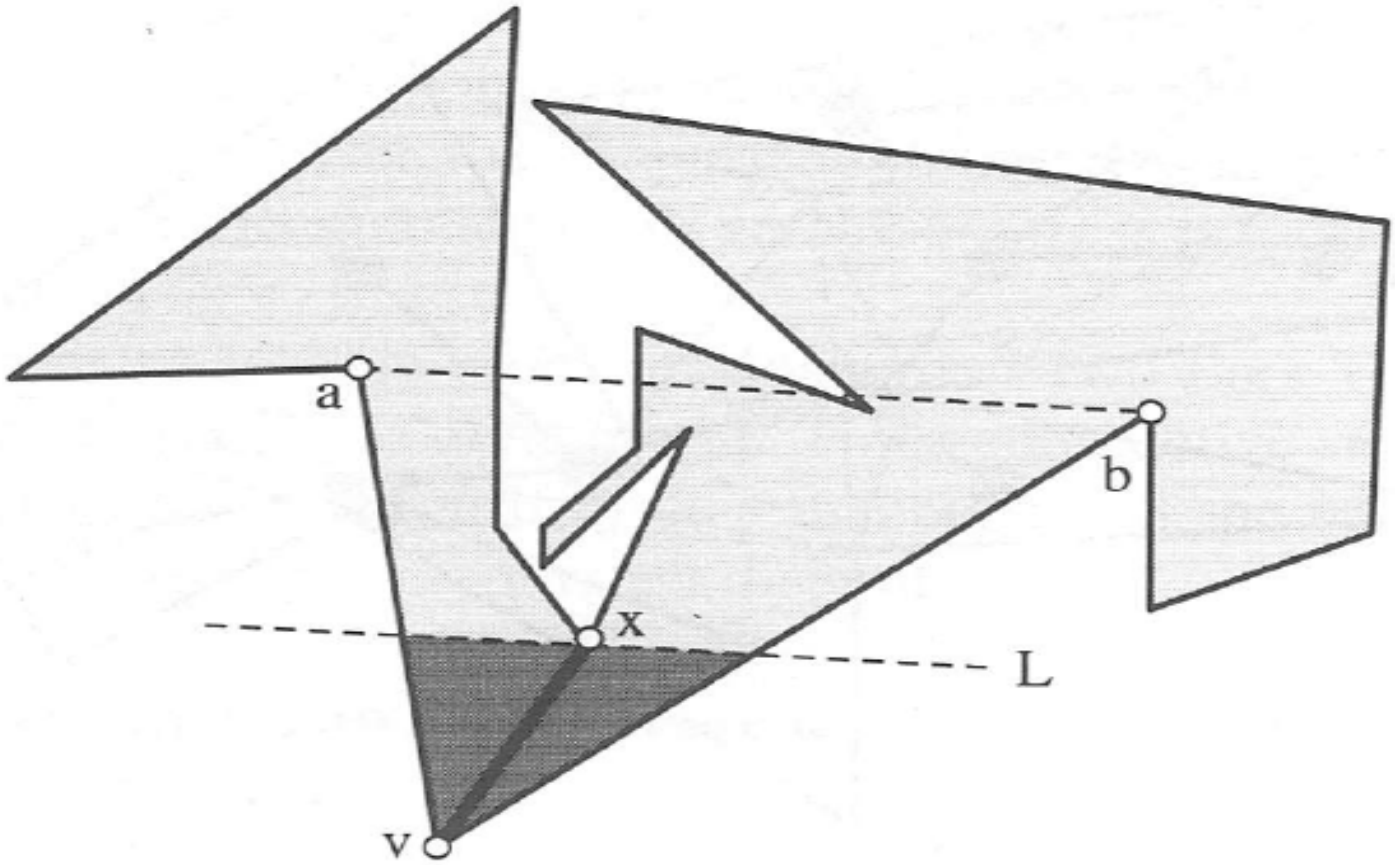


FIGURE 1.12 vx must be a diagonal.

Existence of Triangulation

- Lemma 1.2.3(Triangulation)
  1. Every polygon P of n vertices may be partitioned into triangles by the addition of (zero or more) diagonals.
  2. Proof (by induction)

Existence of Triangulation

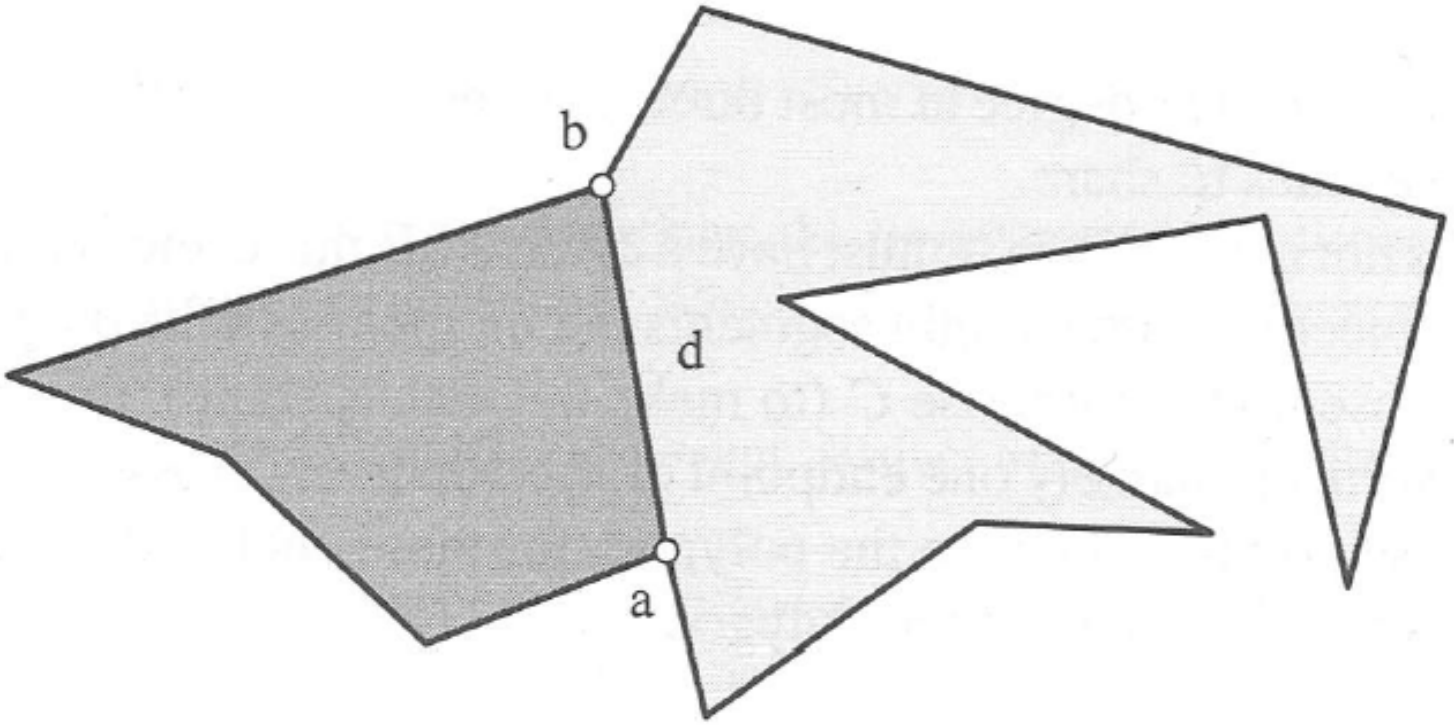


FIGURE 1.13 A diagonal partitions a polygon into two smaller polygons.

Properties of Triangulations

- Lemma 1.2.4 (Number of Diagonals)
  1. Every triangulation of a polygon P of n vertices uses n – 3 diagonals and consists of n – 2 triangles.
  2. Proof (by induction)

Triangulations Dual

1. The *dual* T of triangulation of a polygon is a graph with a node associated with each triangle and an arc between two nodes iff

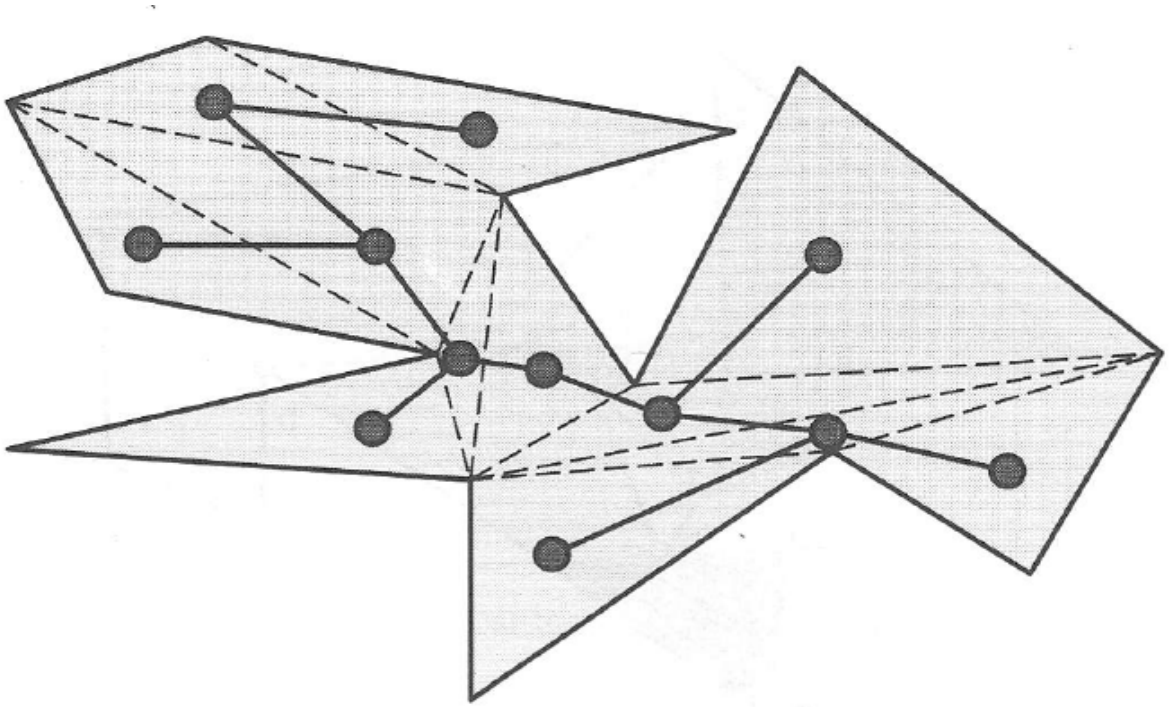


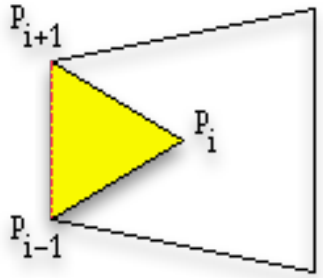
FIGURE 1.14 Triangulation dual.

Triangulation Dual

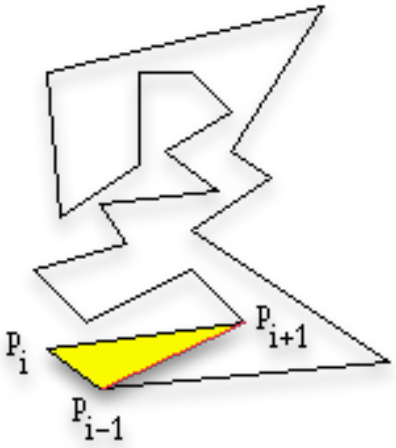
- Lemma 1.2.6
  1. The dual T of a triangulation is a **tree**, with each node of degree at most **three**.
  2. Proof

Triangulation Dual

1. Three consecutive vertices of a polygon *a*, *b*, *c* form an ear of the polygon if *ac* is a diagonal; *b* is the ear tip.



P_i is not an ear

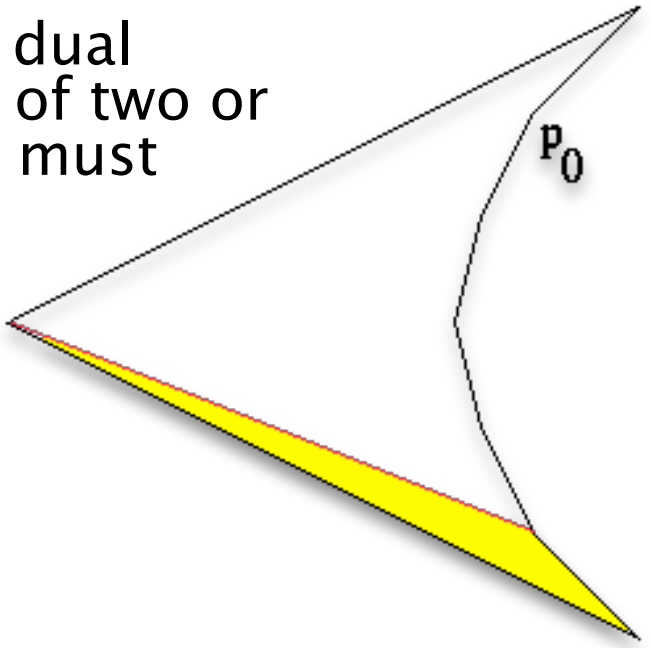


P_i is an ear

Triangulation Dual

- Meisters's Two Ears Thm [1975]
  1. Every polygon of $n \geq 4$ vertices has at least two non-overlapping ears.
  2. Proof
     - A leaf node in a triangulation dual corresponds to an ear. A tree of two or more nodes(by Lemma 1.2.4) must have at least two leaves.

Base Case: A quadrilateral has only 2 ears.



A simple polygon with only 2 ears.

Polygon Triangulation

Area of polygon

Area of a Triangle

- Let us denote this area as $A(T)$
- The area is one half the base times the altitude.
- The base is easy: $|a-b|$ (the length of the vector $a-b$)
- What about the altitude?
  - not easy.

Cross Product

- Recall the magnitude of the cross product of two vectors is the area of the parallelogram.
- A triangle is half of a parallelogram.
- Thus, the area of triangle whose three vertices are arbitrary points a, b, c is half the length of A X B
- A = $b - a$ and B = $c - a$

Area of a Convex Polygon

- Find the area of any polygon by using an expression for the area of a triangle.
- First triangulation
- Every convex polygon may be triangulated as a "fan," with all diagonals incident to a common vertex.
- The area of a polygon with vertices v_0, v_1, ... ,v_{n-1} labeled counterclockwise (Figure 1.16) can be calculated as

$$\mathcal{A}(Q) = \mathcal{A}(a, b, c) + \mathcal{A}(a, c, d) = \mathcal{A}(d, a, b) + \mathcal{A}(d, b, c).$$

Area of a Convex Polygon

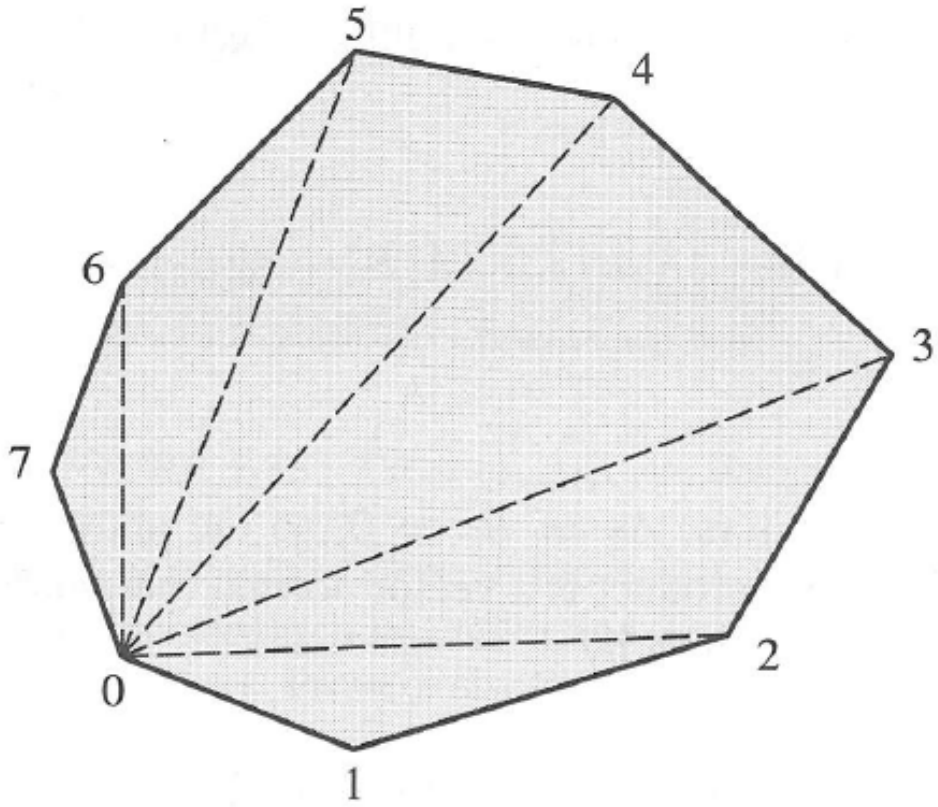


FIGURE 1.16 Triangulation of a convex polygon. The fan center is at 0.

Area of a Convex Quadrilateral

- Two different triangulations of a convex quadrilateral $Q = (a, b, c, d)$

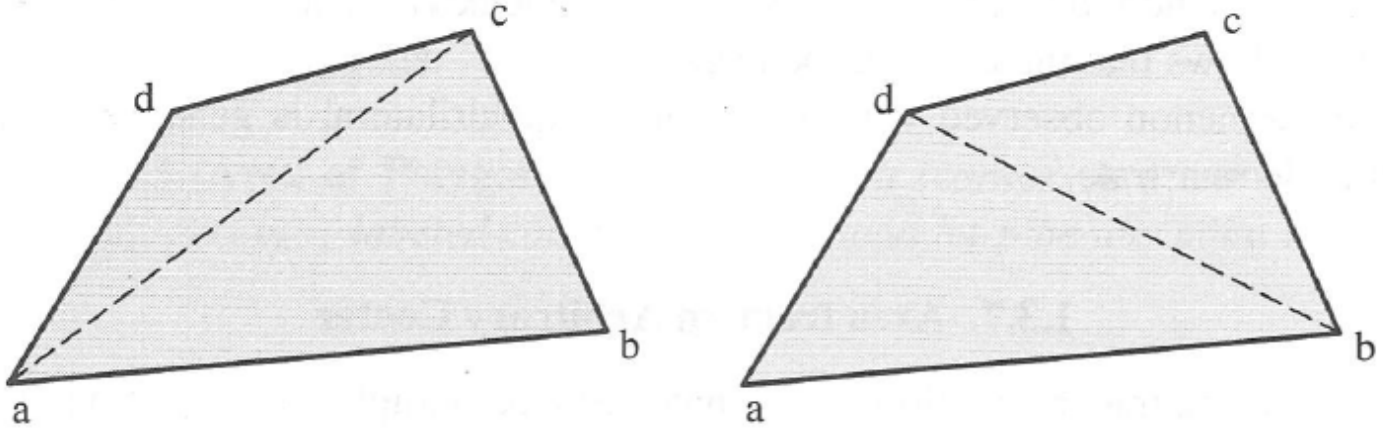


FIGURE 1.17 The two triangulations of a convex quadrilateral.

- The area may be written in two ways.

$$\mathcal{A}(Q) = \mathcal{A}(a, b, c) + \mathcal{A}(a, c, d) = \mathcal{A}(d, a, b) + \mathcal{A}(d, b, c).$$

Area of a Convex Quadrilateral

- Applying Equation (1.2) to

$$\mathcal{A}(Q) = \mathcal{A}(a, b, c) + \mathcal{A}(a, c, d)$$

- We get

$$\begin{aligned} 2\mathcal{A}(Q) = {} & a_0b_1 - a_1b_0 + a_1c_0 - a_0c_1 + b_0c_1 - c_0b_1 \\ & + a_0c_1 - a_1c_0 + a_1d_0 - a_0d_1 + c_0d_1 - d_0c_1. \end{aligned}$$

- Notice ac and db cancel

Area of a Convex Quadrilateral

- Generalizing, we get two terms per polygon edge
- None for internal diagonals.
- So if the coordinates of vertex v_i are x_i and y_i, twice the area of a convex polygon is given by

$$2\mathcal{A}(P) = \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1}).$$

Area of a Nonconvex Quadrilateral

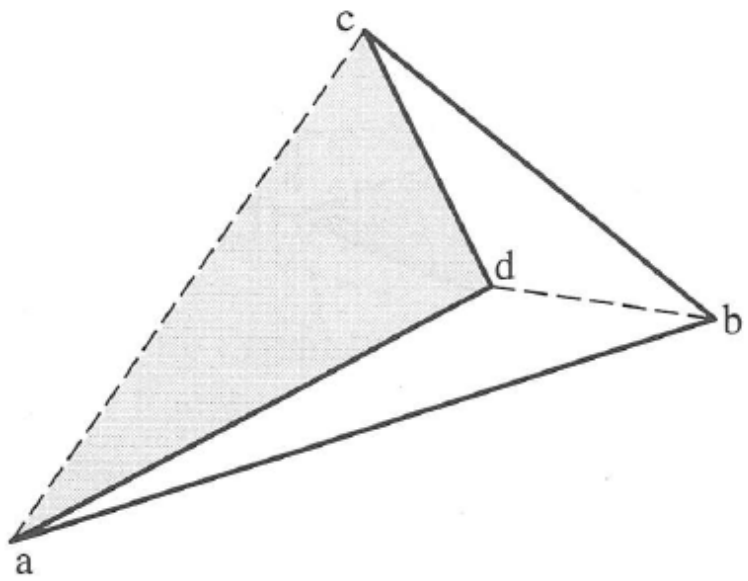


FIGURE 1.18 Triangulation of a nonconvex quadrilateral. The shaded area $\mathcal{A}(a, d, c)$ is negative.

- Can we use the same equation to calculate the area of a nonconvex quadrilateral?

Area of a Nonconvex Quadrilateral

- Suppose we have a nonconvex quadrilateral $Q = (a, b, c, d)$ (Figure 1.18)
- Only one triangulation, using the diagonal db.
- But the algebraic expression obtained is independent of the diagonal chosen

$$\mathcal{A}(Q) = \mathcal{A}(a, b, c) + \mathcal{A}(a, c, d)$$

- The equation is still true, even though the diagonal ac is external to Q.

Area of a Nonconvex Quadrilateral

- Notice $A(a, c, d)$ is negative and
- the area of a nonconvex quadrilateral is $\triangle abc$ minus $A(a, c, d)$.
- Indeed, $A(a, c, d)$ is a clockwise path, so the cross product formulation shows that the area will be negative.
- The phenomenon observe with a nonconvex quadrilateral is general

Area from an Arbitrary Center

- Let $T = \triangle abc$ be a triangle, with the vertices oriented counterclockwise,
- and let p be any external point in the plane.
- Then, we claim

$$\mathcal{A}(T) = \mathcal{A}(p, a, b) + \mathcal{A}(p, b, c) + \mathcal{A}(p, c, a).$$

Area from an Arbitrary Center

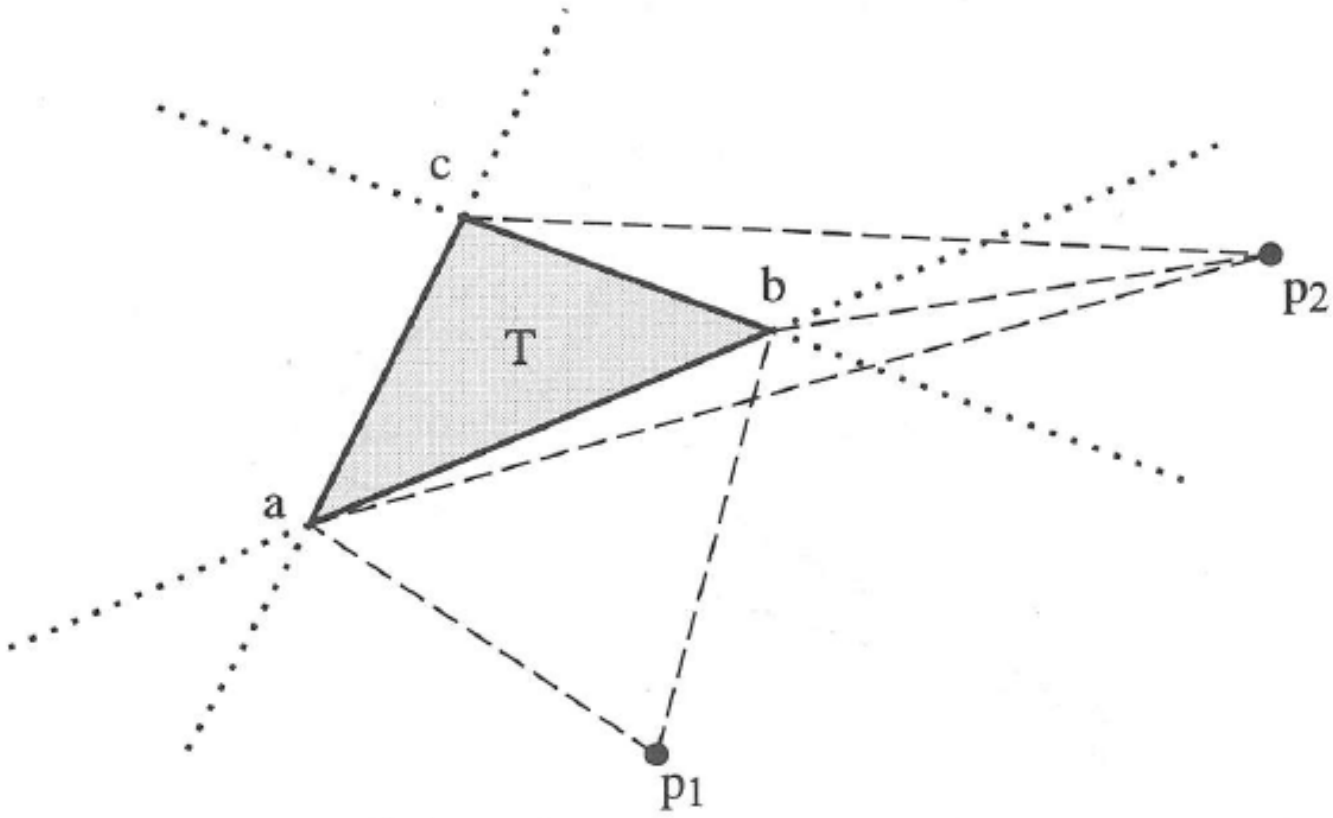


FIGURE 1.19 Area of T based on various external points p_1, p_2.

Area from an Arbitrary Center

- Case 1: $p = p_1$
  1. $A(p_1, a, b)$ is negative (clockwise)
  2. $A(p_1, b, c)$ is positive (counterclockwise)
  3. $A(p_1, c, a)$ is positive (counterclockwise)
- Case 2: $p = p_2$
  1. $A(p_2, a, b)$ is negative (clockwise)
  2. $A(p_2, b, c)$ is negative (clockwise)
  3. $A(p_2, c, a)$ is positive (counterclockwise)
- All other positions for external p in the plane are equivalent to either p_1 or p_2 by symmetry

Area from an Arbitrary Center

- Lemma 1.3.2
  1. If $T = \triangle abc$ is a triangle, with vertices oriented counterclockwise, and p is any point in the plane, then

$$\mathcal{A}(T) = \mathcal{A}(p, a, b) + \mathcal{A}(p, b, c) + \mathcal{A}(p, c, a).$$

Area from an Arbitrary Center

- Lemma 1.3.3 (Area of Polygon)
  1. Let a polygon (convex or nonconvex) P have vertices $v_0, v_1, \ldots, v_{n-1}$ labeled counterclockwise, and let P be any point in the plane. Then

$$\mathcal{A}(P) = \mathcal{A}(p, v_0, v_1) + \mathcal{A}(p, v_1, v_2) + \mathcal{A}(p, v_2, v_3) + \cdots + \mathcal{A}(p, v_{n-2}, v_{n-1}) + \mathcal{A}(p, v_{n-1}, v_0). \tag{1.12}$$

  - If $v_i = (x_i, y_i)$, this expression is equivalent to the equations

$$2\mathcal{A}(P) = \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1}) \tag{1.13}$$

$$= \sum_{i=0}^{n-1} (x_i + x_{i+1})(y_{i+1} - y_i). \tag{1.14}$$

Area from an Arbitrary Center

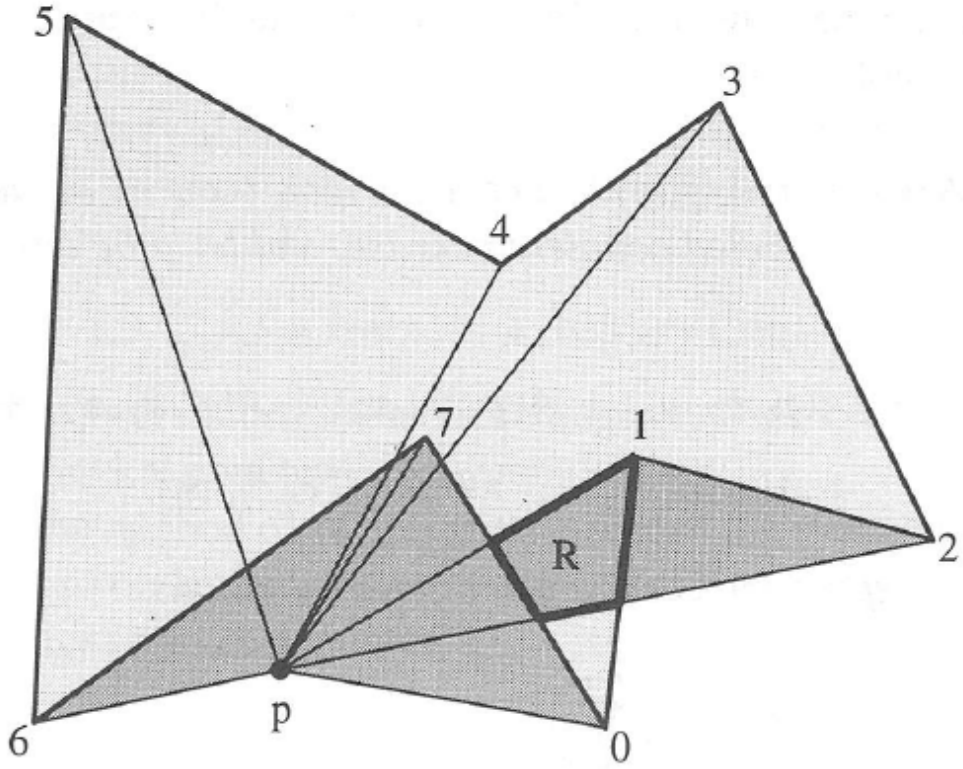


FIGURE 1.20 Computation of the area of a nonconvex polygon from point p. The darker triangles are oriented clockwise, and thus they have negative area.

Volume in Three and Higher Dimensions

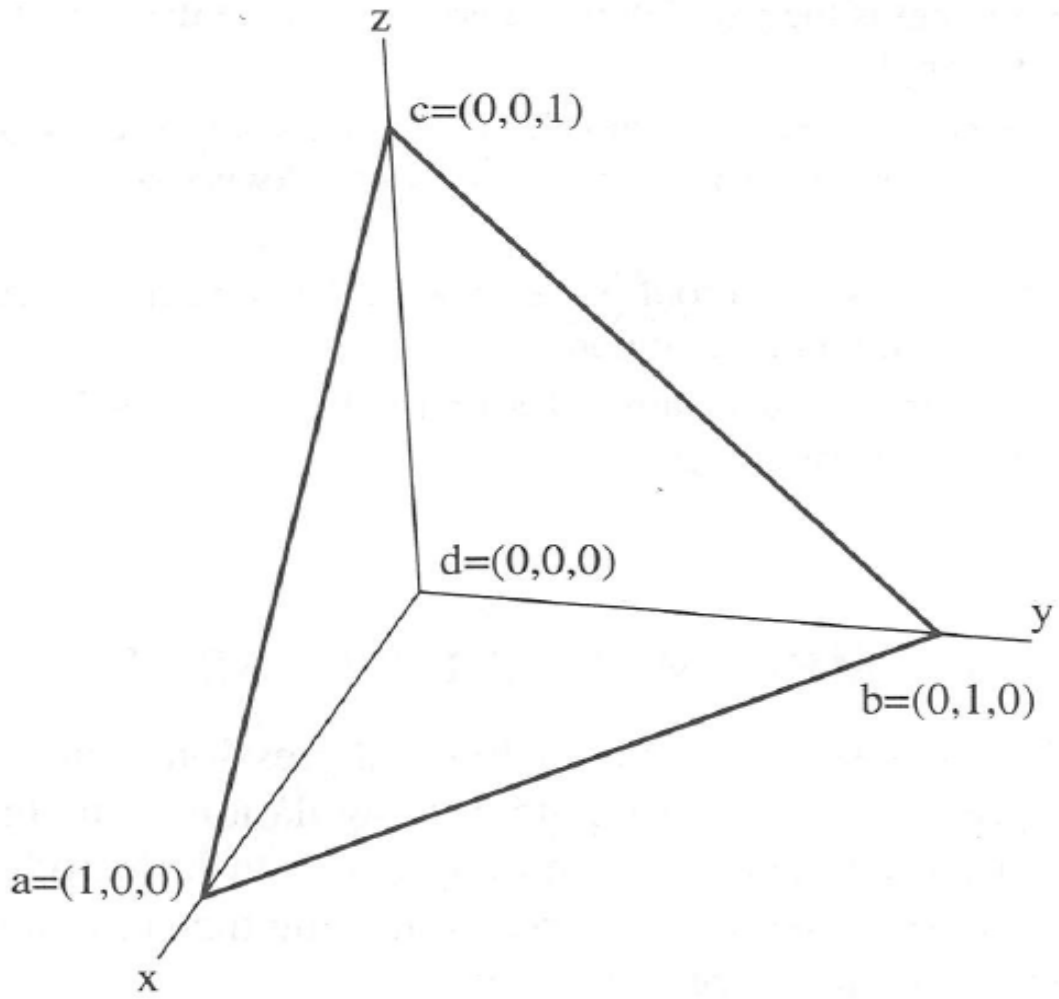


FIGURE 1.21 Tetrahedron at the origin.

Volume in Three and Higher Dimensions

- This volume is signed
- It is positive if (a, b, c) form a counterclockwise viewed from the side away from d,
- So that the face normal determined by the right-hand rule points toward the outside. (Figure 1.21)
- The volume of a polyhedron may be computed by summing the (signed) volumes of tetrahedra formed by an arbitrary point and each triangular face of the polyhedron

Polygon Triangulation

Data representation

Representation of a point

- Array versus Records
  1. Arrays of the appropriate number of coordinates to represent all points
  2. Easier to understand
  3. Generates to higher dimensions more easily
- Integers versus Reals
  1. Integer coordinates rather than floating-point number coordinates
  2. No issue of floating-point round off error
  3. Verifiably correct within a range of coordinate values

Representation of a point

- Point Type Definition
  1. All type identifiers begin with lowercase t
  2. All defined constants entirely uppercase.
  3. The suffixes i and d indicate integer and double types respectively

```
#define X 0
#define Y 1
typedef enum {FALSE, TRUE} bool;

#define DIM 2                   /* Dimension of points */
typedef  int tPointi[DIM];      /* Type integer point */
```

Code 1.1 Point type.

Representation of a Polygon

- Main options
  1. Array or list
  2. Singly or doubly linked list
  3. Linear or circular
- Arrays
  1. Attractive for code clarity
  2. Clumsy with insertion and deletion of points.
- We sacrifice simplicity to gain ease of deletion.
- Use a doubly linked circular list to represent a polygon

Representation of a Polygon

- The data structure for a single vertex
  1. tVertexStructure
  2. next and prev for links to adjacent vertices
  3. vnum is integer index for printout
  4. vertices is global for “head” of the list

```
typedef          struct tVertexStructure tsVertex;          /* Used only in NEW().
*/
typedef          tsVertex *tVertex;
struct tVertexStructure {
   int      vnum;                  /* Index */
   tPointi              v;                  /* Coordinates */
   bool                 ear;                /* TRUE iff an ear */
   tVertex              next,prev;
};
tVertex          vertices  = NULL;  /* "Head" of circular list. */
```

Code 1.2 Vertex structure.

Representation of a Polygon

- Loop to process all vertices

```
tVertex v;
v = vertices;
do {
    /* Process vertex v */
    v = v->next;
} while ( v != vertices );
```

Code 1.3 Loop to process all vertices.

Representation of a Polygon

- Two basic list processing routines
  1. Allocating a new element (NEW)
  2. Adding a new element to the list (ADD)
- As macros, with NEW taking the type as one parameter.
  1. Macros are text based and oblivious to types.
- ADD first checks to see if head is non-NULL
  1. If so, insert the cell prior to head
  2. If not, head points to the added cell
- See Code 1.4

```
#define EXIT FAILURE 1
char *malloc();

#define NEW(p, type)  \
        if ((p=(type *) malloc (sizeof(type))) == NULL) {\
        printf ("NEW: Out of Memory!\n");\
        exit(EXIT_FAILURE);\
   }

#define ADD( head, p )  if ( head )  { \
         p->next = head; \
         p->prev = head->prev; \
         head->prev = p; \
        p->prev->next = p; \
   } \
   else { \
        head = p; \
        head->next = head->prev = p; \
   }
#define FREE(p)      if (p)  {free ((char *) p); p = NULL; }
```

Code 1.4 New and ADD macros. (The backslashes continue the lines so that the preprocessor does not treat those as command lines.) FREE is used in Chapter 3 and 4.

Code for Area

- Computing the area of a polygon is now a straightforward implementation of Equations (1.12) or (1.13)

- Potential problem with Area2
  1. If the coordinates are large, the multiplications of coordinates could cause integer word overflow

```
int          Area2( tPointi a, tPointi b, tPointi c )
{
  return
              (b[X] - a[X]) * (c[Y] - a[Y]) -
              (c[X] - a[X]) * (b[Y] - a[Y]);
}

int      AreaPoly2( void )
{
  int         sum = 0;
  tVertex   p, a;

  p = vertices;       /* Fixed. */
  a = p->next;        /* Moving. */
  do {
      sum += Area2( p->v, a->v, a->next->v );
      a = a->next;
  } while ( a->next != vertices );
  return sum;
}
```

Code 1.5 Area2 and AreaPoly2

Polygon Triangulation

Segment Intersection

Diagonals

- Our goal is to develop code to triangulate a polygon.
- The key step is finding a diagonal
- Lemma 1.5.1
  1. for all edges e of P that are not incident to either v_i or v_j, s and e do not intersect: $s \cap e = \emptyset$
  2. s is internal to P in a neighborhood of v_i and v_j.

Diagonals

- Condition (1) of Lemma 1.5.1
  1. The "diagonalhood" of a segment can be determined without finding the actual point of intersection between s and each e
  2. Only a Boolean segment intersection predicate
- Condition (2) of Lemma 1.5.1
  1. Distinguish internal from external diagonals
  2. Rule out collinear overlap with an incident edge

Problems with Slopes

- The point of intersection
  1. By solving two linear equations in slope-intercept form
  2. Clear method
  3. Not all that difficult to code
- Problem
  1. Code is messy and error prone
  2. Need to deal with two special cases
  3. Vertical segments whose slope is infinite
  4. Parallel segments whose containing lines do not intersect
  5. Both case lead to division by zero

Left turn

- The point of intersection
  1. Can be decided by using a Left predicate
- A directed line is determined by two points given (a, b)
- If c is to the left of the line, then the triple (a,b,c) forms counterclockwise circuit. (see figure 1.22)
- c is to the left of (a,b) iff $A(a,b,c)$ is positive
  - This can be implemented by a single call to Area2 (See code 1.6)
- Straightforward but subject to the special case objections raised earlier.

Left turn

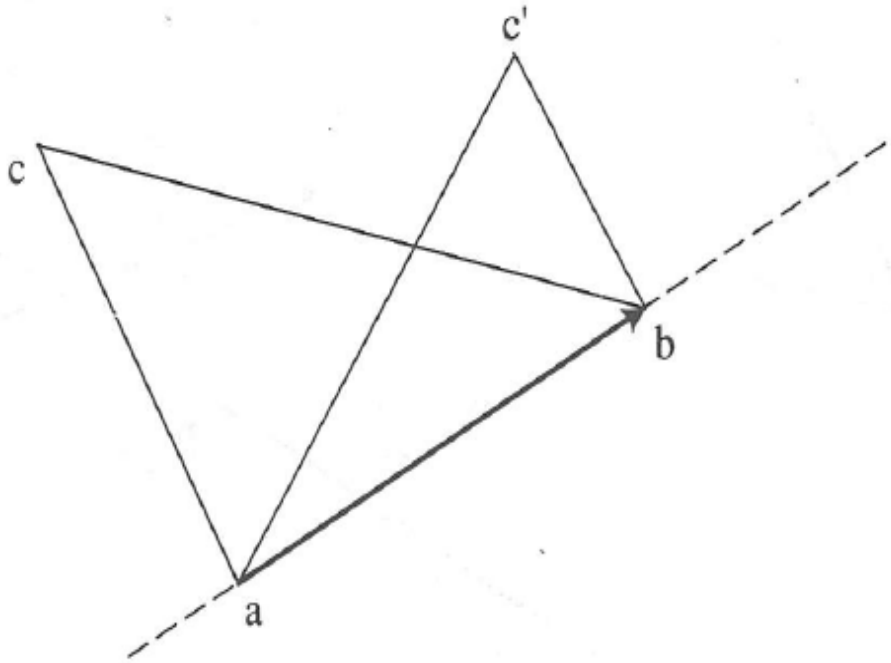


FIGURE 1.22 c is left of ab iff $\triangle abc$ has positive area; $\triangle abc'$ also has positive area.

Left turn

```
bool   Left( tPointi a, tPointi b, tPointi c )
{
  return  Area2( a, b, c ) > 0;
}

bool   LeftOn( tPointi a, tPointi b, tPointi c )
{
  return  Area2( a, b, c ) >= 0;
}

bool   Collinear( tPointi a, tPointi b, tPointi c )
{
  return  Area2( a, b, c ) == 0;
}
```

Code 1.6 Left

- If c is collinear with ab, then the determined triangle has zero area.

Boolean Intersection

- If ab and cd intersect in their interiors,
  1. c and d are split by the line L_1 containing ab
  2. Likewise a and b are split by line L_2 containing cd
- Neither one of these conditions is alone sufficient to guarantee intersection
- When two segments intersect at a point interior to both, if it is known that no three of the four endpoints are collinear

Boolean Intersection

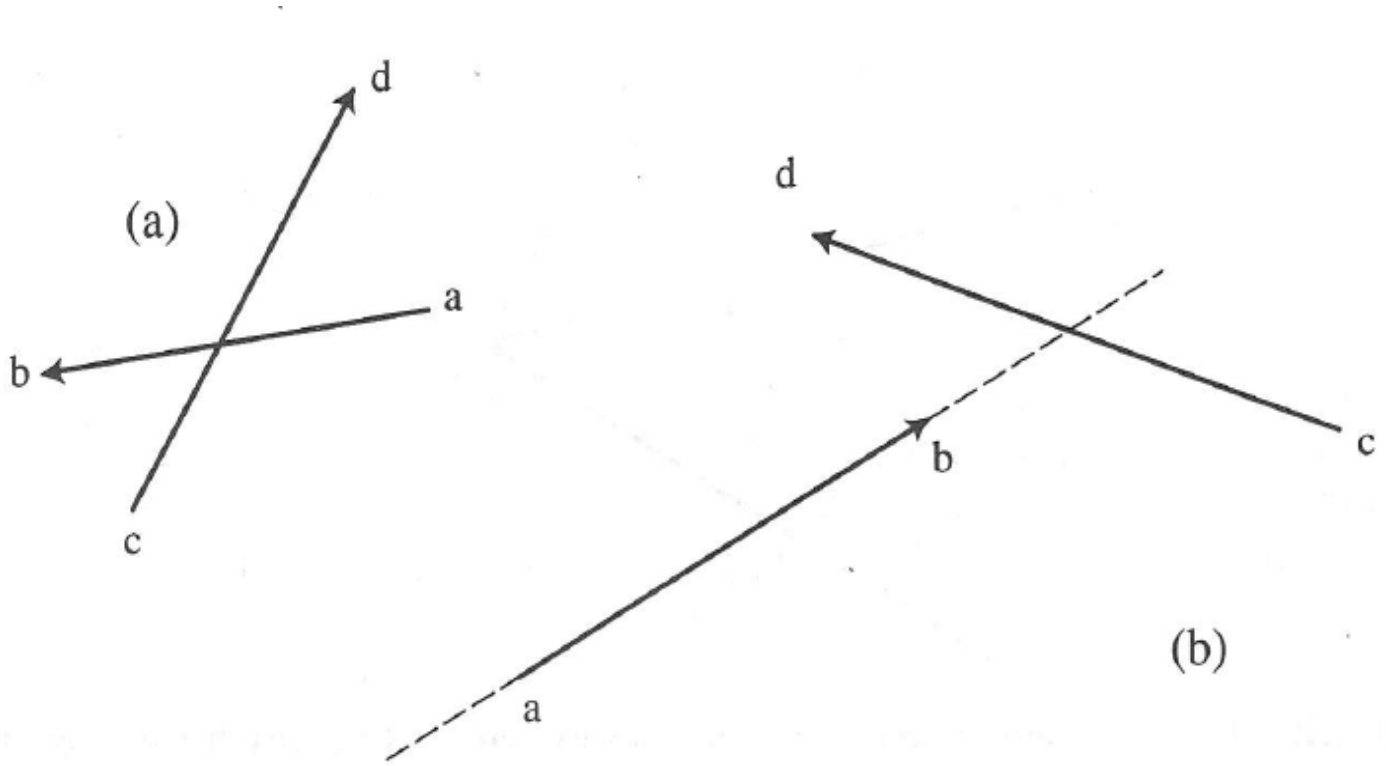


FIGURE 1.23 Two segments intersect (a) iff their endpoints are split by their determined lines; both pair of endpoints must be split (b).

```
bool   IntersectProp( tPointi a, tPointi b, tPointi c, tPointi d )
{
  /* Eliminate improper cases. */
  if (
     Collinear(a,b,c)   ||
     Collinear(a,b,d)   ||
     Collinear(c,d,a)   ||
     Collinear(c,d,b)
     )
     return FALSE;

  return
          Xor( Left(a,b,c), Left(a,b,d) )
     && Xor( Left(c,d,a), Left(c,d,b) );
}
/*Exclusive or: T iff exactly one argument is true. */
bool   Xor( bool x, bool y )
{
  /* The arguments are negated to ensure that they are 0/1 values. */
  return   !x ^ !y;
}
```

Code 1.7 IntersectProp

Boolean Intersection

- Redundancy in this code
  1. Four relevant triangle areas are being computed twice each.
- Two ways to remove redundancy
  1. Storing computed areas in local variables
  2. Designing other primitives that fit the problem better.
- The first if-statement may be removed entirely for the purposes of triangulation
- But sacrifice efficiency for clarity

Boolean Intersection

- It might be tempting to implement the exclusive-or by

```
Area2(a, b, c) * Area2(a, b, c) < 0
&& Area2(c, d, a) * Area2(c, d, b) < 0;
```

- But the product of the areas might cause integer word overflow!
- The problem can be avoided by having Area2 return +1, 0, or -1

Improper Intersection

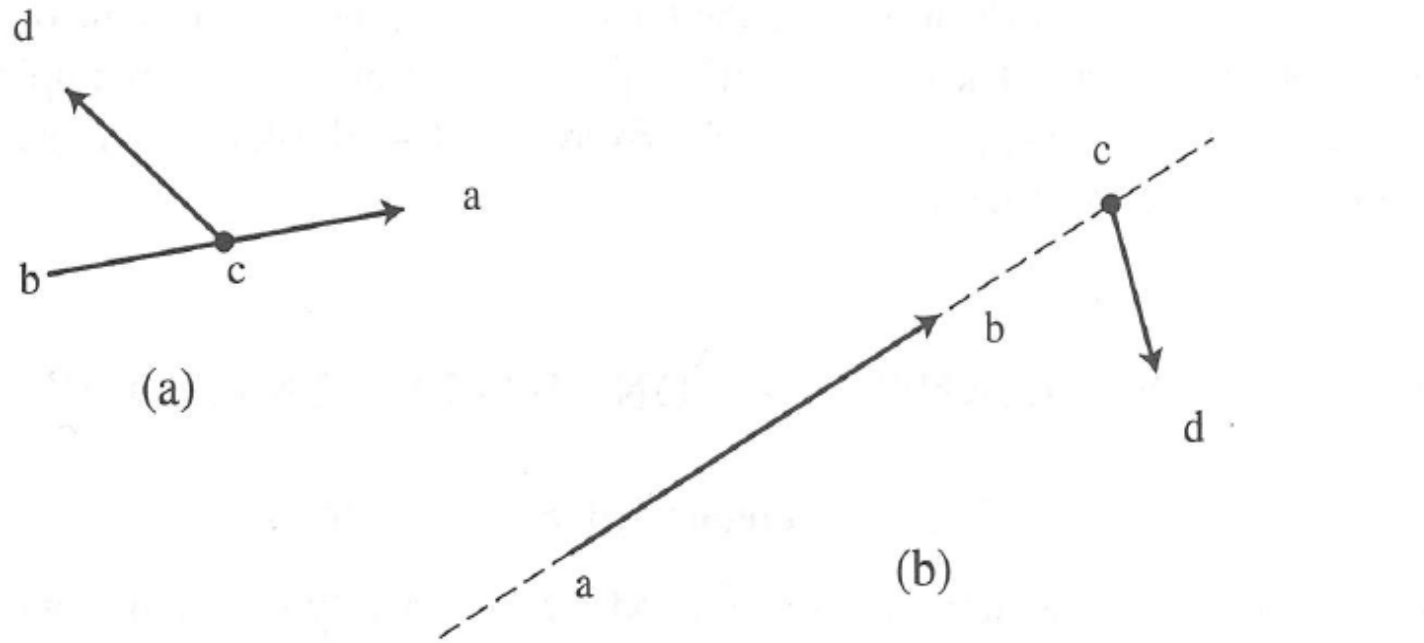


FIGURE 1.24 Improper intersection between two segments (a); collinearity is not sufficient (b).

Improper Intersection

- Lemma 1.5.1 requires that the intersection be completely empty for a segment to be a diagonal
- Special case of improper intersection
  1. An endpoint of one segment (say c) lies somewhere on the other segment ab (Figure 1.24(a))
  2. This only happens if a, b, c are collinear
  3. But collinearity is not a sufficient condition for intersection (Figure 1.24

Betweenness

- Only check betweenness of c when we know it lies on the line containg ab
- If ab is not vertical
  1. c lies on ab iff the x coordinate of c falls in the interval of the x coordinates of a and b
- If ab is vertical
  1. Similarly check on y coordinates
- See code 1.8

Betweenness

```
bool   Between( tPointi a, tPointi b, tPointi c )
{
  tPointi                ba, ca;

  if ( ! Collinear( a, b, c ) )
    return  FALSE;

  /* If ab not vertical, check betweenness on x; else on y. */
  if ( a[X] != b[X] )
    return ((a[X] <= c[X]) && (c[X] <= b[X]))  ||
           ((a[X] >= c[X]) && (c[X] >= b[X]));
  else
    return ((a[Y] <= c[Y]) && (c[Y] <= b[Y]))  ||
           ((a[Y] >= c[Y]) && (c[Y] >= b[Y]));
}
```

Code 1.8 Between

Segment Intersection Code

```
bool   Intersect( tPointi a, tPointi b, tPointi c, tPointi d )
{
  if       ( IntersectProp( a, b, c, d ) )
     return  TRUE;
  else if (   Between( a, b, c )
            || Between( a, b, d )
            || Between( c, d, a )
            || Between( c, d, b )
           )
     return  TRUE;
  else    return  FALSE;
}
```

Code 1.9 Intersect

Segment Intersection Code

```
bool   Diagonalie( tVertex a, tVertex b )
{
  tVertex c, c1;

  /* For each edge (c,c1) of P */
  c = vertices;
  do {
    c1 = c->next;
    /* Skip edges incident to a or b */
    if (      ( c != a ) && ( c1 != a )
         && ( c != b ) && ( c1 != b )
         && Intersect( a->v, b->v, c->v, c1->v )
       )
       return FALSE;
    c = c->next;
  } while ( c != vertices );
  return TRUE;
}
```

Code 1.10 Diagonalie

Polygon Triangulation

Internal or External

InCone

- Goal: distinguish the internal from the external diagonals
- One vector B (along the diagonal) lies strictly in the open cone counterclockwise between two other vectors A and C (along two consecutive edges)
- Need to consider convex and reflex angles

InCone

- Convex case (Figure 1.25 a)
  1. s is internal to P iff it is internal to the cone whose apex is a, and whose sides pass through a_- and a_+
  2. Easily determined by our Left function
  3. a_- must be left of ab and a_+ must be left of ba
- Reflex case (Figure 1.25 b)
  1. reverse of the convex case

InCone

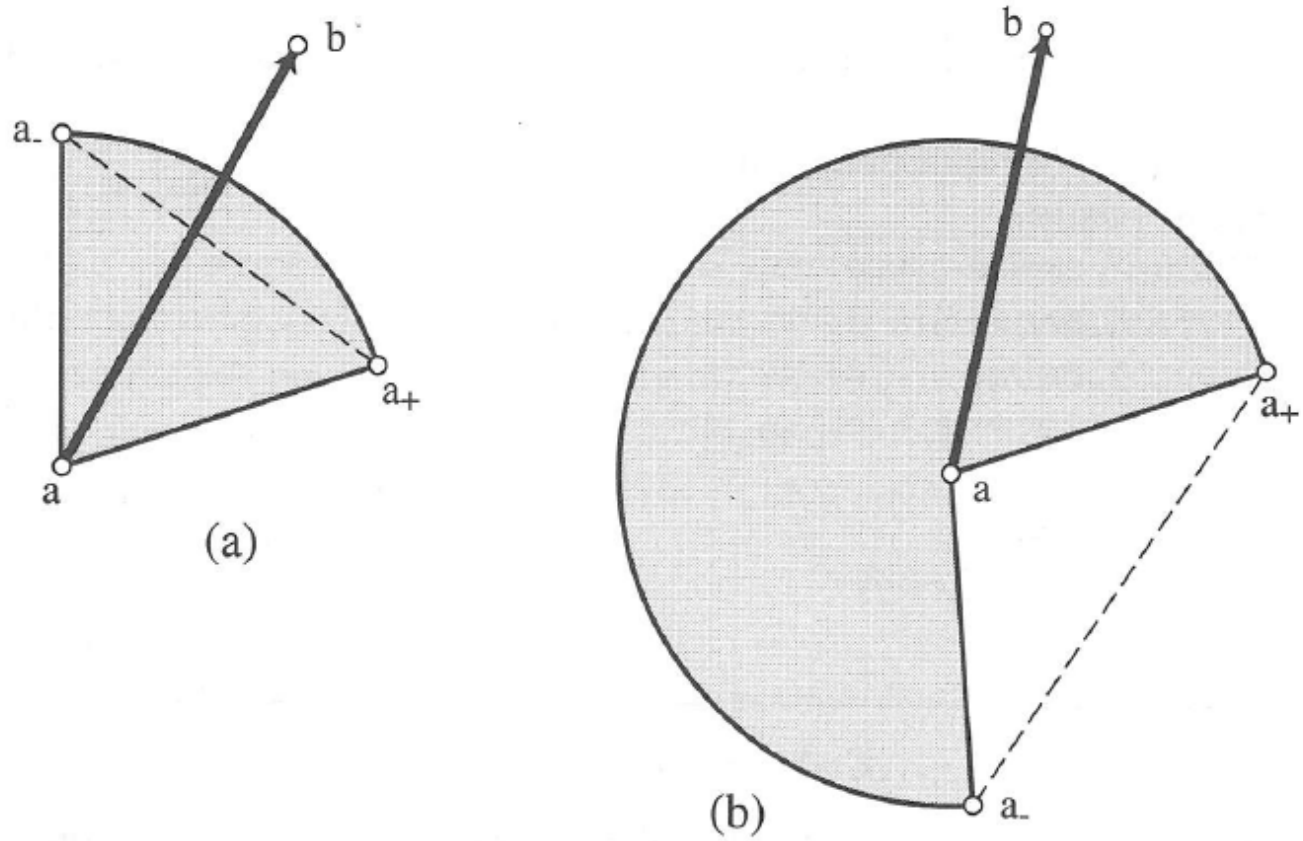


FIGURE 1.25 Diagonal $s = ab$ is in the cone determined by a_-, a, a_+: (a) convex; (b) reflex. In (b), both a_- and a_+ are right of ab.

InCone

- Distinguishing between the convex and reflex cases is accomplished with one invocation of Left
- a is convex *iff* $a\_$ is left or on aa_+
- Note that if ($a\_$, a, a_+) are collinear, the internal angle at a is π, which we define as convex

InCone

```
bool   InCone( tVertex a, tVertex b )
{
  tVertex a0,a1;      /* a0,a,a1 are consecutive vertices. */

  a1 = a->next;
  a0 = a->prev;

  /* If a is a convex vertex ... */
  if( LeftOn( a->v, a1->v, a0->v ) )
     return     Left( a->v, b->v, a0->v )
          && Left( b->v, a->v, a1->v );

  /* Else a is reflex: */
     return !(    LeftOn( a->v, b->v, a1->v )
             && LeftOn( b->v, a->v, a0->v ) );
}
```

Code 1.11 InCone

Diagonal

- ab is a diagonal *iff* Diagonalie(a, b), InCone(a, b), and InCone(b, a) are **ture**
- How to order function calls
  1. InCones should be first
  2. They are each constant-time calculation
  3. Each performs in the neighborhood a and b without regard to the remainder of the polygon, whereas Diagonalie includes a loop over all n polygon edges.

Diagonal

```
bool   Diagonal( tVertex a, tVertex b )
{
  return InCone( a, b ) && InCone( b, a ) && Diagonalie( a, b );
}
```

Code 1.12 Diagonal

Triangulation

- Diagonal-Based Algorithm

- It is an $O(n^4)$ algorithm
  1. (n choose 2) diagonal candidates = $O(n^2)$
  2. Testing each for diagonalhood = $O(n)$
  3. Reapting this $O(n^3)$ computation for each of the n-3 diagonals = $O(n^4)$
- Use the two ears theorem to speed up
- Only $O(n)$ “ear diagonal” candidates
- We can achieve a worst-case complexity of $O(n^3)$ this way

Ear Removal

- Ear Removal

- Improve the above algorithm to $O(n^2)$
  1. Because one call to Diagonal costs $O(n)$, Diagonal may only be called $O(n)$ times
- Key idea
  1. Removal of one ear does not change the polygon very much
  2. Not change whether or not many of its vertices are potential ear tips.
- Determination for potential ear tip of each vertex already uses $O(n^2)$, but is not repeated.

Ear Removal

- Ear Removal
  - Let $(v_0, v_1, v_2, v_3, v_4)$ be five consecutive vertices of P
  - Suppose v_2 is an ear tip and the ear $E_2 = \triangle(v_1, v_2, v_3)$ is deleted (see Figure 1.26)
  - Only v_1 and v_3 change
  - Neighbor vertices remain unchanged

Ear Removal

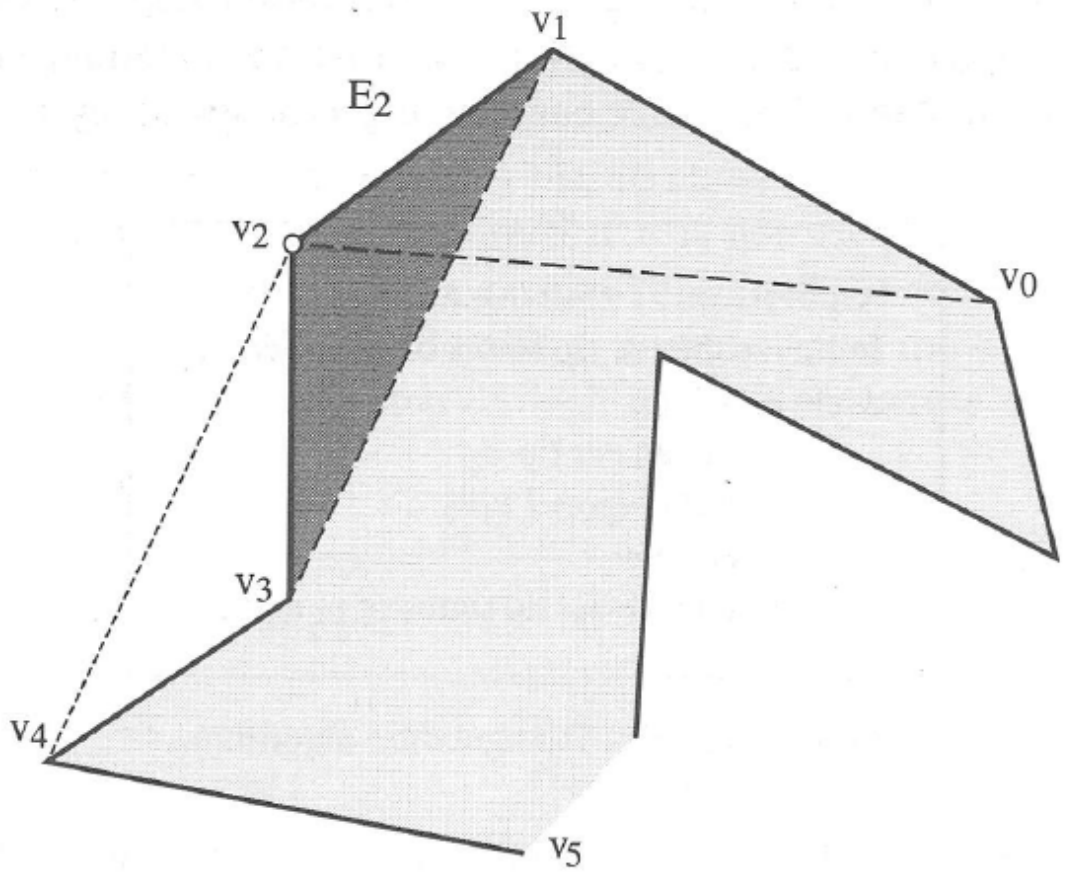


FIGURE 1.26 Clipping an ear $E_2 = \Delta(v_1, v_2, v_3)$. Here the ear status of v_1 changes from TRUE to FALSE.

Ear Removal

- Ear Removal
  - After the expensive initialization step, the ear tip status information can be updated with two calls to Diagonal per iteration

Algorithm: TRIANGULATION
Initialize the ear tip status of each vertex.
while $n > 3$ do
    Locate an ear tip v_2.
    Output diagonal v_1v_3.
    Delete v_2.
    Update the ear tip status of v_1 and v_3.

Algorithm 1.1 Triangulation algorithm

Ear Removal

- Triangulation Code
  - The first task is to initialize the Boolean flag v->ear
  - Need one call to Diagonal per vertex.

Ear Removal

```
void   EarInit( void )
{
  tVertex v0, v1, v2;   /* three consecutive vertices */

  /* Initialize v1->ear for all vertices. */
  v1 = vertices;
  printf("newpath\n");
  do {
    v2 = v1->next;
    v0 = v1->prev;
    v1->ear = Diagonal( v0, v2 );
    v1 = v1->next;
  } while ( v1 != vertices );
}
```

Code 1.13 EarInit

Ear Removal

- Triangulation Code
  - The main Triangulate code consist of a double loop.
  - The outer loop removes one ear per iteration
  - The inner loop searches for an ear by checking the precomputed v2-ear flag, where v_2 is the potential ear tip
  - Status of v_1 and v_3 are updated by calls to Diagonal and the ear is removed by rewiring the next and prev pointers for v_1 and v_3
  - Then, the head point, vertices, is moved to point to v_3

```
void   Triangulate( void )
{
  tVertex v0, v1, v2, v3, v4;   /* five consecutive vertices */
  int   n = nvertices;          /* number of vertices; shrinks to 3. */

  EarInit();
  /* Each step of outer loop removes one ear. */
  while ( n > 3 ) {
    /* Inner loop searches for an ear. */
    v2 = vertices;
    earfound = FALSE;
    do {
      if (v2->ear) {
        /* Ear found. Fill variables. */
        v3 = v2->next; v4 = v3->next;
        v1 = v2->prev; v0 = v1->prev;

        /* (v1,v3) is a diagonal */
        PrintDiagonal( v1, v3 );

        /* Update earity of diagonal endpoints */
        v1->ear = Diagonal( v0, v3 );
        v3->ear = Diagonal( v1, v4 );
```

```
            /* Cut off the ear v2 */
            v1->next = v3;
            v3->prev = v1;
            vertices = v3;              /* In case the head was v2. */
            n--;
            break;   /* out of inner loop; resume outer loop */
          } /* end if ear found */
          v2 = v2->next;
        } while ( v2 != vertices );
      } /* end outer while loop */
}
```

Code 1.14 Triangulate

Example

- Figure 1.27 shows a polygon and the triangulation produced by the simple main program(Code 1.15)

```
main()
{
  ReadVertices();
  PrintVertices();
  Triangulate();
}
```

Code 1.15 main

Example

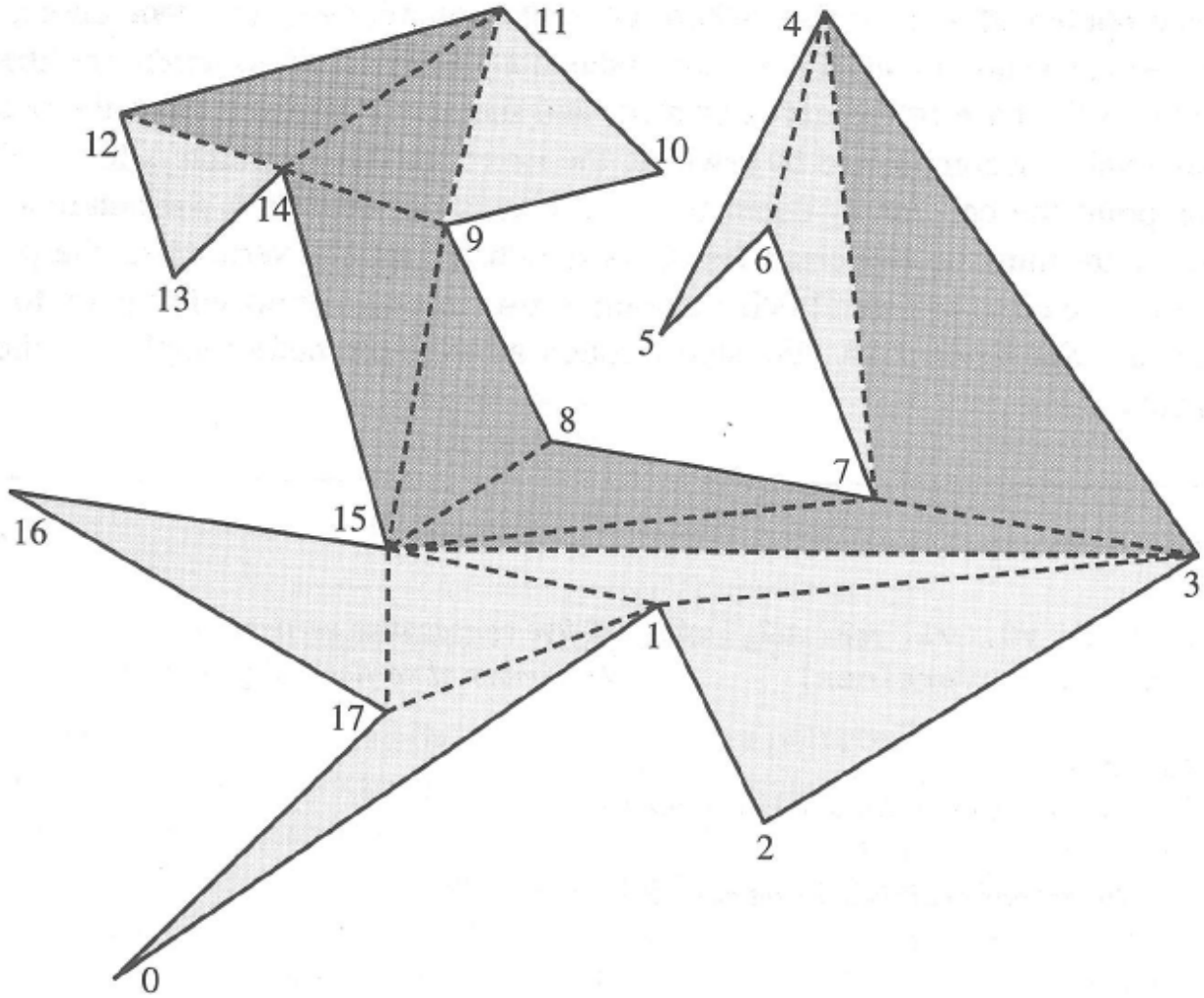


FIGURE 1.27 A polygon of 18 vertices and the triangulation produced by `Triangulate`. The dark subpolygon is the remainder after the 9th diagonal (15, 3) is output. Vertex coordinates are displayed in Table 1.1.

Example

- Now walk through the output of the diagonals
- v_0 is an ear tip, so the first diagonal output is (17,1)
- v_1 is not an ear tip, so v2 pointer moves to v_2
- v_2 is a tip, so print the diagonal (1,3)
- Neither v_3 nor v_4 is an ear tip
- At v_5, the next diagonal is (4,6)
- v_3 v_8 is collinear with v_7, so the next ear detected is not until v_{10}

…

- Another collinearity, v_9 with (v_{11} v_{15}), prevents v_9 from being an ear

…

Example

Table 1.2. The columns show the order in which the diagonals, specified as pairs of endpoint indices, are output

Order	Diagonal Indices	Order	Diagonal Indices
1	(17, 1)	10	(3, 7)
2	(1, 3)	11	(11, 14)
3	(4, 6)	12	(15, 7)
4	(4, 7)	13	(15, 8)
5	(9, 11)	14	(15, 9)
6	(12, 14)	15	(9, 14)
7	(15, 17)		
8	(15, 1)		
9	(15, 3)		

Analysis

- The time complexity of the algorithm.
  1. EarInit costs $O(n^2)$
  2. The outer loop of Triangulate iterates n-3 times = $O(n)$
  3. The inner search-for-an-ear loop is also $O(n)$
  4. The work inside the inner loop is $O(n)$
- Naively we have then a time complexity of $O(n^3)$, falling short of the promised $O(n^2)$

Analysis

- Figure 1.28
- After v_0 is deleted, the inner loop searches past $v_1,\ldots, v_6$ before reaching the next ear tip v_7.
- Then must search past $v_8,\ldots, v_{12}$ before finding the ear tip v_{13}
- Notice the two $O(n)$ Diagonal calls within the loop are only invoked once an ear is found
- Thus although the superficial structure of the code suggest n x n x $n = O(n^3)$
- It is actually n x $(n + n) = O(n^2)$

Analysis

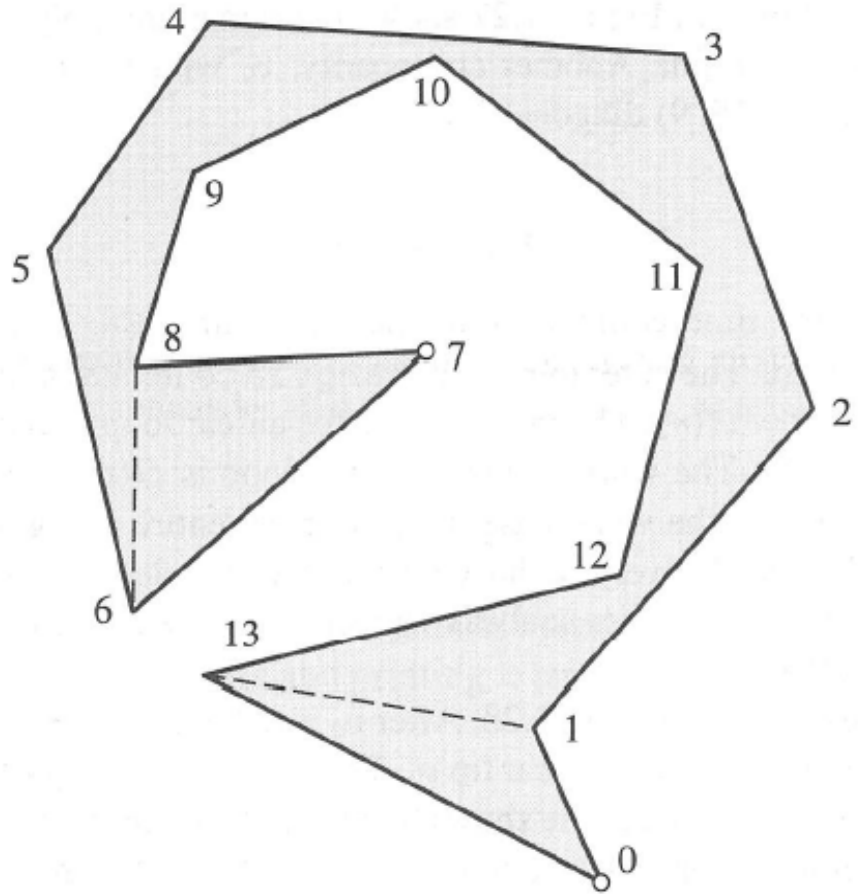


FIGURE 1.28 An example that forces the inner ear loop to search extensively for the next ear.

Conclusion

- We learned:
  1. what is a polygon, diagonal, triangulation
  2. how to determine
     - the area of polygons
     - if 3 points, collinear, turn-left, in-between
     - if two segment intersect
     - if a segment is internal or external a polygon
     - ears in a polygon
- Homework assignment: 1.1.4-1, 1.3.9-4, 1.6.8-2, 1.6.8-3 (due 9/16 before the class)
- Programming assignment: posted online by Friday midnight (due 9/22 midnight)