$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Solution: λ is an eigen value of if and only if

$$\det(\lambda I_n - A) = 0$$

in this case

$$\det\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \lambda-1 & -2 & -3 \\ 0 & \lambda-4 & -5 \\ 0 & 0 & \lambda-6 \end{bmatrix}\right) = 0$$

$$(\lambda-1)(\lambda-4)(\lambda-6) = 0$$

$$(\lambda^2-4\lambda-\lambda+4)(\lambda-6) = 0$$

$$(\lambda^2-5\lambda+4)(\lambda-6) = 0$$

$$\lambda^3-5\lambda^2+4\lambda-6\lambda^2+30\lambda-24 = 0$$

characteristic polynomial → $\boxed{\lambda^3 - 11\lambda^2 + 34\lambda - 24 = 0}$

Now $(\lambda-1)(\lambda-4)(\lambda-6) = 0$

eigen values $\boxed{\lambda = 1, 4, 6}$

For $\lambda = 1$,

$$(\lambda I_n - A)\vec{v} = 0$$

$$\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\right)\vec{v} = 0$$

for $\lambda = 1$ $\Rightarrow$

$$\begin{bmatrix} 0 & -2 & -3 \\ 0 & -3 & -5 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_2 = 0$, $v_3 = 0$ and $v_1 = 1$

for $\lambda = 1$ eigen vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = 4$

$$\left(4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 & -3 \\ 0 & 0 & -5 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2/3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

divide row 1 by 3
divide row 2 by 5
divide row 3 by 2

$$\Rightarrow \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row 3 - row 2
row 1 - row 2

$$v_1 - \frac{2}{3}v_2 = 0$$

$$\Rightarrow \quad v_3 = 0$$

$$v_1 = \frac{2}{3}v_2$$

for $\lambda = 4$ eigen vector $\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}$ if $v_2 = 1$

$E_{\lambda=4} = \text{span}\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}$ to convert unit vector $\begin{bmatrix} 0.5547 \\ 0.8321 \\ 0 \end{bmatrix}$ after dividing by $\sqrt{\left(\frac{2}{3}\right)^2 + 1}$

for $\lambda = 6$

$$\left(6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5 & -2 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \frac{8}{5}v_3$$

$$v_2 = \frac{5}{2}v_3$$

if $v_3 = 1$

for $\lambda = 6$ eigen vector $\begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}$ if $v_3 = 1$

$$E_{\lambda=6} = \text{span}\left(\begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}\right)$$

length of vector $= \sqrt{\left(\frac{8}{5}\right)^2 + \left(\frac{5}{2}\right)^2 + 1} = 3.1324$

$$E_{\lambda=6} = \text{span}\left(\begin{bmatrix} 0.5108 \\ 0.7982 \\ 0.3193 \end{bmatrix}\right)$$