

# Data 609 - Module1

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## Contents

Ex. 1	1
Ex. 2	2
Ex. 3	2

## Ex. 1

Find the minimum of  $f(x, y) = x^2 + xy + y^2$  in  $(x, y) \in \mathbb{R}^2$

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2 + xy + y^2)$$

$$f_y(x, y) = \frac{\partial}{\partial y}(x^2 + xy + y^2)$$

Now the stationary conditions are

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

That means

$$2x + y = 0 \text{ and } x + 2y = 0$$

From second equation above we get  $x = -2y$

replacing  $x$  in first equation gives

$$2 * (-2y) + y = 0 \Rightarrow y = 0$$

putting  $y = 0$  in any of the above equations, we get  $x = 0$

Next is to compute all 4 second partial derivatives.

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(2x + y) = 2$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}(2x + y) = 1$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}(x + 2y) = 1$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(x + 2y) = 2$$

Hessian Matrix

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Thus  $H$  is positive and point  $(0,0)$  is minimum.

## Ex. 2

For  $f(x) = x^4$  in  $\mathbb{R}$ , it has a global minimum at  $x = 0$ . Find its new minimum if a constraint  $x^2 \geq 1$  is added.

Using Penalty method

$$f(x) = x^4 \text{ and } g(x) = x^2 - 1$$

Now using a penalty parameter  $\mu$  we have

$$\Pi = f(x) + \mu[g(x)]^2 = x^4 + \mu(x^2 - 1)^2 = x^4 + \mu(x^4 + 1 - 2x^2)$$

From  $\Pi'(x) = 0$ , we have

$$4x^3 + 4\mu x^3 - 4\mu x = 0$$

on simplifying we get

$$x^2 = \frac{\mu}{(1+\mu)}$$

for very large value of  $\mu$ , we will have

$$x^2 = \frac{\mu}{(1+\mu)} \approx \frac{\mu}{\mu} \approx 1$$

so  $x$  can be 1 or -1 so the new minimum will be at  $x = 1$

## Ex. 3

Use a Lagrange multiplier to solve the optimization problem  $\min f(x, y) = x^2 + 2xy + y^2$ , subject to  $y = x^2 - 2$

$$f(x, y) = x^2 + 2xy + y^2$$

$$h(x, y) = x^2 - y - 2$$

$$\phi = f(x, y) + \lambda h(x, y)$$

$$\phi = (x^2 + 2xy + y^2) + \lambda(x^2 - y - 2)$$

$$\frac{\partial \phi}{\partial x} = 2x + 2y + 2\lambda x = 0$$

$$\frac{\partial \phi}{\partial y} = 2x + 2y - \lambda = 0$$

$$\frac{\partial \phi}{\partial \lambda} = x^2 - y - 2 = 0$$

From the second equation,  $2x + 2y - \lambda = 0 \Rightarrow \lambda = 2(x + y)$

Having  $\lambda = 2(x + y)$  in the first equation gives us

$$2x + 2y + 2(2(x + y))x = 0$$

$$2x + 2y + 4x^2 + 4xy = 0$$

$$2(x + y + 2x^2 + 2xy) = 0 \text{ which gives us } y = \frac{-2x^2 - x}{2x + 1}$$

Substitute  $y = \frac{-2x^2 - x}{2x + 1}$  into the third equation gives us

$$x^2 - 2 - \frac{-2x^2 - x}{2x + 1} = 0 \text{ when simplified gives } x = 1 \text{ or } x = -2.$$

Plugging  $x = 1$  into the third equation gives us  $y = -1$  and plugging in  $x = -2$  gives us  $y = 2$ . We get two points (1, -1) and (-2, 2).

The optimality for  $\min f(x, y) = x^2 + 2xy + y^2$  is (1, -1) with  $f_{\min} = 0$  and (-2, 2) with  $f_{\min} = 0$

Resources:

<https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/lagrange-multipliers-and-constrained-optimization/v/constrained-optimization-introduction>