Data 609 - Module2

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Ex. 1

Show $x^2 + exp(x) + 2x^4 + 1$ is convex.

Solution:

A function is considered convex if $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$, $\forall x, y \in \Omega$, $\alpha \geq 0, \beta \geq 0$, $\alpha + \beta = 1$

$(\alpha x + \beta y)^2 + exp(\alpha x + \beta y) + 2(\alpha x + \beta y)^4 + 1 \leq \alpha(x^2 + exp(x) + 2x^4 + 1) + \beta(y^2 + exp(y) + 2y^4 + 1)$

$(\alpha^2 x^2 + \beta^2 y^2 + 2\alpha\beta xy) + exp(\alpha x + \beta y) + 2(\alpha x + \beta y)^4 + 1 \leq \alpha x^2 + \alpha exp(x) + 2\alpha x^4 + \alpha + \beta y^2 + \beta exp(y) + 2\beta y^4 + \beta$

Using $\alpha + \beta = 1$ we rewrite as

$(\alpha^2 x^2 + \beta^2 y^2 + 2\alpha\beta xy) + exp(\alpha x + \beta y) + 2(\alpha x + \beta y)^4 \leq \alpha x^2 + \alpha exp(x) + 2\alpha x^4 + \beta y^2 + \beta exp(y) + 2\beta y^4$

$2\alpha x^4 + \alpha x^2 + \alpha exp(x) + 2\beta y^4 + \beta y^2 + \beta exp(y) - (\alpha x + \beta y)^2 - exp(\alpha x - \beta y) - 2(\alpha x + \beta y)^4 \geq 0$

The inequality is always true and hence $x^2 + exp(x) + 2x^4 + 1$ is convex.

Ex. 2

Show that the mean of the exponential distribution

$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 (\lambda > 0), \\ 0, & x < 0 \end{cases}$

is $\mu = \frac{1}{\lambda}$ and its variance is $\sigma^2 = \frac{1}{\lambda^2}$.

Solution:

To find the mean we know that $\mu = E[X] = \int xp(x)dx$

$\mu = \int_{-\infty}^{0} x * 0dx + \int_{0}^{\infty} x\lambda e^{-\lambda x} dx$

Using integration by parts $\int uvdx = u \int vdx - \int u'(\int vdx)dx$

replacing $u = x, v = \lambda e^{-\lambda x}$ we will have

$\mu = [-xe^{-\lambda x}]_0^\infty + \int_0^\infty e^{-\lambda x} dx$

$\mu = [-xe^{-\lambda x}]_0^\infty + [-\frac{1}{\lambda}e^{-\lambda x}]_0^\infty$

$\mu = (0-0) + (0+\frac{1}{\lambda}) = \frac{1}{\lambda}$

To find the variance of the given exponential distribution we know that $\sigma^2 = Var[X] = E[X^2] - E[X]^2$

$E[X^2] = \int_0^\infty x^2\lambda e^{-\lambda x}$ which could again be solved by using integration by parts

$E[X^2] = [-x^2e^{-\lambda x}]_0^\infty + \int_0^\infty 2xe^{-\lambda x}dx$

$E[X^2] = [-x^2e^{-\lambda x}]_0^\infty + [-\frac{2}{\lambda}xe^{-\lambda x}dx]_0^\infty + \frac{2}{\lambda}\int_0^\infty e^{-\lambda x}dx$

using integration by parts again

$E[X^2] = [-x^2e^{-\lambda x}]_0^\infty + [-\frac{2}{\lambda}xe^{-\lambda x}dx]_0^\infty + \frac{2}{\lambda}[-\frac{1}{\lambda}xe^{-\lambda x}dx]_0^\infty$

$(\frac{2}{\lambda})(\frac{1}{\lambda}) = \frac{2}{\lambda^2}$

$E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Ex. 3

It is estimated that there is a typo in every 250 data entries in a database, assuming the number of typos can obey the Poisson distribution. For a given 1000 data entries, what is the probability of exactly 4 typos? What is the probability of no typo at all? Use R to draw 1000 samples with $\lambda = 4$ and show their histogram.

Solution:

```
lambda <- 1000 * (1/250)
x <- 4 # 4 typos
p_four_typo <- (lambda ^ x * exp(-lambda)) / factorial(x)
p_four_typo
```

```
## [1] 0.1953668
```

The probability of exactly 4 typos is 19.54%.

```
lambda <- 4
x <- 0 # no typo
p_zero_typo <- (lambda ^ x * exp(-lambda)) / factorial(x)
p_zero_typo
```

```
## [1] 0.01831564
```

The probability of no typos at all is 1.83%.

Now let's confirm our results above using built in R function.

```
# probability of exactly 4 typos
p1 <- dpois(4, lambda = 4)
p1
```

```
## [1] 0.1953668
```

```
# probability of no typo
q2 <- dpois(0, lambda = 4)
q2
```

```
## [1] 0.01831564
```

```
events = 1:1000
lambda = 4
poisson <- rpois(1000, lambda)
```

```r
df <- data.frame(events, poisson)

hist(df$poisson, main = "1000 Samples with lambda = 4")
```