Data 609 - Module1

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Contents

Ex. 1

Find the minimum of $f(x, y) = x^2 + xy + y^2 in(x, y) \in \mathbb{R}^2$

Solution:

$f_x(x, y) = \frac{\partial}{\partial x}(x^2 + xy + y^2)$

$f_y(x, y) = \frac{\partial}{\partial y}(x^2 + xy + y^2)$

Now the stationary conditions are

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

That means

$2x + y = 0$

$x + 2y = 0$

From second equation above we get x = -2y

replacing x in first equation gives

$2 * (-2y) + y = 0 => y = 0$

putting y = 0 in any of the above equations, we get x = 0

Next is to compute all 4 second partial derivatives.

$f_{xx}(x, y) = \frac{\partial}{\partial x}(2x + y) = 2$

$f_{xy}(x, y) = \frac{\partial}{\partial y}(2x + y) = 1$

$f_{yx}(x, y) = \frac{\partial}{\partial y}(x + 2y) = 1$

$f_{yy}(x, y) = \frac{\partial}{\partial y}(x + 2y) = 2$

Hessian Matrix

$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Thus H is positive and point (0,0) is minimum.

Ex. 2

For $f(x) = x^4$ in $\mathbb{R}$, it has a global minimum at x = 0. Find its new minimum if a constraint $x^2 \geq 1$ is added.

Solution:

Using Penalty method

$f(x) = x^4$ and $g(x) = x^2 - 1$

Now using a penalty parameter μ we have

$\Pi = f(x) + \mu[g(x)]^2 = x^4 + \mu(x^2 - 1)^2 = x^4 + \mu(x^4 + 1 - 2x^2)$

From $\Pi'(x) = 0$, we have

$4x^3 + 4\mu x^3 - 4\mu x = 0$

on simplifying we get

$x^2 = \frac{\mu}{(1+\mu)}$

for very large value of μ, we will have

$x^2 = \frac{\mu}{(1+\mu)} \approx \frac{\mu}{\mu} \approx 1$

so x can be 1 or -1 so the new minimum will be at x = 1

Ex. 3

Use a Lagrange multiplier to solve the optimization problem $min f(x, y) = x^2 + 2xy + y^2$, subject to $y = x^2 - 2$

Solution:

$f(x, y) = x^2 + 2xy + y^2$

$h(x, y) = x^2 - y - 2$

$\phi = f(x, y) + \lambda h(x, y)$

$\phi = (x^2 + 2xy + y^2) + \lambda(x^2 - y - 2)$

$\frac{\partial \phi}{\partial x} = 2x + 2y + 2\lambda x = 0$

$\frac{\partial \phi}{\partial y} = 2x + 2y - \lambda = 0$

$\frac{\partial \phi}{\partial \lambda} = x^2 - y - 2 = 0$

From the second equation, $2x + 2y - \lambda = 0 => \lambda = 2(x + y)$

Having $\lambda = 2(x + y)$ in the first equation gives us

$2x + 2y + 2(2(x + y))x = 0$

$2x + 2y + 4x^2 + 4xy = 0$

$2(x + y + 2x^2 + 2xy) = 0$ which gives us $y = \frac{-2x^2 - x}{2x+1}$

Having $y = \frac{-2x^2 - x}{2x+1}$ into the third equation gives us

$x^2 - 2 - (\frac{-2x^2 - x}{2x+1}) = 0$ which after simplifying gives us x = 1 or x = - 2.

Now with x = 1, the third equation gives us y = -1 and with in x = -2 gives us y = 2.

Finally we have two points (1, -1) and (-2, 2).

The optimality for $min f(x, y) = x^2 + 2xy + y^2$ is (1, -1) with $f_{min} = 0$ and (-2, 2) with $f_{min} = 0$

Resources:

https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/lagrange-multipliers-and-constrained-optimization/v/constrained-optimization-introduction