MATH 413
Introduction to Combinatorics

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6.1 L13: The Inclusion-Exclusion principle and applications I

Question 1

How big is $A \cup B \cup C$?

Solution:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

If we just sum up the sizes of the sets, we are counting the elements in common between each of the sets twice. However, after we subtract out the number of elements in each intersection, we are excluding elements in commonb between all three sets.

This is the basis of the inclusion-exclusion principle.

Question 2

How many elements are *not* in $A, B,$ or C?

Solution: Let S be the universal set.

$$\begin{aligned} |A^c \cap B^c \cap C^c| &= |S| - |A \cup B \cup C| \\ &= |S| - \Big(|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\Big) \end{aligned}$$

Example 6.1.1

Find the number of integers between 1 and 1000 inclusive that are not divisible by 5 and not divisible by 6 and not divisible by 8.

Let

- A_1 be the subset of integers divisible by 5,
- A_2 be the subset of integers divisible by 6,
- A_3 be the subset of integers divisible by 8.

Then we want to find $|A_1^c \cap A_2^c \cap A_3^c|$.

$$\begin{aligned}|A_1^c \cap A_2^c \cap A_3^c| &= |S| - |A_1 \cup A_2 \cup A_3| \\ &= |S| - \Big(|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|\Big)\end{aligned}$$

Obviously, $|S| = 1000$. To obtain the number of integers between a and b, where $b \geqslant a$, we can use the formula $\lfloor \frac{b}{a} \rfloor$.

$$\begin{aligned}|A_1| &= \lfloor \frac{1000}{5} \rfloor = 200 \\ |A_2| &= \lfloor \frac{1000}{6} \rfloor = 166 \\ |A_3| &= \lfloor \frac{1000}{8} \rfloor = 125\end{aligned}$$

Similarly,

$$\begin{aligned}|A_1 \cap A_2| &= \lfloor \frac{1000}{\operatorname{lcm} 5, 6} \rfloor = \lfloor \frac{1000}{30} \rfloor = 33 \\ |A_1 \cap A_3| &= \lfloor \frac{1000}{\operatorname{lcm} 5, 8} \rfloor = \lfloor \frac{1000}{40} \rfloor = 25 \\ |A_2 \cap A_3| &= \lfloor \frac{1000}{\operatorname{lcm} 6, 8} \rfloor = \lfloor \frac{1000}{24} \rfloor = 41 \\ |A_1 \cap A_2 \cap A_3| &= \lfloor \frac{1000}{\operatorname{lcm} 5, 6, 8} \rfloor = \lfloor \frac{1000}{120} \rfloor = 8\end{aligned}$$

Thus,

$$\begin{aligned}1000 - \Big(200 + 166 + 125 - 33 - 25 - 41 + 8\Big) &= 1000 - 200 - 166 - 125 + 33 + 25 + 41 - 8 \\ &= 600\end{aligned}$$

Question 3

How many permutations of M,A,T,H,I,S,F,U,N are there where MATH, IS and FUN do not appear as consecutive letters?

Solution: Let

- A_1 be the set of permutations where MATH appears as consecutive letters,
- A_2 be the set of permutations where IS appears as consecutive letters,
- A_3 be the set of permutations where FUN appears as consecutive letters.

Then we want to find $|A_1^c \cap A_2^c \cap A_3^c|$.

$$
\begin{aligned}
|A_1^c \cap A_2^c \cap A_3^c| &= |S| - |A_1 \cup A_2 \cup A_3| \\
&= |S| - \Big(|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|\Big)
\end{aligned}
$$

Obviously, $|S| = 9!$.

$$
\begin{aligned}
|A_1| &= 6! \\
|A_2| &= 8! \\
|A_3| &= 7!
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|A_1 \cap A_2| &= 5! \\
|A_1 \cap A_3| &= 4! \\
|A_2 \cap A_3| &= 6! \\
|A_1 \cap A_2 \cap A_3| &= 3!
\end{aligned}
$$

Thus,

$$
9! - \Big(6! + 8! + 7! - 5! - 4! - 6! + 3!\Big) = 9! - 6! - 8! - 7! + 5! + 4! + 6! - 3!
$$

Theorem 6.1.1 General Form of the Complementary Inclusion-Exclusion Principle

$$
\begin{aligned}
|A_1^c \cap \cdots \cap A_m^c| = |S| - \sum |A_i| + \sum |A_i \cap A_j| \\
- \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^{m+1}|A_1 \cap \cdots \cap A_m|
\end{aligned}
$$

Proof. To begin, we can realize that the LHS counts the number of elements in S that are not in any of the A_i.

For the RHS, Consider an arbitrary $s \in S$. There are two cases:

- Case 1. x is not in any A_i.

  In this case, x would contribute 1 to the LHS and 1 to RHS as it will not appear in any of the summations. Thus, this value will have no impact on the equality.

- Case 2. x is in some $n > 0$ A_i sets.

  Clearly, the contribution to the LHS is 0. For the RHS, the contribution is

$$
1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k}
$$

  This is because we must count the number of ways to choose k sets from the n sets that x is in.

By the binominal theorem, we have

$$\begin{aligned}1 + \sum_{k=1}^{n}(-1)^k\binom{n}{k} &= \sum_{k=0}^{n}(1)^{n-k}(-1)^k\binom{n}{k} \\ &= (1-1)^n \\ &= 0\end{aligned}$$

In both cases, the equality holds. Thus, the LHS and RHS are equal. ■

6.2 L14: The Inclusion-Exclusion principle and applications II: Derangements

Introduction to Dearrangements

Question: You all get up from your chairs, and randomly move to a different chair. What's the probability that no one ends up sitting down in the same chair? What if it were a class of 300 students?

Note:-

I find this definition a little hard to initially interpret.

Derrangements can be thought of through the following example. Suppose a teacher is attempting to pass back a test to four students: A, B, C, D. There are obviously 4! ways to destribute the tests (i.e. there are 4! permutations of A, B, C, D).

Dearrangements are the number of ways to distribute the tests such that no student gets their own test back.

For example A, C, D, B is not a dearragement because A gets their own test back. However, B, A, D, C is a dearrangement because no student gets their own test back.

Our goal is count the number of dearrangements.

Question 4: Abstract version of the original question

Given a permutation $\pi \in S_n$, what is the probability that $\pi(i) \neq i$ for all i. What is the number of D_n for all such permutations?

Solution: Inclusion-Exclusion argument Let A_i be the set of permutations where $\pi(i) = i$.

Then we want to find $|A_1^c \cap A_2^c \cap \cdots \cap A_n^c|$.

$$\begin{aligned}|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| &= |S_n| - |A_1 \cup A_2 \cup \cdots \cup A_n| \\ &= |S_n| - \Big(|A_1| + |A_2| + \cdots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ &\quad + |A_1 \cap A_2 \cap A_3| + \cdots + |A_1 \cap \cdots \cap A_n|\Big)\end{aligned}$$

Clearly, there are $n!$ permutations in S_n.

For each $0 \leq i \leq n$, there are $(n-1)!$ permutations where $\pi(i) = i$. Thus there are $n(n-1)!$ permutations where $\pi(i) = i$ for some i. So,

$$\begin{aligned}\sum_{i=1}^{n} |A_i| &= n(n-1)! \\ &= n!\end{aligned}$$

For each i, j, there are $(n-2)!$ permutations where $\pi(i) = i$ and $\pi(j) = j$. Thus there are $n(n-1)(n-2)!$ permutations where $\pi(i) = i$ and $\pi(j) = j$ for some i and j.

$$\begin{aligned}\sum_{i,j}^{n} |A_i \cap A_j| &= \frac{n(n-1)(n-2)!}{2!} \\ &= \frac{n!}{2}\end{aligned}$$

This makes sense because there are n choices for i and $n-1$ choices for j (since i and j cannot be the same). This leaves $(n-2)!$ ways to arrange the remaining $n-2$ elements. However, i and j are indistinguishable, so we must divide by 2 to account for this.

Following this,

$$\begin{aligned}D_n &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \\ &= n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})\end{aligned}$$

Question 5

What's the probability of picking a derangement as $n \to \infty$?

Solution:

$$\lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{e}$$

Note:-

Just accept that this limit is true right now. Need to prove later.

Question 6

At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that:

- no gentleman receives his own hat?
- at least one gentleman receives own hat?
- at least two gentlemen receive their own hat?

Solution: part a

This is simply the number of derangements for $n = 7$.

$$D_7 = 7!(1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!})$$

Solution: part b

This is just the total number of permutations - the number of derangements for 7 people. So,

$$7! - D_7$$

Solution: part c

This is the number of ways that at least one person receives their own hat - the number of ways exactly one gentlement receives their own hat.

$$7! - D_7 - 7 \cdot D_6$$

6.3 L15: The Inclusion-Exclusion principle and applications II

Consider a problem we have seen before

How many r-combinations are there of a multiset with k distinct objects, each with infinite repetition number?

This is the same as the following question: Find the number of integer solutions to

$$x_1 + x_2 + \cdots + x_k = r$$

subject to $x_i \geq 0$ or each i. We know the answer to this is

$$\binom{r+k-1}{r}$$

Similarly, we have considered the problem where we assume instead $x_i \geq a_i$.

New Problem

Find the number of integer solutions to

$$x_1 + x_2 + \cdots + x_k = r$$

subject to $0 \leq x_i \leq a_i$ for each i.

Example 6.3.1 (How to solve this type of question)

Let S be the set of solutions where we just have $x_i \geq 0$ and let A_i be the set of solutions where $x_i > a_i$. Then we want to find $|A_1^c \cup A_2^c \cdots \cup A_m^c|$.

$$\begin{aligned} |A_1^c \cap \cdots \cap A_m^c| = |S| - \sum |A_i| + \sum |A_i \cap A_j| \\ - \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^{m+1} |A_1 \cap \cdots \cap A_m| \end{aligned}$$

Question 7

Find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 18$$

subject to $5 \geq x_1 \geq 1$, $4 \geq x_2 \geq -2$, $5 \geq x_3 \geq 0$, $9 \geq x_4 \geq 3$.

Solution: First, let us redefine the problem in terms of new variables with equal restrictions but started with 0 as the lowest bound.

The problem becomes:

$$y_1 + y_2 + y_3 + y_4 = 16$$

subject to $4 \geq y_1 \geq 0$, $6 \geq y_2 \geq 0$, $5 \geq y_3 \geq 0$, $6 \geq y_4 \geq 0$.

Now, let S be the set of solutions where each $y_i \geq 0$. There are clearly,

$$\binom{16+4-1}{16} = \binom{19}{16}$$

Now, we must solve each of the following intersections, however, we only need to consider the cases where the intersection is non-empty. This is obviously done by solving this like we have previously studied. Subtract 1+ the upper bound of each y_i from the target value of the sums. And proceed to solve this question as a normal stars and bars problem.

$$\begin{aligned}
|A_1| &= \binom{11+4-1}{11} = \binom{14}{11} \\
|A_2| &= \binom{9+4-1}{9} = \binom{12}{9} \\
|A_3| &= \binom{10+4-1}{10} = \binom{13}{10} \\
|A_4| &= \binom{9+4-1}{9} = \binom{12}{9} \\
|A_1 \cup A_2| &= \binom{4+4-1}{4} = \binom{7}{4} \\
|A_1 \cup A_3| &= \binom{5+4-1}{5} = \binom{8}{5} \\
|A_1 \cup A_4| &= \binom{4+4-1}{4} = \binom{7}{4} \\
|A_2 \cup A_3| &= \binom{3+4-1}{3} = \binom{6}{3} \\
|A_2 \cup A_4| &= \binom{2+4-1}{2} = \binom{5}{2} \\
|A_3 \cup A_4| &= \binom{3+4-1}{3} = \binom{6}{3}
\end{aligned}$$

So the final answer is

$$\begin{aligned}
&= \binom{19}{16} - \Bigg(\binom{14}{11} + \binom{12}{9} + \binom{13}{10} + \binom{12}{9} \\
&- \binom{7}{4} - \binom{8}{5} - \binom{7}{4} - \binom{6}{3} - \binom{5}{2} - \binom{6}{3}\Bigg) \\
&= 55
\end{aligned}$$

6.4 L16: The Inclusion-Exclusion principle and applications IV: Another Forbidden Position Problem

Question 8

Eight people take a walk a walk in a line

$$1, 2, 3, 4, 5, 6, 7, 8$$

where 1 precedes 2 who precedes 3 etc. How many ways are there to rearrange the people so that no one precedes the person he preceded before?

In other words, count w in S_n that avoid the pairs

$$12, 23, \dots, (n-1)n$$

Solution: Let S_n be the set of all permutations of $\{1, 2, \dots, n\}$.
Let A_i be the set of all permutations that contain the pair $ii+1$.

Then we want to find

$$|S_n| - \left(\sum_{i=1}^{n} |A_i| - \sum_{i,j=1}^{n} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| \right)$$

Clearly, $|S_n| = n!$ because there are $n!$ permutations.

Now, we must solve each of the following intersections. Observe the following:

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n-1}{k}(n-k)!$$

This is because we must think of constructing an algorithm that places the paired elements and then places the remaining elements.

This algorithm is represented by the RHS of the above the equation. This is because there are only $n-1$ positions where we can place a "block", i.e., a pair of elements. Given k blocks, there are $\binom{n-1}{k}$ ways to place them. Then, we must place the remaining $n-k$ elements which can be done in $(n-k)!$ ways given their distinct ordering

Thus, this gives us the final form of the answer.

$$n! - \left(\binom{n-1}{1}(n-1)! - \binom{n-1}{2}(n-2)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}(1)! \right)$$

Theorem 6.4.1 Non-attacking rook arragements

The number of non-attacking rook arragements on an $n \times n$ board with forbidden positions is

$$n! - r_1(n-1)! + r_2(n-2)! - \cdots + (-1)^k r_k (n-k)! + \cdots + (-1)^n r_n$$

Here r_k is the number of ways to place k rooks in the forbidden squares.

Proof. This is an application of the inclusion-exclusion principle.

Let A_i be the set of all placements where exactly one of the forbidden squares in row i must be used. Consider the following arbitrary intersection of A_is

$$\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

Note that the sum is over all selections of the k subsets.

This can be observed by constructing an algorithm that places the rooks in the forbidden squares and then places the remaining rooks.

There are r_k ways to place the rooks in the forbidden squares. Then, there are $n - k$ positions to place the remaining rooks, so there are $(n-k)!$ ways to place them. Thus,

$$\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k(n-k)!$$

Substituting this is the inclusion-exclusion principle, we get the theorem.

■

6.5 Practice Problems

Chapter 7

Recurrence Relations and Generating Functions

7.1 L17: Some Number Sequences

Example 7.1.1 (Example 1)

Consider a configuration of n lines where every two lines have a point in common, but no three do. How many regions in the plane are there? Give a recurrence.

$$a_n = a_{n-1} + n$$

TODO: Give an explanation of why this is true.

Example 7.1.2 (Example 2)

Give a simple recurrence for dearragements.

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

TODO: Give an explanation of why this is true. Need to review dearragements from previous lecture.

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$.

Definition 7.1.1: The adjusted Fibonacci sequence: $\hat{F}_n$

This is the number of 1,2 lists of size n. In other words, consider the number of ways a valet can park A cars (size 1) and B cars (size 2) in a parking lot of size n.

$$\hat{F}_n = \begin{cases} 1 & \text{if } n = 0 \\ f_{n+1} & \text{otherwise} \end{cases}$$

Question 9

Prove

$$\sum_{n=0}^{n} f_i = f_{n+2} - 1$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$\sum_{i=0}^{0} f_i = f_0 = 0$$

Similarly,

$$f_{0+2} - 1 = f_2 - 1 = 1 - 1 = 0$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$\sum_{i=0}^{k} f_i = f_{k+2} - 1$$

Indcutive Step: We will prove that the following statement is true for $n = k + 1$.

$$\sum_{i=0}^{k+1} f_i = f_{k+1} + \sum_{i=0}^{k} f_i = f_{k+1} + f_{k+2} - 1 = f_{k+3} - 1$$

Therefore, the statement is true for all n by induction. ■

Question 10

Prove

$$1 + \sum_{i=0}^{n} \hat{F}_i = \hat{F}_{n+2}$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$1 + \sum_{i=0}^{0} \hat{F}_i = 1 + \hat{F}_0 = 1 + 1 = 2$$

Similarly,

$$\hat{F}_{0+2} = \hat{F}_2 = f_{2+1} = f_3 = 2$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$1 + \sum_{i=0}^{k} \hat{F}_i = \hat{F}_{k+2}$$

Inductive Step: We will prove that the following statement is true for $n = k + 1$.

$$
\begin{aligned}
1 + \sum_{i=0}^{k+1} \hat{F}_i &= 1 + \hat{F}_{k+1} + \sum_{i=0}^{k} \hat{F}_i \\
&= \hat{F}_{k+1} + \hat{F}_{k+2} \\
&= f_{k+2} + f_{k+3} \\
&= f_{k+4} \\
&= \hat{F}_{k+3}
\end{aligned}
$$

Therefore, the statement is true for all n by induction. ■

Question 11

Prove that f_n is even if and only if n is divisible by 3.

Solution:

Proof. Given that $f_0 = 0$, $f_1 = 1$, and $f_2 = 1$, we can see that at $n = 3$, $f_3 = 2$, which is even.

This is because the only way to get an even number is to have the parity of the two numbers added togethed (odd + odd or even + even) be the same. So, f_4, must be odd, f_5 must be odd and f_6 must be even.

Given the starting sequence of even, odd, odd. The following sequence must always be even, odd, odd, which repeats every 3 numbers.
Since the first $n = 0$ is the first number in the sequence, every n that is divisible by 3 is even. ■

Note:-

Example problems for later

Guess and prove by induction (you may replace the Fibonnaci number by the adjusted Fibonacci number if it helps you)

- $f_1 + f_3 + \cdots + f_{2n-1} =?$
- $f_0 + f_2 + \cdots + f_{2n} =?$
- $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n =?$
- $(f_0)^2 + (f_1)^2 + \cdots + (f_n)^2 =?$

Obtaining an explicit formula for f_n for linear recurrences

Example 7.1.3

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$. This can be rewritten as a linear recurrence as follows:

$$f_n - f_{n-1} - f_{n-2} = 0$$

We must solve the corresponding characterstic equation. Notice how the largest degree lines up with the ”largest” case of the recurrence.

$$x^2 - x - 1 = 0$$

Let q_1 and q_2 be the roots of the characteristic equation.

It is potentially relevant to note that the following is a solution space of the Fibonacci recurrence (but don't satisfy f_0 – the initial condition):

$$\begin{cases} q_1^n - q_1^{n-1} - q_1^{n-2} = 0 \\ q_2^n - q_2^{n-1} - q_2^{n-2} = 0 \end{cases}$$

The rest of this is based on an ansatz, i.e. we need to make an assumption at the answer and validate it later

$$f_n = c_1 q_1^n + c_2 q_2^n,$$

for *some* $c_1, c_2 \in \mathbb{R}$.

Using the initial conditions of $f_0 = 0$ and $f_1 = 1$, we can solve for c_1 and c_2.

7.2 L18: Introduction to ordinary generating series

Note:-

Up until now, we have solved an instance combinatorial problem at a time. However, in some cases, solving a single combinatorial is too difficult. So, we solve all of the combinatorial problems at once.

Consider a sequence

$$h_0, h_1, h_2, \ldots, h_t, \ldots$$

of natural numbers where h_t is the answer to some counting problem that depends on t.

We can create a generating series of the form:

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_t x^t + \cdots$$

where $h_t = [x^t]g(x)$.

Note:-

The notation $[x^t]g(x)$ is the coefficient of x^t in the polynomial $g(x)$.

Claim 7.2.1 Compositions Generating Series

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Example 7.2.1

Fix k. Let

h_t = number of nonnegative integral solutions to

$$e_1 + \cdots + e_k = t, e_i \in \mathbb{Z}_{\geq 0}$$

We already know from stars and bars that

$$h_t = \binom{t+k-1}{k-1}$$

So obviously, the generating series can be written as

$$g(x) = \sum_{t=0}^{\infty} \binom{t+k-1}{k-1} x^t$$

Note:-

This doesn't really tell us anything. We just combined some definitions and have a genearting series that is hard to utilize in any useful way.

Take our claim and expand it out with Taylor Series k times:

$$\begin{aligned} g(x) &= \left(\frac{1}{1-x}\right)^k \\ &= (1+x+x^2+\cdots)(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots \end{aligned}$$

We can see that the coefficient of x^t is the number of ways to write t as a sum of k nonnegative integers. So,

$$h_t = \binom{t+k-1}{k-1}$$

This combinatorially proves the negative binomial theorem:

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Remark

If you knew that the negative binomial theorem was true, you could use it to algebraically solve the composition problem after choosing the correct generating series $g(x) = \left(\frac{1}{1-x}\right)^k$.

Question 12

What is

$$(1+x+x^2+x^3+x^4+x^5)(x+x^2)(1+x+x^2+x^3+x^4)$$

the generating series for?

Solution: The solutions to the equation

$$e_1 + e_2 + e_3 = t$$

where $0 \le e_1 \le 5, 1 \le e_2 \le 2, 0 \le e_3 \le 4$.

Question 13

Suppose you have 1,5 and 25 cents coins available (infinite supply). Write down the generating series for the number of ways to make change for a dollar. How many ways are there?

Solution:

The generating series for the number of ways to make change for an arbitrary amount of money is:

$$g(x) = (1+x+x^{1+1}+x^{1+1+1}+\cdots)(1+x^5+x^{5+5}+x^{5+5+5}+\cdots)(1+x^{25}+x^{25+25}+x^{25+25+25}+\cdots)$$

This can be simplified as follows:

$$
\begin{aligned}
g(x) &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{25}}\right) \\
&= \frac{1}{(1-x)(1-x^5)(1-x^{25})}
\end{aligned}
$$

The answers is $[x^{100}]g(x)$.

Question 14: Nonstandard Dice

Compute the generating series for the result of rolling two standard dice.
Now factor it, and interpret.

Solution:

$$
\begin{aligned}
g(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6) \\
&= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \\
&= (1 + x)(1 + x)(x + x^3 + x^5)
\end{aligned}
$$

TODO: How does this factor like this?
After factoring, we can see that this is the equivalent of flipping a coin twice and rolling two 1-3-5 dice.

Question 15

Determine the generating series for partitions.

Solution:

$$
\begin{aligned}
g(x) &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^2 + x^{2+2} + x^{2+2+2} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots)\cdots \\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i}\right)
\end{aligned}
$$

In words, this generating series is the product of compositions of just 1's, 2's, 3's, etc.

Definition 7.2.1: Ordinary Generating Series

Let a combinatorial problem be defined as (S, ω) with S a set and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ The ordinary generating series is

$$
g(x) = g_{(S,\omega)}(x) = \sum_{s \in S} x^{\omega(s)}
$$

Theorem 7.2.1 Addition Rule of Generating Series

Suppose $S = A \cup B$ (disjoint union), where (A, ω_A) and (B, ω_B) are combinatorial problems. Moreover $\omega|_A = \omega_A$ and $\omega|_B = \omega_B$.
Then the ordinary generating series for (S, ω) is

$$\begin{aligned} g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) + g_{(B,\omega_B)}(x) \\ &= \sum_{s \in A} x^{\omega_A(s)} + \sum_{s \in B} x^{\omega_B(s)} \\ &= \sum_{s \in S} x^{\omega(s)} \end{aligned}$$

Note:-

The notation $\omega|_A$ means the restriction of ω to A. In this case,

$$\omega|_A = \{\omega_A(s) \mid s \in A\}$$

Theorem 7.2.2 Product rule of Generating Series

Suppose $S = A \times B$ (cartesian product), where (A, ω_A) and (B, ω_B) are combinatorial problems and

$$\omega(a, b) = \omega_A(a) + \omega_B(b)$$

Then,

$$\begin{aligned} g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) g_{(B,\omega_B)}(x) \\ &= \sum_{a \in A} x^{\omega_A(a)} \sum_{b \in B} x^{\omega_B(b)} \\ &= \sum_{a \in A} \sum_{b \in B} x^{\omega(a,b)} \end{aligned}$$

What does it mean for two generating series to be equal?

They are equal coefficient by coefficient.

Let $g(x)$ be the ggenerating series for the number of partitions. What does it mean that $g(x)$ equals the infinite product:

$$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

It means that if we expand/solve enough terms of the RHS, the coefficient of x^t will stabalize (as in no more terms multiplied in will impact the coefficient) and this coefficient will be the same as the coefficient of x^t in the LHS – $g(x)$.

Convergence Issues

Typically, with generating series, we only care about the coefficients and not plugging in any specfic value into x. Because of this, we do not need to worry about convergence. However, in some cases $g(x)$ is a polynomial and in these cases, substitution is fine to perform.

Question 16: Substituting into a generating series

Let $S = \{\text{coins in your pokcet}\}$ and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ be the obvious weight function on coins, i.e. $\omega(\text{nicket}) = 5$. Is $g(x)$ is the corresponding generating series, what is $g(1)$? What is $g'(1)$?

Solution: $g(1)$ will be the number of coins in your pocket. This is trivially true because each term is x to the power of an integer (with no coefficient), so 1 raised to anything will be 1 and so it is just the number of terms. More mathematically,

$$\begin{aligned} g(1) &= \sum_{s \in S} 1^{\omega(s)} \\ &= \sum_{s \in S} 1 \\ &= |S| \end{aligned}$$

$g'(1)$ will the amount of money you have. Based on the previous statement, we can see that the derivative of $g(x)$ is just the sum of the powers which represented the values of the coins in your pocket. More mathematically,

$$g'(1) = \sum_{s \in S} \omega(s)$$

7.3 Practice Problems

Chapter 8

Special Counting Sequences

8.1 L19: Partition identities

Definition 8.1.1: Partition

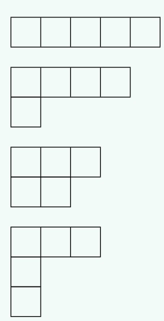
A *partition* is a decreasing sequence of positive integers whose sum is n.

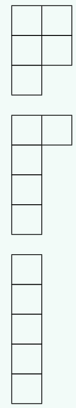
Example 8.1.1

Consider $n = 5$. The partitions of 5 are

$$\begin{aligned}&5\\&4,1\\&3,2\\&3,1,1\\&2,2,1\\&2,1,1,1\\&1,1,1,1,1\end{aligned}$$

However, this can also be represented in terms of the **Ferrers diagram**:
Partitions 5:





Example 8.1.2 (Theorem 8.3.1 from the textbook)

Use Ferrers diagrams to prove that the number of partitions of n in which the largest part is r equals the number of partitions of $n - r$ in which no part is greater than r.

Let P be a partition for n. Remove the first part of P which by definition is the largest part which has an arbitrary size r. This clearly yields a partition of $n - r$ in which no part is greater than r. This process is clearly a bijection and so the number of partitions of n in which the largest part is r equals the number of partitions of $n - r$ in which no part is greater than r.

Note:-

This process for proving two partitions to be of equal size is important.

Generating Series , and partition identities

Recall generating series for partition:

$$g(x) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^k} \right)$$

Question 17

Write down the generating series for:

1. only odd parts
2. no part repeated more than m times

Solution: Only odd parts:

$$\begin{aligned} g(x) &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots) \cdot (1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots) \cdots \\ &= \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^{2k-1}} \right) \end{aligned}$$

Solution: No part repeated more than m times

$$
\begin{aligned}
g(x) &= (1 + x + x^{1+1} + \cdots + x^m)(1 + x^2 + x^{2+2} + \cdots + x^{2m})(1 + x^3 + x^{3+3} + \cdots + x^{3m})\cdots \\
&= (\frac{1}{1-x} - \frac{x^m}{1-x}) \cdot (\frac{1}{1-x^2} - \frac{x^m}{1-x^2}) \cdot (\frac{1}{1-x^3} - \frac{x^m}{1-x^3})\cdots \\
&= \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k} - \frac{x^m}{1-x^k}\right) \\
&= \prod_{k=1}^{\infty} \left(\frac{1-x^m}{1-x^k}\right)
\end{aligned}
$$

Terminology

Definition 8.1.2: Partition size

The *size* of a partition is the sum of its parts.

Definition 8.1.3: Partition length

The *length* of a partition is the number of parts.

Question 18

Write down a two variable generating series that counts partitions by both size and length, using say, a q variable and a z variable respectively. Hence the coefficient of $q^n z^l$ is the number of partitions of size n and length l.

Solution:

$$
\begin{aligned}
g(q,z) &= \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} P(n,l) \cdot q^n z^l \\
&= \prod_{k=0}^{\infty} (1 + zq^k + z^2 q^{2k} + z^3 q^{3k} + \cdots) \\
&= \prod_{k=0}^{\infty} \left(\frac{1}{1-zq^k}\right)
\end{aligned}
$$

Question 19

Consider the identity:

$$\prod_{k=1}^{\infty} 1 + q^k = \prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}}$$

- Interpret both sides of the identity.
- Give an algabraic proof.
- Can you give a combinatorial proof?

Solution: Interpret both sides of the identity

- LHS

$$\prod_{k=1}^{\infty} 1+q^k = (1+q^1)(1+q^2)(1+q^3)\cdots$$

This is clearly the number of partitions with distinct parts that are used at most one time.

- RHS

  This is clearly the generating series for partitions with only odd parts.

Solution: Give an algebraic proof

Proof.

$$\begin{aligned}
\prod_{k=1}^{\infty} 1+q^k &= \prod_{k=1}^{\infty} \frac{1-q^k}{1-q^k}(1+q^k) \\
&= \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^k} \\
&= \prod_{k=1}^{\infty} \frac{\frac{1}{1-q^k}}{\frac{1}{1-q^{2k}}} \\
&= \frac{\prod_{k=1}^{\infty} \frac{1}{1-q^k}}{\prod_{k=1}^{\infty} \frac{1}{1-q^{2k}}} \\
&= \text{ all even partitions divided out of all partitons} \\
&= \prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}}
\end{aligned}$$

∎

Solution: Give a combinatorial proof

Proof. Fix n. Let S be the set of all partitions of n with only distinct parts. Clearly, the LHS represents the generating series that counts S.

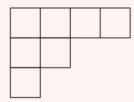
Construct a function f from distinct parts to odd parts by dividing each even number by 2 until every number is odd. This function is clearly well defined as it will only result in odd parts. This must be a bijection because the input to f must have distinct parts so the number of odd parts created is dependent on the even numbers in the input so the output is unique. Lastly, this is clearly reversible by taking the largest subset of repeated odd numbers that is a power of 2 and summing them.

Thus, because we can decompose S into the set of partitions with only odd parts through a bijective function and this decomposition is clearly counted by the RHS. Then, it is clear that the LHS and RHS equivalent. ∎

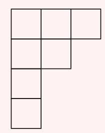
8.2 L20: Partition identities (continued)

Definition 8.2.1: Partition Conjugate

The *conjugate* of a partition is the partition associated to the transpose of the Ferrers diagram.



has a conjugate of



Question 20

Use conjugation to prove the following fact: the number of partitions of n with no part bigger than k equals the number of partitions of n with at most k parts.

Determine the generating series for either of the kinds of partitions from the previous problem.

Solution: Using conjugation...

Proof. This part is trivially true. If no part is bigger than k, then the first row in the Ferrers Diagram, which must be the largest row in the diagram, has at most k parts.

By taking the transpose of the first type of parition, we get a partition with at most k parts as the number of parts will be dictated by the first row.

Thus, the number of partitions of n with no part bigger than k is the same as the number of partitions of n with at most k parts. ■

Solution: Generating Series

$$\begin{aligned} g(x) =& (1 + x + x^{1+1} + \cdots + x^k) \\ & (1 + x^2 + x^{2+2} + \cdots + x^{2k}) \\ & (1 + x^3 + x^{3+3} + \cdots + x^{3k}) \cdots \end{aligned}$$

$$\begin{aligned} g(x) &= \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + \cdots + x^{ki}) \\ &= \prod_{i=1}^{\infty} \left(\frac{1 - x^{ki+1}}{1 - x^i} \right) \end{aligned}$$

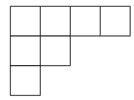
Definition 8.2.2: Durfee Square

A *Durfee square* is the largest square that sits in the NW corner of the Ferrers diagram of a partition.

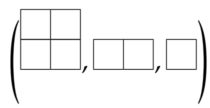
Question 21

Show that every Ferrers shape uniquely decomposes into (D_k, A_k, B_k) where D_k is a Durfee square of size k, A_k is the partition with at most k parts, and B_k is a partition with first part of size at most k.

Solution: Consider an arbitrary partition:



This can obviously be decomoposed into the triple:



There are a few cases to consider:

- First, this is unique. Clearly, this has to be unique because we are not changing any structure from the partition, but rather we are just decomposing it into a triple. So each partition must uniquely map to its own ordered triple.

- Second, A_k is the partition with at most k parts. A_k must have at most k because the Durfee square is of size k. A_k cannot be largely than this because it would contradict either the definition of a partition or the definition of a Durfee square.

- Third, B_k is the partition with first part of size at most k. This is true because if the first part of B_k is larger than k, then this would contradict the definition of a Durfee square or a partition.

Thus, we have shown that every partition can be decomposed into a triple.

Definition 8.2.3: Euler-Gauss Identity

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = 1 + \sum_{j=1}^{\infty} \frac{x^{j^2}}{\left(\prod_{t=1}^{j} 1-x^t\right)^2}$$

Definition 8.2.4: Self-conjugate Partition

A partition is *self-conjugate* i its transpose shape is the same as its original shape (e.g., the partitions (4,1,1,1) and (3,2,1) are self-conjugate, but (3,2) is not).

Question 22

Prove that the number of self conjugate partitions of n equals the number of partitions using only distinct odd parts. Then derive an identity.

Solution: First, we must prove the number of self conjugate partitions of n equals the number of partitions using only distinct odd parts. This can be done by showing a weight preserving bijection between the two sets.

Definition 8.2.5: Weight-preserving function

Define the following function:

$$f : (S, \omega) \mapsto (S', \omega')$$

such that $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ and $\omega' : S' \mapsto \mathbb{Z}_{\geq 0}$ are weight functions. Then, f is a weight-preserving function if and only if:

$$\omega(s) = \omega'(f(s)) \text{ for all } s \in S$$

And so, we can make the following claim

Claim 8.2.1

If $f : (S, \omega) \mapsto (S', \omega')$ is a weight-preserving and bijective then,

$$g_{(S,\omega)}(x) = g_{(S',\omega')}(x) \in \mathbb{Z}[x]$$

In other words, their generating functions are equal.

Now to define the actual bijection. Consider each row of the Ferrers diagram of an arbitrary distinct odd partition. Each row can be bent into a self-conjugate partition. By selecting the middle element, which is gaurunteed to exist because the partition is odd, we can place every square to the left below the middle element and leave the other elements alone. Additionally, this is well defined as each row is distinct. Thus, we have a weight preserving bijection between the set of distinct odd partitions and the set of self-conjugate partitions.

The generating function for the number of distinct odd partitions is trivial and given by:

$$g(x) = \prod_{i=1}^{\infty} 1 - x^{2i-1}$$

as we only want to either select an odd number or not.

The generating function for the number of self-conjugate partitions is more complicated. Consider an arbitrary self-conjugate partition λ. We can decompose it into a triple (D_k, A_k, B_k) through the Durfee square decomposition. We can then apply a second weight-preserving bijection and conjugate the second component of the triple which makes $A_k = B_k$. Lastly, we can apply a third weight-preserving bijection by compressing the triple into a tuple (D_k, C_k) where C_k is A_k merged left-to-right with B_k.

Consequently,

$$C_k \in \bigcup_{j=0}^{\infty} D_j \times \text{Partitions}_{\text{even}_{\leq j \text{ tall}}} = U$$

Clearly,

$$\begin{aligned} g_U(x) &= \sum_{j=0}^{\infty} x^{j^2} \Big(\prod_{t=1}^{j} \frac{1}{1 - x^{2t}} \Big) \\ &= \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{t=1}^{j} \frac{1}{1-x^{2t}}} \end{aligned}$$

So,

$$\prod_{k=1}^{\infty} 1 + x^{2k-1} = \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{t=1}^{j} 1 - x^{2t}}$$

8.3 L21: Exponential generating series

Given a number sequence

$$h_0, h_1, h_2, \ldots$$

consider using the *exponential* series:

Definition 8.3.1: Exponential Generating Series

$$g_{\exp}(x) = \sum_{i=0}^{\infty} h_i \frac{x^i}{i!}$$

Why do we care about using this?

Reason A

Ordinary generating series typically won't have any algabraic properties that are useful such as factorization. However, exponential generating may have these properties.

Example 8.3.1

Fix a positive integer n. Let
$P(n, k)$ = the number of k permutations of an n set. In other words, select k elements from n set where order matters.

Question 23

Compute the ordinary and exponential generating series for $P(n, 0), P(n, 1), \ldots$.

Solution: Let's consider how a single $P(n, k)$ is constructed. There are $\binom{n}{k}$ ways to pick k elements from n set and $k!$ ways to order them. So,

$$\begin{aligned} P(n, k) &= \binom{n}{k} k! \\ &= \frac{n!}{k!(n-k)!} k! \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

Clearly the ordinary generating series is:

$$
\begin{aligned}
g(x) &= \sum_{i=0}^{\infty} P(n,i)x^i \\
&= \sum_{i=0}^{\infty} \frac{n!}{(n-i)!}x^i
\end{aligned}
$$

Additionally, the exponential generating series is:

$$
\begin{aligned}
g_{\text{exp}}(x) &= \sum_{i=0}^{\infty} P(n,i)\frac{x^i}{i!} \\
&= \sum_{i=0}^{\infty} \frac{n!}{(n-i)!}\frac{x^i}{i!} \\
&= \sum_{i=0}^{\infty} \binom{n}{i}x^i \\
&= (1+x)^n
\end{aligned}
$$

Note:-

Note how the exponential series simplifies as compared to the ordinary series.

Question 24

What is the exponential generating series for the sequence $1, 1, 1, \ldots$?

Solution:

$$
\begin{aligned}
g_{\text{exp}}(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\
&= \sum_{i=0}^{\infty} \frac{x^i}{i!} \\
&= e^x
\end{aligned}
$$

Reason B

Ordinary series only work well for *unlabeled* combinatorial sets, whereas exponential generating are better for *labeled* combinatorial sets.

Example 8.3.2

Consider 38 identical balls, how many ways are thre to group them together into 3 different bags? This is a composition problem, and is an unlabeled problem (since all balls are identical). This is a composition problem with a solution of $\binom{38+3-1}{3-1}$

Now consider 38 students in MATH 413. How many ways are there to group them into 3 different groups? This is a *labeled* version of the problem. The answer to this is 3^{38} because there 3 choices for each distint student.

Question 25

What is e^{ax}, for a natural number a, the exponential generating series for?

Solution: We know that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, so

$$\begin{aligned} e^{ax} &= \sum_{i=0}^{\infty} \frac{(ax)^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{x^i}{i!} a^i \end{aligned}$$

a^i is the number of words of length i that can be generated from an alphabet of size a.

Question 26

Find the exponential generating series for the number of ways to place r (distinct) people into three different rooms with at least one person in each room. Repeat the problem with the extra condition of an even number of people.

Solution: One person in each room

$$g_{\exp}(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^3$$

Solution: Even Number of people and one person in each room

$$g_{\exp}(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^3$$

Question 27

Now say $r = 25$ in the above exercise. Compute the actual number of ways to place 25 people in three rooms with at least one person in each room.

Solution: We have already computed the generating series for one person in each room above. This generating series can be simplified even further as:

$$\begin{aligned} g_{\exp}(x) &= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^3 \\ &= (e^x - 1)^3 \\ &= \sum_{i=0}^{3} \binom{3}{i} (e^x)^{3-i} (-1)^i \\ &= \binom{3}{0} e^{3x} - \binom{3}{1} e^{2x} + \binom{3}{2} e^x - \binom{3}{3} \\ &= e^{3x} - 3e^{2x} + 3e^x - 1 \end{aligned}$$

Our desired answer is $[x^{25}]g_{\exp}(x)$, so the answer is as follows:

$$\begin{aligned} [x^{25}]g_{\exp}(x) &= [x^{25}]e^{3x} - 3[x^{25}]e^{2x} + 3[x^{25}]e^x - [x^{25}] \\ &= 3^{25} - 3 \cdot 2^{25} + 3 \cdot 1^{25} - 1 \\ &= 3^{25} - 3 \cdot 2^{25} + 3 \end{aligned}$$

Chapter 9

Homework

9.1 Homework 1

9.2 Homework 2

9.3 Homework 3

9.4 Homework 4

9.5 Homework 5

9.6 Homework 6

9.7 Homework 7

Question 28

Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

that satisfy

$$1 \leq x_1 \leq 6, 0 \leq x_2 \leq 7, 4 \leq x_3 \leq 8, 2 \leq x_4 \leq 6$$

Solution: We first need to reformat the question with adjusted restrictions:

$$\begin{aligned} &y_1 + y_2 + y_3 + y_4 = 13 \\ &0 \leq y_1 \leq 5, 0 \leq y_2 \leq 7, 0 \leq y_3 \leq 4, 0 \leq y_4 \leq 4 \end{aligned}$$

This can be solved using the inclusion-exclusion principle:

Let S be the set of all solutions to the above equations where each $y_i \geq 0$. By stars and bars,

$$|S| = \binom{13+4-1}{13} = \binom{16}{13}$$

Let A_i be the set of all solutions such that $y_i \geq a_i$ where a_i is the exclusive upper integral bound for y_i (e.g. $y_1 \geq 6$)

$$
\begin{aligned}
|A_1| &= \binom{7+4-1}{7} = \binom{10}{7} \\
|A_2| &= \binom{5+4-1}{5} = \binom{8}{5} \\
|A_3| &= \binom{8+4-1}{8} = \binom{11}{8} \\
|A_4| &= \binom{8+4-1}{8} = \binom{11}{8} \\
|A_1 \cap A_2| &= 0 \\
|A_1 \cap A_3| &= \binom{2+4-1}{2} = \binom{5}{2} \\
|A_1 \cap A_4| &= \binom{2+4-1}{2} = \binom{5}{2} \\
|A_2 \cap A_3| &= \binom{0+4-1}{0} = \binom{3}{0} \\
|A_2 \cap A_4| &= \binom{0+4-1}{0} = \binom{3}{0} \\
|A_3 \cap A_4| &= \binom{3+4-1}{3} = \binom{6}{3}
\end{aligned}
$$

Note that the intersection of any 3 sets will have no solution.

Substituting this into the complementary form of the inclusion-exclusion principle, we get:

$$
\begin{aligned}
&\binom{16}{13} - \left(\binom{10}{7} + \binom{8}{5} + \binom{11}{8} + \binom{11}{8}\right. \\
&\qquad \left. - \binom{5}{2} - \binom{5}{2} - \binom{3}{0} - \binom{3}{0} - \binom{6}{3}\right) \\
&\qquad = 96
\end{aligned}
$$

Question 29

Determine the number of permutations of $\{1, 2, \ldots, 9\}$ in which at least one odd integer is in its natural position.

Solution: This is an application of the inclusion-exclusion principle.

Let A_i be the set of all permutations such that i is in its natural position.

We must calculate $|A_1 \cup A_3 \cup A_5 \cup A_7 \cup A_9|$ for the inclusion-exclusion principle.

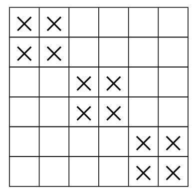
For the sizes of each set individually, the size of each set must be 8! because there is one option for the position that must be in place and then we must place the 8 remaining numbers. There are $\binom{5}{1}$ ways to pick the odd number in place.

Following this pattern we can build the inclusion-exclusion principle as follows:

$$= \binom{5}{1}8! - \binom{5}{2}7! + \binom{5}{3}6! - \binom{5}{4}5! + \binom{5}{5}4!$$
$$= 157824$$

Question 30

What is the number of ways to place six nonattacking rooks on the 6-by-6 boards with forbidden positions as shown?



Solution: This is an application of the inclusion-exclusion principle.

Let S_n be the set of all boards with nonattacking rooks on an $n \times n$ board. Clearly, there are $n!$ boards within S_n. This can be observed by constructing an algorithm to place one rook at a time.

Let A_i be the set of all boards with exactly 1 rook in the forbidden in row i.

We must calculate

$$|A_1^c \cap A_2^c \cap \cdots \cap A_n^c|$$

Note that it is unecessary to consider each subset of A_i individually, but rather we should consider the sum of the selected subsets. This gives the following identity

$$\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k(n-k)!$$

where r_k is the number of ways to place k rooks in forbidden positions.

Substituting this into the complementary form of the inclusion-exclusion principle, we get:

$$\begin{aligned}
&6! - (r_1(6-1)! - r_2(6-2)! + r_3(6-3)! - r_4(6-4)! + r_5(6-5)! - r_6(6-6)!) \\
&= 6! - r_1(5!) + r_2(4!) - r_3(3!) + r_4(2!) - r_5(1!) + r_6(0!) \\
&= 6! - r_1(5!) + r_2(4!) - r_3(3!) + r_4(2!) - r_5 + r_6
\end{aligned}$$

All that remains is to calculate $r_1, r_2, \ldots, r_6$.

$$\begin{aligned}
r_1 &= 12 \\
r_2 &= \binom{3}{1} \cdot (2) + \binom{3}{2}(4 \cdot 4) = 3 \cdot 2 + 3 \cdot 16 = 6 + 48 = 54 \\
r_3 &= \binom{3}{2}\binom{2}{1} \cdot (2 \cdot 4) + \binom{3}{3}(4 \cdot 4 \cdot 4) = 3 \cdot 2 \cdot 2 \cdot 4 + 1 \cdot 64 = 48 + 64 = 112 \\
r_4 &= \binom{3}{2}\binom{2}{1} \cdot 2 + \binom{3}{1} \cdot 2 \cdot 4 \cdot 4 = 3 \cdot 2 \cdot 2 \cdot 2 + 3 \cdot 2 \cdot 4 \cdot 4 = 12 + 96 = 108 \\
r_5 &= \binom{3}{2}\binom{2}{1} \cdot 2 \cdot 4 = 48 \\
r_6 &= 2 \cdot 2 \cdot 2 = 8
\end{aligned}$$

Substituting these values into the formula, we get:

$$6! - (12(5!) - 54(4!) + 112(3!) - 108(2!) + 48 - 8) = 80$$

9.8 Homework 8

Question 31

How many ways are there to arrange the letters in INTELLIGENT with at least two pairs of consecutive identical letters? (For example ITTNELLGENI is an arrangement we want to count, since it has "TT" and "LL".)

Solution: This is an application of the inclusion-exclusion principle.

Let A_i be the set of all arrangements with exactly i pairs of consecutive identical letters.

We must calculate

$$|A_2 \cup A_3 \cup \cdots \cup A_5|$$

For A_2, there are $\binom{5}{2}$ ways to choose the two pairs of consecutive identical letters. There are 7 remaining letters with 3 remaining pairs which can be arranged in $\frac{7!}{2!2!2!}$ ways.

For A_3, there are $\binom{5}{3}$ ways to choose the three pairs of consecutive identical letters. There are 5 remaining letters with 2 remaining pairs which can be arranged in $\frac{5!}{2!2!}$ ways. However, we must consider that the previous calculation A_2 double counted elements of A_3 by the number of pairs that were chosen, which was 2. We must multiply this in to account for this.

Following this pattern, we can see the final answer is

$$\binom{5}{2}\frac{9!}{(2!)^3} - 2 \cdot \binom{5}{3}\frac{8!}{(2!)^2} + 3 \cdot \binom{5}{4}\frac{7!}{(2!)} - 4 \cdot \binom{5}{5}6!$$

Note:-

Could probably use better explanations on why we need to multiply by the number of pairs chosen in the previous calculation.

Question 32

Let n be a positive integer and let $p_1, p_2, \ldots, p_n$ be all of the different prime numbers that divide n. Consider the Euler function ϕ defined by

$$\phi(n) = \left|\{k : 1 \leq k \leq n, \gcd(k, n) = 1\}\right|$$

Use the inclusion-exclusion principle to show that

$$\phi(n) = n \prod_{i=1}^{n} 1 - \frac{1}{p_i}$$

Question 33

Prove that the Fibonacci sequence is the solution of the recurrence relation

$$a_n = 5a_{n-4} + 3a_{n-5}, (n \geq 5)$$

where $a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3$. Then use this formula to show that the Fibonacci numbers satisfy the condition that f_n is divisible by 5 if and only if n is divisible by 5.

Question 34

Let h_n equal the number of different ways in which the squares of a $1 \times n$ chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n (using the characteristic equation method).

Question 35

The *Lucas numbers* $l_0, l_1, l_2, \ldots, l_n, \ldots$ are defined using the same recurrence relation defining the Fibonacci numbers, but with different intial conditions:

$$l_n = l_{n-1} + l_{n-2}, (n \geq 2) \quad l_0 = 2, l_1 = 1$$

Prove that

- $l_n = f_{n-1} + f_{n+1}$ for $n \geq 1$
- $l_0^2 + l_1^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$ for $n \geq 0$

Question 36

Let a_n equal the number of ternary strings of length n made up of 0s, 1s, and 2s such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$a_n = a_{n-1} + 2a_{n-2}$$

for $n \geq 2$ with $a_0 = 1$ and $a_1 = 3$. Then find a formula for a_n.

Question 37

A population grows as follows: in the current generation, each Canadian chooses one American for the next generation, and each American chooses one Canadian and one American for the next generation. Generation 0 consists of one Canadian. What is the size of generation n? (Here we assume all older generations leave when the new generation is chosen.)

9.9 Homework 9

Question 38

Prove that the partition function $p(n)$(= number of partitions of n) satisfies $p(n + 1) > p(n)$.

Solution: By the definition of a partition, we know there are $p(n)$ sequences of decreasing positive integers that sum to n.

For each sequence, we can append a 1 to the end to construct a new sequence, which will still be decreasing and sum to $n+1$. Therefore, we know that $p(n+1)$ is *at least* $p(n)$.

Additionally, we can construct a new partition of the sequence $\{n+1, 0, 0, \dots\}$. This will be a partition of $n+1$ trivially. Additionally, this partition cannot be a partition of any integer less than $n+1$ as the sum is above n. Since, this partition is not in n, we know that $p(n+1) > p(n)$.

Question 39

For each integer $n > 2$ determine a self-conjugate partitoin of n that has at least two parts.

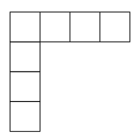
Solution: There are two cases to consider:

- n is odd

  We can construct the follwing partition of n:

$$\{\frac{n-1}{2} + 1, 1_1, 1_2, \dots, 1_{\frac{n-1}{2}}\}$$

  where there are $\frac{n-1}{2}$ 1s. This is self-conjugate as it will create Ferrer Diagrams of the form:

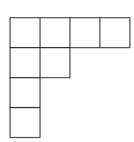


  Since $n > 2$, we know that there are at least two parts. At $n = 3$, $\frac{n-1}{2} + 1 = 2$ and $\frac{n-1}{2} = 1$. So the partition is $\{2, 1\}$.

  This is trivially bijective because $\frac{n-1}{2}$ is bijective.

- n is even

  We know that $n-1$ is odd. So we can construct a sel-conjugate partition for $n-1$ and then modify the generated sequence by changing one of the appended 1s to a 2.
  This generates Ferrer Diagrams of the form:



  We have shown in the odd case that this is a bijection and partition, so we know that this is a self-conjugate partition of n because adding 1 to one of the parts is still a bijective action.

Question 40

Prove that the number of partitions of n in which no part appears exactly once is equal to the number of partitions of n with no parts congruent to 1 or 5 (mod 6).

Question 41

Do part (c) of the last in class exercise of Lecture 19. That is, give a bijective proof that the number of partitions of N using distinct parts equals the number of partitions of N using only odd parts. (Hint: Consider the multiplicity m_p of each part p occuring in partitions of the second type. Now look at the unique expansion of m as powers of 2 and distribute.)

Solution: Consider an arbitrary partition P that only uses odd parts for N:

$$N = o_1 + o_2 + \cdots + o_k$$

Group each odd number to obtain its multiplicity m_p (e.g. $1 + 1 + 1 = 3(1)$):

$$N = m_1(o_1) + m_2(o_2) + \cdots + m_k(o_k)$$

Consider the unique expansion of m_p as powers of 2:

$$m_p = 2^0 + 2^1 + \cdots + 2^r$$

Distribute each o_i across the expansion of m_p:

$$\begin{aligned} &= (2^0 + 2^1 + \cdots + 2^{r_1})o_1 + (2^0 + 2^1 + \cdots + 2^{r_2})o_2 + \cdots + (2^0 + 2^1 + \cdots + 2^{r_k})o_k \\ &= 2^0(o_{\alpha_1} + o_{\alpha_2} + \cdots + o_{\alpha_x}) + 2^1(o_{\beta_1} + o_{\beta_2} + \cdots + o_{n_2}) + \cdots + 2^r(o_{\gamma_y} + o_{\gamma_2} + \cdots + o_{\gamma_z}) \end{aligned}$$

Let a_i be the sum of the o_is that correspond to the 2^i term:

$$= 2^0 a_0 + 2^1 a_1 + \cdots + 2^r a_r$$

We know that each of these terms must be unique as each power of 2 is unique in this sequence.

Thus, we have constructed a bijection between the partitions of N using only odd parts and the partitions of N using distinct parts.

9.10 Homework 10

Chapter 10

Midterms

10.1 Midterm 1

10.1.1 Practice Problems

10.1.2 Solutions

10.1.3 Exam

10.1.4 Exam Solutions

10.2 Midterm 2

10.2.1 Practice Problems

10.2.2 Solutions

10.2.3 Exam

10.2.4 Exam Solutions

10.3 Midterm 3

10.3.1 Practice Problems

Question 42

Find a recurrence relation for the number of different ways to hand out a piece of chewing gum (worth 1 cent) or a candy bar (worth 10 cents) or a donut (worth 20 cents), one item a day, until n cents worth of food has been given away. Don't worry about the initial condition.

Solution: Let $f(n)$ be the number of ways to hand out a piece of chewing gum, a candy bar, or a donut with n cents. At any moment with n cents, we can either give a piece of gum, a candy bar, or a donut. This gives

$$f(n) = f(n-1) + f(n-10) + f(n-20)$$

Question 43

By using generating series, determine the number of integer solutions to the composition problem

$$e_1 + e_2 + e_3 = 22$$

subject to $3 \leq e_1 \leq 8$, $6 \leq e_2 \leq 10$, and $2 \leq e_3 \leq 7$.

Solution: Clearly this can be solved by the following generating series:

$$g_1(x) = (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)(x^6 + x^7 + x^8 + x^9 + x^{10})(x^2 + x^3 + x^4 + x^5 + x^6 + x^7)$$

where the solution is $[x^22]g(x)$. However, this is difficult to extract a coefficient from. Instead, rewrite the problem in terms of new variables:

$$e_1 + e_2 + e_3 = 11$$

where $0 \leq e_1 \leq 5$, $0 \leq e_2 \leq 4$, and $0 \leq e_3 \leq 5$. The generating series now becomes:

$$g_2(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4 + x^5)$$

where we are solving for $[x^{11}]g_2(x)$. This generating series can be simplified as follows:

$$\begin{aligned}
g_2(x) &= (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4 + x^5) \\
&= (\frac{1 - x^6}{1 - x})(\frac{1 - x^5}{1 - x})(\frac{1 - x^6}{1 - x}) \\
&= \frac{(1 - x^6)(1 - x^5)(1 - x^6)}{(1 - x)(1 - x)(1 - x)} \\
&= \frac{(1 - x^6)(1 - x^5)(1 - x^6)}{(1 - x)^3} \\
&= (1 - x^6)(1 - x^5)(1 - x^6)(1 - x)^{-3} \\
&= (1 - x^5 - x^6 + x^{11})(1 - x^6)(1 - x)^{-3} \\
&= (1 - x^5 - x^6 + x^{11} - x^6 + x^{11} + x^{12} - x^{17})(1 - x)^{-3} \\
&= (1 - x^5 - 2x^6 + 2x^{11} + x^{12} - x^{17})(1 - x)^{-3}
\end{aligned}$$

The target solution is $[x^11]g_2(x)$. Using negative binomial theorem, we can find the coefficient of x^{11} as follows:

Reminder that negative binomial theorem is:

$$(1 - x)^{-n} = \sum_{k=0}^{\infty} \binom{n + k - 1}{k} x^k$$

$$\begin{aligned}
[x^{11}]g_2(x) &= [x^{11}]((1 - x^5 - 2x^6 + 2x^{11} + x^{12} - x^{17})(1 - x)^{-3}) \\
&= [x^{11}]\Big((1 - x)^{-3} - x^5(1 - x)^{-3} - 2x^6(1 - x)^{-3} + 2x^{11}(1 - x)^{-3} + x^{12}(1 - x)^{-3} - x^{17}(1 - x)^{-3}\Big) \\
&= [x^{11}](1 - x)^{-3} - [x^{11}]x^5(1 - x)^{-3} + [x^{11}]2x^6(1 - x)^{-3} + [x^{11}]2x^{11}(1 - x)^{-3} + [x^{11}]x^{12}(1 - x)^{-3} - [x^{11}]x^{17}(1 - x)^{-3} \\
&= [x^{11}](1 - x)^{-3} - [x^6](1 - x)^{-3} - 2[x^5](1 - x)^{-3} + 2[x^0](1 - x)^{-3} + [x^{-1}](1 - x)^{-3} - [x^{-6}](1 - x)^{-3} \\
&= \binom{3 + 11 - 1}{3 - 1} - \binom{3 + 6 - 1}{3 - 1} - 2\binom{3 + 5 - 1}{3 - 1} + 2\binom{3 + 0 - 1}{3 - 1} + 0 + 0 \\
&= \binom{13}{2} - \binom{8}{2} - 2\binom{7}{2} + 2\binom{2}{2}
\end{aligned}$$

Question 44

Prove that the number of partitions where no part appears more than two times equals the number of partitions where no part is a multiple of three. (Hint: write down the generating series for the former type of partition and algebraically manipulate to get to the generating series for the latter type.)

Solution: Number of partitions of n where no part appears more than two times:

$$\prod_{i=1}^{\infty}(1 + x^i + x^{2i})$$

Number of partitions of n where no part is a multiple of three:

$$\frac{\prod_{i=1}^{\infty} \frac{1}{1-x^i}}{\prod_{i=1}^{\infty} \frac{1}{1-x^{3i}}} = \prod_{i=1}^{\infty} \frac{\frac{1}{1-x^i}}{\frac{1}{1-x^{3i}}} = \prod_{i=1}^{\infty} \frac{1-x^{3i}}{1-x^i}$$

Algabraic proof:

Proof.

$$\prod_{i=1}^{\infty}(1 + x^i + x^{2i}) = \prod_{i=1}^{\infty} \frac{(1 + x^i + x^{2i})(1 - x^i)}{1 - x^i} = \prod_{i=1}^{\infty} \frac{1 + x^i + x^{2i} - x^i - x^{2i} - x^{3i}}{1 - x^i} = \prod_{i=1}^{\infty} \frac{1 - x^{3i}}{1 - x^i}$$

■

Question 45

Determine the number of permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ in which no even integer is in its natural position. For example, 43258176 is good, but 42358176 is bad because of the position of the 2.

Solution: This is an application of inclusion-exclusion principle.

Let S be the set of all permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Clearly, $|S| = 8!$.

Let A_i be the subset of permutations in S where i is in its natural position. We are looking to calculate the following by the complementary form of the inclusion-exclusion principle:

$$|A_2^c \cap A_4^c \cap A_6^c \cap A_8^c| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_2 \cap A_4 \cap A_6 \cap A_8|$$

Each sum can be calculated as follows:

$$\sum |A_{i_1} \cap A_{i_2} \cdots A_{i_k}| = \binom{4}{k}(8-k)!$$

This is because there are 4 ways to choose the even numbers that are fixed. For each of these choices, there are $8 - k$ ways to choose the remaining numbers.

So, we have:

$$8! - \binom{4}{1}7! + \binom{4}{2}6! - \binom{4}{3}5! + \binom{4}{4}4!$$

Question 46

Define what it means for a pair (S, ω) means to be combinatorial problem. Define generating series in these terms. State and prove the product lemma. (As usual, on the test I might ask a **very precise** question about these proofs (or any proof) so you must be able to understand each line and symbol used in the argument.)

10.3.2 Solutions

10.3.3 Exam

10.3.4 Exam Solutions