MATH 413
Introduction to Combinatorics

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Contents

Chapter 1

What is Combinatorics?

Chapter 2

Permutations and Combinations

2.1 Lecture 2: Four Basic Counting Principles

2.2 Lecture 3: Permutations and selections of sets I

2.3 Lecture 4: Permutations and selections of sets II: binomial identities

2.4 Lecture 5: Permutations and Combinations of multisets I

2.5 Lecture 6: Permutations and Combinations of multisets II

Chapter 3

The Pigeonhole Principle

3.1 Lecture 7: The pigeonhole principle

3.2 Lecture 8: The strong pigeonhole principle

3.3 Lecture 9: Ramsey Theory

Chapter 5

The Binomial Coefficients

5.1 Lecture 10: Binomial coefficients and the binomial theorem I

5.2 Lecture 11: Binomial coefficients and the binomial theorem II

5.3 Lecture 12: Binomial coefficients and the binomial theorem III

Chapter 6

The Inclusion-Exclusion Principle and Applications

6.1 Lecture 13: The Inclusion-Exclusion principle and applications I

6.2 Lecture 14: The Inclusion-Exclusion principle and applications II: Derangements

6.3 Lecture 15: The Inclusion-Exclusion principle and applications II

6.4 Lecture 16: The Inclusion-Exclusion principle and applications IV: Another Forbidden Position Problem

Chapter 7

Recurrence Relations and Generating Functions

7.1 Lecture 17: Some Number Sequences

Example 7.1.1 (Example 1)

Consider a configuration of n lines where every two lines have a point in common, but no three do. How many regions in the plane are there? Give a recurrence.

$$a_n = a_{n-1} + n$$

TODO: Give an explanation of why this is true.

Example 7.1.2 (Example 2)

Give a simple recurrence for dearragements.

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

TODO: Give an explanation of why this is true. Need to review dearragements from previous lecture.

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$.

Definition 7.1.1: The adjusted Fibonacci sequence: $\hat{F}_n$

This is the number of 1,2 lists of size n. In other words, consider the number of ways a valet can park A cars (size 1) and B cars (size 2) in a parking lot of size n.

$$\hat{F}_n = \begin{cases} 1 & \text{if } n = 0 \\ f_{n+1} & \text{otherwise} \end{cases}$$

Question 1

Prove
$$\sum_{n=0}^{n} f_i = f_{n+2} - 1$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.
$$\sum_{i=0}^{0} f_i = f_0 = 0$$

Similarly,
$$f_{0+2} - 1 = f_2 - 1 = 1 - 1 = 0$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.
$$\sum_{i=0}^{k} f_i = f_{k+2} - 1$$

Indcutive Step: We will prove that the following statement is true for $n = k + 1$.
$$\sum_{i=0}^{k+1} f_i = f_{k+1} + \sum_{i=0}^{k} f_i = f_{k+1} + f_{k+2} - 1 = f_{k+3} - 1$$

Therefore, the statement is true for all n by induction. ■

Question 2

Prove
$$1 + \sum_{i=0}^{n} \hat{F}_i = \hat{F}_{n+2}$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.
$$1 + \sum_{i=0}^{0} \hat{F}_i = 1 + \hat{F}_0 = 1 + 1 = 2$$

Similarly,
$$\hat{F}_{0+2} = \hat{F}_2 = f_{2+1} = f_3 = 2$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.
$$1 + \sum_{i=0}^{k} \hat{F}_i = \hat{F}_{k+2}$$

Inductive Step: We will prove that the following statement is true for $n = k + 1$.

$$
\begin{aligned}
1 + \sum_{i=0}^{k+1} \hat{F}_i &= 1 + \hat{F}_{k+1} + \sum_{i=0}^{k} \hat{F}_i \\
&= \hat{F}_{k+1} + \hat{F}_{k+2} \\
&= f_{k+2} + f_{k+3} \\
&= f_{k+4} \\
&= \hat{F}_{k+3}
\end{aligned}
$$

Therefore, the statement is true for all n by induction. ■

Question 3

Prove that f_n is even if and only if n is divisible by 3.

Solution:

Proof. Given that $f_0 = 0$, $f_1 = 1$, and $f_2 = 1$, we can see that at $n = 3$, $f_3 = 2$, which is even.

This is because the only way to get an even number is to have the parity of the two numbers added togethed (odd + odd or even + even) be the same. So, f_4, must be odd, f_5 must be odd and f_6 must be even.

Given the starting sequence of even, odd, odd. The following sequence must always be even, odd, odd, which repeats every 3 numbers.
Since the first $n = 0$ is the first number in the sequence, every n that is divisible by 3 is even. ■

Note:-

Example problems for later

Guess and prove by induction (you may replace the Fibonnaci number by the adjusted Fibonacci number if it helps you)

- $f_1 + f_3 + \cdots + f_{2n-1} = ?$
- $f_0 + f_2 + \cdots + f_{2n} = ?$
- $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n = ?$
- $(f_0)^2 + (f_1)^2 + \cdots + (f_n)^2 = ?$

Obtaining an explicit formula for f_n for linear recurrences

Example 7.1.3

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$. This can be rewritten as a linear recurrence as follows:

$$f_n - f_{n-1} - f_{n-2} = 0$$

We must solve the corresponding characterstic equation. Notice how the largest degree lines up with the "largest" case of the recurrence.

$$x^2 - x - 1 = 0$$

Let q_1 and q_2 be the roots of the characteristic equation.

It is potentially relevant to note that the following is a solution space of the Fibonacci recurrence (but don't satisfy f_0 – the initial condition):

$$\begin{cases} q_1^n - q_1^{n-1} - q_1^{n-2} = 0 \\ q_2^n - q_2^{n-1} - q_2^{n-2} = 0 \end{cases}$$

The rest of this is based on an ansatz, i.e. we need to make an assumption at the answer and validate it later

$$f_n = c_1 q_1^n + c_2 q_2^n,$$

for *some* $c_1, c_2 \in \mathbb{R}$.

Using the initial conditions of $f_0 = 0$ and $f_1 = 1$, we can solve for c_1 and c_2.

7.2 Lecture 18: Introduction to ordinary generating series

Note:-

Up until now, we have solved an instance combinatorial problem at a time. However, in some cases, solving a single combinatorial is too difficult. So, we solve all of the combinatorial problems at once.

Consider a sequence

$$h_0, h_1, h_2, \dots, h_t, \dots$$

of natural numbers where h_t is the answer to some counting problem that depends on t.

We can create a generating series of the form:

$$g(x) = h_0 + h_1 x + h_2 x^2 + \dots + h_t x^t + \dots$$

where $h_t = [x^t]g(x)$.

Note:-

The notation $[x^t]g(x)$ is the coefficient of x^t in the polynomial $g(x)$.

Claim 7.2.1 Compositions Generating Series

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Example 7.2.1

Fix k. Let

h_t = number of nonnegative integral solutions to

$$e_1 + \dots + e_k = t, e_i \in \mathbb{Z}_{\geq 0}$$

We already know from stars and bars that

$$h_t = \binom{t+k-1}{k-1}$$

So obviously, the generating series can be written as

$$g(x) = \sum_{t=0}^{\infty} \binom{t+k-1}{k-1} x^t$$

Note:-

This doesn't really tell us anything. We just combined some definitions and have a genearting series that is hard to utilize in any useful way.

Take our claim and expand it out with Taylor Series k times:

$$\begin{aligned} g(x) &= \left(\frac{1}{1-x}\right)^k \\ &= (1+x+x^2+\cdots)(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots \end{aligned}$$

We can see that the coefficient of x^t is the number of ways to write t as a sum of k nonnegative integers. So,

$$h_t = \binom{t+k-1}{k-1}$$

This combinatorially proves the negative binomial theorem:

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Remark

If you knew that the negative binomial theorem was true, you could use it to algebraically solve the composition problem after choosing the correct generating series $g(x) = \left(\frac{1}{1-x}\right)^k$.

Question 4

What is

$$(1+x+x^2+x^3+x^4+x^5)(x+x^2)(1+x+x^2+x^3+x^4)$$

the generating series for?

Solution: The solutions to the equation

$$e_1 + e_2 + e_3 = t$$

where $0 \le e_1 \le 5, 1 \le e_2 \le 2, 0 \le e_3 \le 4$.

Question 5

Suppose you have 1,5 and 25 cents coins available (infinite supply). Write down the generating series for the number of ways to make change for a dollar. How many ways are there?

Solution:

The generating series for the number of ways to make change for an arbitrary amount of money is:

$$g(x) = (1+x+x^{1+1}+x^{1+1+1}+\cdots)(1+x^5+x^{5+5}+x^{5+5+5}+\cdots)(1+x^{25}+x^{25+25}+x^{25+25+25}+\cdots)$$

This can be simplified as follows:

$$
\begin{aligned}
g(x) &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{25}}\right) \\
&= \frac{1}{(1-x)(1-x^5)(1-x^{25})}
\end{aligned}
$$

The answers is $[x^{100}]g(x)$.

Question 6: Nonstandard Dice

Compute the generating series for the result of rolling two standard dice.
Now factor it, and interpret.

Solution:

$$
\begin{aligned}
g(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6) \\
&= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \\
&= (1 + x)(1 + x)(x + x^3 + x^5)
\end{aligned}
$$

TODO: How does this factor like this?
After factoring, we can see that this is the equivalent of flipping a coin twice and rolling two 1-3-5 dice.

Question 7

Determine the generating series for partitions.

Solution:

$$
\begin{aligned}
g(x) &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^2 + x^{2+2} + x^{2+2+2} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots)\cdots \\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i}\right)
\end{aligned}
$$

In words, this generating series is the product of compositions of just 1's, 2's, 3's, etc.

Definition 7.2.1: Ordinary Generating Series

Let a combinatorial problem be defined as (S, ω) with S a set and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ The ordinary generating series is

$$
g(x) = g_{(S,\omega)}(x) = \sum_{s \in S} x^{\omega(s)}
$$

Theorem 7.2.1 Addition Rule of Generating Series

Suppose $S = A \cup B$ (disjoint union), where (A, ω_A) and (B, ω_B) are combinatorial problems. Moreover $\omega|_A = \omega_A$ and $\omega|_B = \omega_B$.
Then the ordinary generating series for (S, ω) is

$$
\begin{aligned}
g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) + g_{(B,\omega_B)}(x) \\
&= \sum_{s\in A} x^{\omega_A(s)} + \sum_{s\in B} x^{\omega_B(s)} \\
&= \sum_{s\in S} x^{\omega(s)}
\end{aligned}
$$

Note:-

The notation $\omega|_A$ means the restriction of ω to A. In this case,

$$\omega|_A = \{\omega_A(s) \mid s \in A\}$$

Theorem 7.2.2 Product rule of Generating Series

Suppose $S = A \times B$ (cartesian product), where (A, ω_A) and (B, ω_B) are combinatorial problems and

$$\omega(a, b) = \omega_A(a) + \omega_B(b)$$

Then,

$$
\begin{aligned}
g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) g_{(B,\omega_B)}(x) \\
&= \sum_{a\in A} x^{\omega_A(a)} \sum_{b\in B} x^{\omega_B(b)} \\
&= \sum_{a\in A} \sum_{b\in B} x^{\omega(a,b)}
\end{aligned}
$$

What does it mean for two generating series to be equal?

They are equal coefficient by coefficient.

Let $g(x)$ be the ggenerating series for the number of partitions. What does it mean that $g(x)$ equals the infinite product:

$$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

It means that if we expand/solve enough terms of the RHS, the coefficient of x^t will stabalize (as in no more terms multiplied in will impact the coefficient) and this coefficient will be the same as the coefficient of x^t in the LHS – $g(x)$.

Convergence Issues

Typically, with generating series, we only care about the coefficients and not plugging in any specfic value into x. Because of this, we do not need to worry about convergence. However, in some cases $g(x)$ is a polynomial and in these cases, substitution is fine to perform.

Question 8: Substituting into a generating series

Let $S = \{\text{coins in your pokcet}\}$ and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ be the obvious weight function on coins, i.e. $\omega(\text{nicket}) = 5$. Is $g(x)$ is the corresponding generating series, what is $g(1)$? What is $g'(1)$?

Solution: $g(1)$ will be the number of coins in your pocket. This is trivially true because each term is x to the power of an integer (with no coefficient), so 1 raised to anything will be 1 and so it is just the number of terms. More mathematically,

$$\begin{aligned} g(1) &= \sum_{s \in S} 1^{\omega(s)} \\ &= \sum_{s \in S} 1 \\ &= |S| \end{aligned}$$

$g'(1)$ will the amount of money you have. Based on the previous statement, we can see that the derivative of $g(x)$ is just the sum of the powers which represented the values of the coins in your pocket. More mathematically,

$$g'(1) = \sum_{s \in S} \omega(s)$$

Chapter 8

Special Counting Sequences

8.1 Lecture 19: Partition identities

8.2 Lecture 20: Partition identities (continued)

8.3 Lecture 21: Exponential generating series