MATH 413 **Name:** *Amit Sawhney*
Homework 8
11/7/2022

1. Question 2

   Following the inclusion exlclusion principle, we can start by calculating the number of ways to choose 2 double letters and then rearrange the remaining 9 symbols. This is computed as

   $$\binom{5}{2}\frac{9!}{2!2!2!}$$

   However, this computed the case where there are three double letters three times, once for each pair. We must recompute the count to subtract the overcounted cases. So,

   $$\binom{5}{2}\frac{9!}{2!2!2!} - 2\binom{5}{3}\frac{8!}{2!2!}$$

   Similarly, we have now subtracted out too many cases where there are 4 double letters. So we must add those back in.

   $$\binom{5}{2}\frac{9!}{2!2!2!} - 2\binom{5}{3}\frac{8!}{2!2!} + 3\binom{5}{4}\frac{7!}{2!}$$

   Lastly, we have overcounted once more, so following this pattern,

   $$\binom{5}{2}\frac{9!}{2!2!2!} - 2\binom{5}{3}\frac{8!}{2!2!} + 3\binom{5}{4}\frac{7!}{2!} - 4\binom{5}{5}6! = 286920$$

2. Question 32

   Consider an arbitrary n. If you divide by any prime number p, this is the number of positive integers $\leq n$ that are a multiple of p. (e.g. $\frac{12}{2} = 6$ and $2, 4, 6, 8, 10, 12$). Consequently, these numbers cannot be coprime with n as they are a multiple of p. Imagine we want to calculate all of the numbers that are a multiple of two primes p_1 or p_2. Then we can compute $\frac{n}{p_1} + \frac{n}{p_2}$. However, this double counts the numbers that are a multiple of both p_1 and p_2. So to count the number of multiples of p_1 or p_2 this is $\frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{p_1 \cdot p_2}$

   We can expand upon this idea to apply inclusion-exclusion principle to all of the primes up to n.

$$\phi(n) = n - \sum_{p_i \text{ prime } p_i | n} \frac{n}{p_i} + \sum_{p_i, p_j \text{ prime } p_i, p_j | n} \frac{n}{p_i \cdot p_j} - \sum_{p_i, p_j, p_k \text{ prime } p_i, p_j, p_k | n} \frac{n}{p_i, p_j, p_k} + \cdots + (-1)^{|P|} \frac{n}{p_1 \cdot p_2 \cdots p}$$

   For clarification, each summation is over the set of primes that divide n and $|P|$ represents the cardinality of set P where P is the set of primes that are dividing n.

   Simplifying, this expression, we see

$$\phi(n) = n \Bigg(1 - \sum_{p_i \text{ prime } p_i | n} \frac{1}{p_i} + \sum_{p_i, p_j \text{ prime } p_i, p_j | n} \frac{1}{p_i \cdot p_j} - \sum_{p_i, p_j, p_k \text{ prime } p_i, p_j, p_k | n} \frac{1}{p_i, p_j, p_k} + \cdots + (-1)^{|P|} \frac{1}{p_1 \cdot p_2 \cdots p} \Bigg)$$

   Consider the following identity,

$$\prod_{i=1}^{n} (1 - x_i) = 1 - \sum_{i=1}^{n} x_i + \sum_{i,j=1}^{n} x_i \cdot x_j - \sum_{i,j,k=1}^{n} x_i \cdot x_j \cdot x_k + \cdots + (-1)^n x_1 \cdot x_2 \cdots x_n$$
$$= \sum_{I \subset 1,2,\ldots,n} (-1)^{|I|} \prod_{i \in I} x_i$$

   By applying this to the simplified inclusion exclusion principle above, we see that,

$$\phi(n) = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

   Citation: https://math.stackexchange.com/questions/1734597/inclusion-exclusion-principle-challenging-problem

3. Question 4

   At a base case, $f_n = a_n$ for $n = \{0, 1, 2, 3, 4\}$ trivially by the definition of this problem. For $n \geq 5$,

$$f_n = f_{n-1} + f_{n-2} = f_{n-2} + f_{n-3} + f_{n-2} = 2f_{n-2} + f_{n-3}$$
$$f_n = 2 \cdot f_{n-2} + f_{n-3} = 2 \cdot f_{n-2} + f_{n-4} + f_{n-5} = 2 \cdot (f_{n-3} + f_{n-4}) + f_{n-4} + f_{n-5}$$
$$f_n = 2f_{n-3} + 3f_{n-4} + f_{n-5} = 2 \cdot (f_{n-4} + f_{n-5}) + 3f_{n-4} + f_{n-5} = 5f_{n-4} + 3f_{n-5}$$

   So clearly, the Fibonacci sequence is a solution to the recurrence relation a_n.

   To prove the next part, consider $r \in \{0, 1, 2, 3, 4\}$ such that r is a remainder of an arbitrary n divided by 5. We know that because $5 \cdot f_{n-4}$ is trivially divisible by 5, then there must be f_r that is divisible by 5. So,

$$5|f_n \iff 5|f_r \iff r = 0 \iff 5|n$$

   Citation for second part: https://people.math.wisc.edu/ duhlenbr/475-1072Exam2Answers.pdf

4. Question 9

$$h_n = \begin{cases} 1, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ h_{n-1} + h_{n-1} + 2h_{n-2}, & \text{otherwise} \end{cases} \tag{1}$$

   This is because if $n = 0$, there is only 1 possible coloring, don't color. If $n = 1$, there are 3 possible color choices because their is only 1 square. Otherwise, at the current square, if you choose blue or white, there are h_{n-1} possibilities. If you choose red, there are $2 \cdot h_{n-2}$ possibilities because the previous square only has 2 choices and then everything before that can be anything.

5. Question 11

(a) 11a

Proof. We will show the following by induction on n.
Base case: At $n = 1$, $\ell_1 = 1$ and $f_0 + f_2 = 0 + 1 = 1$. So, the base case holds.
Inductive Hypothesis: Assume $\ell_n = f_{n-1} + f_{n+1}, n = \{1, 2, \ldots, k-1\}$.
Inductive Step WTS $\ell_k = f_{k-1} + f_{k+1}$.
By the definition of Lucas numbers,

$$\ell_k = \ell_{k-1} + \ell_{k-2}$$

By the inductive hypothesis,

$$\ell_k = f_{k-2} + f_k + f_{k-3} + f_{k-1}$$

Note that by the definition of Fibonacci numbers,

$$f_{k-1} = f_{k-2} + f_{k-3}$$

$$f_{k+1} = f_k + f_{k-1}$$

Simplifying,

$$\ell_k = f_{k-2} + f_k + f_{k-3} + f_{k-1} = f_{k-1} + f_k + f_{k-1} = f_{k-1} + f_{k+1}$$

So, $\ell_n = f_{n-1} + f_{n+1}$ by induction. □

(b) 11b

Proof. We will prove the following by induction on n.
Base case: At $n = 0$, $(\ell_0)^2 = (2)^2 = 4$. Similarly, $\ell_0\ell_1 + 2 = (2)(1) + 2 = 4$. So, the base case holds.
Inductive Hypothesis: Assume $(\ell_0)^2 + (\ell_1)^2 + \cdots + (\ell_n)^2 = (\ell_n)(\ell_{n+1}) + 2$ for $n = \{0, 1, \ldots, k-1\}$. In other words,

$$\sum_{i=0}^{n} (\ell_i)^2 = (\ell_n)(\ell_{n+1}) + 2$$

for $n = \{0, 1, \ldots, k-1\}$
Inductive Step: We want to show the following:

$$\sum_{i=0}^{k} (\ell_i)^2 = (\ell_k)(\ell_{k+1}) + 2$$

$$\sum_{i=0}^{k} (\ell_i)^2 = (\ell_k)^2 + \sum_{i=0}^{k-1} (\ell_i)^2 = (\ell_k)^2 + (\ell_{k-1})(\ell_k) + 2 = (\ell_k)(\ell_k + \ell_{k-1}) + 2$$

Notice that, $\ell_{k+1} = \ell_k + \ell_{k-1}$.
So,

$$(\ell_k)(\ell_k + \ell_{k-1}) + 2 = (\ell_k)(\ell_{k+1}) + 2$$

.
Hence, $(\ell_0)^2 + (\ell_1)^2 + \cdots + (\ell_n)^2 = (\ell_n)(\ell_{n+1}) + 2$ by induction. $\square$

6. Question 40

   At the base case $n = 0$, clearly there is only 1 string possible and that is the string with no characters.

   At the next base case $n = 1$, there is only 1 character allowed and 3 choices, so there are 3 possible strings.

   At a given position i, there two cases:

   (a) Case 1: You can add a 2 to a string of length $n - 1$. Hence, a_{n-1}

   (b) Case 2: You can add a 20 or 21 to the string of length $n - 2$. Note that 22 is redundant with the previous case. Hence, $2a_{n-2}$.

   So,

   $$a_n = a_{n-1} + 2a_{n-2}$$

   The characteristic equation for this recurrence relation is,

   $$x^2 = x + 2 \iff x^2 - x - 2 = 0$$

   So the roots o this equation are clearly, $x = 2, -1$

   $a_n = a \cdot 2^n + b(-1)^n$

   To solve this system o equations, we plug in the values at $n = 0$ and $n = 2$.

   So,

   $$1 = a + b$$

   $$3 = 2a - b$$

   Solving this system, we get $a = \frac{4}{3}$ and $b = -\frac{1}{3}$

   So the answer is:

   $$a_n = \frac{4}{3}2^n - \frac{1}{3}(-1)^n$$

   Citation: https://discrete.openmathbooks.org/dmoi2/sec_recurrence.html

7. Question A

   Generation 0: 1 Canadian

   Generation 1: 1 American

   Generation 2: 1 Canadian and 1 American

   Generation 3: 2 American and 1 Canadian

   Generation 4: 3 American and 2 Canadian

   Clearly, this follows a Fibonacci sequence. The n-th generation will have f_n people where f_n is the n-th Fibonacci number. The reasoning for this is because each person in generation $n-1$ contributes 1 person to generation n. Additionally, in generation $n-2$ each Canadian generates 1 person for n and each American ALSO generates a person for n. So this explains the recurrence following Fibonacci.

   Citation: Eric Liu