MATH 413
Introduction to Combinatorics

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Contents

Chapter 2

Permutations and Combinations

2.1 L2: Four Basic Counting Principles

Definition 2.1.1: Addition Principle

If S_1, S_2, ..., S_k are disjoint sets, then

$$\left| S = \bigcup_{i=1}^{k} S_i \right| = |S_1| + |S_2| + \cdots + |S_k|$$

Definition 2.1.2: Multiplication Principle

If $S = A \times B$ then $|S| = |A| \cdot |B|$.

Example 2.1.1

How many ways are there to match $2n$ people?

There are clearly $(2n)!$ ways to arrange $2n$ poeople. Let the first two people be on a team, the next two people be on a team, and so on. We can swap the people in each team to generate an equivalent arragement of people. Given that there are n pairs, there are 2^n ways to pick which pairs are being swapped. Moreso, there are n pairs so $n!$ ways to arrange the pairs. Thus, the number of pairs is $\frac{(2n)!}{2^n \cdot n!}$.

Question 1

How many ways are there to form a three-letter sequence using the letters a,b,c,d,e,f?

- with repetition of letters allowed?
- without repetition of any letter?
- without repetition and containing the letter e?
- with repetition and containing the letter e?

Solution:

- with repetition of letters allowed

  Trivially, there are 6 choices at each of the three positions in the sequence, so there are $6^3 = 216$ ways to form the sequence.

- without repetition of any letter

  There are 6 choices for the first letter, 5 choices for the second letter, and 4 choices for the third letter. Thus, there are $6 \cdot 5 \cdot 4 = 120$ ways to form the sequence.

- without repetition and containing the letter e

  There are three posibilities. The first letteer is e, the second letter is e, or the third letter is e. Each of the arragements within each of these three cases are disjoint. So, we can use the Addition Principle. In each case there are 5 choices for the first free position and 4 choices for the second free position. Thus, there are $3 \cdot 5 \cdot 4 = 60$ ways to form the sequence.

- with repetition and containing the letter e

  There are three cases: there is 1 e, 2 e's, or 3 e's

  In case 1, there are $\binom{3}{1}$ ways to pick the position of the 1 e. Then, there are 5 choices for the first free position and 5 choices for the second free position. Thus, there are $\binom{3}{1} \cdot 5 \cdot 5 = 75$ the sequence.

  In case 2, there are $\binom{3}{2}$ ways to pick the position of the 2 e's. Then, there are 5 choices for the remaining free position. Thus, there are $\binom{3}{2} \cdot 5 = 15$ ways to form the sequence.

  In total, there are $75 + 15 + 1 = 91$ ways to form this sequence.

Question 2

A rumor is spread randomly among a group of 10 people by successively choosing one specified person (who will start the rumor) to call someone, who calls someone etc. A person can pass a rumor to anyone except the person who just called and him/herself.

- How many different paths can a rumor travel through the group in three calls? n calls?
- What is the probability that if A starts the rumor, A received the third call?

Solution:

- First question

  There are 10 ways to pick the first person that starts the rumor. The first person can call anyone except himself/herself. So, there are 9 choices. Each the of the next people can call anyone except the person who just called and himself/herself. So, there are 8 choices. Thus, there are $10 \cdot 9 \cdot 8 = 72$ ways This logic applies to the third group of calls. Thus, there are $10 \cdot 9 \cdot 8 \cdot 8 = 5760$ ways to travel through the group in three calls.

  In n calls, there are $10 \cdot 9 \cdot 8 \cdot \dots \cdot 8 = 10 \cdot 9 \cdot 8^{n-1}$ ways to travel through the group.

- Second question

  By our previous logic, there are $9 \cdot 8^2$ ways to travel through the group in 3 calls starting from A. (Note: we fixed the starting person to be A). The number of paths that start at A, travel through an intermediate person B for the first call, travel through an intermediate person C for the second call, and travel through A for the third call is $9 \cdot 8$ as there are 9 choices for the first call, 8 choices for the second call, and 1 choice for the last call. So the probability is $\frac{9 \cdot 8}{9 \cdot 8 \cdot 8} = \frac{1}{8}$.

Definition 2.1.3: Subtraction Principle

If A is contained in U and A^c is the complement then

$$|A^c| = |U| - |A|$$

Question 3

Until recently, area codes were created with the following rules:

1. The first digit cannot be a 0 or 1
2. The secon digit must be a 0 or 1

In 1995 this was abandoned when 360 was used in parts of western Washington state (0 still can't be the first number).

Solution: This is an application of subtraction principle. There are $8 \cdot 2 \cdot 10$ total area codes before the new rule. There are $9 \cdot 10 \cdot 10$ total area codes after the new rule. Thus, the number of area codes that were created is:

$$9 \cdot 10 \cdot 10 - 8 \cdot 2 \cdot 10 = 740$$

2.2 L3: Permutations and selections of sets I

Permutations of sets

Let $S = \{a, b, c\}$. There are $3 \times 2 \times 1 = 3!$ ways to rearrange this set. This can also be thought of as creating a bijection from an n-set to another n-set.

In this class, we think of permutations is to think of them in the context of the set $\{1, 2, \ldots, n\}$.

Example 2.2.1
Let $S = \{\text{A deck of cards}\}$. Then a permutation of S is a shuffling of the deck.

Definition 2.2.1: r-permutation

We can think of this as shuffling a subset $r < n = 52$ of all cards.

Theorem 2.2.1
Let $P(n, r)$ be the number of r-permutations of an n-set. Clearly, by the multiplication principle, we get that if $r \leq n$:

$$P(n, r) = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Clearly, $P(n, n) = n!$ (i.e. when $r = n$).

Question 4

Consider the Movable 15 puzzle problem:

4	15	8	13
5	11		7
1	12	3	10
14	2	6	9

The goal is to order the numbers from 1 to 15 by utilizing the empty space to move the numbers around. How many starting puzzles are possible?

Solution: There are 16! ways to arrange 16 symbols to start $(1, 2, \ldots 15)$ and the empty square.

Example 2.2.2

Consider the TV show episode where James Randi asks an auro reader to guess the order of 5 people behind a curtain without seeing them and simply be reading their auro.

- Why is it less than a 1% chance that the "auro reader" gets it right?
- (Harder) Why is the expectation 1. (Try the case of three people first.)

Solution:

- Probability

  There are 5! ways to arrange the 5 people. So the probability is $\frac{1}{5!} = \frac{1}{120} < 1\%$.

- Expectation

  We need dearragements. The expectation is

$$\sum_{i=0}^{5} i \times \text{ Prob(exactly i correct)}$$

Example 2.2.3 (Circular Permutations)

Suppose n children are arranged in a circle. How many arrangemens are there?

There are $n!$ ways to arrange the children in this circle. However n of the circles generated will just be rotations of each other. So the number of distinct arrangements is $\frac{n!}{n} = (n-1)!$.

Similarly, we can derive the following theorem:

Theorem 2.2.2

The number of circular permutations of a set of n elements is

$$\frac{P(n,r)}{r}$$

Question 5

Ten people, including two who don't want to sit next to one another are seated at a round table. How many arrangements are possible?

Solution: Let A and B be the people that do not want to sit by each other. Construct an algorithm that places A and B at the table and then places the remaining 8 people.

There are 10 choices for A. Because B cannot sit next to A, there are only 7 choices for B. There are 8! ways to arrange the remaining 8 people at the table. However, 10 of the arrangements of this algorithm will be rotations of each other. So the number of ways to arrange the 10 people such that A and B don't sit next to each other is

$$\frac{10 \cdot 7 \cdot 8!}{10}$$

Solution: Another way to approach this problem is to determine the number of possible arragements with no restrictions and take out the bad arragements. Clearly, there are $\frac{10!}{10}$ arragements of 10 people at a round table. To determine the number of bad permutations, join person A and B to be represented under one symbol. There are 9 symbols to seat at the table now. By the same principle as above, there are $\frac{9!}{9} = 8!$ ways to construct this aragement. However, within the joined symbol there could be AB or BA. So the total number of bad aragements is $2 \cdot 8!$. So the total number of aragements where A and B are not sitting next to each other is $9! - 2 \cdot 8!$.

Definition 2.2.2

Define

$$C(n,r) := \binom{n}{k} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

to be the binomial coeicient. This is the number o ways o choosing r elements in an n element set.

Question 6

If a 5-card hand is chosen at random, what is the probability of obtaining a flush (all cards are the same suit?) How about a full house? (Three cards of the same kind, and two of another kind, e.g., three queens and two "4"'s)

Solution: For a flush, there are $\binom{4}{1}$ ways to choose the suit and $\binom{13}{5}$ ways to choose 5 cards of that suit. There are $\binom{52}{5}$ possible hands of 5 cards. So the probability of getting a flush is

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

For a full house, there are $\binom{13}{1}$ ways to choose the three of a kind card and $\binom{4}{3}$ ways to pick the suits from the cards. Similarly, there are $\binom{12}{1}$ remaining ways to pick the two of a kind card and $\binom{4}{2}$ ways to pick the suits. So, the totoal probability is

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

Question 7

How many starting setups are there in Chess 960? In this game, the back row can be rearranged in any way before the game starts as long as it abides by the following two rules:

1. The bishops are placed on opposite colors.
2. The king is between the two rooks.

Solution: There are two ways to solve this question.

1. First way

   We will construct an algorith that places the bishops down, then places the rooks and king together, and then places the remaining pieces.

   There are $\binom{4}{1}$ ways to place one bishop and $\binom{4}{1}$ ways to place the other. Thus, $\binom{4}{1}\binom{4}{1} = 4 \cdot 4 = 16$ ways to place the bishops. We can then imagine the board with 6 remaining places. We then must place the rooks and king together. There are 4 cases where the king can be on any ".":

   - R.R
     There are 4 ways to place the rooks this distance apart. There is only 1 way to place the king in this arrangement. So there are 4 possible arrangements of this form.
   - R..R
     There are 3 ways to place the rooks this distance apart. Thee are 2 ways to place the king in between these rooks. So there are $3 \cdot 2 = 6$ possible arrangements of this form.

- R...R
  There are 2 ways to place the rooks this distance apart. There are 3 ways to place the king in between these rooks. So there are $2 \cdot 3 = 6$ possible arrangements of this form.
- R....R
  There is 1 way to place the rooks this distance apart. There are 4 ways to place the king in between these rooks. So there are $1 \cdot 4 = 4$ possible arrangements of this form.

At this point we have placed 5 of the 8 pieces. So there are $\frac{3!}{2!}$ ways to place the next three pieces, given that their are two knights which are identitcal.

All in all, there are $16 \cdot (4 + 6 + 6 + 4) \cdot 3 = 960$. Hence, why it is called Chess 960.

2. Second way,

   We can start by repeating the start of the previous algorithm. So, we place the bishops down in 16 ways. Then we place the queen. There are 6 ways to do this. Then we place the knights which is possible in $5 choose 2$ ways. At this point, the position of the king and two rooks are forced. So, there are $16 \cdot 6 \cdot \binom{5}{2} = 960$ ways to arrange the pieces.

Note:-

A really important part of this chapter to take away is the process of constructing an algorithm to calculate the number of ways to arrange a set of objects.

2.3 L4: Permutations and selections of sets II: binomial identities

Binomial Identities

These are incredibly useful for combinatorics. Typically, if you are going to try to prove binomial identities algabraically (i.e. directly from the definition), they can be quite difficult, but thinking about them combinatorically can make them easier to understand.

Theorem 2.3.1

For $0 \leq r \leq n$, we have

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof. (Algabraic) Obviously,

$$\begin{aligned}\binom{n}{r} &= \frac{n!}{r!(n-r)!} \\ &= \frac{n!}{(n-r)!r!} \\ &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= \binom{n}{n-r}\end{aligned}$$

However, this only works for simple binomial identities. ■

Proof. (Combinatoric) The LHS counts the number of ways to choose r people from n people. This equivalent to counting which of the n people to exclude from our group. There are $n - r$ people to exclude. So, there are $\binom{n}{n-r}$. Clearly, this is the RHS. So the LHS and the RHS count the same set and thus are equivalent. ■

Theorem 2.3.2 Pascal's Formula

For all integers, n and k with $1 \leq k \leq n - 1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof. Let T be a set of n elements such that $T = \{x_1, x_2, \ldots, x_n\}$. Suppose S is a subset of T with k elements. There are two cases:

- $x_1 \in S$. Then, there are $\binom{n-1}{k-1}$ ways to choose the remaining elements in S.
- $s_1 \notin S$. Then, there are $\binom{n-1}{k}$ ways to to choose the elements in S.

By the addition principle there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ ways to choose k elements from T. So, the equality holds as the LHS and RHS count the same set (i.e. the set of all subsets of T with k elements). ■

Combinatorial Models

Definition 2.3.1: Lattice Paths

Think of $\binom{a+b}{a}$ as the number of paths from $(0,0)$ to (a,b)

Example 2.3.1

Prove that

$$\sum_{j=0}^{b} \binom{a+j-1}{a-1} = \binom{a+b}{a}$$

Proof. Let S be the set of all lattice paths from $(0,0)$ to (a,b) using only right (east) and up (north) moves. Clearly, the RHS counts the the number of paths in S. Now decompose each path in S into the paths at the last time that the path touches the verticle line $x = a-1$. There are $b+1$ places that a path can touch the line $x = a-1$. After the last time it touches this line, the remaining path to (a,b) is fixed (directly up). Each of these elements in the resulting decomposition is disjoint. So, we can apply the addition principle. Suppose j is the y-coordinate that the path touches the line $x = a-1$. Then, there are $\binom{a+j-1}{a-1}$ ways to reach this point. So, there are $\sum_{j=0}^{b} \binom{a+j-1}{a-1}$ ways to count all of the paths that last touch the line $x = a-1$ which is the LHS. So, the LHS and RHS count the same set and thus are equivalent. ■

Question 8

Prove the same identity as above, expressed diferently:

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

Solution:

Proof. Let S be the set of all sequences with $n+1$ elements containing only 0 and 1 and $r+1$ 1's. Clearly, the RHS counts the number of sequences by picking which of the $n+1$ positions will be 1. We can decompose S by removing the last instance of 1 from each sequence. This 1 could be at position $0, 1, \ldots, n+1$. Let k be the position before the last 1. Then, there are $\binom{k}{r}$ ways to choose the remaining 1's in the sequence. Given this, k can range from 0 to n as it cannot be greater than n as that would imply there are 0 1's in the sequence. So, by the addition principle there are

$$\sum_{k=0}^{n} \binom{k}{r}$$

total ways to count this decomposition. So, the LHS and RHS count the same set and thus are equivalent. ■

Question 9

Prove

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}$$

Solution:

Proof. Let S be the set of all lattice paths from $(0,0)$ to (n,n). The RHS clealy counts this set. Consider the line $y = n - x$ and the point $(j, n-j)$. Decompose S into the paths that touch this line at an arbitrary point $(j, n-j)$. Clearly, the number of lattice paths to this point is $\binom{j+(n-j)}{j} = \binom{n}{j}$. Additionally, the number of paths from this point to (n,n) is $\binom{n-j+j}{n-j} = \binom{n}{n-j} = \binom{n}{j}$. This is because we can rewrite the destination point as $(n-j, n-(n-j)) = (n-j, j)$ and think of the starting point as the origin. By the addition principle,

$$\sum_{j=0}^{n} \binom{n}{j}^2$$

So, the LHS and RHS count the same set and thus are equivalent. ■

Question 10

Prove that $\binom{a+b}{a}$ equals the number of partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_b$ satisfying $\lambda_1 \leq a$.

Solution: It may help to read a later chapter to fully understand what a partition is. Essentially, these lattice paths must trace the edge of a Ferrer Diagram. Each lattice path trivially has a bijection with a Ferrer Diagram. The Ferrer Diagrams generated can only have b parts as the lattice path ends at $y = b$. Similarly, the largest part (λ_1) must be $\leq a$ because the lattice path ends at $x = a$.

Because a bijection exists between lattice paths and Ferrer Diagrams with these conditions, the claim must be true.

Committees approach

Another way to view binomial identities is to think of them as two different ways of picking a committee under some constraints.

Question 11

Prove

$$\binom{n}{m}\binom{m}{k} = \binom{n}{m}\binom{n-m}{k-m}$$

Solution:

Proof. Let S be the set of all commitees formed from n people with size k with subcomittees of size m. For the LHS. There are clearly $\binom{n}{k}$ ways to choose the committee members and $\binom{k}{m}$ ways to choose the subcommitee. For the RHS. We can choose the m members of the subcommitee first. After, there are $n - m$ people left to choose from and $k - m$ spots on the entire commitee. So, there are $\binom{n-m}{k-m}$ ways to choose the rest of the commitee. Clearly, the LHS and RHS count the same set and thus are equivalent. ■

Question 12

Prove

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

Solution: Let S be the set of all committees of size r that we can form with m men and n women. Clearly, there are $\binom{m+n}{r}$ ways to choose the commitees. Similarly, we can construct an algorithm that k members from the men and then the remainder from the women. This is represented by $\binom{m}{k}\binom{n}{r-k}$. The number of k men on the team can range from 0 to r. So, by the addition principle,

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

So, the LHS and RHS count the same set and thus are equivalent.

2.4 L5: Permutations and Combinations of multisets I

Multisets: A multiset is a set that allows for repeated elements.

This leads us to a common question regarding permutations: How many permutations are there in a multiset.

Theorem 2.4.1

The number of "words" (meaning permutations) one can generate out of k letters (elements in the multiset) which appears $n_1, n_2, \cdots, n_k$ times is

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

where $n = n_1 + n_2 + \cdots + n_k$.

Proof. Imagine we distinguish the letters the repeated letters in a multiset. For example,

$$\{I_1, L_1, L_2, I_2, N_1, O_1, S_1\}$$

There are $n! = 8!$ ways to arrange the letters. Ignoring the subscripts, we can see how many times we overcount. For example, there are 3 Is. Which means we overcount by a factor of 3!. Similarly, there are 2 Ls, 1 O, 1 N, and 1 S. So, the number of permutations is

$$\frac{8!}{3! \cdot 2! \cdot 1! \cdot 1! \cdot 1!}$$

■

Question 13

Consider the $5! = 120$ permutations of the letters a, b, c, d, e. How do you determine the $40th$ one (in alphabetical order) quickly?

Solution: We know that the first permutation starts with a. Additionally, with 4 remaining letters to permute, there are $4! = 24$ permutations that start with a. So, permutations starting with a are in between 1 and 24. Permutations with b are between 25 and 48. So, we know that the 40th permutation starts with b. Similarly, permutations starting with ba are in the range of 25 to 30, bc are in the range of 31 to 36, bd are in the range of 37 to 42. So, the 40th permutation starts with bd. Contuining on bda are in the range of 37 to 38 and bdc are in the range of 39 to 40. So, the 40th permutation is $bdcea$.

Basic Combination Question

How many ways are there to select an _r-combination_ (size r multi-subset) of a multiset?

Example

For the I, L, L, I, N, O, I, S example, if $r = 3$, possible 3-combinations are:

$$\{I, L, I\}, \{I, I, I\}, \{N, O, S\}, \dots$$

Theorem 2.4.2

The number of r-combinations of k distinct objects, each with unlimited supply is $C(k + r - 1, r)$.

Proof. Let $x_1, x_2, \cdots, x_k$ be the number of times one uses object $A_1, A_2, \cdots, A_k$. So, we know:

$$x_1 + x_2 + \cdots + x_k = r$$

and each $x_i \geq 0$ is a nonnegative integer. This is counted by $C(k + r - 1, r)$. (Keep reading, we need to prove this) ∎

Proof that compositions are counted by $C(k + r - 1, r)$

We need to prove that the number of

- Compositions of r into nonnegative integers
- Number of ways of selecting r things from k objects with repetition

is counted by $C(k + r - 1, r)$.

Proof. Based on our previous problem, we know that each of these is counted by the same thing. Consider the following construction,

Lay out r objects in a row. Then, in order to to form k groups, we can place down $k - 1$ dividers. So, we have $k - 1$ choices of where to place the dividers and $r + k - 1$ positions in which they can be placed. So, the number of ways to place the dividers is

$$\begin{aligned} C(r + k - 1, k - 1) &= \binom{r + k - 1}{k - 1} \\ &= \binom{r + k - 1}{r + k - 1 - (k - 1)} \\ &= \binom{r + k - 1}{r} \end{aligned}$$

∎

Note:-

This proof is known as the Stars and Bars proof.

Question 14

What is the number of integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 20$$

where $x_1 \geq 3$, $x_2 \geq 1$, $x_3 \geq 0$, and $x_4 \geq 5$.

Solution: We can rewrite this problem by redefining the bounds to ensure each $x_i \geq 0$. Our new problem is

$$x_1 + x_2 + x_3 + x_4 = 11$$

Trivially, the solution to this is $\binom{11+4-1}{11}$.

2.5 L6: Permutations and Combinations of multisets II

Question 15

How many ways are there to select six hot dogs if there are three varieties of hot dogs?

Solution: This is the equivalent problem of the number of integral solutions to

$$x_1 + x_2 + x_3 = 6$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.
So, the solution is $\binom{6+3-1}{6} = \binom{8}{6}$.

Question 16

How many ways are there to fill a box with a dozen doughnuts chosen from five varieties with the requirement that at least one doughnut of each kind is picked?

Solution: This is the equivalent problem of the number of integral solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12$$

where $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$, $x_4 \geq 1$, and $x_5 \geq 1$.
The bounds can be rewritten to ensure that each $x_i \geq 0$. So, the new problem is

$$x_1 + x_2 + x_3 + x_4 + x_5 = 7$$

And thus, the answer is $\binom{7+5-1}{7} = \binom{11}{7}$.

Question 17

If there are 10 options of donuts, and one is buying 48 donuts, what is the expected range of options that do not get taken?

Solution: This solution is very long. So I will only show part of it.

To calculate this, we can compute the likelihood that k options are not taken where $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

- $k = 0$: The probability that no options are not taken is calculated by the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 48$$

where each $x_i \geq 1$.
There are $\binom{38+10-1}{38}$ solutions to this equation. There are $\binom{10}{0}$ ways to pick the 0 elements that are not taken. So, the probability is

$$\frac{\binom{38+10-1}{38} \cdot \binom{10}{0}}{\binom{48+10-1}{48}}$$

- Continue this exact template by taking a way k x_i's and computing the number of integral solutions.

Chapter 3

The Pigeonhole Principle

3.1 L7: The pigeonhole principle

Definition 3.1.1

The *Pigeonhole Principle*: If $n+1$ objects are distributed into n boxes then at least one box contains two or more objects.

Question 18

There are n married couples. How many of the $2n$ people must be selected to guarantee that a married couple is selected?

Solution: Imagine each box is a married couple. There are n boxes. So, by the pigeonhole principle, if we select $n+1$ people there must be at least one box with both married counterparts in it.

Question 19

Show that i $n+1$ integers are chosen from $\{1, 2, \ldots, 2n\}$ then there are always two which differ by 1.

Solution: Create a bucket for each pair of consecutive integers (i.e. (1,2), (3,4), etc) There are n buckets. Each time we select an integer, put it in the corresponding bucket. By the pigeonhole principle, after we select $n+1$ integers, two of the buckets must have at least two integers. Since each bucket contains two consecutive integers, there must be two integers that differ by 1.

Question 20

A chess master has 11 weeks to prepare for a tournament plays at least one game per day, but not more than 12 games during any calendar week. Prove there is a succession of consecutive days where the master plays exactly 21 games.

Solution: Let a_i be the number of cummulative hours the chess master plays on day i. Clearly,

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_{77} \leq 132$$

Extending upon this, we see that:

$$22 \leq a_1 + 21 \leq a_2 + 21 \leq \cdots \leq a_{77} + 21 \leq 153$$

We need to show that there exists some a_i such that $a_i = a_j + 21$ for some $j \leq i$.

There are 154 possible numbers from a_1 to a_{77} and $a_1 + 22$ to $a_{77} + 21$. However, there are only 153 values these numbers can be assigned to. Therefore, there must be two numbers that are the same by the pigeonhole principle.

Note that these numbers cannot appear in the same sequence (i.e. a_1 to a_{77} or $a_1 + 22$ to $a_{77} + 21$). So, there must be a number a_i such that $a_i = a_j + 21$ for some $j \leq i$.

Question 21

A student views TikTok at least one hour each day for 7 weeks but not more than 11 hours in any one week. Prove there is some period of consecutive days where the student watches exactly 20 hours of TikTok. (Assume a whole number of hours of TikTok watched each day.)

Solution: Let a_i be the cummulative hours that a student watchings TikTok on day i. Clearly,

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_{49} \leq 77$$

Extending upon this, we see that:

$$21 \leq a_1 + 20 \leq a_2 + 20 \leq \cdots \leq a_{49} + 20 \leq 97$$

We need to show that there exists some a_i such that $a_i = a_j + 20$ for some $j \leq i$.

There are 98 possible numbers from a_1 to a_{49} and $a_1 + 21$ to $a_{49} + 20$. However, there are only 97 values these numbers can be assigned to. Therefore, there must be two numbers that are the same by the pigeonhole principle.

Question 22

From the integers $1, 2, \cdots, 200$ we choose 101 integers. Show that two of the chosen integers have the property that one divides the other.

Solution: Write each of the in the form $2^s \cdot a$ where a is an odd integer. This is a unique expansion of each integer. So, a must be in the set$\{1, 3, 5, \cdots, 199\}$. So, a has 100 possible values. After selecting 101 integers, two of the integres must share the same a. Let these be $2^s \cdot a$ and $2^t \cdot a$. Then, $2^s \cdot a = 2^t \cdot a$. Assume $s \leq t$. Without loss of generality, the first number divides the second.

Question 23

Suppose you are given any nine 3-dimensional lattice points (nine points in 3-space with integer coordinates). Prove that between some two points, the open line segment connecting these points must again pass through some other lattice point.

Philosophy of the Pigeonhole Principle

By nature, the pigeonhole principle creates structure in randomness. In other words, if there is a long enough sequence of random numbers, there must be a random structure. The $n + 1$ of the pigeonhole principle, is where the structure transitions from solid to fluid, a freezing point of sorts. More philsophically, out of chaos comes order.

Question 24

In the past thousand years, YOU had ancestors A and P such that P was an ancestor to both the father and mother of A. (That is, no-one's family tree is really a tree.) [You can make some assumptions about the total population of the world etc.]

Solution: Assume that it takes 25 years for a generation to create a new generation. This means there have been 40 generations. So, the number of nodes in the tree is

$$\sum_{i=0}^{40} 2^i = 2^{41} - 1$$

Currently, there are less than 10^{10} people on Earth and there were less people in previous generations. Thus, we know that an upper bound for the number of people who have lived on Earth in the past 40 generations is $40 \cdot 10^{10}$.

However, we know that $40 \cdot 10^{10} < 2^{41} - 1$. So, by the pigeonhole principle, two of the nodes in the ancestral must be the same and thus, you are inbred.

3.2 L8: The strong pigeonhole principle

Definition 3.2.1: Strong Pigeonhole Principle

Let $q_1, q_2, \cdots, q_n$ be positive integers. If

$$q_1 + q_2 + \cdots + q_n - n + 1$$

obects are distributed into n boxes, the either box 1 contains q_1 objects, or box 2 contains q_2 objects, or $\cdots$, or box n contains q_n objects.

Proof. Suppose not. That is suppose that no box contains q_i objects. Then there are at most

$$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n$$

objects in the n boxes. However this is a contradiction because we distributed $q_1 + q_2 + \cdots + q_n - n + 1$ objects into n boxes. ∎

Special cases

There are two special cases of the strong pigeonhole principle.

1. If $q_i = 2$ for each i, this implies the weak pigeonhole principle.

   For clarity, let $q_i = 2$ for each i. Then, the strong pigeonhole principle says that if

   $$q_1 + q_2 + \cdots + q_n - n + 1 = 2n - n + 1 = n + 1$$

   objects are distributed into n boxes, then either box 1 contains 2 objects, or box 2 contains 2 objects, or $\ldots$, or box n contains 2 objects. In other words, one of the boxes contains 2 objects.

2. Suppose $q_i = r$ for each i. This implies that if you distribute $rn - n + 1 = n(r - 1) + 1$ objects into n boxes, then one of the boxes contains at least r objects.

   A small reformation of this case is that if the average of $a_1, a_2, \cdots, a_n > r - 1$ then at least one $a_i \geq r$. This is pretty easy to see.

   Proof. Suppose not. That is suppose that no $a_i \geq r$. Then each $a_i \leq r - 1$. So, at most the average of $a_1, a_2, \cdots, a_n$ is $r - 1$. However, this is a contradiction because the average is greater than $r - 1$. ∎

Question 25

Grades A,B,C,D,F are to be given in a class. What's the smallest size of a class to ensure that there is at least 5 A's, 5 B's, 4 C's, 2 D's OR 1 F?

Solution: This is an application of the strong pigeonhole principle. Let $q_1 = 5$, $q_2 = 5$, $q_3 = 4$, $q_4 = 2$, $q_5 = 1$. In order to guarantee at least q_i objects in each box i, by the strong pigeonhole principle to show we need at least $q_1 + q_2 + q_3 + q_4 + q_5 - 5 + 1$. This is $5 + 5 + 4 + 2 + 1 - 5 + 1 = 13$. So, we need at least 13 students.

Question 26

Show that any sequence of $n^2 + 1$ real numbers $a_1, a_2, \dots, a_{n^2+1}$ has an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

Solution:

Proof. Without loss of generality, assume that $a_1, a_2, \dots, a_{n^2+1}$ does not have an increasing subsequence of length $n + 1$. Let L_k be the longest increasing subsequence starting at a_k. Clearly, $1 \le L_k \le n$ as there is no increasing subsequence of length $n + 1$.

Consider the sequence, $L_1, L_2, \dots, L_{n^2+1}$. Note that $n^2 + 1 = n((n + 1) - 1) + 1$. By the strong pigeonhole principle, where $r = n + 1$, this implies that $n + 1$ of the L_k's are equal which can be denoted by $L_{k_1} = L_{k_2} = \cdots = L_{k_{n+1}}$. From this, we can construct a sequence:

$$a_{k_1} \ge a_{k_2} \ge \cdots \ge a_{k_{n+1}}$$

which is a decreasing subsequence of length $n + 1$.

It may not be clear that this is a decreasing subsequence. Suppose not. That is, suppose that $a_{k_i} < a_{k_{i+1}}$ for some i. Then, $L_{k_i} > L_{k_{i+1}}$ because we can construct a longer increasing subsequence than $L_{k_{i+1}}$ by adding a_{k_i} to the start of the sequence that starts at $a_{k_{i+1}}$. However, this is a contradiction because $L_{k_i} = L_{k_{i+1}}$. So, this must be a decreasing subsequence. ■

3.3 L9: Ramsey Theory

Example 3.3.1

Consider a party with six people. For any two of these people, either they've met before (acquaintances) or not (strangers). Prove that there is either three people who are mutual acquaintances or three people who are mutual strangers.

Proof. Draw a graph with six vertices with each vertex representing a person blue. Draw a red edge between two vertices if they are acquaintances and a blue edge if they are strangers. We want to show that there is red triangle or a blue triangle.

Consider a vertex v. There are 5 edges incoming to v. We need to distrubte 2 colors across 5 edges. By the strong pigeonhole principle where $r = 3$, $2(3 - 1) + 1 = 5$ which implies that there are either 3 red edges or 3 blue edges. Without loss of generality, assume that there are 3 blue edges connecting to vertices x, y, z. If any of $(x, y), (y, z), (z, x)$ are blue, then we have a blue triangle. (e.g. if (x, y) is blue then our triangle is $(x, y), (y, v), (x, v)$). If none of them are red, then they form a red triangle among themselves, and so in either case, there is a red triangle or blue triangle. ■

Definition 3.3.1: Ramsey Numbers

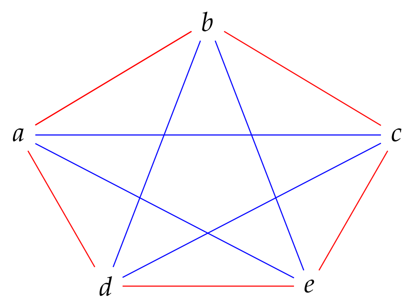
Let $R(r,s)$ be the least integer such that a complete graph with $R(r,s)$ vertices (i.e. one that has all possible edges) and is colored with red and blue, MUST either have a red complete subgraph on r vertices or a blue complete subgraph on s vertices.

Question 27

Show that $R(3,3) \neq 5$.

Solution: We have already shown $R(3,3) = 6$. In order to show that $R(3,3) \neq 5$, we need to show that there is a complete graph with 5 vertices that does not have a red complete subgraph on 3 vertices or a blue complete subgraph on 3 vertices.

Consider the following graph:



Clearly, this does not have a red complete subgraph on 3 vertices nor a blue complete subgraph on 3 vertices.

Note:-

No one knows what $R(5,5)$ is. It is expected that no one will ever know what $R(6,6)$ is.

Theorem 3.3.1

$R(r,s)$ always exists. In fact,

$$R(r,s) \leq R(r-1,s) + R(r,s-1)$$

Proof. This is done by double induction on $r,s \geq 1$.

Base Case

$$R(r,1) = R(1,s) = 1$$

This is clearly true, because a complete graph with 1 vertex is gaurunteed to have a subgraph on 1 vertex that is blue or red. The color chosen is irrelevant because there are no edges so the color is implicit.

Trivially the inequality (i.e. the base case holds true) holds because:

$$R(r,1) \leq R(r-1,1) + R(r,1-1) \iff 1 \leq 1+0 \iff 1 \leq 1$$
$$R(1,s) \leq R(1-1,s) + R(1,s-1) \iff 1 \leq 0+1 \iff 1 \leq 1$$

Inductive Step

Suppose that $R(r-1,s)$ and $R(r,s-1)$ exist and consider the complete graph with $R(r-1,s)+R(r,s-1)$ vertices. Pick a vertex v. There are $R(r-1,s)+R(r,s-1)-1$ edges incident to v. Split the adjacent vertices w into two classes, RED and BLUE, depending on if (v,w) is a red or blue edge. Clearly,

$$R(r-1,s) + R(r,s-1) = |RED| + |BLUE| + 1$$

The LHS represnts the total number of vertices in the original graph we constructed. The RHS is the decomposition of this graph based on an arbitrary vertex v based on the adjacent edge colors. Because the graph is complete, RED and BLUE will contain every vertex in the original graph except the arbitrary vertex v which is why we add 1. Thus, I claim that one of the following must be true:

$$|RED| \geq R(r-1,s)$$
$$|BLUE| \geq R(r,s-1)$$

Suppose not. That is suppose that both $|RED| < R(r-1,s)$ and $|BLUE| < R(r,s-1)$. Then, $|RED|+1 \leq R(r-1,s)$ and $|BLUE|+1 \leq R(r,s-1)$

$$|RED| + |BLUE| + 2 \leq R(r-1,s) + R(r,s-1)$$
$$|RED| + |BLUE| + 1 < R(r-1,s) + R(r,s-1)$$

But this is a contradiction as $|RED|+|BLUE|+1 = R(r-1,s)+R(r,s-1)$. So either $|RED| \geq R(r-1,s)$ or $|BLUE| \geq R(r,s-1)$

Without loss of generality, suppose that $|RED| = R(r-1,s)$. Then, the vertices in RED, must have a red K_{r-1} or a blue K_s. If the former is true, then this K_{r-1} with v forms a K_r within the original graph (as each vertex in RED was connected to v with a red edge). Otherwise, the latter is true and thus the original graph contains a K_s. In either case, the original graph contains a K_r or a K_s. Thus, not only does $R(r,s)$ exist but the inequality holds true as we either need one less vertices than what is in $R(r-1,s)+R(r,s-1)$ or we need to include the same number of vertices as $R(r-1,s)+R(r,s-1)$.

$$R(r,s) \leq R(r-1,s) + R(r,s-1)$$

■

Chapter 5

The Binomial Coefficients

5.1 L10: Binomial coefficients and the binomial theorem I

Recall the following two facts:

- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Note:-

One can think o Pascal's formula (the second fact) as the recursive definition the compute binomial coefficients with the following initial conditions:

$$\binom{n}{0} = \binom{n}{n} = 1$$

Properties of the Binomial Coefficients

- The row sums give powers of 2
- The symmetry of the binomial coefficient.
- The number $\binom{n}{k}$ is the number of paths SW-NE "/" and NW-SE "\\" steps from $\binom{0}{0}$ to (n, k).

Proof. Proof of third property:

The only way to get to $\binom{n}{k}$ is to pass through $\binom{n-1}{k}$ or $\binom{n-1}{k-1}$. From here, we can apply induction and pascal's formula to show that this is true. ■

Question 28

Write Pascal's triangle as follows:

$$\begin{array}{llllll}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}$$

Now consider the sums along SW to NE: $1, 1, 1+1, 1+2, 1+3+1, 1+4+3, \ldots$ We've seen these numbers beore. What are they? (Can you give a proof?)

Solution: Clearly this sequence is $1, 1, 2, 3, 5, 8, \ldots$. This is the Fibonacci sequence.

The Fibonacci sequence is defined as:

$$a_n = a_{n-1} + a_{n-2}$$

While this can be solved with recursion we can craft a combinatorial model to represent this situation. Consider the n-the Fibonacci number and a strip of length n. Imagine we are going to fill this strip with Rabbits (size 1) and Cadillacs (size 2). At a point n, we have two choices,

1. Place a Rabbit and lose one space
2. Place a caddilac and lose two spaces

Clearly, this is defined by the recurrence above. We can use a Rabbit and thus there are a_{n-1} ways to fill the strip or we can use a Cadillac and thus there are a_{n-2} ways to fill the strip.

Definition 5.1.1: The Binomial Theorem

Let $n \in \mathbb{Z}_{>0}$. Then,

$$\begin{aligned}(x+y)^n &= x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\end{aligned}$$

Proof. Consider the expression $(x+y)^n$ exanded such that:

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots$$

Consider the process of FOIL to exapnd this expression. The term $x^i y^j$ The coefficient of this term is the number of ways to choose i xs and j ys where $i + j = n$ which is:

$$\begin{aligned}\binom{i+j}{i} &= \binom{n}{i} \\ &= \binom{n}{n-i} \\ &= \binom{i+j}{(i+j)-i} \\ &= \binom{i+j}{j} \\ &= \binom{n}{j}\end{aligned}$$

■

Binomial Identities

- $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

  Proof. Set $x = y = 1$ and apply binomial theorem. ■

- $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$

  Proof. Set $x = 1$ and $y = -1$. Apply Binomial Theorem. ∎

- $\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$

  Proof. Use the previous property and move the negative coefficients to the RHS. ∎

- $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1} = \binom{n}{1} + \binom{n}{3} + \cdots$

  Proof. Using the first property, we know the total sum of the coefficients to be 2^n. In each of these equations, we have removed half of the terms so the new sum is $\frac{2^n}{2} = 2^{n-1}$. ∎

Question 29

Evaluate:

$$\binom{n}{0} - 2\binom{n}{1} + 3\binom{n}{2} + \cdots + (-1)^n(n+1)\binom{n}{n}$$

Solution:

Proof. Consider a simplified binomial theorem where $y = 1$:

$$(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

By taking the derivative of both sides:

$$n(x+1)^{n-1} = \sum_{i=1}^{n} i\binom{n}{i} x^{i-1}$$

Let $x = -1$.

$$\begin{aligned} 0 &= \sum_{i=1}^{n} i\binom{n}{i}(-1)^{i-1} \\ (-1)\cdot 0 &= (-1)\sum_{i=1}^{n} i\binom{n}{i}(-1)^{i-1} \\ 0 &= \sum_{i=1}^{n} i\binom{n}{i}(-1)^{i} \end{aligned}$$

Lastly, we can add this with property 2 above to obtain the desired result. ∎

Question 30

Prove that the sequence of numbers in each row of Pascal's triangle is a power of 11. I.e., $\{1, 2, 1\} \rightarrow 121 = 112$. For this you need to "carry over" numbers bigger than 9 to the left. So for example $\{1, 5, 10, 10, 5, 1\}$ is $161051 = 115$.

Solution: Consider the binomial expansion of $(1 + x)^n$ and let $x = 10$:

$$
\begin{aligned}
(x+1)^n &= \sum_{i=0}^{n} \binom{n}{i} x^i \\
11^n &= \sum_{i=0}^{n} \binom{n}{i} 10^i \\
11^n &= \binom{n}{0} + \binom{n}{1}10 + \binom{n}{2}100 + \cdots + \binom{n}{n}10^n
\end{aligned}
$$

Question 31

Give a combinatorial proof that the number of ways to select even sized subsets equals the number of ways to select odd sized subsets (equals 2^{n-1}).

Solution: We can encode a an even sized subset by recording which numbers from 1 to $n-1$ are being used. If this encoded subset is odd, then it can be inferred that n must have been used (since this is a encoding of an even sized subset). If this encoded subset is even, then it can be inferred that n was not used. Because of this, the number of even subsets is simply the number of subsets that we can create using elements of the set $\{1, 2, \ldots, n-1\}$ which is 2^{n-1}. The same reasoning and connclusion applies to the odd sized subsets.

5.2 L11: Binomial coefficients and the binomial theorem II

Note:-

Note that the rows of Pascal's triangle increase to some maximum and then decrease. This is called *unimodality*. Observe the following sequences:

- $1, 2, 5, 7, 5, 2, 1$ is unimodal and symmetric
- $1, 6, 8, 10, 9, 3, 2$ is unimodal but not symmetric
- $1, 5, 3, 8, 3, 5, 1$ is symmetric but not unimodal
- $43, 2, 1, 33, 2$ is just nothing

Theorem 5.2.1 Unimodality of Binomial Coefficients

Binomial Coefficients are unimodal.

Intuition/Philosophy

Consider a unimodal sequence of natural numbers

$$a_1 \leq a_2 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$$

This sequence should lead you to some conclusions:

- Let $a_1 = \#S_1$, $a_2 = \#S_2$, …, $a_n = \#S_n$. So this original sequence counts the size of some sets:

$$S_1, S_2, \ldots, S_m, \ldots, S_n$$

  Our main motivation for this representation is because of what follows:

- Recall that if there exists an injection from a set $A \to B$, then $A \leq B$. Using this logic, it is clear that there are injective maps:

$$S_1 \hookrightarrow S_2, S_2 \hookrightarrow S_3, \ldots, S_{m-1} \hookrightarrow S_m, \ldots, S_{n-1} \hookrightarrow S_n$$

- Similarly, recall that if there is a surjection from a set $A \longrightarrow B$, then $A \geq B$. Using this logic, it is clear that there are surjective maps:

$$S_m \twoheadrightarrow S_{m+1}, S_{m+1} \twoheadrightarrow S_{m+2}, \ldots, S_{n-1} \twoheadrightarrow S_n$$

Thus, to clarify some syntax:

$$|A| \leq |B| \iff A \hookrightarrow B$$
$$|A| \geq |B| \iff A \twoheadrightarrow B$$

Proof of Binomial Coefficient Unimodality

Proof. Consider the fact that $\binom{n}{k}$ counts the lattice paths from $(0,0)$ to $v = (k, n-k)$. using "N" and "E" steps. Similarly, $\binom{n}{k+1}$ (the next binomial coefficient in the sequence/pascal's triangle) counts the lattice paths from $(0,0)$ to $w = (k+1, n-k-1)$. Assume that $k < \frac{n}{2}$. We can draw a perpendicular bisector between $\overline{vw}$. At this point, note that this bisector must intersect the lattice path from $(0,0)$ to v at some point. This is because this lattice path must pass through the points $(k, n-k-1)$, $(k-1, n-k-2)$, $\ldots$, to some point on the y-axis below $n-k$. This is because the perpendicular bisector must have a slope of -1. Thus, the lattice path must intersect with one of these points to get to $k, n-k$. For every lattice path from $(0,0)$ to v, we can construct a new lattice path from $(0,0)$ to w by applying a reflexive property, reflex the lattice path across the perpendicular bisector for every step after the last time the lattice path touches the perpendicular bisector. Lastly, this reflected path must reach w. Consider the last point a that the lattice path touches the bisector. At a, the lattice path must take 2 steps up for every step to the right in order to reach v. Similarly, at this point the lattice path would need to take 2 steps to the right for every step up in order to reach w Thus, when reflected at this point, the path that was going to reach v must reach w. Hence, there exists a injection between the binomial coefficients while $k < \frac{n}{2}$. This function must be injective because given two paths that reach w, we can simply reflect the paths across the perpendicular bisector from the last point of intersection to the end of the path and reveal that the paths are the same lattice path to v. So, for each k where $k < \frac{n}{2}$: We trivially have that $\binom{n}{k} \neq \binom{n}{k+1}$ because we can consider lattice paths that go directly to w. There is no lattice path that goes directly to v that can be a pre-image of this path.

$$\binom{n}{k} < \binom{n}{k+1}$$

Similarly, take $k > \frac{n}{2}$. We can construct the same situtation, except prove that there is an injection from point w to v which would imply that:

$$\binom{n}{k+1} < \binom{n}{k} \iff \binom{n}{k} > \binom{n}{k+1}$$

Hence, we have proven that the binomial coefficients are unimodal. ■

Corollary 5.2.1

The largest binomial coefficient among

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$$

is $\binom{n}{\frac{n}{2}}$ if n is even and $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$ if n is odd.

5.3 L12: Binomial coefficients and the binomial theorem III

Applied and Abstract Context

Applied

Suppose that you're a designer of houses that you plan to put in a newly constructed neighbourhood. Each house is basically the same, but can be adorned with some of the following n upgrades:

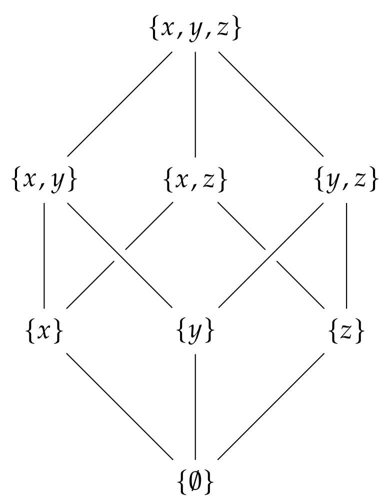
- Hard wood floors
- Skylight
- Granite Countertops
- Stainless stell appliances
- ...

In order to be unique, you want to give each house a different set of options, BUT, no house A has upgrades that are a subset of another house B's upgrades (otherwise the owner of house A would be majorized by house B). By doing this, no two houses are comparable (one house may have hardwood floors, and other a sun roof).

What is the maximum number of houses you can design under these conditions?

Abstract

Consider the following poset (partially ordered set) for $S = \{x, y, z\}$:



Definition 5.3.1: Anti-chains

Let S be an arbitrary set. This is a collection of subsets that no subset is a subset of another subset.

Consider a set $S = \{a, b, c, d, e\}$. Then:

- $\Big\{\{a, b\}, \{b, c, d\}, \{a, d\}, \{a, c\}\Big\}$ is an anti-chain.
- $\Big\{\{a, b\}, \{a, b, c\}, \{c, d\}\Big\}$ is not.

What is the largest size of any antichain on n elements? Note that this is the same question we posed before in the Applied section above. The answer to this question is given by Sperner's Theorem.

Definition 5.3.2: Chains

A *chain* is a collection C of subsets of $\{1, 2, \ldots, n\}$ such that if $A, B \in C$, the either:

- $A \subseteq B$ or
- $B \subseteq A$

A chain is considered *maximal* if the chain contains one subset of each possible size.

Question 32

How many maximal chains of $\{1, 2, \ldots, n\}$ are there?

Solution: By the properties of constructing a chain, we can think of building a chain one element at a time. There are n choices for the first element in the the chain. Let the first choice be called a. Then the start of our chain is $\{a\}$. Continuing on there are $n-1$ choices for the next part of the chain. Call this b. Clearly, our chain will be $\Big\{\{a\}, \{a, b\}, \ldots\Big\}$. Following this, there are cleary $n!$ possible chains.

Theorem 5.3.1 Sperner's Theorem

Let S be a set of n elements. Then the maximum size of an antichain is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$

Proof. Fix an antichain $\mathcal{A}$. Let S be a set defined by

$$S = \{(a, C) : a \in \mathcal{A}, \text{ where } C \text{ is a maximal chain with } a \in C\}$$

Note that $a \in C$ and $a \in \mathcal{A}$. Every maximal chain can only intersect with an antichain in one place. Suppose not. That is suppose that a maximal chain C and antichain intersect in more than one place. Without loss of generality assume that the chain and anti-chain intersect at two subsets, T_1, T_2. By the definition of a chain, we have that either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$. Assume $T_1 \subseteq T_2$. Then, this is a contradiction. Since T_1 and T_2 are elements of the antichain, they cannot be subsets of each other. So, chains and anti-chains can only intersect *once*. Because of this it is clear that the number of maximal chains is an upperbound on the size of set S. Thus,

$$\#S \leq n!$$

Fix $a \in \mathcal{A}$ of size k. Exactly $k!(n-k)!$ maximal chains must contain a. This is because in a maximal chain, it must contain subsets of each size so there must be subsets of sizes $1, 2, \ldots, k-1$. Each of these subsets must be a subset of a. Thus, with k elements, there are $k!$ ways to build a maximal chain up to a. Similarly, there are $n-k$ elements left to build the remaining maximal chain.

Define a_k such that a_k is the number of subsets in $\mathcal{A}$ that are of size k.

$$a_k = \#\{a \in \mathcal{A} : \#a = k\}$$

It follows that:

$$\#A = \sum_{k \geq 0} a_k$$

This sum counts the number of subsets in $\mathcal{A}$ that are of size k for all sizes k so it must be the number of elements in the antichain. At this point, it can be recognized that:

$$\#S = \sum_{k \geq 0} a_k k!(n-k)!$$

This is because for an arbitrary k, there are a_k subsets of size k in $\mathcal{A}$ and each of these subsets can be paired with $k!(n-k)!$ maximal chains. Thus, the total number of elements in S (i.e. the total number of elments in the antichain that are in any number of maximal chains) is the sum over all k. Thus,

$$\begin{aligned}
\#S = \sum_{k\geq 0} a_k k!(n-k)! &\leq n! \\
\iff \frac{1}{n!}\sum_{k\geq 0} a_k k!(n-k)! &\leq \frac{1}{n!}n! \\
\iff \sum_{k\geq 0} a_k \frac{k!(n-k)!}{n!} &\leq 1 \\
\iff \sum_{k\geq 0} a_k \frac{1}{\binom{n}{k}} &\leq 1 \\
\iff \binom{n}{\lfloor \frac{n}{n} \rfloor}\sum_{k\geq 0} a_k \frac{1}{\binom{n}{k}} &\leq \binom{n}{\lfloor \frac{n}{n} \rfloor} \\
\iff \sum_{k\geq 0} a_k &\leq \binom{n}{\lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

Thus,

$$\#\mathcal{A} = \sum_{k\geq 0} a_k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

So $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is an upperbound on the size of an antichain. ■

Chapter 6

The Inclusion-Exclusion Principle and Applications

6.1 L13: The Inclusion-Exclusion principle and applications I

Question 33

How big is $A \cup B \cup C$?

Solution:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

If we just sum up the sizes of the sets, we are counting the elements in common between each of the sets twice. However, after we subtract out the number of elements in each intersection, we are excluding elements in commonb between all three sets.

This is the basis of the inclusion-exclusion principle.

Question 34

How many elements are *not* in $A, B,$ or C?

Solution: Let S be the universal set.

$$\begin{aligned} |A^c \cap B^c \cap C^c| &= |S| - |A \cup B \cup C| \\ &= |S| - \Big(|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\Big) \end{aligned}$$

Example 6.1.1

Find the number of integers between 1 and 1000 inclusive that are not divisible by 5 and not divisible by 6 and not divisible by 8.

Let

- A_1 be the subset of integers divisible by 5,
- A_2 be the subset of integers divisible by 6,
- A_3 be the subset of integers divisible by 8.

Then we want to find $|A_1^c \cap A_2^c \cap A_3^c|$.

$$\begin{aligned} |A_1^c \cap A_2^c \cap A_3^c| &= |S| - |A_1 \cup A_2 \cup A_3| \\ &= |S| - \Big(|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|\Big) \end{aligned}$$

Obviously, $|S| = 1000$. To obtain the number of integers between a and b, where $b \geqslant a$, we can use the formula $\lfloor \frac{b}{a} \rfloor$.

$$\begin{aligned} |A_1| &= \lfloor \frac{1000}{5} \rfloor = 200 \\ |A_2| &= \lfloor \frac{1000}{6} \rfloor = 166 \\ |A_3| &= \lfloor \frac{1000}{8} \rfloor = 125 \end{aligned}$$

Similarly,

$$\begin{aligned} |A_1 \cap A_2| &= \lfloor \frac{1000}{\operatorname{lcm} 5, 6} \rfloor = \lfloor \frac{1000}{30} \rfloor = 33 \\ |A_1 \cap A_3| &= \lfloor \frac{1000}{\operatorname{lcm} 5, 8} \rfloor = \lfloor \frac{1000}{40} \rfloor = 25 \\ |A_2 \cap A_3| &= \lfloor \frac{1000}{\operatorname{lcm} 6, 8} \rfloor = \lfloor \frac{1000}{24} \rfloor = 41 \\ |A_1 \cap A_2 \cap A_3| &= \lfloor \frac{1000}{\operatorname{lcm} 5, 6, 8} \rfloor = \lfloor \frac{1000}{120} \rfloor = 8 \end{aligned}$$

Thus,

$$\begin{aligned} 1000 - \Big(200 + 166 + 125 - 33 - 25 - 41 + 8\Big) &= 1000 - 200 - 166 - 125 + 33 + 25 + 41 - 8 \\ &= 600 \end{aligned}$$

Question 35

How many permutations of M,A,T,H,I,S,F,U,N are there where MATH, IS and FUN do not appear as consecutive letters?

Solution: Let

- A_1 be the set of permutations where MATH appears as consecutive letters,
- A_2 be the set of permutations where IS appears as consecutive letters,
- A_3 be the set of permutations where FUN appears as consecutive letters.

Then we want to find $|A_1^c \cap A_2^c \cap A_3^c|$.

$$
\begin{aligned}
|A_1^c \cap A_2^c \cap A_3^c| &= |S| - |A_1 \cup A_2 \cup A_3| \\
&= |S| - \Big(|A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|\Big)
\end{aligned}
$$

Obviously, $|S| = 9!$.

$$
\begin{aligned}
|A_1| &= 6! \\
|A_2| &= 8! \\
|A_3| &= 7!
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|A_1 \cap A_2| &= 5! \\
|A_1 \cap A_3| &= 4! \\
|A_2 \cap A_3| &= 6! \\
|A_1 \cap A_2 \cap A_3| &= 3!
\end{aligned}
$$

Thus,

$$
9! - \Big(6! + 8! + 7! - 5! - 4! - 6! + 3!\Big) = 9! - 6! - 8! - 7! + 5! + 4! + 6! - 3!
$$

Theorem 6.1.1 General Form of the Complementary Inclusion-Exclusion Principle

$$
\begin{aligned}
|A_1^c \cap \cdots \cap A_m^c| = |S| - \sum |A_i| + \sum |A_i \cap A_j| \\
- \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^{m+1} |A_1 \cap \cdots \cap A_m|
\end{aligned}
$$

Proof. To begin, we can realize that the LHS counts the number of elements in S that are not in any of the A_i.

For the RHS, Consider an arbitrary $s \in S$. There are two cases:

- Case 1. x is not in any A_i.

  In this case, x would contribute 1 to the LHS and 1 to RHS as it will not appear in any of the summations. Thus, this value will have no impact on the equality.

- Case 2. x is in some $n > 0$ A_i sets.

  Clearly, the contribution to the LHS is 0. For the RHS, the contribution is

$$
1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k}
$$

  This is because we must count the number of ways to choose k sets from the n sets that x is in.

By the binominal theorem, we have

$$1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (1)^{n-k} (-1)^k \binom{n}{k}$$
$$= (1-1)^n$$
$$= 0$$

In both cases, the equality holds. Thus, the LHS and RHS are equal. ∎

6.2 L14: The Inclusion-Exclusion principle and applications II: Derangements

Introduction to Dearrangements

Question: You all get up from your chairs, and randomly move to a different chair. What's the probability that no one ends up sitting down in the same chair? What if it were a class of 300 students?

Note:-

I find this definition a little hard to initially interpret.

Derrangements can be thought of through the following example. Suppose a teacher is attempting to pass back a test to four students: A, B, C, D. There are obviously 4! ways to destribute the tests (i.e. there are 4! permutations of A, B, C, D).

Dearrangements are the number of ways to distribute the tests such that no student gets their own test back.

For example A, C, D, B is not a dearragement because A gets their own test back. However, B, A, D, C is a dearrangement because no student gets their own test back.

Our goal is count the number of dearrangements.

Question 36: Abstract version of the original question

Given a permutation $\pi \in S_n$, what is the probability that $\pi(i) \neq i$ for all i. What is the number of D_n for all such permutations?

Solution: Inclusion-Exclusion argument Let A_i be the set of permutations where $\pi(i) = i$.

Then we want to find $|A_1^c \cap A_2^c \cap \cdots \cap A_n^c|$.

$$|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = |S_n| - |A_1 \cup A_2 \cup \cdots \cup A_n|$$
$$= |S_n| - \Big(|A_1| + |A_2| + \cdots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n|$$
$$+ |A_1 \cap A_2 \cap A_3| + \cdots + |A_1 \cap \cdots \cap A_n| \Big)$$

Clearly, there are $n!$ permutations in S_n.

For each $0 \leq i \leq n$, there are $(n-1)!$ permutations where $\pi(i) = i$. Thus there are $n(n-1)!$ permutations where $\pi(i) = i$ for some i. So,

$$\begin{aligned}\sum_{i=1}^{n} |A_i| &= n(n-1)! \\ &= n!\end{aligned}$$

For each i, j, there are $(n-2)!$ permutations where $\pi(i) = i$ and $\pi(j) = j$. Thus there are $n(n-1)(n-2)!$ permutations where $\pi(i) = i$ and $\pi(j) = j$ for some i and j.

$$\begin{aligned}\sum_{i,j}^{n} |A_i \cap A_j| &= \frac{n(n-1)(n-2)!}{2!} \\ &= \frac{n!}{2}\end{aligned}$$

This makes sense because there are n choices for i and $n-1$ choices for j (since i and j cannot be the same). This leaves $(n-2)!$ ways to arrange the remaining $n-2$ elements. However, i and j are indistinguishable, so we must divide by 2 to account for this.

Following this,

$$\begin{aligned}D_n &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \\ &= n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})\end{aligned}$$

Question 37

What's the probability of picking a derangement as $n \to \infty$?

Solution:

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}$$

Note:-

Just accept that this limit is true right now. Need to prove later.

Question 38

At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that:

- no gentleman receives his own hat?
- at least one gentleman receives own hat?
- at least two gentlemen receive their own hat?

Solution: part a

This is simply the number of derangements for $n = 7$.

$$D_7 = 7!(1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!})$$

Solution: part b

This is just the total number of permutations - the number of derangements for 7 people. So,

$$7! - D_7$$

Solution: part c

This is the number of ways that at least one person receives their own hat - the number of ways exactly one gentlement receives their own hat.

$$7! - D_7 - 7 \cdot D_6$$

6.3 L15: The Inclusion-Exclusion principle and applications II

Consider a problem we have seen before

How many r-combinations are there of a multiset with k distinct objects, each with infinite repetition number?

This is the same as the following question: Find the number of integer solutions to

$$x_1 + x_2 + \cdots + x_k = r$$

subject to $x_i \geq 0$ or each i. We know the answer to this is

$$\binom{r+k-1}{r}$$

Similarly, we have considered the problem where we assume instead $x_i \geq a_i$.

New Problem

Find the number of integer solutions to

$$x_1 + x_2 + \cdots + x_k = r$$

subject to $0 \leq x_i \leq a_i$ for each i.

Example 6.3.1 (How to solve this type of question)

Let S be the set of solutions where we just have $x_i \geq 0$ and let A_i be the set of solutions where $x_i > a_i$. Then we want to find $|A_1^c \cup A_2^c \cdots \cup A_m^c|$.

$$\begin{aligned} |A_1^c \cap \cdots \cap A_m^c| &= |S| - \sum |A_i| + \sum |A_i \cap A_j| \\ &\quad - \sum |A_i \cap A_j \cap A_k| + \cdots + (-1)^{m+1} |A_1 \cap \cdots \cap A_m| \end{aligned}$$

Question 39

Find the number of integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 18$$

subject to $5 \geq x_1 \geq 1$, $4 \geq x_2 \geq -2$, $5 \geq x_3 \geq 0$, $9 \geq x_4 \geq 3$.

Solution: First, let us redefine the problem in terms of new variables with equal restrictions but started with 0 as the lowest bound.

The problem becomes:

$$y_1 + y_2 + y_3 + y_4 = 16$$

subject to $4 \geq y_1 \geq 0$, $6 \geq y_2 \geq 0$, $5 \geq y_3 \geq 0$, $6 \geq y_4 \geq 0$.

Now, let S be the set of solutions where each $y_i \geq 0$. There are clearly,

$$\binom{16+4-1}{16} = \binom{19}{16}$$

Now, we must solve each of the following intersections, however, we only need to consider the cases where the intersection is non-empty. This is obviously done by solving this like we have previously studied. Subtract 1+ the upper bound of each y_i from the target value of the sums. And proceed to solve this question as a normal stars and bars problem.

$$\begin{aligned}
|A_1| &= \binom{11+4-1}{11} = \binom{14}{11} \\
|A_2| &= \binom{9+4-1}{9} = \binom{12}{9} \\
|A_3| &= \binom{10+4-1}{10} = \binom{13}{10} \\
|A_4| &= \binom{9+4-1}{9} = \binom{12}{9} \\
|A_1 \cup A_2| &= \binom{4+4-1}{4} = \binom{7}{4} \\
|A_1 \cup A_3| &= \binom{5+4-1}{5} = \binom{8}{5} \\
|A_1 \cup A_4| &= \binom{4+4-1}{4} = \binom{7}{4} \\
|A_2 \cup A_3| &= \binom{3+4-1}{3} = \binom{6}{3} \\
|A_2 \cup A_4| &= \binom{2+4-1}{2} = \binom{5}{2} \\
|A_3 \cup A_4| &= \binom{3+4-1}{3} = \binom{6}{3}
\end{aligned}$$

So the final answer is

$$\begin{aligned}
&= \binom{19}{16} - \left(\binom{14}{11} + \binom{12}{9} + \binom{13}{10} + \binom{12}{9}\right. \\
&\left. - \binom{7}{4} - \binom{8}{5} - \binom{7}{4} - \binom{6}{3} - \binom{5}{2} - \binom{6}{3}\right) \\
&= 55
\end{aligned}$$

6.4 L16: The Inclusion-Exclusion principle and applications IV: Another Forbidden Position Problem

Question 40

Eight people take a walk a walk in a line

$$1, 2, 3, 4, 5, 6, 7, 8$$

where 1 precedes 2 who precedes 3 etc. How many ways are there to rearrange the people so that no one precedes the person he preceded before?

In other words, count w in S_n that avoid the pairs

$$12, 23, \ldots, (n-1)n$$

Solution: Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$.
Let A_i be the set of all permutations that contain the pair $ii+1$.

Then we want to find

$$|S_n| - \left(\sum_{i=1}^{n} |A_i| - \sum_{i,j=1}^{n} |A_i \cap A_j| + \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_n|\right)$$

Clearly, $|S_n| = n!$ because there are $n!$ permutations.

Now, we must solve each of the following intersections. Observe the following:

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n-1}{k}(n-k)!$$

This is because we must think of constructing an algorithm that places the paired elements and then places the remaining elements.

This algorithm is represented by the RHS of the above the equation. This is because there are only $n-1$ positions where we can place a "block", i.e., a pair of elements. Given k blocks, there are $\binom{n-1}{k}$ ways to place them. Then, we must place the remaining $n-k$ elements which can be done in $(n-k)!$ ways given their distinct ordering

Thus, this gives us the final form of the answer.

$$n! - \left(\binom{n-1}{1}(n-1)! - \binom{n-1}{2}(n-2)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}(1)!\right)$$

Theorem 6.4.1 Non-attacking rook arragements

The number of non-attacking rook arragements on an $n \times n$ board with forbidden positions is

$$n! - r_1(n-1)! + r_2(n-2)! - \cdots + (-1)^k r_k(n-k)! + \cdots + (-1)^n r_n$$

Here r_k is the number of ways to place k rooks in the forbidden squares.

Proof. This is an application of the inclusion-exclusion principle.

Let A_i be the set of all placements where exactly one of the forbidden squares in row i must be used. Consider the following arbitrary intersection of A_is

$$\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

Note that the sum is over all selections of the k subsets.

This can be observed by constructing an algorithm that places the rooks in the forbidden squares and then places the remaining rooks.

There are r_k ways to place the rooks in the forbidden squares. Then, there are $n - k$ positions to place the remaining rooks, so there are $(n - k)!$ ways to place them. Thus,

$$\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k(n - k)!$$

Substituting this is the inclusion-exclusion principle, we get the theorem.

■

Chapter 7

Recurrence Relations and Generating Functions

7.1 L17: Some Number Sequences

Example 7.1.1 (Example 1)

Consider a configuration of n lines where every two lines have a point in common, but no three do. How many regions in the plane are there? Give a recurrence.

$$a_n = a_{n-1} + n$$

TODO: Give an explanation of why this is true.

Example 7.1.2 (Example 2)

Give a simple recurrence for dearragements.

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

TODO: Give an explanation of why this is true. Need to review dearragements from previous lecture.

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$.

Definition 7.1.1: The adjusted Fibonacci sequence: $\hat{F}_n$

This is the number of 1,2 lists of size n. In other words, consider the number of ways a valet can park A cars (size 1) and B cars (size 2) in a parking lot of size n.

$$\hat{F}_n = \begin{cases} 1 & \text{if } n = 0 \\ f_{n+1} & \text{otherwise} \end{cases}$$

Question 41

Prove

$$\sum_{n=0}^{n} f_i = f_{n+2} - 1$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$\sum_{i=0}^{0} f_i = f_0 = 0$$

Similarly,

$$f_{0+2} - 1 = f_2 - 1 = 1 - 1 = 0$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$\sum_{i=0}^{k} f_i = f_{k+2} - 1$$

Indcutive Step: We will prove that the following statement is true for $n = k + 1$.

$$\sum_{i=0}^{k+1} f_i = f_{k+1} + \sum_{i=0}^{k} f_i = f_{k+1} + f_{k+2} - 1 = f_{k+3} - 1$$

Therefore, the statement is true for all n by induction. ∎

Question 42

Prove

$$1 + \sum_{i=0}^{n} \hat{F}_i = \hat{F}_{n+2}$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$1 + \sum_{i=0}^{0} \hat{F}_i = 1 + \hat{F}_0 = 1 + 1 = 2$$

Similarly,

$$\hat{F}_{0+2} = \hat{F}_2 = f_{2+1} = f_3 = 2$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$1 + \sum_{i=0}^{k} \hat{F}_i = \hat{F}_{k+2}$$

Inductive Step: We will prove that the following statement is true for $n = k + 1$.

$$\begin{aligned} 1 + \sum_{i=0}^{k+1} \hat{F}_i &= 1 + \hat{F}_{k+1} + \sum_{i=0}^{k} \hat{F}_i \\ &= \hat{F}_{k+1} + \hat{F}_{k+2} \\ &= f_{k+2} + f_{k+3} \\ &= f_{k+4} \\ &= \hat{F}_{k+3} \end{aligned}$$

Therefore, the statement is true for all n by induction. ■

Question 43

Prove that f_n is even if and only if n is divisible by 3.

Solution:

Proof. Given that $f_0 = 0$, $f_1 = 1$, and $f_2 = 1$, we can see that at $n = 3$, $f_3 = 2$, which is even.

This is because the only way to get an even number is to have the parity of the two numbers added togethed (odd + odd or even + even) be the same. So, f_4, must be odd, f_5 must be odd and f_6 must be even.

Given the starting sequence of even, odd, odd. The following sequence must always be even, odd, odd, which repeats every 3 numbers.
Since the first $n = 0$ is the first number in the sequence, every n that is divisible by 3 is even. ■

Note:-

Example problems for later

Guess and prove by induction (you may replace the Fibonnaci number by the adjusted Fibonacci number if it helps you)

- $f_1 + f_3 + \cdots + f_{2n-1} =?$
- $f_0 + f_2 + \cdots + f_{2n} =?$
- $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n =?$
- $(f_0)^2 + (f_1)^2 + \cdots + (f_n)^2 =?$

Obtaining an explicit formula for f_n for linear recurrences

Example 7.1.3

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$. This can be rewritten as a linear recurrence as follows:

$$f_n - f_{n-1} - f_{n-2} = 0$$

We must solve the corresponding characterstic equation. Notice how the largest degree lines up with the "largest" case of the recurrence.

$$x^2 - x - 1 = 0$$

Let q_1 and q_2 be the roots of the characteristic equation.

It is potentially relevant to note that the following is a solution space of the Fibonacci recurrence (but don't satisfy f_0 – the initial condition):

$$\begin{cases} q_1^n - q_1^{n-1} - q_1^{n-2} = 0 \\ q_2^n - q_2^{n-1} - q_2^{n-2} = 0 \end{cases}$$

The rest of this is based on an ansatz, i.e. we need to make an assumption at the answer and validate it later

$$f_n = c_1 q_1^n + c_2 q_2^n,$$

for *some* $c_1, c_2 \in \mathbb{R}$.

Using the initial conditions of $f_0 = 0$ and $f_1 = 1$, we can solve for c_1 and c_2.

7.2 L18: Introduction to ordinary generating series

Note:-

Up until now, we have solved an instance combinatorial problem at a time. However, in some cases, solving a single combinatorial is too difficult. So, we solve all of the combinatorial problems at once.

Consider a sequence

$$h_0, h_1, h_2, \ldots, h_t, \ldots$$

of natural numbers where h_t is the answer to some counting problem that depends on t.

We can create a generating series of the form:

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_t x^t + \cdots$$

where $h_t = [x^t]g(x)$.

Note:-

The notation $[x^t]g(x)$ is the coefficient of x^t in the polynomial $g(x)$.

Claim 7.2.1 Compositions Generating Series

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Example 7.2.1

Fix k. Let

h_t = number of nonnegative integral solutions to

$$e_1 + \cdots + e_k = t, e_i \in \mathbb{Z}_{\geq 0}$$

We already know from stars and bars that

$$h_t = \binom{t+k-1}{k-1}$$

So obviously, the generating series can be written as

$$g(x) = \sum_{t=0}^{\infty} \binom{t+k-1}{k-1} x^t$$

Note:-

This doesn't really tell us anything. We just combined some definitions and have a genearting series that is hard to utilize in any useful way.

Take our claim and expand it out with Taylor Series k times:

$$\begin{aligned} g(x) &= \left(\frac{1}{1-x}\right)^k \\ &= (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)\cdots \end{aligned}$$

We can see that the coefficient of x^t is the number of ways to write t as a sum of k nonnegative integers. So,

$$h_t = \binom{t+k-1}{k-1}$$

This combinatorially proves the negative binomial theorem:

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Remark

If you knew that the negative binomial theorem was true, you could use it to algebraically solve the composition problem after choosing the correct generating series $g(x) = \left(\frac{1}{1-x}\right)^k$.

Question 44

What is

$$(1 + x + x^2 + x^3 + x^4 + x^5)(x + x^2)(1 + x + x^2 + x^3 + x^4)$$

the generating series for?

Solution: The solutions to the equation

$$e_1 + e_2 + e_3 = t$$

where $0 \leq e_1 \leq 5, 1 \leq e_2 \leq 2, 0 \leq e_3 \leq 4$.

Question 45

Suppose you have 1,5 and 25 cents coins available (infinite supply). Write down the generating series for the number of ways to make change for a dollar. How many ways are there?

Solution:

The generating series for the number of ways to make change for an arbitrary amount of money is:

$$g(x) = (1 + x + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^5 + x^{5+5} + x^{5+5+5} + \cdots)(1 + x^{25} + x^{25+25} + x^{25+25+25} + \cdots)$$

This can be simplified as follows:

$$
\begin{aligned}
g(x) &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{25}}\right) \\
&= \frac{1}{(1-x)(1-x^5)(1-x^{25})}
\end{aligned}
$$

The answers is $[x^{100}]g(x)$.

Question 46: Nonstandard Dice

Compute the generating series for the result of rolling two standard dice.
Now factor it, and interpret.

Solution:

$$
\begin{aligned}
g(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6) \\
&= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \\
&= (1 + x)(1 + x)(x + x^3 + x^5)
\end{aligned}
$$

TODO: How does this factor like this?
After factoring, we can see that this is the equivalent of flipping a coin twice and rolling two 1-3-5 dice.

Question 47

Determine the generating series for partitions.

Solution:

$$
\begin{aligned}
g(x) &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^2 + x^{2+2} + x^{2+2+2} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots)\cdots \\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i}\right)
\end{aligned}
$$

In words, this generating series is the product of compositions of just 1's, 2's, 3's, etc.

Definition 7.2.1: Ordinary Generating Series

Let a combinatorial problem be defined as (S, ω) with S a set and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ The ordinary generating series is

$$
g(x) = g_{(S,\omega)}(x) = \sum_{s \in S} x^{\omega(s)}
$$

Theorem 7.2.1 Addition Rule of Generating Series

Suppose $S = A \cup B$ (disjoint union), where (A, ω_A) and (B, ω_B) are combinatorial problems. Moreover $\omega|_A = \omega_A$ and $\omega|_B = \omega_B$.
Then the ordinary generating series for (S, ω) is

$$
\begin{aligned}
g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) + g_{(B,\omega_B)}(x) \\
&= \sum_{s\in A} x^{\omega_A(s)} + \sum_{s\in B} x^{\omega_B(s)} \\
&= \sum_{s\in S} x^{\omega(s)}
\end{aligned}
$$

Note:-

The notation $\omega|_A$ means the restriction of ω to A. In this case,

$$\omega|_A = \{\omega_A(s) \mid s \in A\}$$

Theorem 7.2.2 Product rule of Generating Series

Suppose $S = A \times B$ (cartesian product), where (A, ω_A) and (B, ω_B) are combinatorial problems and

$$\omega(a,b) = \omega_A(a) + \omega_B(b)$$

Then,

$$
\begin{aligned}
g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) g_{(B,\omega_B)}(x) \\
&= \sum_{a\in A} x^{\omega_A(a)} \sum_{b\in B} x^{\omega_B(b)} \\
&= \sum_{a\in A} \sum_{b\in B} x^{\omega(a,b)}
\end{aligned}
$$

What does it mean for two generating series to be equal?

They are equal coefficient by coefficient.

Let $g(x)$ be the generating series for the number of partitions. What does it mean that $g(x)$ equals the infinite product:

$$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

It means that if we expand/solve enough terms of the RHS, the coefficient of x^t will stabalize (as in no more terms multiplied in will impact the coefficient) and this coefficient will be the same as the coefficient of x^t in the LHS – $g(x)$.

Convergence Issues

Typically, with generating series, we only care about the coefficients and not plugging in any specfic value into x. Because of this, we do not need to worry about convergence. However, in some cases $g(x)$ is a polynomial and in these cases, substitution is fine to perform.

Question 48: Substituting into a generating series

Let $S = \{\text{coins in your pokcet}\}$ and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ be the obvious weight function on coins, i.e. $\omega(\text{nicket}) = 5$. Is $g(x)$ is the corresponding generating series, what is $g(1)$? What is $g'(1)$?

Solution: $g(1)$ will be the number of coins in your pocket. This is trivially true because each term is x to the power of an integer (with no coefficient), so 1 raised to anything will be 1 and so it is just the number of terms. More mathematically,

$$\begin{aligned} g(1) &= \sum_{s \in S} 1^{\omega(s)} \\ &= \sum_{s \in S} 1 \\ &= |S| \end{aligned}$$

$g'(1)$ will the amount of money you have. Based on the previous statement, we can see that the derivative of $g(x)$ is just the sum of the powers which represented the values of the coins in your pocket. More mathematically,

$$g'(1) = \sum_{s \in S} \omega(s)$$

7.3 Practice Problems

Question 49

Find the coefficient of x^{16} in $(x^2 + x^3 + \cdots)^5$.

Solution: This is the generating series for the following composition problem:

$$e_1 + e_2 + e_3 + e_4 + e_5 = 16$$

where $e_1 \geq 2$, $e_2 \geq 2$, $e_3 \geq 2$, $e_4 \geq 2$, and $e_5 \geq 2$.
This can be rewritten as:

$$e_1 + e_2 + e_3 + e_4 + e_5 = 6$$

where each $e_i \geq 0$. Which represents the generating series:

$$\begin{aligned} g(x) &= (1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)^5 \\ &= \left(\frac{1}{1-x}\right)^5 \\ &= (1-x)^{-5} \end{aligned}$$

where we are solving for $[x^6]g(x) = \binom{5+6-1}{5-1} = \binom{10}{4}$.

Chapter 8

Special Counting Sequences

8.1 L19: Partition identities

Definition 8.1.1: Partition

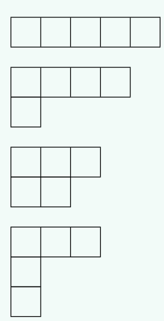
A *partition* is a decreasing sequence of positive integers whose sum is n.

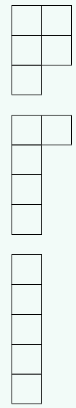
Example 8.1.1

Consider $n = 5$. The partitions of 5 are

$$\begin{aligned}
&5\\
&4,1\\
&3,2\\
&3,1,1\\
&2,2,1\\
&2,1,1,1\\
&1,1,1,1,1
\end{aligned}$$

However, this can also be represented in terms of the **Ferrers diagram**:
Partitions 5:





Example 8.1.2 (Theorem 8.3.1 from the textbook)

Use Ferrers diagrams to prove that the number of partitions of n in which the largest part is r equals the number of partitions of $n - r$ in which no part is greater than r.

Let P be a partition for n. Remove the first part of P which by definition is the largest part which has an arbitrary size r. This clearly yields a partition of $n - r$ in which no part is greater than r. This process is clearly a bijection and so the number of partitions of n in which the largest part is r equals the number of partitions of $n - r$ in which no part is greater than r.

Note:-

This process for proving two partitions to be of equal size is important.

Generating Series , and partition identities

Recall generating series for partition:

$$g(x) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^k} \right)$$

Question 50

Write down the generating series for:

1. only odd parts
2. no part repeated more than m times

Solution: Only odd parts:

$$\begin{aligned} g(x) &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots) \cdot (1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots) \cdots \\ &= \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^{2k-1}} \right) \end{aligned}$$

Solution: No part repeated more than m times

$$
\begin{aligned}
g(x) &= (1 + x + x^{1+1} + \cdots + x^m)(1 + x^2 + x^{2+2} + \cdots + x^{2m})(1 + x^3 + x^{3+3} + \cdots + x^{3m})\cdots \\
&= (\frac{1}{1-x} - \frac{x^m}{1-x}) \cdot (\frac{1}{1-x^2} - \frac{x^m}{1-x^2}) \cdot (\frac{1}{1-x^3} - \frac{x^m}{1-x^3})\cdots \\
&= \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k} - \frac{x^m}{1-x^k}\right) \\
&= \prod_{k=1}^{\infty} \left(\frac{1-x^m}{1-x^k}\right)
\end{aligned}
$$

Terminology

Definition 8.1.2: Partition size

The *size* of a partition is the sum of its parts.

Definition 8.1.3: Partition length

The *length* of a partition is the number of parts.

Question 51

Write down a two variable generating series that counts partitions by both size and length, using say, a q variable and a z variable respectively. Hence the coefficient of $q^n z^l$ is the number of partitions of size n and length l.

Solution:

$$
\begin{aligned}
g(q,z) &= \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} P(n,l) \cdot q^n z^l \\
&= \prod_{k=0}^{\infty} (1 + zq^k + z^2 q^{2k} + z^3 q^{3k} + \cdots) \\
&= \prod_{k=0}^{\infty} \left(\frac{1}{1-zq^k}\right)
\end{aligned}
$$

Question 52

Consider the identity:

$$
\prod_{k=1}^{\infty} 1 + q^k = \prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}}
$$

- Interpret both sides of the identity.
- Give an algbraic proof.
- Can you give a combinatorial proof?

Solution: Interpret both sides of the identity

- LHS

$$\prod_{k=1}^{\infty} 1+q^k = (1+q^1)(1+q^2)(1+q^3)\cdots$$

This is clearly the number of partitions with distinct parts that are used at most one time.

- RHS

  This is clearly the generating series for partitions with only odd parts.

Solution: Give an algebraic proof

Proof.

$$\begin{aligned}
\prod_{k=1}^{\infty} 1+q^k &= \prod_{k=1}^{\infty} \frac{1-q^k}{1-q^k}(1+q^k) \\
&= \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^k} \\
&= \prod_{k=1}^{\infty} \frac{\frac{1}{1-q^k}}{\frac{1}{1-q^{2k}}} \\
&= \frac{\prod_{k=1}^{\infty} \frac{1}{1-q^k}}{\prod_{k=1}^{\infty} \frac{1}{1-q^{2k}}} \\
&= \text{ all even partitions divided out of all partitons} \\
&= \prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}}
\end{aligned}$$

■

Solution: Give a combinatorial proof

Proof. Fix n. Let S be the set of all partitions of n with only distinct parts. Clearly, the LHS represents the generating series that counts S.

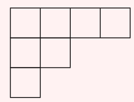
Construct a function f from distinct parts to odd parts by dividing each even number by 2 until every number is odd. This function is clearly well defined as it will only result in odd parts. This must be a bijection because the input to f must have distinct parts so the number of odd parts created is dependent on the even numbers in the input so the output is unique. Lastly, this is clearly reversible by taking the largest subset of repeated odd numbers that is a power of 2 and summing them.

Thus, because we can decompose S into the set of partitions with only odd parts through a bijective function and this decomposition is clearly counted by the RHS. Then, it is clear that the LHS and RHS equivalent. ■

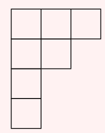
8.2 L20: Partition identities (continued)

Definition 8.2.1: Partition Conjugate

The *conjugate* of a partition is the partition associated to the transpose of the Ferrers diagram.



has a conjugate of



Question 53

Use conjugation to prove the following fact: the number of partitions of n with no part bigger than k equals the number of partitions of n with at most k parts.

Determine the generating series for either of the kinds of partitions from the previous problem.

Solution: Using conjugation...

Proof. This part is trivially true. If no part is bigger than k, then the first row in the Ferrers Diagram, which must be the largest row in the diagram, has at most k parts.

By taking the transpose of the first type of parition, we get a partition with at most k parts as the number of parts will be dictated by the first row.

Thus, the number of partitions of n with no part bigger than k is the same as the number of partitions of n with at most k parts. ∎

Solution: Generating Series

$$\begin{aligned} g(x) =&(1 + x + x^{1+1} + \cdots + x^k) \\ &(1 + x^2 + x^{2+2} + \cdots + x^{2k}) \\ &(1 + x^3 + x^{3+3} + \cdots + x^{3k}) \cdots \end{aligned}$$

$$\begin{aligned} g(x) &= \prod_{i=1}^{\infty}(1 + x^i + x^{2i} + \cdots + x^{ki}) \\ &= \prod_{i=1}^{\infty}\left(\frac{1 - x^{ki+1}}{1 - x^i}\right) \end{aligned}$$

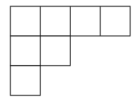
Definition 8.2.2: Durfee Square

A *Durfee square* is the largest square that sits in the NW corner of the Ferrers diagram of a partition.

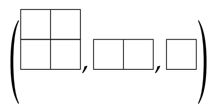
Question 54

Show that every Ferrers shape uniquely decomposes into (D_k, A_k, B_k) where D_k is a Durfee square of size k, A_k is the partition with at most k parts, and B_k is a partition with first part of size at most k.

Solution: Consider an arbitrary partition:



This can obviously be decomoposed into the triple:



There are a few cases to consider:

- First, this is unique. Clearly, this has to be unique because we are not changing any structure from the partition, but rather we are just decomposing it into a triple. So each partition must uniquely map to its own ordered triple.

- Second, A_k is the partition with at most k parts. A_k must have at most k because the Durfee square is of size k. A_k cannot be largely than this because it would contradict either the definition of a partition or the definition of a Durfee square.

- Third, B_k is the partition with first part of size at most k. This is true because if the first part of B_k is larger than k, then this would contradict the definition of a Durfee square or a partition.

Thus, we have shown that every partition can be decomposed into a triple.

Definition 8.2.3: Euler-Gauss Identity

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = 1 + \sum_{j=1}^{\infty} \frac{x^{j^2}}{\left(\prod_{t=1}^{j} 1-x^t\right)^2}$$

Definition 8.2.4: Self-conjugate Partition

A partition is *self-conjugate* i its transpose shape is the same as its original shape (e.g., the partitions (4,1,1,1) and (3,2,1) are self-conjugate, but (3,2) is not).

Question 55

Prove that the number of self conjugate partitions of n equals the number of partitions using only distinct odd parts. Then derive an identity.

Solution: First, we must prove the number of self conjugate partitions of n equals the number of partitions using only distinct odd parts. This can be done by showing a weight preserving bijection between the two sets.

Definition 8.2.5: Weight-preserving function

Define the following function:

$$f : (S, \omega) \mapsto (S', \omega')$$

such that $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ and $\omega' : S' \mapsto \mathbb{Z}_{\geq 0}$ are weight functions. Then, f is a weight-preserving function if and only if:

$$\omega(s) = \omega'(f(s)) \text{ for all } s \in S$$

And so, we can make the following claim

Claim 8.2.1

If $f : (S, \omega) \mapsto (S', \omega')$ is a weight-preserving and bijective then,

$$g_{(S,\omega)}(x) = g_{(S',\omega')}(x) \in \mathbb{Z}[x]$$

In other words, their generating functions are equal.

Now to define the actual bijection. Consider each row of the Ferrers diagram of an arbitrary distinct odd partition. Each row can be bent into a self-conjugate partition. By selecting the middle element, which is gaurunteed to exist because the partition is odd, we can place every square to the left below the middle element and leave the other elements alone. Additionally, this is well defined as each row is distinct. Thus, we have a weight preserving bijection between the set of distinct odd partitions and the set of self-conjugate partitions.

The generating function for the number of distinct odd partitions is trivial and given by:

$$g(x) = \prod_{i=1}^{\infty} 1 - x^{2i-1}$$

as we only want to either select an odd number or not.

The generating function for the number of self-conjugate partitions is more complicated. Consider an arbitrary self-conjugate partition λ. We can decompose it into a triple (D_k, A_k, B_k) through the Durfee square decomposition. We can then apply a second weight-preserving bijection and conjugate the second component of the triple which makes $A_k = B_k$. Lastly, we can apply a third weight-preserving bijection by compressing the triple into a tuple (D_k, C_k) where C_k is A_k merged left-to-right with B_k.

Consequently,

$$C_k \in \bigcup_{j=0}^{\infty} D_j \times \text{Partitions}_{\text{even}_{\leq j \text{ tall}}} = U$$

Clearly,

$$\begin{aligned} g_U(x) &= \sum_{j=0}^{\infty} x^{j^2} \Big(\prod_{t=1}^{j} \frac{1}{1 - x^{2t}} \Big) \\ &= \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{t=1}^{j} \frac{1}{1-x^{2t}}} \end{aligned}$$

So,

$$\prod_{k=1}^{\infty} 1 + x^{2k-1} = \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{t=1}^{j} 1 - x^{2t}}$$

8.3 L21: Exponential generating series (7.3)

Given a number sequence

$$h_0, h_1, h_2, \ldots$$

consider using the *exponential* series:

Definition 8.3.1: Exponential Generating Series

$$g_{\exp}(x) = \sum_{i=0}^{\infty} h_i \frac{x^i}{i!}$$

Why do we care about using this?

Reason A

Ordinary generating series typically won't have any algabraic properties that are useful such as factorization. However, exponential generating may have these properties.

Example 8.3.1

Fix a positive integer n. Let
$P(n, k)$ = the number of k permutations of an n set. In other words, select k elements from n set where order matters.

Question 56

Compute the ordinary and exponential generating series for $P(n, 0), P(n, 1), \ldots$.

Solution: Let's consider how a single $P(n, k)$ is constructed. There are $\binom{n}{k}$ ways to pick k elements from n set and $k!$ ways to order them. So,

$$\begin{aligned} P(n, k) &= \binom{n}{k} k! \\ &= \frac{n!}{k!(n-k)!} k! \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

Clearly the ordinary generating series is:

$$\begin{aligned} g(x) &= \sum_{i=0}^{\infty} P(n,i)x^i \\ &= \sum_{i=0}^{\infty} \frac{n!}{(n-i)!}x^i \end{aligned}$$

Additionally, the exponential generating series is:

$$\begin{aligned} g_{\text{exp}}(x) &= \sum_{i=0}^{\infty} P(n,i)\frac{x^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{n!}{(n-i)!}\frac{x^i}{i!} \\ &= \sum_{i=0}^{\infty} \binom{n}{i}x^i \\ &= (1+x)^n \end{aligned}$$

Note:-

Note how the exponential series simplifies as compared to the ordinary series.

Question 57

What is the exponential generating series for the sequence $1, 1, 1, \ldots$?

Solution:

$$\begin{aligned} g_{\text{exp}}(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ &= \sum_{i=0}^{\infty} \frac{x^i}{i!} \\ &= e^x \end{aligned}$$

Reason B

Ordinary series only work well for *unlabeled* combinatorial sets, whereas exponential generating are better for *labeled* combinatorial sets.

Example 8.3.2

Consider 38 identical balls, how many ways are thre to group them together into 3 different bags? This is a composition problem, and is an unlabeled problem (since all balls are identical). This is a composition problem with a solution of $\binom{38+3-1}{3-1}$

Now consider 38 students in MATH 413. How many ways are there to group them into 3 different groups? This is a *labeled* version of the problem. The answer to this is 3^{38} because there 3 choices for each distint student.

Question 58

What is e^{ax}, for a natural number a, the exponential generating series for?

Solution: We know that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, so

$$\begin{aligned} e^{ax} &= \sum_{i=0}^{\infty} \frac{(ax)^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{x^i}{i!} a^i \end{aligned}$$

a^i is the number of words of length i that can be generated from an alphabet of size a.

Question 59

Find the exponential generating series for the number of ways to place r (distinct) people into three different rooms with at least one person in each room. Repeat the problem with the extra condition of an even number of people.

Solution: One person in each room

$$g_{\exp}(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^3$$

Solution: Even Number of people and one person in each room

$$g_{\exp}(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^3$$

Question 60

Now say $r = 25$ in the above exercise. Compute the actual number of ways to place 25 people in three rooms with at least one person in each room.

Solution: We have already computed the generating series for one person in each room above. This generating series can be simplified even further as:

$$\begin{aligned} g_{\exp}(x) &= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^3 \\ &= (e^x - 1)^3 \\ &= \sum_{i=0}^{3} \binom{3}{i} (e^x)^{3-i} (-1)^i \\ &= \binom{3}{0} e^{3x} - \binom{3}{1} e^{2x} + \binom{3}{2} e^x - \binom{3}{3} \\ &= e^{3x} - 3e^{2x} + 3e^x - 1 \end{aligned}$$

Our desired answer is $[x^{25}]g_{\exp}(x)$, so the answer is as follows:

$$\begin{aligned} [x^{25}]g_{\exp}(x) &= [x^{25}]e^{3x} - 3[x^{25}]e^{2x} + 3[x^{25}]e^x - [x^{25}] \\ &= 3^{25} - 3 \cdot 2^{25} + 3 \cdot 1^{25} \\ &= 3^{25} - 3 \cdot 2^{25} + 3 \end{aligned}$$

8.4 L23: Catalan Numbers

Definition 8.4.1: Rooted binary Tree

A *rooted binary tree* is a tree with a root such that each vertex has a left and right subtree. Let G be the combinatorial set of these trees and $\omega : G \longrightarrow \mathbb{Z}_{\geq 0}$ be the weight function that counts the number of vertices in the tree. Then a decomposition is as follows:

$$G \longrightarrow \{\emptyset\} \cup \{\bullet\} \times G^2$$

this is a weight preserving bijection. Hence we can obtain the functional equation:

$$g(x) = 1 + xg(x)^2$$

Solving this formula for the quadratic equation gives us:

$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

However, when exanding these equations we get that $\frac{1+\sqrt{1-4x}}{2x}$ has negative coefficients and thus is not a valid solution. Therefore, we have (by expansion of taylor series)

$$g(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 \cdots$$

Note that by our studies of generating series, $[x^n]g(x)$ is the number of rooted binary trees with n vertices. The coefficients of this expansion are the Catalan numbers.

Dyck Paths

Note that catalan numbers also count the number of Dyck paths. A Dyck path is a path that starts at $(0, 0)$ and ends at some arbitrary $(x, 0)$. It will never fall below the x-axis and will only use / and \ to move from point (x, y) to $(x + 1, y + 1)$ or $(x + 1, y - 1)$.

Question 61

Prove that the number of Dyck paths from $(0, 0)$ to $(2n, 0)$ is

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

in steps as follows:

(a) Count the number of paths from $(0, 0)$ to $(2n, 0)$.

(b) Show the number of bad paths is equal to the number of paths from $(0, 0)$ to $(2n, -2)$. Hint: consider the first time a path dips below the x-axis and "reflect" every step after that first step below the x-axis.

(c) Use the subtraction principle and simplify.

Solution:

(a) We will first begin by counting the number of paths from $(0, 0)$ to $(2n, 0)$. There are two ways to approach this. Since, we have to end at the same "level" as we start we need the same amount of steps up as we have down. Thus, there are $\binom{2n}{n}$ ways to pick the steps up and thus, the steps down are fixed. Second, we can consider all of the steps between these points as one big set. There must be n steps up and n steps down. These steps can happen in any arragement. So, this question is just the number of permutations of this set which is $\frac{(2n)!}{n!n!} = \binom{2n}{n}$. Note that we must divide by $n!$ twice to account for the n up steps that are identical and the n down steps that are identical.

(b) We will now count the number of bad paths. A bad path is defined as any path that crosses below the x-axis. We can count this by creating a bijection bijection for a bad path from the first point that it crosses below the x-axis and taking a reflected step for every step after that below the x-axis. This path must end at $(2n, -2)$ because after it takes the bad step, there must have been more down steps than up steps. Because of this in the original graph there must be 1 more up step than down steps in the remaining path. Because we are reflecting each step, the bijected path will end up with 1 more down step from the point at which it crossed the x-axis which is at $y = -2$. To count the number of these paths, we can choose the $n + 1$ steps that had to be taken down so $\binom{2n}{n+1}$.

(c) Lastly, using the subtraction principle, we have that the number of Dyck paths is

$$\begin{aligned}
\binom{2n}{n} - \binom{2n}{n+1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1!)} \\
&= \frac{(2n!)}{n(n-1)!n!} - \frac{(2n)!}{(n+1)(n)!(n-1)!} \\
&= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{(2n)!}{n!(n-1)!}\left(\frac{n+1-n}{n(n+1)}\right) \\
&= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n(n+1)}\right) \\
&= \frac{(2n!)}{n!(n-1)!n(n+1)} \\
&= \frac{1}{n+1}\left(\frac{(2n)!}{n!(n-1)!n}\right) \\
&= \frac{1}{n+1}\left(\frac{(2n)!}{n!n!}\right) \\
&= \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

Question 62

A modification of the same argument shows the following: suppose A and B are being voted upon and A gets a votes and B gets b votes with $a \geq b$. Now suppose we count the votes one at a time, tracking who is leading at each step. Then the total number of ways A is always leading (or tied with) B throughout the count is $C(a+b, a) - C(a+b, a+1)$. How?

Solution: This situation is counting the number of paths from $(0, 0)$ to $(a + b, a - b)$. Similar to the previous question we can first calculate the number of total paths to $(a + b, a - b)$ and then subtract out the bad paths. The number of paths from $(0, 0)$ to $(a + b, a - b)$ is $\binom{a+b}{a}$. To find the number of bad paths we can apply the reflection principle. Suppose they are tied at k votes each meaning there have been $2k$ votes. Assume that the $2k + 1$ vote goes to b. There are $a - k$ votes left for A and $b - k - 1$ votes left for B. Construct a bijection that reflects every vote (i.e. gives votes for A to B and vis-versa) after vote $2k + 1$. Thus we will end with $b - 1$ votes for A and $a + 1$ votes for B and thus the graph will end at $(a + b, b - a - 2)$ because $b - 1 - (a + 1) = b - a - 2$. Thus, the number of bad paths is the number of ways to choose the $a + 1$ votes for B which is $\binom{a+b}{a+1}$. Thus, the number of good paths is

$$\binom{a+b}{a} - \binom{a+b}{a+1}$$

Question 63

Show

$$C_n = \frac{4n-2}{n+1}C_{n-1}$$

Solution:

$$\begin{aligned}
C_n &= \frac{4n-2}{n+1}C_{n-1} \\
&= \frac{4n-2}{n+1}\left(\frac{1}{n}\binom{2n-2}{n-1}\right) \\
&= \frac{4n-2}{n+1}\left(\frac{1}{n}\frac{(2n-2)!}{(n-1)!(n-1)!}\right) \\
&= \frac{4n-2}{n+1}\left(\frac{(2n-2)!}{(n-1)!n!}\right) \\
&= \frac{2(2n-1)(2n-2)!}{(n+1)(n-1)!n!} \\
&= \frac{n}{n}\left(\frac{2(2n-1)(2n-2)!}{(n+1)(n-1)!n!}\right) \\
&= \frac{2n(2n-1)(2n-2)!}{(n+1)(n!)(n!)} \\
&= \frac{1}{n+1}\left(\frac{2n!}{n!n!}\right) \\
&= \frac{1}{n+1}\binom{2n}{n} \\
&= C_n
\end{aligned}$$

Question 64

Show

$$C_n = \sum_{k=0}^{n} C_k C_{n-k+1}$$

Solution: Let S be the set of all Dyck paths from $(0,0)$ to $(2n,0)$. The LHS trivially counts the number of elements in S. Let $2k+2$ be the arbitrary point on the x-axis where the path first touches. The path can be decomposed into two paths, one of length $2k$ (i.e. the part) and one of length $2n-2k-2$ (i.e. the tail). Thus, this is counted by $C_k C_{n-k+1}$, so in total,

$$\sum_{k=0}^{n} C_k C_{n-k+1}$$

Thus, the LHS and RHS count the same set in two different ways and are equal.

Question 65

Let $2n$ equally spaced points on a circle be chosen. Show the number of ways to join them in pairs with lines, with no lines crossing is C_n. Hint: Use the recursion. (Can you give a bijection?)

Solution: Construct a bijection such that the first timee a line is touched (i.e. a connector between two points within the circle), you move up in the Dyck path and the second time you see the line, you move down. This is well-defined, injective, and bijective. It is clearly, well-defined because you can only get Dyck paths from this function. It isn't possible to see the end of the line without seeing the start of the line. Because of this, it is impossible to see more down moves than up moves at any point. It is injective because given a two Dyck paths that are the same, you can construct two circles that are connected by reversing this process to build the circle. Lastly, it is surjective because if you have an arbitrary Dyck path, there will always be a circle that has $2n$ points that will map to the Dyck path.

Question 66

Show that the number of SYT (Standard Young Tableau) of shape (n, n) is C_n

For clarity, a SYT of shape (n, n) is a Ferrer Diagram with 2 rows with n columns each. In this shape, the boxes are labeled $1, 2, \ldots, 2n$ and each row and column must be increasing.

Solution: Construct a bijection between the set of SYT of shape (n, n) and Dyck paths. If the k-th element is in the first row, move up, else move down. This is well-defined because in order to be a valid SYT, each row and column must be increasing. If an invalid Dyck path is generated (i.e. goes below the x-axis, then there must have been a row or column that was not increasing.) The process of creating Dyck paths using this function is clearly injective and bijective because each number is unique to a Dyck path.

Chapter 9

Homework

9.1 Homework 1

1. How many nonempty words can be formed from three As and five B's? (The "words" are "mathematical words" not necessarily "dictionary words". For example B, BAAB, ABABABBB are possible words you need to count.)

   Solution: We must consider each possible length from 1 to 8. At each length we can decompose by the number of A's in the word. So, at each length:

$$
\begin{aligned}
\text{length 1: } & \binom{1}{0} + \binom{1}{1} \\
\text{length 2: } & \binom{2}{0} + \binom{2}{1} + \binom{2}{2} \\
\text{length 3: } & \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} \\
\text{length 4: } & \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} \\
\text{length 5: } & \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} \\
\text{length 6: } & \binom{6}{1} + \binom{6}{2} + \binom{6}{3} \\
\text{length 7: } & \binom{7}{2} + \binom{7}{3} \\
\text{length 8: } & \binom{8}{3}
\end{aligned}
$$

   Thus, the total number of words is: 200

2. How many ternary $(0, 1, 2)$ sequences of length 10 are there without any consecutive digits the same?

   Solution: For the first digit in the sequence, there are 3 choices. For every choice after this, there can be no repeats with the previous digit. Hence there are only 2 choices. Thus, the total number of sequences is: $3 \cdot 2^9 = 1536$

3. What is the probability that if one letter is chosen at random from the word RECURRENCE and one letter is chosen from RELATION, the two letters are the same?

   Solution: We need to consider the probability of drawing the same letter from both words. There are only 3 letters in common: R, E, N. So, the probablity must be

$$\frac{3}{10}\cdot\frac{1}{8}+\frac{3}{10}\cdot\frac{1}{8}+\frac{1}{10}\cdot\frac{1}{8}=\frac{7}{80}=\frac{3}{28}$$

4. Corrupt professor Z has a class of 50 students. He needs to give exactly 10 A's. However five students already have a special deal (they are professor Z's nephews and nieces) and will get A's for sure. How many ways can the 10 A's be distributed?

   Solution: Given that 5 students have fixed grades of A, there are only 5 A's that can be distributed among the remaining 45 students. So there are $\binom{45}{5}$ ways to distribute the A's in this situation.

5. Prove the identity:

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

   COMBINATORIALLY by counting the same combinatorial set in two different ways (in other words, "double count"). (Do NOT give the standard induction proof, nor the "Gauss argument" of summing $1+2+\cdots+n$ and layering with $n+\cdots+2+1$.)

   Solution: Let S be a set of distinct handshakes between n people. We will count this in two ways:

   - There are clearly $\binom{n}{2}$ ways to form pairs in the room which much give each other handshakes. Following this, $\binom{n}{2} = \frac{n!}{2!(n-2)! = \frac{n(n-1)}{2}}$. This is exactly the RHS.
   - Decompose the set S by individuals initiating the handshake. The first person has $n-1$ choices to initiate a handshake. The second has $n-2$ choices, and so on. By the addition principle, we have $(n-1)+(n-2)+\cdots+1 = \sum_{i=1}^{n-1} i$. which is exactly the LHS.

   Thus, because the LHS and RHS count the same set. It is clear that they are equivalent.

9.2 Homework 2

1. In how many ways can two red and four blue rooks be placed on an 8-by-8 board so that no two rooks can attack one another?

   Solution: Construct an algorithm that places the rooks on the board. First, place the red rooks individually on the board. Then, place the remaining blue rooks. Each time a rook is placed, the board loses one row and column of possible placements. Thus, there are $\frac{8^2\cdot 7^2}{2!}$. We must divide by 2! because the rooks are identical. Similarly for the blue rooks, there are $\frac{6^2\cdot 5^2\cdot 4^2\cdot 3^2}{4!}$. Thus, the total number of ways is

$$\frac{64\cdot 49\cdot 36\cdot 25\cdot 16\cdot 9}{2!\cdot 4!} = 8467200$$

2. Prove that the number of permutations of m A's and at most n B's equals

$$\binom{m+n+1}{m+1}$$

   Solution: Consider the number of permutations of $m+1$ $A's$ and n B's. This is trivially counted by

$$\binom{m+n+1}{m+1}$$

   This is because there are $m+n+1$ positions to place the A's and B's. There are $m+1$ positions to place the A's which fixes the positions of B's. Consider a bijective function f from set the set of permutations of $m+1$ A's and n B's to the set of permutations of m A's and at most n B's.

f can be defined by the following process. Find the last occurance of A in the permutation and truncate the permutation at that point, removing every letter after this A. We must show that this function is well-defined, injective, and surjective.

This function is obviously *well-defined.* There is gaurunteed to be at least 1 A. Thus, the output of this function will have $m+1-1=m$ A's. Similarly, there could have been 0 to n B's after the last A. So, there are at most n B's in the output.

This function is *injective* because this process is reversible. Consider $f(a_1) = f(a_2)$ where a_1 and a_2 are permutations of $m+1$ A's and n B's. This means that they each had an A truncated at the same position and the same number of B's removed. Thus, $a_1 = a_2$ which makes f injective.

This function is *surjective.* Consider a permutation π with m A's and at most n B's. We can append an A to the end of π and then append $n - |\pi|$. This must be a permutation with $m+1$ A's and n B's. So, f is surjective.

3. Determine the number of towers of the form $\emptyset \subseteq A \subseteq B \subseteq \{1, 2, \ldots, n\}$. Note: there is no meaning to the word "towers" other than what is given.

   Solution: Consider $x \in \{1, 2, \ldots, n\}$. There are three cases for x:

   - $x \in A$
   - $x \in B$ (implying it is not in A)
   - $x \in \{1, 2, \ldots, n\}$

   With n possibilities for x, there are 3^n possible towers.

4. Show by a combinatorial argument that
$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

   Solution: Consider an arbitrary group n boys and n girls. Let S be the set of all pairs that we can form from these two groups.

   The LHS precisely counts this because there are $\binom{2n}{2}$ ways to pick pairs from $2n$ total people. Decompose S into the girl-girl pairs, boy-girl pairs, and boy-boy pairs. Clearly, there are $\binom{n}{2}$ ways to form pairs in both the girl-girl pairs and boy-boy pairs individually. In the case of boy-girl pairs, we have n choices for the first selection and n choices for the second. Thus, there are n^2 ways to form boy-girl pairs. In total this is $2\binom{n}{2} + n^2$ which is the RHS.

   Since the LHS and RHS count the same set, they are equivalent.

5. By counting the same set in two ways, prove the following identity:
$$\sum_{k=0}^{n} \binom{k}{m}\binom{n}{k} = \binom{n}{m} 2^{n-m}$$

   Solution: Let S be the set of all commitees we can form with a subcomittee of m members. The LHS counts the number of ways to form a committee and then form a subcomittee of m members across all n which is exactly S. The RHS forms the subcomittee first and then for the remaining $n-m$ people they are either in the commitee or not which is counted by $\binom{n}{m}2^{n-m}$. Since the LHS and RHS count the same set, they are equivalent.

9.3 Homework 3

1. You are an amazingly successful hotelier; one of your properties is "Chez Four-Thirteen". Suppose there are 8 distinguishable hotel rooms (numbered $101, 102, \ldots, 108$), each with maximum occupancy of 4 people. A bus comes in with anywhere from 1 to 32 (distinguishable) people, but you do not know exactly how many. While waiting, you decide to calculate (making use of your UIUC Math 413 education) the number of possible ways the people on the bus can spend the night in those hotel rooms so you can make a welcome sign. Determine this number. Use a computer to express the answer with all the digits (e.g., 31415926535897932384626433832795); write down for your check-in clerk the order of magnitude (e.g., 10^{31}). I recommend python to do the computation.

2. What is the probability that a poker hand contains exactly one pair (that is, a poker hand with exactly four different ranks?)

   Solution: There are $\binom{52}{5}$ possible poker hands of size 5. We can construct an algorithm to count the number of hands with exactly one pair.

   There are $\binom{13}{1}$ ways to choose the rank of the pair and $\binom{4}{2}$ ways to pick the actual cards in the hand. For the remaing cards in the hand, there are $\binom{12}{3}$ ways to pick ranks and $\binom{4}{1}$ choices for each rank selected. In total, the probability is:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}}$$

3. A bagel store sells six different kinds of bagels. Suppose you choose 15 bagels at random. What is the probability that your choice contains at least one bagel of each kind? If one of the bagels types is Sesame, what is the probability that your choice contains at least three Sesame bagels AND AT LEAST ONE BAGEL OF EACH KIND?

   Solution: In the first situation we are solving the question:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$$

   where each $x_i \geq 1$. This problem can equivalently rewritten such that

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 9$$

   where each $y_i \geq 0$. This has the solution:

$$\binom{9+6-1}{6-1} = \binom{14}{5}$$

   Similarly, for the second question we have

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$$

   where $x_1 \geq 3$ and $x_i \geq 1$ for $i \neq 1$. Equivalently written:

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 7$$

where each $y_i \geq 0$. This has a solution of

$$\binom{7+6-1}{6-1} = \binom{12}{5}$$

There are a total of $\binom{15+6-1}{6-1} = \binom{20}{5}$ ways to choose 15 bagels without restrictions. Thus, the probabilities, respectively, are:

$$\frac{\binom{14}{5}}{\binom{20}{5}} \quad \frac{\binom{12}{5}}{\binom{20}{5}}$$

4. Let n be a positive integer. Suppose we choose a sequence $i_1, i_2, \ldots, i_n$ of integers between 1 and n at random. What is the probability that the sequence contains exactly $n-2$ different integers?

   Solution: If $n-2$ integers are different that means either 1 digit is repeated 3 times or 2 digits are repeated 2 times. We can count the number of ways there are to form sequences of each type.

   - 1 digit is repeated 3 times:

   $$\binom{n}{1}\binom{n-1}{n-3}\frac{n!}{3!}$$

   - 2 digits are repeated 2 times:

   $$\binom{n}{2}\binom{n-2}{n-4}\frac{n!}{2!2!}$$

   In total, there are obviously $n!$ ways to create sequence. So, the probabilities is:

   $$\frac{\binom{n}{1}\binom{n-1}{n-3}\frac{n!}{3!} + \binom{n}{2}\binom{n-2}{n-4}\frac{n!}{2!2!}}{n!}$$

5. Suppose n people line up to get into q different clubs. How many ways are there to do it? (The people are distinguishable and the order people are in line matters.)

   Solution: This is simply stars and bars/string cutting problem. We have n people total distributed among q "boxes" where each box has at least 0 people in it. Thus, the number of ways to split this group up is:

   $$\binom{n+q-1}{q-1}$$

   However, the order of the line matters, so we must account for all permutations of the line:

   $$\binom{n+q-1}{q-1}n!$$

9.4 Homework 4

1. Use the pigeonhole principle to prove that the decimal expansion of a rational number $\frac{m}{n}$ eventually is repeating. For example,
$$\frac{34,478}{99,900} = 0.34512512512512512\cdots$$

2. A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 13 hours.

   Consider the sequence of days where she has studied for a_i hours on day i (cummulative).
$$1 \leq a_1 < a_2 < \cdots a_{37} \leq 60$$
   It follows then that:
$$14 \leq a_1 + 13 < a_2 + 13 < \cdots a_{37} + 13 \leq 73$$
   We want to show that there exists $a_i = a_j + 13$ such that $j \leq i$ as this would imply a sequence of days where she studied for 13 hours.

   Between the two sequences, there are clearly, $37 \cdot 2 = 74$ numbers ranging in value from 1 to 73. By the pigeonhole principle, there are 74 variables (pigeons) and 73 numbers (pigeonholes) so there must be two "variables", i.e. an a_i and $a_j + 13$ such that $a_i = a_j + 13$.

   Thus, the student must have studied for exactly 13 hours on some sequence of days.

3. Prove that, for any $n + 1$ integers $a_1, a_2, \cdots, a_{n+1}$, there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

   When dividing numbers by n, there are only n possible remainders $0, 1, 2, \cdots, n - 1$. Since we have $n + 1$ numbers, two of them must share the same remainder. Define the following:
$$\begin{aligned} a_i &= c_1 n + r \\ a_j &= c_2 n + r \end{aligned}$$
   It follows that:
$$\begin{aligned} a_i - b_j &= (c_1 n + r) - (c_2 n + r) \\ &= c_1 n - c_2 n \\ &= n(c_1 - c_2) \end{aligned}$$
   Thus, by the definition of division, n divides some $a_i - a_j$.

4. There are 100 people at a party. Each person has an even number (possibly zero) of acquaintances. Prove that there are three people at the party with the same number of acquaintances.

5. Suppose that the mn people of a marching band are standing in a rectangular formation of m rows and n columns in such a way that in each row each person is taller than the one to his or her left. Suppose that the leader rearranges the people in each column in increasing order of height from front to back. Show that the rows are still arranged in increasing order of height from left to right.

9.5 Homework 5

1. Use *combinatorial* reasoning to prove the identity (in the form given)

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

(*Hint*: Let S be a set with three distinguished elements a, b and c and count certain k-subsets o S.)

2. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m \binom{2m}{m} & \text{if } n = 2m \end{cases}$$

(*Hint*: For $n = 2m$, consider the coefficient of x^n in $(1-x^2)^n = (1+x)^n(1-x)^{-n}$.)

3. By integrating the binomial expansion, prove that, for a positive integer n,

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$

4. Let n and k be positive integers. Give a combinatorial proof of the identity

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k}$$

5. Let nand k be positive integers. Give a combinatorial proof that

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

9.6 Homework 6

1. Prove that the only antichain of $S = 1, 2, 3, 4$ of size 6 is the antichain of all 2-subsets of S.

2. Construct a partition of the subsets of $1, 2, 3, 4, 5$ into symmetric chains.

3. A talk show host has just bought 10 new jokes. Each night he tells some of the jokes. What is the largest number of nights on which you can tune in so that you never hear on one night at least all the jokes you heard on one of the other nights? (Thus, for instance, it is acceptable that you hear jokes 1, 2, and 3 on one night, jokes 3 and 4 on another, and jokes 1, 2, and 4 on a third. It is not acceptable that you hear jokes 1 and 2 on one night and joke 2 on another night.)

4. Prove the identity of Exercise 25 using the binomial theorem and the relation $(1+x)^{m_1}(1+x)^{m_2} = (1+x)^{m1+m2}$.

5. Prove that

$$\sum_{n_1+n_2+n_3=n} \binom{n}{n_1 \cdot n_2 \cdot n_3} (-1)^{n_1+n_2+n_3} = 1$$

where the summation extends over all nonnegative integral solutions of $n_1 + n_2 + n_3 = n$.

9.7 Homework 7

1. Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

that satisfy

$$1 \le x_1 \le 6, 0 \le x_2 \le 7, 4 \le x_3 \le 8, 2 \le x_4 \le 6$$

Solution: We first need to reformat the question with adjusted restrictions:

$$\begin{aligned} &y_1 + y_2 + y_3 + y_4 = 13 \\ &0 \le y_1 \le 5, 0 \le y_2 \le 7, 0 \le y_3 \le 4, 0 \le y_4 \le 4 \end{aligned}$$

This can be solved using the inclusion-exclusion principle:

Let S be the set of all solutions to the above equations where each $y_i \ge 0$. By stars and bars,

$$|S| = \binom{13+4-1}{13} = \binom{16}{13}$$

Let A_i be the set of all solutions such that $y_i \ge a_i$ where a_i is the exclusive upper integral bound for y_i (e.g. $y_1 \ge 6$)

$$\begin{aligned} |A_1| &= \binom{7+4-1}{7} = \binom{10}{7} \\ |A_2| &= \binom{5+4-1}{5} = \binom{8}{5} \\ |A_3| &= \binom{8+4-1}{8} = \binom{11}{8} \\ |A_4| &= \binom{8+4-1}{8} = \binom{11}{8} \\ |A_1 \cap A_2| &= 0 \\ |A_1 \cap A_3| &= \binom{2+4-1}{2} = \binom{5}{2} \\ |A_1 \cap A_4| &= \binom{2+4-1}{2} = \binom{5}{2} \\ |A_2 \cap A_3| &= \binom{0+4-1}{0} = \binom{3}{0} \\ |A_2 \cap A_4| &= \binom{0+4-1}{0} = \binom{3}{0} \\ |A_3 \cap A_4| &= \binom{3+4-1}{3} = \binom{6}{3} \end{aligned}$$

Note that the intersection of any 3 sets will have no solution.

Substituting this into the complementary form of the inclusion-exclusion principle, we get:

$$\begin{aligned} &\binom{16}{13} - \Bigg(\binom{10}{7} + \binom{8}{5} + \binom{11}{8} + \binom{11}{8} \\ &\quad - \binom{5}{2} - \binom{5}{2} - \binom{3}{0} - \binom{3}{0} - \binom{6}{3}\Bigg) \\ &= 96 \end{aligned}$$

2. Determine the number of permutations of $\{1, 2, \ldots, 9\}$ in which at least one odd integer is in its natural position. ***Solution:*** This is an application of the inclusion-exclusion principle.

   Let A_i be the set of all permutations such that i is in its natural position.

   We must calculate $|A_1 \cup A_3 \cup A_5 \cup A_7 \cup A_9|$ for the inclusion-exclusion principle.

   For the sizes of each set individually, the size of each set must be 8! because there is one option for the position that must be in place and then we must place the 8 remaining numbers. There are $\binom{5}{1}$ ways to pick the odd number in place.

   Following this pattern we can build the inclusion-exclusion principle as follows:

$$\begin{aligned}&= \binom{5}{1}8! - \binom{5}{2}7! + \binom{5}{3}6! - \binom{5}{4}5! + \binom{5}{5}4! \\ &= 157824\end{aligned}$$

3. What is the number of ways to place six nonattacking rooks on the 6-by-6 boards with forbidden positions as shown?

×	×				
×	×				
		×	×		
		×	×		
				×	×
				×	×

Solution: This is an application of the inclusion-exclusion principle.

Let S_n be the set of all boards with nonattacking rooks on an $n \times n$ board. Clearly, there are $n!$ boards within S_n. This can be observed by constructing an algorithm to place one rook at a time.

Let A_i be the set of all boards with exactly 1 rook in the forbidden in row i.

We must calculate

$$|A_1^c \cap A_2^c \cap \cdots \cap A_n^c|$$

Note that it is unecessary to consider each subset of A_i individually, but rather we should consider the sum of the selected subsets. This gives the following identity

$$\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k(n-k)!$$

where r_k is the number of ways to place k rooks in forbidden positions.

Substituting this into the complementary form of the inclusion-exclusion principle, we get:

$$\begin{aligned}&6! - (r_1(6-1)! - r_2(6-2)! + r_3(6-3)! - r_4(6-4)! + r_5(6-5)! - r_6(6-6)!) \\ &= 6! - r_1(5!) + r_2(4!) - r_3(3!) + r_4(2!) - r_5(1!) + r_6(0!) \\ &= 6! - r_1(5!) + r_2(4!) - r_3(3!) + r_4(2!) - r_5 + r_6\end{aligned}$$

All that remains is to calculate $r_1, r_2, \ldots, r_6$.

$$\begin{aligned}
r_1 &= 12 \\
r_2 &= \binom{3}{1} \cdot (2) + \binom{3}{2}(4 \cdot 4) = 3 \cdot 2 + 3 \cdot 16 = 6 + 48 = 54 \\
r_3 &= \binom{3}{2}\binom{2}{1} \cdot (2 \cdot 4) + \binom{3}{3}(4 \cdot 4 \cdot 4) = 3 \cdot 2 \cdot 2 \cdot 4 + 1 \cdot 64 = 48 + 64 = 112 \\
r_4 &= \binom{3}{2}\binom{2}{1} \cdot 2 + \binom{3}{1} \cdot 2 \cdot 4 \cdot 4 = 3 \cdot 2 \cdot 2 \cdot 2 + 3 \cdot 2 \cdot 4 \cdot 4 = 12 + 96 = 108 \\
r_5 &= \binom{3}{2}\binom{2}{1} \cdot 2 \cdot 4 = 48 \\
r_6 &= 2 \cdot 2 \cdot 2 = 8
\end{aligned}$$

Substituting these values into the formula, we get:

$$6! - (12(5!) - 54(4!) + 112(3!) - 108(2!) + 48 - 8) = 80$$

9.8 Homework 8

1. How many ways are there to arrange the letters in INTELLIGENT with at least two pairs of consecutive identical letters? (For example ITTNELLGENI is an arrangement we want to count, since it has "TT" and "LL".)

Solution: This is an application of the inclusion-exclusion principle.

Let A_i be the set of all arrangements with exactly i pairs of consecutive identical letters.

We must calculate

$$|A_2 \cup A_3 \cup \cdots \cup A_5|$$

For A_2, there are $\binom{5}{2}$ ways to choose the two pairs of consecutive identical letters. There are 7 remaining letters with 3 remaining pairs which can be arranged in $\frac{7!}{2!2!2!}$ ways.

For A_3, there are $\binom{5}{3}$ ways to choose the three pairs of consecutive identical letters. There are 5 remaining letters with 2 remaining pairs which can be arranged in $\frac{5!}{2!2!}$ ways. However, we must consider that the previous calculation A_2 double counted elements of A_3 by the number of pairs that were chosen, which was 2. We must multiply this in to account for this.

Following this pattern, we can see the final answer is

$$\binom{5}{2}\frac{9!}{(2!)^3} - 2 \cdot \binom{5}{3}\frac{8!}{(2!)^2} + 3 \cdot \binom{5}{4}\frac{7!}{(2!)} - 4 \cdot \binom{5}{5}6!$$

Note:-

Could probably use better explanations on why we need to multiply by the number of pairs chosen in the previous calculation.

2. Let n be a positive integer and let $p_1, p_2, \ldots, p_n$ be all of the different prime numbers that divide n. Consider the Euler function ϕ defined by

$$\phi(n) = \left|\{k : 1 \leq k \leq n, \gcd(k, n) = 1\}\right|$$

Use the inclusion-exclusion principle to show that

$$\phi(n) = n \prod_{i=1}^{n} 1 - \frac{1}{p_i}$$

3. Prove that the Fibonacci sequence is the solution of the recurrence relation

$$a_n = 5a_{n-4} + 3a_{n-5}, (n \geq 5)$$

where $a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3$. Then use this formula to show that the Fibonacci numbers satisfy the condition that f_n is divisble by 5 if and only if n is divisible by 5.

4. Let h_n equal the number of different ways in which the squares of a $1 \times n$ chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n (using the characteristic equation method).

5. The *Lucas numbers* $l_0, l_1, l_2, \ldots, l_n, \ldots$ are defined using the same recurrence relation defining the Fibonacci numbers, but with different intial conditions:

$$l_n = l_{n-1} + l_{n-2}, (n \geq 2) \quad l_0 = 2, l_1 = 1$$

Prove that

- $l_n = f_{n-1} + f_{n+1}$ for $n \geq 1$
- $l_0^2 + l_1^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$ for $n \geq 0$

6. Let a_n equal the number of ternary strings of length n made up of 0s, 1s, and 2s such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$a_n = a_{n-1} + 2a_{n-2}$$

for $n \geq 2$ with $a_0 = 1$ and $a_1 = 3$. Then find a formula for a_n.

7. A population grows as follows: in the current generation, each Canadian chooses one American for the next generation, and each American chooses one Canadian and one American for the next generation. Generation 0 consists of one Canadian. What is the size of generation n? (Here we assume all older generations leave when the new generation is chosen.)

9.9 Homework 9

1. Prove that the partition function $p(n)$(= number of partitions of n) satisfies $p(n + 1) > p(n)$.

Solution: By the definition of a partition, we know there are $p(n)$ sequences of decreasing positive integers that sum to n.

For each sequence, we can append a 1 to the end to construct a new sequence, which will still be decreasing and sum to $n + 1$. Therefore, we know that $p(n + 1)$ is *at least* $p(n)$.

Additionally, we can construct a new partition of the sequence $\{n + 1, 0, 0, \ldots\}$. This will be a partition of $n + 1$ trivially. Additionally, this partition cannot be a partition of any integer less than $n + 1$ as the sum is above n. Since, this partition is not in n, we know that $p(n + 1) > p(n)$.

2. For each integer $n > 2$ determine a self-conjugate partitoin of n that has at least two parts.

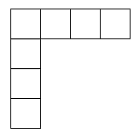
   Solution: There are two cases to consider:

   - n is odd

     We can construct the follwing partition of n:

     $$\{\frac{n-1}{2} + 1, 1_1, 1_2, \dots, 1_{\frac{n-1}{2}}\}$$

     where there are $\frac{n-1}{2}$ 1s. This is self-conjugate as it will create Ferrer Diagrams of the form:

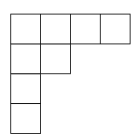
     

     Since $n > 2$, we know that there are at least two parts. At $n = 3$, $\frac{n-1}{2} + 1 = 2$ and $\frac{n-1}{2} = 1$. So the partition is $\{2, 1\}$.

     This is trivially bijective because $\frac{n-1}{2}$ is bijective.

   - n is even

     We know that $n - 1$ is odd. So we can construct a sel-conjugate partition for $n - 1$ and then modify the generated sequence by changing one of the appended 1s to a 2.
     This generates Ferrer Diagrams of the form:

     We have shown in the odd case that this is a bijection and partition, so we know that this is a self-conjugate partition of n because adding 1 to one of the parts is still a bijective action.

3. Prove that the number of partitions of n in which no part appears exactly once is equal to the number of partitions of n with no parts congruent to 1 or 5 (mod 6).

4. Do part (c) of the last in class exercise of Lecture 19. That is, give a bijective proof that the number of partitions of N using distinct parts equals the number of partitions of N using only odd parts. (Hint: Consider the multiplicity m_p of each part p occuring in partitions of the second type. Now look at the unique expansion of m as powers of 2 and distribute.)

   Solution: Consider an arbitrary partition P that only uses odd parts for N:

   $$N = o_1 + o_2 + \cdots + o_k$$

   Group each odd number to obtain its multiplicity m_p (e.g. $1 + 1 + 1 = 3(1)$):

   $$N = m_1(o_1) + m_2(o_2) + \cdots + m_k(o_k)$$

   Consider the unique expansion of m_p as powers of 2:

$$m_p = 2^0 + 2^1 + \cdots + 2^r$$

Distribute each o_i across the expansion of m_p:

$$\begin{aligned} &= (2^0 + 2^1 + \cdots + 2^{r_1})o_1 + (2^0 + 2^1 + \cdots + 2^{r_2})o_2 + \cdots + (2^0 + 2^1 + \cdots + 2^{r_k})o_k \\ &= 2^0(o_{\alpha_1} + o_{\alpha_2} + \cdots + o_{\alpha_x}) + 2^1(o_{\beta_1} + o_{\beta_2} + \cdots + o_{n_2}) + \cdots + 2^r(o_{\gamma_y} + o_{\gamma_2} + \cdots + o_{\gamma_z}) \end{aligned}$$

Let a_i be the sum of the o_is that correspond to the 2^i term:

$$= 2^0 a_0 + 2^1 a_1 + \cdots + 2^r a_r$$

We know that each of these terms must be unique as each power of 2 is unique in this sequence.

Thus, we have constructed a bijection between the partitions of N using only odd parts and the partitions of N using distinct parts.

9.10 Homework 10

1. By considering partitions with distinct (that is, non-repeated) parts, prove that

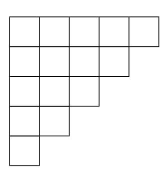
$$\prod_{k\geqslant 1}(1 + x^k) = 1 + \sum_{m\geqslant 1} \frac{x^{m(m+1)/2}}{\prod_{k=1}^{m}(1 - x^k)}$$

(Hint: look for a "maximal triangle" rather than a Durfee square.)

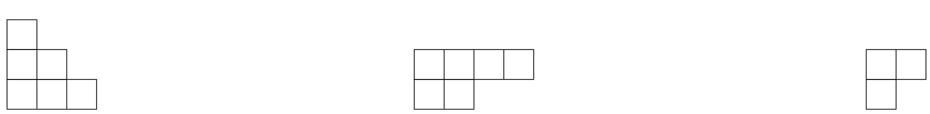
Consider a similar approach to the Durfee Square decomposition. We know the generating series for distinct parts is

$$\prod_{k\geq 1}(1 + x^k)$$

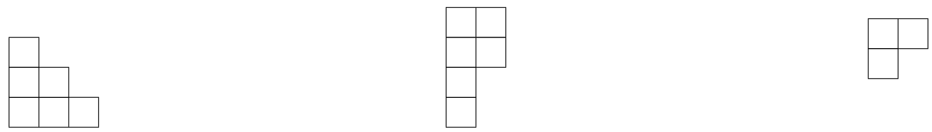
Consider an arbitrary partition P with distinct parts:



This can clearly be decomposed into a unique triple with a maximal triangle with side lengths m:



We can construct a weight preserving bijection by conjugating the second component.



We can construct another weight preserving bijection by merging the second and third component as follows

Suppose this tuple is TD (triangle-decomposed).

$$TD \in \left(T_m, \text{Partitions}_{\leq m \text{ wide}}\right) = S$$

A maximal triangle of size m will have $\frac{m(m+1)}{2}$ "parts" as it can be constructed by constructing a square of $m \times m+1$ and removing the top right part to leave the maximal triangle. Additionally, partitions with no part bigger than m (i.e. a Ferrer Diagram with a maximum width of m) is represented by the generating series $\prod_{k=1}^{m} \frac{1}{1-x^k}$. In total, it is clear that

$$\begin{aligned} g(x) &= \sum_{m\geq 1}\left(\left(x^{\frac{m(m+1)}{2}}\right)\left(\prod_{k=1}^{m}\frac{1}{1-x^k}\right)\right) \\ &= \sum_{m\geq 1}\frac{x^{\frac{m(m+1)}{2}}}{\prod_{k=1}^{m} 1-x^k} \end{aligned}$$

However, this bijection process does not hold for when $n = 1$ as there will not be any element in the second part of the tuple. So, we must account for the 1 missing element that is not being counted in this generating series. Thus,

$$\prod_{k\geq 1}(1+x^k) = 1 + \sum_{m\geq 1}\frac{x^{\frac{m(m+1)}{2}}}{\prod_{k=1}^{m} 1-x^k}$$

2. Determine the number of n digit numbers with all digits even, such that 2 and 4 each occur a nonzero, odd number of times.

   We can treat this as number of ways to n objects (digits) into 5 buckets ($\{0, 2, 4, 6, 8\}$ where the number of objects must be nonzero and odd. This can be represented by the generating series:

$$g_{exp}(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^2$$

   This can be simplified as follows:

$$\begin{aligned}
g_{exp}(x) &= \left(e^x\right)^3\left(\frac{e^{-x}-e^x}{-2}\right)^2 \\
&= \left(e^{3x}\right)\frac{(e^{-x}-e^x)^2}{4} \\
&= \frac{(e^{3x})(e^{-2x}-e^0-e^0+e^{2x})}{4} \\
&= \frac{e^{5x}+e^x-2e^{3x}}{4}
\end{aligned}$$

We are solving for $[x^n]g_{exp}(x)$.

$$[x^n]g(x) = \frac{5^n+1-2\cdot 3^n}{4}$$

3. Determine the exponential generating series for the number of ways to put (distinct) people into n rooms where each room has at least five people. Now determine the actual answer for 30,50,100 people in $n = 5$ rooms (hint: use a computer or Wolfram Alpha).

   This is clearly recognized by the generating series:

$$\begin{aligned}
g_{exp}(x) &= \left(\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots\right)^n \\
&= \left(\left(1+x+\frac{x^2}{2!}+\cdots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\right)^n \\
&= \left(e^x-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\right)^n
\end{aligned}$$

   Let $n = 5$.

   - 30 people: 110145542812228123200
   - 50 people: 806780866916589778834164156134970000
   - 100 people: 100! * Number1/Number2 (See last page) (Number is too large, didn't include on this document)

4. Show that $\frac{e^x}{(1-x)^n}$ is the exponential generating function for the number of ways to choose some subset (possibly empty) of r distinct objects and distribute them into n different boxes with the order in each box counted.

   Let A be the set of choosing r distinct objects and placing them into one box. This is clearly represented by the generating series:

$$g_1(x) = e^x$$

   Similarly, let B be the set of the chosen r elements and the ways they can be placed into n different boxes. This is clearly represented by the generating series

$$g_2(x) = \frac{1}{(1-x)}$$

Clearly, $S = A \star B$ which is the star product and thus, we can clearly see that the number of ways to choose a subset of r and distribute r elements into n boxes is counted by S which has a generating series:

$$g(x) = \frac{e^x}{(1-x)^n}$$

5. Find the exponential generating series for the number of functions $f : A \to B$ where $A = \{1, 2, ..., n\}$ and $B = \{1, 2, ..., m\}$ such that the preimage set of every element of B is at least size four, i.e., $\#f^{-1}(j) \geqslant 4$ for each $j \in B$.

$$g(x) = \left(e^x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})\right)^m$$

Chapter 10

Midterms

Note: The listed questions on exams are more questions than what was on the exam. Typically, the exams have 5 questions, 6 points each, however between two sections of the course, there are two variants of the exam.

10.1 Midterm 1

10.1.1 Practice Problems

10.1.2 Solutions

10.1.3 Exam

1. Determine the number of integral solutions

$$x_1 + x_2 + x_3 + x_4 + x_5 = 413$$

   subject to

$$x_1 \geq 40, x_2 \geq 0, x_3 \geq 20, x_4 \geq -15, x_5 \geq -35$$

2. Give a COMBINATORIAL PROOF of the following identity (i.e., double count a set). A non-combinatorial proof will receive no credit.

$$\sum_{k=0}^{n} \binom{n}{k} 412^{n-k} = 413^n$$

3. Count the number of ways to place four nonattacking rooks on a 8×8 chessboard such that neither the first row nor the first column is empty.

4. Remember this problem from class: *Let S be the set of permutations of at most m A's and at most n B's. The claim is*

$$\#S = \binom{m+n+2}{m+1} - 1$$

   Precisely define S' which "obviously" has cardinality $\binom{m+n+2}{m-1} - 1$ and giving a bijection $f : S \to S'$. Explain why your claimed bijection is WELL-DEFINED, i.e., given $w \in S'$ that indeed $f(w) \in S$. You do NOT need to show f is injective or surjective here.

5. Yihong and Zhuo are solving the Homework 2:

   > *In how many ways can two red and four blue rooks be placed on an 8×8 chessboard so that no two rooks can attack each other?*

6. What fraction of all of the arrangements of EFFLORESCENCE has consecutive Cs, consecutive Fs but no consecutive Es? For example ECCLROESFFENE is one such arrangement.

7. Give a COMBINATORIAL proof of the following identity:

$$\binom{2}{2}\binom{n}{2} + \binom{3}{2}\binom{n-1}{2} + \binom{4}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\binom{2}{2} = \binom{n+3}{5}$$

10.1.4 Exam Solutions

10.2 Midterm 2

10.2.1 Exam

10.2.2 Exam Solutions

10.3 Midterm 3

10.3.1 Practice Problems

1. Find a recurrence relation for the number of different ways to hand out a piece of chewing gum (worth 1 cent) or a candy bar (worth 10 cents) or a donut (worth 20 cents), one item a day, until n cents worth of food has been given away. Don't worry about the initial condition.

   Solution: Let $f(n)$ be the number of ways to hand out a piece of chewing gum, a candy bar, or a donut with n cents. At any moment with n cents, we can either give a piece of gum, a candy bar, or a donut. This gives

$$f(n) = f(n-1) + f(n-10) + f(n-20)$$

2. By using generating series, determine the number of integer solutions to the composition problem

$$e_1 + e_2 + e_3 = 22$$

subject to $3 \leq e_1 \leq 8$, $6 \leq e_2 \leq 10$, and $2 \leq e_3 \leq 7$.

Solution: Clearly this can be solved by the following generating series:

$$g_1(x) = (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)(x^6 + x^7 + x^8 + x^9 + x^{10})(x^2 + x^3 + x^4 + x^5 + x^6 + x^7)$$

where the solution is $[x^{22}]g(x)$. However, this is difficult to extract a coefficient from. Instead, rewrite the problem in terms of new variables:

$$e_1 + e_2 + e_3 = 11$$

where $0 \leq e_1 \leq 5$, $0 \leq e_2 \leq 4$, and $0 \leq e_3 \leq 5$. The generating series now becomes:

$$g_2(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4 + x^5)$$

where we are solving for $[x^{11}]g_2(x)$. This generating series can be simplified as follows:

$$
\begin{aligned}
g_2(x) &= (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4 + x^5) \\
&= (\frac{1 - x^6}{1 - x})(\frac{1 - x^5}{1 - x})(\frac{1 - x^6}{1 - x}) \\
&= \frac{(1 - x^6)(1 - x^5)(1 - x^6)}{(1 - x)(1 - x)(1 - x)} \\
&= \frac{(1 - x^6)(1 - x^5)(1 - x^6)}{(1 - x)^3} \\
&= (1 - x^6)(1 - x^5)(1 - x^6)(1 - x)^{-3} \\
&= (1 - x^5 - x^6 + x^{11})(1 - x^6)(1 - x)^{-3} \\
&= (1 - x^5 - x^6 + x^{11} - x^6 + x^{11} + x^{12} - x^{17})(1 - x)^{-3} \\
&= (1 - x^5 - 2x^6 + 2x^{11} + x^{12} - x^{17})(1 - x)^{-3}
\end{aligned}
$$

The target solution is $[x^{11}]g_2(x)$. Using negative binomial theorem, we can find the coefficient of x^{11} as follows:

Reminder that negative binomial theorem is:

$$
(1 - x)^{-n} = \sum_{k=0}^{\infty} \binom{n + k - 1}{k} x^k
$$

$$
\begin{aligned}
[x^{11}]g_2(x) &= [x^{11}]((1 - x^5 - 2x^6 + 2x^{11} + x^{12} - x^{17})(1 - x)^{-3}) \\
&= [x^{11}]\Big((1 - x)^{-3} - x^5(1 - x)^{-3} - 2x^6(1 - x)^{-3} + 2x^{11}(1 - x)^{-3} + x^{12}(1 - x)^{-3} - x^{17}(1 - x)^{-3}\Big) \\
&= [x^{11}](1 - x)^{-3} - [x^{11}]x^5(1 - x)^{-3} + [x^{11}]2x^6(1 - x)^{-3} + [x^{11}]2x^{11}(1 - x)^{-3} + [x^{11}]x^{12}(1 - x)^{-3} - [x^{11}]x^{17}(1 - x \\
&= [x^{11}](1 - x)^{-3} - [x^6](1 - x)^{-3} - 2[x^5](1 - x)^{-3} + 2[x^0](1 - x)^{-3} + [x^{-1}](1 - x)^{-3} - [x^{-6}](1 - x)^{-3} \\
&= \binom{3 + 11 - 1}{3 - 1} - \binom{3 + 6 - 1}{3 - 1} - 2\binom{3 + 5 - 1}{3 - 1} + 2\binom{3 + 0 - 1}{3 - 1} + 0 + 0 \\
&= \binom{13}{2} - \binom{8}{2} - 2\binom{7}{2} + 2\binom{2}{2}
\end{aligned}
$$

3. Prove that the number of partitions where no part appears more than two times equals the number of partitions where no part is a multiple of three. (Hint: write down the generating series for the former type of partition and algebraically manipulate to get to the generating series for the latter type.)

   Solution: Number of partitions of n where no part appears more than two times:

$$
\prod_{i=1}^{\infty}(1 + x^i + x^{2i})
$$

Number of partitions of n where no part is a multiple of three:

$$
\begin{aligned}
\frac{\prod_{i=1}^{\infty} \frac{1}{1-x^i}}{\prod_{i=1}^{\infty} \frac{1}{1-x^{3i}}} &= \prod_{i=1}^{\infty} \frac{\frac{1}{1-x^i}}{\frac{1}{1-x^{3i}}} \\
&= \prod_{i=1}^{\infty} \frac{1 - x^{3i}}{1 - x^i}
\end{aligned}
$$

Algabraic proof:

Proof.

$$\begin{aligned}\prod_{i=1}^{\infty}(1+x^i+x^{2i}) &= \prod_{i=1}^{\infty}\frac{(1+x^i+x^{2i})(1-x^i)}{1-x^i}\\ &= \prod_{i=1}^{\infty}\frac{1+x^i+x^{2i}-x^i-x^{2i}-x^{3i}}{1-x^i}\\ &= \prod_{i=1}^{\infty}\frac{1-x^{3i}}{1-x^i}\end{aligned}$$

■

4. Determine the number of permutations of $\{1,2,3,4,5,6,7,8\}$ in which no even integer is in its natural position. For example, 43258176 is good, but 42358176 is bad because of the position of the 2.

   Solution: This is an application of inclusion-exclusion principle.

   Let S be the set of all permutations of $\{1,2,3,4,5,6,7,8\}$. Clearly, $|S| = 8!$.

   Let A_i be the subset of permutations in S where i is in its natural position. We are looking to calculate the following by the complementary form of the inclusion-exclusion principle:

$$|A_2^c \cap A_4^c \cap A_6^c \cap A_8^c| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_2 \cap A_4 \cap A_6 \cap A_8|$$

   Each sum can be calculated as follows:

$$\sum |A_{i_1} \cap A_{i_2} \cdots A_{i_k}| = \binom{4}{k}(8-k)!$$

   This is because there are 4 ways to choose the even numbers that are fixed. For each of these choices, there are $8-k$ ways to choose the remaining numbers.

   So, we have:

$$8! - \binom{4}{1}7! + \binom{4}{2}6! - \binom{4}{3}5! + \binom{4}{4}4!$$

5. Define what it means for a pair (S,ω) means to be combinatorial problem. Define generating series in these terms. State and prove the product lemma. (As usual, on the test I might ask a **very precise** question about these proofs (or any proof) so you must be able to understand each line and symbol used in the argument.)

10.3.2 Exam

10.3.3 Exam Solutions