MATH 413 **Name:** *Amit Sawhney*
Homework 7
October 2022

1. Question 9

   We introduce new variables

   $$y_1 = x_1 - 1, y_2 = x_2, y_3 = x_3 - 4, y_4 = x_4 - 2$$

   and our equation becomes

   $$y_1 + y_2 + y_3 + y_4 = 20 - 1 - 4 - 2 = 13.$$

   The inequalities on the x_i's side are satisfied if and only if

   $$0 \leq y_1 \leq 5, 0 \leq y_2 \leq 7, 0 \leq y_3 \leq 4, 0 \leq y_4 \leq 4.$$

   Let S be the set of all nonnegative integral solutions of the previous equation. The size of S is

   $$|S| = \binom{13+4-1}{13} = 560.$$

   Let P_1 be the property that $y_1 \geq 6$, P_2 the property that $y_2 \geq 8$, P_3 the property that $y_3 \geq 5$, and P_4 the property that $y_4 \geq 5$. Let A_i denote the subset of S consisting of the solutions satisfying property P_i, $(i = 1, 2, 3, 4)$. We wish to evaluate the size of the set $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}$, and we do so by applying the inclusion-exclusion principle. The set of A_1 consists of all those solutions in S for which $y_1 \geq 6$. Performing a change in variable $(z_1 = y_1 - 6, z_2 = y_2, z_3 = y_3, z_4 = y_4)$, we see that the number of solutions in A_1 is the same as the number of nonnegative integral solutions of

   $$z_1 + z_2 + z_3 + z_4 = 7$$

   Hence,

   $$|A_1| = \binom{10}{7} = 120$$

   In a similar way, we obtain

   $$|A_2| = \binom{8}{5} = 56, |A_3| = \binom{11}{8} = 165, |A_4| = \binom{11}{8} = 165.$$

   The set of $A_1 \cap A_2$ consists of all those solutions in S for which $y_1 \geq 6$ and $y_2 \geq 8$. Performing a change in variable $(u_1 = y_1 - 6, u_2 = y_2 - 8, u_3 = y_3, u_4 = y_4$, we see that the number of solutions in $A_1 \cap A_2$ is the same as the number of nonnegative integral solutions of

   $$u_1 + u_2 + u_3 + u_4 = -1.$$

Hence,

$$|A_1 \cap A_2| = \binom{2}{-1} = 0.$$

Similarly, we get

$$|A_1 \cap A_3| = \binom{5}{2} = 1-, |A_1 \cap A_4| = \binom{5}{2} = 10$$
$$|A_2 \cap A_3| = \binom{3}{0} = 1, |A_2 \cap A_4| = \binom{3}{0} = 1$$
$$\text{and } |A_3 \cap A_4| = \binom{6}{3} = 20.$$

The intersection of any three of the sets A_1, A_2, A_3, A_4 is empty. We now apply the inclusion-exclusion principle to obtain

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 560 - (120 + 56 + 165 + 165) + (0 + 10 + 10 + 1 + 1 + 20) = 96.$$

Citation: Page 171-172 of the book.

2. Question 13

   Let A_1, A_3, A_5, A_7, A_9 be the sets of permutations where $1, 3, 5, 7, 9$ are respectively in their natural position. We then apply the complementary form of the inclusion-exclusion principle. In this situation it reads

$$|A_1^c \cup A_3^c \cup A_5^c \cup A_7^c \cup A_9^c| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k|$$
$$- \sum |A_i \cap A_j \cap A_k \cap A_l| + |A_1 \cap A_3 \cap A_5 \cap A_7 \cap A_9|$$

   where the sums are taken in the obvious way.

   We have

$$|A_i| = (9-1)!$$
$$|A_i \cap A_j| = (9-2)!$$
$$|A_i \cap A_j \cap A_k| = (9-3)!$$
$$|A_i \cap A_j \cap A_k \cap A_l| = (9-4)!$$

   and

$$|A_1 \cap A_3 \cap A_5 \cap A_7 \cap A_9| = (9-5)!$$

   Hence the answer is $\binom{5}{1}8! - \binom{5}{2}7! + \binom{5}{3}6! - \binom{5}{4}5! + 4!$

3. Question 24b

The answer is based on the formula:

$$n! - r_1(n-1)! + r_2(n-2)! - \cdots + (-1)^k r_l(n-k)! + \cdots (-1)^n r_n$$

$$r_1 = 12$$
$$r_2 = 2 + 4(4) + 4(4) + 2 + 4(2) + 4(2) + 2 = 54$$
$$r_3 = 112$$
$$r_4 = 44$$
$$r_5 = 48$$
$$r_6 = 8$$

Answer: $6! - 12 \cdot 5! + 54 \cdot 4! - 102 \cdot 3! + 44 \cdot 2! - 48 + 8$

Citation: https://www.math.hkust.edu.hk/ mabfchen/Math3343/Homework3.pdf

(I couldn't derive all of the $r_1 - -r_6$ in time so take off points as necessary). I made an attempt to break up each set of numbers into their cases (e.g. there are $\binom{6}{3}$ ways to pick 3 unordered numbers from 6 options $(1, 2, 3, 4, 5, 6)$. And then I tried to break up how many of those cases generated 4 possibilities and then how many generated 8 and create a sum, but the closest I get from r_2 is 112 which appears to be wrong. Attempted the same idea for $4, 5$ and couldn't get the right answer for those either so for r_3, r_4, r_5 I wasn't able to count those myself).