MATH 413 **Name:** *Amit Sawhney*
Homework 5
10/10/2022

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1. Question 11

   Let S be a subset with three distinguished elements a, b, c. The number of k subsets in S is $\binom{n}{k}$. Additionally, the number of k subsets in $S \setminus \{a, b, c\}$ is $\binom{n-3}{k}$. Clearly, the number of subsets that contain at least $\{a, b, c\}$ is $\binom{n}{k} - \binom{n-3}{k}$ which is the LHS. Decompose S into three types of subsets. The first type of subset only contains c, The second type of subset can contain b but not a. And the the third type of subset can contain a. For type 1, there are $\binom{n-3}{k-1}$ subsets, For type 2, there are $\binom{n-2}{k-1}$ subsets. For type 3, there are $\binom{n-1}{k-1}$ subsets. When added together, this is the RHS.

2. Question 12

3. Question 16

   Consider the formula for binomial expansion:

   $$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

   Integrate both sides of this equation from 0 to 1 over x as follows:

   LHS:

   $$\int_0^1 (1+x)^n = \frac{(1+x)^{n+1}}{n+1}\bigg|_0^1 = \frac{(1+1)^{n+1}}{n+1} - \frac{(1)^{n+1}}{n+1} = \frac{2^{n+1}-1}{n+1}$$

   RHS:

   $$\int_0^1 \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} \int_0^1 x^k = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x^{k+1}}{k+1}\bigg|_0^1 \right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{(1)^{k+1}}{k+1} - \frac{(0)^{k+1}}{k+1} \right)$$

   $$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{k+1} \right) = \binom{n}{0} + \frac{1}{2} \cdot \binom{n}{1} + \frac{1}{3} \cdot \binom{n}{2} + \cdots \binom{n}{n} \cdot \frac{1}{n+1}$$

   Clearly, this proves the identity.

4. Question 27

   Consider S to be a set of all of the ways to pick a team of size k from n people with 1 director or 2 directors.

   (a) We can count this by picking the teams first and picking directors from the team.

   Construct an algorithm that picks a team with k people. For each k member team, there are k ways to pick a singular director and $k(k-1)$ ways to pick co-directors. Considering all possible sizes of k, there are:

$$\sum_{k=1}^{n} \binom{n}{k} \cdot k(k-1) + k = \sum_{k=1}^{n} \binom{n}{k} \cdot k^2 - k + k = \sum_{k=1}^{n} k^2 \binom{n}{k}$$

ways to do this. This is the RHS.

(b) We can count this by picking the directors first and then picking who is on each team of directors.

There are n ways to choose a singular director. For the remaining people, they are either on the directors team or not, so there are 2^{n-1} possibilities for the remaining people. If there are 2 directors, there are $n(n-1)$ ways to form 2 co-directors with n people. From the remaining people they, are either on the directors team or not, so there are 2^{n-2} ways to do this for the remaining $n-2$ people.

In total,

$$n2^{n-1} + n(n-1)2^{n-2} = n(n+1)2^{n-2}$$

ways to choose these teams, which is the LHS.

5. Question 28

For clarity, this question can be written as,

$$\sum_{k=1}^{n} k \binom{n}{k} \binom{n}{k} = n \binom{2n-1}{n-1}$$

Imagine there are $2n$ people trying out for a football team where only juniors and seniors can try out. The coach must select n people and 1 captain (where only a senior can be captain). Let S be the set of teams with a distinct captain that can be formed.

(a) LHS

Decompose this set S into the set of n juniors and n seniors. For each size k from 0 to n, we can select k juniors and then k seniors. From this selection, there are k choices for the captain. Mathematically, there are $\binom{n}{k}$ possibilities for the juniors, then $\binom{n}{k}$ possibilities for the seniors. Lastly, there are k possibilities for the captain as there are k seniors, so

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k} k$$

(b) RHS

Once again, decompose this set into n juniors and n seniors. There are n seniors and so there are n choices for the captain. After the captain is selected, select the remaining $n-1$ people. Mathematically, there are $\binom{2n-1}{n-1}$ ways to choose the remaining people. So,

$$n \binom{2n-1}{n-1}$$

ways to count S with this algorithm. This is the RHS.