MATH 413
Introduction to Combinatorics

Amit Sawhney

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7.1 Lecture 17: Some Number Sequences

Example 7.1.1 (Example 1)

Consider a configuration of n lines where every two lines have a point in common, but no three do. How many regions in the plane are there? Give a recurrence.

$$a_n = a_{n-1} + n$$

TODO: Give an explanation of why this is true.

Example 7.1.2 (Example 2)

Give a simple recurrence for dearragements.

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

TODO: Give an explanation of why this is true. Need to review dearragements from previous lecture.

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$.

Definition 7.1.1: The adjusted Fibonacci sequence: $\hat{F}_n$

This is the number of 1,2 lists of size n. In other words, consider the number of ways a valet can park A cars (size 1) and B cars (size 2) in a parking lot of size n.

$$\hat{F}_n = \begin{cases} 1 & \text{if } n = 0 \\ f_{n+1} & \text{otherwise} \end{cases}$$

Question 1

Prove

$$\sum_{n=0}^{n} f_i = f_{n+2} - 1$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$\sum_{i=0}^{0} f_i = f_0 = 0$$

Similarly,

$$f_{0+2} - 1 = f_2 - 1 = 1 - 1 = 0$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$\sum_{i=0}^{k} f_i = f_{k+2} - 1$$

Indcutive Step: We will prove that the following statement is true for $n = k + 1$.

$$\sum_{i=0}^{k+1} f_i = f_{k+1} + \sum_{i=0}^{k} f_i = f_{k+1} + f_{k+2} - 1 = f_{k+3} - 1$$

Therefore, the statement is true for all n by induction. ∎

Question 2

Prove

$$1 + \sum_{i=0}^{n} \hat{F}_i = \hat{F}_{n+2}$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$1 + \sum_{i=0}^{0} \hat{F}_i = 1 + \hat{F}_0 = 1 + 1 = 2$$

Similarly,

$$\hat{F}_{0+2} = \hat{F}_2 = f_{2+1} = f_3 = 2$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$1 + \sum_{i=0}^{k} \hat{F}_i = \hat{F}_{k+2}$$

Inductive Step: We will prove that the following statement is true for $n = k + 1$.

$$\begin{aligned} 1 + \sum_{i=0}^{k+1} \hat{F}_i &= 1 + \hat{F}_{k+1} + \sum_{i=0}^{k} \hat{F}_i \\ &= \hat{F}_{k+1} + \hat{F}_{k+2} \\ &= f_{k+2} + f_{k+3} \\ &= f_{k+4} \\ &= \hat{F}_{k+3} \end{aligned}$$

Therefore, the statement is true for all n by induction. ■

Question 3

Prove that f_n is even if and only if n is divisible by 3.

Solution:

Proof. Given that $f_0 = 0$, $f_1 = 1$, and $f_2 = 1$, we can see that at $n = 3$, $f_3 = 2$, which is even.

This is because the only way to get an even number is to have the parity of the two numbers added togethed (odd + odd or even + even) be the same. So, f_4, must be odd, f_5 must be odd and f_6 must be even.

Given the starting sequence of even, odd, odd. The following sequence must always be even, odd, odd, which repeats every 3 numbers.
Since the first $n = 0$ is the first number in the sequence, every n that is divisible by 3 is even. ■

Note:-

Example problems for later

Guess and prove by induction (you may replace the Fibonnaci number by the adjusted Fibonacci number if it helps you)

- $f_1 + f_3 + \cdots + f_{2n-1} =?$
- $f_0 + f_2 + \cdots + f_{2n} =?$
- $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n =?$
- $(f_0)^2 + (f_1)^2 + \cdots + (f_n)^2 =?$

Obtaining an explicit formula for f_n for linear recurrences

Example 7.1.3

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$. This can be rewritten as a linear recurrence as follows:

$$f_n - f_{n-1} - f_{n-2} = 0$$

We must solve the corresponding characterstic equation. Notice how the largest degree lines up with the "largest" case of the recurrence.

$$x^2 - x - 1 = 0$$

Let q_1 and q_2 be the roots of the characteristic equation.

It is potentially relevant to note that the following is a solution space of the Fibonacci recurrence (but don't satisfy f_0 – the initial condition):

$$\begin{cases} q_1^n - q_1^{n-1} - q_1^{n-2} = 0 \\ q_2^n - q_2^{n-1} - q_2^{n-2} = 0 \end{cases}$$

The rest of this is based on an ansatz, i.e. we need to make an assumption at the answer and validate it later

$$f_n = c_1 q_1^n + c_2 q_2^n,$$

for *some* $c_1, c_2 \in \mathbb{R}$.

Using the initial conditions of $f_0 = 0$ and $f_1 = 1$, we can solve for c_1 and c_2.

7.2 Lecture 18: Introduction to ordinary generating series

Chapter 8

Special Counting Sequences

8.1 Lecture 19: Partition identities

8.2 Lecture 20: Partition identities (continued)

8.3 Lecture 21: Exponential generating series