MATH 413
Introduction to Combinatorics

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Chapter 7

Recurrence Relations and Generating Functions

7.1 Lecture 17: Some Number Sequences

Example 7.1.1 (Example 1)

Consider a configuration of n lines where every two lines have a point in common, but no three do. How many regions in the plane are there? Give a recurrence.

$$a_n = a_{n-1} + n$$

TODO: Give an explanation of why this is true.

Example 7.1.2 (Example 2)

Give a simple recurrence for dearragements.

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

TODO: Give an explanation of why this is true. Need to review dearragements from previous lecture.

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$.

Definition 7.1.1: The adjusted Fibonacci sequence: $\hat{F}_n$

This is the number of 1,2 lists of size n. In other words, consider the number of ways a valet can park A cars (size 1) and B cars (size 2) in a parking lot of size n.

$$\hat{F}_n = \begin{cases} 1 & \text{if } n = 0 \\ f_{n+1} & \text{otherwise} \end{cases}$$

Question 1

Prove

$$\sum_{n=0}^{n} f_i = f_{n+2} - 1$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$\sum_{i=0}^{0} f_i = f_0 = 0$$

Similarly,

$$f_{0+2} - 1 = f_2 - 1 = 1 - 1 = 0$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$\sum_{i=0}^{k} f_i = f_{k+2} - 1$$

Indcutive Step: We will prove that the following statement is true for $n = k + 1$.

$$\sum_{i=0}^{k+1} f_i = f_{k+1} + \sum_{i=0}^{k} f_i = f_{k+1} + f_{k+2} - 1 = f_{k+3} - 1$$

Therefore, the statement is true for all n by induction. ■

Question 2

Prove

$$1 + \sum_{i=0}^{n} \hat{F}_i = \hat{F}_{n+2}$$

Solution:

Proof. We will prove this by induction on n.

Base case: $n = 0$.

$$1 + \sum_{i=0}^{0} \hat{F}_i = 1 + \hat{F}_0 = 1 + 1 = 2$$

Similarly,

$$\hat{F}_{0+2} = \hat{F}_2 = f_{2+1} = f_3 = 2$$

So, the base case is true.

Inductive Hypothesis: Assume that the following statement is true for $n = k$.

$$1 + \sum_{i=0}^{k} \hat{F}_i = \hat{F}_{k+2}$$

Inductive Step: We will prove that the following statement is true for $n = k + 1$.

$$\begin{aligned} 1 + \sum_{i=0}^{k+1} \hat{F}_i &= 1 + \hat{F}_{k+1} + \sum_{i=0}^{k} \hat{F}_i \\ &= \hat{F}_{k+1} + \hat{F}_{k+2} \\ &= f_{k+2} + f_{k+3} \\ &= f_{k+4} \\ &= \hat{F}_{k+3} \end{aligned}$$

Therefore, the statement is true for all n by induction. ■

Question 3

Prove that f_n is even if and only if n is divisible by 3.

Solution:

Proof. Given that $f_0 = 0$, $f_1 = 1$, and $f_2 = 1$, we can see that at $n = 3$, $f_3 = 2$, which is even.

This is because the only way to get an even number is to have the parity of the two numbers added togethed (odd + odd or even + even) be the same. So, f_4, must be odd, f_5 must be odd and f_6 must be even.

Given the starting sequence of even, odd, odd. The following sequence must always be even, odd, odd, which repeats every 3 numbers.
Since the first $n = 0$ is the first number in the sequence, every n that is divisible by 3 is even. ■

Note:-

Example problems for later

Guess and prove by induction (you may replace the Fibonnaci number by the adjusted Fibonacci number if it helps you)

- $f_1 + f_3 + \cdots + f_{2n-1} =?$
- $f_0 + f_2 + \cdots + f_{2n} =?$
- $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n =?$
- $(f_0)^2 + (f_1)^2 + \cdots + (f_n)^2 =?$

Obtaining an explicit formula for f_n for linear recurrences

Example 7.1.3

Consider the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$ and $f_1 = 1$. This can be rewritten as a linear recurrence as follows:

$$f_n - f_{n-1} - f_{n-2} = 0$$

We must solve the corresponding characterstic equation. Notice how the largest degree lines up with the "largest" case of the recurrence.

$$x^2 - x - 1 = 0$$

Let q_1 and q_2 be the roots of the characteristic equation.

It is potentially relevant to note that the following is a solution space of the Fibonacci recurrence (but don't satisfy f_0 – the initial condition):

$$\begin{cases} q_1^n - q_1^{n-1} - q_1^{n-2} = 0 \\ q_2^n - q_2^{n-1} - q_2^{n-2} = 0 \end{cases}$$

The rest of this is based on an ansatz, i.e. we need to make an assumption at the answer and validate it later

$$f_n = c_1 q_1^n + c_2 q_2^n,$$

for *some* $c_1, c_2 \in \mathbb{R}$.

Using the initial conditions of $f_0 = 0$ and $f_1 = 1$, we can solve for c_1 and c_2.

7.2 Lecture 18: Introduction to ordinary generating series

Note:-

Up until now, we have solved an instance combinatorial problem at a time. However, in some cases, solving a single combinatorial is too difficult. So, we solve all of the combinatorial problems at once.

Consider a sequence

$$h_0, h_1, h_2, \ldots, h_t, \ldots$$

of natural numbers where h_t is the answer to some counting problem that depends on t.

We can create a generating series of the form:

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_t x^t + \cdots$$

where $h_t = [x^t]g(x)$.

Note:-

The notation $[x^t]g(x)$ is the coefficient of x^t in the polynomial $g(x)$.

Claim 7.2.1 Compositions Generating Series

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Example 7.2.1

Fix k. Let

h_t = number of nonnegative integral solutions to

$$e_1 + \cdots + e_k = t, e_i \in \mathbb{Z}_{\geq 0}$$

We already know from stars and bars that

$$h_t = \binom{t+k-1}{k-1}$$

So obviously, the generating series can be written as

$$g(x) = \sum_{t=0}^{\infty} \binom{t+k-1}{k-1} x^t$$

Note:-

This doesn't really tell us anything. We just combined some definitions and have a genearting series that is hard to utilize in any useful way.

Take our claim and expand it out with Taylor Series k times:

$$\begin{aligned} g(x) &= \left(\frac{1}{1-x}\right)^k \\ &= (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)\cdots \end{aligned}$$

We can see that the coefficient of x^t is the number of ways to write t as a sum of k nonnegative integers. So,

$$h_t = \binom{t+k-1}{k-1}$$

This combinatorially proves the negative binomial theorem:

$$g(x) = \left(\frac{1}{1-x}\right)^k$$

Remark

If you knew that the negative binomial theorem was true, you could use it to algebraically solve the composition problem after choosing the correct generating series $g(x) = \left(\frac{1}{1-x}\right)^k$.

Question 4

What is

$$(1 + x + x^2 + x^3 + x^4 + x^5)(x + x^2)(1 + x + x^2 + x^3 + x^4)$$

the generating series for?

Solution: The solutions to the equation

$$e_1 + e_2 + e_3 = t$$

where $0 \le e_1 \le 5, 1 \le e_2 \le 2, 0 \le e_3 \le 4$.

Question 5

Suppose you have 1,5 and 25 cents coins available (infinite supply). Write down the generating series for the number of ways to make change for a dollar. How many ways are there?

Solution:

The generating series for the number of ways to make change for an arbitrary amount of money is:

$$g(x) = (1 + x + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^5 + x^{5+5} + x^{5+5+5} + \cdots)(1 + x^{25} + x^{25+25} + x^{25+25+25} + \cdots)$$

This can be simplified as follows:

$$
\begin{aligned}
g(x) &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^{25}}\right) \\
&= \frac{1}{(1-x)(1-x^5)(1-x^{25})}
\end{aligned}
$$

The answers is $[x^{100}]g(x)$.

Question 6: Nonstandard Dice

Compute the generating series for the result of rolling two standard dice.
Now factor it, and interpret.

Solution:

$$
\begin{aligned}
g(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6) \\
&= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \\
&= (1 + x)(1 + x)(x + x^3 + x^5)
\end{aligned}
$$

TODO: How does this factor like this?
After factoring, we can see that this is the equivalent of flipping a coin twice and rolling two 1-3-5 dice.

Question 7

Determine the generating series for partitions.

Solution:

$$
\begin{aligned}
g(x) &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^2 + x^{2+2} + x^{2+2+2} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots)\cdots \\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i}\right)
\end{aligned}
$$

In words, this generating series is the product of compositions of just 1's, 2's, 3's, etc.

Definition 7.2.1: Ordinary Generating Series

Let a combinatorial problem be defined as (S, ω) with S a set and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ The ordinary generating series is

$$
g(x) = g_{(S,\omega)}(x) = \sum_{s \in S} x^{\omega(s)}
$$

Theorem 7.2.1 Addition Rule of Generating Series

Suppose $S = A \cup B$ (disjoint union), where (A, ω_A) and (B, ω_B) are combinatorial problems. Moreover $\omega|_A = \omega_A$ and $\omega|_B = \omega_B$.
Then the ordinary generating series for (S, ω) is

$$
\begin{aligned}
g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) + g_{(B,\omega_B)}(x) \\
&= \sum_{s\in A} x^{\omega_A(s)} + \sum_{s\in B} x^{\omega_B(s)} \\
&= \sum_{s\in S} x^{\omega(s)}
\end{aligned}
$$

Note:-

The notation $\omega|_A$ means the restriction of ω to A. In this case,

$$\omega|_A = \{\omega_A(s) \mid s \in A\}$$

Theorem 7.2.2 Product rule of Generating Series

Suppose $S = A \times B$ (cartesian product), where (A, ω_A) and (B, ω_B) are combinatorial problems and

$$\omega(a, b) = \omega_A(a) + \omega_B(b)$$

Then,

$$
\begin{aligned}
g_{(S,\omega)}(x) &= g_{(A,\omega_A)}(x) g_{(B,\omega_B)}(x) \\
&= \sum_{a\in A} x^{\omega_A(a)} \sum_{b\in B} x^{\omega_B(b)} \\
&= \sum_{a\in A} \sum_{b\in B} x^{\omega(a,b)}
\end{aligned}
$$

What does it mean for two generating series to be equal?

They are equal coefficient by coefficient.

Let $g(x)$ be the ggenerating series for the number of partitions. What does it mean that $g(x)$ equals the infinite product:

$$g(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

It means that if we expand/solve enough terms of the RHS, the coefficient of x^t will stabalize (as in no more terms multiplied in will impact the coefficient) and this coefficient will be the same as the coefficient of x^t in the LHS – $g(x)$.

Convergence Issues

Typically, with generating series, we only care about the coefficients and not plugging in any specfic value into x. Because of this, we do not need to worry about convergence. However, in some cases $g(x)$ is a polynomial and in these cases, substitution is fine to perform.

Question 8: Substituting into a generating series

Let $S = \{\text{coins in your pokcet}\}$ and $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ be the obvious weight function on coins, i.e. $\omega(\text{nicket}) = 5$. Is $g(x)$ is the corresponding generating series, what is $g(1)$? What is $g'(1)$?

Solution: $g(1)$ will be the number of coins in your pocket. This is trivially true because each term is x to the power of an integer (with no coefficient), so 1 raised to anything will be 1 and so it is just the number of terms. More mathematically,

$$\begin{aligned} g(1) &= \sum_{s\in S} 1^{\omega(s)} \\ &= \sum_{s\in S} 1 \\ &= |S| \end{aligned}$$

$g'(1)$ will the amount of money you have. Based on the previous statement, we can see that the derivative of $g(x)$ is just the sum of the powers which represented the values of the coins in your pocket. More mathematically,

$$g'(1) = \sum_{s\in S} \omega(s)$$

Chapter 8

Special Counting Sequences

8.1 Lecture 19: Partition identities

Definition 8.1.1: Partition

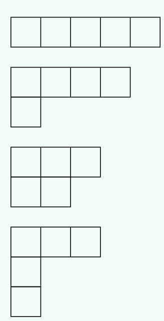
A *partition* is a decreasing sequence of positive integers whose sum is n.

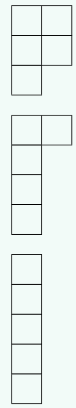
Example 8.1.1

Consider $n = 5$. The partitions of 5 are

$$\begin{aligned}&5\\&4,1\\&3,2\\&3,1,1\\&2,2,1\\&2,1,1,1\\&1,1,1,1,1\end{aligned}$$

However, this can also be represented in terms of the **Ferrers diagram**:
Partitions 5:





Example 8.1.2 (Theorem 8.3.1 from the textbook)

Use Ferrers diagrams to prove that the number of partitions of n in which the largest part is r equals the number of partitions of $n - r$ in which no part is greater than r.

Let P be a partition for n. Remove the first part of P which by definition is the largest part which has an arbitrary size r. This clearly yields a partition of $n - r$ in which no part is greater than r. This process is clearly a bijection and so the number of partitions of n in which the largest part is r equals the number of partitions of $n - r$ in which no part is greater than r.

Note:-

This process for proving two partitions to be of equal size is important.

Generating Series , and partition identities

Recall generating series for partition:

$$g(x) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^k} \right)$$

Question 9

Write down the generating series for:

1. only odd parts
2. no part repeated more than m times

Solution: Only odd parts:

$$\begin{aligned} g(x) &= (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots) \cdot (1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots) \cdots \\ &= \prod_{k=1}^{\infty} \left(\frac{1}{1 - x^{2k-1}} \right) \end{aligned}$$

Solution: No part repeated more than m times

$$\begin{aligned} g(x) &= (1 + x + x^{1+1} + \cdots + x^m)(1 + x^2 + x^{2+2} + \cdots + x^{2m})(1 + x^3 + x^{3+3} + \cdots + x^{3m}) \cdots \\ &= \left(\frac{1}{1-x} - \frac{x^m}{1-x}\right) \cdot \left(\frac{1}{1-x^2} - \frac{x^m}{1-x^2}\right) \cdot \left(\frac{1}{1-x^3} - \frac{x^m}{1-x^3}\right) \cdots \\ &= \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k} - \frac{x^m}{1-x^k}\right) \\ &= \prod_{k=1}^{\infty} \left(\frac{1-x^m}{1-x^k}\right) \end{aligned}$$

Terminology

Definition 8.1.2: Partition size

The *size* of a partition is the sum of its parts.

Definition 8.1.3: Partition length

The *length* of a partition is the number of parts.

Question 10

Write down a two variable generating series that counts partitions by both size and length, using say, a q variable and a z variable respectively. Hence the coefficient of $q^n z^l$ is the number of partitions of size n and length l.

Solution:

$$\begin{aligned} g(q, z) &= \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} P(n, l) \cdot q^n z^l \\ &= \prod_{k=0}^{\infty} (1 + zq^k + z^2 q^{2k} + z^3 q^{3k} + \cdots) \\ &= \prod_{k=0}^{\infty} \left(\frac{1}{1 - zq^k}\right) \end{aligned}$$

Question 11

Consider the identity:

$$\prod_{k=1}^{\infty} 1 + q^k = \prod_{k=1}^{\infty} \frac{1}{1 - q^{2k-1}}$$

- Interpret both sides of the identity.
- Give an algabraic proof.
- Can you give a combinatorial proof?

Solution: Interpret both sides of the identity

- LHS

$$\prod_{k=1}^{\infty} 1 + q^k = (1+q^1)(1+q^2)(1+q^3)\cdots$$

This is clearly the number of partitions with distinct parts that are used at most one time.

- RHS

  This is clearly the generating series for partitions with only odd parts.

Solution: Give an algebraic proof

Proof.

$$\begin{aligned}
\prod_{k=1}^{\infty} 1 + q^k &= \prod_{k=1}^{\infty} \frac{1-q^k}{1-q^k}(1+q^k) \\
&= \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^k} \\
&= \prod_{k=1}^{\infty} \frac{1}{1-q^k} - \frac{q^{2k}}{1-q^k} \\
&= \prod_{k=1}^{\infty} \frac{1}{1-q^k} - \prod_{k=1}^{\infty} \frac{q^{2k}}{1-q^k} \\
&= \text{ All partitions - partitions with even parts} \\
&= \text{ Partitions with odd parts} \\
&= \prod_{k=1}^{\infty} \frac{1}{1-q^{2k-1}}
\end{aligned}$$

■

Solution: Give a combinatorial proof

Proof. Fix n. Let S be the set of all partitions of n with only distinct parts. Clearly, the LHS represents the generating series that counts S.

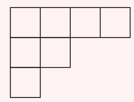
Construct a function f from distinct parts to odd parts by dividing each even number by 2 until every number is odd. This function is clearly well defined as it will only result in odd parts. This must be a bijection because the input to f must have distinct parts so the number of odd parts created is dependent on the even numbers in the input so the output is unique. Lastly, this is clearly reversible by taking the largest subset of repeated odd numbers that is a power of 2 and summing them.

Thus, because we can decompose S into the set of partitions with only odd parts through a bijective function and this decomposition is clearly counted by the RHS. Then, it is clear that the LHS and RHS equivalent. ■

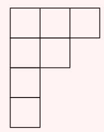
8.2 Lecture 20: Partition identities (continued)

Definition 8.2.1: Partition Conjugate

The *conjugate* of a partition is the partition associated to the transpose of the Ferrers diagram.



has a conjugate of



Question 12

Use conjugation to prove the following fact: the number of partitions of n with no part bigger than k equals the number of partitions of n with at most k parts.

Determine the generating series for either of the kinds of partitions from the previous problem.

Solution: Using conjugation...

Proof. This part is trivially true. If no part is bigger than k, then the first row in the Ferrers Diagram, which must be the largest row in the diagram, has at most k parts.

By taking the transpose of the first type of parition, we get a partition with at most k parts as the number of parts will be dictated by the first row.

Thus, the number of partitions of n with no part bigger than k is the same as the number of partitions of n with at most k parts. ∎

Solution: Generating Series

$$
\begin{aligned}
g(x) = &(1 + x + x^{1+1} + \cdots + x^{k}) \\
&(1 + x^2 + x^{2+2} + \cdots + x^{2k}) \\
&(1 + x^3 + x^{3+3} + \cdots + x^{3k}) \cdots
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + \cdots + x^{ki}) \\
&= \prod_{i=1}^{\infty} \left(\frac{1 - x^{ki+1}}{1 - x^i} \right)
\end{aligned}
$$

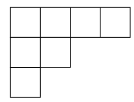
Definition 8.2.2: Durfee Square

A *Durfee square* is the largest square that sits in the NW corner of the Ferrers diagram of a partition.

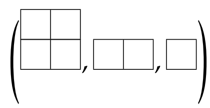
Question 13

Show that every Ferrers shape uniquely decomposes into (D_k, A_k, B_k) where D_k is a Durfee square of size k, A_k is the partition with at most k parts, and B_k is a partition with first part of size at most k.

Solution: Consider an arbitrary partition:



This can obviously be decomoposed into the triple:



There are a few cases to consider:

- First, this is unique. Clearly, this has to be unique because we are not changing any structure from the partition, but rather we are just decomposing it into a triple. So each partition must uniquely map to its own ordered triple.

- Second, A_k is the partition with at most k parts. A_k must have at most k because the Durfee square is of size k. A_k cannot be largely than this because it would contradict either the definition of a partition or the definition of a Durfee square.

- Third, B_k is the partition with first part of size at most k. This is true because if the first part of B_k is larger than k, then this would contradict the definition of a Durfee square or a partition.

Thus, we have shown that every partition can be decomposed into a triple.

Definition 8.2.3: Euler-Gauss Identity

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = 1 + \sum_{j=1}^{\infty} \frac{x^{j^2}}{\left(\prod_{t=1}^{j} 1 - x^t\right)^2}$$

Definition 8.2.4: Self-conjugate Partition

A partition is *self-conjugate* i its transpose shape is the same as its original shape (e.g., the partitions (4,1,1,1) and (3,2,1) are self-conjugate, but (3,2) is not).

Question 14

Prove that the number of self conjugate partitions of n equals the number of partitions using only distinct odd parts. Then derive an identity.

Solution: First, we must prove the number of self conjugate partitions of n equals the number of partitions using only distinct odd parts. This can be done by showing a weight preserving bijection between the two sets.

Definition 8.2.5: Weight-preserving function

Define the following function:

$$f : (S, \omega) \mapsto (S', \omega')$$

such that $\omega : S \mapsto \mathbb{Z}_{\geq 0}$ and $\omega' : S' \mapsto \mathbb{Z}_{\geq 0}$ are weight functions. Then, f is a weight-preserving function if and only if:

$$\omega(s) = \omega'(f(s)) \text{ for all } s \in S$$

And so, we can make the following claim

Claim 8.2.1

If $f : (S, \omega) \mapsto (S', \omega')$ is a weight-preserving and bijective then,

$$g_{(S,\omega)}(x) = g_{(S',\omega')}(x) \in \mathbb{Z}[x]$$

In other words, their generating functions are equal.

Now to define the actual bijection. Consider each row of the Ferrers diagram of an arbitrary distinct odd partition. Each row can be bent into a self-conjugate partition. By selecting the middle element, which is gaurunteed to exist because the partition is odd, we can place every square to the left below the middle element and leave the other elements alone. Additionally, this is well defined as each row is distinct. Thus, we have a weight preserving bijection between the set of distinct odd partitions and the set of self-conjugate partitions.

The generating function for the number of distinct odd partitions is trivial and given by:

$$g(x) = \prod_{i=1}^{\infty} 1 - x^{2i-1}$$

as we only want to either select an odd number or not.

The generating function for the number of self-conjugate partitions is more complicated. Consider an arbitrary self-conjugate partition λ. We can decompose it into a triple (D_k, A_k, B_k) through the Durfee square decomposition. We can then apply a second weight-preserving bijection and conjugate the second component of the triple which makes $A_k = B_k$. Lastly, we can apply a third weight-preserving bijection by compressing the triple into a tuple (D_k, C_k) where C_k is A_k merged left-to-right with B_k.

Consequently,

$$C_k \in \bigcup_{j=0}^{\infty} D_j \times \text{Partitions}_{\text{even}_{\leq j \text{ tall}}} = U$$

Clearly,

$$\begin{aligned} g_U(x) &= \sum_{j=0}^{\infty} x^{j^2} \Big(\prod_{t=1}^{j} \frac{1}{1 - x^{2t}} \Big) \\ &= \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{t=1}^{j} \frac{1}{1-x^{2t}}} \end{aligned}$$

So,

$$\prod_{k=1}^{\infty} 1 + x^{2k-1} = \sum_{j=0}^{\infty} \frac{x^{j^2}}{\prod_{t=1}^{j} \frac{1}{1-x^{2t}}}$$

8.3 Lecture 21: Exponential generating series

Midterm 3