MAE 261B – Computational Mechanics of Solids and Structures

Lecture 20:
Mixed Finite Elements for Shell Theory

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Mixed Variational Principles

Mixed Variational Principles

- **Key idea** Enforce strain-displacement and/or stress-strain relations weakly by Lagrange Multipliers.
- Stresses and/or strains are treated as independent fields, along with displacements.
- Thee main principles:
  1. **Hellinger-Reissner Principle**: $\boldsymbol{\sigma}, \boldsymbol{u}$ independent
  2. **Reissner strain-displacement Principle**: $\boldsymbol{\epsilon}, \boldsymbol{u}$ independent
  3. **Hu-Washizu Principle**: $\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \boldsymbol{u}$ independent

Minimum Potential Energy (MPE) for Linear Elasticity

Potential Energy:

$$\Pi[\boldsymbol{u}] = \int_V \underbrace{\frac{1}{2}\nabla_s \boldsymbol{u} \cdot \boldsymbol{C} \nabla_s \boldsymbol{u}}_{w(\nabla_s \boldsymbol{u})} \, dV \underbrace{- \int_V \boldsymbol{b} \cdot \boldsymbol{u} dV - \int_{\partial V} \bar{\boldsymbol{t}} \cdot \boldsymbol{u} dA}_{\Pi^{\text{ext}}},$$

$$\nabla_s \boldsymbol{u} \equiv \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

MPE Principle:

$$\inf_{\boldsymbol{u} \text{ admissible}} \Pi[\boldsymbol{u}] \Rightarrow \delta\Pi = 0$$

$$\Rightarrow \quad \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0 \text{ + boundary conditions}$$

where $\boldsymbol{\sigma} \equiv \boldsymbol{C}\boldsymbol{\epsilon}$, and $\boldsymbol{\epsilon} \equiv \nabla_s \boldsymbol{u}$

Hu-Washizu Principle

Concept:

- Treat $\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \boldsymbol{u}$ as independent fields
- $\boldsymbol{\sigma}$ serves as multiplier for strain-displacement relation

$$\Pi^{\mathsf{HW}}[\boldsymbol{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}] \equiv \int_V \left[w(\boldsymbol{\epsilon}) - \boldsymbol{\sigma} : (\boldsymbol{\epsilon} - \nabla_s \boldsymbol{u})\right] dV + \Pi^{\mathsf{ext}} - \int_{\partial V_u} (\boldsymbol{\sigma}\boldsymbol{n}) \cdot (\boldsymbol{u} - \bar{\boldsymbol{u}}) dA$$

H-W:

$$\delta\Pi^{\mathsf{HW}} = 0 \Rightarrow \begin{cases} \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0 \\ \boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon} \\ \boldsymbol{\epsilon} = \nabla_s \boldsymbol{u} \\ + \text{ boundary conditions} \end{cases}$$

Admissibility for Hu-Washizu

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Variational Foundation

Mixed FE Approximation for Linear Elasticity

Applications to Shells

$$\Pi^{\mathsf{HW}}[\boldsymbol{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}] \equiv \int_V \left[w(\boldsymbol{\epsilon}) - \boldsymbol{\sigma} : (\boldsymbol{\epsilon} - \nabla_s \boldsymbol{u})\right] dV + \Pi^{\mathsf{ext}} - \int_{\partial V_u} (\boldsymbol{\sigma}\boldsymbol{n}) \cdot (\boldsymbol{u} - \bar{\boldsymbol{u}}) dA$$

- $\boldsymbol{\sigma}, \boldsymbol{\epsilon}$ appear without derivatives, $\Rightarrow \boldsymbol{\sigma}, \boldsymbol{\epsilon}$ only need to be $\mathcal{C}^{-1}$ (piecewise constant OK)
- $\boldsymbol{u}$ differentiated one time $\Rightarrow \boldsymbol{u} \in \mathcal{C}^0$.
- Only $\boldsymbol{u}$ needs to satisfy essential boundary conditions

Mixed FE Approximation for Linear Elasticity

Some Options for Approximating H-W

1. Interpolate all three fields explicitly.
   Note: leads to **lots** of DOF: 6 per node for $\boldsymbol{\sigma}, \boldsymbol{\epsilon}$; 3 per node for $\boldsymbol{u}$.
2. Assumed Natrual Strain (ANS) method:
   - Design some form of *assumed* strains to alleviate locking, etc.
   - Design *assumed* stress field to be *orthogonal* to difference of *assumed* strains and displacement strains.
   - Condense out *assumed* strains at element level
   - $\Rightarrow$ only $\boldsymbol{u}$ DOF are global
3. Enhanced Assumed Strain (EAS) method:
   - Design assumed strains by *enhancing* the displacement strains with additional terms (typically to *remove* specific terms that cause locking).
   - Design *assumed* stress field to be *orthogonal* to enhancement.

Note: *Key advantage of these methods over reduced integration is avoidance of spurious (degenerate) modes.*

Assumed Strain Example: “B-bar” method

- Define displacement strains using shape functions in the normal way

$$\boldsymbol{\epsilon}^h = \nabla_s \boldsymbol{u} = \underbrace{\boldsymbol{B}}_{\text{strain-displacement “B” matrix}} \underbrace{\boldsymbol{d}}_{\text{nodal displacement vector}}$$

- Define assumed strains also in terms of nodal displacements

$$\bar{\boldsymbol{\epsilon}} = \underbrace{\bar{\boldsymbol{B}}}_{\text{Assumed strain-displacement operator}} \boldsymbol{d}$$

- Design assumed stress to be orthogonal to strain difference

$$\int_V \bar{\boldsymbol{\sigma}} : (\boldsymbol{\epsilon} - \bar{\boldsymbol{\epsilon}}) dV = 0$$

Can always do this, since we have total freedom in using as many polynomial terms as we like to define $\bar{\boldsymbol{\sigma}}$.

B-bar method

$$\Pi^{\text{HW}}[\boldsymbol{u}^h, \bar{\boldsymbol{\epsilon}}, \bar{\boldsymbol{\sigma}}] = \int_V \left[w(\bar{\boldsymbol{\epsilon}}) - \bar{\boldsymbol{\sigma}} : (\bar{\boldsymbol{\epsilon}} - \boldsymbol{\epsilon}^h) \right] dV + \Pi^{\text{ext}} - \int_{\partial V_u} (\bar{\boldsymbol{\sigma}} \boldsymbol{n}) \cdot (\boldsymbol{u}^h - \bar{\boldsymbol{u}}) dA$$

Insert: $\boldsymbol{u}^h = \boldsymbol{N}\boldsymbol{d}, \quad \boldsymbol{\epsilon}^h = \boldsymbol{B}\boldsymbol{d}, \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{B}}\boldsymbol{d},$

$$\Pi^{\text{HW},h}(\boldsymbol{d}, \bar{\boldsymbol{\sigma}}) = \int_V w(\bar{\boldsymbol{\epsilon}}) dV - \underbrace{\int_V \bar{\boldsymbol{\sigma}} : (\bar{\boldsymbol{B}}\boldsymbol{d} - \boldsymbol{B}\boldsymbol{d}) dV}_{=0} + \Pi^{\text{ext}} - \int_{\partial V_u} (\bar{\boldsymbol{\sigma}} \boldsymbol{n}) \cdot (\boldsymbol{u}^h - \bar{\boldsymbol{u}}) dA$$

Internal forces

$$\boldsymbol{f}^{\text{int}} = \frac{\partial}{\partial \boldsymbol{d}} \int_V w(\bar{\boldsymbol{\epsilon}}) dV = \int_V \bar{\boldsymbol{B}}^T \underbrace{\boldsymbol{\sigma}(\bar{\boldsymbol{\epsilon}})}_{\frac{\partial w}{\partial \boldsymbol{\epsilon}}(\bar{\boldsymbol{\epsilon}})} dV$$

Neat trick: Expand DOF to include $\bar{\boldsymbol{\sigma}}$; *but since it is never used, we don't need to actually construct it!*

- $\boldsymbol{d}$ are the only true DOF.
- Simply replace B-matrix by B-bar in code.

EAS formulation

- Define assumed strains as

$$\bar{\boldsymbol{\epsilon}} = \underbrace{\boldsymbol{\epsilon}}_{\boldsymbol{B}\boldsymbol{d}} + \underbrace{\tilde{\boldsymbol{\epsilon}}}_{\textit{enhanced}\text{ strains}}$$

$$\delta\Pi^{\mathsf{HW}} = \int_V \delta(\boldsymbol{\epsilon} + \tilde{\boldsymbol{\epsilon}}) : \boldsymbol{\sigma}(\boldsymbol{\epsilon} + \tilde{\boldsymbol{\epsilon}})dV - \int_V \delta(\bar{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\epsilon}})dV + \ldots$$

- Euler equations $\Rightarrow \tilde{\boldsymbol{\epsilon}} = 0$ (strong form).
- In approximation, carefully designed $\tilde{\boldsymbol{\epsilon}}$ won't vanish, but rather will improve element (eliminate locking, etc.).
- Typically express enhancement in polynomial expansion

$$\tilde{\boldsymbol{\epsilon}} = \sum_{a=1}^{N} \underbrace{\psi_a(\xi)}_{\text{polynomials}} \underbrace{\boldsymbol{\alpha}_a}_{\text{coefficients}} .$$

- Solve for $\boldsymbol{d}$ and $\boldsymbol{\alpha}$: $\partial\Pi^{\mathsf{HW}}/\partial\boldsymbol{d} = \mathbf{0}, \quad \partial\Pi^{\mathsf{HW}}/\partial\boldsymbol{\alpha} = \mathbf{0}$

Applications to Shells

Dvorkin-Bathe ANS Element

A continuum mechanics based four-node shell element for general non-linear analysis

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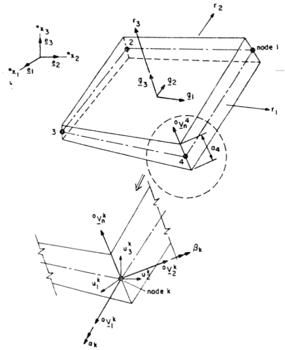
$$ {}^t x_i = \sum_{k=1}^{4} h_k\, {}^t x_i^k + \frac{r_3}{2} \sum_{k=1}^{4} a_k h_k\, {}^t V_{ni}^k \qquad (1) $$



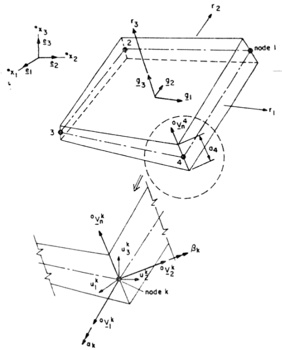
$$ {}^0\mathbf{V}_1^k = \frac{\mathbf{e}_2 \times {}^0\mathbf{V}_n^k}{|\mathbf{e}_2 \times {}^0\mathbf{V}_n^k|} $$

$$ {}^0\mathbf{V}_2^k = {}^0\mathbf{V}_n^k \times {}^0\mathbf{V}_1^k $$

Figure 1 Four-node shell element

Dvorkin-Bathe ANS Element

$$ {}^l x_i = \sum_{k=1}^{4} h_k \, {}^l x_i^k + \frac{r_3}{2} \sum_{k=1}^{4} a_k h_k \, {}^l V_{ni}^k \qquad (1) $$



$$ {}^0\mathbf{V}_1^k = \frac{\mathbf{e}_2 \times {}^0\mathbf{V}_n^k}{|\mathbf{e}_2 \times {}^0\mathbf{V}_n^k|} $$

$$ {}^0\mathbf{V}_2^k = {}^0\mathbf{V}_n^k \times {}^0\mathbf{V}_1^k $$

Figure 1 Four-node shell element

Assumed transverse shear strains:

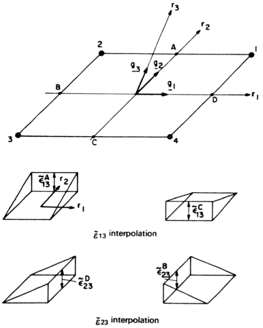


Figure 2 Interpolation functions for the transverse shear strains

$$ \begin{aligned} \tilde{\varepsilon}_{13} &= \tfrac{1}{2}(1 + r_2)\tilde{\varepsilon}_{13}^{A} + \tfrac{1}{2}(1 - r_2)\tilde{\varepsilon}_{13}^{C} \\ \tilde{\varepsilon}_{23} &= \tfrac{1}{2}(1 + r_1)\tilde{\varepsilon}_{23}^{D} + \tfrac{1}{2}(1 - r_1)\tilde{\varepsilon}_{23}^{B} \end{aligned} \qquad (3) $$

H-W functional:

$$ \Pi^* = \frac{1}{2}\int_V \tilde{\tau}^{ij}\tilde{\varepsilon}_{ij}\,dV + \int_V \lambda^{13}(\tilde{\varepsilon}_{13} - \tilde{\varepsilon}_{13}^{DI})\,dV + \int_V \lambda^{23}(\tilde{\varepsilon}_{23} - \tilde{\varepsilon}_{23}^{DI})\,dV - \mathcal{W} \qquad (4) $$

Dvorkin-Bathe ANS Element

- Assumed transverse shear strains:

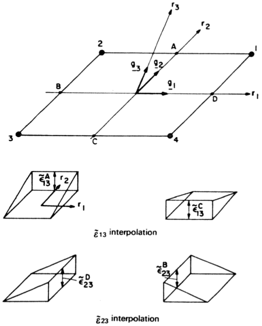


Figure 2 Interpolation functions for the transverse shear strains

$$\tilde{\varepsilon}_{13} = \tfrac{1}{2}(1+r_2)\tilde{\varepsilon}^A_{13} + \tfrac{1}{2}(1-r_2)\tilde{\varepsilon}^C_{13}$$
$$\tilde{\varepsilon}_{23} = \tfrac{1}{2}(1+r_1)\tilde{\varepsilon}^D_{23} + \tfrac{1}{2}(1-r_1)\tilde{\varepsilon}^B_{23} \qquad (3)$$

- H-W functional:

$$\Pi^* = \tfrac{1}{2}\int_V \tilde{\tau}^{ij}\tilde{\varepsilon}_{ij}\,dV + \int_V \lambda^{13}(\tilde{\varepsilon}_{13} - \tilde{\varepsilon}^{DI}_{13})\,dV + \qquad (4)$$
$$\int_V \lambda^{23}(\tilde{\varepsilon}_{23} - \tilde{\varepsilon}^{DI}_{23})\,dV - \mathcal{W}$$

- Assumed stresses (multipliers):

$$\lambda^{13} = \lambda^A\delta(r_1)\delta(1-r_2) + \lambda^C\delta(r_1)\delta(1+r_2)$$
$$\lambda^{23} = \lambda^D\delta(r_2)\delta(1-r_1) + \lambda^B\delta(r_2)\delta(1+r_1) \qquad (5)$$

- Discrete constraints, equating assumed strains with displacement strains at sampling points:

$$\tilde{\varepsilon}_{13}\Big|_{\text{at A}} = \tilde{\varepsilon}^{DI}_{13}\Big|_{\text{at A}} \qquad \tilde{\varepsilon}_{13}\Big|_{\text{at C}} = \tilde{\varepsilon}^{DI}_{13}\Big|_{\text{at C}}$$
$$\tilde{\varepsilon}_{23}\Big|_{\text{at D}} = \tilde{\varepsilon}^{DI}_{23}\Big|_{\text{at D}} \qquad \tilde{\varepsilon}_{23}\Big|_{\text{at B}} = \tilde{\varepsilon}^{DI}_{23}\Big|_{\text{at B}} \qquad (6)$$

- Local condensation: solve these constraint equations for assumed strains in terms of nodal displacements.
- Use simplified functional to compute residual and stiffness:

$$\Pi^* = \tfrac{1}{2}\int_V \tilde{\tau}^{ij}\tilde{\varepsilon}_{ij}\,dV - \mathcal{W} \qquad (7)$$

Properties

- All element matrices evaluated using full Gauss quadrature.
- Insensitive to rigid body rotations (because shear strains are linked to displacements).
- No spurious (degenerate) modes.
- ***Eliminates shear locking***.