TENSORS IN GENERAL COORDINATES

A 2^{nd}-order tensor is defined as a linear operator on a vector space.
In $\mathbb{R}^3$, $\underset{\sim}{A}$ is a 2-tensor if

$$\underset{\sim}{v} = \underset{\sim}{A}\,\underset{\sim}{u} \in \mathbb{R}^3 \quad \forall\, u \in \mathbb{R}^3.$$

Use the covariant expressions for $\underset{\sim}{u}$ & $\underset{\sim}{v}$,

$$v^i g_i = \underset{\sim}{A}(u^j \underset{\sim}{g}_j)$$

and contract with g_k

$$\begin{aligned} v^i g_i \cdot g_k &= g_k \cdot (A g_j)\, u^j \\ v^i g_{ik} &= (g_k \cdot A g_j)\, u^j \\ v_k &= A_{kj}\, u^j \end{aligned}$$

where we've defined the covariant components of A as

$$A_{kj} \equiv g_k \cdot A g_j \,. \tag{1}$$

This expression motivates the following

$$\underset{\sim}{A} = A_{kj}\, \underset{\sim}{g}^k \otimes \underset{\sim}{g}^j \tag{2}$$

where $\otimes$ denotes the outer or "dyadic" product of two vectors, s.t.

$$(u \otimes v)\, w = u\,(v \cdot w)$$

for $u, v, w \in \mathbb{R}^3$.

EXERCISE: Use eqn (2) to compute $g_k \cdot A g_j$ and show that eqn (1) is recovered.

Likewise we can express A in terms of the tangent vectors also,

$$A = A^{ij} g_i \otimes g_j = A^{i}_{\cdot j}\, g_i \otimes g^j = A_i^{\cdot j}\, g^i \otimes g_j \tag{3}$$

where

$$A^{ij} \equiv g^i \cdot A g^j, \quad A^{i}_{\cdot j} \equiv g^i \cdot A g_j, \quad A_i^{\cdot j} \equiv g_i \cdot A g^j \tag{4}$$

contravariant components   mixed components

NOTE: The mixed components $A^{i}_{\cdot j}$ & $A_{j}^{\cdot i}$ are generally distinct. They are equal if and only if A is symmetric:

$$A^{i}_{\cdot j} = A_{j}^{\cdot i} \iff A^{ij} = A^{ji} \text{ and } A_{ij} = A_{ji}.$$

EXERCISE: Use eqns. (1)–(4) to demonstrate the following identities

a) $(AB)_{ij} = A_{ij} B^{k}_{\cdot j} = A_{i}^{\cdot k} B_{kj}$

b) $\operatorname{tr} A = g^{i} \cdot A g_{i} = A^{i}_{\cdot i} = A_{i}^{\cdot i} = g^{ij} A_{ij} = g_{ij} A^{ij}$

c) $A : B \equiv \operatorname{tr}(A^{T} B) = A_{ij} B^{ij} = A^{ij} B_{ij} = A^{i}_{\cdot j} B_{i}^{\cdot j}$

d) $A_{ij} = g_{ik} g_{jl} A^{kl} = g_{jk} A_{i}^{\cdot k} = g_{ik} A^{k}_{\cdot j}$

e) $A^{ij} = g^{ik} g^{jl} A_{kl} = g^{ik} A_{k}^{\cdot j} = g^{jk} A^{i}_{\cdot k}$

f) $A_{i}^{\cdot j} = g^{jk} A_{ik} = g_{ik} A^{kj}$

g) $A^{i}_{\cdot j} = g^{ik} A_{kj} = g_{jk} A^{ik}$

h) $I = g^{i} \otimes g_{i} = g_{i} \otimes g^{i} = g^{ij} g_{i} \otimes g_{j} = g_{ij} g^{i} \otimes g^{j}$ (the identity tensor)

HIGHER ORDER TENSORS

Easy to generalize from 2nd order.

$$\begin{aligned}
\underset{\sim}{C} &= C^{ijkl\cdots} g_{i} \otimes g_{j} \otimes g_{k} \otimes g_{l} \otimes \cdots \\
&= C_{pqrs\cdots} g^{p} \otimes g^{q} \otimes g^{r} \otimes g^{s} \otimes \cdots \\
&= C^{ij\cdots k}{}_{lm\cdots n}{}^{pq\cdots r} g_{i} \otimes g_{j} \otimes \cdots \otimes g_{k} \otimes g^{l} \otimes g^{m} \otimes \cdots \otimes g^{n} \otimes g_{p} \otimes g_{q} \otimes \cdots \otimes g_{r}
\end{aligned}$$