TENSOR CALCULUS IN GENERAL COORDINATES

Let $\Phi : \mathbb{R}^3 \to \mathbb{R}$ be some scalar field. Thus

$$\nabla\Phi = \text{grad}\,\Phi(\underset{\sim}{r}) \equiv \frac{\partial\Phi}{\partial X_i}\underset{\sim}{E_i}$$

where $\underset{\sim}{r}(\theta^1,\theta^2,\theta^3) = x_i E_i$ is the position vector expressed in Cartesian frame $\mathbb{X} = \{E_1, E_2, E_3\}$.

Then by the chain rule

$$d\Phi = \nabla\Phi\cdot d\underset{\sim}{r} = \frac{\partial\Phi}{\partial\theta^\alpha}d\theta^\alpha$$

$$= \nabla\Phi\cdot g_i\, d\theta^i$$

$$\Rightarrow \quad \nabla\Phi\cdot g_i = \frac{\partial\Phi}{\partial\theta^i} \quad \longrightarrow \quad \boxed{\nabla\Phi = \frac{\partial\Phi}{\partial\theta^i}g^i}$$

For a tensor field $\underset{\sim}{T}(\underset{\sim}{r})$ of arbitrary order, define

$$\boxed{\text{grad}\,T = \nabla\underset{\sim}{T} \equiv \frac{\partial T}{\partial\theta^i}\otimes\underset{\sim}{g}^i}$$

Likewise we can define the divergence

$$\boxed{\text{div}\,\underset{\sim}{T} = \nabla\cdot\underset{\sim}{T} \equiv \frac{\partial T}{\partial\theta^i}\cdot g^i}$$

EXAMPLE Let $\underset{\sim}{v} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. $\to \underset{\sim}{v} = v^i\underset{\sim}{g}_i$

$$\Rightarrow \nabla\cdot\underset{\sim}{v} = (v^i\underset{\sim}{g}_i)_{,j}\cdot g^j = v^i|_j\, g_i\cdot g^j$$

$$= v^i|_j\,\delta_i^j = v^i|_i$$

$$\therefore \quad \nabla\cdot\underset{\sim}{v} = v^i|_i$$

Recall, $\frac{1}{\sqrt{g}}(\sqrt{g})_{,i} = \Gamma^j_{ji}$. So,

$$v^i|_i = \delta^j_i v^i|_j = \delta^j_i(v^i_{,j} + \Gamma^i_{kj}v^k) = v^i_{,i} + \Gamma^i_{ki}v^k$$

$$= v^i_{,i} + \frac{1}{\sqrt{g}}(\sqrt{g})_{,k}v^k = \frac{1}{\sqrt{g}}(v^i\sqrt{g})_{,i}$$

$\therefore$ we have an alternate expression for div $\underset{\sim}{v}$ (without Γ^k_{ij}'s)

$$\boxed{\nabla\cdot\underset{\sim}{v} = v^i|_i = \frac{1}{\sqrt{g}}(v^i\sqrt{g})_{,i}}$$