INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 18, 1077–1089 (1982)

AN EXPLICIT FORMULATION FOR AN EFFICIENT TRIANGULAR PLATE-BENDING ELEMENT

JEAN-LOUIS BATOZ

Département de Génie Mécanique, Université de Technologie de Compiègne, 60206 Compiègne, France

SUMMARY

In this paper a new explicit expression of the stiffness matrix of the 9 degrees-of-freedom plate-bending triangular element called DKT in Reference 1 is presented. Some additional interesting numerical results assessing the behaviour of this element are presented and discussed.

INTRODUCTION

In a recent paper[1] an assessment study on triangular plate-bending elements with 3 degrees-of-freedom (DOF) at the corner nodes only (Figure 1) has been presented. A detailed formulation of three elements, namely DKT (Discrete Kirchoff Triangle), HSM (Hybrid Stress Model) and SRI (Selective Reduced Integration), was given together with the results of several static and free vibration plate problems. The results were compared with those found in the literature and dealing with other 9 DOF triangular elements. It was found that the DKT and HSM elements were the most efficient, cost-effective and reliable elements of their class to predict displacements as well as stresses and frequencies. The DKT element (a displacement model) is particularly attractive due to the availability of the interpolation functions of the two components of the normal rotation in terms of the 9 nodal variables.

The purpose of this paper is to present an *explicit* expression of the stiffness matrix of the DKT element (without numerical integration). This new presentation allows a significant reduction of algebraic operations in the evaluation of the stiffness matrices and bending moments. This formulation should be attractive for finite element programs on desktop and micro computers. The paper includes hitherto unpublished numerical results on 9 DOF triangular elements related to the element evaluation and assessment criteria suggested by Robinson.[2,3] The problems considered include the patch-test, the analysis of a rectangular cantilever plate under differential loads and the analysis of the same plate under twisting moments. Additional results on the behaviour of DKT and HSM for the analysis of rectangular clamped and simply-supported plates subjected to concentrated load are also given.

EXPLICIT STIFFNESS MATRIX OF THE DKT ELEMENT

The stiffness matrix of the DKT element can be expressed in the standard form (Reference 1, equation 31):

$$[k] = \int_A [B]^{\mathrm{T}}[D_b][B]\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

0029–5981/82/071077–13$01.30
© 1982 by John Wiley & Sons, Ltd.

Received 10 July 1981
Revised 20 August 1981

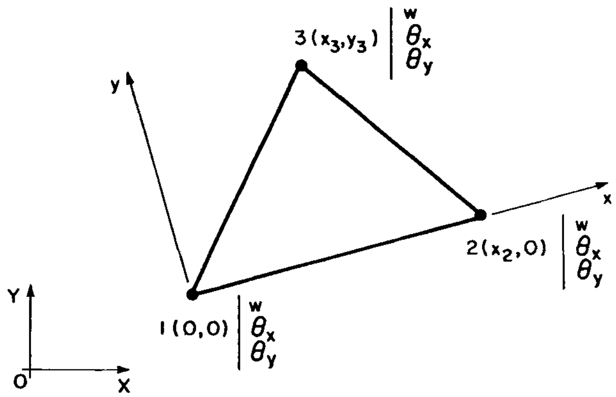

Figure 1. Nine degrees-of-freedom triangular plate-bending element

If we consider a local xy co-ordinate system such that the x axis coïncides with the side defined by nodes 1–2 (Figure 1) we have

$$[B]=\frac{1}{2A}\begin{bmatrix} y_3\langle H_{x,\xi}\rangle \\ -x_3\langle H_{y,\xi}\rangle+x_2\langle H_{y,\eta}\rangle \\ -x_3\langle H_{x,\xi}\rangle+x_2\langle H_{x,\eta}\rangle+y_3\langle H_{y,\xi}\rangle \end{bmatrix} \tag{2}$$

where $A=0{\cdot}5x_2y_3$ is the triangle area. $\langle H_x(\xi,\eta)\rangle$, $\langle H_y(\xi,\eta)\rangle$ are the interpolation functions (Reference 1, equation 26) of the rotations β_x and β_y of the normal in the x–z and y–z planes, respectively, in terms of the 9 element nodal variables:

$$\langle U_n\rangle=\langle w_1\theta_{x_1}\theta_{y_1}w_2\theta_{x_2}\theta_{y_2}w_3\theta_{x_3}\theta_{y_3}\rangle \tag{3}$$

where θ_x and θ_y are rotations around the *local* axis x and y, and w is the normal displacement along z normal to the xy plane. The components of $\langle H_{x,\xi}\rangle$, $\langle H_{x,\eta}\rangle$, $\langle H_{y,\xi}\rangle$ and $\langle H_{y,\eta}\rangle$ are given in Reference 1, Appendix A, in the case of a general local co-ordinate system. ξ and η are two area co-ordinates (Figure 2). Equation (2) is a special form of the general expression given in Reference 1, equation 30.

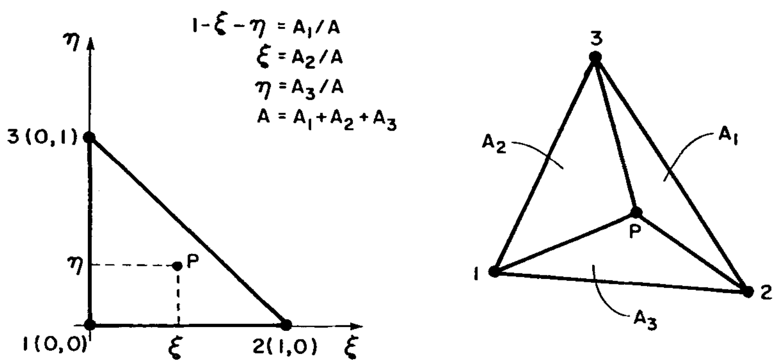

Figure 2. Area co-ordinates ξ and η

The $[B]$ matrix can be decomposed into the product of two matrices $[L]$ and $[\alpha]$ of order 3×9 and 9×9, respectively, where $[\alpha]$ is independent of ξ and η. This decomposition follows that presented in Reference 4:

$$[B]=\frac{1}{2A}[L][\alpha] \tag{4}$$

with

$$[L]=\begin{bmatrix}\langle l\rangle & \langle 0\rangle & \langle 0\rangle\\ \langle 0\rangle & \langle l\rangle & \langle 0\rangle\\ \langle 0\rangle & \langle 0\rangle & \langle l\rangle\end{bmatrix} \tag{5}$$

where

$$\langle 0\rangle=\langle 0\quad 0\quad 0\rangle \quad\text{and}\quad \langle l\rangle=\langle 1-\xi-\xi\eta\rangle \tag{6}$$

The components of the 9×9 $[\alpha]$ matrix are given in Table I (with x corresponding with the 1–2 side). The components α_{ij} are obtained by considering each term of $[B]$, say B_{ij} in the form

$$B_{ij}(\xi,\eta)=(1-\xi-\eta)\alpha_{kj}+\xi\alpha_{k+1,j}+\eta\alpha_{k+2,j} \tag{7}$$

where $k=3(i-1)+1$. The α_{ij}'s are solely dependent on the geometry of the element (x_2, x_3, y_3).

Introducting equation (4) in equation (1) (and using $dx\,dy=2A\,d\xi\,d\eta$) one obtains an *explicit* form for the stiffness matrix:

$$[k]=\frac{1}{2A}[\alpha]^{\mathrm{T}}[DL][\alpha] \tag{8}$$

with

$$[DL]=\int_0^1\int_0^{1-\xi}[L]^{\mathrm{T}}[D_b][L]\,d\xi\,d\eta \tag{9}$$

In the particular case of a homogenous orthortropic material (where the local axes x and y coïncide with the axes of orthotropy) we have

$$[D_b]=\begin{bmatrix}E_1 & E_2 & 0\\ E_2 & E_3 & 0\\ 0 & 0 & E_4\end{bmatrix} \tag{10}$$

where E_1, E_2, E_3, E_4 are function of the thickness and of the material constants. For a homogeneous isotropic plate $E_1=E_3=Eh^3/12(1-\nu^2)$, $E_2=\nu E_1$ and $E_4=E_1(1-\nu)/2$, where E, ν, and h are Young's modulus, Poisson's ratio and thickness, respectively.

Due to the nature of the $[D_b]$ and $[L]$ matrices, the 9×9 $[DL]$ matrix is given by

$$[DL]=\frac{1}{24}\begin{bmatrix}E_1[R] & E_2[R] & 0\\ E_2[R] & E_3[R] & 0\\ 0 & 0 & E_4[R]\end{bmatrix} \tag{11}$$

Table I. $[\alpha]$ matrix

$$[\alpha]=\begin{bmatrix}
y_3 p_6 & 0 & -4y_3 & -y_3 p_6 & 0 & -2y_3 & 0 & 0 & 0 \\
-y_3 p_6 & 0 & 2y_3 & y_3 p_6 & 0 & 4y_3 & 0 & 0 & 0 \\
y_3 p_5 & -y_3 q_5 & y_3(2-r_5) & y_3 p_4 & y_3 q_4 & y_3(r_4-2) & -y_3(p_4+p_5) & y_3(q_4-q_5) & y_3(r_4-r_5) \\
-x_2 t_5 & x_{23}+x_2 r_5 & -x_2 q_5 & 0 & x_3 & 0 & x_2 t_5 & x_2(r_5-1) & -x_2 q_5 \\
0 & x_{23} & 0 & x_2 t_4 & x_3+x_2 r_4 & -x_2 q_4 & -x_2 t_4 & x_2(r_4-1) & -x_2 q_4 \\
x_{23} t_5 & x_{23}(1-r_5) & x_{23} q_5 & -x_3 t_4 & x_3(1-r_4) & x_3 q_4 & -x_{23} t_5+x_3 t_4 & -x_{23} r_5-x_3 r_4-x_2 & x_3 q_4+x_{23} q_5 \\
-x_3 p_6-x_2 p_5 & x_2 q_5+y_3 & -4x_{23}+x_2 r_5 & x_3 p_6 & -y_3 & 2x_3 & x_2 p_5 & x_2 q_5 & (r_5-2)x_2 \\
-x_{23} p_6 & y_3 & 2x_{23} & x_{23} p_6+x_2 p_4 & -y_3+x_2 q_4 & -4x_3+x_2 r_4 & -x_2 p_4 & x_2 q_4 & (r_4-2)x_2 \\
\begin{matrix} x_{23} p_5 \\ +y_3 t_5 \end{matrix} & \begin{matrix} -x_{23} q_5 \\ +(1-r_5)y_3 \end{matrix} & \begin{matrix} (2-r_5)x_{23} \\ +y_3 q_5 \end{matrix} & \begin{matrix} -x_3 p_4 \\ +y_3 t_4 \end{matrix} & \begin{matrix} (r_4-1)y_3 \\ -x_3 q_4 \end{matrix} & \begin{matrix} (2-r_4)x_3 \\ -y_3 q_4 \end{matrix} & \begin{matrix} -x_{23} p_5+x_3 p_4 \\ -(t_4+t_5)y_3 \end{matrix} & \begin{matrix} -x_{23} q_5-x_3 q_4 \\ +(r_4-r_5)y_3 \end{matrix} & \begin{matrix} -x_{23} r_5-x_3 r_4 \\ +4x_2+(q_5-q_4)y_3 \end{matrix}
\end{bmatrix}$$

with $x_{ij}=x_i-x_j \qquad y_{ij}=y_i-y_j \qquad l_{ij}^2=x_{ij}^2+y_{ij}^2$

$p_4=-6x_{23}/l_{23}^2 \qquad p_5=-6x_3/l_{31}^2 \qquad p_6=-6x_{12}/l_{12}^2$

$t_4=-6y_{23}/l_{23}^2 \qquad t_5=-6y_3/l_{31}^2 \qquad q_4=3x_{23}y_{23}/l_{23}^2$

$q_5=3x_3y_3/l_{31}^2 \qquad r_4=3y_{23}^2/l_{23}^2 \qquad r_5=3y_{31}^2/l_{31}^2$

with

$$[R]=\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix} \tag{12}$$

We observe that the $[DL]$ matrix is dependent on the element mechanical properties (and on the thickness). This matrix is constant for all the elements in the case of a constant thickness isotropic homogeneous plate.

With the definition of the $[DL]$ and $[\alpha]$ matrices, as given by equation (11) and Table I, respectively, we can directly compute $[k]$ using equation (8). However, the particular nature of the $[DL]$ matrix can be exploited to reduce the number of operations involved in $[\alpha]^{\mathrm{T}}[DL]$ $[\alpha]$. If we consider a partioned $[\alpha]$ matrix in the form

$$[\alpha]=\begin{bmatrix}[\alpha_{11}] & [\alpha_{12}] & [\alpha_{13}]\\ [\alpha_{21}] & [\alpha_{22}] & [\alpha_{23}]\\ [\alpha_{31}] & [\alpha_{32}] & [\alpha_{33}]\end{bmatrix} \tag{13}$$

the product $[Q]=[\alpha]^{\mathrm{T}}[DL]$ can be expressed as

$$[Q]=\frac{1}{24}\left[\begin{array}{c:c:c}(E_1[\alpha_{11}]^{\mathrm{T}}+E_2[\alpha_{21}]^{\mathrm{T}})[R] & (E_2[\alpha_{11}]^{\mathrm{T}}+E_3[\alpha_{21}]^{\mathrm{T}})[R] & E_4[\alpha_{31}]^{\mathrm{T}}[R]\\ (E_1[\alpha_{12}]^{\mathrm{T}}+E_2[\alpha_{22}]^{\mathrm{T}})[R] & (E_2[\alpha_{12}]^{\mathrm{T}}+E_3[\alpha_{22}]^{\mathrm{T}})[R] & E_4[\alpha_{32}]^{\mathrm{T}}[R]\\ (E_1[\alpha_{13}]^{\mathrm{T}}+E_2[\alpha_{23}]^{\mathrm{T}})[R] & (E_2[\alpha_{13}]^{\mathrm{T}}+E_3[\alpha_{23}]^{\mathrm{T}})[R] & E_4[\alpha_{33}]^{\mathrm{T}}[R]\end{array}\right] \tag{14}$$

The components of the above $[Q]$ matrix are nine 3×3 matrices of the type $[a_{ij}][R]$. The matrix $[R]$ is such that[4]

$$[a_{ij}][R]=[a_{ij}]+[b_{ii}]$$

with

$$b_{ii}=\sum_{k=1}^{3} a_{ik} \tag{15}$$

The use of equation (15) leads to the evaluation of $[Q]=[\alpha]^{\mathrm{T}}[DL]$ with a minimum of algebraic operations. Thus the stiffness matrix $[k]$ in the local co-ordinate system is obtained by

$$[k]=\frac{1}{2A}[Q][\alpha] \tag{16}$$

The stiffness matrix in the global co-ordinate system is defined in a standard manner using the direction of the local axis in the global ones.

The knowledge of the nodal variables $\langle U_n\rangle$ in the local co-ordinates of each element allows the computation of the bending moments M_x, M_y and M_{xy} (Figure 3), at any point (ξ, η) in the interior or on the boundary of the element, by the expression

$$\{M(\xi,\eta)\}=\begin{Bmatrix}M_x\\ M_y\\ M_{xy}\end{Bmatrix}=[D_b][B(\xi,\eta)]\{U_n\} \tag{17}$$

Using equation (4):

$$\{M\}=\frac{1}{24}[D_b][L][\alpha]\{U_n\}$$

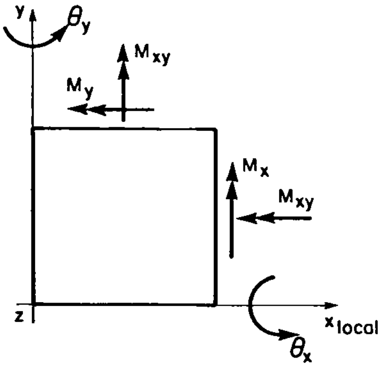

Figure 3. Bending moments M_x, M_y, M_{xy}

The vector $[\alpha]\ \{U_n\}$ is *constant* for an element. The matrix $[D_b][L]$ depends upon ξ and η but is particularly simple due to the nature of both $[D_b]$ and $[L]$. Thus the computation of $\{M\}$ at several points on an element is done in a minimum number of operations. The bending moments are not continuous along the inter element sides.

The number of algebraic operations (multiplications and additions) is reduced by nearly 20 per cent if the explicit formulation is used instead of the standard method (presented in Reference 1) for the evaluation of the stiffness matrix in the global plate co-ordinate system. For the calculation of bending moments at four points the number of operations is reduced to 50 per cent of the standard formulation. It should also be mentioned, however, that the FORTRAN code is slightly longer, for the explicit formulation, than the standard one.

ADDITIONAL NUMERICAL RESULTS

The numerical results presented in the following section deal with some plate-bending problems suggested by Robinson[2,3] to assess the engineering plate-bending elements. These results reinforce the conclusions presented in Reference 1.

Patch test

The validity of the formulation of the DKT element is (partly) checked by considering a rectangular plate with boundary conditions and concentrated loads that give a theoretical constant state of bending moments all over the plate (Figure 4). This patch test problem is defined in Reference 2. The correct values $M_x = M_y = M_{xy} = 1$ are obtained everywhere in the plate considering four DKT elements with the central node 5 located arbitrarily.

The patch test is also passed satisfactorily for two other 9 DOF plate-bending elements, namely HCT,[5] a displacement model, and HSM a hybrid stress element the formulation of which is reconsidered in Reference 1. The results obtained with HSM are the same with those using the A–9 element of Reference 6. For meshes of the type given in Figure 4, however, the patch test is not passed with the BCIZ1,[7] which is a 9 DOF incompatible displacement triangular element. The details of the formulation are given in Reference 8, chapter 10. This element is still implemented in some computer codes. The patch test results confirm the dangerous use of BCIZ1 in practical applications. In Table II we give the values of the bending moments obtained at the centroids of the four elements with node 5 located at $x = \frac{3}{4}a$ and $y = \frac{3}{2}b$ ($E = 1000$; $h = 1$; $\nu = 0{\cdot}3$; $a = 20$; $b = 10$). The error reaches 30 per cent for M_x, element 4. Greater errors are obtained if the evaluation of stresses is done at other points in the elements!

Table II. Values of M_x, M_y, M_{xy} at centroids for the patch test problem of Figure 4 using DKT, HSM, HCT and BCIZ1†

Nine DOF triangular element	w_3	w_5	M_x, M_y, M_{xy} at centroid element 1	element 2	element 3	element 4
DKT			+1·00	+1·00	+1·00	+1·00
HSM	−12·480	−1·137	+1·00	+1·00	+1·00	+1·00
HCT			+1·00	+1·00	+1·00	+1·00
			+0·944	+0·832	+1·007	+1·298
BCIZ1	−12·475	−0·925	+0·840	+1·111	+0·867	+1·200
			+1·019	+1·018	+0·992	+0·957

† $E = 1000$; $\nu = 0{\cdot}3$; $h = 1$; $a = 20$; $b = 10$.

Behaviour of a cantilever rectangular plate under twisting loads

As stated in References 2 and 3 the critical test for a single quadrilateral element or two triangular elements is usually the case of a plate under twist moments with one side fully clamped which activates differential bending. This differential bending can be obtained using differential loads (Figures 5 and 6) or twisting moments at the free corner nodes (Figures 7 and 8). The plate is discretized with two triangular elements. Two mesh orientations are possible (Figures 5–8). The values of the deflection w at one free corner is plotted for increasing length L for the two mesh orientations and for the two loading cases. Figures 5 and 6 deal with the case of differential loads and Figures 7 and 8 deal with the case of twisting moments. The data are given on Figure 5. The results obtained for DKT, BCIZ1, HCT and HSM are

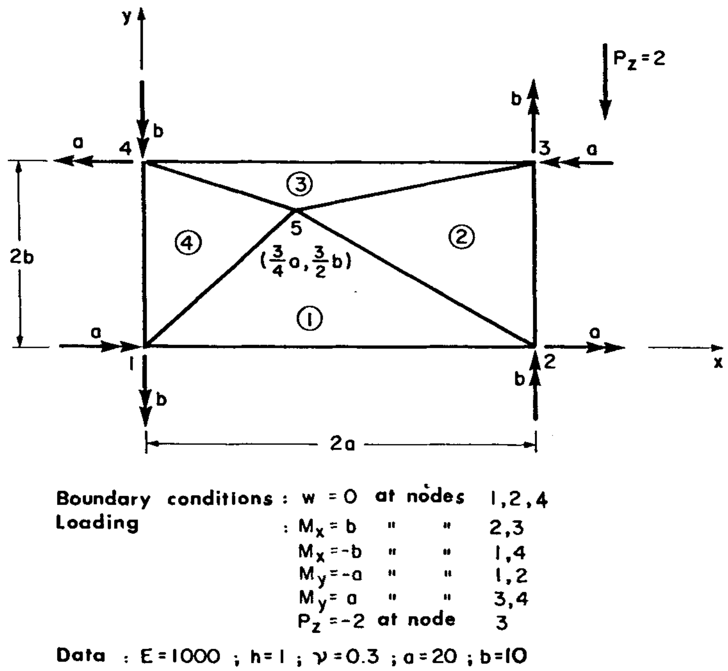

Figure 4. Patch test problem

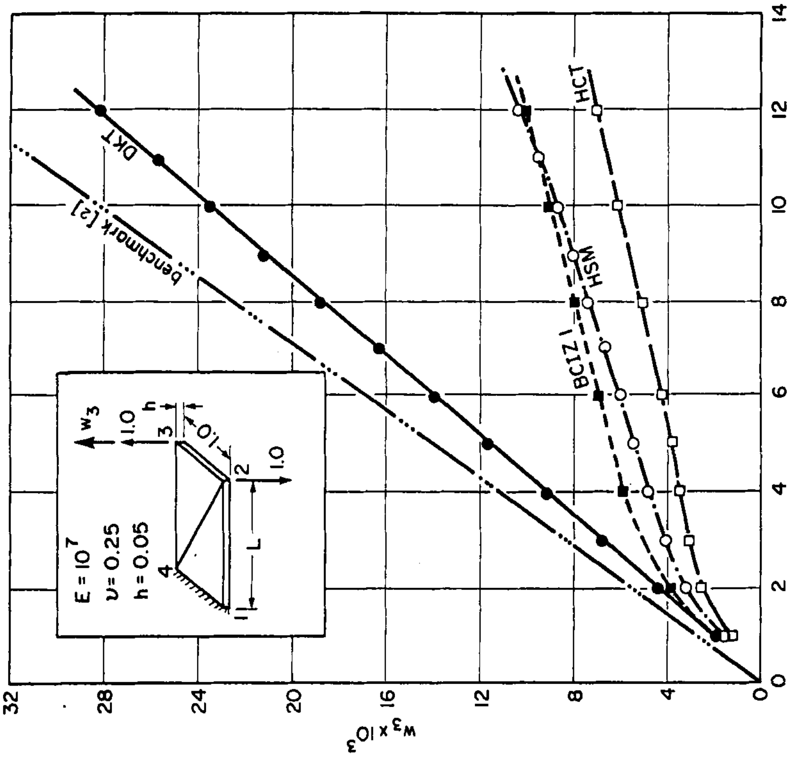

Figure 6. Rectangular cantilever plate under differential loads—w_3 vs. L for mesh B

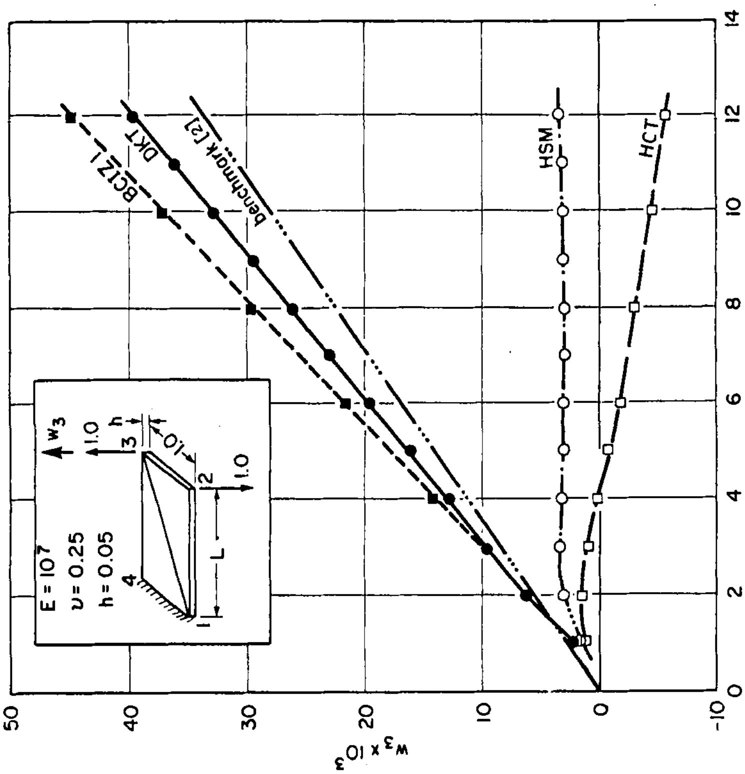

Figure 5. Rectangular cantilever plate under differential loads—w_3 vs. L for mesh A

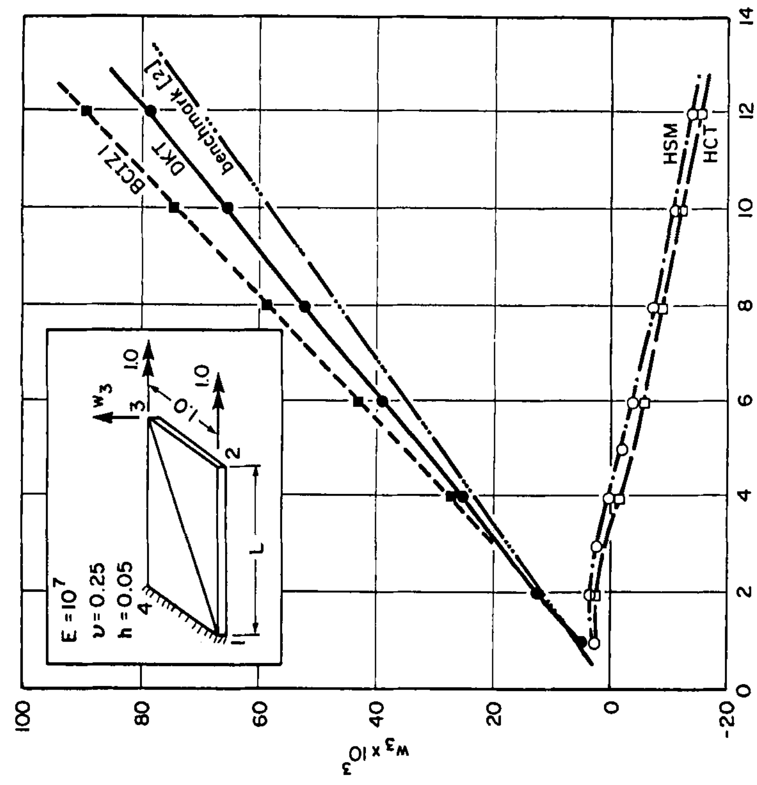

Figure 7. Rectangular cantilever plate under twisting momen vs. L for mesh A

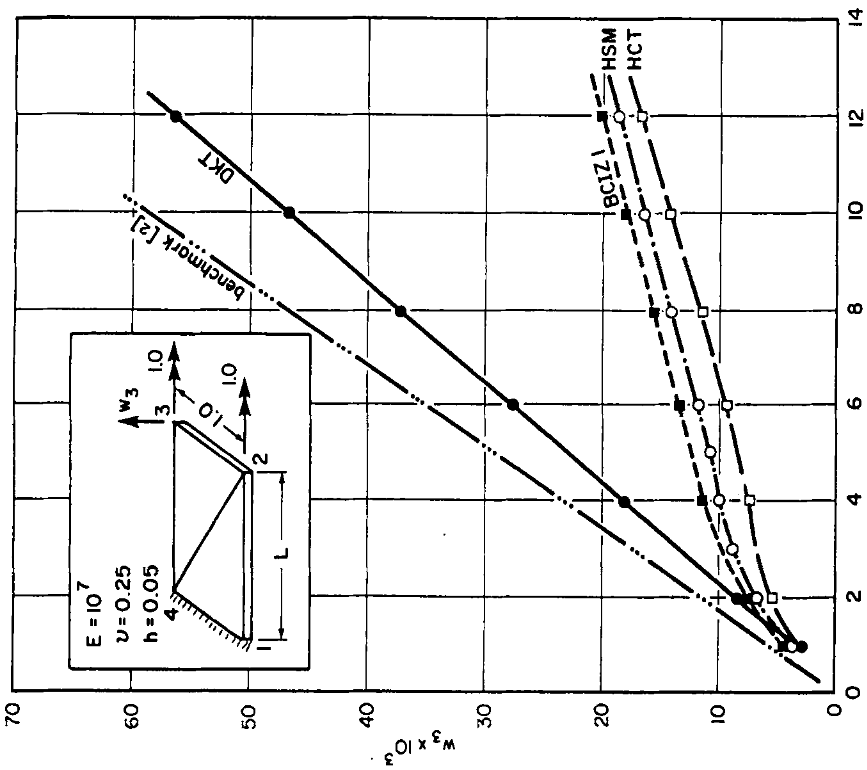

Figure 8. Rectangular cantilever plate under twisting moments—w_3 vs. L for mesh B

compared with the *benchmark* values obtained using 16 compatible rectangular elements with 16 DOF,[2,3] based on the classical thin plate theory. Figures 5–8 show that the DKT element behaves much better than the other three 9 DOF elements. The results using HSM and HCT are valid for aspect ratios less than two. The effect of mesh orientation is eliminated if the results (w versus L) are plotted for four overlapping triangles. Figures 9 and 10 show that the DKT element gives results that closely follow the benchmark for an aspect ratio of 12. The incompatible BCIZ1 element performs much better than the HCT and HSM element. The results obtained in this study with HSM correspond to the results associated with TBC3(DS)-ASAS in Reference 2, Figure A1.1.

Convergence studies for rectangular plates

A detailed study on the behaviour of several 9 DOF triangular plate-bending elements for the analysis of square plates has been presented in Reference 1. In this note we present the results of the analysis of clamped and simply-supported rectangular plates (with aspect ratios of 1, 2, 3) subjected to a central concentrated load. The results obtained with DKT and HSM are compared for the two edge conditions. Figures 11 and 12 deal with the error on the central displacement versus the number of subdivisions along each side. Four different quarter-plate meshes are considered involving 2, 8, 32 and 128 elements, respectively. Figures 13 deals with the error in corner reaction for the simply-supported plate. Figure 14 deals with the error in bending moment at the middle of the longest side. The reference value are those taken from Timoshenko[9] with Poisson's ratio equal to 0·3.

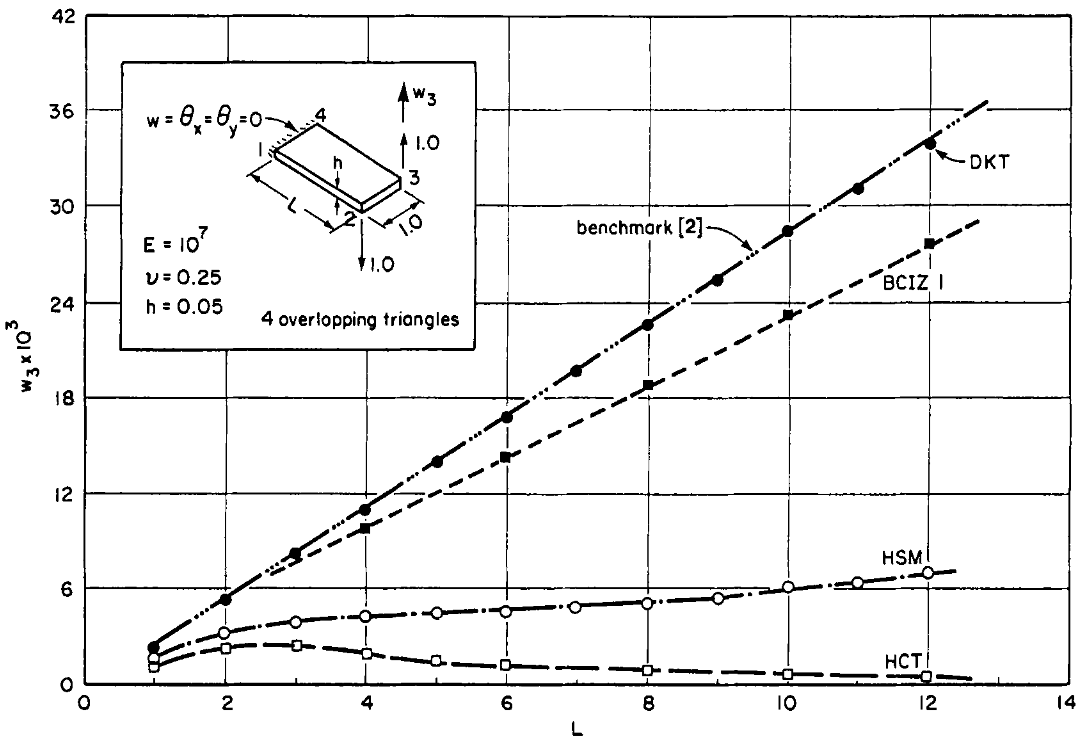

Figure 9. Rectangular plate under differential loads—w_3 vs. L for four overlapping triangles

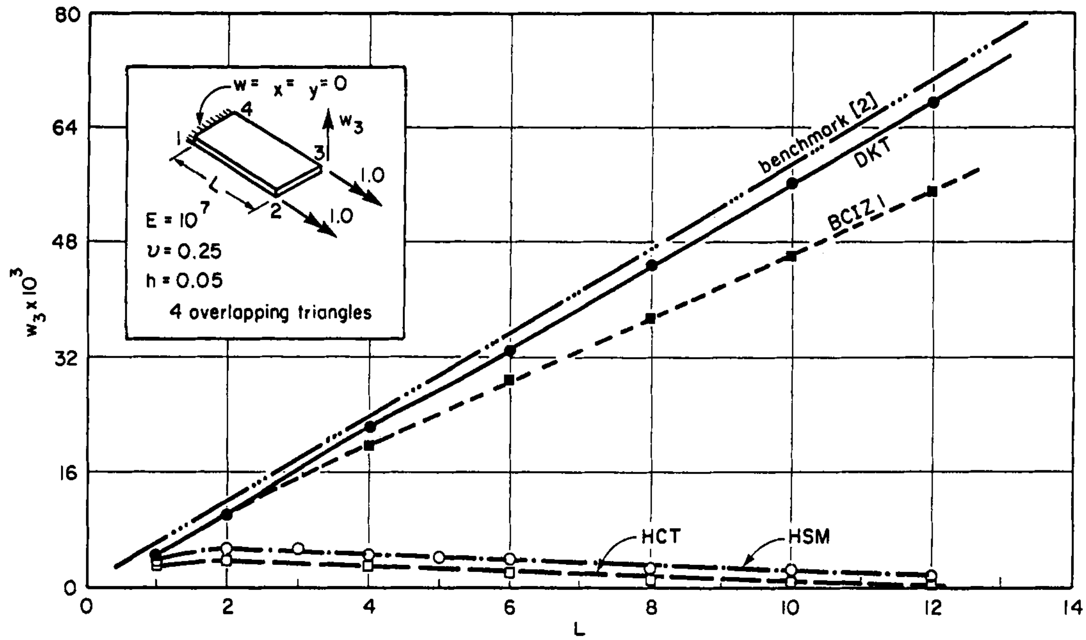

Figure 10. Rectangular cantilever plate under twisting moments—w_3 vs. L for four overlapping triangles

Figures 11–14 show that the good convergence properties of the DKT element do not deteriorate with the element aspect ratio. For the evaluation of the central displacement the convergence rate is much lower with HSM than for DKT for the longest rectangular plate ($b/a = 3$). The convergence characteristics for the stresses evaluated at one point are less marked. For the corner reaction (Figure 13) the DKT element gives better results for the rectangular plates than the HSM element, but the opposite is true for the bending moment (Figure 14).

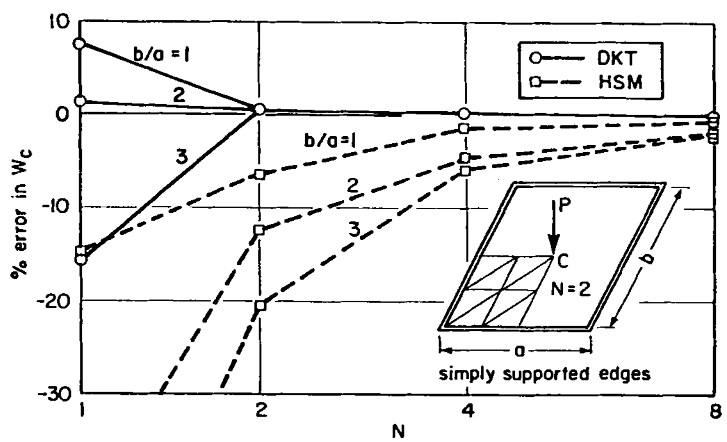

Figure 11. Simply-supported rectangular plate under concentrated load—error in central displacement for various meshes

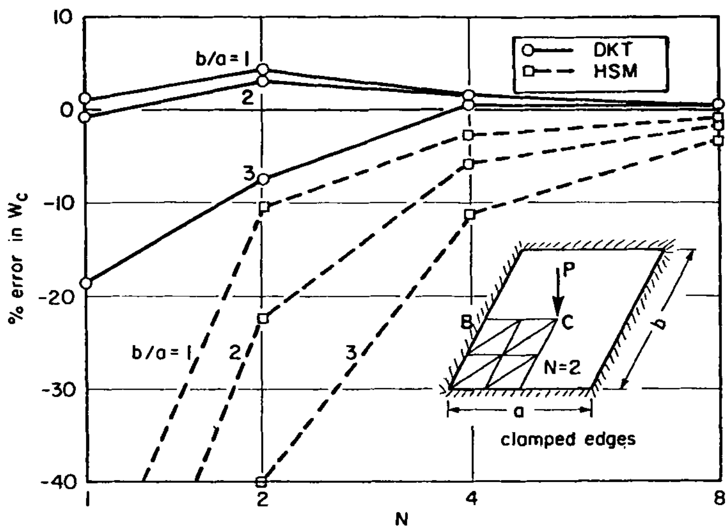

Figure 12. Clamped rectangular plate under concentrated load—error in central displacement for various meshes

CONCLUDING REMARKS

A new explicit expression (without numerical integration) for the stiffness matrix of the triangular plate-bending element named DKT is presented. The formulation based on that of Reference 4 allows a significant reduction in computer time compared with the standard evaluation using three numerical integration points. The results presented in this paper, together with the previous results published in Reference 1, confirm the excellent behaviour of the DKT element compared to the other 9 DOF plate-bending triangular elements. The

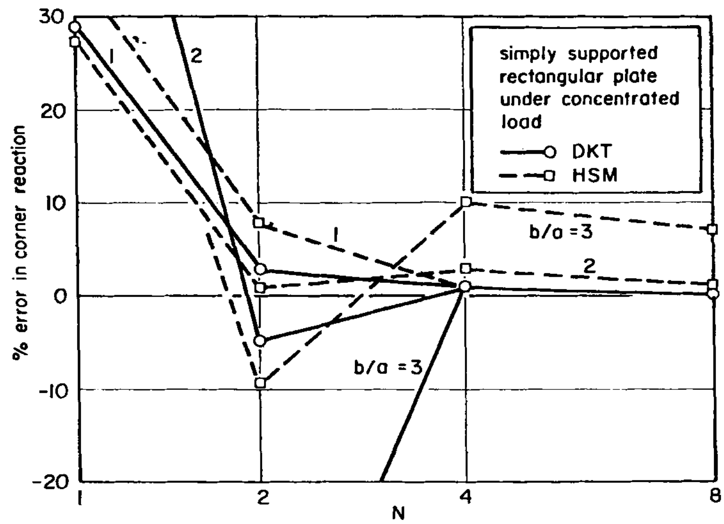

Figure 13. Simply-supported rectangular plate under concentrated load—error in corner reaction for various meshes

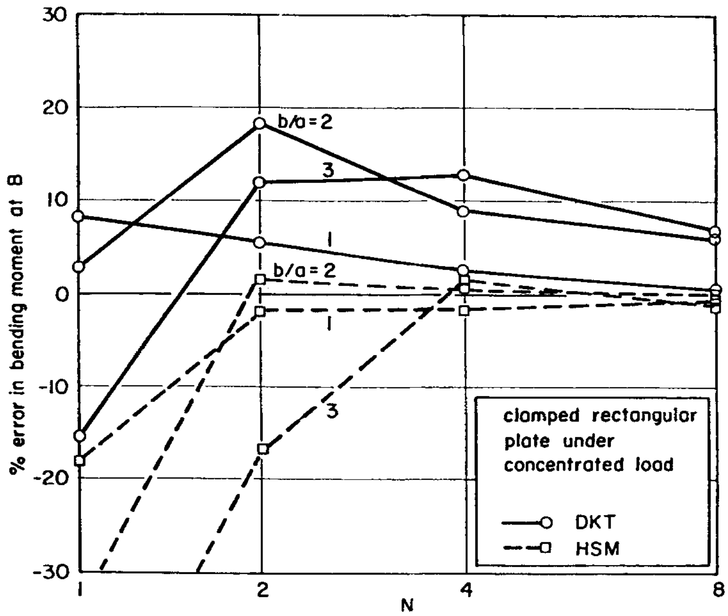

Figure 14. Clamped rectangular plate under concentrated load—error in bending moment at mid-side for various meshes

formulation of this displacement element is simple and is such that the continuity of all essential variables along the element sides is ensured. The convergence towards the thin plate solutions is guaranteed. Results are accurate and reliable with a very good performance with respect to element aspect ratio.

ACKNOWLEDGEMENTS

The author expresses his gratitude to G. Cantin, G. Dhatt and J. P. Jameux for their helpful comments.

REFERENCES

1. J. L. Batoz, K. J. Bathe and L. W. Ho, 'A study of three-node triangular plate bending elements', *Int. J. num. Meth. Engng*, **15**, 1771–1812 (1980).
2. J. Robinson, 'Element evaluation. A set of assessment points and standard tests', *Proc. F.E.M. in the Commercial Environment*, vol. 1, pp. 217–248 (Oct. 1978).
3. J. Robinson and G. Haggenmacher, 'Lora—An accurate four nodes stress plate bending element', *Int. num. Meth. Engng*, **14**(2), 296–306 (1979).
4. K. T. Joseph and K. S. Rao, 'A fast algorithm for triangular plate bending element', *Int. J. num. Meth. Engng*, **14**, 1100–1104 (1979).
5. R. W. Clough and J. L. Tocher, 'Finite element stiffness matrices for analysis of plate bending', *Proc. Conf. on Matrix Methods in Structural Mechanics*, WPAFB, Ohio, 1965, pp. 515–545.
6. A. Razzaque, 'Program for triangular bending elements with derivative smoothing', *Int. J. num. Meth. Engng*, **6**, 333–343 (1973).
7. G. P. Bazeley, Y. K. K. Cheung, B. M. Irons and O. C. Zienkiewicz, 'Triangular elements in plate bending—conforming and non-conforming solutions', *Proc. Conf. on Matrix Methods in Structural Mechanics*, WPAFB, Ohio, 1965, pp. 547–576.
8. O. C. Zienkiewicz, '*The Finite Element Method*', 3rd edn, McGraw-Hill, London, 1977.
9. S. Timoshenko and S. Woinowsky-Krieger, '*Theory of Plates and Shells*', 2nd edn, McGraw-Hill, New York, 1969.