Lecture 7
1-D FEA — Bars

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Introduction to Finite Element Methods
MAE M168/CEE 135C

Section 7.1

Interpolation and Shape Functions in 1-D

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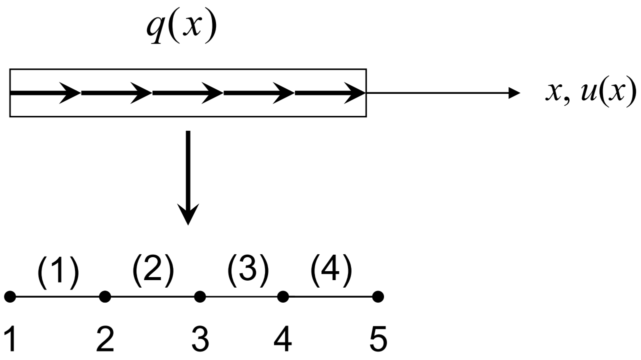
Recall basic procedure:

1. **Discretize** a continuum into discrete pieces each with simple geometry (*finite elements*).
2. **Approximate** the *continuous PDE solution* over each piece by interpolating among *discrete nodal values*.
3. **Assemble** discrete element approximation into global *algebraic equations* (matrices, vectors)

1. Discretize



Approximation by Interpolation

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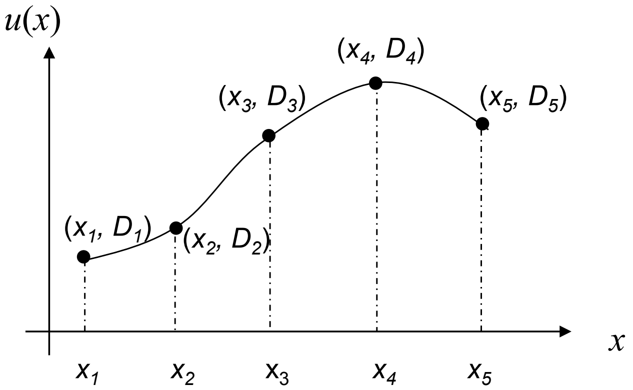
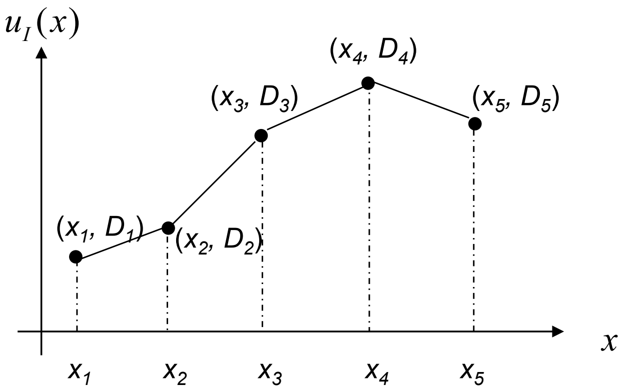
2. Approximate local behavior using DOF
   - Define $u_I(x)$ by local polynomial interpolation of $u(x)$.
   - Let $x_1, x_2, \ldots\ldots, x_5$ be nodal positions.
   - Define: $D_i = u(x_i)\ i = 1, 2, \ldots\ldots$ Nodal Displacements





Linear Interpolation

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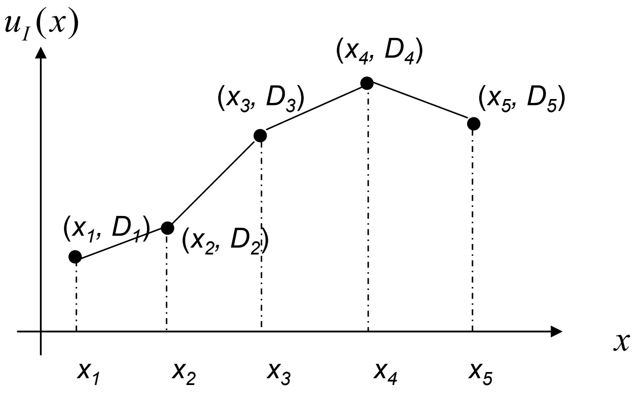

- Interpolation: Connecting the dots, to define $u_I(x)$ in between the D_i's.
- Use interpolation as approximation in R-R or Galerkin. $\hat{u}(x) = u_I(x)$
- Convergence requirements:
  - Approximation must be smooth (energy & weak form finite).
  - EBC's must be met.
  - Basis functions should be complete

Section 7.2

Linear Shape Functions

Linear Displacement Over Element

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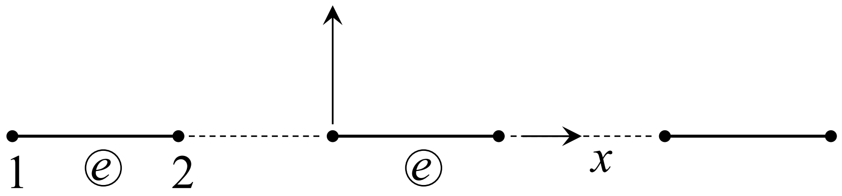
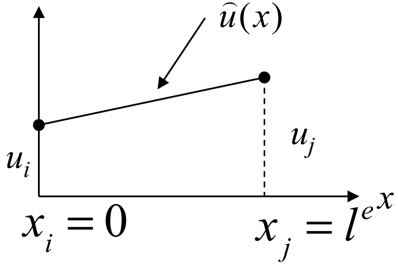


Local coordinate system
for element (e)

- Want $\hat{u}(x)$ linear over element (e) $\Rightarrow \hat{u}(x) = a + bx$
- But $\hat{u}(x)$ interpolates between u_i and u_j

$$\Rightarrow \begin{aligned} u_i &= \hat{u}(x_i) = a + bx_i \\ u_j &= \hat{u}(x_j) = a + bx_j \end{aligned}$$



Shape Functions

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$$\Rightarrow \begin{array}{ll} u_i = \hat{u}(x_i) = a + bx_i & x_i = 0 \\ u_j = \hat{u}(x_j) = a + bx_j & x_j = l^e \end{array} \qquad \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l^e \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}$$

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = \frac{1}{l^e} \begin{bmatrix} l^e & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} u_i / l^e \\ (u_j - u_i)/l^e \end{Bmatrix}$$

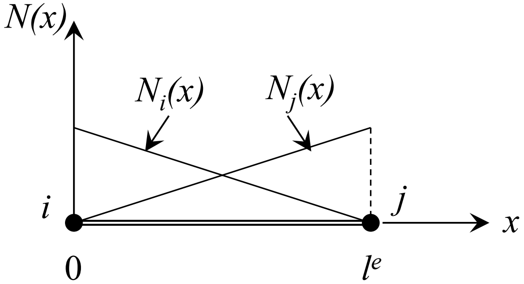
$$\hat{u}(x) = \underbrace{\left(1 - \frac{x}{l^e}\right)}_{N_i(x)} u_i + \underbrace{\frac{x}{l^e}}_{N_j(x)} u_j \quad = N_i(x)u_i + N_j(x)u_j$$

Basis or ***shape*** functions

Shape Functions: Matrix Form

$$\hat{u}(x) = N_i(x)u_i + N_j(x)u_j$$

$$N_i(x) = 1 - \frac{x}{l^e} \; ; \; N_j(x) = \frac{x}{l^e}$$



$$\hat{u}(x) = \begin{bmatrix} N_i^e(x) & N_j^e(x) \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \qquad \mathbf{N^e(x)} \qquad \mathbf{d^e}$$

Important Properties of Shape Functions

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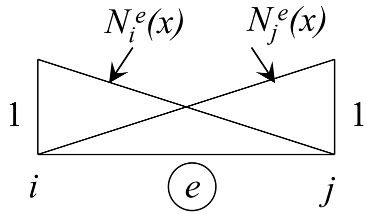
$$N_i(x_i) = 1$$

$$N_i(x_j) = 0$$



$$N_j(x_j) = 1$$

$$N_j(x_i) = 0$$

$$N_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kronecker Delta Property

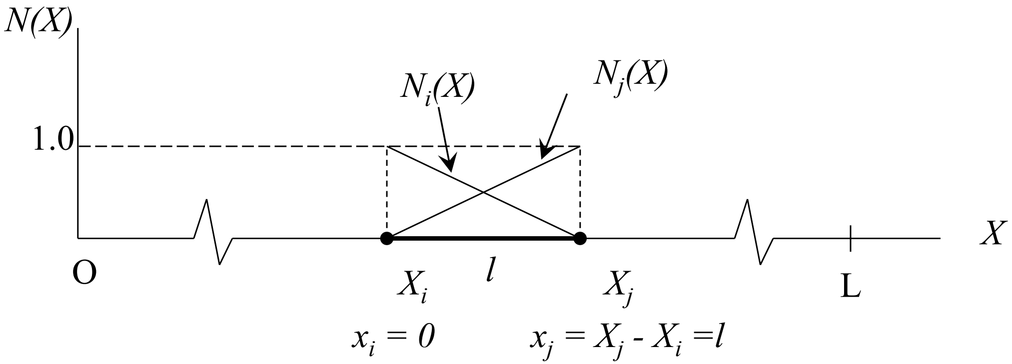
$$N_i^e(x) + N_j^e(x) = 1$$

Partition of Unity Property

Global Axial Coordinate

- Rewrite shape function in global axial coordinate X using:

$$x = X - X_i \;\text{ for }\; 0 \le x \le l$$



Shape Functions: Global Axial Coordinate

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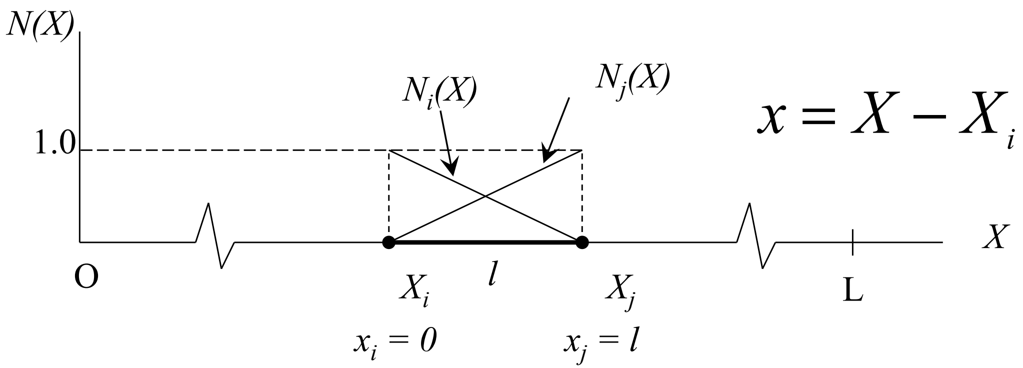


$$x = X - X_i \text{ for } 0 \le x \le l$$

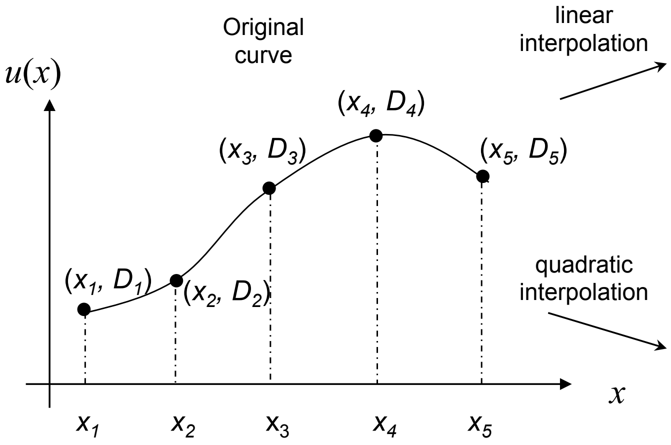
$$N_i = 1 - \frac{x}{l} \quad \Rightarrow N_i = 1 - \left(\frac{X - X_i}{l} \right) = \frac{l - X + X_i}{l} = \frac{(X_i + l) - X}{l} = \frac{X_j - X}{l}$$

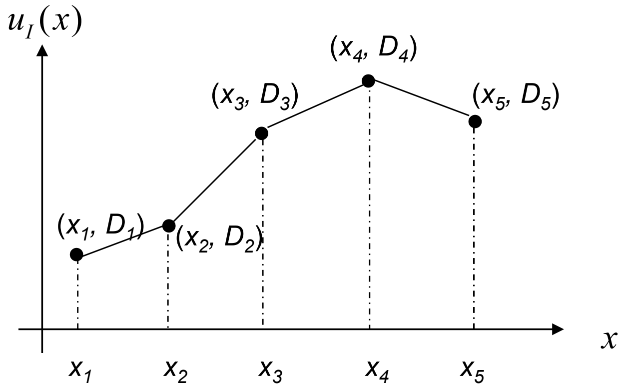
$$N_j = \frac{x}{l} = \frac{X - X_i}{l}$$

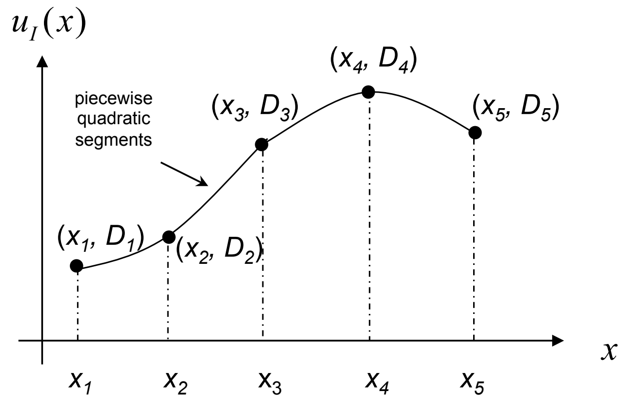
Section 7.3

Quadratic Shape Functions

Quadratic Interpolation







$$D_i = u(x_i) \; i = 1, 2, \ldots..$$

Quadratic Shape Functions

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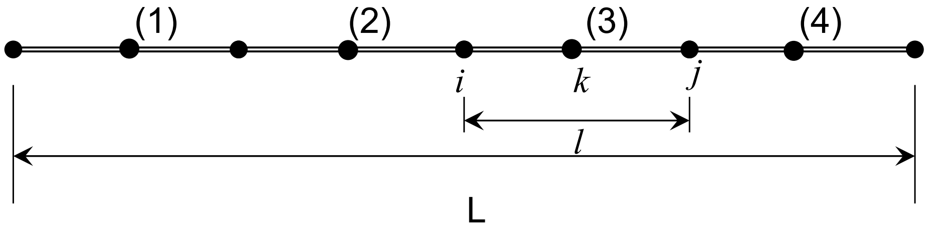
- Consider four-element model of the bar.
- Three data points define a unique quadratic polynomial.
- Elements are required to have three nodes i, j and k.



Quadratic Function

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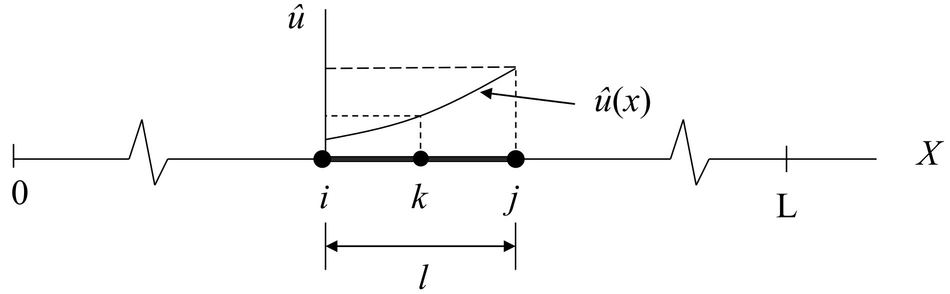
- Piecewise approximation of u in each element is to be a quadratic function.
- In any element u is a linear combination of three displacement modes

$$u \approx \hat{u}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad \left(0 \le x \le l\right)$$



Deriving Shape Functions

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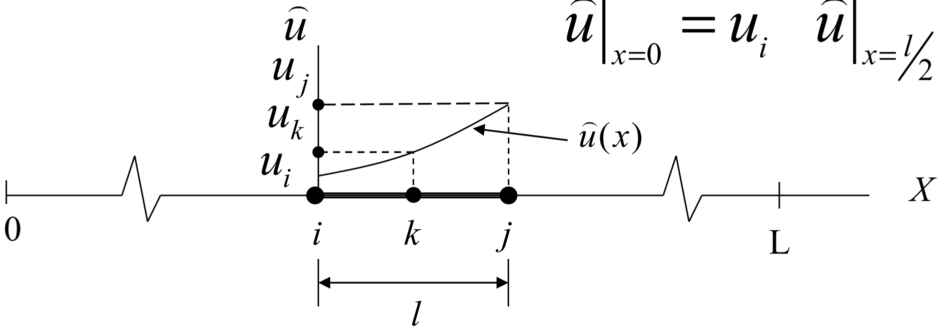
- Rewrite $\hat{u}(x)$ in terms of shape functions and nodal displacements.
  - Solve amplitudes α_1, α_2 and α_3 in terms of nodal displacements u_i, u_j and u_k .
  - Interpolation conditions:

$$\hat{u}\Big|_{x=0} = u_i \quad \hat{u}\Big|_{x=l/2} = u_k \quad \hat{u}\Big|_{x=l} = u_j$$



$$u \approx \hat{u}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad \left(0 \le x \le l\right)$$

Deriving Shape Functions

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$$u \approx \hat{u}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad (0 \le x \le l)$$

$$\hat{u}\big|_{x=0} = u_i \quad \hat{u}\big|_{x=l/2} = u_k \quad \hat{u}\big|_{x=l} = u_j$$

These above condition provides the following:

$$\alpha_1 + \alpha_2(0) + \alpha_3(0)^2 = u_i$$
$$\alpha_1 + \alpha_2(l/2) + \alpha_3(l/2)^2 = u_k$$
$$\alpha_1 + \alpha_0(l) + \alpha_3(l)^2 = u_j$$

Solve for α_i's

Quadratic Shape Functions

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- Substitute α_1, α_2 and α_3 in $\hat{u}(x)$

$$\hat{u}(x) = u_i + \frac{4}{l}\left[u_k - u_i - \frac{1}{4}\left(u_j - u_i\right)\right]x + \frac{4}{l^2}\left[-u_k + u_i - \frac{1}{2}\left(u_j - u_i\right)\right]x^2$$

$$\Rightarrow \hat{u}(x) = \underbrace{\frac{2}{l^2}\left(x - \frac{l}{2}\right)(x - l)}_{N_i(x)}u_i + \underbrace{\frac{2}{l^2}x\left(x - \frac{l}{2}\right)}_{N_j(x)}u_j \underbrace{- \frac{4}{l^2}x(x - l)}_{N_k(x)}u_k$$

- $N_i(x), N_j(x)$ and $N_k(x)$ are defined as coefficients of u_i, u_j and u_k

Basic Properties of Shape Functions

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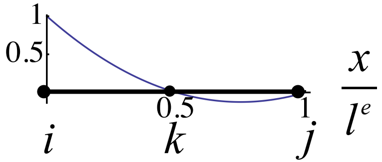
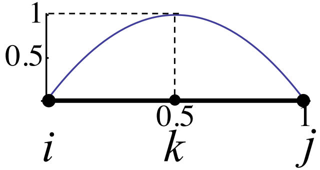
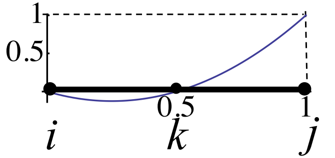
$$\hat{u}(x) = N_i(x)u_i + N_j(x)u_j + N_k(x)u_k$$

$$N_i(x) = \frac{2}{l^2}\left(x - \frac{l}{2}\right)\left(x - l\right) \qquad N_k(x) = -\frac{4}{l^2}x\left(x - l\right) \qquad N_j(x) = \frac{2}{l^2}x\left(x - \frac{l}{2}\right)$$







$$N_i(0) = 1 \qquad N_k(0) = 0 \qquad N_j(0) = 0$$

$$N_i(l/2) = 0 \qquad N_k(l/2) = 1 \qquad N_j(l/2) = 0$$

$$N_i(l) = 0 \qquad N_k(l) = 0 \qquad N_j(l) = 1$$

Kronecker delta and partition of unity properties satisfied

Section 7.4

N-th Order Interpolation

Higher Order Interpolation in 1-D ("P-Elements")

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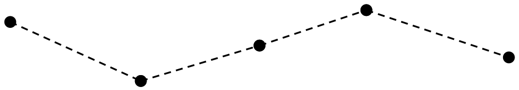
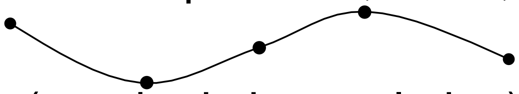
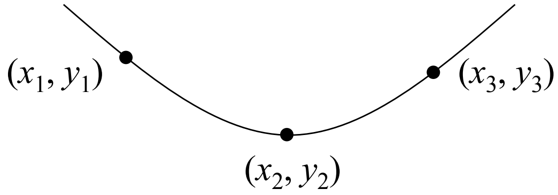
- Linear Interpolation



- Want Smoother approximation: piecewise quadratic, cubic, etc….



- Example: can fit a unique parabola (quadratic interpolation) to 3 points



Higher Order Polynomial

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- In general (polynomial of degree n)

$$y(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

- In Matrix form

$$y(x) = \underbrace{\begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix}}_{\mathbf{X}(x)} \underbrace{\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}}_{\mathbf{a}}$$

- Want interpolation of $n+1$ point pairs $(x_1, y_1)\cdots(x_{n+1}, y_{n+1})$

$$\Rightarrow y(x_i) = y_i \quad \text{for } i = 1,\ldots,n+1$$

Interpolate n+1 Point Pairs

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- Want interpolation of $n+1$ point pairs $(x_1, y_1)\cdots(x_{n+1}, y_{n+1})$

$$\Rightarrow y(x_i) = y_i \quad \text{for } i = 1,\ldots,n+1$$

$$\Rightarrow y_1 = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_n \end{Bmatrix} \ldots \; y_i = \begin{bmatrix} 1 & x_i & x_i^2 & \cdots & x_i^n \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_n \end{Bmatrix}$$

$$\ldots \; y_{n+1} = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_n \end{Bmatrix}$$

Interpolate n+1 Point Pairs

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$$y_1 = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_n \end{Bmatrix} \quad y_2 = \begin{bmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^n \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_n \end{Bmatrix} \quad \ldots \quad y_{n+1} = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{bmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_n \end{Bmatrix}$$

$$\underbrace{\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{n+1} \end{Bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \\ \vdots & & & & \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_n \end{Bmatrix}}_{\mathbf{a}} \qquad \Rightarrow \mathbf{a} = \mathbf{A}^{-1}\mathbf{y}$$

$$y(x) = \mathbf{X}(x)\mathbf{a} \qquad \Rightarrow y(x) = \mathbf{X}(x)\mathbf{A}^{-1}\mathbf{y} = \mathbf{N}(x)\mathbf{y}$$

$$y(x) = \underbrace{\begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix}}_{\mathbf{X}(x)} \underbrace{\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}}_{\mathbf{a}}$$

Shape Function

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$$\Rightarrow y(x) = \mathbf{X}(x)\mathbf{A}^{-1}\mathbf{y} = \mathbf{N}(x)\mathbf{y}$$

$$\mathbf{N}(x) = \mathbf{X}(x)\mathbf{A}^{-1} = \begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \\ \vdots & & & & \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} N_1(x) & N_2(x) & \cdots & N_{n+1}(x) \end{bmatrix}$$

Linear Shape Functions (n=1)

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- For n =1 (linear)

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \frac{1}{x_2 - x_1}$$

$$\mathbf{N} = \mathbf{X}\mathbf{A}^{-1} = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \frac{1}{x_2 - x_1} = \begin{bmatrix} \underbrace{\dfrac{x_2 - x}{x_2 - x_1}}_{N_1} & \underbrace{\dfrac{x - x_1}{x_2 - x_1}}_{N_2} \end{bmatrix}$$

Quadratic Shape Functions (n=2)

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- For n =2 (quadratic)

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

$$\mathbf{N} = \mathbf{X}\mathbf{A}^{-1} = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix}$$

Quadratic Shape Functions (n=2)

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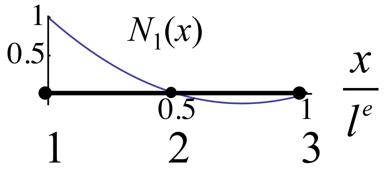
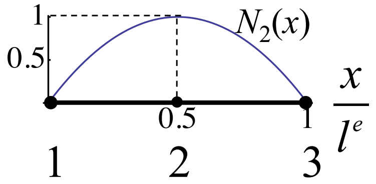
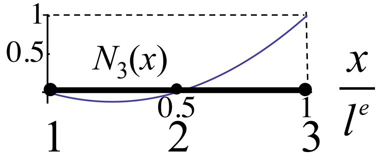
$$N_1(x) = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}$$



$$N_2(x) = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}$$



$$N_3(x) = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$



Note: Partition of unity &
Kronecker delta properties

Higher Order Shape Functions

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$$n=1$$

$$N_1(x)=\frac{(x_2-x)}{(x_2-x_1)}$$

$$N_2(x)=\frac{(x_1-x)}{(x_1-x_2)}$$

$$n=2$$

$$N_1(x)=\frac{(x_2-x)(x_3-x)}{(x_2-x_1)(x_3-x_1)}$$

$$N_2(x)=\frac{(x_1-x)(x_3-x)}{(x_1-x_2)(x_3-x_2)}$$

$$N_3(x)=\frac{(x_1-x)(x_2-x)}{(x_1-x_3)(x_2-x_3)}$$

$$n=?$$

$$N_i(x)=\frac{(x_1-x)(x_2-x)\cdots(x_{i-1}-x)(x_{i+1}-x)\cdots(x_{n+1}-x)}{(x_1-x_i)(x_2-x_i)\cdots(x_{i-1}-x_i)(x_{i+1}-x_i)\cdots(x_{n+1}-x_i)}$$

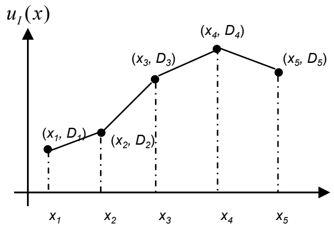
Section 7.5

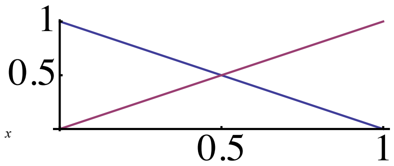
Elastic Bar Elements: Rayleigh-Ritz

Summary & Review

- Piecewise Linear interpolation

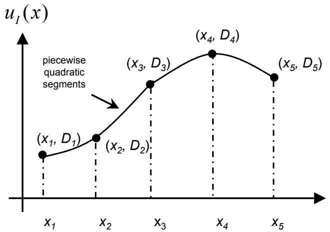
$$\hat{u}(x) = N_i(x)u_i + N_j(x)u_j$$

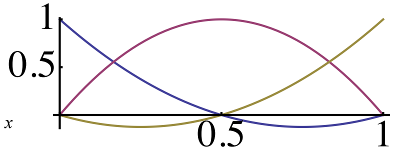




- Piecewise Quadratic interpolation

$$\hat{u}(x) = N_i(x)u_i + N_j(x)u_j + N_k(x)u_k$$



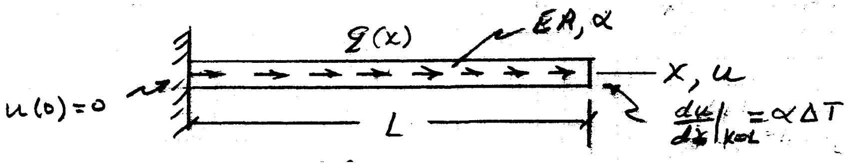


- Higher-order…

- Global approximation

$$\hat{u}(X) = D_1 N_1(X) + D_2 N_2(X) + \ldots = \sum_{i=1}^{N} D_i N_i(X)$$

1-D Elasticity with Thermal Strains



- Separate Strain into Elastic and Thermal Parts:

$$\underbrace{\frac{du}{dx}}_{\text{Total Strain}} = \underbrace{\varepsilon}_{\text{Elastic Strain}} + \underbrace{\overbrace{\alpha}^{\text{CTE}}\ \Delta T}_{\text{Thermal Strain}}$$

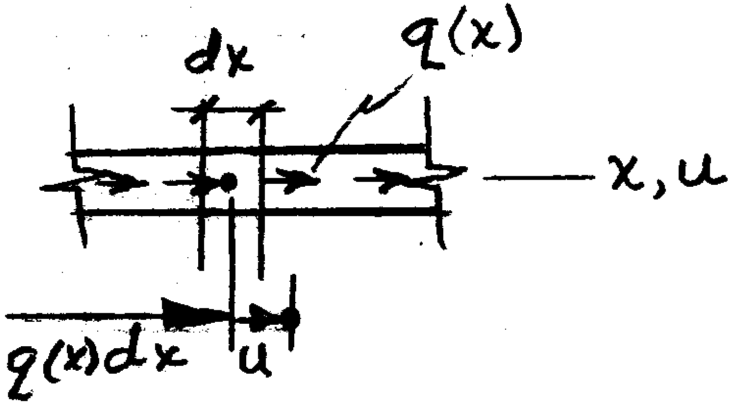
- Elastic Stress:

$$\sigma = E\varepsilon = E\left(\frac{du}{dx} - \alpha\Delta T\right) = \underbrace{E\frac{du}{dx}}_{\text{"Total Stress"}} - \underbrace{E\alpha\Delta T}_{\text{"Thermal Stress"}}$$

- Strain Energy:

$$U = \int_0^L \frac{1}{2}\sigma\varepsilon A dx = \int_0^L \frac{1}{2}EA\varepsilon^2 dx = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx} - \alpha\Delta T\right)^2 dx$$

Include Work Done by Distributed Loads

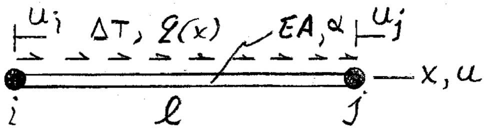


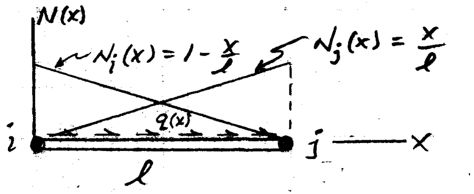
$$dV = -q(x)u\,dx \qquad V = \int_0^L dV = -\int_0^L q(x)u\,dx$$

Total Potential Energy:

$$\Pi[u] = U - V = \int_0^L \left[\frac{1}{2} EA \left(\frac{du}{dx} - \alpha \Delta T \right)^2 - q(x)u \right] dx$$

Linear Element

Displacement dof	Two-Node Linear Finite Element Subjected to Distributed Load $q(x)$
$\mathbf{d} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$	



$$u(x) \approx \hat{u}(x) = N_i(x)u_i + N_j(x)u_j$$

$$\hat{u}(x) = \begin{bmatrix} N_i^e(x) & N_j^e(x) \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \mathbf{N^e(x)} \, \mathbf{d^e}$$

$$\frac{du}{dx} = u_i \frac{dN_i^e}{dx} + u_j \frac{dN_j^e}{dx} = \underbrace{\begin{bmatrix} \frac{dN_i^e}{dx} & \frac{dN_j^e}{dx} \end{bmatrix}}_{\mathbf{B}^e(x)} \underbrace{\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}}_{\mathbf{d^e}}$$

$\mathbf{B}^e(x)$: Strain-displacement operator

Element Stiffness Matrix

UCLA

$$\varepsilon^2 = \left(\mathbf{B}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}} - \alpha\Delta T\right)^2 = \left(\mathbf{B}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}}\right)^2 - 2\alpha\Delta T\mathbf{B}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}} + (\alpha\Delta T)^2$$

$$= \mathbf{d}^{\mathrm{e}^T}\mathbf{B}^{\mathrm{e}^T}\mathbf{B}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}} - 2\alpha\Delta T\mathbf{B}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}} + (\alpha\Delta T)^2$$

$$\therefore U^e = \int_0^{l^e} \frac{EA(x)}{2}\left[\mathbf{d}^{\mathrm{e}^T}\mathbf{B}^{\mathrm{e}^T}\mathbf{B}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}} - 2\alpha\Delta T\mathbf{B}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}} + (\alpha\Delta T)^2\right]dx$$

$$\therefore U^e = \frac{1}{2}\mathbf{d}^{\mathrm{e}^T}\left(\int_0^{l^e} EA(x)\mathbf{B}^{\mathrm{e}^T}\mathbf{B}^{\mathrm{e}}\,dx\right)\mathbf{d}^{\mathrm{e}} - \left(\int_0^{l^e} EA\alpha\Delta T\mathbf{B}^{\mathrm{e}}\,dx\right)\mathbf{d}^e + const$$

$$= \frac{1}{2}\mathbf{d}^{\mathrm{e}^T}\mathbf{k}^e\mathbf{d}^{\mathrm{e}} - \mathbf{q}_{\Delta T}^{e^T}\mathbf{d}^e = \frac{1}{2}\begin{bmatrix} d_i^e & d_j^e \end{bmatrix}\begin{bmatrix} k_{ii}^e & k_{ij}^e \\ k_{ji}^e & k_{jj}^e \end{bmatrix}\begin{Bmatrix} d_i^e \\ d_j^e \end{Bmatrix} - \begin{bmatrix} q_{\Delta Ti}^e & q_{\Delta Tj}^e \end{bmatrix}\begin{Bmatrix} d_i^e \\ d_j^e \end{Bmatrix}$$

$$\mathbf{k}^e = \int_0^{l^e} EA(x)\mathbf{B}^{\mathrm{e}^T}\mathbf{B}^{\mathrm{e}}\,dx$$ Element Stiffness Matrix

$$\mathbf{q}_{\Delta T}^e = \int_0^{l^e} EA\alpha\Delta T\mathbf{B}^{\mathrm{e}^T}\,dx$$ Element Thermal Load Vector

Element External Loads

UCLA

$$V_e = -\int_0^{l^e} qu\,dx = -\int_0^{l^e} q\mathbf{N}^{\mathbf{e}}(x)\mathbf{d}^{\mathbf{e}}dx \quad = -\underbrace{\left(\int_0^{l^e} q\mathbf{N}^{\mathbf{e}}(x)dx\right)}_{\mathbf{q}^{\mathbf{e}^T}}\mathbf{d}^{\mathbf{e}}$$

$$= -\mathbf{q}^{\mathbf{e}^T}\mathbf{d}^{\mathbf{e}} = -\begin{bmatrix} f_i^e & f_j^e \end{bmatrix}\begin{Bmatrix} u_i^e \\ u_j^e \end{Bmatrix}$$

$$\mathbf{q}^{\mathbf{e}^T} = -\int_0^{l^e} q(x)\mathbf{N}^{\mathbf{e}}(x)dx = -\int_0^{l^e} q(x)\begin{bmatrix} N_i^e(x) & N_j^e(x) \end{bmatrix}dx$$

$$\text{or } \mathbf{q}^{\mathbf{e}} = -\int_0^{l^e} q(x)\begin{Bmatrix} N_i^e(x) \\ N_j^e(x) \end{Bmatrix}dx = \begin{Bmatrix} \int_0^{l^e} q(x)N_i^e(x)dx \\ \int_0^{l^e} q(x)N_j^e(x)dx \end{Bmatrix}$$

Total Potential Energy (Global)

$$\Pi = \sum_{e=1}^{4} \left(U^e + V^e \right) - P_0 u(0) - P_L u(L)$$

$$\Pi = \sum_{e=1}^{4} \left\{ \frac{1}{2} \mathbf{d}^{\mathbf{e}^T} \mathbf{k}^{\mathbf{e}} \mathbf{d}^{\mathbf{e}} - \mathbf{q}^{\mathbf{e}^T} \mathbf{d}^{\mathbf{e}} \right\} - P_0 u(0) - P_L u(L)$$

Global Assembly:

$$\Pi = \frac{1}{2} \mathbf{D}^T \mathbf{K} \mathbf{D} - \mathbf{Q}^T \mathbf{D}$$

$$\delta\Pi = 0 \Rightarrow \mathbf{K}\mathbf{D} - \mathbf{Q} = \mathbf{0}$$

Global Equilibrium Equations

Summary and Review

UCLA

$$\hat{u}(X) = \sum_{i=1}^{N} D_i N_i(X) = \mathbf{N}(X)\mathbf{D}$$

Weak form

Potential Energy

$$\Pi = \int_0^L \left(\frac{EA}{2}\varepsilon^2 - qu \right) dx - P_0 u(0) - P_L u(L)$$

$$= \frac{1}{2}\mathbf{D}^T\mathbf{K}\mathbf{D} - \mathbf{Q}^T\mathbf{D}$$

Rayleigh-Ritz

$$G[u(X), v(X)] = \int_0^L \left(EA \frac{du}{dX}\frac{dv}{dX} + qv \right) dX$$

$$-P_0 v(0) - P_L v(L) = 0$$

Galerkin

$$\mathbf{K}\mathbf{D} = \mathbf{Q} \Rightarrow \mathbf{D} = \mathbf{K}^{-1}\mathbf{Q}$$

Section 7.6

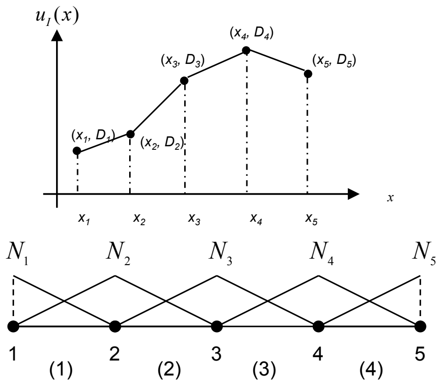
Elastic Bar Elements: Galerkin

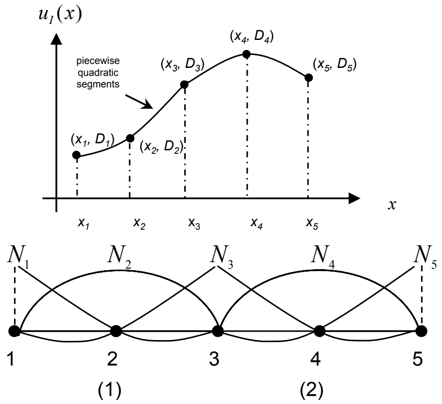
Galerkin approach to FEM

UCLA

- Discretize into elements
- Interpolation with piecewise polynomial basis (shape) functions

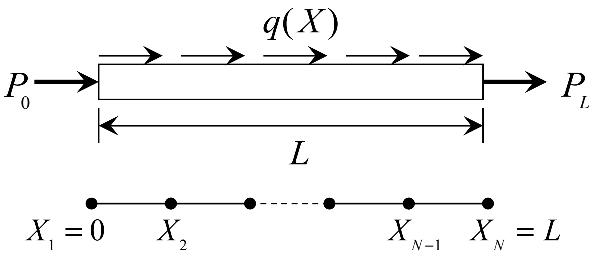
$$\hat{u}(X) = D_1 N_1(X) + D_2 N_2(X) + \ldots = \sum_{i=1}^{N} D_i N_i(X)$$





- Plug approximation into weak form

Galerkin FE formulation for 1-D Elasticity



- Approximate displacement

$$\hat{u}(X) = \sum_{i=1}^{N} D_i N_i(X)$$

- Plug into weak form

$$G[u(X),v(X)] = \int_0^L \left(EA \frac{du}{dX}\frac{dv}{dX} - qv \right) dX - P_0 v(0) - P_L v(L) = 0$$

$$u(X) \to \hat{u}(X) = \sum_{i=1}^{N} N_i(X) \qquad v(X) \to N_k(X)$$

$$N_k(X_i) = \delta_{ki} \Rightarrow N_k(0) = \delta_{k1} \; N_k(L) = \delta_{kN}$$

$$\Rightarrow \int_0^L \left\{ EA \frac{d\hat{u}}{dX}\frac{dN_k}{dX} - qN_k \right\} dX - P_L \delta_{kN} - P_0 \delta_{k1} = 0$$

Galerkin Approach

UCLA

$$\hat{u}(X) = \sum_{i=1}^{N} D_i N_i(X) = \underbrace{\begin{bmatrix} N_1 & N_2 & \cdots & N_N \end{bmatrix}}_{\mathbf{N}(X)} \underbrace{\begin{Bmatrix} D_1 \\ D_2 \\ \vdots \\ D_N \end{Bmatrix}}_{\mathbf{D}} = \mathbf{N}(X)\mathbf{D}$$

$$\hat{\varepsilon}(X) = \frac{d}{dX}\hat{u}(X) = \underbrace{\begin{bmatrix} \frac{dN_1}{dX} & \frac{dN_2}{dX} & \cdots & \frac{dN_N}{dX} \end{bmatrix}}_{\substack{\mathbf{B}(X) = \frac{d}{dX}\mathbf{N}(X) \\ \text{Strain-Displacement Operator}}} \begin{Bmatrix} D_1 \\ D_2 \\ \vdots \\ D_N \end{Bmatrix} = \mathbf{B}(X)\mathbf{D}$$

Galerkin Approach

UCLA

$$\int_0^L \left\{ EA\frac{d\hat{u}}{dX}\frac{dN_k}{dX} - qN_k \right\} dX - P_L\delta_{kN} - P_0\delta_{k1} = 0$$

$$\int_0^L \left\{ EA\mathbf{BD}\frac{dN_k}{dX} - qN_k \right\} dX - P_L\delta_{kN} - P_0\delta_{k1} = 0$$

$$\int_0^L EA\mathbf{BD}\frac{dN_1}{dX}dX = \int_0^L q(X)N_1 dX + P_0$$

$$\int_0^L EA\mathbf{BD}\frac{dN_k}{dX}dX = \int_0^L q(X)N_k dX \qquad k = 2,\ldots,N-1$$

$$\int_0^L EA\mathbf{BD}\frac{dN_n}{dX}dX = \int_0^L q(X)N_n\, dX + P_L$$

Galerkin Approach

UCLA

$$\Rightarrow \left(\int_0^L EA\mathbf{B}\frac{dN_k}{dX}dX \right)\mathbf{D} = \int_0^L q(X)N_k dX + P_L\delta_{kN} + P_0\delta_{k1}$$

$$\underbrace{\begin{bmatrix} \int_0^L EA\frac{dN_1}{dX}\mathbf{B}dX \\ \int_0^L EA\frac{dN_2}{dX}\mathbf{B}dX \\ \vdots \\ \int_0^L EA\frac{dN_N}{dX}\mathbf{B}dX \end{bmatrix}}_{\mathbf{K}}\mathbf{D} = \underbrace{\begin{bmatrix} \int_0^L qN_1 dX + P_0 \\ \int_0^L qN_2 dX \\ \vdots \\ \int_0^L qN_N dX + P_L \end{bmatrix}}_{\mathbf{Q}} \Rightarrow \left(\int_0^L EA \underbrace{\begin{bmatrix} \frac{dN_1}{dX} \\ \frac{dN_2}{dx} \\ \vdots \\ \frac{dN_N}{dX} \end{bmatrix}}_{\mathbf{B}^T}\mathbf{B}dX \right)\mathbf{D} = \mathbf{Q}$$

$$\mathbf{KD} = \mathbf{Q} \;\Rightarrow\; \mathbf{D} = \mathbf{K}^{-1}\mathbf{Q}$$

Galerkin Approach

UCLA

$$\mathbf{K} = \int_0^L EA\mathbf{B}^T\mathbf{B}dX$$

$$\mathbf{Q} = \underbrace{\begin{Bmatrix} \int_0^L qN_1\,dX \\ \int_0^L qN_2\,dX \\ \vdots \\ \vdots \\ \int_0^L qN_N\,dX \end{Bmatrix}}_{\mathbf{Q}^{dist}} + \underbrace{\begin{Bmatrix} P_0 \\ 0 \\ \vdots \\ 0 \\ P_L \end{Bmatrix}}_{\mathbf{Q}^{conc}}$$

Lumped nodal forces consistent with $q(x)$

Forces applied directly at nodes

Consistent Nodal Loads

UCLA

$$\mathbf{Q}^{dist} = \begin{Bmatrix} \int_0^L qN_1 dX \\ \int_0^L qN_2 dX \\ \vdots \\ \vdots \\ \int_0^L qN_N dX \end{Bmatrix} = \int_0^L q \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_N \end{bmatrix} dX = \int_0^L q\mathbf{N}^T dX$$

Galerkin Approach

UCLA

- Using nodal indices

$$\mathbf{Q} = \mathbf{Q}^{dist} + \mathbf{Q}^{conc}$$

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ k_{N1} & \cdots & \cdots & k_{NN} \end{bmatrix} \qquad \mathbf{Q}^{dist} = \begin{Bmatrix} q_1 \\ \vdots \\ \vdots \\ q_N \end{Bmatrix} \qquad \mathbf{Q}^{conc} = \begin{Bmatrix} P_0 \\ 0 \\ \vdots \\ 0 \\ P_L \end{Bmatrix}$$

$$k_{ij} = \int_0^L EA \frac{dN_i}{dX} \frac{dN_j}{dX} dX \qquad q_i = \int_0^L q(X) N_i dX$$

UCLA

- Integrate over elements and sum up

$$X = X_i + x \Rightarrow dX = dx; \; \frac{d(\bullet)}{dX} = \frac{d(\bullet)}{dx}$$

$$k_{ij} = \int_0^L EA \frac{dN_i}{dX}\frac{dN_j}{dX} dX = \sum_{e=1}^{\#elements} \underbrace{\int_0^{l^e} EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx}_{k_{ij}^{(e)}} = \sum_{e=1}^{\#elements} k_{ij}^{(e)}$$

$$\mathbf{K} = \int_0^L EA\mathbf{B}^T\mathbf{B}dX = \sum_{e=1}^{\#elements} \underbrace{\int_0^{l^e} EA\mathbf{B}^T\mathbf{B}dx}_{\mathbf{k}^{(e)}} = \sum_{e=1}^{\#elements} \mathbf{k}^{(e)}$$

$$q_i = \int_0^L qN_i dX = \sum_{e=1}^{\#elements} \underbrace{\int_0^{l^e} qN_i dx}_{q_i^{(e)}} = \sum_{e=1}^{\#elements} q_i^{(e)}$$

$$\mathbf{Q}^{dist} = \int_0^L q\mathbf{N}^T dX = \sum_{e=1}^{\#elements} \underbrace{\int_0^{l^e} q\mathbf{N}^T dx}_{\mathbf{q}^{(e)}} = \sum_{e=1}^{\#elements} \mathbf{q}^{(e)}$$

Global Assembly

UCLA

- Compute element matrices (e.g., 2x2 for linear)

$$\mathbf{k}^{(e)} = \int_0^{l^e} EA\mathbf{B}^{(e)T}\mathbf{B}^{(e)}dx = \begin{bmatrix} k_{ii}^{(e)} & k_{ij}^{(e)} \\ k_{ij}^{(e)} & k_{jj}^{(e)} \end{bmatrix}; \ \mathbf{B}^{(e)} = \begin{bmatrix} \frac{dN_i}{dx} & \frac{dN_j}{dx} \end{bmatrix}$$

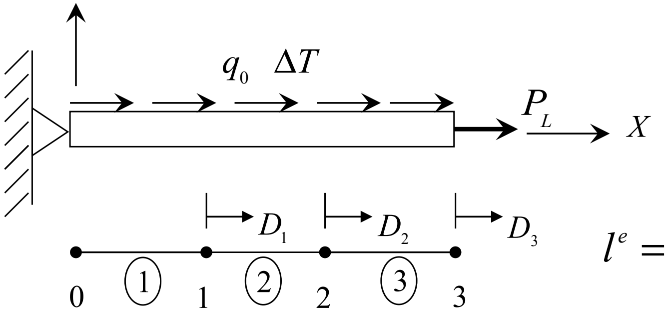
$$\mathbf{q}^{(e)} = \int_0^{l^e} q\mathbf{N}^{(e)T}dx = \begin{Bmatrix} q_i^{(e)} \\ q_j^{(e)} \end{Bmatrix}; \ \mathbf{N}^{(e)} = \begin{bmatrix} N_i & N_j \end{bmatrix}$$

- Assemble with code numbers $\mathbf{Q}^{dist}, \mathbf{K}$
- Add end forces at boundary nodes directly to $\mathbf{Q}^{conc}$

Section 7.7

Example 1

Example 1: Linear Element



E, A, L, α const.

$$l^e = \frac{L}{3}$$

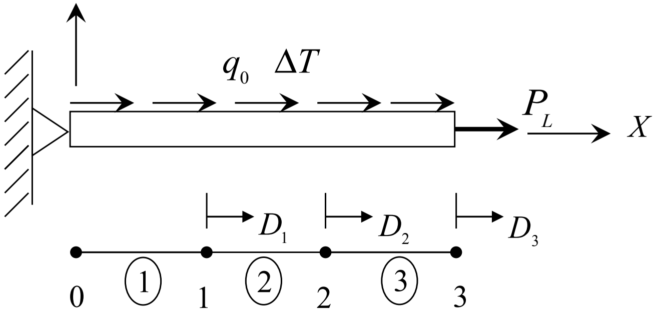
$$\mathbf{N}^e = \left[\begin{array}{cc} 1-\dfrac{x}{l^e} & \dfrac{x}{l^e} \end{array}\right] \quad \mathbf{B}^e = \frac{d\mathbf{N}^e}{dx} = \left[\begin{array}{cc} -\dfrac{1}{l^e} & \dfrac{1}{l^e} \end{array}\right]$$

$$\Rightarrow \mathbf{k^e} = \int_0^{l^e} EA\mathbf{B^e}^T\,\mathbf{B^e}\,dx = \int_0^{l^e} \frac{EA}{\left(l^e\right)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

$$\Rightarrow \mathbf{k^e} = \frac{1}{\left(L/3\right)^2}\int_0^{l^e} EA\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \frac{EA}{L/3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

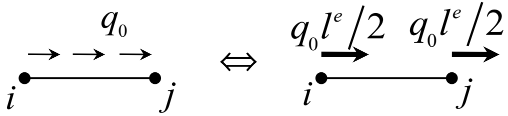
External Loads

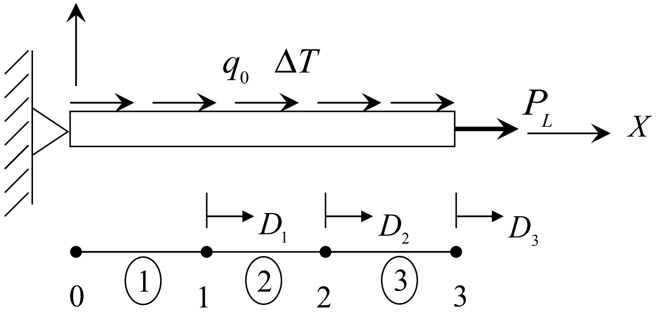
UCLA



$E,\ A,\ L,\ \alpha$ const.

$$\mathbf{q^e} = \int_0^{l^e} q_0 \mathbf{N}^{e^T}\, dx = \int_0^{l^e} q_0 \begin{Bmatrix} \dfrac{\left(l^e - x\right)}{l^e} \\ \dfrac{x}{l^e} \end{Bmatrix} dx \quad = \begin{Bmatrix} q_0 l^e/2 \\ q_0 l^e/2 \end{Bmatrix} = \frac{q_0 L}{6} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

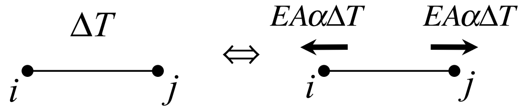




E, A, L, α const.

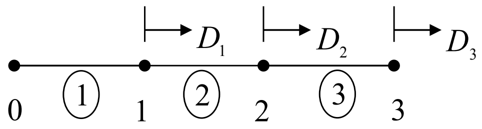
$$\mathbf{q}^{\mathbf{e}} = \int_0^{l^e} EA\alpha\Delta T \mathbf{B}^{e^T}\, dx \qquad \mathbf{N}^e = \begin{bmatrix} 1 - \dfrac{x}{l^e} & \dfrac{x}{l^e} \end{bmatrix} \quad \mathbf{B}^e = \frac{d\mathbf{N}^e}{dx} = \begin{bmatrix} -\dfrac{1}{l^e} & \dfrac{1}{l^e} \end{bmatrix}$$

$$= EA\alpha\Delta T \int_0^{l^e} \begin{Bmatrix} -1/l^e \\ 1/l^e \end{Bmatrix} dx = EA\alpha\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$



Assemble Global Stiffness

UCLA



e	Code #
1	0 1
2	1 2
3	2 3

$$\mathbf{k}^{(1)} = \frac{EA}{l^e} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \end{array}$$

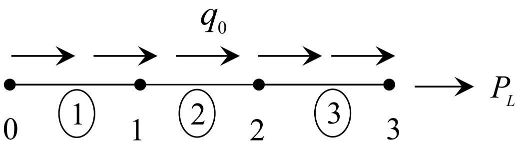
$$\mathbf{k}^{(2)} = \frac{EA}{l^e} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

$$\mathbf{k}^{(3)} = \frac{EA}{l^e} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

$$\mathbf{K} = \frac{3EA}{L} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

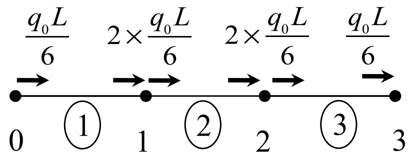
Assemble Global Forces

$$l^e = L/3$$



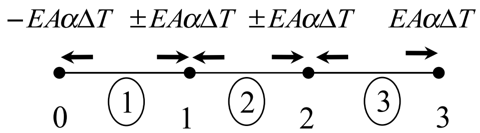
Distributed Load

$$\mathbf{q}^{(e)} = \frac{q_0 l^e}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{q_0 L}{6}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



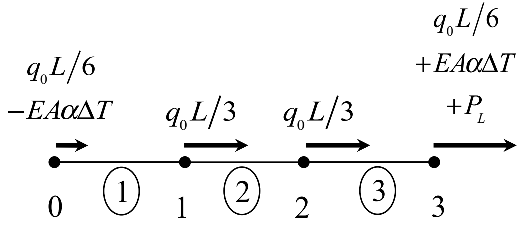
Thermal Load

$$\mathbf{q}^{(e)}_{\Delta T} = EA\alpha\Delta T\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Add Point Load Globally

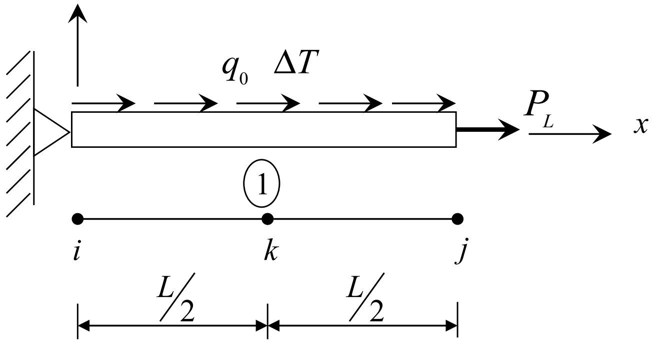
$$\mathbf{Q} = \mathbf{Q}^{dist} + \mathbf{Q}^{conc} + \mathbf{Q}^{\Delta T} = \begin{Bmatrix} \dfrac{q_0 L}{3} \\ \dfrac{q_0 L}{3} \\ \dfrac{q_0 L}{6} + EA\alpha\Delta T + P_L \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$



Section 7.8

Example 2

Example 2: Single Quadratic Element



$$E,\ A,\ L,\ \alpha \text{ const.}$$

$$N_i(x) = \frac{2}{L^2}\left(x - \frac{L}{2}\right)(x - L)$$

$$N_j(x) = \frac{2}{L^2}x\left(x - \frac{L}{2}\right)$$

$$N_k(x) = -\frac{4}{L^2}x(x - L)$$

Element Stiffness

UCLA

$$\mathbf{k^e} = \int_0^L EA\mathbf{B^e}^T\mathbf{B^e}dx$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{dN_i}{dx} & \dfrac{dN_k}{dx} & \dfrac{dN_j}{dx} \end{bmatrix}$$

$$\frac{dN_i}{dx} = \frac{1}{L}\left(\frac{4x}{L} - 3\right)$$

$$\frac{dN_j}{dx} = \frac{1}{L}\left(\frac{4x}{L} - 1\right)$$

$$\frac{dN_k}{dx} = -\frac{4}{L}\left(\frac{2x}{L} - 1\right)$$

Element Stiffness

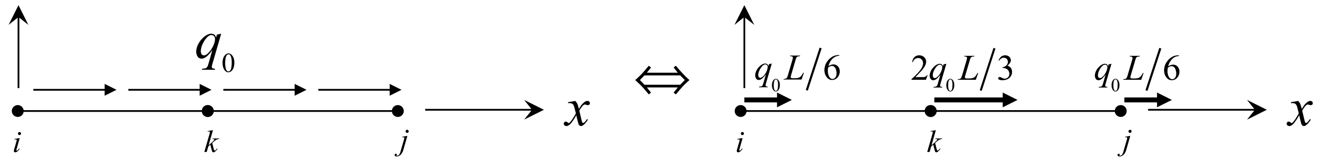
UCLA

$$\mathbf{B}^T\mathbf{B} = \begin{bmatrix} \dfrac{dN_i}{dx} \\ \dfrac{dN_k}{dx} \\ \dfrac{dN_j}{dx} \end{bmatrix} \begin{bmatrix} \dfrac{dN_i}{dx} & \dfrac{dN_k}{dx} & \dfrac{dN_j}{dx} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{dN_i}{dx}\right)^2 & \dfrac{dN_i}{dx}\dfrac{dN_k}{dx} & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \ddots \end{bmatrix}$$

$$k_{ik} = \int_0^L EA \frac{dN_i}{dx}\frac{dN_k}{dx}\,dx = -\int_0^L EA \frac{1}{L}\left(\frac{4x}{L} - 3\right)\frac{4}{L}\left(\frac{2x}{L} - 1\right)dx$$

$$\Rightarrow k_{ik} = -\frac{2}{3}\frac{EA}{L}$$

Distributed Load



$$q_i = \int_0^L q_0 N_i dx = \int_0^L q_0 \frac{2}{L^2}\left(x - \frac{L}{2}\right)(x-L)\,dx = \frac{q_0 L}{6}$$

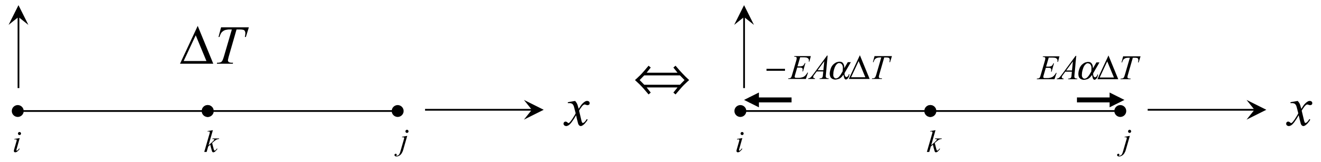
$$q_j = \int_0^L q_0 N_j dx = \int_0^L q_0 \frac{2}{L^2} x\left(x - \frac{L}{2}\right)dx = \frac{q_0 L}{6}$$

$$q_k = \int_0^L q_0 N_k dx = -\int_0^L q_0 \frac{4}{L^2} x(x-L)\,dx = \frac{2}{3} q_0 L$$

$$q_i + q_j + q_k = q_0 L = \int_0^L q(x)\,dx$$

Statically Equivalent

Thermal Load



$$q_i = \int_0^L EA\alpha\Delta T \frac{dN_i}{dx} dx = EA\alpha\Delta T \int_0^L \frac{dN_i}{dx} dx = EA\alpha\Delta T \left[N_i \right]_0^L = -EA\alpha\Delta T$$

$$q_k = \int_0^L EA\alpha\Delta T \frac{dN_k}{dx} dx = EA\alpha\Delta T \left[N_k \right]_0^L = 0$$

$$q_j = \int_0^L EA\alpha\Delta T \frac{dN_j}{dx} dx = EA\alpha\Delta T \left[N_j \right]_0^L = EA\alpha\Delta T$$

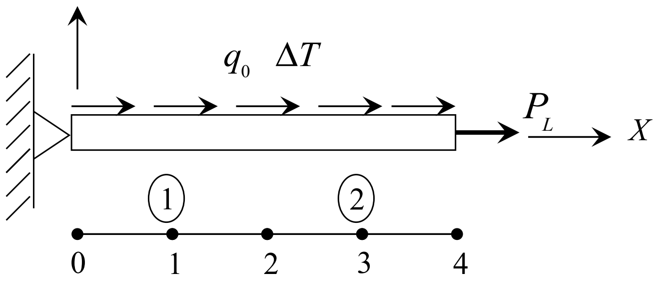
$$q_i + q_j + q_k = 0$$

Thermal Loads Should
Self-equilibrate!

Section 7.9

Examples 3

Example 3: Two Quadratic Elements



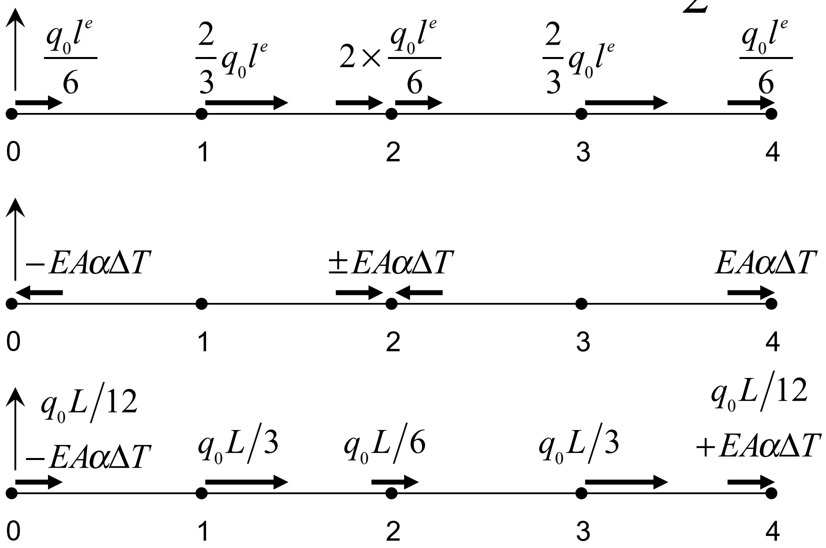
E, A, L, α const.

$$l^e = \frac{L}{2}$$

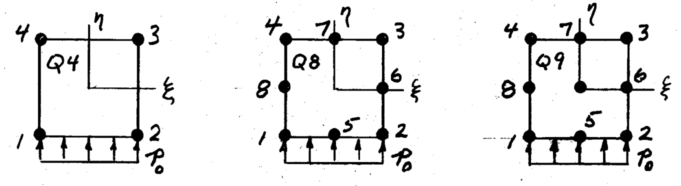
Distributed Load

Thermal Load

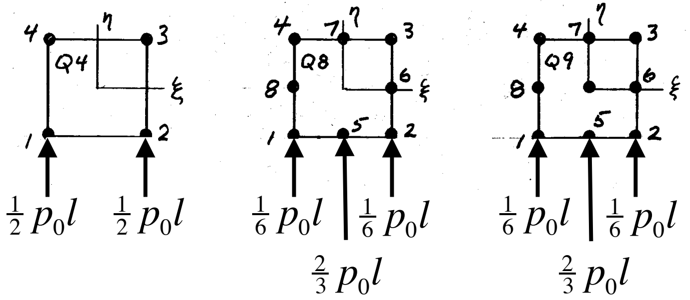
Total



Skipping ahead to 2-D



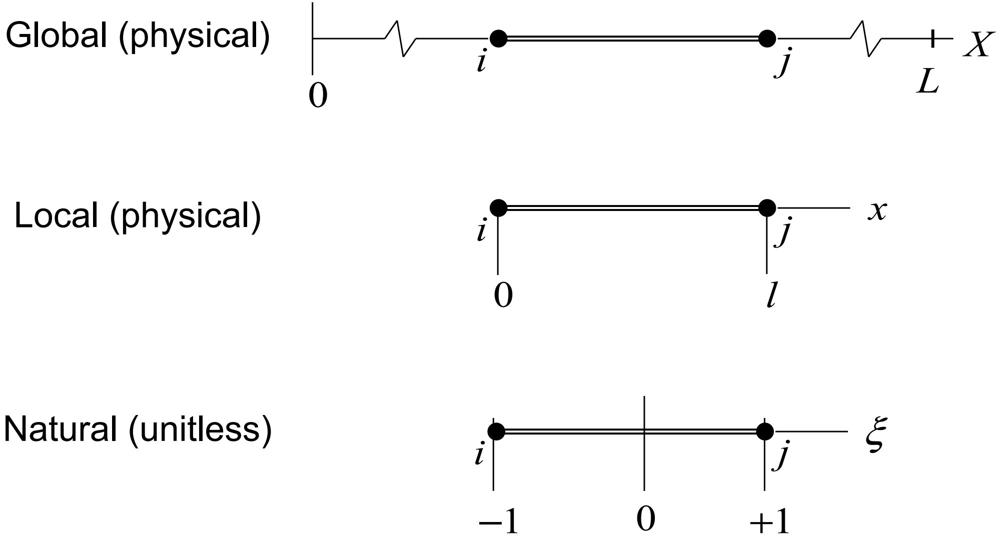
Shall we guess what nodal loads are?



Section 7.10

Coordinate Systems

Coordinate Systems

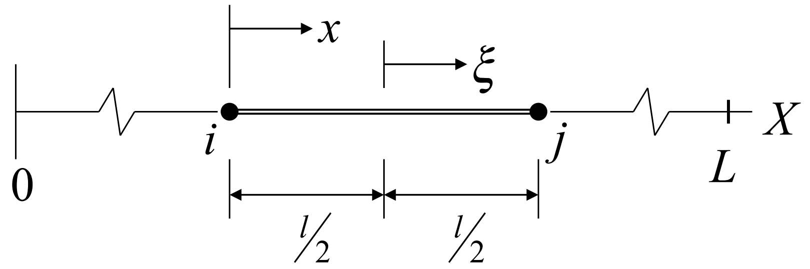


Natural Coordinate System 1-D Element

UCLA

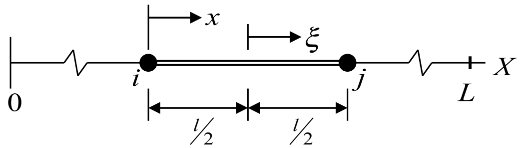
- Two-node bar element of length l.

- Origin of natural coordinate system is located at the center.



Natural Coordinate System 1-D Element

UCLA

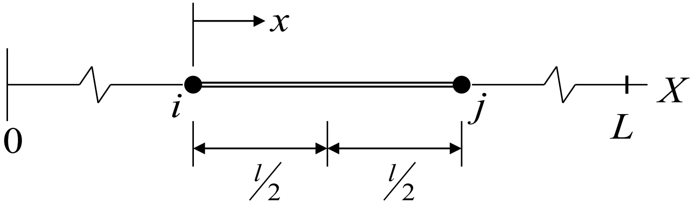


- Compare global coordinate X, local coordinate x, and natural coordinate ξ

Coordinate System	Node i Coordinate	Node j Coordinate	Range
Global cartesian coordinate	$X = X_i$	$X = X_j$	$(0 \leq X \leq L)$
Local cartesian coordinate	$x = 0$	$x = l$	$(0 \leq X \leq l)$
Natural cartesian coordinate	$\xi = -1$	$\xi = 1$	$(-1 \leq \xi \leq 1)$

Linear Shape Functions in Natural Coordinate

UCLA



- Use linear coordinate transformations,

$$X = X_i + x \text{ or } x = X - X_i \quad \text{and} \quad \xi = \frac{2}{l}\left(x - \frac{l}{2}\right) \text{ or } x = \frac{l}{2}(\xi + 1)$$

- Substitute $x = \frac{l}{2}(\xi + 1)$ in $N_i(x)$ and $N_j(x)$

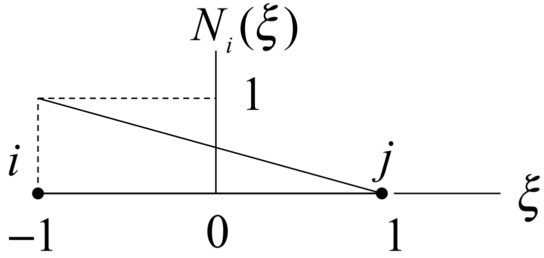
$$N_i = 1 - \frac{x}{l} = \frac{1}{2}(1 - \xi) \qquad \text{and} \qquad N_j = \frac{x}{l} = \frac{1}{2}(1 + \xi)$$

Linear Shape Functions in Natural Coordinate

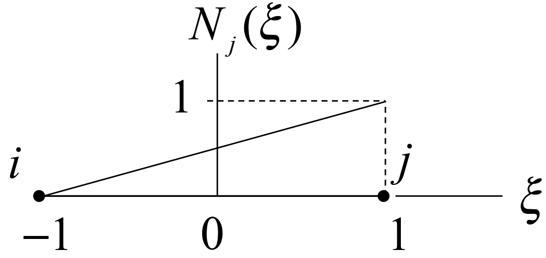
UCLA

- Graphs of two shape functions retain same linear shapes.
- Shape functions retain same key properties.

$$N_i = 1 - \frac{x}{l} = \frac{1}{2}(1-\xi)$$



$$N_j = \frac{x}{l} = \frac{1}{2}(1+\xi)$$



Axial displacement

UCLA

- $\hat{u}(x)$ in the linear element is interpolated between u_{i} and u_{j} using N_{i} and N_{j}.

- Express axial displacement in natural coordinate,

$$\hat{u}(x) = \underbrace{\frac{1}{2}\left(1-\xi\right)}_{N_i(\xi)} u_i + \underbrace{\frac{1}{2}\left(1+\xi\right)}_{N_j(\xi)} u_j = \begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \mathbf{Nd}$$

Element Geometry in Global Coordinate

UCLA

- X interpolates between X_i and X_j using $N_i(\xi)$ and $N_j(\xi)$.

$$X = X_i + x = X_i + \frac{l}{2}(\xi+1) = X_i + \frac{(X_j - X_i)}{2}(\xi+1)$$

$$\Rightarrow X = \underbrace{\frac{l}{2}(1-\xi)}_{N_i(\xi)} X_i + \underbrace{\frac{l}{2}(1+\xi)}_{N_j(\xi)} X_j = \begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \begin{Bmatrix} X_i \\ X_j \end{Bmatrix} = \mathbf{NX}$$

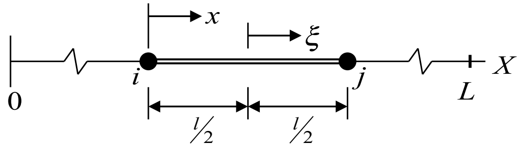
Section 7.11

Isoparametric Elements

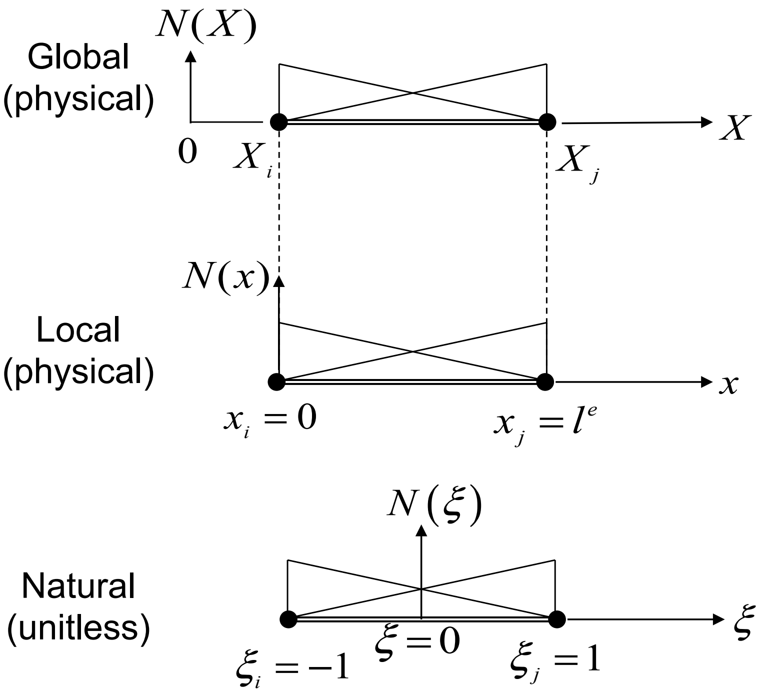
The Key Idea

UCLA

- Problems with “standard” elements:
  - Interior nodes evenly spaced.
  - Shape functions depend on geometry.
- The solution: Isoparametric mapping
  - Convenient coordinate transformation between Natural and Physical coordinate systems.



Linear Shape Functions



$$N_i(X) = \frac{X_j - X}{l^e}$$

$$N_j(X) = \frac{X - X_i}{l^e}$$

$$x = X - X_i$$

$$N_i(x) = 1 - \frac{x}{l^e} \qquad N_j(x) = \frac{x}{l^e}$$

$$\xi = \frac{2}{l^e}\left(x - \frac{l^e}{2}\right)$$

$$N_i(\xi) = \frac{1}{2}(1 - \xi) \qquad N_j(\xi) = \frac{1}{2}(1 + \xi)$$

Geometry is gone!

Redefine Interpolation

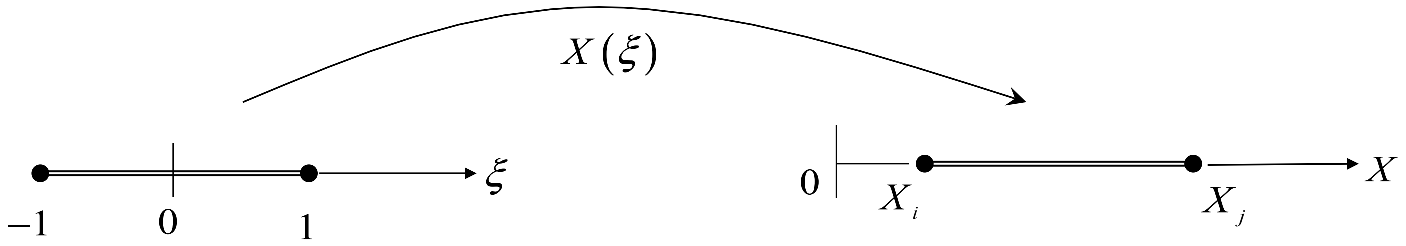
UCLA

$$u(\xi) = u\left(X(\xi)\right) = \sum_{i=1}^{N} u_i N_i \left(X(\xi)\right) = \sum_{i=1}^{N} u_i N_i(\xi)$$

$$u(\xi) = u_i N_i(\xi) + u_j N_j(\xi) = \begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$u(\xi) = u_i \underbrace{\frac{1}{2}(1-\xi)}_{N_i(\xi)} + u_j \underbrace{\frac{1}{2}(1+\xi)}_{N_j(\xi)}$$

Coordinate Transformation



Natural Geometry

Physical Geometry

$$X = X(\xi) = X_i + \frac{X_j - X_i}{2}(\xi + 1)$$

$$= X_i \underbrace{\frac{1}{2}(1-\xi)}_{N_i(\xi)} + X_j \underbrace{\frac{1}{2}(1+\xi)}_{N_j(\xi)}$$

Transformation *interpolates* between nodal positions

Isoparametric Elements

UCLA

- Interpolate **both** displacement and position using the **same** shape functions
  - Linear

$$u(\xi) = u_i N_i(\xi) + u_j N_j(\xi) = \mathbf{N}(\xi)\mathbf{d}$$ ← Nodal displacements

$$X(\xi) = X_i N_i(\xi) + X_j N_j(\xi) = \mathbf{N}(\xi)\mathbf{X}$$ ← Nodal positions

  - Quadratic

$$u(\xi) = u_i N_i(\xi) + u_j N_j(\xi) + u_k N_k(\xi) = \mathbf{N}(\xi)\mathbf{d}$$

$$X(\xi) = X_i N_i(\xi) + X_j N_j(\xi) + X_k N_k(\xi) = \mathbf{N}(\xi)\mathbf{X}$$

Now we can move mid-nodes away from center

Compute Derivatives Using the Chain Rule

UCLA

- Need $\dfrac{du}{dX}$ for mechanics: $\int EA\left(\dfrac{du}{dX}\right)^2 dX\ldots$

$$u(\xi) = u_i \underbrace{\frac{1}{2}(1-\xi)}_{N_i(\xi)} + u_j \underbrace{\frac{1}{2}(1+\xi)}_{N_j(\xi)} \qquad X(\xi) = X_i \underbrace{\frac{1}{2}(1-\xi)}_{N_i(\xi)} + X_j \underbrace{\frac{1}{2}(1+\xi)}_{N_j(\xi)}$$

$$\frac{du}{dX} = \frac{du}{d\xi}\frac{d\xi}{dX} = \left(u_i \frac{dN_i}{d\xi} + u_j \frac{dN_j}{d\xi}\right)\left(\frac{dX}{d\xi}\right)^{-1}$$

$$\frac{dX}{d\xi} = \left(X_i \frac{dN_i}{d\xi} + X_j \frac{dN_j}{d\xi}\right) \qquad \Rightarrow \frac{du}{dX} = \frac{u_j - u_i}{X_j - X_i}$$

Integration

UCLA

$$\int_{X_i}^{X_j} f(X)\,dX \qquad \left(f = EA\left(\frac{du}{dX}\right)^2 \right)$$

$$dX = \frac{dX}{d\xi}d\xi \qquad \rightarrow \int_{-1}^{1} f\left(X(\xi)\right)\frac{dX}{d\xi}d\xi = \int_{-1}^{1} f\left(X(\xi)\right)J(\xi)\,d\xi$$

$$J(\xi) = \frac{dX(\xi)}{d\xi} = \frac{d\mathbf{N}(\xi)}{d\xi}\mathbf{X}$$

"Jacobian" of mapping $X(\xi)$

Energy and Stiffness

UCLA

$$U^e = \int_{X_i}^{X_j} \frac{EA}{2}\left(\frac{du}{dX}\right)^2 dX = \int_{-1}^{1} \frac{EA}{2}\left[\frac{du}{d\xi}\left(\frac{dX}{d\xi}\right)^{-1}\right]^2 J d\xi$$

$$\mathbf{k} = \int_{X_i}^{X_j} EA\mathbf{B}^T\mathbf{B}dX = \int_{-1}^{1} EA\mathbf{B}^T\mathbf{B}Jd\xi$$

For linear shape functions $\quad \mathbf{B} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\frac{2}{l^e} \qquad J = \frac{l^e}{2}$

$$\Rightarrow \mathbf{k} = \int_{-1}^{1} EA\begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix}\frac{4}{l^{e^2}}\frac{l^e}{2}d\xi = \frac{EA}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Higher Order Elements

UCLA

$$\mathbf{k}^e = \int_0^{l^e} EA\mathbf{B}^T\mathbf{B}dX = \int_{-1}^{1} EA\mathbf{B}^T\mathbf{B}Jd\xi$$

$$\mathbf{B} = \frac{d\mathbf{N}(\xi)}{dX} = \frac{d\mathbf{N}(\xi)}{d\xi}\left(\frac{dX}{d\xi}\right)^{-1} = \frac{1}{J}\frac{d\mathbf{N}(\xi)}{d\xi}$$

$$\mathbf{B}^T\mathbf{B} = \frac{1}{J^2}\left(\frac{d\mathbf{N}(\xi)}{d\xi}\right)^T\frac{d\mathbf{N}(\xi)}{d\xi}$$

$$\Rightarrow \mathbf{k}^e = \int_{-1}^{1} EA\left(\frac{d\mathbf{N}(\xi)}{d\xi}\right)^T\frac{d\mathbf{N}(\xi)}{d\xi}\frac{1}{J}d\xi$$

Summary and Review

UCLA

- Interpolate **both** displacement and position using the **same** shape functions

$$u(\xi) = \sum_{i=1}^{N} u_i N_i(\xi) = \mathbf{N}(\xi)\mathbf{d} \leftarrow \text{Nodal displacements}$$

$$X(\xi) = \sum_{i=1}^{N} X_i N_i(\xi) = \mathbf{N}(\xi)\mathbf{X} \leftarrow \text{Nodal positions}$$

$$\frac{dX}{d\xi} = \frac{d\mathbf{N}}{d\xi}\mathbf{X} \qquad \mathbf{B} = \frac{d\mathbf{N}(\xi)}{dX} = \frac{d\mathbf{N}(\xi)}{d\xi}\left(\frac{dX}{d\xi}\right)^{-1} = \frac{1}{J}\frac{d\mathbf{N}(\xi)}{d\xi}$$

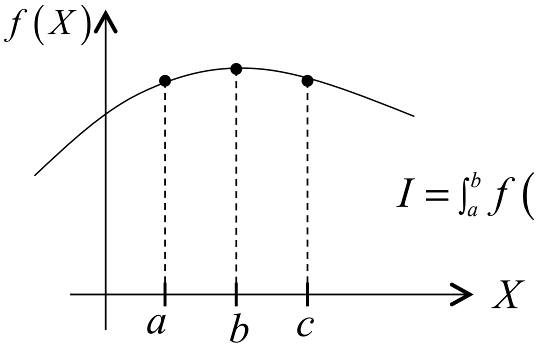
$$\mathbf{B}^T\mathbf{B} = \frac{1}{J^2}\left(\frac{d\mathbf{N}(\xi)}{d\xi}\right)^T \frac{d\mathbf{N}(\xi)}{d\xi} \qquad \mathbf{k}^e = \int_{-1}^{1} EA\left(\frac{d\mathbf{N}(\xi)}{d\xi}\right)^T \frac{d\mathbf{N}(\xi)}{d\xi}\frac{1}{J}\,d\xi$$

Section 7.12

Numerical Integration: Gaussian Quadrature

Numerical Integration

- Quadrature:
  - Approximate integrals by sampling.
- Recall: Midpoint rule, Trapezoidal rule, Simpson's rule



$$I = \int_a^b f(X)\,dX$$

$$I_{MR} = f\left(\frac{a+b}{2}\right)(b-a)$$

$$I_{TR} = \frac{f(a)+f(b)}{2}(b-a)$$

$$I_{SR} = \frac{2I_{MR}+I_{TR}}{3}$$

Generic Quadrature Formula

UCLA

$$I = \int_{X_1}^{X_2} f\, dX = \int_{-1}^{1} \phi\, d\xi$$

$$f dX = f\left(X(\xi)\right) J d\xi = \phi(\xi)\, d\xi$$

Idea:

$$I = \int_{-1}^{1} \phi\, d\xi \approx w_1 \phi(\xi_1) + w_2 \phi(\xi_2) + \ldots + w_n \phi(\xi_n) = \sum_{p=1}^{n} w_p \phi(\xi_p)$$

$n =$ selected order of numerical integration

$\xi_p =$ quadrature points

$w_p =$ quadrature weights

Gauss Quadrature

UCLA

- A method to exactly integrate polynomials of order k, denoted $P_k(\xi)$, where $k \le (2n-1)$
  - Example: cubic or less gets computed exactly

$$n = 2 \quad k \le (2n-1) = 3$$

$$P_k(\xi) = c_0 + c_1\xi + \ldots + c_k\xi^k$$

$$\int_{-1}^{1} d\xi$$

$$\int_{-1}^{1} \xi \, d\xi$$

$$\int_{-1}^{1} \xi^2 \, d\xi$$

$$\int_{-1}^{1} \xi^3 \, d\xi$$

Quadrature Weights and Quadrature Points

UCLA

- For n = 1

$$k = (2n-1) = 1 \qquad \rightarrow \text{Linear}$$

$$\phi(\xi) = a_0 + a_1\xi$$

$$I = \int_{-1}^{1} \phi(\xi)\, d\xi = \left[a_0\xi + \frac{1}{2}a_1\xi^2 \right]_{-1}^{1} = 2a_0$$

$$I = w_1\phi(\xi_1)$$

$$\Rightarrow 2a_0 = w_1(a_0 + a_1\xi_1) = w_1a_0 + w_1a_1\xi_1$$

$$\therefore w_1 = 2 \ \ \& \ \ \xi_1 = 0$$

Quadrature Weights and Quadrature Points

UCLA

- For n = 2

$$k=(2n-1)=3 \qquad \rightarrow \text{Cubic}$$

$$\phi(\xi)=a_0+a_1\xi+a_2\xi^2+a_3\xi^3$$

$$I=\int_{-1}^{1}\phi(\xi)\,d\xi=\left[a_0\xi+\frac{1}{2}a_1\xi^2+\frac{1}{3}a_2\xi^3+\frac{1}{4}a_3\xi^4\right]_{-1}^{1}$$

$$=2a_0+\frac{2}{3}a_2$$

$$I=w_1\phi(\xi_1)+w_2\phi(\xi_2)$$

Quadrature Weights and Quadrature Points

UCLA

$$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$$

$$I = \int_{-1}^{1} \phi(\xi)\, d\xi = 2a_0 + \frac{2}{3}a_2$$

$$I = w_1\phi(\xi_1) + w_2\phi(\xi_2)$$

$$\Rightarrow 2a_0 + \frac{2}{3}a_2 = w_1\left(a_0 + a_1\xi_1 + a_2\xi_1^2 + a_3\xi_1^3\right)$$
$$+ w_2\left(a_0 + a_1\xi_2 + a_2\xi_2^2 + a_3\xi_2^3\right)$$

Quadrature Weights and Quadrature Points

UCLA

$$2a_0 + \frac{2}{3}a_2 = w_1\left(a_0 + a_1\xi_1 + a_2\xi_1^2 + a_3\xi_1^3\right)$$
$$+ w_2\left(a_0 + a_1\xi_2 + a_2\xi_2^2 + a_3\xi_2^3\right)$$

$$w_1 a_1 \xi_1 + w_2 a_1 \xi_2 = 0 \Rightarrow w_1 \xi_1 = -w_2 \xi_2 \qquad (1)$$

$$w_1 a_0 + w_2 a_0 = 2a_0 \Rightarrow w_1 + w_2 = 2 \qquad (2)$$

$$w_1 a_2 \xi_1^2 + w_2 a_2 \xi_2^2 = \frac{2}{3}a_2 \Rightarrow w_1 \xi_1^2 + w_2 \xi_2^2 = \frac{2}{3} \qquad (3)$$

$$w_1 a_3 \xi_1^3 + w_2 a_3 \xi_2^3 = 0 \Rightarrow \xi_1 = -\xi_2 \qquad (4)$$

Quadrature Weights and Quadrature Points

UCLA

$$w_1 a_1 \xi_1 + w_2 a_1 \xi_2 = 0 \Rightarrow w_1 \xi_1 = -w_2 \xi_2 \qquad (1)$$

$$w_1 a_0 + w_2 a_0 = 2a_0 \Rightarrow w_1 + w_2 = 2 \qquad (2)$$

$$w_1 a_2 \xi_1^2 + w_2 a_2 \xi_2^2 = \frac{2}{3} a_2 \Rightarrow w_1 \xi_1^2 + w_2 \xi_2^2 = \frac{2}{3} \qquad (3)$$

$$w_1 a_3 \xi_1^3 + w_2 a_3 \xi_2^3 = 0 \Rightarrow \xi_1 = -\xi_2 \qquad (4)$$

$$(1)\,\&\,(4) \Rightarrow w_1 = w_2$$

$$(2) \Rightarrow w_1 = w_2 = 1$$

$$(3) \Rightarrow \xi_1^2 = \xi_2^2 = \frac{1}{3}$$

$$(4) \Rightarrow -\xi_1 = \xi_2 = \frac{1}{\sqrt{3}} = 0.57735\ldots$$

Quadrature Weights and Quadrature Points

UCLA

n	ξ_1	w_1	$k \leq 2n-1$
1	0	2	1
2	-0.57735027 0.57735027	1.0 1.0	3
3	-0.77459667 0.0 0.77459667	0.55555555 0.88888889 0.55555555	5
4	-0.86113631 -0.33998104 0.33998104 0.86113631	0.34785485 0.65214515 0.65214515 0.34785485	7
5	-0.90617975 -0.53846931 0.0 0.53846931 0.90617975	0.23692689 0.47862867 0.56888889 0.47862867 0.23692689	9

Section 7.13

Quadrature Exercises

Quadrature for Element Stiffness

UCLA

How many quadrature points do we need to integrate exactly?

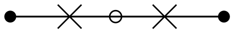
$$\mathbf{k}^e = \int_0^{l^e} EA\mathbf{B}^T\mathbf{B}\,dx = \int_{-1}^{1} EA\mathbf{B}^T\mathbf{B}\frac{l}{2}\,d\xi$$

$\mathbf{N}$ linear

$\mathbf{B}$ constant

$\mathbf{B}^T\mathbf{B}$ constant

$\Rightarrow$ 1 pt. rule



$\mathbf{N}$ quadratic

$\mathbf{B}$ linear

$\mathbf{B}^T\mathbf{B}$ quadratic

$\Rightarrow$ 2 pt. rule

$\mathbf{N}$ cubic

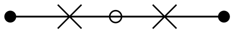
$\mathbf{B}$ quadratic

$\mathbf{B}^T\mathbf{B}$ quadratic

$4 \le 2n - 1 \Rightarrow n = 3$

$\Rightarrow$ 3 pt. rule

Exercise 1: Stiffness for Quadratic Element



n	ξ_p	w_p	$k \le 2n-1$
2	$\pm 1/\sqrt{3}$	1.0 1.0	3

$$\mathbf{N} = \begin{bmatrix} \frac{1}{2}(\xi-1)\xi & (1-\xi)(1+\xi) & \frac{1}{2}(\xi+1)\xi \end{bmatrix} \qquad \mathbf{B} = \frac{d\mathbf{N}}{dX} = \frac{d\mathbf{N}}{d\xi}\left(\frac{dX}{d\xi}\right) = \frac{2}{l}\begin{bmatrix} \xi-\frac{1}{2} & -2\xi & \xi+\frac{1}{2} \end{bmatrix}$$

$$\mathbf{k}^e = \int_{-1}^{1} EA\mathbf{B}^T\mathbf{B}\frac{l}{2}d\xi = EA\frac{l}{2}\int_{-1}^{1}\begin{bmatrix} N_1'N_1' & N_1'N_2' & N_1'N_3' \\ N_2'N_1' & N_2'N_2' & N_2'N_3' \\ N_3'N_1' & N_3'N_2' & N_3'N_3' \end{bmatrix} d\xi = \frac{EA}{l}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ & \frac{16}{3} & -\frac{8}{3} \\ \text{symm.} & & \frac{7}{3} \end{bmatrix} \quad \text{(Exact)}$$

$$k_{12} = EA\frac{l}{2}\int_{-1}^{1}\frac{dN_1}{dX}\frac{dN_2}{dX}d\xi = EA\frac{l}{2}\int_{-1}^{1}\frac{2}{l}\left(\xi-\frac{1}{2}\right)\frac{2}{l}(-2\xi)d\xi$$

$$\approx EA\frac{2}{l}\sum_{p=1}^{2}\left[\left(\xi-\frac{1}{2}\right)(-2\xi)\right]_{\xi_p} w_p \quad = EA\frac{2}{l}\left[\left(1/\sqrt{3}-1/2\right)\left(-2/\sqrt{3}\right)+\left(-1/\sqrt{3}-1/2\right)\left(2/\sqrt{3}\right)\right]$$

$$= EA\frac{2}{l}\left(2/\sqrt{3}\right)\left[\left(-1/\sqrt{3}+1/2\right)+\left(-1/\sqrt{3}-1/2\right)\right]$$

$$= EA\frac{2}{l}\left(2/\sqrt{3}\right)\left(-2/\sqrt{3}\right) = -\frac{8EA}{3l}$$

Exercise 2: “Reduced Integration” with 1 quadrature point

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$$\mathbf{k}^e \approx \frac{EAl}{2}\left(\mathbf{B}^T\mathbf{B}\right)\Big|_{\xi=0} 2 \qquad \mathbf{B}\Big|_{\xi=0} = \frac{4}{l^2}\begin{bmatrix} 1/4 & 0 & -1/4 \\ 0 & 0 & 0 \\ -1/4 & 0 & 1/4 \end{bmatrix}$$

$$\mathbf{k}^e = \frac{EA}{l}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Can Find modes with zero restoring forces $\quad \mathbf{k}^e\mathbf{d}^e = \mathbf{0}$

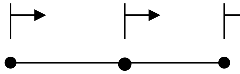
$$\frac{EA}{l}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

"Reduced Integration" Can Lead to Unstable Elements

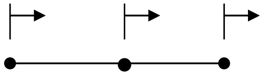
Zero energy $U^e = \frac{1}{2}\mathbf{d}^{e^T}\mathbf{k}^e\mathbf{d}^e = \frac{EA}{2l}\begin{Bmatrix} d_1 & d_2 & d_3 \end{Bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$

$$\Rightarrow \frac{EA}{2l}(d_1 - d_3)^2 = 0$$

$d_1 = 0 \quad d_2 \neq 0 \quad d_3 = 0$



$d_1 = d_2 = d_3$



Rigid

$$\varepsilon\big|_{\xi=0} = \mathbf{Bd}\big|_{\xi=0} = \frac{2}{l}\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \frac{2}{l}(d_3 - d_1)$$

$$\text{Strain energy density} = \frac{1}{2}\sigma\varepsilon$$

$$= \frac{1}{2}E\varepsilon^2 = \frac{E}{2}\frac{4}{l^2}(d_3 - d_1)^2$$

Summary and Review

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$$I = \int_{X_1}^{X_2} f\, dX = \int_{-1}^{1} \phi\, d\xi$$

$$fdX = f\left(X\left(\xi\right)\right) J d\xi = \phi\left(\xi\right) d\xi$$

Idea:

$$I = \int_{-1}^{1} \phi\, d\xi \approx w_1 \phi\left(\xi_1\right) + w_2 \phi\left(\xi_2\right) + \ldots + w_n \phi\left(\xi_n\right) = \sum_{i=1}^{n} w_i \phi\left(\xi_i\right)$$

$n =$ selected order of numerical integration

$\xi_{\mathrm{i}} =$ quadrature points

$w_i =$ quadrature weights