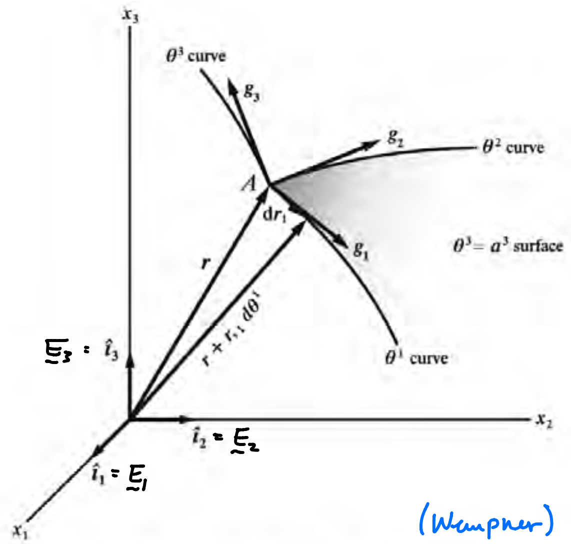
VECTORS IN GENERAL COORDINATES

The idea: track position in space using some coordinates that are <u>not</u> the standard Cartesian projections. But still may want to express the position (& other fields) relative to some Cartesian frame.



(Wempner)

Figure 2.1 Curvilinear coordinate lines and surfaces

Denote:

$(\theta^1, \theta^2, \theta^3) \equiv$ Arbitrary Curvilinear Coordinates

$\underset{\sim}{X} = \{E_1, E_2, E_3\} \equiv$ A Cartesian frame ("Lab Frame")

$\underset{\sim}{r}(\theta^1, \theta^2, \theta^3) = x_i E_i \equiv$ Position of point with coords $(\theta^1, \theta^2, \theta^3)$

$x_i(\theta^1, \theta^2, \theta^3) = r \cdot E_i \equiv$ Cartesian Components of position relative to frame $\underset{\sim}{X}$.

To say that position can be tracked by the θ^i coords $\Rightarrow$ functions $x_i(\theta^1, \theta^2, \theta^3)$ can be defined (along with the inverses $\theta^i(x_1, x_2, x_3)$).

BASE VECTORS

Now consider infinitesimal changes in position accompanying change in coord. $d\theta^i$

$$dr = r(\theta^i + d\theta^i) - r(\theta^i) = \frac{\partial r}{\partial \theta^i} d\theta^i = \underset{\sim}{g}_i\, d\theta^i$$

$\boxed{g_i \equiv \frac{\partial r}{\partial \theta^i}}$ = BASE VECTOR tangent to θ^i curve. (Also "tangent base vector," or "covariant basis vector")

Since θ^i curves are not parallel, the set $\{g_1, g_2, g_3\} = \{g_i\}$ forms a BASIS for $\mathbb{R}^3$ (Tangent/Covariant Basis)

Properties of $\{g_i\}$

1a) Can use it to express vectors: $\underset{\sim}{v} \in \mathbb{R}^3 \Rightarrow \exists\ v^i \in \mathbb{R}$ s.t. $\underset{\sim}{v} = v^i \underset{\sim}{g}_i$ (covariant components rel. to $\{g_i\}$)

1b) May not be unit length (or even unitless!): $|g_i| \neq 1$

1c) May not be $\perp$: $g_i \cdot g_j \neq \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$\Big\}$ Not orthonormal in general.

NORMAL BASE VECTORS

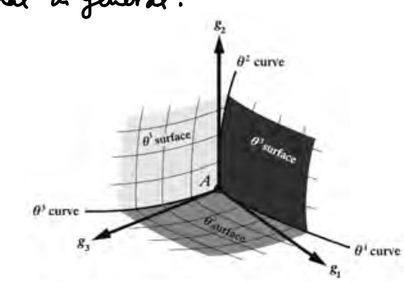


Figure 2.2 Network of coordinate curves and surfaces

θ^i-surface $\equiv$ Surface defined by positions with fixed value of θ^i.

$\underset{\sim}{g}_i$ not generally $\perp$ to θ^i-surface $(g_i \cdot g_j \neq \delta_{ij})$ but we can define a set of complementary basis vectors $\underset{\sim}{g}^i$ s.t.

$$\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

g^i's often called normal/reciprocal/contravariant basis vectors.

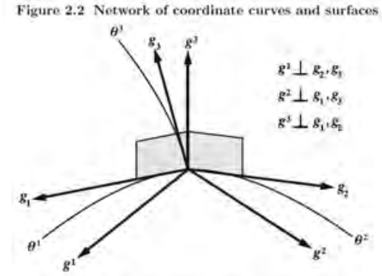


Figure 2.3 Tangent and normal base vectors

Properties of $\{\underset{\sim}{g}^i\}$:

2a) Form a basis : $\forall \underset{\sim}{v} \in \mathbb{R}^3 \ \exists \ v_i \in \mathbb{R}$ s.t. $\underset{\sim}{v} = v_i \underset{\sim}{g}^i$

2b) Not orthonormal : $\underset{\sim}{g}^i \cdot \underset{\sim}{g}^j \neq \delta^{ij}$ (generally)

(similar to $\{\underset{\sim}{g}_i\}$)

2c) Reciprocal to $\{\underset{\sim}{g}_i\}$: $\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j$

Because $\{\underset{\sim}{g}_i\}$ and $\{\underset{\sim}{g}^i\}$ are bases (properties 1a & 2a) we must be able to express them in terms of each other $\Rightarrow$ define g_{ij} & g^{ij} s.t.

$$\underset{\sim}{g}_i = g_{ij} \underset{\sim}{g}^j \qquad \text{and} \qquad \underset{\sim}{g}^i = g^{ij} \underset{\sim}{g}_j$$

Now take dot products,

$$\underset{\sim}{g}_i \cdot \underset{\sim}{g}_k = (g_{ij} \underset{\sim}{g}^j) \cdot \underset{\sim}{g}_k = g_{ij} \underbrace{(\underset{\sim}{g}^j \cdot \underset{\sim}{g}_k)}_{=\delta^j_k} = g_{ij} \delta^j_k = g_{ik}$$

$$\underset{\sim}{g}^i \cdot \underset{\sim}{g}^k = (g^{ij} \underset{\sim}{g}_j) \cdot \underset{\sim}{g}^k = g^{ij} \delta^k_j = g^{ik}$$

That is, $g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$ and $g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j$.

Also,

$$\delta^j_i = \underset{\sim}{g}_i \cdot \underset{\sim}{g}^j = g_{ip} \underset{\sim}{g}^p \cdot g^{jq} \underset{\sim}{g}_q = g_{ip} g^{jq} \delta^q_p = g_{ip} g^{jp}$$

$\therefore \ g_{ip} g^{jp} = \delta^j_i \ \Rightarrow \ [g^{ij}] = [g_{ij}]^{-1}$ (Matrices are inverses)

METRICS

Differential position

$$d\underset{\sim}{r} = \underset{\sim}{r}_{,i}\, d\theta^i = \underset{\sim}{g}_i\, d\theta^i$$

Differential length

$$dl^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = (\underset{\sim}{g}_i d\theta^i) \cdot (\underset{\sim}{g}_j d\theta^j) = g_{ij}\, d\theta^i d\theta^j$$

g_{ij} are the components of the METRIC TENSOR,

$$g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j = \underset{\sim}{r}_{,i} \cdot \underset{\sim}{r}_{,j} = x_{k,i}\, x_{k,j}$$

Differential Volume

$$dV = d\underset{\sim}{r}_1 \cdot (d\underset{\sim}{r}_2 \times d\underset{\sim}{r}_3) = \underset{\sim}{r}_{,1} \cdot (\underset{\sim}{r}_{,2} \times \underset{\sim}{r}_{,3})\, d\theta^1 d\theta^2 d\theta^3$$

$$\Rightarrow \quad dV = \underset{\sim}{g}_1 \cdot (\underset{\sim}{g}_2 \times \underset{\sim}{g}_3)\, d\theta^1 d\theta^2 d\theta^3 = \sqrt{g}\, d\theta^1 d\theta^2 d\theta^3$$

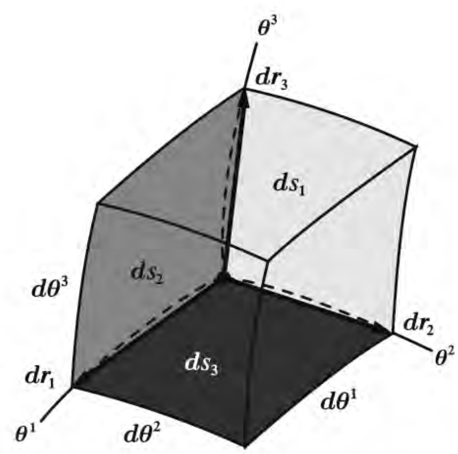


Figure 2.6 Elemental parallelepiped

where (see Wempner, §2.3)

$$\underset{\sim}{g}_i \times \underset{\sim}{g}_j = e_{ijk}\, \underset{\sim}{g}^k \qquad \text{and} \qquad \underset{\sim}{g}^i \times \underset{\sim}{g}^j = e^{ijk}\, \underset{\sim}{g}_k$$

$$e_{ijk} = \sqrt{g}\, \epsilon_{ijk} \qquad \text{and} \qquad e^{ijk} = \frac{1}{\sqrt{g}} \epsilon^{ijk}$$

and

$$g = |g_{ij}| = \det[g_{ij}] = (\det[x_{k,i}])^2$$

and $\epsilon^{ijk} = \epsilon_{ijk}$ is the Levi-Civita/alternating/permutation tensor.

Differential Area

Define unit normal to θ^i-surface

$$\hat{e}_i = \frac{g^i}{|g^i|}$$

Then area of element of θ^3-surface (for instance) is

$$ds_3 = \hat{e}_3 \cdot (dr_1 \times dr_2) = \hat{e}_3 \cdot (g_1 \times g_2)\, d\theta^1 d\theta^2$$

Note, $\hat{e}_3 = \frac{1}{|g^3|} g^3 = \frac{1}{\sqrt{g^3 \cdot g^3}} g^3 = \frac{1}{\sqrt{g^{33}}} g^3$ and $g_1 \times g_2 = \sqrt{g}\, g^3$, so

$$ds_3 = \left(\frac{1}{\sqrt{g^{33}}} g^3\right) \cdot \left(\sqrt{g}\, g^3\right) d\theta^1 d\theta^2$$

$$\boxed{ds_3 = \sqrt{g^{33}}\sqrt{g}\, d\theta^1 d\theta^2}$$

This will be useful for membranes & shells.

COMPONENTS OF VECTORS

$\{g_i\}$ and $\{g^i\}$ are both bases so we can represent any vector in either one:

$$v = v^i g_i = v_i g^i$$

Explicitly we get these components by taking dot products with base vectors

$$v \cdot g^i = (v^j g_j) \cdot g^i = v^j \delta_j^i = v^i$$

$$v \cdot g_i = (v_j g^j) \cdot g_i = v_j \delta_i^j = v_i$$

Can also get relation between the two sets of components in a similar way

$$v^i = v \cdot g^i = (v_j g^j) \cdot g^i = g^{ij} v_j$$

$$v_i = v \cdot g_i = (v^j g_j) \cdot g_i = g_{ij} v^j$$

NOMENCLATURE: $v_i = v \cdot g_i$ = COVARIANT COMPONENTS OF v

$v^i = v \cdot g^i$ = CONTRAVARIANT COMPONENTS OF v

(See Wempner, §2.7 for explanation of these names)

EXERCISE: Use component representations to prove the following:

$$\boldsymbol{V} \cdot \boldsymbol{U} = V^i U_i = V_i U^i = V_i U_j g^{ij} = V^i U^j g_{ij},$$

$$\boldsymbol{V} \times \boldsymbol{U} = e_{ijk} V^i U^j \boldsymbol{g}^k = e^{ijk} V_i U_j \boldsymbol{g}_k,$$

$$\boldsymbol{U} \cdot (\boldsymbol{V} \times \boldsymbol{W}) = e_{ijk} U^i V^j W^k = e^{ijk} U_i V_j W_k,$$

$$\boldsymbol{U} \times (\boldsymbol{V} \times \boldsymbol{W}) = (U^i W_i)(V^j \boldsymbol{g}_j) - (U^i V_i)(W^j \boldsymbol{g}_j).$$

DERIVATIVES

Differentiate a vector → both components and basis vectors depend on coords:

$$\underset{\sim}{v}_{,i} = (v^k \underset{\sim}{g}_k)_{,i} = v^k_{,i}\underset{\sim}{g}_k + v^k \underset{\sim}{g}_{k,i}$$

or

$$= (v_k \underset{\sim}{g}^k)_{,i} = v_{k,i}\underset{\sim}{g}^k + v_k \underset{\sim}{g}^k_{,i}$$

∴ We need to differentiate the basis vectors. Start with covariant basis

$$\underset{\sim}{g}_{k,i} = (\underset{\sim}{r}_{,k})_{,i} = \underset{\sim}{r}_{,ki} = \underset{\sim}{g}_{i,k}$$

Express these in components rel. to the two bases,

$$\underset{\sim}{g}_{i,j} = \Gamma^k_{ij}\underset{\sim}{g}_k = \Gamma_{ijk}\underset{\sim}{g}^k$$

where the coefficients are defined

$$\Gamma_{ijk} \equiv \underset{\sim}{g}_k \cdot \underset{\sim}{g}_{i,j} \qquad \Gamma^k_{ij} \equiv \underset{\sim}{g}^k \cdot \underset{\sim}{g}_{i,j}$$

and are called the CHRISTOFFEL SYMBOLS of the first & second kind.

Note the symmetries,

$$\Gamma_{ijk} = \Gamma_{jik} \qquad \Gamma^k_{ij} = \Gamma^k_{ji}$$

Can also show (exercise)

$$\Gamma_{ijk} = \tfrac{1}{2}\left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right)$$

and

$$\underset{\sim}{g}^i_{,k} = -\Gamma^i_{jk}\underset{\sim}{g}^j .$$

From these results we then have

$$\underset{\sim}{v}_{,i} = v^j_{,i}\underset{\sim}{g}_j + v^j \underset{\sim}{g}_{j,i} = (v^j_{,i} + v^k \Gamma^j_{ki})\underset{\sim}{g}_j \equiv v^j|_i \underset{\sim}{g}_j$$

$$\underset{\sim}{v}_{,i} = v_{j,i}\underset{\sim}{g}^j + v_j \underset{\sim}{g}^j_{,i} = (v_{j,i} - v_k \Gamma^k_{ji})\underset{\sim}{g}^j \equiv v_j|_i \underset{\sim}{g}^j$$

Where we define the COVARIANT DERIVATIVES of v^j & v_j as

$$v^j|_i \equiv v^j_{,i} + v^k \Gamma^j_{ki} \qquad v_j|_i \equiv v_{j,i} - v_k \Gamma^k_{ji}$$

Other useful results (exercise)

$$\frac{\partial g}{\partial g_{ij}} = g\, g^{ij} \qquad \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial \theta^i} = \Gamma^j_{ji}$$