FINITE ELEMENT APPROXIMATION

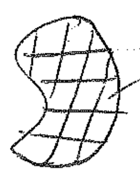
MPE : $\inf_{\varphi \in X} I[\varphi]$

X = Space of admissible solutions

R-R : $\inf_{\varphi \in X_h} I[\varphi]$

X_h = Finite Dim'l supspace of X

FE interpolation : $\underset{\sim}{x} = \underset{\sim}{\varphi}_h(\underset{\sim}{X}) = \sum_{a=1}^{N} \underset{\sim}{x}_a N_a(\underset{\sim}{X})$



Ω^e

$B = \bigcup_{e=1}^{E} \Omega^e$

$$= \sum_{e=1}^{E} \sum_{a=1}^{N} \underset{\sim}{x}_a^{(e)} N_a^e(\underset{\sim}{X})$$

$\underset{\sim}{x}_a \equiv$ coordinates of nodes in $\varphi(B)$

$N_a(X) \equiv$ Material Shape fns on B

R-R' : $\inf_{\underline{x} \in \mathbb{R}^{3N}} I(\underline{x})$

Elasticity :

$$\cancel{\frac{\partial I}{\partial x_{ia}}} \quad I[\varphi] = \int_B w(\nabla\varphi)\, dV - \int_B R B_i \varphi_i\, dV - \int_{S_2} \bar{T}_i \varphi_i\, dS$$

R-R' : $\frac{\partial I}{\partial x_{ia}} = 0$

$$\cancel{\Rightarrow 0 = \int_B \frac{\partial W}{\partial F_{kJ}} \frac{\partial F_{kJ}}{\partial x_{ia}}\, dV - \int_B R B_k \frac{\partial \varphi_k}{\partial x_{ia}}\, dV - \int_{S_2} \bar{T}_i \frac{\partial \varphi_i}{\partial x_{ia}}\, dS}$$

$$(\varphi_h)_{i} = \sum_{a=1}^{N} x_{ia} N_a(X)$$

$$\Rightarrow (F_h)_{iJ} = (\varphi_h)_{i,J} = \sum_{a=1}^{N} x_{ia} N_{a,J}(X)$$

$$\therefore \; I_h = \int_B W(F_h)\, dV - \int_B R B_i \Big(\sum_{a=1}^{N} x_{ia} N_a \cancel{(X)}\Big) dV - \int_{S_2} T_i \Big(\sum_a x_{ia} N_a\Big) dS$$

$$\boxed{I_h = \int_B W(F_h)\, dV - \sum_{a=1}^{N} f_{ia}^{ext} x_{ia}}$$

$$f_{ia}^{ext} = \int_B R B_i N_a \, dV + \int_{S_2} T_i N_a \, dS$$

E-L eqns of RR': $f_h(x_h) \equiv DI_h(x_h) = 0$

$$f_{ia} = \frac{\partial I_h}{\partial x_{ia}} = \underbrace{\frac{\partial}{\partial x_{ia}} \int_B W(F_h)\, dV}_{\equiv f_{ia}^{int}} - f_{ia}^{ext}$$

$$f_{ia} = f_{ia}^{int} - f_{ia}^{ext} = 0 \Rightarrow f_{ia}^{int} = f_{ia}^{ext}$$

$$\boxed{\underline{f}_h^{int} = \underline{f}_h^{ext}} \qquad (*)$$

$$f_{ia}^{int} = \frac{\partial}{\partial x_{ia}} \int_B W(F_h)\, dV = \int_B \frac{\partial W}{\partial F_{kJ}}(F_h) \frac{\partial}{\partial x_{ia}} \Big(\sum_{b=1}^{N} x_{kb} N_{b,J}\Big) dV$$

$$= \int_B P_{kJ}(F_h)\, \delta_{ik} \delta_{ab} N_{b,J} \, dV$$

$$\boxed{f_{ia}^{int}(x_h) = \int_B P_{iJ}(F_h) N_{a,J} \, dV}$$

So, how do we solve $(*)$, set of nonlinear equations for x_{ia}'s? NEWTON

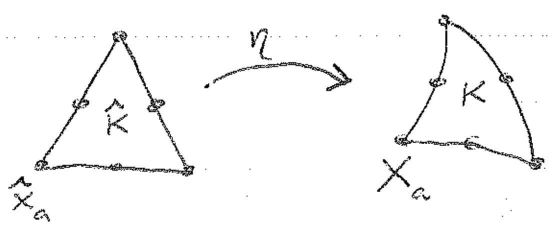
So far $\int$'s are over $\mathcal{B}$. But we could write over $\varphi(\mathcal{B})$. would need spatial shape fns $n_a = N_a \circ \varphi_h^{-1}$.

To construct isoparametric mappings between $\hat{K}$ and K or $\varphi(K)$ we need coords of nodes in K or $\varphi(K)$, i.e. X_{Ia} or x_{ia}.

$$\underset{\text{Fixed}}{X_{Ia}} \qquad \underset{\text{Changing}}{x_{ia}}$$

$\therefore$ $\int_{\mathcal{B}} \Rightarrow$ compute N_a once and never again

$\int_{\varphi(\mathcal{B})} \Rightarrow$ " n_a every time x_{ia} changes



isoparametric : $\eta^K_I(\lambda) = \sum_a X_{Ia} \hat{N}_a(\lambda)$

~~Actual~~ Material/Reference Shape Fns $N_a = \hat{N}_a \circ \eta^{-1}$

$$N_{a,I} = \hat{N}_{a,\alpha} J^{-1}_{\alpha I} \qquad [J_{I\alpha}] = \begin{bmatrix} \eta_{I,\alpha} \\ 1 \cdots 1 \end{bmatrix}$$

$$dV = J\, d\hat{V} \qquad J = \det[J_{I\alpha}]$$

Same as before ... INDEPENDENT OF φ_h (i.e., x_{ia})

see any other FE text for spatial approach.

Newton-Raphson Solution Procedure

$$f_h(x_h + u_h) \sim f_h(x_h) + \underline{Df_h(x_h)}\, u_h$$

$$Df_h = D(DI_h) = D^2 I_h$$

$$K_h(x_h) = Df_h(x_h) = D^2 I_h(x_h)$$

$$\boxed{K_{iakb} = \frac{\partial f^{int}_{ia}}{\partial x_{kb}}(x_h)}$$

$$= \frac{\partial^2 I_h}{\partial x_{ia}\,\partial x_{kb}}(x_h) = K_{kbia}$$

(symmetric!)
but non constant

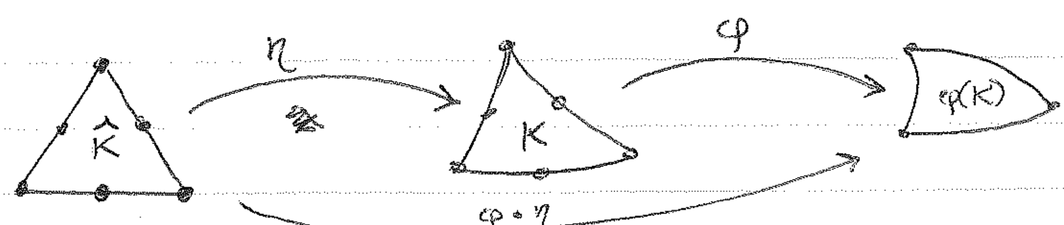
$$K_{iakb} = \frac{\partial}{\partial x_{ia}} \int_B P_{iJ}(F_h)\, N_{a,J}\, dV$$

$$= \int_B \frac{\partial P_{iJ}}{\partial F_{lL}}(F_h)\, N_{a,J}\, \frac{\partial}{\partial x_{kb}}\left(\sum_{c=1}^{N} x_{lc} N_{c,L}\right) dV$$

$$= \int_B C_{iJlL}(F_h)\, N_{a,J}\, \delta_{kl}\, \delta_{bc}\, N_{c,L}\, dV$$

$$\boxed{K_{iakb} = \int_B C_{iJkL}(F_h)\, N_{a,J}\, N_{b,L}\, dV}$$

Isoparametric Elements



Standard Domain — Reference Domain (Material) — Current Domain (Spatial)

Evaluate integrals w/ Quadrature

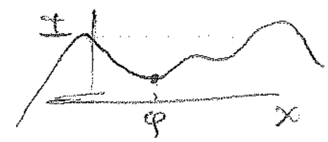
$$f^{int,E}_{ia} = \sum_{p=1}^{Q} P_{iJ}(F_h^E(\xi^{(p)}))\, N^E_{a,J}(\xi^{(p)})\, \hat{w}_p\, J(\xi^{(p)})\, \hat{V}$$

&

$$K^E_{iakb} = \sum_{p=1} C_{iJkL}(F_h^E(\xi^{(p)}))\, N^E_{a,J}(\xi^{(p)})\, N^E_{b,L}(\xi^{(p)})\, J(\xi^{(p)})\, \hat{w}_p\, \hat{V}$$

$\therefore$ Shape Fn Derivs & Jacobian and Constitutive relations are evaluated at each quad pt.

Solution Strategies For Nonlinear Eqs $\underline{f}^{int} - \underline{f}^{ext} = \underline{0}$



Nonlinear $\Rightarrow$ Possible bifurcations instabilities

INCREMENTAL Solution Procedure:

Gradually increment applied loads / prescribed displacements

$$\underline{f}^{ext}_0,\ \underline{f}^{ext}_1,\ \ldots,\ \underline{f}^{ext}_n,\ \underline{f}^{ext}_{n+1},\ \ldots$$

GIVEN / SPECIFIED force arrays.

n plays the role of pseudo-time

$\hookrightarrow\ \underline{x}_0,\ \underline{x}_1,\ \ldots,\ \underline{x}_n,\ \underline{x}_{n+1},\ \ldots$

$\underline{x}_n \equiv$ solution @ $\underline{f}^{ext}_n$

Method of Continuation: Assume $\underline{x}_n$ is known given $\underline{f}_{n+1}$ compute $\underline{x}_{n+1}$.

ITERATIVE Equation Solving Methods:

- Gradient methods: Steepest descent, Conj. Grad.
- Newton methods "
- Hybrid "

(cf. Nocedal & Wright)

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~~Also~~

NEWTON-RAPHSON (most popular in FE community)

Start w/ $\underline{x}_n$, solve for $\underline{x}_{n+1}$ iteratively by linearization

Denote: $\underline{x}_{n+1}^{(k)} \equiv k^{th}$ iteration $k = 0, 1, \ldots$

Set iteration: $\underline{x}_{n+1}^{(k+1)} = \underline{x}_{n+1}^{(k)} + \underline{u}$

linearize w.r.t. $\underline{u}$

$$f^{int}(\underline{x}_{n+1}^{(k)} + u) - \underline{f}^{ext} = 0$$

linearize

$$\langle Df^{int}(\underline{x}_{n+1}^{(k)}), \underline{u} \rangle + \underline{f}^{int}(x_n^{(k)}) - \underline{f}_{n+1}^{ext} = 0$$

$$K_{iakb}(\underline{x}_{n+1}^{(k)})\underline{u}_{kb} = f_{ia}^{ext} - f_{ia}^{int}(x_n^{(k)}) \equiv r_{ia}^{(k)}$$

residual

$$\underline{u}_{kb} = K^{-1}_{kbia}(x_{n+1}^{(k)})\, r_{ia}^{(k)}$$

~~Algorithm~~

NR Algorithm

(0) initialize iteration

set $k=0$ $\quad x_{n+1}^{(0)} = x_n$

(i) set up system : compute $r_{n+1}^{(k)}$ $\quad K(x_{n+1}^{(k)})$

(ii) Solve : $K_{n+1}^{(k)}\, u = r_{n+1}^{(k)} \longrightarrow u$

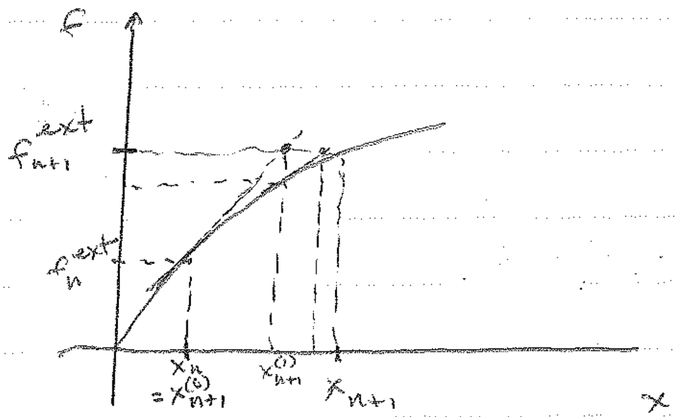
(iii) update : $x_{n+1}^{(k+1)} = x_{n+1}^{(k)} + u$

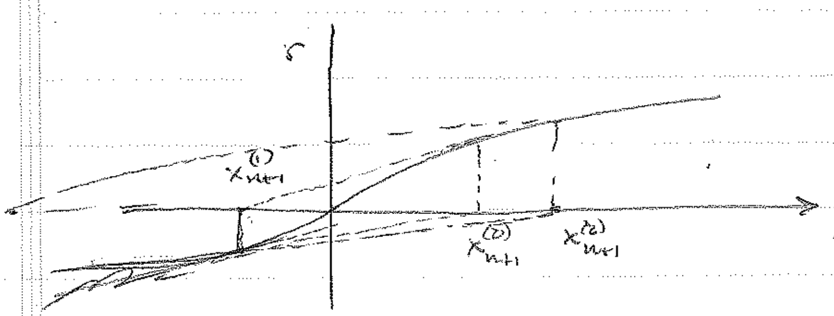
(iv) Convergence test :

$$\frac{\| x_{n+1}^{(k+1)} - x_{n+1}^{(k)} \|}{\| x_{n+1}^{(k+1)} - x_{n+1}^{(0)} \|} < \text{TOL} \; ?$$

YES $\Rightarrow$ EXIT

NO $\Rightarrow$ $k \leftarrow k+1$ Goto (i)





Divergent

Properties of N-R

(1) Domain of Attraction

If K nonsingular $\Rightarrow$ $\exists\, \rho > 0$ s.t.
if $\| x_{n+1} - x_n \| < \rho$ NR will converge.

max $\rho \equiv$ radius of convergence.

$\{ x_n$ s.t. $\| x_{n+1} - x_n \| < \rho \} \equiv$ Domain of attraction

(2) If NR converges & $K(x_{n+1})$ is nonsingular, the convergence rate is quadratic

Error : $$e^{(k)} \equiv \frac{\| x_{n+1}^{(k)} - x_{n+1} \|}{\| x_{n+1} - x_n \|}$$

Then $\exists\, C > 0$ st. $e^{(k+1)} < C \left(e^{(k)}\right)^2$

i.e., $e^{(k)} \to 10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}$