VARIATIONAL VISCOPLASTICITY (Ref., Ortiz and Stainier, CMAME 171 (1999): 419–444)

Goal: Want to extend hyperelasticity to include irreversible constitutive behavior

- viscosity, rate-dependent stresses
- permanent (plastic) deformation

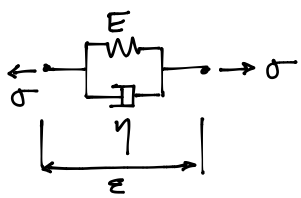
But we'd like to retain variational (minimum) structure.

Key idea: Define an incremental BVP for a viscoplastic update to the state of the material. Formulate a variational principle for the update.

Viscoelasticity

Start simple w/ rate-dependance. Deal w/ plasticity later. Use time-discretization to define the update.

Consider a Kelvin-Voit model



$$\sigma = \sigma^e + \sigma^v = E\varepsilon + \eta\dot{\varepsilon}$$

Finite-deformation Newtonian viscosity:

$$\underset{\sim}{\sigma}^v = 2\eta\, \underset{\sim}{d}^{dev} \qquad \underset{\sim}{P}^v = J\underset{\sim}{\sigma}^v \underset{\sim}{F}^{-T}$$

$$\underset{\sim}{d}^{dev} = \underset{\sim}{d} - \tfrac{1}{3}\operatorname{tr}(\underset{\sim}{d})\underset{\sim}{I} \qquad \underset{\sim}{d} = \tfrac{1}{2}(\underset{\sim}{\ell} + \underset{\sim}{\ell}^T) = \text{rate of deformation tensor}$$

$$\underset{\sim}{\ell} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} \equiv \text{velocity gradient} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}}$$

$$\ell_{ik} = \dot{F}_{iJ} F^{-1}_{Jk}$$

$$\therefore \quad \underset{\sim}{P} = \underbrace{\underset{\sim}{P}^e(F)}_{\text{elastic}} + \underbrace{\underset{\sim}{P}^v(\dot{F}; F)}_{\text{viscous}} \qquad \underset{\sim}{P}^e(\underset{\sim}{F}) = \frac{\partial W^e}{\partial \underset{\sim}{F}} \quad P^v(\dot{F}, F) = ?$$

Assume $\exists$ a dissipative potential $\phi(\dot{F}, F)$ such that

$$\underset{\sim}{P}^v = \frac{\partial \phi}{\partial \underset{\sim}{\dot{F}}}(\dot{F}, F) \qquad P^v_{iJ} = \frac{\partial \phi}{\partial \dot{F}_{iJ}}(\dot{F}, F)$$

E.g., for Newtonian viscosity: $\quad \phi = \eta J |d^{dev}|^2$

Verify: $\displaystyle \frac{\partial \phi}{\partial \dot{F}_{iJ}} = 2\eta J d^{dev}_{mn} \frac{\partial d^{dev}_{mn}}{\partial \dot{F}_{iJ}} \qquad A^{dev} B^{dev} = A^{dev}(B - B^{vol}) = A^{dev} B \;)$

$$= 2\eta J d^{dev}_{mn} \frac{\partial d_{mn}}{\partial \dot{F}_{iJ}}$$

$$= 2\eta J d^{dev}_{mn} \frac{\partial}{\partial \dot{F}_{iJ}} \left[\tfrac{1}{2}\left(\dot{F}_{mP} F^{-1}_{Pn} + \dot{F}_{nP} F^{-1}_{Pm}\right)\right]$$

$$= \eta J d^{dev}_{mn} \left(\delta_{im} F^{-1}_{Jn} + \delta_{in} F^{-1}_{Pm}\right)$$

$$= 2\eta J d^{dev}_{im} F^{-1}_{Jm} = P^v_{iJ} \;\checkmark$$

Use potential structure + time discretization to define an effective (incremental) strain energy:

$$W(F_{n+1}) = W^e(F_{n+1}) + \Delta t\, \phi\left(\frac{F_{n+1} - F_n}{\Delta t}; F_n\right)$$

$$\frac{\partial W}{\partial F_{n+1}} = \underbrace{\frac{\partial W^e(F_{n+1})}{\partial F_{n+1}}}_{P^e_{n+1}} + \Delta t \underbrace{\frac{\partial \phi}{\partial \dot{F}}\left(\frac{F_{n+1} - F_n}{\Delta t}; F_n\right)}_{P^v_{n+1}} \underbrace{\frac{\partial}{\partial F_{n+1}}\left(\frac{F_{n+1} - F_n}{\Delta t}\right)}_{I \frac{1}{\Delta t}}$$

$$= P^e_{n+1} + P^v_{n+1} = P_{n+1}$$

Incremental Potential Energy (inertia + viscosity):

$$I[\chi_{n+1}] = \int_{\Omega_0} \left\{ \rho_0 \left|\frac{\chi_{n+1} - \chi_n}{\sqrt{\beta}\, \Delta t}\right|^2 + W(\nabla \chi_{n+1}) \right\} dV + \text{F.T.}$$

Same story as before...

Alternative discretization of ϕ: decouple volumetric & isochoric deformations exactly

$$\Delta\phi \longrightarrow \Delta t \eta J_{n+1} \Bigg| \underbrace{\frac{\log \sqrt{C^{dev}_{n+1,n}}}{\Delta t}}_{\sim d^{dev}} \Bigg|^2$$

$$C^{dev} = (F^{dev})^T (F^{dev}) \quad F^{dev} = J^{-1/3} F \quad \left(\text{s.t. } \det F^{dev} = (J^{-1/3})^3 \cdot \det F = 1\right)$$

$$F_{n+1,n} = F_{n+1} F_n^{-1} \longrightarrow F^{dev}_{n+1,n} = J_{n+1}^{-1/3} J_n^{1/3} F_{n+1,n}$$