Elastic Materials

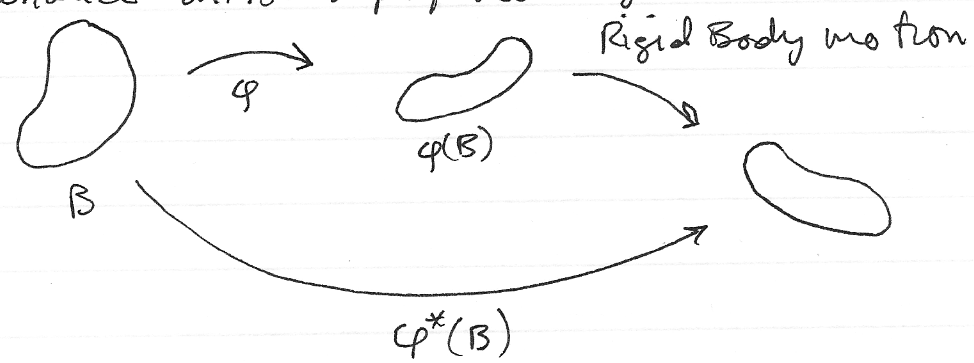
To model specific materials:

- General principles:
  - Material Frame Indifference (Objectivity)
  - Material symmetry (isotropy, etc.)
  - internal constraints (incompress.)

- Modeling:
  - Phenomenological (fit experiments)
  - Statistical Mechanics
  - multiscale simulation/modeling
  - homogenization

Material Frame Indifference

Invariance w.r.t. superposed rigid rotations..



~~Ex.~~

i.e., nothing important should change if you tilt your head while watching the deformed body.

Rigid Motion $\quad x = QX + b = \varphi^R(X)$

$b \in \mathbb{R}^3$

$Q \in SO(3) = \{ Q \in \mathbb{R}^{3\times 3} \text{ s.t. } Q^T Q = 1 \ \& \ \det Q = +1 \}$

$$\therefore \quad F^* = \nabla \varphi^* = \nabla \varphi^R(\varphi(X)) = Q F$$

$w(F)$ is Frame indifferent / objective $\Longleftrightarrow$

$$\boxed{w(F) = w(QF) \quad \forall\, Q \in SO(3) \quad \forall F}$$

Can show (~~Ho~~ cf. Holzapfel or Mal's notes)

w is MFI $\Longleftrightarrow$ $w(C)$ $\quad C = F^T F \quad C_{IJ} = F_{kI} F_{kJ}$

What about ~~B~~ 1st Piola–Kirchhoff Stress P_{iJ}?

$$P_{iJ} = \frac{\partial w}{\partial F_{iJ}}(F)$$

$$w(F) = w(QF) \quad \forall\, Q \in SO(3)$$

$$P^*_{iJ} = \frac{\partial w}{\partial F_{iJ}}(QF)$$

$$F \to F + dF$$

$$w(Q(F+dF)) = w(F+dF)$$

$$\cancel{w(QF)} + \frac{\partial w}{\partial F_{iJ}}(QF)\, Q_{ij}\, dF_{jJ} = \cancel{w(F)} + \frac{\partial w(F)}{\partial F_{iJ}} dF_{iJ}$$

$$\Rightarrow P_{iJ}(QF)\, Q_{ij}\, dF_{jJ} = \cancel{P_{iJ}\, dF_{iJ}}\; P_{jJ}(F)\, dF_{jJ}$$

$\forall\, dF_{jJ}$

$\hookrightarrow$

$$P_{iJ}(QF)\, Q_{ij} = P_{jJ}(F)$$

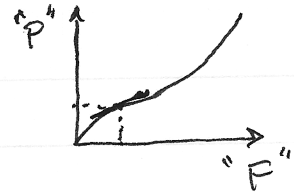
$$Q^T P^* = P$$

or

$$\boxed{P_{iJ}(QF) = Q_{ij} P_{jJ}}$$

$$P^* = Q P$$

$$\boxed{P^*_{iJ}\, Q_{ij}\, Q_{kj} = P_{jJ}(F)\, Q_{kj}}$$



in general $P(F)$ will not be linear.
→ incremental constitutive relations

$$w(F) \rightarrow \quad P_{iJ} = \frac{\partial w}{\partial F_{iJ}}(F)$$

Perturb $F \rightarrow F + dF \Rightarrow P + dP$

$$dP_{iJ} = \frac{\partial P_{iJ}}{\partial F_{kL}}(F)\, dF_{kL} = \underbrace{\frac{\partial^2 w}{\partial F_{iJ}\partial F_{kL}}(F)}_{\cancel{\equiv C_{iJkL}(F)}} dF_{kL}$$

$$C_{iJkL}(F) \equiv \frac{\partial^2 w}{\partial F_{iJ}\partial F_{kL}}(F) \quad \text{Lagrangian (tangent) Elastic Moduli}$$

MFI for C_{iJkL}: $\quad P_{iJ}(Q(F+dF))\, Q_{ij} = P_{jJ}(F+dF)$

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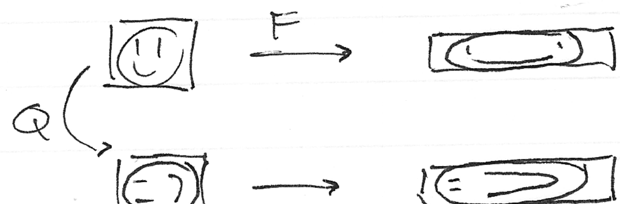
$$\boxed{C_{iJkL}(QF) = Q_{im} Q_{kn} C_{mJnL}(F)}$$

<u>Material Symmetry</u>

$Q \in SO(3)$ is a symmetry of $w \Leftrightarrow w(FQ) = w(F) \ \forall F$

Pre-multiply w/ $Q \Rightarrow$ Rotate, then deform.
MFI = post-mult. $\Rightarrow$ Deform, then rotate.



$$\left.\begin{matrix} w(F) \\ w(FQ) \end{matrix}\right\} \text{equal} \Rightarrow Q \text{ is a symmetry}$$

If any $Q \in SO(3)$ is a symmetry → Isotropic

Elastic Material is ~~the~~ isotropic $\Leftrightarrow$ $W = W(I_1, I_2, I_3)$

$$\begin{cases} I_1 = \mathrm{tr}(C) \\ I_2 = \frac{1}{2}\left[\mathrm{tr}(C^2) - \mathrm{tr}(C)^2\right] \\ I_3 = \det(C) = J^2 \end{cases} \qquad C = F^T F$$

invariants of $\underset{\sim}{C}$ form complete & irreducible set of invariants for isotropic material

Examples of Hyperelastic Materials

Biomaterials and polymers are only materials which exhibit large strain reversible behavior.

Incompressible Mats. $\quad J=1$

$$W(F) = w(F) + p(J-1) \qquad \frac{\partial J}{\partial F_{iJ}} = J F^{-1}_{Ji}$$

$$\hookrightarrow \quad P_{iJ} = \frac{\partial w}{\partial F_{iJ}} + pF^{-1}_{Ji}$$

Ex Neo-Hookean Mat.

$$w = \frac{\mu}{2} tr(C)$$

$$\frac{\partial w}{\partial F_{iJ}} = \frac{\mu}{2}\frac{\partial}{\partial F_{iJ}} tr(C) = \frac{\mu}{2}\left(C_{KL}\delta_{KL}\right) = \frac{\mu}{2}\frac{\partial}{\partial F_{iJ}}\left(F_{mK}F_{mL}\delta_{KL}\right)$$

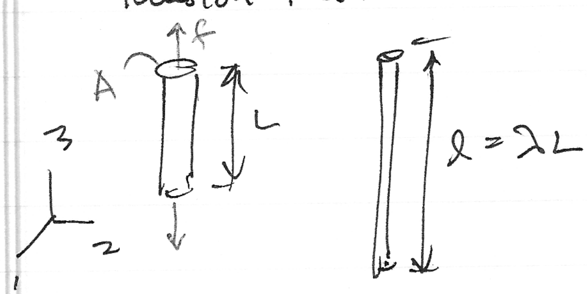
$$= \frac{\mu}{2}\left(\delta_{im}\delta_{JK}F_{mL}\delta_{KL} + F_{mK}\delta_{im}\delta_{JL}\delta_{KL}\right)$$

$$= \frac{\mu}{2}\left(F_{iJ} + F_{iJ}\right) = \mu F_{iJ}$$

$$P_{iJ} = \mu F_{iJ} + pJF^{-1}_{Ji}$$

$$J - 1 = 0$$

Tension test



λ = stretch ratio $= \frac{\partial x_3}{\partial X_3} = F_{33}$

$P = f/A$ nominal stress

$$F = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad J = \det(F) = 1$$

$$\underset{\sim}{P} = \mu \underset{\sim}{F} + pJ\underset{\sim}{F}^{-T} = \mu\begin{pmatrix} \lambda^{-1/2} & & \\ & \lambda^{-1/2} & \\ & & \lambda \end{pmatrix} + pJ\begin{pmatrix} \lambda^{1/2} & & \\ & \lambda^{1/2} & \\ & & \lambda^{-1} \end{pmatrix}$$

$$P_{11} = P_{22} = 0 = \mu\lambda^{-1/2} + pJ\lambda^{1/2} = \mu/\sqrt{\lambda} + p\cancel{J}^{1}\sqrt{\lambda}$$

$$\Rightarrow \quad p = -\frac{\mu}{\lambda}$$

$$P_{33} = P = \mu\lambda + \left(-\frac{\mu}{\lambda}\right)\frac{1}{\lambda} = \mu\lambda - \frac{\mu}{\lambda^2}$$

Ex 2 St. Venant – Kirchhoff Model

"Linear" hyperelastic model.

$$S_{IJ} = \lambda_0 (E_{KK}) \delta_{IJ} + 2\mu_0 E_{IJ} = C_{IJKL} E_{KL}$$

$$\Rightarrow \quad W(C) = \tfrac{1}{2} S_{IJ} E_{IJ} = \tfrac{1}{2} C_{IJKL} E_{IJ} E_{KL}$$

Uniaxial Response $\quad \underset{\sim}{F} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\underset{\sim}{E} = \tfrac{1}{2}(\underset{\sim}{C} - \underset{\sim}{1}) = \frac{1}{2}\begin{pmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

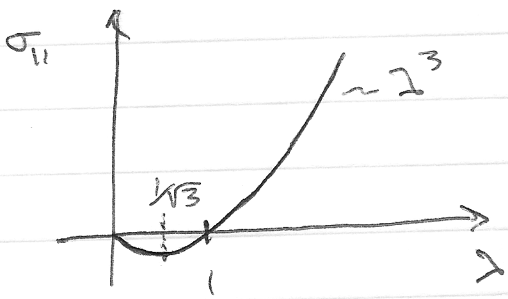
$$S_{11} = (\lambda_0 + 2\mu_0) E_{11} = \frac{\lambda_0 + 2\mu_0}{2} (\lambda^2 - 1)$$

$\cancel{E_{11} = \tfrac{1}{2}(\lambda^2 - 1)}$

$\underset{\sim}{\sigma} = \frac{1}{J} F S F^T \Longrightarrow \quad \sigma_{11} = \lambda S_{11} = (\lambda_0 + 2\mu_0) \frac{\lambda}{2}(\lambda^2 - 1)$

$\lambda \to \infty \Rightarrow \sigma_{11} \to \infty$ Good

$\lambda \to 0 \Rightarrow \sigma_{11} \to 0$ No Good



unstable in compression!

Dangerous!

Ex 3 Mooney Rivlin Materials

$$W(C) = \tfrac{1}{2}[\alpha_1 (I_1 - 3) + \alpha_2 (I_2 - 3)]$$

$\alpha_1 = \text{const}, \quad \alpha_2 = \text{const}$

$J = 1$ incompressible

Ex4 NeoHookean extended to compressible Range

$$w(C) = \frac{1}{2}\lambda_0 (\log(J))^2 - \mu_0 \log(J) + \frac{\mu_0}{2}(I_1 - 3)$$

$$I_1 = \operatorname{tr} \underset{\sim}{C} \qquad J = \det \underset{\sim}{F} \qquad \underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}$$

Usefull formulae:
$$\frac{\partial F^{-1}_{Ji}}{\partial F_{kL}} = -F^{-1}_{Jk} F^{-1}_{Li}$$

$$\frac{\partial J}{\partial F_{iJ}} = J F^{-1}_{Ji}$$

$$P_{iJ} = \frac{\partial w}{\partial F_{iJ}} = (\lambda_0 \log(J) - \mu_0) F^{-1}_{Ji} + \mu_0 F_{iJ}$$

$$C_{iJkL} = \frac{\partial P_{iJ}}{\partial F_{kL}} = \lambda_0 F^{-1}_{Ji} F^{-1}_{Lk} + \mu_0 \delta_{ik} \delta_{JL}$$
$$- (\lambda_0 \log(J) - \mu_0) F^{-1}_{Jk} F^{-1}_{Li}$$

Uniaxial Stress-Strain Curve

$$F = \begin{pmatrix} \lambda' & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \lambda = l/L \qquad \lambda' = \text{transverse stretch ratio}$$

$$J = (\lambda')^2 \lambda$$

$$P = (\lambda_0 \log(\lambda'^2 \lambda) - \mu_0) \begin{pmatrix} 1/\lambda' & & \\ & 1/\lambda' & \\ & & 1/\lambda \end{pmatrix} + \mu_0 \begin{pmatrix} \lambda' & & \\ & \lambda' & \\ & & \lambda \end{pmatrix}$$

Boundary Conditions: Transverse tractions = 0

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P \end{pmatrix} = (\lambda_0 \log(\lambda'^2 \lambda) - \mu_0) \begin{pmatrix} 1/\lambda' & & \\ & 1/\lambda' & \\ & & 1/\lambda \end{pmatrix} + \mu_0 \begin{pmatrix} \lambda' & & \\ & \lambda' & \\ & & \lambda \end{pmatrix}$$

$J \neq 0 \Rightarrow \lambda' \neq 0 \Rightarrow$
$$0 = [\lambda_0 \log(\lambda'^2 \lambda) - \mu_0] + \mu_0 (\lambda')^2$$
$$P = [\lambda_0 \log(\lambda'^2 \lambda) - \mu_0]/\lambda + \mu_0 \lambda$$

Cauchy Stress $\quad \underset{\sim}{\sigma} = J^{-1} \underset{\sim}{P} \, \underset{\sim}{F}^T$

$$\sigma = P/(\lambda')^2$$

$$\Rightarrow \quad 0 = [\lambda_0 \log(\lambda'^2 \lambda) - \mu_0] + \mu_0 (\lambda')^2 \qquad (*)$$

$$\sigma = \frac{\mu_0}{\lambda}\left[\left(\frac{\lambda}{\lambda'}\right)^2 - 1\right]$$

(*) is a nonlinear equation for λ' as a fn. of λ.
say $\quad f(\lambda') = 0$. Can solve with a Newton iterative procedure.

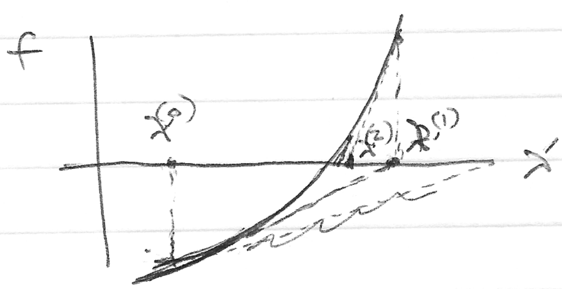
Starting $\lambda' \to f(\lambda') \neq 0$

$f(\lambda + d\lambda) \approx f(\lambda') + \frac{df}{d\lambda}(\lambda') \, d\lambda'$

find $d\lambda$ s.t. $f(\lambda' + d\lambda') = 0$

$\hookrightarrow \quad f(\lambda') + \frac{df}{d\lambda}(\lambda') \, d\lambda' = 0 \quad \Rightarrow d\lambda' = -\left(\frac{df}{d\lambda}(\lambda')\right)^{-1} f(\lambda')$

since f nonlinear, have to iterate.



Solving (*) find