DKT ELEMENTS

Two papers to read on DKT formulation:

- Batoz, Bathe & Ho (1980) → PLATES
- Bathe & Ho (1981). → SHELLS

BASIC IDEA: Enforce the transverse-shear condition ($\gamma = 0$) as a constraint at isolated points in the element.

Batoz, Bathe, & Ho, IJNME (1980)

- They study 3 elements: a DKT element, a "hybrid stress" element, and a "reduced integration" element.
- Following the approach of superimposing a "bending element" and a "membrane element."
- Emphasis on Reliability. Want an element that always yields stable solutions.

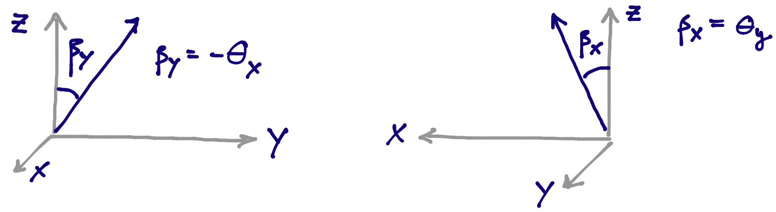
REVIEW OF PRIOR LITERATURE

- Early 3-node, 9-dof elements → issues with incompleteness, incompatibility, orientation invariance, & singularity.
- Realized that single-field polynomial expansion was not possible
- "HTC element" = Clough-Tocher
- HSM elements: hybrid/mixed formulation of displacements & stresses/moments
  - still a Kirchhoff theory element.
- DKT elements – use Mindlin kinematics, with bending energy/stresses only and impose the Kirchhoff constraint locally at discrete points.
  - early literature a bit disorganized it seems, but they claim the element behaves quite well nonetheless.

DKT FORMULATION

Kinematics: $\quad u = z\,\beta_x(x,y) \qquad v = z\,\beta_y(x,y) \qquad w = w(x,y)$

They call β_x & β_y "rotations" but they are not precisely the components of the rotation vector



Looks like $\vec{\beta} = \hat{e}_z \times \vec{\theta}$.

$$\hat{e}_z \times (\theta_x \hat{e}_x + \theta_y \hat{e}_y) = \theta_x(\hat{e}_z \times \hat{e}_x) + \theta_y(\hat{e}_z \times \hat{e}_y)$$

$$= \theta_x \hat{e}_y - \theta_y \hat{e}_x \Rightarrow \beta_y = \theta_x \quad \beta_x = -\theta_y \quad \text{opposite}$$

$\therefore \; \vec{\beta} = -\hat{e}_z \times \vec{\theta}$.

But also, note that (from the sketches) we have in the linear approx.

$$\beta_x \simeq -w_{,x} \qquad \beta_y \simeq -w_{,y}$$

such that $\vec{\beta} = -\nabla w$ and thus $\nabla w = \hat{e}_z \times \vec{\theta}$. Then note also that

$$\hat{e}_z \times \nabla w = \hat{e}_z \times (\hat{e}_z \times \vec{\theta}) = (\overset{0}{\cancel{\hat{e}_z \cdot \vec{\theta}}})\,\hat{e}_z - (\overset{1}{\cancel{\hat{e}_z \cdot \hat{e}_z}})\,\vec{\theta}$$

$$= -\vec{\theta}$$

So $\quad \vec{\theta} = -\hat{e}_z \times \nabla w = \nabla w \times \hat{e}_z = \hat{e}_z \times \vec{\beta}$.

SUMMARY OF RELATIONS:

$$\vec{\beta} = -\hat{e}_z \times \vec{\theta} \qquad\qquad \vec{\theta} = \hat{e}_z \times \vec{\beta}$$

$$\vec{\beta} = -\nabla w \qquad\qquad \vec{\theta} = -\hat{e}_z \times \nabla w$$

$$\nabla w = \hat{e}_z \times \vec{\theta}$$

Strains: $\quad \varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \qquad u_\alpha = z\beta_\alpha \rightarrow \varepsilon_{\alpha\beta} = \frac{1}{2} z (\beta_{\alpha,\beta} + \beta_{\beta,\alpha})$

$= z\,\beta_{(\alpha,\beta)}$ curvature tensor $\quad \kappa_{\alpha\beta} = \frac{1}{2}(\beta_{\alpha,\beta} + \beta_{\beta,\alpha})$

Shear constraint: $\vec{\beta} + \nabla w = \vec{\gamma} = 0$

Constitutive theory: $\mathcal{U}_{bend} = \frac{1}{2}\int_A D_{\alpha\beta\gamma\delta}\, \rho_{\alpha\beta}\, \rho_{\gamma\delta}\, dx\, dy \qquad \rho_{\alpha\beta} = \frac{1}{2}(\theta_{\alpha,\beta} + \theta_{\beta,\alpha})$

$\beta_1 = -\theta_2 \quad \beta_2 = \theta_1 \quad \rightarrow \quad \beta_{1,1} = -\theta_{2,1} \quad \beta_{2,2} = \theta_{1,2} \quad \beta_{1,2} = -\theta_{2,2} \quad \beta_{2,1} = \theta_{1,1}$

Inconvenient to relate $\underset{\sim}{\kappa}$ and $\underset{\sim}{\rho}$. Better to stick with κ, I think, because $\beta_{\alpha,\beta} = -w_{,\alpha\beta}$ in the case when the constraint is enforced. So if we want to express the energy in terms of the surface curvature tensor

$$b_{\alpha\beta} = \frac{1}{2}(w_{,\alpha\beta} + w_{,\beta\alpha}) = w_{,\alpha\beta},$$

then we should just substitute $\kappa_{\alpha\beta}$ for $b_{\alpha\beta}$.

Write the energy in the form

$$U_b = \frac{1}{2}\int_A \underset{\sim}{\kappa}^T \underset{\sim}{D}\, \underset{\sim}{\kappa}\, dx\, dy \qquad \underset{\sim}{\kappa} = \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{bmatrix} = \begin{bmatrix} \beta_{1,1} \\ \beta_{2,2} \\ (\beta_{1,2} + \beta_{2,1}) \end{bmatrix}$$

DKT ELEMENT

Conditions:

(a) 3 dof: (w, θ_x, θ_y) at each node.

(b) Kirchhoff Constraint $\vec{\theta} = e \times \vec{\beta} = -e \times \nabla w$ imposed discretely. ($e \equiv$ normal to triangle)

Interpolation:

(1) Quadratic Rotations: $\vec{\beta}(x,y) = \sum_{a=1}^{6} \vec{\beta}_a N_a(x,y) \qquad \vec{\beta}_a = -e \times \vec{\theta}_a$

(2a) Kirchhoff constraint imposed at corner nodes $\nabla w_a = -\vec{\beta}_a$

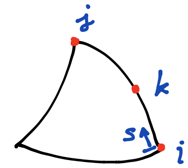
Since $\vec{\beta}_a$ is a DOF at each node, this simply gives us a way to specify ∇w at the corners.

(2b) Kirchhoff constraint imposed partially at midnodes $w_{,s} = -\beta_s \Rightarrow (\nabla w + \vec{\beta}) \cdot e_s = 0$

Not clear yet if we use this to eliminate β_s or $w_{,s}$.

(3) Cubic interpolation of $w(s)$ along edges

$$w_{/s}\big|_k = \frac{3}{2\ell_{ij}}\left(w\big|_j - w\big|_i\right) - \frac{1}{4}\left(w_{/s}\big|_i + w_{/s}\big|_j\right)$$



So this gives us $w_{/s}$ at midnodes $\rightarrow$ β_s at midnodes by (2a)

$\therefore$ β_s is defined in terms of w & $w_{/s}$ at corners.

(4) β_n @ midnodes is linearly interpolated from the corners.

$$\beta_n\big|_k = \left(\beta_n\big|_i + \beta_n\big|_j\right)/2$$

$\therefore$ Now have $\vec{\beta}_k$ in terms of βs and ws at the corners.

$$(\beta_n)_k = \frac{1}{2}\left[(\beta_n)_i + (\beta_n)_j\right] \qquad (\beta_s)_k = -(w_{/s})_k = \frac{3}{2\ell_{ij}}(w_i - w_j) - \frac{1}{4}\left[(\beta_s)_i + (\beta_s)_j\right]$$

THAT'S IT! Now just need to insert these into the shape function expressions for $\vec{\beta}(x_\alpha)$ to derive effective shape functions for $\vec{\beta}$ as fn of corner w's & β's.

Aside: Cubic interpolation in 1-D

Problem: interpolate $w(s)$ between points i & j such that

$$w(s_i = 0) = w_i \quad \& \quad w(s_j = \ell_{ij}) = w_j \qquad \text{(position)}$$

$$w_{/s}(0) = \beta_{s_i} \quad \& \quad w_{/s}(\ell_{ij}) = \beta_{s_j} \qquad \text{(slope)}$$

$\Rightarrow$ Curve interpolates both position & slope.

4 conditions $\longrightarrow$ 4-term polynomial:

$$w(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3$$

Conditions:

$w(0) = \boxed{w_i = c_0}$ ✓ $\qquad\qquad$ $w_{/s}(0) = \boxed{c_1 = \beta_i}$

$w(\ell) = w_j = c_1\ell + c_2\ell^2 + c_3\ell^3$ $\qquad$ $w_{/s}(\ell) = \beta_j = c_1 + 2c_2\ell + 3c_3\ell^2$

$\qquad = \beta_i\ell + c_2\ell^2 + c_3\ell^3$ $\qquad\qquad\qquad\qquad = \beta_i + 2c_2\ell + 3c_3\ell^2$

Solve for $c_2, c_3 \ldots$

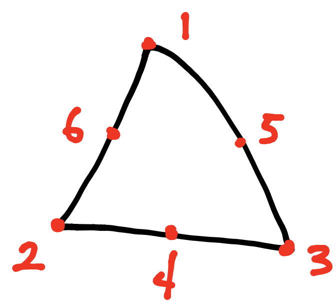
$$w(s) = \left[1 - 3\left(\frac{s}{\ell}\right)^2 + 2\left(\frac{s}{\ell}\right)^3\right] w_i + s\left[1 - 2\frac{s}{\ell} + \left(\frac{s}{\ell}\right)^2\right] \beta_i$$
$$+ \left[3\left(\frac{s}{\ell}\right)^2 - 2\left(\frac{s}{\ell}\right)^3\right] w_j + \left(\frac{s}{\ell} - 1\right)\frac{s^2}{\ell} \beta_j$$

Evaluate at midpoint

$$w(\ell/2) = \frac{1}{2}(w_i + w_j) + \frac{\ell}{8}(\beta_i - \beta_j)$$
$$w_{,s}(\ell/2) = \frac{3}{2\ell}(w_j - w_i) - \frac{1}{4}(\beta_i + \beta_j)$$

Same as above.

Work out shape functions



NODE 6: $i=1 \quad j=2 \qquad x_{12} = \vec{x}_2 - \vec{x}_1 \quad \ell_{12} = |\vec{x}_{12}| \quad e_s = \frac{\vec{x}_{12}}{\ell_{12}} \quad e_n = e_s \times e$

$$\beta_s = \vec{\beta} \cdot e_s \quad \beta_n = \beta \cdot e_n$$

$$\vec{\beta}_6 = (\beta_s)_6 e_s + (\beta_n)_6 e_n$$
$$= -\frac{1}{4}(e_s \otimes e_s)(\vec{\beta}_1 + \vec{\beta}_2) + \frac{3}{2\ell_{12}}(w_1 - w_2)\, e_s + \frac{1}{2}(e_n \otimes e_n)(\vec{\beta}_1 + \vec{\beta}_2)$$

NOTE: $e_n \otimes e_n + e_s \otimes e_s = I \Rightarrow e_n \otimes e_n = I - (e_s \otimes e_s)$

$$\therefore \quad \vec{\beta}_6 = -\frac{1}{4}(e_s \otimes e_s)(\vec{\beta}_1 + \vec{\beta}_2) + \frac{1}{2}(I - e_s \otimes e_s)(\vec{\beta}_1 + \vec{\beta}_2) + \frac{3}{2\ell_{12}}(w_1 - w_2)\, e_s$$
$$= \left(\frac{1}{2}I - \frac{3}{4} e_s \otimes e_s\right)(\vec{\beta}_1 + \vec{\beta}_2) + \frac{3}{2\ell_{12}}(w_1 - w_2)\, e_s$$
$$= \underbrace{\frac{1}{2}\left(I - \frac{3}{2} e_s \otimes e_s\right)}_{= \underset{\sim}{P}_{12}}(\vec{\beta}_1 + \vec{\beta}_2) + \underbrace{\frac{3}{2\ell_{12}} e_s}_{= \vec{d}_{12}}(w_1 - w_2)$$
$$\vec{\beta}_6 = \underset{\sim}{P}_{12} \vec{\beta}_1 + \underset{\sim}{P}_{12} \vec{\beta}_2 + \vec{d}_{12}\, w_1 - \vec{d}_{12}\, w_2$$

Should end up w/ similar results for 4 & 5

$$\vec{\beta}_4 = \underset{\sim}{P}_{23}(\vec{\beta}_2 + \vec{\beta}_3) + \vec{d}_{23}(w_2 - w_3)$$

$$\vec{\beta}_5 = \underset{\sim}{P}_{31}(\vec{\beta}_3 + \vec{\beta}_1) + \vec{d}_{31}(w_3 - w_1)$$

$$\vec{\beta}(\theta^\alpha) = \sum_{a=1}^{6} \vec{\beta}_a N_a(\theta^\alpha) = w_1 (N_6 \vec{d}_{12} - N_5 \vec{d}_{31})$$
$$+ \; w_2 (N_4 \vec{d}_{23} - N_6 \vec{d}_{12})$$
$$+ \; w_3 (N_5 \vec{d}_{31} - N_4 \vec{d}_{23})$$
$$+ (N_1 I + N_5 P_{31} + N_6 P_{12}) \vec{\beta}_1$$
$$+ (N_2 I + N_4 P_{23} + N_6 P_{12}) \vec{\beta}_2$$
$$+ (N_3 I + N_4 P_{23} + N_5 P_{31}) \vec{\beta}_3$$

But $\vec{\beta} = - e \times \vec{\Theta} = - [\hat{e}] \vec{\Theta}$ $\quad$ ($\hat{e} = - \hat{e}^T$ and $\hat{e} v = e \times v \;\; \forall v$)

$$\Rightarrow \vec{\beta}(\theta^\alpha) = w_1 (\dots) + w_2 (\dots) + w_3 (\dots)$$
$$- (N_1 I + N_5 P_{31} + N_6 P_{12}) \hat{e} \vec{\Theta}_1$$
$$- (N_2 I + N_4 P_{23} + N_6 P_{12}) \hat{e} \vec{\Theta}_2$$
$$- (N_3 I + N_4 P_{23} + N_5 P_{31}) \hat{e} \vec{\Theta}_3 = \underline{H} \, \underline{u}$$

$$\underline{u} = (w_1, \vec{\Theta}_1^T, w_2, \Theta_2^T, w_3, \Theta_3^T)$$