Surface Geometry

Tangent basis vectors on Ω_0 & Ω

$$\underset{\sim}{A}_\alpha = \underset{\sim}{X}_{,\alpha} \qquad \underset{\sim}{A}_3 = \frac{\underset{\sim}{A}_1 \times \underset{\sim}{A}_2}{\sqrt{A}} \qquad\qquad \underset{\sim}{a}_\alpha = \underset{\sim}{x}_{,\alpha} \qquad \underset{\sim}{a}_3 = \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{\sqrt{a}}$$

where

$$\sqrt{A} = |\underset{\sim}{A}_1 \times \underset{\sim}{A}_2| = \sqrt{\det A_{\alpha\beta}} \qquad\qquad \sqrt{a} = |\underset{\sim}{a}_1 \times \underset{\sim}{a}_2| = \sqrt{\det a_{\alpha\beta}}$$

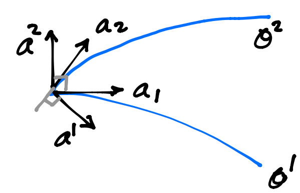
are the surface area metrics and

$$A_{\alpha\beta} = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{A}_\beta \qquad\qquad a_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta$$

are the surface metric tensors. Denote the area elements

$$d^2S = A_{\alpha\beta}\, d\theta^\alpha d\theta^\beta \qquad\qquad d^2s = a_{\alpha\beta}\, d\theta^\alpha d\theta^\beta$$

(Referred to as the 1st fundamental forms on Ω_0 & Ω.)



Dual Basis Vectors : Define $\{A^\alpha\}$ & $\{a^\alpha\}$ s.t.

$$\underset{\sim}{A}^\alpha \cdot \underset{\sim}{A}_i = \delta^\alpha_i \quad \text{and} \quad \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_i = \delta^\alpha_i$$

Note $A^\alpha \cdot A_3 = 0$ $a^\alpha \cdot a_3 = 0 \Rightarrow A^\alpha$ & a^α are still tangent to Ω_0 & Ω

Normal Vector & Curvature

Important thing to note about A_3 and a_3 : they are normal to the surfaces at every point. Normals will change from one pt. to another. Rate of change of A_3, a_3 with respect to θ^α is measure of curvature of the surface, e.g.,

$$d\underset{\sim}{a}_3 = \frac{\partial \underset{\sim}{a}_3}{\partial \theta^\alpha} d\theta^\alpha$$

But since $a_3 \cdot a_3 = 1 \Rightarrow \frac{\partial a_3}{\partial \theta^\alpha} \cdot a_3 = 0 \Rightarrow \underset{\sim}{a}_{3,\alpha}$ are tangent vectors.

$\longrightarrow$ Describe them in terms of their projections on $\{\underset{\sim}{a}_\beta\}$

$$\boxed{\underset{\sim}{a}_{3,\alpha} \cdot \underset{\sim}{a}_\beta \equiv -b_{\alpha\beta}} = \text{curvature tensor}$$

$\Rightarrow \underset{\sim}{a}_{3,\alpha} = -b_{\alpha\beta}\, \underset{\sim}{a}^\beta$

Can also define $\quad a_{3,\alpha} = -b_\alpha^\beta a_\beta \longrightarrow \boxed{b_\alpha^\beta = -a_{3,\alpha} \cdot a^\beta}$ mixed components of $\underset{\sim}{b}$

$$\rightarrow b_{\alpha\beta} = -a_{3,\alpha} \cdot a_\beta = (b_\alpha^\gamma a_\gamma) \cdot a_\beta = b_\alpha^\gamma a_{\gamma\beta}$$

$$\Rightarrow b_{\alpha\beta} = b_\alpha^\gamma a_{\gamma\beta}, \quad b_\alpha^\beta = b_{\alpha\gamma} a^{\gamma\beta}$$

<u>Second fundamental form</u>

Recall $\quad a_3 \cdot a_\alpha = a_3 \cdot \underset{\sim}{x}_{,\alpha} = 0. \Rightarrow (a_3 \cdot x_{,\alpha})_{,\beta} = 0$

$$\Rightarrow \underbrace{a_{3,\beta} \cdot a_\alpha}_{= -b_{\alpha\beta}} + a_3 \cdot x_{,\alpha\beta} = 0 \qquad (*)$$

$$\Rightarrow \boxed{b_{\alpha\beta} = a_3 \cdot x_{,\alpha\beta}}$$ $\longrightarrow$ Curvature tensor is <u>symmetric</u>.

$\therefore$ Symmetrize $(*)$: $\quad b_{\alpha\beta} = -\frac{1}{2}(a_{3,\alpha} \cdot a_\beta + a_{3,\beta} \cdot a_\alpha)$

$\times$ by $d\theta^\alpha d\theta^\beta$:

$$-b_{\alpha\beta} d\theta^\alpha d\theta^\beta = \frac{1}{2}(a_{3,\alpha} d\theta^\alpha \cdot x_{,\beta} d\theta^\beta + a_{3,\beta} d\theta^\beta \cdot x_{,\alpha} d\theta^\alpha)$$

$$\boxed{-b_{\alpha\beta} d\theta^\alpha d\theta^\beta = da_3 \cdot dx} \equiv 2^{nd} \text{ fundamental form of } \Omega$$

Principal, Mean, & Gauss Curvatures

$b_{\alpha\beta}$ symmetric $\longrightarrow$ real eigenvalues $\quad (b_{\alpha\beta} - b\, a_{\alpha\beta}) \lambda^\beta = 0 \longrightarrow b_1, b_2$

$\qquad (b_\alpha^\beta - b\, \delta_\alpha^\beta) \lambda_\beta = 0 \qquad \underset{\sim}{\lambda}_1, \underset{\sim}{\lambda}_2$

$b_\alpha \equiv$ principal curvatures

$\underset{\sim}{\lambda}_\alpha$ = curvature directions $\quad |\lambda_\alpha| = 1$

Mean Curvature: $\; H = \frac{1}{2}(b_1 + b_2) = \frac{1}{2} \mathrm{tr}\, \underset{\sim}{b} = \frac{1}{2} b_{\alpha\beta} a^{\alpha\beta} = \frac{1}{2} b_\alpha^\alpha = \frac{1}{2} b^{\alpha\beta} a_{\alpha\beta}$

Gauss Curvature: $\quad K = b_1 b_2 = \det(b_{\alpha\beta}) / \det(a_{\alpha\beta}) = \det(b_\alpha^\beta)$

Derivatives of Basis Vectors

$$\frac{\partial a_\alpha}{\partial \theta^\beta} = a_{\alpha,\beta} = (a_{\alpha,\beta}\cdot a_i)\, a^i = \underbrace{(a_{\alpha,\beta}\cdot a_\gamma)}_{\equiv \Gamma_{\alpha\beta\gamma}} a^\gamma + \underbrace{(a_{\alpha,\beta}\cdot a_3)}_{\equiv b_{\alpha\beta}} a_3$$

$$\Rightarrow \quad a_{\alpha,\beta} = \underbrace{\Gamma_{\alpha\beta\gamma}\, a^\gamma}_{\text{Tangent part}} + \underbrace{b_{\alpha\beta}\, a_3}_{\text{normal part}}$$

CS of 1st kind

Can show: $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(a_{\alpha\gamma,\beta} + a_{\beta\gamma,\alpha} - a_{\alpha\beta,\gamma} \right)$

Also define $\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta\delta}\, a^{\delta\gamma}$ ($a^{\alpha\beta}$ = dual/inverse metric tensor)

CS of 2nd kind

Then $\quad a_{\alpha,\beta} = \underbrace{\Gamma_{\alpha\beta}^{\gamma}\, a_\gamma}_{\text{tangent part}} + \underbrace{b_{\alpha\beta}\, a_3}_{\text{normal part}} = \Gamma_{\alpha\beta}^{i}\, a_i \longrightarrow \Gamma_{\alpha\beta}^{3} = b_{\alpha\beta}$

Note difference rel. to a_3,

$$a_{3,\alpha} = -b_{\alpha\beta}\, a^\beta = -b_\alpha^{\ \beta} a_\beta \longrightarrow \text{only tangent part because } a_3\cdot a_3 = 1$$