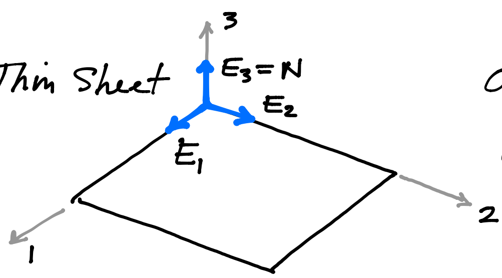
Plane Stress

Main Idea: For a thin sheet of material stresses in the direction normal to the sheet are easily relaxed by allowing the sheet to deform through the thickness. $\Rightarrow$ We want to reduce the problem from 3-D to 2-D by elimination of the thickness stretch such that normal stress is relaxed.

Thin Sheet



Orient the lab frame such that $\underset{\sim}{E}_3 = \underset{\sim}{N}$ is the undeformed sheet normal. Then a planar deformation is

$$\underset{\sim}{F} = F_{\alpha\beta}\, \underset{\sim}{E}_\alpha \otimes \underset{\sim}{E}_\beta + \lambda\, \underset{\sim}{N} \otimes \underset{\sim}{N}$$

$$\frac{\partial x_\alpha}{\partial X_\beta} = F_{\alpha\beta} \equiv \text{In-plane deformation components} \quad \alpha, \beta \in \{1,2\}$$

$$\frac{\partial x_3}{\partial X_3} = F_{33} = \lambda \equiv \text{Thickness Stretch}$$

In components,

$$[F_{iJ}] = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Plane Stress Condition: $T_3^{(N)} = 0 \rightarrow \underset{\sim}{E}_3 \cdot \underset{\sim}{P} \cdot \underset{\sim}{N} = 0 \Rightarrow P_{33} = 0.$

(Note: really we should do $\underset{\sim}{n} \cdot \underset{\sim}{P} \cdot \underset{\sim}{N} = 0$ but since we are assuming planar deformation, here $\underset{\sim}{n} = \underset{\sim}{N} = \underset{\sim}{E}_3$. In general, since $\underset{\sim}{n} \sim \underset{\sim}{F}^{-T} \underset{\sim}{N}$, we want $\underset{\sim}{N} \cdot \underset{\sim}{F}^{-1} \underset{\sim}{P} \cdot \underset{\sim}{N} = 0.$)

3-D Constitutive Law: $\underset{\sim}{P} = \underset{\sim}{P}(\underset{\sim}{F}) = \underset{\sim}{P}(F_{\alpha\beta}, \lambda) \Rightarrow \begin{cases} P_{\alpha\beta} = P_{\alpha\beta}(F_{\gamma\delta}, \lambda) \\ P_{33} = P_{33}(F_{\alpha\beta}, \lambda) \end{cases}$

$\therefore$ $F_{\alpha\beta}$ & $\lambda = F_{33}$ are the "natural" independent variables in the 3-D law.

But the plane stress assumption trades λ for P_{33} and specifies $P_{33} = 0$.

$\Rightarrow$ need to solve for λ as a fn of $F_{\alpha\beta}$ imposing $P_{33}(F_{\alpha\beta}, \lambda) = 0$.

Strategy:

(i) Solve $P_{33}(F_{\alpha\beta},\lambda)=0 \longrightarrow \lambda=\lambda(F_{\alpha\beta})$

(ii) Define:
$$w^{2D}(F) \equiv w^{3D}(F_{\alpha\beta},\lambda(F_{\alpha\beta}))$$
$$P^{2D}_{\alpha\beta}(F) \equiv \frac{\partial w^{2D}}{\partial F_{\alpha\beta}}$$
$$C^{2D}_{\alpha\beta\gamma\delta}(F) \equiv \frac{\partial P^{2D}_{\alpha\beta}}{\partial F_{\gamma\delta}} = \frac{\partial^2 w^{2D}}{\partial F_{\alpha\beta}\partial F_{\gamma\delta}}$$

Consistent Linearization

$$P^{2D}_{\alpha\beta}(F) = \frac{\partial w^{2D}}{\partial F_{\alpha\beta}} = \frac{\partial}{\partial F_{\alpha\beta}}\left[w(F,\lambda(F))\right] = \underbrace{\frac{\partial w}{\partial F_{\alpha\beta}}}_{P_{\alpha\beta}} + \underbrace{\frac{\partial w}{\partial \lambda}}_{P_{33}=0!}\frac{\partial \lambda}{\partial F_{\alpha\beta}}$$

$$\Rightarrow \boxed{P_{\alpha\beta} = \frac{\partial w}{\partial F_{\alpha\beta}}(F_{\alpha\beta},\lambda)}$$

$$C^{2D}_{\alpha\beta\gamma\delta} = \frac{\partial}{\partial F_{\gamma\delta}}\left[P_{\alpha\beta}(F,\lambda)\right] = \underbrace{\frac{\partial P_{\alpha\beta}}{\partial F_{\gamma\delta}}}_{C_{\alpha\beta\gamma\delta}} + \underbrace{\frac{\partial P_{\alpha\beta}}{\partial F_{33}}}_{C_{\alpha\beta33}}\frac{\partial \lambda}{\partial F_{\gamma\delta}}$$

How do we get $\frac{\partial \lambda}{\partial F_{\alpha\beta}}$? From plane stress constraint!

$$P_{33}(F_{\alpha\beta},\lambda)=0 \longrightarrow dP_{33}=0=\frac{\partial P_{33}}{\partial F_{\alpha\beta}}dF_{\alpha\beta} + \frac{\partial P_{33}}{\partial F_{33}}d\lambda$$
$$\Rightarrow C_{33\alpha\beta}\,dF_{\alpha\beta} + C_{3333}\,d\lambda = 0$$
$$C_{33\alpha\beta}\,dF_{\alpha\beta} + C_{3333}\frac{\partial \lambda}{\partial F_{\alpha\beta}}dF_{\alpha\beta} = 0$$
$$\left(C_{33\alpha\beta} + C_{3333}\frac{\partial \lambda}{\partial F_{\alpha\beta}}\right)dF_{\alpha\beta} = 0$$
$$\rightarrow \boxed{\frac{\partial \lambda}{\partial F_{\alpha\beta}} = -\frac{C_{33\alpha\beta}}{C_{3333}}}$$

$$\therefore \quad C^{2D}_{\alpha\beta\gamma\delta} = \frac{\partial}{\partial F_{\gamma\delta}}\left[P_{\alpha\beta}(F,\lambda)\right] = C_{\alpha\beta\gamma\delta} + C_{\alpha\beta33}\frac{\partial \lambda}{\partial F_{\gamma\delta}}$$
$$\Rightarrow \boxed{C^{2D}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - C_{\alpha\beta33}\,C_{33\gamma\delta}\,\frac{1}{C_{3333}}}$$

This is the linearization of the 1st P-K stress consistent with the plane stress constraint $P_{33}(F_{\alpha\beta},\lambda)=0$. This gives the correction to the 3D tangent moduli to account for $\lambda(F_{\alpha\beta})$ s.t. $dP_{\alpha\beta} = C^{2D}_{\alpha\beta\gamma\delta}\,dF_{\alpha\beta}$.