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AN EXCURSION INTO LARGE ROTATIONS*

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The present discourse develops an enlarged exploration of the matrix formulation of finite rotations in space initiated in [1]. It is shown how a consistent but subtle matrix calculus inevitably leads to a number of elegant expressions for the transformation or rotation matrix $\boldsymbol{T}$ appertaining to a rotation about an arbitrary axis. Also analysed is the case of multiple rotations about fixed or follower axes. Particular attention is paid to an explicit derivation of a single compound rotation vector equivalent to two consecutive arbitrary rotations. This theme is discussed in some detail for a number of cases. Semitangential rotations—for which commutativity holds—first proposed in [2, 3] are also considered. Furthermore, an elementary geometrical analysis of large rotations is also given. Finally, we deduce in an appendix, using a judicious reformulation of quarternions, the compound pseudovector representing the combined effect of n rotations.

In the author's opinion the present approach appears preferable to a pure vectorial scheme—and even more so to an indicial formulation—and is computationally more convenient.

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Chinese proverb

1. On the matrix formulation of large rotations

1.1. *Prolegomena*

While infinitesimal rotations in space may be assigned a vectorial identity this does not hold for finite rotations [4, 5]. To confirm this, we need only recall one of the three essential properties of vector quantities, namely their commutativity which permits the synthesis of two vectors via the parallelogram law. This is, as we know, not true for two or more finite rotations about arbitrary axes in space since the sequential order of their imposition determines in each case a different result. This non-commutativity of finite rotations is, however, not only characteristic of truly large rotations but holds even when second order effects have to be considered.

In what follows we present a novel matrix formulation of compound finite rotations in space. We refer in this connection also to [1] where a brief outline of some aspects of the theory is presented.

In a mathematical sense a finite rotation ϑ about an axis—defined say by a unit vector $\boldsymbol{e}$—leads to a linear (matrix) one-to-one vector transformation [4]. To formulate this relationship it is useful to introduce as in Fig. 1[1] a so-called rotational *pseudo*vector

*Dedicated to Professor Udo Wegner, D.Sc., D.Sc.h.c. on the occasion of his 80th anniversary.

[1]The figures in this paper are all in colour. They have been inserted at the end of this paper (pp. 127–142) in a separate section.

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$$\boldsymbol{\vartheta} = \vartheta \boldsymbol{e} \tag{1}$$

along the axis of rotation OR. Now $\boldsymbol{\vartheta}$ may be written as a column matrix in terms of its components φ, χ, ψ in a cartesian system $Oxyz$.

$$\boldsymbol{\vartheta} = \{\varphi \quad \chi \quad \psi\} \tag{2}$$

with

$$\vartheta = (\varphi^2 + \chi^2 + \psi^2)^{1/2}\,. \tag{2a}$$

The reader should note that for finite (in contrast to infinitely small) rotations φ, χ, ψ *cannot* be interpreted as component rotations about the cartesian axes x, y, z.

Consider now a vector $\boldsymbol{p}$ which as a result of the application of $\boldsymbol{\vartheta}$ is transported to a new position $\hat{\boldsymbol{p}}$; see Fig. 1. Our task is to establish the transformation

$$\hat{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{\vartheta})\boldsymbol{p} \tag{3}$$

where the so-called transformation rotation matrix $\boldsymbol{T}(\boldsymbol{\vartheta})$ is a nonlinear function of $\boldsymbol{\vartheta}$.

Before proceeding to the detailed theory of large rotations and the central problem of the formation of the rotation matrix $\boldsymbol{T}(\boldsymbol{\vartheta})$ it is appropriate to pinpoint two well-known characteristics [4] of $\boldsymbol{T}(\boldsymbol{\vartheta})$ the formal proof of which is recapitulated below.
In fact,

(a) The rotation matrix is orthogonal, i.e.,

$$\boldsymbol{T}\boldsymbol{T}^{\mathrm{t}} = \boldsymbol{I}_3 = \boldsymbol{T}^{\mathrm{t}}\boldsymbol{T}\,. \tag{4}$$

This follows also from the interpretation of $\boldsymbol{T}$ in the form of direction cosines.

(b) The rotation matrix $\boldsymbol{T}$ appertaining to a sequence of n successive rotations

$$\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \ldots, \boldsymbol{\vartheta}_i, \ldots, \boldsymbol{\vartheta}_{n-1}, \boldsymbol{\vartheta}_n$$

about axes *fixed* in space is formed by the matrix product of the individual transformation matrices $\boldsymbol{T}_i$ arrayed in the inverse order of the application of the rotations $\boldsymbol{\vartheta}_i$. Thus,

$$\boldsymbol{T} = \boldsymbol{T}_n\boldsymbol{T}_{n-1} \cdots \boldsymbol{T}_i \cdots \boldsymbol{T}_2\boldsymbol{T}_1\,. \tag{5}$$

On the other hand, the inverse order

$$\boldsymbol{T} = \boldsymbol{T}_1\boldsymbol{T}_2 \cdots \boldsymbol{T}_i \cdots \boldsymbol{T}_{n-1}\boldsymbol{T}_n \tag{5a}$$

applies if the rotations $\boldsymbol{\vartheta}_i$ take place in the sequential order

$$\boldsymbol{\vartheta}_n, \boldsymbol{\vartheta}_{n-1}, \ldots, \boldsymbol{\vartheta}_i, \ldots, \boldsymbol{\vartheta}_2, \boldsymbol{\vartheta}_1\,.$$

In what follows we denote these rotation orders as sequences I and II, respectively.

1.2. On the formation of the rotation matrix $\boldsymbol{T}$

We enter now into the development of the main theory concerning the most judicious derivation of the rotation matrix $\boldsymbol{T}$. To this purpose we return to Fig. 1 and observe the vector $\boldsymbol{p}$ in its initial and final positions $OP = \boldsymbol{p}$ and $O\hat{P} = \hat{\boldsymbol{p}}$. We also read

$$\hat{\boldsymbol{p}} = \boldsymbol{p} + \boldsymbol{p}_{\Delta}. \tag{6}$$

Our task is to transform the right-hand side of (6) into the matrix product of (3). To this end we first deduce from the inlet Fig. 1(a)

$$\boldsymbol{p}_{\Delta} = \overline{PD} + \overline{D\hat{P}} \tag{7}$$

where $\overline{D\hat{P}}$ is drawn normal to $\overline{PC}$. We also note that the vector $\overline{D\hat{P}}$ stands perpendicular to the plane OPC and points hence in the direction $(\boldsymbol{e} \times \boldsymbol{p})$. To find its magnitude we observe that

$$D\hat{P} = a \sin \vartheta. \tag{8}$$

On the other hand, we observe that the magnitude of $(\boldsymbol{e} \times \boldsymbol{p})$ is

$$|\boldsymbol{e} \times \boldsymbol{p}| = 1 \cdot p \sin \alpha = p\frac{a}{p} = a.$$

It follows in conjunction with (7) and (8) that

$$\overline{D\hat{P}} = (\boldsymbol{e} \times \boldsymbol{p}) \sin \vartheta = \frac{\sin \vartheta}{\vartheta}(\boldsymbol{\vartheta} \times \boldsymbol{p}). \tag{9}$$

Next we proceed to the determination of the vector $\overline{PD}$. Fig. 1 demonstrates immediately that it is not only perpendicular to $(\boldsymbol{e} \times \boldsymbol{p})$ but also to $\boldsymbol{e}$ since it lies in the plane $PC\hat{P}$ normal to $\boldsymbol{e}$. Hence it may be assigned the direction of $\boldsymbol{e} \times (\boldsymbol{e} \times \boldsymbol{p})$. Now the absolute value of the last vector is clearly again a since $\boldsymbol{e}$ is a unit vector and is also normal to $(\boldsymbol{e} \times \boldsymbol{p})$. At the same time Fig. 1 yields

$$PD = a - a \cos \vartheta = (1 - \cos \vartheta)a = 2 \sin^2 \frac{\vartheta}{2} a. \tag{10}$$

Hence we deduce, using (1),

$$\overline{PD} = 2 \sin^2 \frac{\vartheta}{2} (\boldsymbol{e} \times (\boldsymbol{e} \times \boldsymbol{p})) = \tfrac{1}{2}\frac{\sin^2(\vartheta/2)}{(\vartheta/2)^2}(\boldsymbol{\vartheta} \times (\boldsymbol{\vartheta} \times \boldsymbol{p})). \tag{11}$$

Applying (9), (11) in (6), (7) we may construe the vector $\hat{\boldsymbol{p}}$ in the form

$$\hat{\boldsymbol{p}} = \boldsymbol{p} + \frac{\sin\vartheta}{\vartheta} - (\boldsymbol{\vartheta}\times\boldsymbol{p}) + \tfrac{1}{2}\frac{\sin^2(\vartheta/2)}{(\vartheta/2)^2}(\boldsymbol{\vartheta}\times(\boldsymbol{\vartheta}\times\boldsymbol{p}))\,. \tag{12}$$

Now we are in a position to rewrite (12) in a matrix form which is the alpha and omega of modern computational mechanics. Introducing first the (3×3) auxiliary matrices

$$\boldsymbol{S} = \begin{bmatrix} 0 & -\psi & \chi \\ \psi & 0 & -\varphi \\ -\chi & \varphi & 0 \end{bmatrix} \quad\text{and}\quad \boldsymbol{S}^2 = \boldsymbol{S}\boldsymbol{S} = \begin{bmatrix} -(\chi^2+\psi^2) & \chi\varphi & \varphi\psi \\ \chi\varphi & -(\psi^2+\varphi^2) & \psi\chi \\ \varphi\psi & \psi\chi & -(\varphi^2+\chi^2) \end{bmatrix} \tag{13}$$

we confirm by an elementary argument the equivalence of the following vector and matrix operations

$$\boldsymbol{\vartheta}\times\boldsymbol{p} = \boldsymbol{S}\boldsymbol{p} \quad\text{and}\quad \boldsymbol{\vartheta}\times(\boldsymbol{\vartheta}\times\boldsymbol{p}) = \boldsymbol{S}^2\boldsymbol{p}\,. \tag{14}$$

The reader will observe that $\boldsymbol{S}$ is in its form identical to the well-known *antisymmetrical* matrix representing infinitely small rotations about orthogonal axes. However, in the present, more general, context $\boldsymbol{S}$ contains the cartesian components of a finite rotational pseudovector $\boldsymbol{\vartheta}$.

We are now in a position to set up the transformation matrix $\boldsymbol{T}$. Substituting (14) in (12) and noting (3), we immediately find

$$\hat{\boldsymbol{p}} = \boldsymbol{p} + \frac{\sin\vartheta}{\vartheta}\boldsymbol{S}\boldsymbol{p} + \tfrac{1}{2}\frac{\sin^2(\vartheta/2)}{(\vartheta/2)^2}\boldsymbol{S}^2\boldsymbol{p} = \boldsymbol{T}(\boldsymbol{\vartheta})\boldsymbol{p} \tag{15}$$

where

$$\boldsymbol{T}(\boldsymbol{\vartheta}) = \boldsymbol{I}_3 + \frac{\sin\vartheta}{\vartheta}\boldsymbol{S} + \tfrac{1}{2}\left(\frac{\sin\vartheta/2}{\vartheta/2}\right)^2\boldsymbol{S}^2\,. \tag{16}$$

Naturally, this matrix expression can also be established directly without any recourse to vector calculus.

Recollecting that $\boldsymbol{T}$ is an orthogonal matrix, the inverse relation to (15) reads

$$\boldsymbol{p} = \boldsymbol{T}(-\boldsymbol{\vartheta})\hat{\boldsymbol{p}} = \boldsymbol{T}^{\mathrm{t}}\hat{\boldsymbol{p}} \tag{15a}$$

where

$$\boldsymbol{T}^{\mathrm{t}} = \boldsymbol{I}_3 - \frac{\sin\vartheta}{\vartheta}\boldsymbol{S} + \tfrac{1}{2}\left(\frac{\sin\vartheta/2}{\vartheta/2}\right)^2\boldsymbol{S}^2 = \boldsymbol{T}(-\boldsymbol{\vartheta}) \tag{16a}$$

since $\boldsymbol{S}$ is antisymmetrical and $\boldsymbol{S}^{\mathrm{t}} = -\boldsymbol{S}$. Thus, the rotation matrix $\boldsymbol{T}^{\mathrm{t}}$ is seen to return the vector $\hat{\boldsymbol{p}}$ to its original position $\boldsymbol{p}$. The formal proof of the orthogonality of $\boldsymbol{T}$ is given below.

Eq. (16) is in the writer's opinion the most concise and elegant formulation of the rotation matrix $\boldsymbol{T}$, oddly enough never before to his knowledge quoted in the literature but for a brief reference in [1]. Indeed (16) allows to recover previous limiting results. In fact, considering the case $\vartheta \to 0$ we obtain the standard small rotation formula

$$\boldsymbol{T} = \boldsymbol{I}_3 + \boldsymbol{S} . \tag{17}$$

As stated before, the components of $\boldsymbol{\vartheta}$ represent in the present limit the individual rotations about the Cartesian axes $Oxyz$. The next higher approximation in $\boldsymbol{\vartheta}$ is of the second order and reads

$$\boldsymbol{T} = \boldsymbol{I}_3 + \boldsymbol{S} + \tfrac{1}{2}\boldsymbol{S}^2 \tag{18}$$

This expression was first evolved in [7].

At this stage it is useful to look at the classical literature; see, e.g., [4, 5]. Most authors start with relations of the type (7), (8) and (12) but instead of proceeding immediately to (16) they apply an involved argument leading ultimately to much more complex expressions. For example, starting at random with the excellent textbook of Rosenberg [4] we note the introduction of a parameter vector along the $\boldsymbol{e}$ or $\boldsymbol{\vartheta}$ axis in the form

$$\boldsymbol{\omega} = \omega \boldsymbol{e} = \{\omega_x \quad \omega_y \quad \omega_z\} = \tan\frac{\vartheta}{2}\,\boldsymbol{e} \tag{19}$$

with

$$\omega = \tan\frac{\vartheta}{2} = [\omega_x^2 + \omega_y^2 + \omega_z^2]^{1/2} \tag{20}$$

where the entries ω_x, ω_y, ω_z are the Cartesian components of the vector $\boldsymbol{\omega}$.

Having defined (19), Rosenberg proceeds to the modification of (15) and derives the relatively involved expression of the rotation matrix (see also [5]),

$$\boldsymbol{T} = \frac{1}{1+\omega^2}\begin{bmatrix} 1+\omega_x^2-\omega_y^2-\omega_z^2 & 2(\omega_x\omega_y-\omega_z) & 2(\omega_x\omega_z+\omega_y) \\ 2(\omega_x\omega_y+\omega_z) & 1-\omega_x^2+\omega_y^2-\omega_z^2 & 2(\omega_y\omega_z-\omega_x) \\ 2(\omega_x\omega_z-\omega_y) & 2(\omega_y\omega_z+\omega_x) & 1-\omega_x^2-\omega_y^2+\omega_z^2 \end{bmatrix}. \tag{21}$$

It is left as an exercise to the reader to transform (21) back to the concise form of (16).

For our subsequent developments in Section 3 it is pertinent to establish at this stage expressions analogous to (12), (15) and (16) but with $\boldsymbol{\omega}$ in place of $\boldsymbol{\vartheta}$.

Starting with (12) we confirm immediately the formula,

$$\hat{\boldsymbol{p}} = \boldsymbol{p} + 2\cos^2\frac{\vartheta}{2}(\boldsymbol{\omega}\times\boldsymbol{p}) + 2\cos^2\frac{\vartheta}{2}(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{p})) .$$

Noting next (20), we observe that

$$\cos^2\frac{\vartheta}{2} = \frac{1}{1+\tan^2(\vartheta/2)} = \frac{1}{1+\boldsymbol{\omega}^t\boldsymbol{\omega}}\,.$$

Hence

$$\hat{\boldsymbol{p}} = \boldsymbol{p} + \frac{2}{1+\boldsymbol{\omega}^t\boldsymbol{\omega}}[(\boldsymbol{\omega}\times\boldsymbol{p}) + (\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{p}))]\,. \tag{22}$$

We introduce now the auxiliary matrix $\boldsymbol{R}$ which takes here the place of $\boldsymbol{S}$ in (13) and write

$$\boldsymbol{R} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \frac{\tan\vartheta/2}{\vartheta}\begin{bmatrix} 0 & -\psi & \chi \\ \psi & 0 & -\varphi \\ -\chi & \varphi & 0 \end{bmatrix} = \tfrac{1}{2}\frac{\tan\vartheta/2}{\vartheta/2}\boldsymbol{S}\,. \tag{23}$$

In analogy to (14) we note the vector and matrix rules

$$\boldsymbol{\omega}\times\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p} \quad \text{and} \quad \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{p}) = \boldsymbol{R}^2\boldsymbol{p}\,. \tag{24}$$

These yield in pursuance of (22) the matrix expression

$$\hat{\boldsymbol{p}} = \boldsymbol{p} + \frac{2}{1+\boldsymbol{\omega}^t\boldsymbol{\omega}}[\boldsymbol{R}+\boldsymbol{R}^2]\boldsymbol{p} = \boldsymbol{T}(\boldsymbol{\omega})\boldsymbol{p} \tag{25}$$

with the rotation matrix

$$\boldsymbol{T}(\boldsymbol{\omega}) = \boldsymbol{I}_3 + \frac{2}{1+\boldsymbol{\omega}^t\boldsymbol{\omega}}[\boldsymbol{R}+\boldsymbol{R}^2]\,. \tag{26}$$

This relation is possibly simpler than (16) but not so convenient when considering an alternative and theoretically most important series expansion further below.

The inverse transformation matrix $\boldsymbol{T}(-\boldsymbol{\omega})$ is clearly

$$\boldsymbol{T}(-\boldsymbol{\omega}) = \boldsymbol{I}_3 + \frac{2}{1+\boldsymbol{\omega}^t\boldsymbol{\omega}}[-\boldsymbol{R}+\boldsymbol{R}^2] = \boldsymbol{T}^t(\boldsymbol{\omega})\,. \tag{27}$$

Another complex form of $\boldsymbol{T}$ based on the components φ, χ, ψ of $\boldsymbol{\vartheta}$ and quoted in the literature may be put in the form,

$$\boldsymbol{T} = \begin{bmatrix} 1-\left[1-\left(\frac{\varphi}{\vartheta}\right)^2\right](1-\cos\vartheta) & \frac{\varphi\chi}{\vartheta^2}(1-\cos\vartheta)-\frac{\psi}{\vartheta}\sin\vartheta & \frac{\varphi\psi}{\vartheta^2}(1-\cos\vartheta)+\frac{\chi}{\vartheta}\sin\vartheta \\ \frac{\varphi\chi}{\vartheta^2}(1-\cos\vartheta)+\frac{\psi}{\vartheta}\sin\vartheta & 1-\left[1-\left(\frac{\chi}{\vartheta}\right)^2\right](1-\cos\vartheta) & \frac{\psi\chi}{\vartheta^2}(1-\cos\vartheta)-\frac{\varphi}{\vartheta}\sin\vartheta \\ \frac{\varphi\psi}{\vartheta^2}(1-\cos\vartheta)-\frac{\chi}{\vartheta}\sin\vartheta & \frac{\psi\chi}{\vartheta^2}(1-\cos\vartheta)+\frac{\varphi}{\vartheta}\sin\vartheta & 1-\left[1-\left(\frac{\psi}{\vartheta}\right)^2\right](1-\cos\vartheta) \end{bmatrix} \tag{28}$$

which may easily—but unnecessarily so—be reduced once more to the form of (16).

The reader will have observed that the Eulerian definition of large rotations is not adopted in the present paper. This is deliberate since Eulerian angles lead to an undesirable lack of symmetry.

At this stage the author may be permitted a historical aside. It is surprising to observe the complex approach to finite rotations adopted over many decades in the past in otherwise excellent textbooks of mechanics. This may usually be attributed to a reluctance in applying a consistent vector approach or more preferably an advanced matrix scheme. As an example, we may quote Whittaker's justly famous treatise [7] which adheres strictly to a classical interpretation in components although vectors and matrices were at the time of writing readily available. This remark does in no way refer to the many excellent textbooks which appeared in the post-second-world-war era especially in the United States (see e.g. [4, 5]).

Expression (16) triggered off in the author's mind a train of thoughts which crystallised into a hunch that it might prove feasible to transform it into a theoretically more convenient series in $\boldsymbol{S}$. This is in fact not only possible but also remarkably straightforward to achieve. We deduce it here in two steps. In the first the trigonometric functions in (16) are expanded in series in ϑ. Then in the second step by a judicious consideration of the powers in $\boldsymbol{S}$ we transform the series finally into one in $\boldsymbol{S}$.

We commence with the first step and obtain

$$\begin{aligned}\boldsymbol{T} = \boldsymbol{I}_3 &+ \left(1 - \frac{\vartheta^2}{3!} + \frac{\vartheta^4}{5!} + \cdots + (-1)^n \frac{\vartheta^{2n}}{(2n+1)!} \pm \cdots\right)\boldsymbol{S} \\ &+ \left(\frac{1}{2!} - \frac{\vartheta^2}{4!} + \frac{\vartheta^4}{6!} - \cdots + (-1)^n \frac{\vartheta^{2n}}{(2n+2)!} \pm \cdots\right)\boldsymbol{S}^2 .\end{aligned} \tag{29}$$

Next we observe the powers of $\boldsymbol{S}$ and confirm by simple matrix multiplications the interesting relations

$$\begin{aligned}\boldsymbol{S}^3 &= -\vartheta^2\boldsymbol{S}, \qquad \boldsymbol{S}^4 = -\vartheta^2\boldsymbol{S}^2, \\ \boldsymbol{S}^5 &= +\vartheta^4\boldsymbol{S}, \qquad \boldsymbol{S}^6 = +\vartheta^4\boldsymbol{S}^2,\end{aligned}$$

leading to the recurrence formulae

$$\boldsymbol{S}^{2n-1} = (-1)^{n-1}\vartheta^{2(n-1)}\boldsymbol{S}, \qquad \boldsymbol{S}^{2n} = (-1)^{n-1}\vartheta^{2(n-1)}\boldsymbol{S}^2 . \tag{30}$$

Applying (30) in (29), we deduce immediately for $\boldsymbol{T}$ in the series in $\boldsymbol{S}$,

$$\boldsymbol{T} = \boldsymbol{I}_3 + \boldsymbol{S} + \frac{1}{2!}\boldsymbol{S}^2 + \frac{1}{3!}\boldsymbol{S}^3 + \cdots + \frac{1}{n!}\boldsymbol{S}^n + \cdots , \tag{31}$$

which represents, in fact, the expansion of $\boldsymbol{T}$,

$$\boldsymbol{T} = \mathrm{e}^{\boldsymbol{S}} = \exp \boldsymbol{S} . \tag{32}$$

As the author has now been made aware by the referee, the surprisingly concise and elegant result of (32) may also be deduced by arguments based on Lie's theory of groups as is done in quantum mechanics. However, one may be permitted to ask: why use a steamhammer to crack a nut?

The simplicity of expression (32) leads an inquiring mind to pose the question: Is it possible to establish (32) by a direct argument? This proves indeed surprisingly simple. Consider a finite rotation $\boldsymbol{\vartheta} = \{\varphi \;\chi \;\psi\}$ with an associated auxiliary matrix $\boldsymbol{S}$. Now the finite rotation can be composed of an infinite number of infinitely small rotations each assigned an auxiliary matrix $\lim_{n\to\infty}(1/n\,\boldsymbol{S})$ and a corresponding rotation matrix, see (17), (31),

$$\boldsymbol{T}_i = \lim_{n\to\infty}\left[\boldsymbol{I}_3 + \frac{1}{n}\boldsymbol{S}\right], \quad i = 1 \text{ to } \infty . \tag{31a}$$

Application of (5) to the sequence $n \to \infty$ of such rotations yields clearly

$$\boldsymbol{T} = \lim_{n\to\infty}\left[\boldsymbol{I}_3 + \frac{1}{n}\boldsymbol{S}\right]^n . \tag{33}$$

Developing this expression by the binomial theorem and taking the limit we confirm immediately the series of (31) and hence (32). The present argument was first announced in [1].

Consider now a rigid body subject to two consecutive rotations—presently about fixed axes—in the alternative sequential orders $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_2$ or I^{r} and $\boldsymbol{\vartheta}_2$, $\boldsymbol{\vartheta}_1$ or II^{r}, respectively; the superscript r stands for rigid or fixed axes. Let these rotations be defined by the auxiliary matrices $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, respectively. The associated rotation matrices are denoted by $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$. Consider next an arbitrary vector in its initial position $\boldsymbol{p}$. Applying the sequential order I^{r} we observe that $\boldsymbol{p}$ moves first as a result of the rotation $\boldsymbol{\vartheta}_1$ to

$$\boldsymbol{p}_1 = \boldsymbol{T}_1\boldsymbol{p} . \tag{34}$$

Following the subsequent application of the rotation $\boldsymbol{\vartheta}_2$, the vector $\boldsymbol{p}_1$ is transported to

$$\boldsymbol{p}_2 = \boldsymbol{T}_2\boldsymbol{p}_1 = \boldsymbol{T}_2\boldsymbol{T}_1\boldsymbol{p} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}\boldsymbol{p} = \hat{\boldsymbol{p}} . \tag{35}$$

Thus the overall or compound rotation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$ is for sequence I^{r} simply

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{T}_2\boldsymbol{T}_1 . \tag{36}$$

Extending an analogous argument to sequence II^{r} or applying merely a cyclic change to the suffices leads immediately to the corresponding result

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}} = \boldsymbol{T}_1\boldsymbol{T}_2 . \tag{37}$$

An obvious generalisation to n rotations $\boldsymbol{\vartheta}$ in either sequence yields ultimately expressions (5), (5a) for $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$, $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}}$, respectively.

Now the alternative final positions I^{r} and II^{r} are, in general, clearly different since compound finite rotations are as a rule non-commutative. (An exception to this rule occurs only when the axes of rotation are parallel.) This emerges immediately from the evident observation that, in general,

$$\boldsymbol{S}_1\boldsymbol{S}_2 \neq \boldsymbol{S}_2\boldsymbol{S}_1 \tag{38}$$

unless the axes are parallel.

Thus, as a rule,

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{T}_2\boldsymbol{T}_1 \neq \boldsymbol{T}_1\boldsymbol{T}_2 = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}} \tag{39}$$

and

$$\mathrm{e}^{\boldsymbol{S}_2}\,\mathrm{e}^{\boldsymbol{S}_1} \neq \mathrm{e}^{\boldsymbol{S}_1+\boldsymbol{S}_2} \neq \mathrm{e}^{\boldsymbol{S}_1}\,\mathrm{e}^{\boldsymbol{S}_2} \tag{39a}$$

unless the matrices $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are commutative! See also the discussion on commutative rotations in Section 3.3. This corrects an inadvertent error in [1]. Since, in general,

$$\boldsymbol{S} \neq \boldsymbol{S}_1 + \boldsymbol{S}_2\,, \tag{39b}$$

the question arises how the correct matrix $\boldsymbol{S}$ can be determined as an explicit function of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. The solution to this problem is given in Section 3.

We are now in a position to present a very simple proof of the orthogonality of $\boldsymbol{T}$ using (26). Noting that $\boldsymbol{S}$ and $\boldsymbol{S}^{\mathrm{t}} = -\boldsymbol{S}$ are commutative we immediately obtain

$$\boldsymbol{T}\boldsymbol{T}^{\mathrm{t}} = \mathrm{e}^{\boldsymbol{S}}\,\mathrm{e}^{-\boldsymbol{S}} = \mathrm{e}^{\boldsymbol{O}_3} = \boldsymbol{I}_3 = \boldsymbol{T}^{\mathrm{t}}\boldsymbol{T}\,. \tag{4a}$$

On the other hand, the confirmation of (4) when $\boldsymbol{T}$ is written as in (16) is somewhat more elaborate. Using (16a) for $\boldsymbol{T}$ we obtain

$$\boldsymbol{T}\boldsymbol{T}^{\mathrm{t}} = \boldsymbol{I}_3 - \left(\frac{\sin\vartheta}{\vartheta}\right)^2 \boldsymbol{S}^2 + \left(\frac{\sin\vartheta/2}{\vartheta/2}\right)^2 \boldsymbol{S}^2 + \tfrac{1}{4}\left(\frac{\sin\vartheta/2}{\vartheta/2}\right)^4 \boldsymbol{S}^4\,. \tag{40}$$

Expressing $\sin\vartheta$ in $\vartheta/2$, bearing in mind the expression for $\boldsymbol{S}^4$ in (24), and rearranging, we establish

$$\boldsymbol{T}\boldsymbol{T}^{\mathrm{t}} = \boldsymbol{I}_3 + \left(\frac{\sin\vartheta/2}{\vartheta/2}\right)^2 \left[-\left(\sin^2\frac{\vartheta}{2} + \cos^2\frac{\vartheta}{2}\right) + 1\right]\boldsymbol{S}^2\,, \tag{40a}$$

which immediately reduces to (4).

2. Compound rotation about follower axes

In the preceding section the proposed theory presumes that the individual axes of rotation are fixed in space. Consequently, our argument has to be modified if the sequence of rotations takes place about follower axes, i.e. about axes attached to the body itself. In this case each rotation is affected by all preceding rotations. In what follows we first present a rather pedestrian solution of this problem but return subsequently to the subject with a more imaginative approach.

Limiting ourselves initially to two consecutive rotations we consider once more the sequential order I^{f} consisting of two rotations (1), (2) this time about follower axes; see Fig. 2. In setting up the rotation matrix $\boldsymbol{T}_1$ for the first rotation $\boldsymbol{\vartheta}_1$ the procedure is, of course, in no way different from that for a fixed axis. Thus, it is only necessary to enter in the basic matrix $\boldsymbol{S}_1$ the components φ_1, χ_1, ψ_1 of $\boldsymbol{\vartheta}_1$ with respect to the fixed coordinate system $Oxyz$; Fig. 2. In contrast, when constructing $\boldsymbol{T}_2$ we have to remember that the pseudovector $\boldsymbol{\vartheta}_2$ is also rotated away from its initial position through the prior action of $\boldsymbol{\vartheta}_1$. In fact, denoting the final vectorial value of $\boldsymbol{\vartheta}_2$ by $\boldsymbol{\vartheta}_{21}$ we have

$$\boldsymbol{\vartheta}_{21} = \boldsymbol{T}_1 \boldsymbol{\vartheta}_2 = \{\varphi_{21} \quad \chi_{21} \quad \psi_{21}\}, \tag{41}$$

the components of which are measured in the fixed system $Oxyz$ and are different from the entries in the initial vector $\boldsymbol{\vartheta}_2 = \{\varphi_2 \;\; \chi_2 \;\; \psi_2\}$. Having established φ_{21}, χ_{21}, ψ_{21} we can set up the associated auxiliary matrix $\boldsymbol{S}_{21}$ and hence using (16) or any other corresponding expression the relevant rotation matrix $\boldsymbol{T}_{21}$. Thus, in this case the final rotation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ of the sequence I^{f} becomes

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_{21} \boldsymbol{T}_1 \tag{42}$$

where the superior index f refers to the follower nature of the axes. The opposite procedure has to be adopted if we apply the sequential order II^{f} or (2), (1). Here we start with the rotation vector

$$\boldsymbol{\vartheta}_2 = \{\varphi_2 \quad \chi_2 \quad \psi_2\}$$

as originally given in the $Oxyz$ system and use the standard technique of Section 1 to set up $\boldsymbol{S}_2$ and $\boldsymbol{T}_2$. Subsequently we establish

$$\boldsymbol{\vartheta}_{12} = \boldsymbol{T}_2 \boldsymbol{\vartheta}_1 = \{\varphi_{12} \quad \chi_{12} \quad \psi_{12}\} \tag{43}$$

and deduce hence $\boldsymbol{S}_{12}$ and the rotation matrix $\boldsymbol{T}_{12}$. Ultimately we obtain

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{T}_{12} \boldsymbol{T}_2 . \tag{44}$$

To conclude this section we generalise our analysis by considering a compound rotation composed of a sequential imposition of n rotational pseudovectors $\boldsymbol{\vartheta}_i$ arrayed as $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_2$, $\boldsymbol{\vartheta}_3, \ldots, \boldsymbol{\vartheta}_i, \ldots, \boldsymbol{\vartheta}_{n-1}$, $\boldsymbol{\vartheta}_n$ (sequence I^{f}) or alternatively $\boldsymbol{\vartheta}_n$, $\boldsymbol{\vartheta}_{n-1}$, $\boldsymbol{\vartheta}_{n-2}, \ldots, \boldsymbol{\vartheta}_i, \ldots, \boldsymbol{\vartheta}_2$, $\boldsymbol{\vartheta}_1$ (sequence II^{f}) in each case about follower axes. The pertinent rotation matrices $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ or $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}}$ are then set up in the form

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_{n,n-1} \boldsymbol{T}_{n-1,n-2} \cdots \boldsymbol{T}_{i,i-1} \cdots \boldsymbol{T}_{32} \boldsymbol{T}_{21} \boldsymbol{T}_1 \tag{45}$$

or

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{T}_{12}\boldsymbol{T}_{23}\boldsymbol{T}_{34} \cdots \boldsymbol{T}_{i-1,i} \cdots \boldsymbol{T}_{n-2,n-1}\boldsymbol{T}_{n-1,n}\boldsymbol{T}_{n} . \tag{46}$$

To elucidate the adopted indicial notation for the sequence I^{f} (II^{f}) we observe that a typical rotation matrix $\boldsymbol{T}_{i,i-1}$ ($\boldsymbol{T}_{i-1,i}$) can be deduced from a standard expression like (16) once the *current* position of the affiliated pseudovector $\boldsymbol{\vartheta}_i$ ($\boldsymbol{\vartheta}_{i-1}$), resulting from the preceding rotation $\boldsymbol{\vartheta}_{i-1}$ ($\boldsymbol{\vartheta}_i$), is determined by application of an expression like (41) (eq. (43)). Of course each $\boldsymbol{\vartheta}_i$ ($\boldsymbol{\vartheta}_{i-1}$) is affected by *all preceding* rotations.

We next reconsider the above problem in an alternative light. But before proceeding to this more interesting interpretation it is useful to draw attention to some evident but relevant reformulations of (15) or (25). In particular, we recollect that the rotation matrix $\boldsymbol{T}(-\boldsymbol{\vartheta}) = \boldsymbol{T}^{\mathrm{t}}$ if applied as an operand on $\hat{\boldsymbol{p}}$ must restore it to its original position $\boldsymbol{p}$; see e.g. (15a), (16a). If the given vector $\boldsymbol{p}$ coincides now with the direction of the prescribed axis of rotation $\boldsymbol{e}$ or with $\boldsymbol{\vartheta}(\boldsymbol{\omega})$, then (15), (15a), (25) reduce to the identities

$$\boldsymbol{T}\boldsymbol{p} = \boldsymbol{p} \quad \text{and} \quad \boldsymbol{T}^{\mathrm{t}}\boldsymbol{p} = \boldsymbol{p}$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (47)

$$\boldsymbol{T}\boldsymbol{\vartheta} = \boldsymbol{\vartheta} \quad \text{and} \quad \boldsymbol{T}^{\mathrm{t}}\boldsymbol{\vartheta} = \boldsymbol{\vartheta} .$$

This leads to another interpretation which we take up in the subsequent section.

Returning to our current problem we reconsider the sequence I^{f} composed of the two rotations $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_{21}$ and remind the reader of the ultimate rotation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ of (42) which we assume for the moment as known. Let us now suppose that following the completion of sequence I^{f} we wish to annul or remove posteriorly the effect of $\boldsymbol{\vartheta}_1$. This is clearly achieved by premultiplying (42) by $\boldsymbol{T}_1(-\boldsymbol{\vartheta}_1) = \boldsymbol{T}_1^{\mathrm{t}}$. At the same time we observe that this operation suppresses *all* effects of $\boldsymbol{\vartheta}_1$ and must hence reduce the right-hand side to the transformation matrix $\boldsymbol{T}_2$ for an axis fixed in space! Thus, we have

$$\boldsymbol{T}_1^{\mathrm{t}}\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_1^{\mathrm{t}}\boldsymbol{T}_{21}\boldsymbol{T}_1 = \boldsymbol{T}_2 . \tag{48}$$

The congruent operation with $\boldsymbol{T}_1$ applied on $\boldsymbol{T}_{21}$ is seen to transform the latter simply into $\boldsymbol{T}_2$. Eq. (48) produces an immediate explicit expression for $\boldsymbol{T}_{21}$. In fact, pre- and postmultiplying with $\boldsymbol{T}_1$ and $\boldsymbol{T}_1^{\mathrm{t}}$, respectively, we obtain

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}\boldsymbol{T}_1^{\mathrm{t}} = \boldsymbol{T}_{21} = \boldsymbol{T}_1\boldsymbol{T}_2\boldsymbol{T}_1^{\mathrm{t}} . \tag{49}$$

This formula for $\boldsymbol{T}_{21}$ may now be substituted in (42) and reduces $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ to the at first surprising alternative expression for $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ viz.

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_1\boldsymbol{T}_2 = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}} \tag{50}$$

where the superscript r indicates axes fixed (rigid) in space. Eq. (50) follows also immediately from (49) by postmultiplication with $\boldsymbol{T}_1$. Thus, the follower sequence I^{f} yields the same ultimate transformation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}}$ as the sequence II^{r} does for axes fixed in space. A little physical and geometrical insight illustrated in Fig. 2 confirms immediately the result. This interpretation does not of course mean that the paths followed on the one hand by the rotations $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_{21}$, and on the other hand by the rotations $\boldsymbol{\vartheta}_2$, $\boldsymbol{\vartheta}_{12}$ are coincident. It merely indicates that the starting and final locations of the two compound rotations are identical; see also the confirmation in Fig. 4.

Let us next extend our analysis to three follower rotations. Following (45) we have

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_{321}\boldsymbol{T}_{21}\boldsymbol{T}_1 \tag{51}$$

where we adopt here for $\boldsymbol{T}_{321}$ a more suggestive indicial notation than $\boldsymbol{T}_{32}$ in order to display the full sequence of (follower) rotations 1, 21, 321. It is evident that an analogous operation to that leading to (48) but extending this time to the sequential removal of the effects of the pseudovectors $\boldsymbol{\vartheta}_{21}$, $\boldsymbol{\vartheta}_1$ using the respective transformation matrices $\boldsymbol{T}_{21}^{\mathrm{t}}$, $\boldsymbol{T}_1^{\mathrm{t}}$ leads to

$$\boldsymbol{T}_1^{\mathrm{t}}\boldsymbol{T}_{21}^{\mathrm{t}}\boldsymbol{T}_{\mathrm{I}} = \boldsymbol{T}_1^{\mathrm{t}}\boldsymbol{T}_{21}^{\mathrm{t}}\boldsymbol{T}_{321}\boldsymbol{T}_{21}\boldsymbol{T}_1 = \boldsymbol{T}_3\,. \tag{52}$$

The result of (52) is evident since the removal of $\boldsymbol{\vartheta}_{21}$, $\boldsymbol{\vartheta}_1$ must produce the state $\boldsymbol{\vartheta}_3$ and the transformation matrix $\boldsymbol{T}_3$ for a fixed axis. Substituting expression (49) for $\boldsymbol{T}_{21}$ in (52) and recollecting the orthogonality relation (4) we find

$$\boldsymbol{T}_2^{\mathrm{t}}\boldsymbol{T}_1^{\mathrm{t}}\boldsymbol{T}_{\mathrm{I}} = \boldsymbol{T}_2^{\mathrm{t}}\boldsymbol{T}_1^{\mathrm{t}}\boldsymbol{T}_{321}\boldsymbol{T}_1\boldsymbol{T}_2 = \boldsymbol{T}_3\,. \tag{53}$$

Applying now a double congruent transformation involving in turn $\boldsymbol{T}_2$ and $\boldsymbol{T}_1$ we obtain in place of (53),

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}\boldsymbol{T}_2^{\mathrm{t}}\boldsymbol{T}_1^{\mathrm{t}} = \boldsymbol{T}_{321} = \boldsymbol{T}_1\boldsymbol{T}_2\boldsymbol{T}_3\boldsymbol{T}_2^{\mathrm{t}}\boldsymbol{T}_1^{\mathrm{t}}\,, \tag{54}$$

which proves a most suggestive expression for the follower transformation $\boldsymbol{T}_{321}$. Postmultiplication $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ yields the expected formula

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_1\boldsymbol{T}_2\boldsymbol{T}_3 = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}}\,. \tag{55}$$

The generalisation to n rotations proceeds automatically and confirms the assertion that n follower rotations in the sequence $\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \ldots, \boldsymbol{\vartheta}_i, \ldots, \boldsymbol{\vartheta}_{n-1}, \boldsymbol{\vartheta}_n$ produce the same transformation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}}$ as n fixed rotation in the sequence $\boldsymbol{\vartheta}_n, \boldsymbol{\vartheta}_{n-1}, \ldots, \boldsymbol{\vartheta}_i, \ldots, \boldsymbol{\vartheta}_2, \boldsymbol{\vartheta}_1$.

By an analogous argument or simply by cyclic changes of suffices $1, 2, 3, \ldots, i, \ldots, n-1, n$, we confirm immediately the inverse proposition

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}\,. \tag{56}$$

Formulae (51), (52) permit immediately to convert a follower problem into one with axes fixed in space.

For the interested reader we quote also the generalisation of (54) for any number i of follower rotations in the form

$$T_i^{\mathrm{f}} = T_1 T_2 \cdots T_{i-1} T_i T_{i-1}^{\mathrm{t}} \cdots T_2^{\mathrm{t}} T_1^{\mathrm{t}} \tag{57}$$

for sequence I^{f}. Analogously we have for sequence II^{f},

$$T_i^{\mathrm{f}} = T_n T_{n-1} \cdots T_{i+1} T_i T_{i+1}^{\mathrm{t}} \cdots T_{n-1}^{\mathrm{t}} T_n^{\mathrm{t}} \,. \tag{58}$$

We use again on the left-hand side of (57), (58) the superior f to indicate the follower nature of the respective rotation matrices. The reader should remember on the other hand that all rotation matrices on the right-hand side are specified for axes fixed in space.

3. The single equivalent pseudovector $\boldsymbol{\vartheta}$ in the place of a series of rotation $\boldsymbol{\vartheta}_i$

We first return to the second equation in (47), which we rewrite in the form

$$[\boldsymbol{T} - \boldsymbol{I}_3]\boldsymbol{\vartheta} = \boldsymbol{o}_3 \,. \tag{59}$$

Now this expression can be interpreted as a spectral problem of the kind

$$[\boldsymbol{T} - \lambda \boldsymbol{I}_3]\boldsymbol{\vartheta} = \boldsymbol{o}_3 \,, \tag{60}$$

in which the only pertinent eigenvalue is $\lambda = 1$ and $\boldsymbol{\vartheta}$ is the associated eigenvector. The reader will observe the uniqueness of the posed problem since the other two eigenvalues of (52) are non-real.

Our above statement is trivial for a single rotation $\boldsymbol{\vartheta}$. However, for a sequential application of multiple rotations $\boldsymbol{\vartheta}_i$—be they about fixed or follower axes—the eigenvector formulation of (60) may be the most economical proposition to obtain the equivalent single pseudovector $\boldsymbol{\vartheta}$ which replaces the given compound rotation. Prior to the eigenvector analysis it is, in fact, only necessary to set up $\boldsymbol{T}$ by formulae of the type of (5), (5a), (45), (46).

There remains, of course, the interesting question if there exists a simple explicit method of establishing the substitute pseudovector $\boldsymbol{\vartheta}$. The answer is affirmative and is demonstrated below for a compound rotation consisting of two consecutive individual rotations (1) and (2), but can be generalised to any number of rotations; see in this connection the presentation in appendix A. To formulate the theory concisely it is helpful to adopt here in place of the rotation vector $\boldsymbol{\vartheta}$ the parameter pseudovector $\boldsymbol{\omega}$ in the form of (19) viz.

$$\boldsymbol{\omega} = \{\omega_x \quad \omega_y \quad \omega_z\} = \tan\frac{\vartheta}{2}\, \boldsymbol{e} \tag{61}$$

and use also the associated auxiliary matrix $\boldsymbol{R}$ of (23).

Consider next the rotation of a rigid body as depicted in Fig. 3, the notation of which differs slightly from that in Fig. 1. Noting that M is the midpoint of $\overline{PP_1} = \boldsymbol{p}_\Delta$ we have

$$\overline{AM} = \tfrac{1}{2}(\boldsymbol{p} + \boldsymbol{p}_1) \quad \text{or} \quad \boldsymbol{p} + \boldsymbol{p}_1 = 2\overline{AM}\,.$$

Also (62)

$$\boldsymbol{p} + \boldsymbol{p}_\Delta = \boldsymbol{p}_1 \quad \text{or} \quad \boldsymbol{p}_1 - \boldsymbol{p} = \boldsymbol{p}_\Delta\,.$$

Since $\overline{CM}$ is perpendicular to $\overline{PP_1}$, we have

$$PM = \tfrac{1}{2}p_\Delta = CM \tan \vartheta/2\,. \tag{63}$$

Furthermore, we observe in Fig. 3 that $\boldsymbol{p}_\Delta$ stands normal on $\boldsymbol{\omega}$ and $2\overline{AM} = \underline{\boldsymbol{p} + \boldsymbol{p}_1}$ and is hence concurrent with $\boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1)$. Noting (63) and the angle α between $\boldsymbol{\omega}$ and $\overline{AM}$, the magnitude of

$$|\boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1)|$$

is seen to be

$$\sin\alpha \cdot \tan\frac{\vartheta}{2} \cdot 2AM = \frac{CM}{AM}\tan\frac{\vartheta}{2} \cdot 2AM = 2CM \cdot \tan\frac{\vartheta}{2} = |\boldsymbol{p}_\Delta|\,. \tag{64}$$

Thus, we obtain the simple relation

$$\boldsymbol{p}_\Delta = \boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1) = \boldsymbol{p}_1 - \boldsymbol{p}\,. \tag{65}$$

Bearing in mind the definition of $\boldsymbol{R}$ in (23), (65) can also be written in the more convenient matrix form

$$\boldsymbol{p}_1 - \boldsymbol{p} = \boldsymbol{R}[\boldsymbol{p} + \boldsymbol{p}_1]\,. \tag{66}$$

The reader should note again the vectorial equivalence of the operations

$$\boldsymbol{\omega} \times \boldsymbol{p} \quad \text{and} \quad \boldsymbol{R}\boldsymbol{p}$$

(see also (14) and (24)). In the subsequent analysis we also require the relation

$$\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p}_1 = \boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p}\,, \tag{67}$$

which follows immediately from the observation that the angles enclosed by $\boldsymbol{\omega}$, $\boldsymbol{p}$ and $\boldsymbol{\omega}$, $\boldsymbol{p}_1$ are equal and that furthermore $|\boldsymbol{p}_1| = |\boldsymbol{p}|$.

We are now in a position to examine a compound rotation extending over two parameter vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and assume first that the axes are fixed in space.

3.1. Compound rotation about fixed axes

Applying first rotation $\boldsymbol{\omega}_1$, (66) becomes

$$p_1 - p = R_1[p + p_1]\,. \tag{68}$$

The subsequent rotation ω_2 transports p_1 to p_2 for which we have

$$p_2 - p_1 = R_2[p_1 + p_2]\,. \tag{69}$$

Our aim is to establish a single compound rotation pseudovector ω_{I} and its auxiliary matrix R_{I} which transfer the vector p in one go to p_2. Following (65), (66) the necessary condition reads

$$p_2 - p = \omega_{\mathrm{I}}^{\mathrm{r}} \times [p_2 + p] \quad \text{or} \quad p_2 - p = R_{\mathrm{I}}^{\mathrm{r}}[p_2 + p] \tag{70}$$

where the superscript r refers once more to axes fixed in space. We adopt in the analysis the more convenient matrix notation and obtain by addition of (68) and (69),

$$p_2 - p = R_1[p + p_1] + R_2[p_1 + p_2]$$

which may be rearranged into

$$p_2 - p = [R_1 + R_2][p_2 + p] - R_1[p_2 - p_1] + R_2[p_1 - p]\,.$$

Using now (68) and (69) in the last two terms we obtain

$$p_2 - p = [R_1 + R_2][p_2 + p] - R_1R_2[p_2 + p_1] + R_2R_1[p_1 + p]$$

or

$$p_2 - p = [R_1 + R_2 - [R_1R_2 - R_2R_1]][p_2 + p] - R_1R_2[p_1 - p] - R_2R_1[p_2 - p_1]\,. \tag{71}$$

At this stage and for subsequent reference the reader may easily confirm that the operations

$$[R_1R_2 - R_2R_1]p \quad \text{and} \quad (\omega_1 \times \omega_2) \times p \tag{72}$$

are vectorially equivalent. Recollecting also the antisymmetry of the R operation we note that

$$[R_1R_2]^{\mathrm{t}} = R_2^{\mathrm{t}}R_1^{\mathrm{t}} = (-R_2)(-R_1) = R_2R_1\,. \tag{73}$$

It is now straightforward to prove that the last two terms of (71) transform into

$$\begin{aligned} &-R_1R_2[p_1 - p] - R_2R_1[p_2 - p_1] = \\ &\quad = \omega_1^{\mathrm{t}}\omega_2(p_1 - p) - \omega_1^{\mathrm{t}}(p_1 - p)\omega_2 + \omega_1^{\mathrm{t}}\omega_2(p_2 - p_1) - \omega_2^{\mathrm{t}}(p_2 - p_1)\omega_1\,, \end{aligned} \tag{A}$$

the second and fourth terms of which vanish as a result of

$$\omega_1^{\mathrm{t}}p_1 = \omega_1^{\mathrm{t}}p \quad \text{and} \quad \omega_2^{\mathrm{t}}p_2 = \omega_2^{\mathrm{t}}p_1 \tag{74}$$

which follow from (67). The first and third terms of (A) reduce to

$$\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2(\boldsymbol{p}_2-\boldsymbol{p}) .$$

Substituting this expression into (71), we find the relatively simple formula

$$\boldsymbol{p}_2-\boldsymbol{p}=\frac{1}{1-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2}[\boldsymbol{R}_1+\boldsymbol{R}_2-[\boldsymbol{R}_1\boldsymbol{R}_2-\boldsymbol{R}_2\boldsymbol{R}_1]][\boldsymbol{p}_2+\boldsymbol{p}] . \tag{75}$$

Comparison with the second equation in (70) yields the desired expression for the equivalent auxiliary matrix

$$\boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}}=\frac{1}{1-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2}[\boldsymbol{R}_1+\boldsymbol{R}_2-[\boldsymbol{R}_1\boldsymbol{R}_2-\boldsymbol{R}_2\boldsymbol{R}_1]] . \tag{76}$$

Remembering the equivalence of $\boldsymbol{R}$ and $\boldsymbol{\omega}\times$ operations and their extension to the rule of (72) we also deduce the compound pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ in the form

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}=\frac{1}{1-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1+\boldsymbol{\omega}_2-\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2] \tag{77}$$

or more conveniently

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}=\frac{1}{1-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1+\boldsymbol{\omega}_2-\boldsymbol{R}_1\boldsymbol{\omega}_2] . \tag{78}$$

Expressions (76), (77), (78) confirm once more that the superposition principle does not apply to large rotations except in the trivial case of parallel axes. In fact, in the latter case $\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2=\boldsymbol{o}_3$, $\boldsymbol{R}_1\boldsymbol{\omega}_2=\boldsymbol{o}_3$ and $\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2=\tan\vartheta_1/2\tan\vartheta_2/2$ and (77), (78) reduce to

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}=\tan\frac{\vartheta_{\mathrm{I}}}{2}\boldsymbol{e}=\frac{\tan\vartheta_1/2+\tan\vartheta_2/2}{1-\tan\vartheta_1/2\tan\vartheta_2/2}\boldsymbol{e}=\tan\frac{(\vartheta_1+\vartheta_2)}{2}\boldsymbol{e} .$$

Thus

$$\vartheta_{\mathrm{I}}=\vartheta_1+\vartheta_2 . \tag{79}$$

However, (76) to (78) solve at the same time the problem posed by the inequality of (40). In fact, (76) can be rewritten via (23) in terms of the $\boldsymbol{S}_{\mathrm{I}}$, $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ matrices and yields hence $\boldsymbol{S}_{\mathrm{I}}$ as a nonlinear function of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$. Only for parallel axes does this expression reduce to the limiting case

$$\boldsymbol{S}_{\mathrm{I}}=\boldsymbol{S}_1+\boldsymbol{S}_2 , \tag{80}$$

which is the pendant of (79).

At this stage it is pertinent to refer to two characteristic properties of the operator $\boldsymbol{R}$ valid for arbitrary axes viz.

$$\boldsymbol{R}_1\boldsymbol{\omega}_2 = -\boldsymbol{R}_2\boldsymbol{\omega}_1 \tag{81}$$

and

$$\boldsymbol{R}_1\boldsymbol{\omega}_1 = \boldsymbol{R}_2\boldsymbol{\omega}_2 = \boldsymbol{o}_3\,, \tag{82}$$

which reflect the matrix equivalent of the wellknown properties of a crossproduct with respect to reversal of the constituent vectors and parallelism of the vectors. Expression (81) shows that (78) can also be written in the form

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{R}_2\boldsymbol{\omega}_1]\,. \tag{78a}$$

We are now in a position to establish an explicit expression for the magnitude of the compound pseudovector $\boldsymbol{\vartheta}_{\mathrm{I}}$ or $\boldsymbol{\omega}_{\mathrm{I}}$. To this purpose we deduce the intensity or length $\tan \vartheta_{\mathrm{I}}/2$ of $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ from the scalar product $(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$. It is thereby particularly attractive to use in one of the two factors (78) and in the other (78a). However, for briefness of the argument, we apply here the same expression (78). Defining the angle formed by the vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ by γ, recollecting (80), noting the important devices of (81), (82), and the easily established relations (see also (115))

$$\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2 = \tan\frac{\vartheta_1}{2}\tan\frac{\vartheta_2}{2}\cos\gamma\,, \tag{83}$$

$$\boldsymbol{R}_1^2 = -\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1\boldsymbol{I}_3 + \boldsymbol{\omega}_1\boldsymbol{\omega}_1^{\mathrm{t}}\,, \tag{84}$$

as also

$$\begin{aligned}\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{R}_1^2\boldsymbol{\omega}_2 &= -(\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2)(\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1) + (\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_1)^2\\ &= -\tan^2\frac{\vartheta_1}{2}\tan^2\frac{\vartheta_2}{2} + \tan^2\frac{\vartheta_1}{2}\tan^2\frac{\vartheta_2}{2}\cos^2\gamma\,,\end{aligned} \tag{85}$$

we ultimately obtain,

$$\begin{aligned}(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} &= \tan^2\frac{\vartheta_{\mathrm{I}}}{2}\\ &= \frac{\tan^2\vartheta_1/2 + \tan^2\vartheta_2/2 + 2\tan\vartheta_1/2\tan\vartheta_2/2\cos\gamma + \tan^2\vartheta_1/2\tan^2\vartheta_2/2(1-\cos^2\gamma)}{(1-\tan\vartheta_1/2\tan\vartheta_2/2\cos\gamma)^2}\,.\end{aligned} \tag{86}$$

Note that as a result of (81) and (82) all expressions of the form $\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{R}\boldsymbol{\omega}$ vanish irrespective of the indices. It is clearly more appropriate to form

$$\cos^2\frac{\vartheta_{\mathrm{I}}}{2}=\frac{1}{1+\tan^2\vartheta_{\mathrm{I}}/2}=\frac{(1-\tan\vartheta_1/2\tan\vartheta_2/2\cos\gamma)^2}{1+\tan^2\vartheta_1/2+\tan^2\vartheta_2/2+\tan^2\vartheta_1/2\tan^2\vartheta_2/2}$$
$$=\frac{(\cos\vartheta_1/2\cos\vartheta_2/2-\sin\vartheta_1/2\sin\vartheta_2/2\cos\gamma)^2}{\cos^2\vartheta_1/2\cos^2\vartheta_2/2(1+\tan^2\vartheta_1/2)(1+\tan^2\vartheta_2/2)}$$
$$=(\cos\vartheta_1/2\cos\vartheta_2/2-\sin\vartheta_1/2\sin\vartheta_2/2\cos\gamma)^2\,. \tag{86a}$$

We deduce the simple relation

$$\cos\vartheta_{\mathrm{I}}/2=\cos\vartheta_1/2\cos\vartheta_2/2-\sin\vartheta_1/2\sin\vartheta_2/2\cos\gamma \tag{87}$$

which will be confirmed in the following section by an elementary geometrical argument.

We now consider briefly the reversed sequence II^{r} involving the order of rotations $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_1$. The corresponding expressions for $\boldsymbol{R}_{\mathrm{II}}^{\mathrm{r}}$ and $\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}$ are immediately deduced from (76), (77), (78), (78a) by cyclic changes of the indices 1 and 2. For completeness we reproduce here

$$\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}=\frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_2+\boldsymbol{\omega}_1-\boldsymbol{R}_2\boldsymbol{\omega}_1]=\frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_2+\boldsymbol{\omega}_1+\boldsymbol{R}_1\boldsymbol{\omega}_2]$$

and

$$\boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}}=\frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_2+\boldsymbol{R}_1+[\boldsymbol{R}_1\boldsymbol{R}_2-\boldsymbol{R}_2\boldsymbol{R}_1]]\,. \tag{88}$$

We observe that the reversal of the sequential order from I^{r} to II^{r} and vice versa merely changes the sign but not the magnitude of the nonlinear terms in the expressions of $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$, $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}}$ and their pendants $\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}$, $\boldsymbol{R}_{\mathrm{II}}^{\mathrm{r}}$. Moreover, these nonlinear terms are normal to the pair of vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$. Consequently, the reversal of the sequential order from I^{r} to II^{r} does not affect the magnitude of the compound pseudovector but only its direction. Thus

$$|\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}|=|\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}| \tag{89}$$

which follows also from (87) which applies as it stands to sequence II^{r}. The results of the limiting cases in (79) and (80) are equally true for sequence II^{r}.

Having the auxiliary matrix $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}}$ ($\boldsymbol{R}_{\mathrm{II}}^{\mathrm{r}}$) of the compound rotation, the associated rotation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$ ($\boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}}$) can be established in one go using (26) or (16). Note that $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ ($\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}$) is now the eigenvector of (60) with $\boldsymbol{T}=\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$ ($\boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}}$). In this connection it is relevant to have an explicit expression for $(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ in terms of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. To establish it we need only follow the argument leading to (86) and obtain

$$(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}=\frac{1}{(1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)^2}[\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1+\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2+2\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2+(\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1)(\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2)-(\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)^2]\,. \tag{86b}$$

This is clearly symmetrical in $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and hence yields the same expression for $(\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}$ as also confirmed by (89). For the purpose of setting up $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}$ ($\boldsymbol{T}^{\mathrm{r}}_{\mathrm{II}}$) we require

$$1/(1+(\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}})^{\mathrm{t}}\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}})$$

which simplifies to

$$\frac{1}{1+(\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}})^{\mathrm{t}}\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}}=\frac{(1-\boldsymbol{\omega}^{\mathrm{t}}_{1}\boldsymbol{\omega}_2)^2}{(1+\boldsymbol{\omega}^{\mathrm{t}}_{1}\boldsymbol{\omega}_1)(1+\boldsymbol{\omega}^{\mathrm{t}}_{2}\boldsymbol{\omega}_2)}\,. \tag{86c}$$

This is in direct agreement with (86a).

We now summarise the relevant formulae of $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}$, $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{II}}$ for fixed axes,

$$\boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}=\boldsymbol{T}(\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}})=\boldsymbol{T}_2\boldsymbol{T}_1\,,\qquad \boldsymbol{T}^{\mathrm{r}}_{\mathrm{II}}=\boldsymbol{T}(\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{II}})=\boldsymbol{T}_1\boldsymbol{T}_2\,, \tag{90}$$

which are applicable to two rotations and are special cases of (5) and (5a). This completes our analysis for axes fixed in space and we may turn our attention to follower axes. The reader should note that a generalisation to more than two rotations is straightforward but omitted here for economy of space; see, however, Appendix A.

An appended note to Subsection 3.1

To set the mind of the critical reader at rest it is of some relevance to develop an independent verification of the correctness of (78) for $\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}$ and of (76) for $\boldsymbol{R}^{\mathrm{r}}_{\mathrm{I}}$. One possible check consists of using $\boldsymbol{R}^{\mathrm{r}}_{\mathrm{I}}$ in (16) or (26) to form $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}$ as an explicit function in $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ and prove hence that it may be reduced to the factorised expression $\boldsymbol{T}_2\boldsymbol{T}_1$. To construct this proof in an effective manner a subtle and judicious interpretation of the nature of the $\boldsymbol{R}$ and in particular the $(\boldsymbol{R}_1\boldsymbol{R}_2-\boldsymbol{R}_2\boldsymbol{R}_1)$ operators is called for. Also the reader should recollect relations (81), (82) in order to shortcut continuously the argument.

However, a more elegant solution consists in injecting from the right to the left the pseudovector $\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}$ in $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}=\boldsymbol{T}_2\boldsymbol{T}_1$,

$$\boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}=\boldsymbol{T}_2\boldsymbol{T}_1\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}\,,$$

and carrying out the operations in the sequential order

$$\boldsymbol{T}_1\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}=\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}+\boldsymbol{\omega}_{\Delta 1}\rightarrow \boldsymbol{T}_2[\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}+\boldsymbol{\omega}_{\Delta 1}]=\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}+\boldsymbol{\omega}_{\Delta 1}+\boldsymbol{\omega}_{\Delta 2}=\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}\,, \tag{91}$$

leading hopefully to the ejection of $\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}$ at the end of the operations.

To establish this proof we may proceed as follows.

First transformation

$$\boldsymbol{T}_1\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}=\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}+\boldsymbol{\omega}_{\Delta 1} \tag{92}$$

in which clearly

$$\boldsymbol{\omega}_{\Delta 1} = \frac{2}{1+\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}[\boldsymbol{R}_1 + \boldsymbol{R}_1^2]\boldsymbol{\omega}_1^{\mathrm{r}} = \frac{2}{1+\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}\frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_1 + \boldsymbol{R}_1^2][\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1\boldsymbol{\omega}_2]$$
$$= \frac{2}{1+\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}\frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_1\boldsymbol{\omega}_2 - \boldsymbol{R}_1^3\boldsymbol{\omega}_2]\,. \tag{93}$$

We now observe that in extension of (30),

$$\boldsymbol{R}_1^3 = -\tan^2\frac{\vartheta_1}{2}\boldsymbol{R}_1 = -\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1\boldsymbol{R}_1\,. \tag{94}$$

Consequently (93) reduces to

$$\boldsymbol{\omega}_{\Delta 1} = \frac{2}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}\boldsymbol{R}_1\boldsymbol{\omega}_2\,. \tag{95}$$

Second transformation

$$\boldsymbol{T}_2[\boldsymbol{\omega}_1^{\mathrm{r}} + \boldsymbol{\omega}_{\Delta 1}] = \boldsymbol{\omega}_1^{\mathrm{r}} + \boldsymbol{\omega}_{\Delta 1} + \frac{2}{1+\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_2 + \boldsymbol{R}_2^2][\boldsymbol{\omega}_1^{\mathrm{r}} + \boldsymbol{\omega}_{\Delta 1}]\,.$$

Now using (95),

$$\boldsymbol{\omega}_1^{\mathrm{r}} + \boldsymbol{\omega}_{\Delta 1} = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1\boldsymbol{\omega}_2 + 2\boldsymbol{R}_1\boldsymbol{\omega}_2] = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{R}_1\boldsymbol{\omega}_2]\,, \tag{96}$$

or applying (81),

$$\boldsymbol{\omega}_1^{\mathrm{r}} + \boldsymbol{\omega}_{\Delta 1} = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_2\boldsymbol{\omega}_1]\,. \tag{96a}$$

Application of the operator

$$\frac{2}{1+\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_2 + \boldsymbol{R}_2^2]$$

to expression (96a) is exactly analogous to the corresponding step in (93) of the first transformation and leads to the second incremental pseudovector of (91). To this purpose we only require the pendant of (94),

$$\boldsymbol{R}_2^3 = -\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2\boldsymbol{R}_2 \tag{94a}$$

and find

$$\boldsymbol{\omega}_{\Delta 2} = \frac{2}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}\boldsymbol{R}_2\boldsymbol{\omega}_1\,. \tag{95a}$$

Hence we confirm

$$\boldsymbol{\omega}_{\Delta 1}+\boldsymbol{\omega}_{\Delta 2}=\frac{2}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_1\boldsymbol{\omega}_2+\boldsymbol{R}_2\boldsymbol{\omega}_1]=\boldsymbol{o}_3 \tag{97}$$

as to be expected.

Thus, the postmultiplication with or injection from the right of $\boldsymbol{\omega}_1^{\mathrm{r}}$ into $\boldsymbol{T}_2\boldsymbol{T}_1$ leads at the end of the operations to the ejection of $\boldsymbol{\omega}_1^{\mathrm{r}}$ from the left and proves thereby the assertion of (90).

3.2. *Compound rotation about follower axes*

Bearing in mind the main conclusion of Section 2 on a compound rotation $\mathrm{I}^{\mathrm{f}}(\mathrm{II}^{\mathrm{f}})$ about follower axes and its equivalence to a compound rotation $\mathrm{II}^{\mathrm{r}}(\mathrm{I}^{\mathrm{r}})$ about fixed axes, one may consider the developments of this subsection as superfluous. However, an independent analysis and confirmation has its attraction and yields some additional insight.

Starting with the sequence I^{f} or $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ we observe that the kinematic effect of the first rotation $\boldsymbol{\omega}_1$ remains unaffected as in (68). Nevertheless, the pseudovector $\boldsymbol{\omega}_1$ transports at the same time the pseudovector $\boldsymbol{\omega}_2$ to the new position $\boldsymbol{\omega}_2^{\mathrm{f}}=\boldsymbol{\omega}_{21}$, where the superscript f refers as in Section 2 to the follower character of the movement 2. In fact, applying (25) to the initial vector $\boldsymbol{p}=\boldsymbol{\omega}_2$ we obtain

$$\boldsymbol{\omega}_2^{\mathrm{f}}=\boldsymbol{\omega}_{21}=\boldsymbol{T}_1(\boldsymbol{\omega}_1)\boldsymbol{\omega}_2$$

or

$$\boldsymbol{\omega}_{21}=\boldsymbol{\omega}_2+\frac{2}{1+\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}[\boldsymbol{R}_1+\boldsymbol{R}_1^2]\boldsymbol{\omega}_2\,. \tag{98}$$

It is of some interest and subsequent relevance to set up the corresponding expression $\boldsymbol{R}_{21}$ for the auxiliary matrix $\boldsymbol{R}$. This is straightforward for the component $\boldsymbol{R}_1\boldsymbol{\omega}_2$ of (98) since in accordance with (72) it becomes $\boldsymbol{R}_1\boldsymbol{R}_2-\boldsymbol{R}_2\boldsymbol{R}_1$. However, the analogy is at first not so obvious when considering the operation $\boldsymbol{R}_1^2\boldsymbol{\omega}_2$. Be it as it may, a little thought shows that

$$\boldsymbol{R}_1^2\boldsymbol{\omega}_2=\boldsymbol{R}_1\boldsymbol{R}_1\boldsymbol{\omega}_2$$

must, in principle, transform as $\boldsymbol{R}_1\boldsymbol{\omega}_2$ but with

$$\boldsymbol{R}_{\mathrm{r}}=\boldsymbol{R}_1\boldsymbol{R}_2-\boldsymbol{R}_2\boldsymbol{R}_1=\boldsymbol{R}_1\boldsymbol{R}_2-[\boldsymbol{R}_1\boldsymbol{R}_2]^{\mathrm{t}} \tag{99}$$

taking the place of $\boldsymbol{R}_2$. Thus,

$$\boldsymbol{R}_1^2\boldsymbol{\omega}_2\rightarrow\boldsymbol{R}_1\boldsymbol{R}_{\mathrm{r}}-\boldsymbol{R}_{\mathrm{r}}\boldsymbol{R}_1\,. \tag{100}$$

It follows that the required expression for $\boldsymbol{R}_{21}=\boldsymbol{R}_2^{\mathrm{f}}$ is

$$\boldsymbol{R}_{21} = \boldsymbol{R}_2 + \frac{2}{1 + \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}[\boldsymbol{R}_{\mathrm{r}} + \boldsymbol{R}_1\boldsymbol{R}_{\mathrm{r}} - \boldsymbol{R}_{\mathrm{r}}\boldsymbol{R}_1]\,. \tag{101}$$

In order not to interrupt the main flow of the argument, we postpone until the end of this subsection a more formal proof of (101).

Following the activation of the second rotation in the form of $\boldsymbol{\omega}_{21}$ we may apply (78) to determine the compound rotation $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$ noting, however, that $\boldsymbol{\omega}_{21}$ now takes the place of $\boldsymbol{\omega}_2$. Consequently we have

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_{21}}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_{21} - \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_{21}] = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_{21}}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_{21} - \boldsymbol{R}_1\boldsymbol{\omega}_{21}]\,. \tag{102}$$

We first note the familiar expression

$$\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2 \tag{103}$$

which follows from (67) or (74). Substituting (98) in (102), we obtain

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}\left[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + 2\frac{\boldsymbol{R}_1 + \boldsymbol{R}_1^2}{1 + \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}\boldsymbol{\omega}_2 - \boldsymbol{R}_1\boldsymbol{\omega}_2 - 2\frac{\boldsymbol{R}_1^2 + \boldsymbol{R}_1^3}{1 + \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}\boldsymbol{\omega}_2\right]. \tag{104}$$

Using (94) for $\boldsymbol{R}_1^3$, $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$ reduces to

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{R}_1\boldsymbol{\omega}_2] = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_2\boldsymbol{\omega}_1] = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}\,, \tag{105}$$

in which the second formula for $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$ derives from (81). Eq. (105) is in line with our assertion in (50). Alternatively, in vector notation

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2] = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}\,. \tag{106}$$

Expressions (105), (106) are the pendants to (78), (77).

The auxiliary matrix $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}$ is clearly,

$$\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_1 + \boldsymbol{R}_2 + [\boldsymbol{R}_1\boldsymbol{R}_2 - \boldsymbol{R}_2\boldsymbol{R}_1]] = \boldsymbol{R}_{\mathrm{II}}^{\mathrm{r}}\,. \tag{107}$$

Eqs. (105), (107) confirm our assertion in (50) that the follower sequence I^{f} defined by $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$ or $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}$ is necessarily identical with the rigid sequence II^{r} defined by $\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}$ or $\boldsymbol{R}_{\mathrm{II}}^{\mathrm{r}}$.

Again the reversal of the order of rotation into the follower sequence II^{f} or $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_1$ is immediately taken care of by cyclic changes of the indices 1, 2. In fact, as for the rigid sequences, the change from I^{f} to II^{f} merely involves a change of sign of the nonlinear term. For example, $\boldsymbol{\omega}_{21}$ of (98) and $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$ of (105) become

$$\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_1 + \frac{2}{1+\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_2 + \boldsymbol{R}_2^2]\boldsymbol{\omega}_1$$

and (108)

$$\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}} = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 + \boldsymbol{R}_2\boldsymbol{\omega}_1] = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 - \boldsymbol{R}_1\boldsymbol{\omega}_2] = \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}\,.$$

Similarly $\boldsymbol{R}_{12}$ and $\boldsymbol{R}_{\mathrm{II}}^{\mathrm{f}}$ are simply

$$\boldsymbol{R}_{12} = \boldsymbol{R}_2 - \frac{2}{1+\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_{\mathrm{r}} + \boldsymbol{R}_2\boldsymbol{R}_{\mathrm{r}} - \boldsymbol{R}_{\mathrm{r}}\boldsymbol{R}_2]$$

and (109)

$$\boldsymbol{R}_{\mathrm{II}}^{\mathrm{f}} = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_2 + \boldsymbol{R}_1 - [\boldsymbol{R}_1\boldsymbol{R}_2 - \boldsymbol{R}_2\boldsymbol{R}_1]] = \boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}}\,.$$

Eq. (87) for the overall angle of rotation ϑ_{I} or ϑ_{II} in a rigid sequence applies equally to a follower succession. Also the magnitude $|\boldsymbol{\omega}|$ of the follower compound rotation is clearly fixed as in (89) by

$$|\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}| = |\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}| = |\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}| = |\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}}|\,. \tag{110}$$

Finally, recollecting (42), (44) and (50), the compound rotation or transformation matrices read

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}) = \boldsymbol{T}_{21}\boldsymbol{T}_1 = \boldsymbol{T}_1\boldsymbol{T}_2 = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}}\,, \qquad \boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{T}(\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}}) = \boldsymbol{T}_{12}\boldsymbol{T}_2 = \boldsymbol{T}_2\boldsymbol{T}_1 = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}\,. \tag{111}$$

It goes without saying that $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ ($\boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}}$) can be established in one go using $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}$ ($\boldsymbol{R}_{\mathrm{II}}^{\mathrm{f}}$) of (107) (eq. (109)) in (16) or (26). The attentive reader should note that while the final locations and compound pseudovectors of (110) are the same for I^{f} (II^{f}) and II^{r} (I^{r}) this does not apply of course for the complete intermediate movements or paths.

On the proof of (101) *for* $\boldsymbol{R}_{21}$

It is now instructive to produce an alternative deduction of (101). To this purpose we consider (49), $\boldsymbol{T}_{21} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}\boldsymbol{T}_1^{\mathrm{t}}$, and observe that the state 21 can be interpreted as a compound 'rigid' rotation composed of the first rotation $(-\boldsymbol{\omega}_1)$ (for which $\boldsymbol{T}_1(-\boldsymbol{\omega}_1) = \boldsymbol{T}_1^{\mathrm{t}}(\boldsymbol{\omega}_1)$) and the subsequent rotation given by $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$. Hence the auxiliary matrix $\boldsymbol{R}_{21}$ can be deduced from (76) for $\boldsymbol{R}$ subject to the substitutions

$$\boldsymbol{\omega}_1 \to -\boldsymbol{\omega}_1\,, \qquad \boldsymbol{R}_1 \to -\boldsymbol{R}_1\,, \qquad \boldsymbol{R}_2 \to \boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}\,, \qquad \boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}} \to \boldsymbol{R}_{21}\,.$$

This may be expressed as

$$\begin{aligned}\boldsymbol{R}_{21} &= \frac{1}{1+\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}}[-\boldsymbol{R}_1 + \boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}} - [-\boldsymbol{R}_1\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}} - \boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}(-\boldsymbol{R}_1)]]\\ &= \frac{1}{1+\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}}[-\boldsymbol{R}_1 + \boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}} + [\boldsymbol{R}_1\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}} - \boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}\boldsymbol{R}_1]]\end{aligned} \tag{112}$$

in which $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}$ is given by (107).

To simplify the subsequent reading it is helpful to interrupt for the moment the main argument and to list first some of the required transformations. Thus, using (105) we find

$$\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1^f = (\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1 + \boldsymbol{\omega}_1^t\boldsymbol{\omega}_2)/(1-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2)\,. \tag{113}$$

Hence

$$1/(1+\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1^f) = (1-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2)/(1+\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1)\,. \tag{114}$$

We also call for a number of expressions, the complete set of which reads,

$$\begin{aligned}
&\text{(a)} \quad \boldsymbol{R}_1^2 = -\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1\boldsymbol{I}_3 + \boldsymbol{\omega}_1\boldsymbol{\omega}_1^t\,,\\
&\text{(b)} \quad \boldsymbol{R}_2^2 = -\boldsymbol{\omega}_2^t\boldsymbol{\omega}_2\boldsymbol{I}_3 + \boldsymbol{\omega}_2\boldsymbol{\omega}_2^t\,,\\
&\text{(c)} \quad \boldsymbol{R}_1\boldsymbol{R}_2 = -\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2\boldsymbol{I}_3 + \boldsymbol{\omega}_2\boldsymbol{\omega}_1^t\,,\\
&\text{(d)} \quad \boldsymbol{R}_2\boldsymbol{R}_1 = -\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2\boldsymbol{I}_3 + \boldsymbol{\omega}_1\boldsymbol{\omega}_2^t\,.
\end{aligned} \tag{115}$$

Formula (a) was given previously in (84). Eq. (115) can be verified very easily if we subtract the respective second expression on the right-hand side from the overall expression on the left-hand side.

Before proceeding to the proposed deduction of $\boldsymbol{R}_{21}$ we present an incidental application of the last two expressions in (115) which will come in useful in Appendix A. Considering in particular an arbitrary vector $\boldsymbol{p}$ we observe that

$$\boldsymbol{R}_r\boldsymbol{p} = [\boldsymbol{R}_1\boldsymbol{R}_2 - \boldsymbol{R}_2\boldsymbol{R}_1]\boldsymbol{p} = (\boldsymbol{\omega}_1^t\boldsymbol{p})\boldsymbol{\omega}_2 - (\boldsymbol{\omega}_2^t\boldsymbol{p})\boldsymbol{\omega}_1\,. \tag{115a}$$

We are now in a position to continue the main argument. Substituting (107) into (112) we obtain, using (114), after some simple operations

$$\boldsymbol{R}_{21} = \boldsymbol{R}_2 + \frac{1}{1+\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1}[\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2\boldsymbol{R}_1 - \boldsymbol{\omega}_1^t\boldsymbol{\omega}_1\boldsymbol{R}_2 + 2\boldsymbol{R}_r + \boldsymbol{R}_1\boldsymbol{R}_r - \boldsymbol{R}_r\boldsymbol{R}_1]\,, \tag{116}$$

which on first reading does not quite agree with (101) we set out to prove. However, to overcome this apparent discrepancy expressions (115) may be called upon. In fact,

(1) Postmultiply (a) with $\boldsymbol{R}_2$ and (d) with $\boldsymbol{R}_1$, use in the first or second expression the identity (81) in the transposed form $\boldsymbol{\omega}_1^t\boldsymbol{R}_2 = -\boldsymbol{\omega}_2^t\boldsymbol{R}_1$ and add the resulting expressions to form

$$-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2\boldsymbol{R}_1 - \boldsymbol{\omega}_1^t\boldsymbol{\omega}_1\boldsymbol{R}_2 = \boldsymbol{R}_1^2\boldsymbol{R}_2 + \boldsymbol{R}_2\boldsymbol{R}_1^2\,. \tag{117}$$

(2) Postmultiply (c) with $-2\boldsymbol{R}_1$ or premultiply (d) with $-2\boldsymbol{R}_1$, use identity (82) to form

$$+2\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1\boldsymbol{R}_1 = -2\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_1\,. \tag{118}$$

Addition of (117) and (118) yields the missing link,

$$\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2\boldsymbol{R}_1 - \boldsymbol{\omega}_1^t\boldsymbol{\omega}_1\boldsymbol{R}_2 = \boldsymbol{R}_1\boldsymbol{R}_r - \boldsymbol{R}_r\boldsymbol{R}_1\,. \tag{119}$$

Substitution of (119) in (116) confirms ultimately (101). An alternative expression for $\boldsymbol{R}_{21}$ may be deduced from (119) in the form

$$\boldsymbol{R}_{21} = \boldsymbol{R}_2 + \frac{2}{1 + \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}[(\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)\boldsymbol{R}_1 - (\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1)\boldsymbol{R}_2 + \boldsymbol{R}_{\mathrm{r}}]\,, \tag{120}$$

which is not so physically suggestive as (101) but may be computationally preferable. This completes our analysis of $\boldsymbol{R}_{21} = \boldsymbol{R}_2^{\mathrm{f}}$.

3.3. Compound semitangential rotations

An effective application of large rotations in space within a nonlinear finite element theory is an intellectually demanding undertaking due to the nonconservative character of moments about fixed axes. This leads to a series of difficulties culminating in strictly nonsymmetrical tangent stiffness matrices of the elements. Associated with this 'Schönheitsfehler' is the noncommutativity of rotations about fixed or follower axes. These irritating difficulties are circumvented in an elegant manner by the specification of semitangential moments and corresponding (in the virtual work sense) semitangential rotations. Semitangential moments, which are conservative, were initially conceived by Ziegler for the more esoteric stability analysis of struts [8]. The associated semitangential rotations were first proposed in [2, 3] especially in order to ensure commutativity of large rotations. As a result successive semitangential rotations lead—independently of the adopted sequential order—to an identical final position. Hence the potential function of conservative moments can be specified by a unique function of the semitangential rotations, even if these are finite. This consequence immediately ensures a simple and consistent derivation of the tangent stiffness matrix of the elements and must be considered to be in principle a far more important achievement than the symmetry of the elemental tangent stiffness matrix which is a natural corollary of the use of semitangential rotations.

In what follows we present a brief resumé of the important concept of semitangential rotations. For a more complete exposition of this subject and the corresponding semitangential moments the reader may consult [2, 3].

Consider a rigid body subject within a sequence I to two consecutive pseudovectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ for the moment taken to act about axes either of the fixed or follower type; see also Fig. 4. As stated in Subsection 3.2 the first rotation vector $\boldsymbol{\omega}_1$ is clearly unaffected by the specification (fixed or follower) of its axis. On the other hand, the pseudovector $\boldsymbol{\omega}_2$ remains stationary for fixed axes but is transported to the position $\boldsymbol{\omega}_2^{\mathrm{f}} = \boldsymbol{\omega}_{21}$ of (98) for follower axes. Now the definition of semitangential rotations provides that the second rotation is operated not by $\boldsymbol{\omega}_2 = \boldsymbol{\omega}_2^{\mathrm{F}}$ or $\boldsymbol{\omega}_2^{\mathrm{f}}$ but within the sequential order I^{s} by

$$\boldsymbol{\omega}_2^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{\omega}_2^{\mathrm{F}} + \boldsymbol{\omega}_2^{\mathrm{f}}] = \tfrac{1}{2}[\boldsymbol{\omega}_2 + \boldsymbol{\omega}_{21}]\,, \tag{121}$$

which stipulates for $\boldsymbol{\omega}_2^{\mathrm{s}}$ the mean vectorial values of the pseudovectors $\boldsymbol{\omega}_2^{\mathrm{F}}$ and $\boldsymbol{\omega}_2^{\mathrm{f}}$ about fixed and follower axes, respectively. The procedure is illustrated in the example of Fig. 4.

In order to understand the physics of the process it is useful to assume for the moment that the angles of rotation are small, strictly infinitely small; see also Fig. 5. Bearing in mind this restriction on the angles, it is convenient to operate with $\boldsymbol{\vartheta}$ of (2) in place of $\boldsymbol{\omega}$ as pseudovector. Its components φ, χ, ψ reduce in the present limiting case to the individual rotations about the x, y, z axes. In the example of Fig. 5 the assumed angles of rotation are $\vartheta_1 = \varphi$ and $\vartheta_2 = \psi$. As a result of our assumption on the magnitudes of φ and χ, the transformation matrix $\boldsymbol{T}_1(\varphi)$ of (16) reduces to (17) viz.

$$\boldsymbol{T}_1(\varphi) = \boldsymbol{I}_3 + \boldsymbol{S}_1(\varphi)\,. \tag{122}$$

We deduce for the follower angle,

$$\boldsymbol{\chi}^{\mathrm{f}} = \boldsymbol{T}_1\boldsymbol{\chi} = \boldsymbol{\chi} + \boldsymbol{S}_1\boldsymbol{\chi} = \boldsymbol{\chi} + \boldsymbol{\varphi} \times \boldsymbol{\chi}\,. \tag{123}$$

Consequently the semitangential rotation is given by

$$\boldsymbol{\chi}^{\mathrm{s}} = \tfrac{1}{2}(\boldsymbol{\chi} + \boldsymbol{\chi}^{\mathrm{f}}) = \boldsymbol{\chi} + \tfrac{1}{2}\boldsymbol{\varphi} \times \boldsymbol{\chi}\,. \tag{124}$$

Eqs. (123) and (124) demonstrate that while $\boldsymbol{\chi}^{\mathrm{f}}$ is derived from $\boldsymbol{\chi}$ by a rotation $\boldsymbol{\varphi}$, $\boldsymbol{\chi}^{s}$ itself is obtained by a rotation $\boldsymbol{\varphi}/2$.

For an arbitrary vector $\boldsymbol{\vartheta}_1 = \{\varphi_1\ \chi_1\ \psi_1\}$ this rule implies that $\boldsymbol{\vartheta}_2^{\mathrm{s}}$ is deduced from $\boldsymbol{\vartheta}_2 = \{\varphi_2\ \chi_2\ \psi_2\}$ by rotations φ_1, χ_1, ψ_1 about the $Oxyz$ axes. In fact, (121) reduces in the present case to

$$\boldsymbol{\vartheta}_2^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{\vartheta}_2^{\mathrm{r}} + \boldsymbol{\vartheta}_2^{\mathrm{f}}]\,. \tag{125}$$

However, our above description does *not* hold for finite rotations in which case the full nonlinear expressions for $\boldsymbol{T}(\boldsymbol{\vartheta})$ or $\boldsymbol{T}(\boldsymbol{\omega})$ of (16) or (26) must be applied and the pseudovector $\boldsymbol{\omega}_2^{\mathrm{s}}$ derived from the *vectorial* relation of (121).

Next we establish the compound vector $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}}$ corresponding to the sequential order I^{s} and equivalent to two rotations $\boldsymbol{\omega}_1^{\mathrm{s}} = \boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2^{\mathrm{s}}$. To this purpose we need only apply (78) for $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ in the form

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{s}}}\,[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2^{\mathrm{s}} - \boldsymbol{R}_1\boldsymbol{\omega}_2^{\mathrm{s}}]\,. \tag{126}$$

We first observe that

$$\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{s}} = \tfrac{1}{2}(\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{f}}) = \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2\,. \tag{126a}$$

Using now the definition of (121) for $\boldsymbol{\omega}_2^{\mathrm{s}}$, (126) may be rewritten in the form

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}} = \frac{1}{2(1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)}\,[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1\boldsymbol{\omega}_2] + \frac{1}{2(1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)}\,[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2^{\mathrm{f}} - \boldsymbol{R}_1\boldsymbol{\omega}_2^{\mathrm{f}}]\,. \tag{127}$$

Observing (78), (102) and (105) as well as the identity $\boldsymbol{\omega}_2^{\mathrm{f}} = \boldsymbol{\omega}_{21}$, (127) reduces to

$$\begin{aligned}\boldsymbol{\omega}_{I}^{s} &= \frac{1}{2(1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)}\,[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1\boldsymbol{\omega}_2] + \frac{1}{2(1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)}\,[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{R}_1\boldsymbol{\omega}_2]\\ &= \tfrac{1}{2}[\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} + \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}] = \tfrac{1}{2}[\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}} + \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}] = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}\,[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2] = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{s}} = \boldsymbol{\omega}^{\mathrm{s}}\,.\end{aligned} \tag{128}$$

Thus, we have established a formal proof that for compound semitangential rotations the equivalent pseudovector

$$\boldsymbol{\omega}^{s} = \boldsymbol{\omega}^{s}_{\mathrm{I}} = \boldsymbol{\omega}^{s}_{\mathrm{II}} \tag{129}$$

is indeed independent of the sequential order, be it 1, 2 (I^{s}) or 2, 1 (II^{s}). The reader should note that in the above argument $\boldsymbol{\omega}^{\mathrm{f}}_{\mathrm{I}}$ is in turn given by (102) and (105). Of course, the result of our reasoning is in a way self-evident, bearing in mind the adoption of (121) for $\boldsymbol{\omega}^{s}_{2}$ which induces the elimination of the nonlinear term $\boldsymbol{R}_1\boldsymbol{\omega}_2$.

The final expression for $\boldsymbol{\omega}^{s}$ in (128) is seen to display some kind of commutativity but with a change of scale due to the factor $1/(1-\boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_2)$. However, full commutativity is achieved when $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are orthogonal and $\boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_2 = 0$. Then the pseudovector $\boldsymbol{\omega}^{s}$ becomes simply

$$\boldsymbol{\omega}^{s} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \,. \tag{130}$$

In conclusion we have established that semitangential pseudovectors $\boldsymbol{\omega}^{s}_{i}$ ($i = 1, 2$) can be composed if orthogonal as true vectors leading to the desired commutativity of large rotations in space. Clearly, the argument can immediately be extended to any number and sequential order of semitangential rotations and leads to

$$\boldsymbol{\omega}^{s} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \cdots + \boldsymbol{\omega}_i + \cdots + \boldsymbol{\omega}_{n-1} + \boldsymbol{\omega}_n \,. \tag{131}$$

Eq. (131) as well as (130) presume of course that rotations take place about orthogonal axes.

The auxiliary matrix $\boldsymbol{R}^{s}$ associated with $\boldsymbol{\omega}^{s}$ of (128) is given for arbitrary axes by

$$\boldsymbol{R}^{s} = \frac{1}{1-\boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_2}[\boldsymbol{R}_1 + \boldsymbol{R}_2] \tag{132}$$

and for an arbitrary number of rotations about orthogonal axes

$$\boldsymbol{R}^{s} = \boldsymbol{R}_1 + \boldsymbol{R}_2 + \cdots + \boldsymbol{R}_i + \cdots + \boldsymbol{R}_{n-1} + \boldsymbol{R}_n \,. \tag{133}$$

Eqs. (132) and (133) permit the immediate construction of the rotation matrix $\boldsymbol{T}^{s}$ appertaining to $\boldsymbol{\omega}^{s}$, as per (16) or (26), in accordance with

$$\boldsymbol{T}^{s} = \boldsymbol{T}(\boldsymbol{\omega}^{s}) = \boldsymbol{I}_3 + \frac{2}{1+(\boldsymbol{\omega}^{s})^{t}\boldsymbol{\omega}^{s}}[\boldsymbol{R}^{s} + (\boldsymbol{R}^{s})^2] \,. \tag{134}$$

For three orthogonal pseudovectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$, (131) yields

$$(\boldsymbol{\omega}^{s})^{t}\boldsymbol{\omega}^{s} = \boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_1 + \boldsymbol{\omega}^{t}_{2}\boldsymbol{\omega}_2 + \boldsymbol{\omega}^{t}_{3}\boldsymbol{\omega}_3 \,. \tag{135}$$

For the restricted case of two nonorthogonal pseudovectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ we have

$$(\boldsymbol{\omega}^{s})^{t}\boldsymbol{\omega}^{s} = \frac{1}{(1-\boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_2)^2}(\boldsymbol{\omega}^{t}_{1} + \boldsymbol{\omega}^{t}_{2})(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) = \frac{1}{(1-\boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_2)^2}(\boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_1 + \boldsymbol{\omega}^{t}_{2}\boldsymbol{\omega}_2 + 2\boldsymbol{\omega}^{t}_{1}\boldsymbol{\omega}_2) \,. \tag{136}$$

When building up $\boldsymbol{T}^s$ we require in accordance with (26), $2/(1+(\boldsymbol{\omega}^s)^t\boldsymbol{\omega}^s)$, which reduces for the restricted case of (136) to

$$\frac{2}{1+(\boldsymbol{\omega}^s)^t\boldsymbol{\omega}^s}=\frac{2(1-\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2)^2}{1+\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1+\boldsymbol{\omega}_2^t\boldsymbol{\omega}_2+(\boldsymbol{\omega}_1^t\boldsymbol{\omega}_2)^2}\,. \tag{136a}$$

At this stage it is of some interest to look closer at the auxiliary matrix $\boldsymbol{S}^s$ of (13) and the exponential expression (33) for $\boldsymbol{T}(\boldsymbol{\omega}^s)$. We observe that as a result of the commutativity of the semitangential rotations $\boldsymbol{\omega}_i^s$ the associated $\boldsymbol{S}_i^s$ must also display some kind of commutativity. The point is subtle and demands some exploration. In fact, each individual $\boldsymbol{S}_i^s$, as also $\boldsymbol{\omega}_i^s$, depends, of course, on the preceding rotations and their order, but the ultimate $\boldsymbol{\omega}^s$ and $\boldsymbol{S}^s$ determining $\boldsymbol{T}^s=\boldsymbol{T}(\boldsymbol{\omega}^s)=\exp\boldsymbol{S}^s$ must remain unaffected. In effect, we may derive immediately from (134) and (23), relating the $\boldsymbol{R}$ and $\boldsymbol{S}$ matrices, the relevant expression

$$\boldsymbol{R}^s=\frac{\tan\vartheta^s/2}{\vartheta^s}\boldsymbol{S}^s=\frac{\tan\vartheta_1/2}{\vartheta_1}\boldsymbol{S}_1+\frac{\tan\vartheta_2/2}{\vartheta_2}\boldsymbol{S}_2+\cdots+\frac{\tan\vartheta_n/2}{\vartheta_n}\boldsymbol{S}_n \tag{137}$$

from which $\boldsymbol{S}^s$ may be deduced. Thus, the $\boldsymbol{S}_i$ matrices are seen to satisfy some kind of weighted commutativity. Finally, the penultimate $\boldsymbol{T}^s$ matrix due to $\boldsymbol{S}^s$ may be written following (31) as

$$\boldsymbol{T}^s=\boldsymbol{T}(\boldsymbol{\omega}^s)=\exp\boldsymbol{S}^s=\mathrm{e}^{\boldsymbol{S}_n^s}\,\mathrm{e}^{\boldsymbol{S}_{n-1}^s}\cdots\mathrm{e}^{\boldsymbol{S}_i^s}\cdots\mathrm{e}^{\boldsymbol{S}_2^s}\,\mathrm{e}^{\boldsymbol{S}_1^s}\,, \tag{138}$$

the right-hand side of which may be expressed in any other sequential order. It goes without saying that alternative paths do not coincide but their terminal points are in all cases identical and solely determined by $\boldsymbol{\omega}^s$.

In order to extend our understanding of semitangential rotations we now analyse in detail the sequential path I^s and follow this up with the path II^s. We start by considering again $\boldsymbol{\omega}_2^s$ and its associated $\boldsymbol{R}_2^s$. To this purpose we use (98) of Subsection 3.2 in (121) and obtain immediately

$$\boldsymbol{\omega}_2^s=\boldsymbol{\omega}_2+\frac{1}{1+\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1}[\boldsymbol{R}_1+\boldsymbol{R}_1^2]\boldsymbol{\omega}_2 \tag{139}$$

where we drop the superscript r for economy in printing.

Following (121), the corresponding relation for $\boldsymbol{R}_2^s$ must read

$$\boldsymbol{R}_2^s=\tfrac{1}{2}[\boldsymbol{R}_2^r+\boldsymbol{R}_2^f]\,. \tag{140}$$

Applying now (101) for $\boldsymbol{R}_2^f=\boldsymbol{R}_{21}$ and writing again $\boldsymbol{R}_2^r=\boldsymbol{R}_2$, we find

$$\boldsymbol{R}_2^s=\boldsymbol{R}_2+\frac{1}{1+\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1}[\boldsymbol{R}_r+\boldsymbol{R}_1\boldsymbol{R}_r-\boldsymbol{R}_r\boldsymbol{R}_1] \tag{141}$$

where $\boldsymbol{R}_r$ is defined in (99).

Having $\boldsymbol{\omega}_2^{\mathrm{s}}$ and $\boldsymbol{R}_2^{\mathrm{s}}$, the associated rotation matrix $\boldsymbol{T}_2^{\mathrm{s}}$ is easily established using (26) in the form

$$\boldsymbol{T}_2^{\mathrm{s}} = \boldsymbol{I}_3 + \frac{2}{1 + (\boldsymbol{\omega}_2^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{s}}}[\boldsymbol{R}_2^{\mathrm{s}} + (\boldsymbol{R}_2^{\mathrm{s}})^2] . \tag{142}$$

It is helpful to obtain an explicit expression for $(\boldsymbol{\omega}_2^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{s}}$. To this end it is simplest to apply (121) and to recollect that $(\boldsymbol{\omega}_2^{\mathrm{f}})^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{f}} = \boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2$ since the $\boldsymbol{\omega}_2^{\mathrm{f}}$ vector must retain the same length as $\boldsymbol{\omega}_2^{\mathrm{r}} = \boldsymbol{\omega}_2$. There remains an expression of the form $(\boldsymbol{\omega}_2^{\mathrm{r}})^2\boldsymbol{\omega}_2^{\mathrm{f}}$ which upon substitution of (98) for $\boldsymbol{\omega}_2^{\mathrm{f}}$ may easily be reduced using the devices of (81), (82). We find, for example

$$\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{R}_1\boldsymbol{\omega}_2 = -\,\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{R}_2\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{R}_2^{\mathrm{t}}\boldsymbol{\omega}_1 = \boldsymbol{o}_3 . \tag{82a}$$

Also entering the first equation in (115),

$$\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{R}_1^2\boldsymbol{\omega}_2 = -(\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1)(\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2) + (\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)^2 .$$

We ultimately confirm the simple formula

$$(\boldsymbol{\omega}_2^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{s}} = \frac{1}{1 + \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1}[\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2 + (\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)^2] . \tag{143}$$

We are now in a position to derive the ultimate rotation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}} = \boldsymbol{T}^{\mathrm{s}}$, which must be independent of the sequential order. In fact,

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}} = \boldsymbol{T}(\boldsymbol{\omega}^{\mathrm{s}}) = \boldsymbol{T}_2^{\mathrm{s}}\boldsymbol{T}_1 \tag{144}$$

where $\boldsymbol{T}_1$ is the standard matrix for a fixed $\boldsymbol{\omega}_1$.

We now leave the sequential order I^{s} and turn our attention to the sequence II^{s} composed of the rotations $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_1^{\mathrm{s}}$. Here only $\boldsymbol{\omega}_1^{\mathrm{s}}$ interests us. Following a reasoning analogous to that for sequence I^{s} or simply by a cyclic change of suffices 1 and 2 we obtain

$$\boldsymbol{\omega}_1^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{\omega}_1^{\mathrm{r}} + \boldsymbol{\omega}_1^{\mathrm{f}}] = \tfrac{1}{2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_{12}] , \tag{145}$$

$$\boldsymbol{\omega}_1^{\mathrm{s}} = \boldsymbol{\omega}_1 + \frac{1}{1 + \boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_2 + \boldsymbol{R}_2^2]\boldsymbol{\omega}_1 , \tag{146}$$

$$\boldsymbol{R}_1^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{R}_1^{\mathrm{r}} + \boldsymbol{R}_1^{\mathrm{f}}] , \tag{147}$$

$$\boldsymbol{R}_1^{\mathrm{s}} = \boldsymbol{R}_1 - \frac{1}{1 + \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{R}_{\mathrm{r}} + \boldsymbol{R}_2\boldsymbol{R}_{\mathrm{r}} - \boldsymbol{R}_{\mathrm{r}}\boldsymbol{R}_2] , \tag{148}$$

$$(\boldsymbol{\omega}_1^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_1^{\mathrm{s}} = \frac{1}{1 + \boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1 + (\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)^2] . \tag{149}$$

We are now in a position to establish the rotation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}}$ and write

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}} = \boldsymbol{I}_3 + \frac{2}{1+(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}}}[\boldsymbol{R}_{\mathrm{I}}^{\mathrm{s}} + (\boldsymbol{R}_{\mathrm{I}}^{\mathrm{s}})^2] . \tag{150}$$

Thus, an alternative expression for $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{s}} = \boldsymbol{T}^{\mathrm{s}}$ may be set up in the form

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{s}} = \boldsymbol{T}(\boldsymbol{\omega}^{\mathrm{s}}) = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}}\boldsymbol{T}_2 . \tag{151}$$

It is relatively straightforward but demands some skill to prove formally the identity of $\boldsymbol{T}(\boldsymbol{\omega}^{\mathrm{s}})$ of (134) with $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}}$ and $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{s}}$ of (144) and (151). However, the most elegant procedure is that in the appended note to Subsection 3.1 involving the injection and ejection of $\boldsymbol{\omega}^{\mathrm{s}}$ as an eigenvector of $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}}$ and $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{s}}$, respectively.

The reader will observe that while the terminal points of I^{s} and II^{s} as well as that deduced from a direct application of $\boldsymbol{\omega}^{\mathrm{s}}$ are the same, this does not hold, of course, for the complete intermediate movements or paths.

We conclude this section by considering how $\boldsymbol{T}^{\mathrm{s}}$ can be related to $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$ $(= \boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}})$ and $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ $(= \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}})$ as derived from $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}}$ $(= \boldsymbol{R}_{\mathrm{II}}^{\mathrm{f}})$ and $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{f}}$ $(= \boldsymbol{R}_{\mathrm{II}}^{\mathrm{r}})$ of (76) and (107), respectively. To achieve this deduction in an efficient manner, at least for the restricted case of (128), we expand (78) and (108) for $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ and $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$ by inserting within the square brackets a zero column expressed by $\boldsymbol{R}_1\boldsymbol{\omega}_1 = \boldsymbol{R}_2\boldsymbol{\omega}_2 = \boldsymbol{o}_3$ of (82) to obtain

$$\begin{aligned}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} &= \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2]] = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{I}_3 - \boldsymbol{R}_1][\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2] ,\\ \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} &= \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{R}_1[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2]] = \frac{1}{1-\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{I}_3 + \boldsymbol{R}_1][\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2] ,\end{aligned} \tag{152}$$

which are not only attractive but physically suggestive expressions; note that one may substitute in any of the formulae $-\boldsymbol{R}_2$ for $\boldsymbol{R}_1$. Using (128), (152) becomes

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}} = [\boldsymbol{I}_3 - \boldsymbol{R}_1]\boldsymbol{\omega}^{\mathrm{s}} , \qquad \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}} = [\boldsymbol{I}_3 + \boldsymbol{R}_1]\boldsymbol{\omega}^{\mathrm{s}} . \tag{153}$$

On the other hand we have the identities

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} \quad \text{and} \quad \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} . \tag{154}$$

Adding the equations in (154) and applying (128) on the left-hand side and (153) on the right-hand side, we obtain

$$\boldsymbol{\omega}^{\mathrm{s}} = \boldsymbol{T}^{\mathrm{s}}\boldsymbol{\omega}^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}[\boldsymbol{I}_3 - \boldsymbol{R}_1] + \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}[\boldsymbol{I}_3 + \boldsymbol{R}_1]]\boldsymbol{\omega}^{\mathrm{s}} . \tag{155}$$

We immediately deduce

$$\boldsymbol{T}^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}[\boldsymbol{I}_3 - \boldsymbol{R}_1] + \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}[\boldsymbol{I}_3 + \boldsymbol{R}_1]] , \tag{156}$$

which impresses itself upon one's mind by its neatness. The reader may, of course, substitute $\boldsymbol{T}^{\mathrm{f}}_{\mathrm{II}}$ and $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{II}}$ for $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}$ and $\boldsymbol{T}^{\mathrm{f}}_{\mathrm{I}}$, respectively.

4. A geometrical interpretation of compound finite rotations

We round off our discourse on large rotations with an elementary geometrical interpretation of compound rotations. To this purpose it is helpful to follow up such motions on a unit sphere. In fig. 6 we show a unit sphere with its centre at O and two intersecting follower axes of rotation OR_1 and OR_2—welded to the sphere and enclosing an angle γ—to which we assign finite rotations $\boldsymbol{\vartheta}_1$ and $\boldsymbol{\vartheta}_2$, respectively. It suffices incidentally to restrict our attention to intersecting axes. As detailed in Subsections 3.1 and 3.2 the sequential order of rotations may be either $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_2$ (i.e. I^{f}) or $\boldsymbol{\vartheta}_2$, $\boldsymbol{\vartheta}_1$ (i.e. II^{f}) and yields in each case a compound pseudovector $\boldsymbol{\vartheta}$ pointing in a different direction. In what follows we concentrate on sequential order I^{f}.

As depicted in Fig. 6 the axes OR_1 and OR_2 pierce the unit sphere at Γ and Δ, respectively. Let $\Gamma\Delta = \gamma$ be an arc of a great circle through Γ and Δ. Applying initially a rotation $\boldsymbol{\vartheta}_1$ about OR_1 we observe that Δ moves to Δ_{I} along a circular path in a plane normal to OR_1. At the same time the great circular arc $\Gamma\Delta$ moves to $\Gamma\Delta_{\mathrm{I}}$ and the axis OR_1 is transferred to OR_{21}. If we next impose a rotation $\boldsymbol{\vartheta}_2$ about the axis OR_{21} we note that the point Γ proceeds to Γ_{I} and the arc $\Gamma\Delta_{\mathrm{I}}$ to $\Gamma_{\mathrm{I}}\Delta_{\mathrm{I}} = \gamma$. In order to locate the compound axis of rotation OE_{I} we perceive that the triangle $\Delta E_{\mathrm{I}}\Delta_{\mathrm{I}}$ must be isosceles since a rotation about OE_{I} must bring the arc $E_{\mathrm{I}}\Delta$ into coincidence with the arc $E_{\mathrm{I}}\Delta_{\mathrm{I}}$. (For the same reason the triangle $\Delta\Gamma\Delta_{\mathrm{I}}$ is isosceles since it is generated by a rotation about $O\Gamma$.) It follows that the axis OE_{I} must lie in a plane $O\Gamma Z$ through O bisecting the angle ϑ_1. By the same argument the axis OE_{I} must reside also in a plane $O\Delta_{\mathrm{I}}H$ through O bisecting the angle $\Gamma\Delta_{\mathrm{I}}\Gamma_{\mathrm{I}} = \vartheta_2$ at Δ_{I}. Otherwise a rotation about OE_{I} could not merge the arc $E_{\mathrm{I}}\Gamma$ with $E_{\mathrm{I}}\Gamma_{\mathrm{I}}$. Thus OE_{I} must coincide with the line of intersection of the planes $O\Gamma Z$ and $O\Delta H$. If the order of rotations is reversed into the order II^{f}, the kinematic result is different as reproduced in Fig. 7. Figs. 6 and 7 indicate also the unknown compound angle of rotation ϑ (clearly the same for cases I^{f} and II^{f}) operating at the root E_{I} (E_{II}) of the isosceles spherical triangles $\Delta E_{\mathrm{I}}\Delta_{\mathrm{I}}$ ($\Delta E_{\mathrm{II}}\Delta_{\mathrm{II}}$) and $\Gamma E_{\mathrm{I}}\Gamma_{\mathrm{I}}$ ($\Gamma E_{\mathrm{II}}\Gamma_{\mathrm{II}}$). In each case the planes $O\Gamma Z$ ($O\Gamma_{\mathrm{II}}Z$) and $O\Delta_{\mathrm{I}}H$ ($O\Delta H$) bisect the angle ϑ.

Reverting again to the compound rotation I^{f} of Fig. 6 we observe that arcs $\Gamma E_{\mathrm{I}}Z$, $\Delta_{\mathrm{I}}E_{\mathrm{I}}H$, $\Gamma\Delta$ are arcs of great circles since they are generated by planes through the centre O. We also note that the spherical triangle $\Gamma E_{\mathrm{I}}\Delta \equiv \Gamma_{\mathrm{I}}E_{\mathrm{I}}\Delta_{\mathrm{I}}$ includes at Γ (Γ_{I}) and Δ (Δ_{I}) the angles $\vartheta_1/2$ and $\vartheta_2/2$ and at E_{I} the angle $\pi - \vartheta/2$. For conciseness we introduce the following connotations of the arcs,

$$\Delta E = \alpha\,, \qquad E\Gamma = \beta\,, \qquad \Gamma\Delta = \gamma\,. \tag{157}$$

The reader will observe that in the present context the angles ϑ_1, ϑ_2 and γ are known.

Applying now the sine proposition to the spherical triangle $\Gamma E_{\mathrm{I}}\Delta$ and noting that $\sin(\pi - \vartheta/2) = \sin \vartheta/2$ we obtain

$$\frac{\sin \vartheta_1/2}{\sin \alpha} = \frac{\sin \vartheta_2/2}{\sin \beta} = \frac{\sin \vartheta/2}{\sin \gamma}\,. \tag{158}$$

Hence we deduce the angles α, β and the position of the axis OE_I from

$$\sin \alpha = \frac{\sin \vartheta_1/2}{\sin \vartheta/2} \sin \gamma \,, \qquad \sin \beta = \frac{\sin \vartheta_2/2}{\sin \vartheta/2} \sin \gamma \tag{159}$$

in which ϑ is still unknown. In order to determine ϑ we consider the cosine proposition in the spherical triangle $\Gamma\Delta E_\mathrm{I}$. Noting $\cos(\pi - \vartheta/2) = -\cos \vartheta/2$ we have

$$\cos \vartheta/2 = \cos \vartheta_1/2 \cos \vartheta_2/2 - \sin \vartheta_1/2 \sin \vartheta_2/2 \cos \gamma \tag{160}$$

which yields the last unknown ϑ and confirms (87).

This completes the elementary analysis of the sequential order I^f. The reader will observe that (157) to (160) apply also to the sequential order II^f of Fig. 7 but some or all of the angles α, β are complementary to those of order I^f. As observed in Figs. 6 and 7 and discussed also in Subsections 3.1 and 3.2, the *magnitude* of $\boldsymbol{\vartheta}$ is unaffected by the adopted sequence.

To simplify the reading of Figs. 6 and 7 we apply a special colour scheme. Thus, blue denotes the first and red the second rotation whilst green indicates the movement of the compound rotation $\boldsymbol{\vartheta}$ which transports the sphere in one go to its final position.

To conclude this section we consider briefly the special problem arising when the axes OR_1 and OR_2 are parallel. The argument proceeds as in the more general previous case but is clearly simpler and is reproduced for either sequence I^f or II^f in Fig. 8. Considering the sequential order I we observe immediately from the depicted construction that

$$\text{angle } \Gamma E_\mathrm{I} H = \vartheta_1/2 + \vartheta_2/2 \,,$$

$$\text{angle } \Gamma E_\mathrm{I} \Gamma_\mathrm{I} = 2(\text{angle } \Gamma E_\mathrm{I} H) = \vartheta_1 + \vartheta_2 = \vartheta \,.$$

Thus, as stated in (73), two rotations about parallel axes are composed into a single rotation equal to their algebraic sum. Clearly the sequential order does not affect again the result. However, the axis of rotation is in the two cases *not* identical. Considering the sine rule for the triangle $\Gamma E_\mathrm{I} \Delta$ we have

$$\frac{E_\mathrm{I}\Delta}{\sin \vartheta_1/2} = \frac{E_\mathrm{I}\Gamma}{\sin \vartheta_2/2} = \frac{\Gamma\Delta}{\sin(\vartheta_1 + \vartheta_2)/2} \tag{161}$$

from which we deduce the distances $E_\mathrm{I}\Delta$ and $E_\mathrm{I}\Gamma$ and hence the centre of rotation E_I. Fig. 8 includes also the construction for the sequential order II^f and indicates that E_II is the mirror image of E_I with respect to $\Gamma\Delta$. The adopted colour scheme is once more that of Figs. 6 and 7.

5. Some illustrations of the theory

In this section we present a brief account of some simple applications of the preceding theory. Three distinct specifications of axes are considered being in turn, fixed in space, follower and semitangential. In each case two sequences of rotations $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_2$ and $\boldsymbol{\vartheta}_2$, $\boldsymbol{\vartheta}_1$ are analysed. In the concluding Section 6 we analyse a more ambitious example.

In the present example the prescribed rotations about fixed axes are

$$\boldsymbol{\vartheta}_1 = \tfrac{1}{2}\pi\{0 \quad 0 \quad 1\} = \tfrac{1}{2}\pi\, \boldsymbol{e}_z\,, \qquad \boldsymbol{\vartheta}_2 = \tfrac{1}{2}\pi\{0 \quad 1 \quad 0\} = \tfrac{1}{2}\pi\, \boldsymbol{e}_y\,, \tag{162}$$

in which $\boldsymbol{e}_z$ and $\boldsymbol{e}_y$ define the unit vectors in the Oz and Oy directions. In the present case

$$\tan \vartheta_1/2 = \tan \vartheta_2/2 = 1\,. \tag{163}$$

The alternative definition $\boldsymbol{\omega}$ of the rotations is hence

$$\boldsymbol{\omega}_1 = \{0 \quad 0 \quad 1\} = \boldsymbol{e}_z\,, \qquad \boldsymbol{\omega}_2 = \{0 \quad 1 \quad 0\} = \boldsymbol{e}_y\,. \tag{164}$$

We observe that

$$\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2 = 1\,, \qquad \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2 = 0\,. \tag{165}$$

Before proceeding to the analysis of the various examples it is helpful to list here the auxiliary matrices $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ and the rotation matrices $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ associated with the pseudovectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ ($\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_2$).

In Figs. 9a, 9b we observe in black the initial or fixed coordinate system $Oxyz$ and in green its final position following the rotation $\boldsymbol{\vartheta}_1$ or $\boldsymbol{\vartheta}_2$. As a result of the simple assumptions in (162) it is in the present case a trivial task to set up the rotation matrices by inspection. For example, we observe that the rows of $\boldsymbol{T}$ contain the direction cosines of the black with respect to the green axes or alternatively the columns of $\boldsymbol{T}$ represent the direction cosines of green with respect to black axes. Finally, it suffices also to note the correspondence of the black and green axes. In any case we obtain

$$\boldsymbol{T}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{T}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{166}$$

To exemplify the more formal process we may apply (16) or (26) using $\boldsymbol{S}$ or $\boldsymbol{R}$ matrices. For example, we deduce from (164)

$$\boldsymbol{R}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{R}_1^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{167}$$

$$\boldsymbol{R}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{R}_2^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{168}$$

Substitution in (26) confirms (166). We also show in Figs. 9a and 9b in black a vector $\boldsymbol{p}$ in its initial position

$$\boldsymbol{p} = \{0 \quad -1 \quad 1\} \tag{169}$$

and in green in its final position; p_1 and p_2, respectively. We have

$$p_1 = T_1 p = \{1 \quad 0 \quad 1\} \tag{170}$$

and

$$p_2 = T_2 p = \{1 \quad -1 \quad 0\}\,. \tag{171}$$

5.1. Compound rotation about fixed axes

We start with the simplest of all rotational provisions in which the pseudovectors $\boldsymbol{\vartheta}$ and $\boldsymbol{\omega}$ act along axes fixed in space. Figs. 10 and 11 demonstrate in a colour scheme the sequences $\mathrm{I}^{\mathrm{r}}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ and $\mathrm{II}^{\mathrm{r}}(\boldsymbol{\omega}_2, \boldsymbol{\omega}_1)$ of the system $Oxyz$ and of the vector $\boldsymbol{p}$ of (169). In particular, we show in black the initial position of $\boldsymbol{p}$ and of $Oxyz$, which is assigned at the same time the role of the fixed system. The subsequent two positions are marked in turn in blue and red, respectively, the blue representing the movement and terminal due to the first rotation $\boldsymbol{\vartheta}_1$ ($\boldsymbol{\vartheta}_2$) and the red the movement and terminal due to the second rotation $\boldsymbol{\vartheta}_2$ ($\boldsymbol{\vartheta}_1$) each time for sequence I^{r} (II^{r}). Finally, we also display in green the path followed by the tip of $\boldsymbol{p}$ if it were to reach the final position in one go. This is also shown in more detail in Figs. 12 and 13. Note that the above colour scheme applies to all Figs. 10 to 26; see also Figs. 6, 7, 8.

We proceed now to a detailed discussion of sequences I^{r} and II^{r}.

5.1.1. Sequence I^{r}

To find the rotation matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$ we apply (36),

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{T}_2 \boldsymbol{T}_1\,, \tag{36}$$

and use (166) for the constituent matrices. We obtain

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{172}$$

By inspection of Fig. 9 the matrix $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$ may also be set up directly using, for example, the direction cosines of 'black' with respect to 'red' axes. Alternatively, we may establish $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}$ in a single operation by eliciting first the terminal pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ of (78),

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}} \boldsymbol{\omega}_2} [\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1 \boldsymbol{\omega}_2]\,. \tag{173}$$

Noting (164), (165) and (166) we find

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \{1 \quad 1 \quad 1\} \tag{174}$$

and hence

$$(\boldsymbol{\omega}_{\rm I}^{\rm r})^{\rm t}(\boldsymbol{\omega}_{\rm I}^{\rm r}) = 3\,, \tag{175}$$

which is confirmed by (86a).

Next we rewrite $\boldsymbol{\omega}_{\rm I}^{\rm r}$ of (174) in the standard form

$$\boldsymbol{\omega}_{\rm I}^{\rm r} = \tan\frac{\vartheta_{\rm I}}{2}\,\boldsymbol{e} = \sqrt{3}\{\tfrac{1}{3}\sqrt{3} \quad \tfrac{1}{3}\sqrt{3} \quad \tfrac{1}{3}\sqrt{3}\} \tag{176}$$

and find for the compound angle of rotation $\vartheta_{\rm I}$,

$$\tan \vartheta_{\rm I}/2 = \sqrt{3}\,, \qquad \vartheta_{\rm I}/2 = \pi/3\,, \qquad \vartheta_{\rm I} = 2\pi/3\,. \tag{177}$$

Incidentally, to refresh our memory we also note that the pseudovector $\boldsymbol{\vartheta}_{\rm I}^{\rm r}$ reads

$$\boldsymbol{\vartheta}_{\rm I}^{\rm r} = \tfrac{2}{3}\pi\boldsymbol{e} = \tfrac{2}{9}\pi\sqrt{3}\{1 \quad 1 \quad 1\}\,. \tag{178}$$

The auxiliary matrix $\boldsymbol{R}_{\rm I}^{\rm r}$ derives immediately from (174) as

$$\boldsymbol{R}_{\rm I}^{\rm r} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \tag{179}$$

which may also be deduced from (76) in association with (167) and (168).

Applying now (26) in the form

$$\boldsymbol{T}_{\rm I}^{\rm r} = \boldsymbol{I}_3 + \frac{2}{1 + (\boldsymbol{\omega}_{\rm I}^{\rm r})^{\rm t}\boldsymbol{\omega}_{\rm I}^{\rm r}}[\boldsymbol{R}_{\rm I}^{\rm r} + (\boldsymbol{R}_{\rm I}^{\rm r})^2] \tag{180}$$

and using (175) and (179) we confirm expression (172).

5.1.2. Sequence $\mathrm{II}^{\rm r}$

We apply in the present case (37), which reads

$$\boldsymbol{T}_{\rm II}^{\rm r} = \boldsymbol{T}_1\boldsymbol{T}_2 \tag{37}$$

and use (166) to obtain

$$\boldsymbol{T}_{\rm II}^{\rm r} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \tag{181}$$

which may also be confirmed by inspection of Fig. 10. Alternatively we proceed via $\boldsymbol{\omega}_{\rm II}^{\rm r}$ as exemplified above for the sequence $\mathrm{I}^{\rm r}$. Applying (88) in the form

$$\boldsymbol{\omega}_{\rm II}^{\rm r} = \frac{1}{1 - \boldsymbol{\omega}_1^{\rm t}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{R}_1\boldsymbol{\omega}_2] \tag{182}$$

we find, using (164), (165) and (166),

$$\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{II}} = \{-1 \quad 1 \quad 1\} \tag{183}$$

and

$$(\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{II}})^{\mathrm{t}}\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{II}} = 3\,, \tag{184}$$

in agreement with (173). Eq. (183) indicates immediately that

$$\tan \vartheta^{\mathrm{r}}_{\mathrm{II}}/2 = \tan \vartheta^{\mathrm{r}}_{\mathrm{I}}/2$$

and hence, as to be expected,

$$\vartheta^{\mathrm{r}}_{\mathrm{II}} = \vartheta^{\mathrm{r}}_{\mathrm{I}} = 2\pi/3\,. \tag{185}$$

There follows for the vector $\boldsymbol{\vartheta}^{\mathrm{r}}_{\mathrm{II}}$,

$$\boldsymbol{\vartheta}^{\mathrm{r}}_{\mathrm{II}} = \tfrac{2}{9}\pi\sqrt{3}\{-1 \quad 1 \quad 1\}\,. \tag{178a}$$

The associated auxiliary matrix $\boldsymbol{R}^{\mathrm{r}}_{\mathrm{II}}$ reads

$$\boldsymbol{R}^{\mathrm{r}}_{\mathrm{II}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}. \tag{186}$$

This is in line with (76) rewritten for the sequential order II^{r}.

Applying now (26) in the form

$$\boldsymbol{T}^{\mathrm{r}}_{\mathrm{II}} = \boldsymbol{I}_3 + \frac{2}{1 + (\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{II}})^{\mathrm{t}}\boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{II}}}\,[\boldsymbol{R}^{\mathrm{r}}_{\mathrm{II}} + (\boldsymbol{R}^{\mathrm{r}}_{\mathrm{II}})^2] \tag{187}$$

we obtain once more the matrix (181).

5.2. Compound rotation about follower axes

We consider again the same example but assume this time a follower character of the axes. As a result the axis of the second rotation is transported to its operating location by the first rotation. Following our theory in Section 2 and (50), (56),

$$\boldsymbol{\omega}^{\mathrm{f}}_{\mathrm{I}} = \boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{II}}\,, \quad \boldsymbol{T}^{\mathrm{f}}_{\mathrm{I}} = \boldsymbol{T}^{\mathrm{r}}_{\mathrm{II}}\,, \qquad \boldsymbol{\omega}^{\mathrm{f}}_{\mathrm{II}} = \boldsymbol{\omega}^{\mathrm{r}}_{\mathrm{I}}\,, \quad \boldsymbol{T}^{\mathrm{f}}_{\mathrm{II}} = \boldsymbol{T}^{\mathrm{r}}_{\mathrm{I}}\,. \tag{188}$$

Hence, the conclusions of Subsection 5.1 are immediately applicable to the present case. Thus, it may suffice if we demonstrate here the equivalence by using the alternative (42), (44) of Section 2. In order not to overload Figs. 10 and 11 we reproduce the sequences I^{f} and II^{f}

separately in Figs. 14 and 15. The same kind of presentation and colour scheme are adopted. The reader's attention is also drawn to Figs. 12, 13 which are equally valid for II^{f} and I^{f}.

In what follows we refer briefly and separately to the application of (42) and (44).

5.2.1. Sequence I^{f}

We apply here (42),

$$\boldsymbol{T}^{\mathrm{f}}_{\mathrm{I}} = \boldsymbol{T}_{21}\boldsymbol{T}_1\,; \tag{42}$$

to evaluate it we only have to establish $\boldsymbol{T}_{21}$. To this purpose we require $\boldsymbol{\omega}^{\mathrm{f}}_2 = \boldsymbol{\omega}_{21}$ or directly $\boldsymbol{R}^{\mathrm{f}}_2 = \boldsymbol{R}_{21}$. Using (98) or (41) we find

$$\boldsymbol{\omega}^{\mathrm{f}}_2 = \boldsymbol{\omega}_{21} = \{-1 \quad 0 \quad 0\} \tag{189}$$

which is evident from Fig. 12 since the original rotation $\boldsymbol{\omega}_2$ about the y-axis becomes, as a result of the action of $\boldsymbol{\omega}_1$, an equal and opposite rotation about the original (black) x-axis.

Clearly,

$$\boldsymbol{R}^{\mathrm{f}}_2 = \boldsymbol{R}_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad \boldsymbol{R}^2_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{190}$$

(see also (101) for a direct derivation of $\boldsymbol{R}_{21}$). Application of $\boldsymbol{R}_{21}$ in $\boldsymbol{T}_{21}$ in accordance with (26) yields

$$\boldsymbol{T}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{191}$$

Substitution of (191) for $\boldsymbol{T}_{21}$ and (166) for $\boldsymbol{T}_1$ in (42) confirms immediately the result of $\boldsymbol{T}^{\mathrm{r}}_{\mathrm{II}} = \boldsymbol{T}^{\mathrm{f}}_{\mathrm{I}}$ in (181).

It should be observed that while the final locations and green paths of sequences II^{r} and I^{f} are the same, this does not hold, in general, for the intermediate paths in blue and red.

5.2.2. Sequence II^{f}

We apply here (44),

$$\boldsymbol{T}^{\mathrm{f}}_{\mathrm{II}} = \boldsymbol{T}_{12}\boldsymbol{T}_2\,. \tag{44}$$

Using for $\boldsymbol{\omega}_{12}$ (108) or (43) we find

$$\boldsymbol{\omega}^{\mathrm{f}}_1 = \boldsymbol{\omega}_{12} = \{1 \quad 0 \quad 0\} \tag{192}$$

which is confirmed in Fig. 13.

Clearly,

$$\boldsymbol{R}_1^{\mathrm{f}} = \boldsymbol{R}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \boldsymbol{R}_{12}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{193}$$

(see also (109) for a direct derivation of $\boldsymbol{R}_{12}$). Application of $\boldsymbol{R}_{12}$ in $\boldsymbol{T}_{12}$ in accordance with (26) yields

$$\boldsymbol{T}_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{194}$$

Substitution of (194) for $\boldsymbol{T}_{12}$ and (106) for $\boldsymbol{T}_2$ confirms immediately the result for $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}}$ in (172).

The same observations made in Subsection 5.2.1 concerning final locations and green paths on the one hand and blue and red paths on the other hand apply here also for sequences I^{r} and II^{f}.

5.3. Compound rotations with semitangential rotations

In this subsection we take up once more the example of Subsections 5.1 and 5.2 but assume this time that the rotations are applied in the semitangential manner proposed in Subsection 3.3. As a result of this specification of rotations we must obtain the same ultimate location however we proceed, be it on one of the paths I^{s} and II^{s} or the direct approach via $\boldsymbol{\omega}^{\mathrm{s}}$. It goes without saying that while the terminal point and the final $\boldsymbol{\omega}^{\mathrm{s}}$ are in each case the same, this does not hold for the intermediate paths of the sequences I^{s} and II^{s}.

In view of the characteristic properties of semitangential rotations it appears preferable to give an integrated account and not one under separate headings as in Subsections 5.1 and 5.2. All numerical results may be observed in Figs. 16, 17 and 18 in which the same presentation and colour schemes are applied as in the preceding subsections. In particular, we start with the analysis of path I^{s}, corroborate its findings via the path II^{s} and obtain the ultimate confirmation via the compound pseudovector $\boldsymbol{\omega}^{\mathrm{s}}$.

When proceeding along I^{s}, the essential additional information concerns the pseudovector $\boldsymbol{\omega}_2^{\mathrm{s}}$ of (121) and its associated auxiliary matrix $\boldsymbol{R}_2^{\mathrm{s}}$ of (140) or (141).

Using $\boldsymbol{\omega}_2 = \boldsymbol{\omega}_2^{\mathrm{r}}$ of the second equation in (164) and $\boldsymbol{\omega}_2^{\mathrm{f}}$ of (189) we find

$$\boldsymbol{\omega}_2^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{\omega}_2^{\mathrm{r}} + \boldsymbol{\omega}_2^{\mathrm{f}}] = \tfrac{1}{2}\{-1 \quad 1 \quad 0\} \tag{195}$$

which may also be confirmed from (139).

Hence, either directly from (140) in association with (168) and (190) or alternatively from (141),

$$\boldsymbol{R}_2^{\mathrm{s}} = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}. \tag{196}$$

We also obtain from (143) or directly from (195),

$$(\boldsymbol{\omega}_2^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{s}} = \tfrac{1}{2}\,. \tag{197}$$

We are now in a position to establish the rotation matrix $\boldsymbol{T}_2^{\mathrm{s}}$ via (26) in the form

$$\boldsymbol{T}_2^{\mathrm{s}} = \boldsymbol{I}_3 + \frac{2}{1+(\boldsymbol{\omega}_2^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_2^{\mathrm{s}}}[\boldsymbol{R}_2^{\mathrm{s}} + (\boldsymbol{R}_2^{\mathrm{s}})^2]\,. \tag{198}$$

Substituting (196) and (197) in (198) we find

$$\boldsymbol{T}_2^{\mathrm{s}} = \tfrac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ -2 & -2 & 1 \end{bmatrix}. \tag{199}$$

There follows from (166) and (199),

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}} = \boldsymbol{T}_2^{\mathrm{s}}\boldsymbol{T}_1 = \tfrac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}. \tag{200}$$

Before leaving the path I^{s} we rewrite $\boldsymbol{\omega}_2^{\mathrm{s}}$ of (195) in the standard form

$$\boldsymbol{\omega}_2^{\mathrm{s}} = \tan \vartheta_2^{\mathrm{s}}/2\; \boldsymbol{e} = \tfrac{1}{2}\sqrt{2}\{-\tfrac{1}{2}\sqrt{2} \quad \tfrac{1}{2}\sqrt{2} \quad 0\}\,. \tag{201}$$

Thus,

$$\tan \vartheta_2^{\mathrm{s}}/2 = \tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \vartheta_2^{\mathrm{s}} = 0.39183\pi\,. \tag{202}$$

We now proceed to the path II^{s} and the derivation of $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{s}}$ which must agree with $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}}$ of (200). Having $\boldsymbol{T}_2$ in (166) it is only necessary to establish $\boldsymbol{T}_1^{\mathrm{s}}$ of (150).

Following (145) we have

$$\boldsymbol{\omega}_1^{\mathrm{s}} = \tfrac{1}{2}[\boldsymbol{\omega}_1^{\mathrm{r}} + \boldsymbol{\omega}_1^{\mathrm{f}}] = \tfrac{1}{2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2]\,. \tag{203}$$

Applying the first equations in (164) and (192), or directly from (146), we obtain

$$\boldsymbol{\omega}_1^{\mathrm{s}} = \tfrac{1}{2}\{1 \quad 0 \quad 1\}\,. \tag{204}$$

Hence, we find (alternatively we may apply (147) or (148)),

$$\boldsymbol{R}_1^{\mathrm{s}} = \tfrac{1}{2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{205}$$

We also derive from (204) or directly from (149),

$$(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}} = \tfrac{1}{2}\,. \tag{206}$$

Next we write in extension of (26),

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}} = \boldsymbol{I}_3 + \frac{2}{1+(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}}}[\boldsymbol{R}_{\mathrm{I}}^{\mathrm{s}} + (\boldsymbol{R}_{\mathrm{I}}^{\mathrm{s}})^2]\,. \tag{207}$$

Substituting (205) and (206) in (207) we determine

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}} = \tfrac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}. \tag{208}$$

There follows from (166) and (208) that

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{s}} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}}\boldsymbol{T}_2 = \tfrac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}, \tag{209}$$

which is in agreement with (200).

By writing (204) in the standard form we note that

$$\tan \vartheta_{\mathrm{I}}^{\mathrm{s}}/2 = \tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \vartheta_{\mathrm{I}}^{\mathrm{s}} = 0.39183\pi\,. \tag{210}$$

We finally rehearse the simplest of all procedures involving the direct evaluation of the compound pseudovector $\boldsymbol{\omega}^{\mathrm{s}}$ via (130). We find

$$\boldsymbol{\omega}^{\mathrm{s}} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \{0 \quad 1 \quad 1\}\,. \tag{211}$$

Hence, or directly from (133), (167) and (168),

$$\boldsymbol{R}^{\mathrm{s}} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{212}$$

Thence using (134),

$$\boldsymbol{T}^{\mathrm{s}} = \boldsymbol{I}_3 + \frac{2}{1+(\boldsymbol{\omega}^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}^{\mathrm{s}}}[\boldsymbol{R}^{\mathrm{s}} + (\boldsymbol{R}^{\mathrm{s}})^2]\,, \tag{213}$$

in association with

$$(\boldsymbol{\omega}^{\mathrm{s}})^{\mathrm{t}}\boldsymbol{\omega}^{\mathrm{s}} = 2\,, \tag{214}$$

which derives from (135) or (211), we obtain by applying (212),

$$\boldsymbol{T}^{s} = \tfrac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \tag{215}$$

ensuring a triple agreement with (200) and (209).

It is of some interest to establish the auxiliary matrix $\boldsymbol{S}^{s}$ referred to in (137). To this purpose we first determine $\boldsymbol{\vartheta}^{s}$ by writing $\boldsymbol{\omega}^{s}$ of (211) in the standard form

$$\boldsymbol{\omega}^{s} = \tan \vartheta^{s}/2\, \boldsymbol{e} = \sqrt{2}\{0 \quad \tfrac{1}{2}\sqrt{2} \quad \tfrac{1}{2}\sqrt{2}\}\,. \tag{216}$$

Thus, $\tan \vartheta^{s}/2 = \sqrt{2}$ and $\vartheta^{s} = 0.60817\pi$. Writing (211) in the presently relevant form

$$\boldsymbol{R}^{s} = \frac{\tan \vartheta^{s}/2}{\vartheta^{s}}\boldsymbol{S}^{s} = \boldsymbol{R}_1 + \boldsymbol{R}_2\,, \tag{217}$$

we find

$$\boldsymbol{S}^{s} = 0.60817\pi\, \tfrac{1}{2}\sqrt{2}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{218}$$

There follows

$$\boldsymbol{T}^{s} = \exp \boldsymbol{S}^{s} = \exp \boldsymbol{S}_2^{s} \exp \boldsymbol{S}_1 = \exp \boldsymbol{S}_1^{s} \exp \boldsymbol{S}_2\,. \tag{219}$$

5.4. Pure geometrical approach of establishing $\boldsymbol{\vartheta}_{\mathrm{I}}^{\mathrm{f}}$, $\boldsymbol{\vartheta}_{\mathrm{II}}^{\mathrm{f}}$

We conclude the discussion of the example analysed in Subsections 5.1, 5.2 and 5.3 by referring briefly to the geometrical approach initiated in Section 4. For economy of space we restrict our considerations to the follower type of axes and the derivation of the compound pseudovectors $\boldsymbol{\vartheta}_{\mathrm{I}}^{\mathrm{f}}$, $\boldsymbol{\vartheta}_{\mathrm{II}}^{\mathrm{f}}$. The applied notation is basically that of Section 4.

We have for the present example

$$\vartheta_1 = \vartheta_2 = \tfrac{1}{2}\pi \quad \text{and} \quad \gamma = \tfrac{1}{2}\pi\,. \tag{220}$$

To establish the compound angle of rotation $\vartheta = \vartheta_{\mathrm{I}}^{\mathrm{f}} = \vartheta_{\mathrm{II}}^{\mathrm{f}}$ we apply (160) which reduces in the current case to

$$\cos \vartheta/2 = \cos \vartheta_1/2 \cos \vartheta_2/2 = \tfrac{1}{2}\sqrt{2}\,\tfrac{1}{2}\sqrt{2} = \tfrac{1}{2}\,. \tag{221}$$

Hence,

$$\tfrac{1}{2}\vartheta = \tfrac{1}{3}\pi \quad \text{and} \quad \vartheta_{\mathrm{I}}^{\mathrm{f}} = \vartheta_{\mathrm{II}}^{\mathrm{f}} = \tfrac{2}{3}\pi \tag{222}$$

which agrees with (177) and (185).

Also from (159),

$$\sin\alpha = \frac{\sin\vartheta_1/2}{\sin\vartheta/2}\sin\gamma = \sqrt{2/3}\,, \qquad \cos\alpha = \tfrac{1}{3}\sqrt{3}\,,$$
$$\sin\beta = \frac{\sin\vartheta_2/2}{\sin\vartheta/2}\sin\gamma = \sqrt{2/3}\,, \qquad \cos\beta = \tfrac{1}{3}\sqrt{3}\,. \tag{223}$$

There follows that α, β are either 54.73561° or the complementary 125.26349°.

In particular, we observe from the geometrical construction in Figs. 19 and 20 that the inclination of the compound pseudovectors $\boldsymbol{\vartheta}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{\vartheta}_{\mathrm{II}}^{\mathrm{r}}$, $\boldsymbol{\vartheta}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{\vartheta}_{\mathrm{I}}^{\mathrm{r}}$ with respect to the fixed axes $Oxyz$ are

$$\begin{cases} \pi-\alpha,\ \alpha,\ \alpha\,, & \text{for sequence } \mathrm{I}^{\mathrm{f}}\,,\\ \alpha,\ \alpha,\ \alpha\,, & \text{for sequence } \mathrm{II}^{\mathrm{f}}\,. \end{cases} \tag{224}$$

We deduce from (223) and (224) that

$$\begin{cases} \boldsymbol{\vartheta}_{\mathrm{I}}^{\mathrm{f}} = \tfrac{2}{9}\pi\sqrt{3}\{-1 \quad 1 \quad 1\}\,,\\ \boldsymbol{\vartheta}_{\mathrm{II}}^{\mathrm{f}} = \tfrac{2}{9}\pi\sqrt{3}\{1 \quad 1 \quad 1\}\,, \end{cases} \tag{225}$$

in agreement with the findings of Subsections 5.1 and 5.2 (see also (178a) and (178)).

6. A slightly more ambitious example

Following the very simple illustrations in Section 5 the reader should appreciate the complexities of a more ambitious example. Of particular importance is a nonorthogonality of the pseudovectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and the relative ease of applying even in a more intricate case the concept of semitangential rotations. It should suffice by now to limit the presentation to the essential points of the argument.

The prescribed pseudovectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ first taken about fixed axes are

$$\begin{aligned} \boldsymbol{\omega}_1 &= \tan\frac{\vartheta_1}{2}\,\boldsymbol{e}_1 = \tfrac{1}{3}\sqrt{3}\,\{\tfrac{1}{3}\sqrt{3} \quad \tfrac{1}{3}\sqrt{3} \quad -\tfrac{1}{3}\sqrt{3}\} = \tfrac{1}{3}\{1 \quad 1 \quad -1\}\,,\\ \boldsymbol{\omega}_2 &= \tan\frac{\vartheta_2}{2}\,\boldsymbol{e}_2 = 2\{\tfrac{3}{5} \quad -\tfrac{4}{5} \quad 0\} = \tfrac{2}{5}\{3 \quad -4 \quad 0\}\,. \end{aligned} \tag{226}$$

We observe that

$$\tan\frac{\vartheta_1}{2} = \tfrac{1}{3}\sqrt{3}\,, \quad \vartheta_1 = \tfrac{1}{3}\pi\,, \qquad \tan\frac{\vartheta_2}{2} = 2\,, \quad \vartheta_2 = 0.70483\pi\,. \tag{227}$$

Also

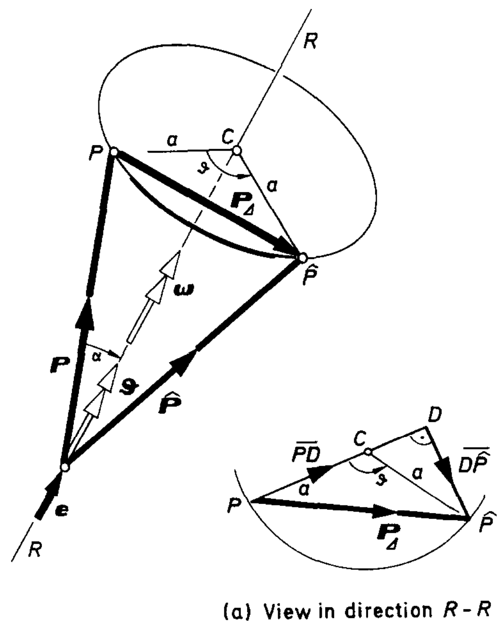

Fig. 1. Construction of transformation matrix $\boldsymbol{T}(\boldsymbol{\vartheta})$ for an arbitrary rotational pseudovector $\boldsymbol{\vartheta}$.

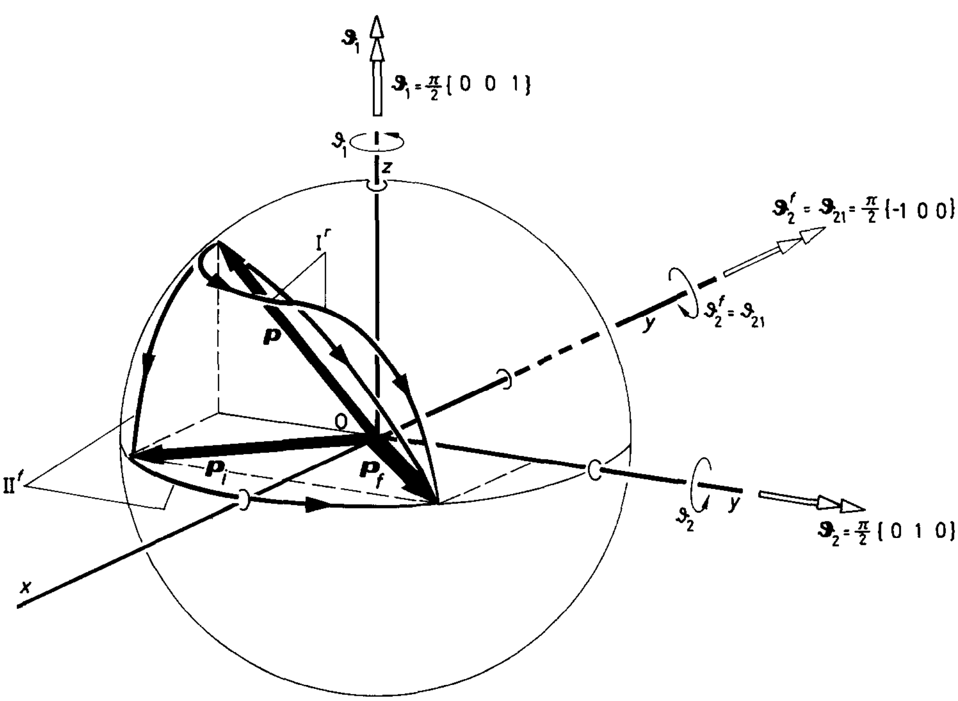

Fig. 2. Compound rotation I^f about follower axes composed of rotations $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_2^f = \boldsymbol{\vartheta}_{21}$. The point O is centre of sphere.

For subsequent reference also shown is compound rotation II^r composed of rotations $\boldsymbol{\vartheta}_2$, $\boldsymbol{\vartheta}_1$. Terminal point is for both sequences the same.

Kinematic effects are observed *within* each individual rotation by the transportation of a vector $\boldsymbol{p}$ initially given by $\boldsymbol{p} = \{0\ -1\ 1\}$. Axes vanishing *behind* the sphere are depicted with dotted lines. Adopted colour scheme:

First rotation is depicted in each case by blue, second rotation by red. Green indicates the path followed by the vector $\boldsymbol{p}$ in response to the equivalent single compound pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^f = \boldsymbol{\omega}_{\mathrm{II}}^r$ (discussed at subsequent stages of the article).

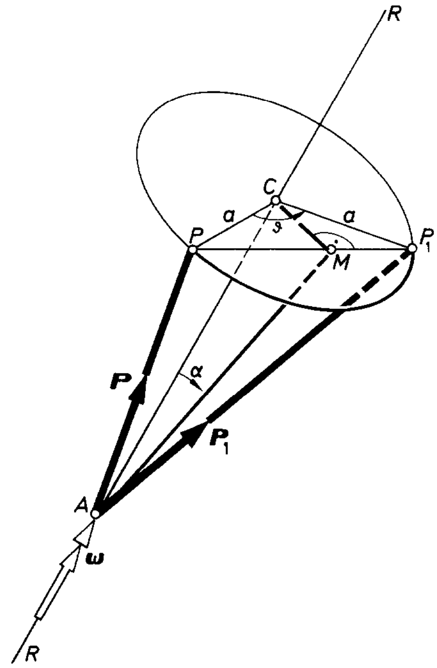

Fig. 3. On the vectorial-geometrical proof of $\boldsymbol{p}_1 - \boldsymbol{p} = \boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1)$ or $\boldsymbol{p}_1 - \boldsymbol{p} = \boldsymbol{R}[\boldsymbol{p} + \boldsymbol{p}_1]$.

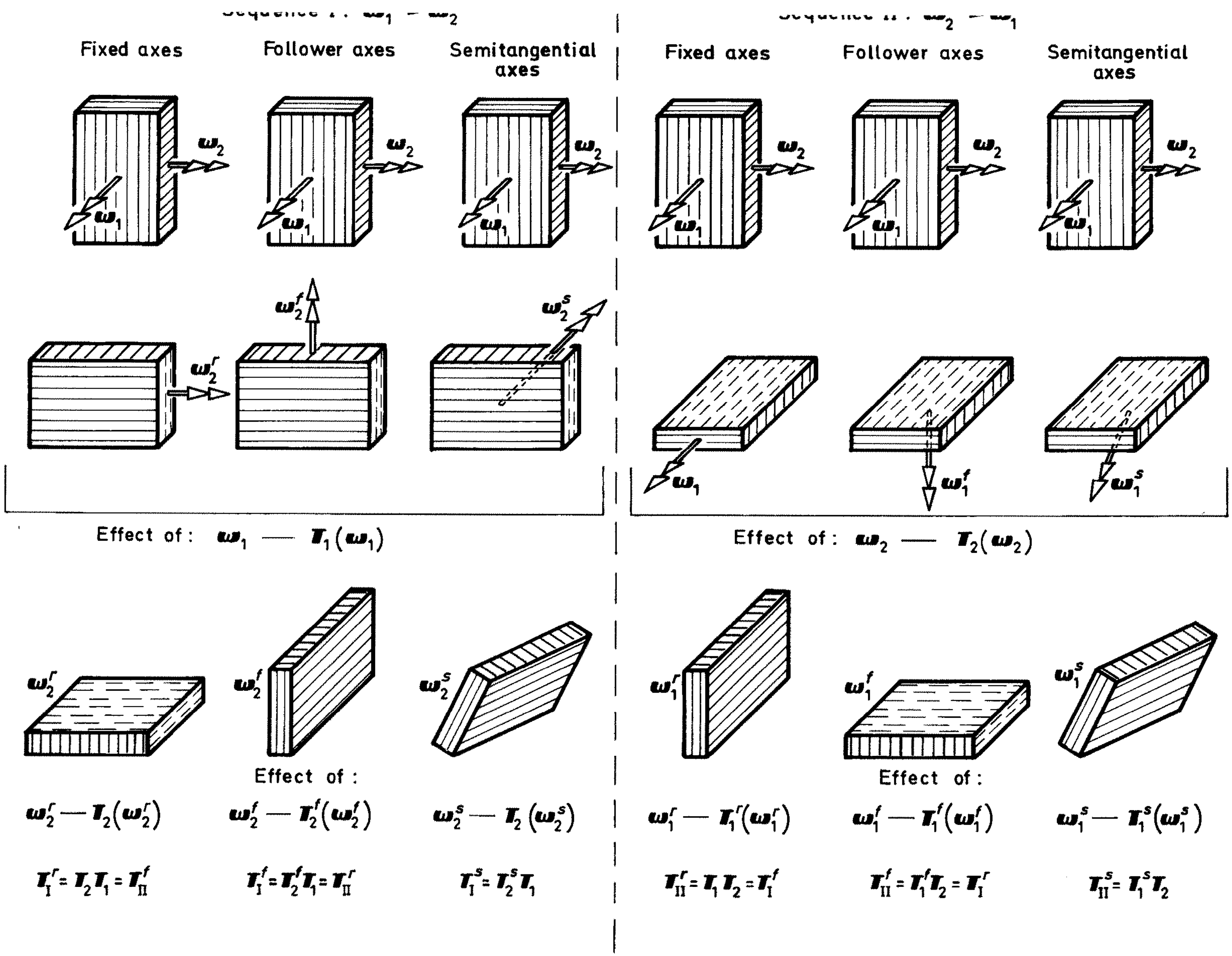

Fig. 4. Rotation of a brick under a compound rotation composed of prescribed rotations ϑ_1, ϑ_2 or ϑ_2, ϑ_1 (sequences I or II). Alternative systems and sequences of rotations for fixed, follower and semitangential type of axes. Continuous and discontinuous shading in the *same* colour depict *opposite* faces of the brick. Demonstration of $T_I^f = T_{II}^r$ and $T_{II}^f = T_I^r$.

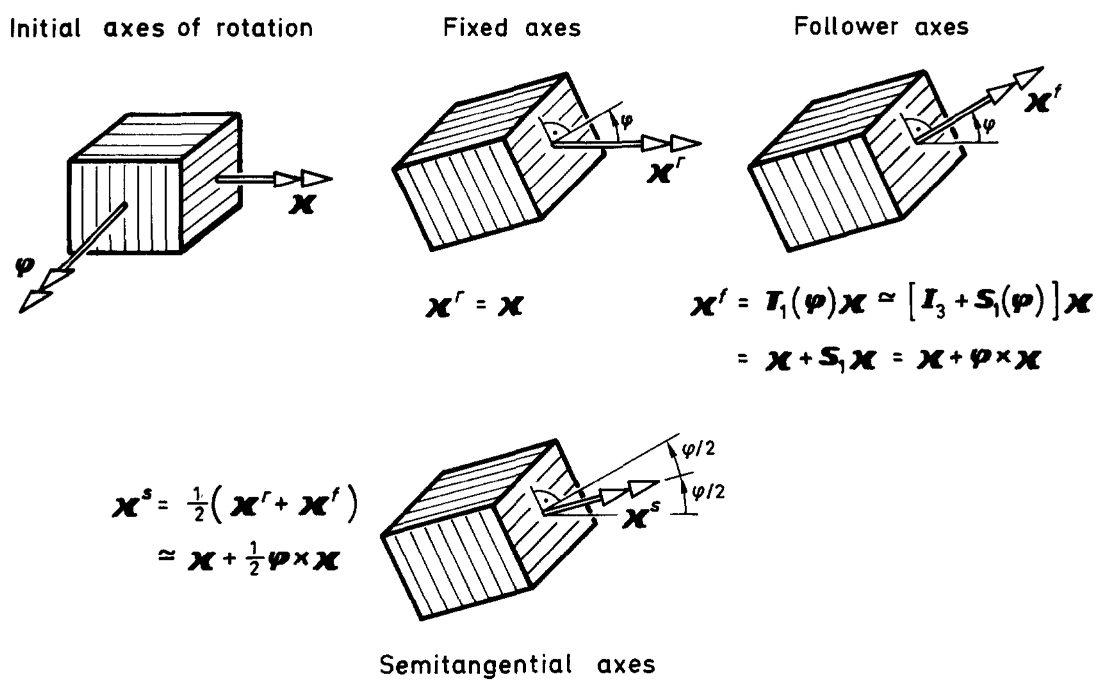

Fig. 5. On the concept of semitangential rotations (illustration on a brick for small angles).

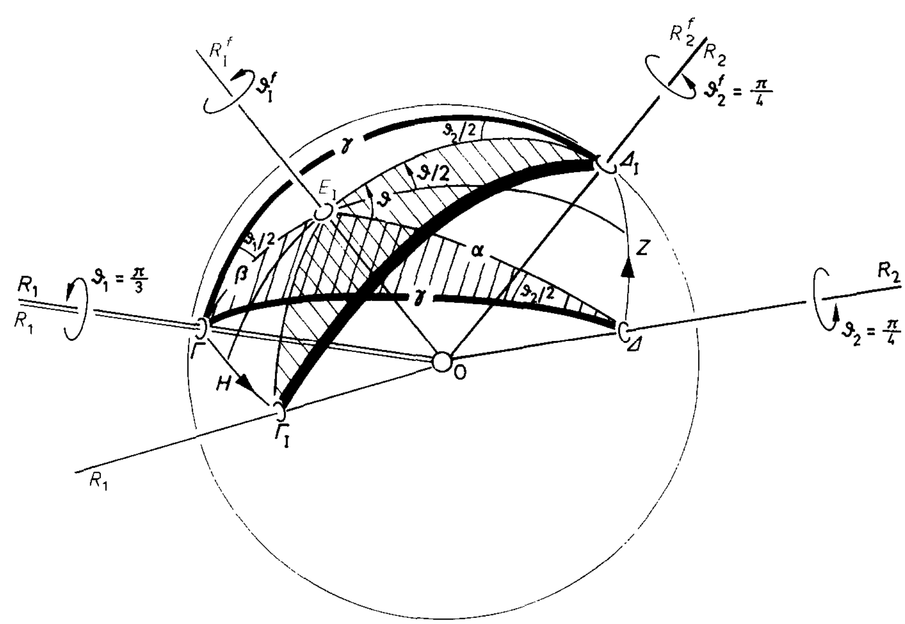

Fig. 6. Geometrical construction on a unit sphere of a follower sequence I^f. Effect of rotations is depicted by the movement of an arc $\Gamma\Delta$. Adopted colour scheme:

First rotation is depicted in each case in blue, second rotation in red. Green indicates the path followed by a vector $\boldsymbol{p}$ in response to the equivalent single compound pseudovector $\boldsymbol{\omega}_1^f$.

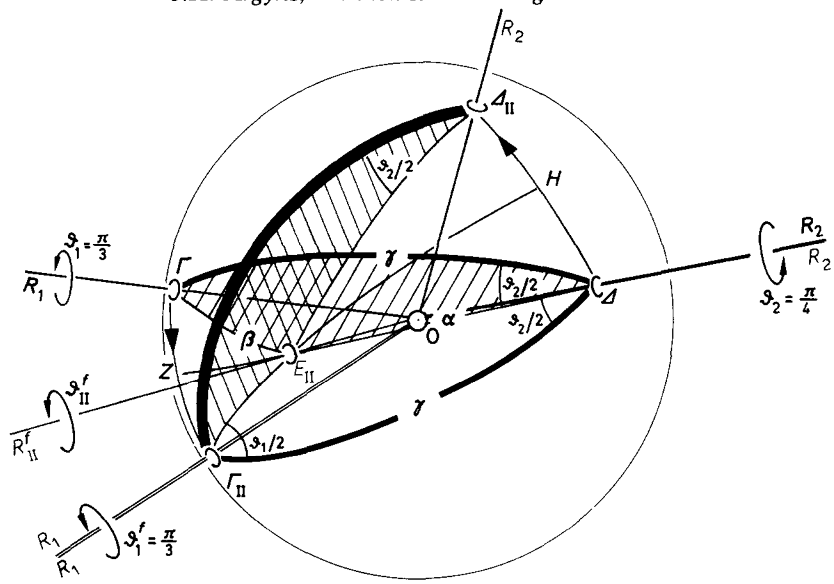

Fig. 7. Geometrical construction on a unit sphere of a follower sequence II^f. Effect of rotations is depicted by the movement of an arc $\Gamma\Delta$. Adopted colour scheme:

First rotation is depicted in each case in blue, second rotation in red. Green indicates the path followed by a vector $\boldsymbol{p}$ in response to the equivalent single compound pseudovector $\boldsymbol{\omega}^f_{II}$.

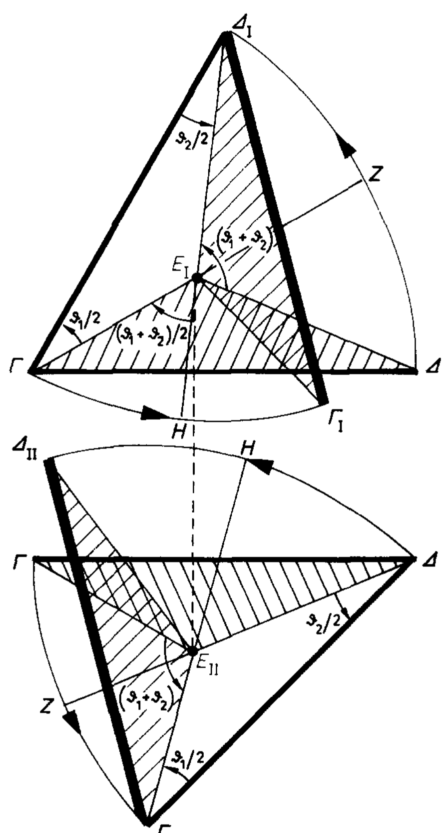

Fig. 8. Geometrical construction of paths I^f and II^f when axes are parallel. Same colour scheme adopted as in Figs. 6 and 7.

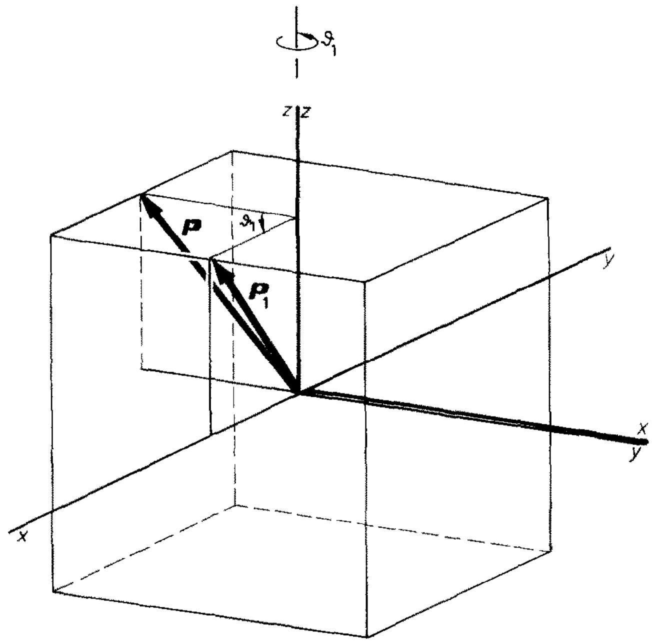

Fig. 9a. Effect of rotation $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ \ 0\ \ 1\}$ on a system of axes $Oxyz$. Initial axes are shown in black and final ones in green.

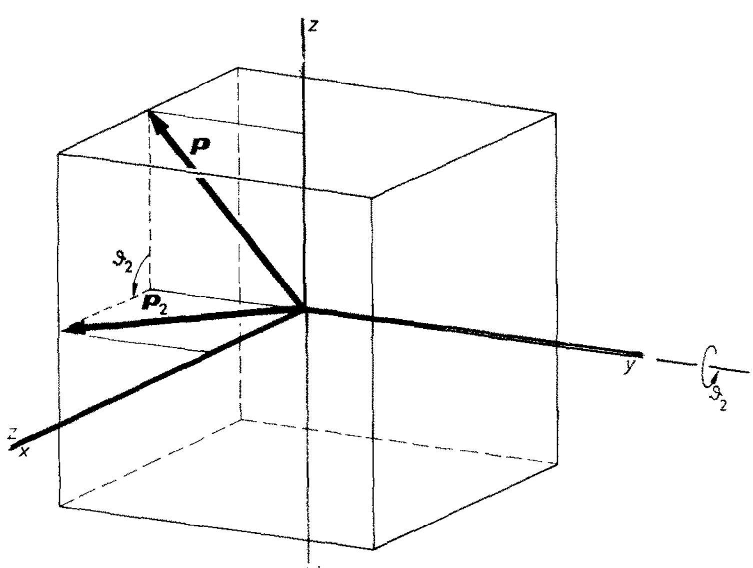

Fig. 9b. Effect of rotation $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ \ 1\ \ 0\}$ on a system of axes $Oxyz$. Initial axes are shown in black and final ones in green.

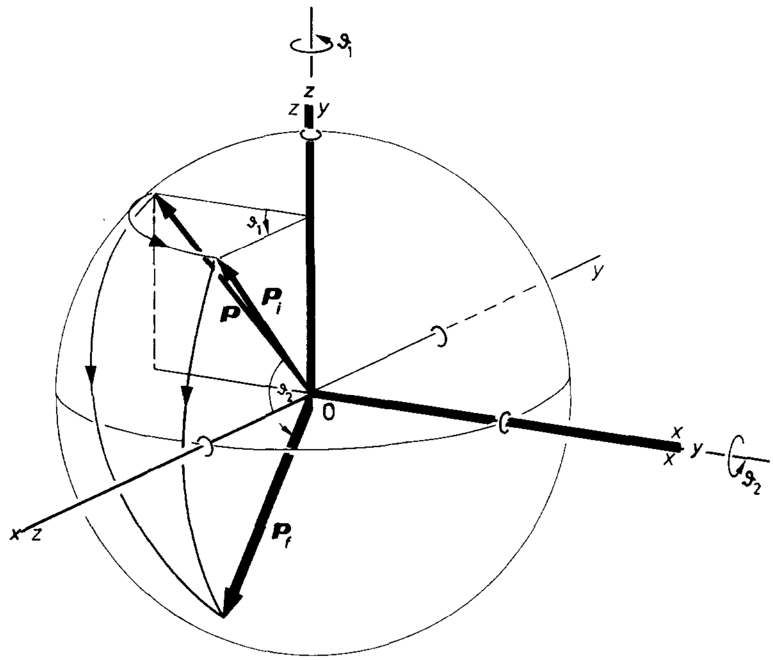

Fig. 10. Compound rotation I^r with initial rotational pseudovectors $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ 0\ 1\}$, $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ 1\ 0\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ -1\ 1\}$. Axes vanishing *behind* the sphere are depicted with dotted lines.

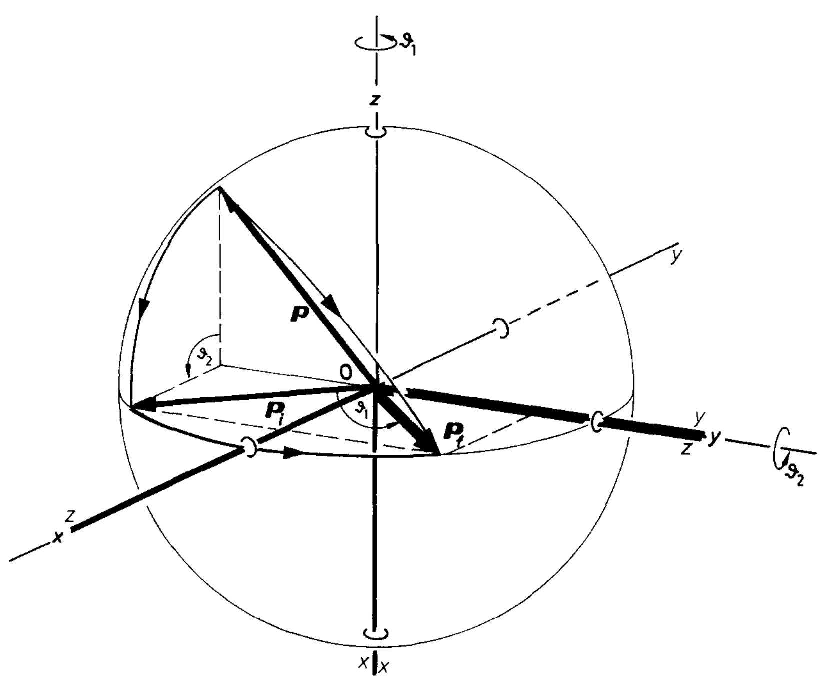

Fig. 11. Compound rotation II^r with initial rotational pseudovectors $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ 1\ 0\}$, $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ 0\ 1\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ -1\ 1\}$. Axes vanishing *behind* the sphere are depicted with dotted line.

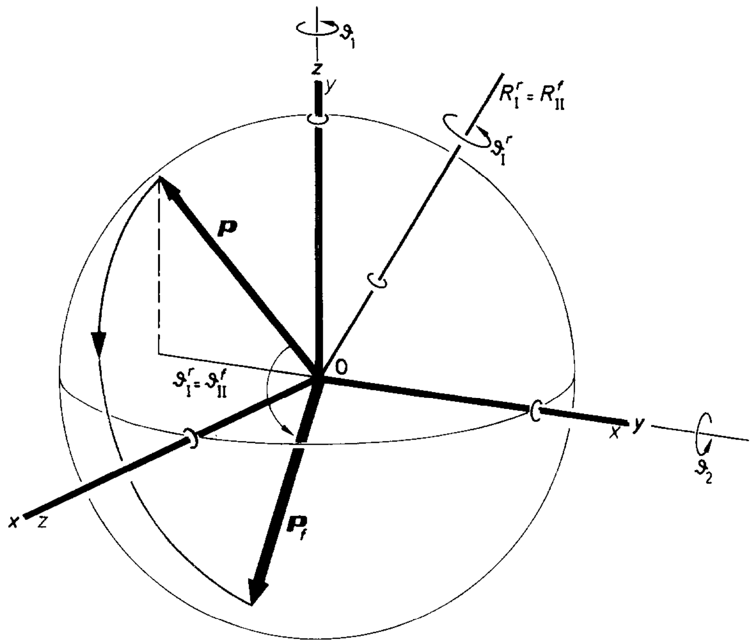

Fig. 12. Rotation sequences I^r and II^f. Transportation of $\boldsymbol{p}$ into terminal position through application of equivalent pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^r = \boldsymbol{\omega}_{\mathrm{II}}^f$ on initial vector $\boldsymbol{p} = \{0 \;\; -1 \;\; 1\}$.

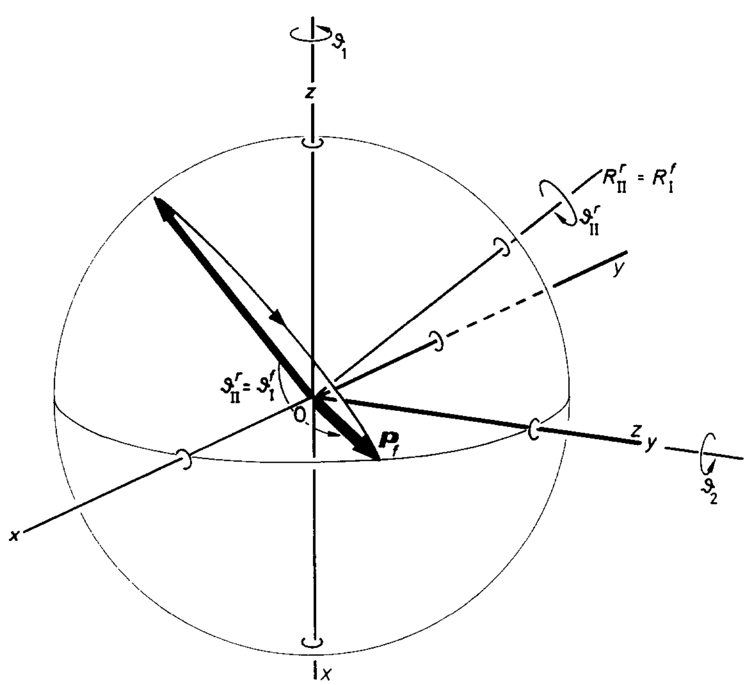

Fig. 13. Rotation sequences II^r and I^f. Transportation of $\boldsymbol{p}$ into terminal position through application of equivalent pseudovector $\boldsymbol{\omega}_{\mathrm{II}}^r = \boldsymbol{\omega}_{\mathrm{I}}^f$ on initial vector $\boldsymbol{p} = \{0 \;\; -1 \;\; 1\}$. Axes vanishing *behind* the sphere are depicted with dotted line.

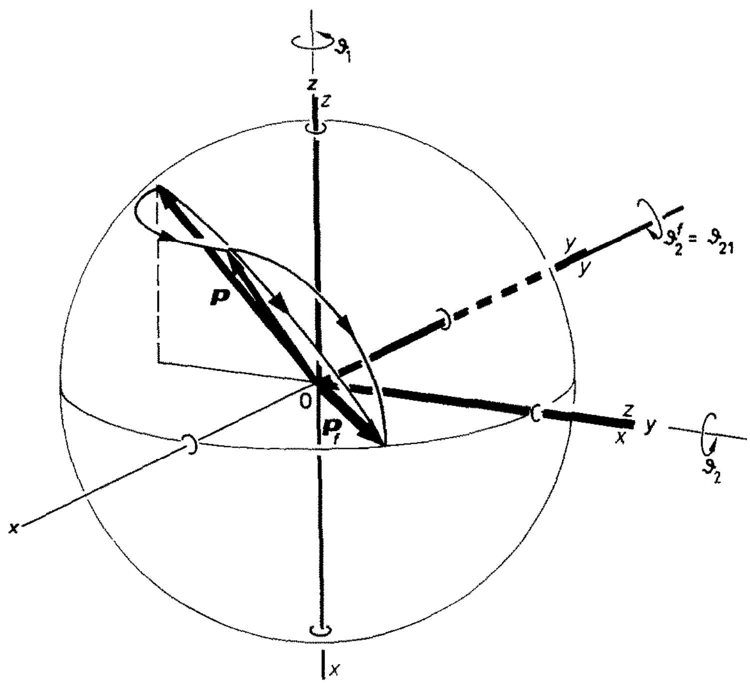

Fig. 14. Compound follower rotation I^{f} with initial rotation pseudovectors $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ \ 0\ \ 1\}$, $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ \ 1\ \ 0\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ \ -1\ \ 1\}$. Axes vanishing *behind* the sphere are depicted with dotted line.

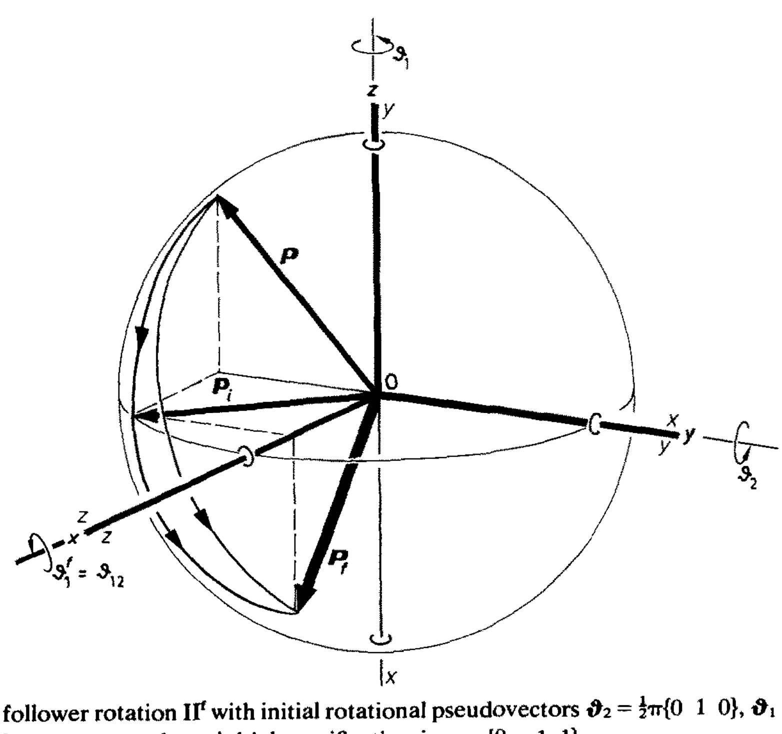

Fig. 15. Compound follower rotation II^{f} with initial rotational pseudovectors $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ \ 1\ \ 0\}$, $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ \ 0\ \ 1\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ \ -1\ \ 1\}$.

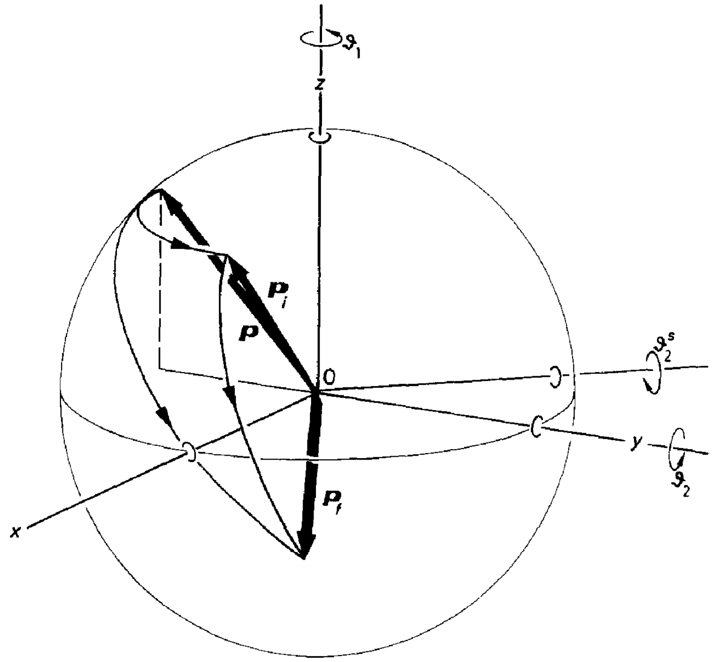

Fig. 16. Compound semitangential rotation I^s with initial rotational pseudovectors $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ \ 0\ \ 1\}$, $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ \ 1\ \ 0\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ \ -1\ \ 1\}$.

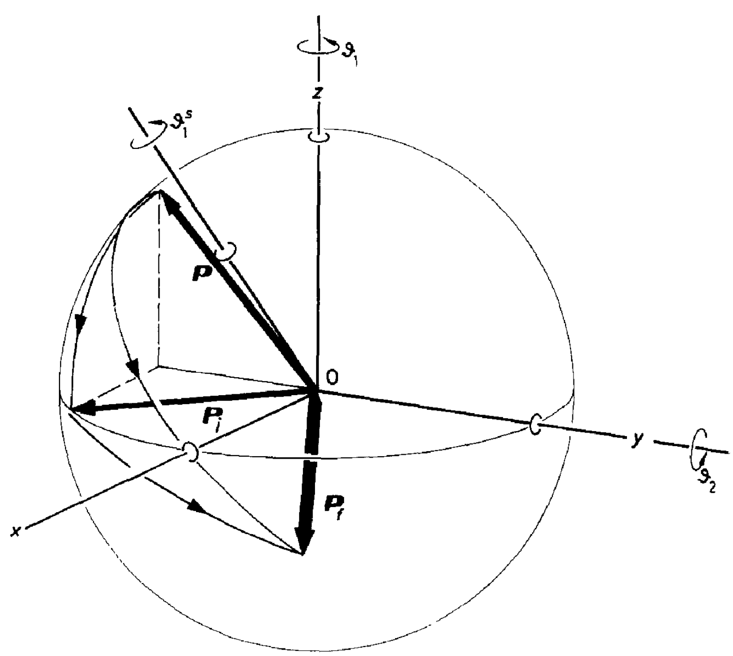

Fig. 17. Compound semitangential rotation II^s with initial rotational pseudovectors $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ \ 1\ \ 0\}$, $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ \ 0\ \ 1\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ \ -1\ \ 1\}$.

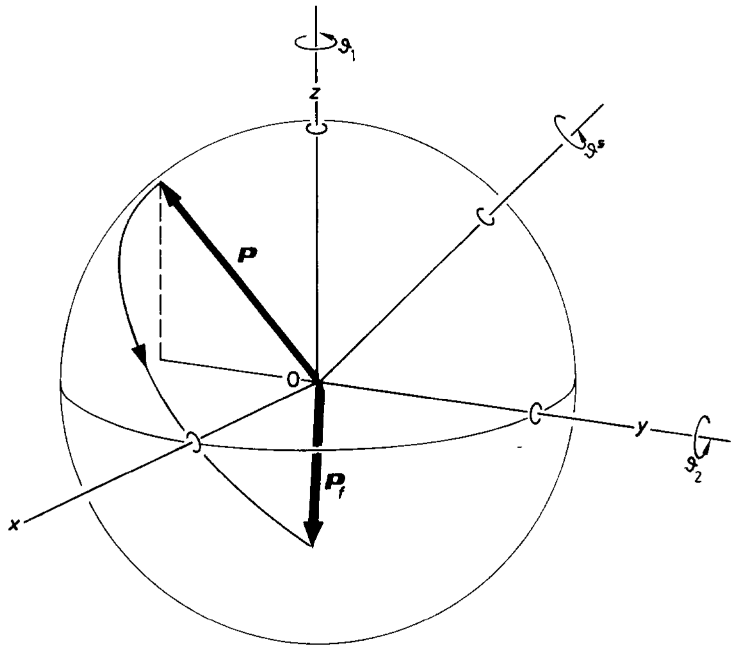

Fig. 18. Semitangential rotational sequence. Transportation of p into terminal position through application of equivalent semitangential pseudovector $\omega^s = \omega^s_I = \omega^s_{II}$.

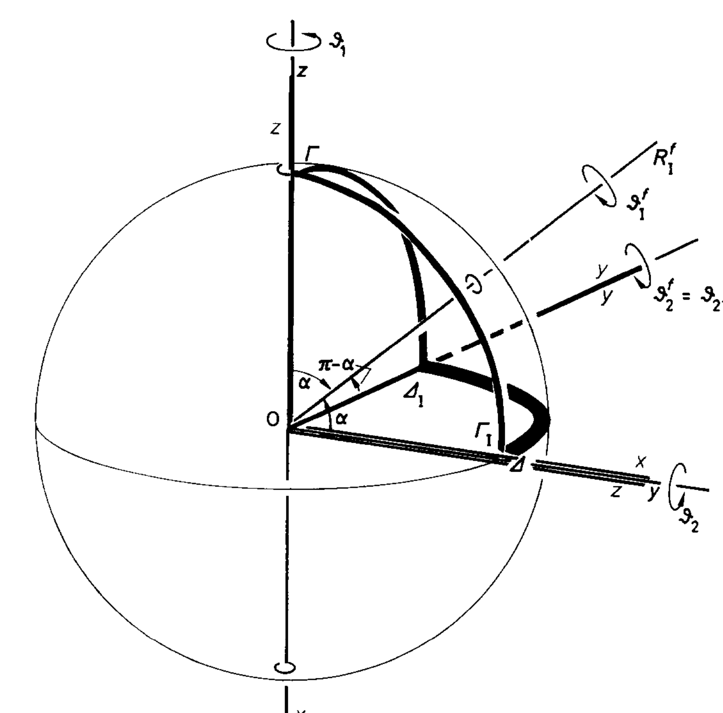

Fig. 19. Geometrical construction on a unit sphere of a follower sequence Γ^f. Effect of rotations is depicted by the movement of an arc $\Gamma\Delta$. Initial rotational pseudovectors $\vartheta_1 = \frac{1}{2}\pi\{0 \ 0 \ 1\}$, $\vartheta_2 = \frac{1}{2}\pi\{0 \ 1 \ 0\}$.

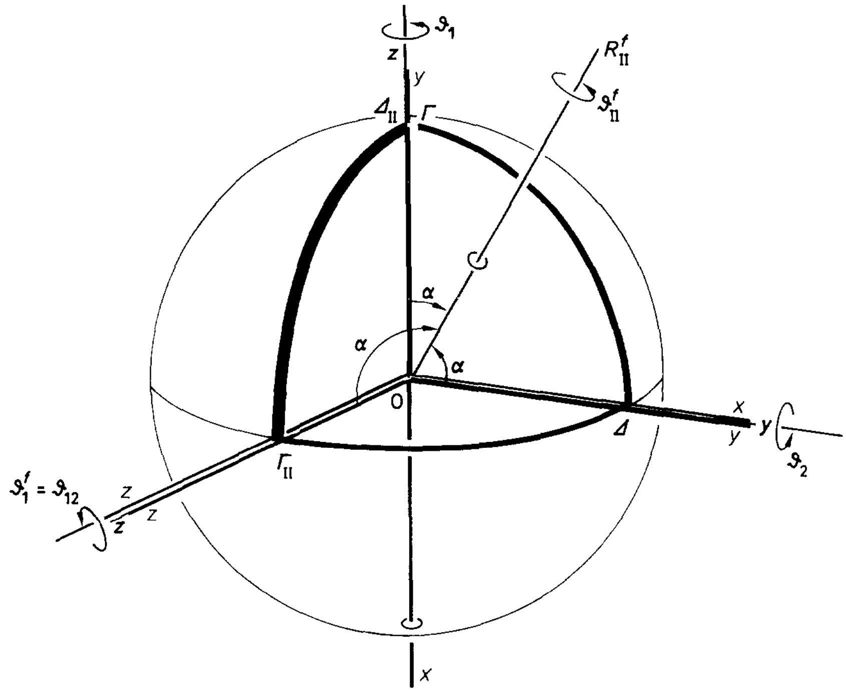

Fig. 20. Geometrical construction on a unit sphere of a follower sequence II^f. Effect of rotations is depicted by the movement of an arc $\Gamma\Delta$. Initial rotational pseudovectors $\boldsymbol{\vartheta}_2 = \frac{1}{2}\pi\{0\ \ 1\ \ 0\}$, $\boldsymbol{\vartheta}_1 = \frac{1}{2}\pi\{0\ \ 0\ \ 1\}$.

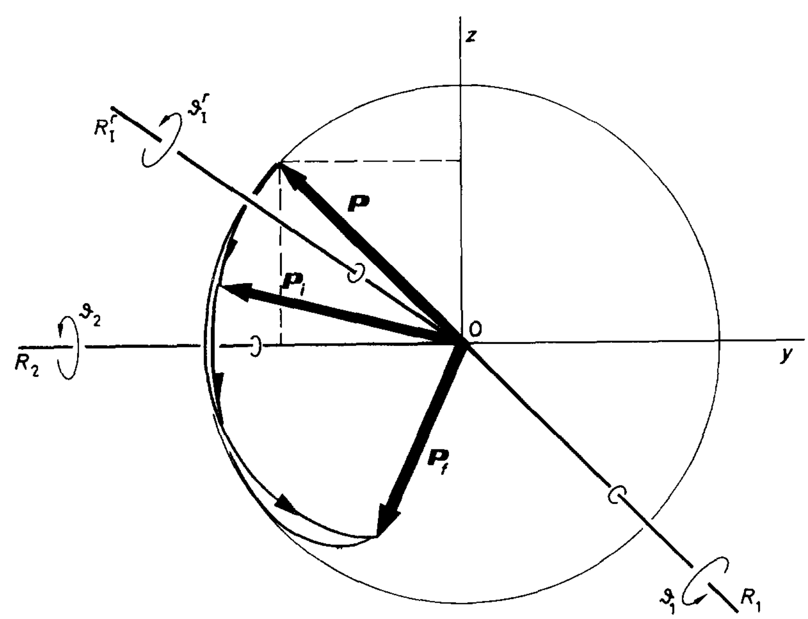

Fig. 21. Compound rotation I^r with initial rotational pseudovectors $\boldsymbol{\omega}_1 = \frac{1}{3}\{1\ \ 1\ \ -1\}$, $\boldsymbol{\omega}_2 = \frac{2}{5}\{3\ \ -4\ \ 5\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ \ -1\ \ 1\}$. Ox-axis perpendicular to plane of figure.

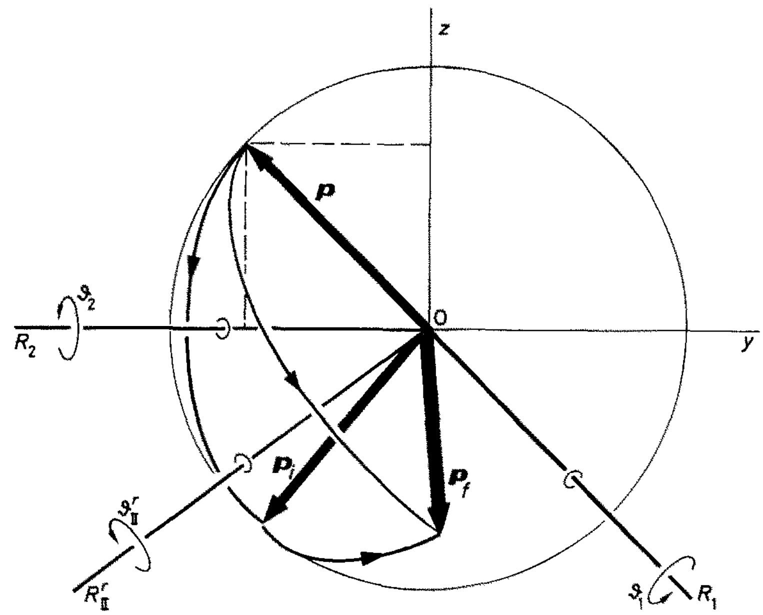

Fig. 22. Compound rotation II^r with initial rotational pseudovectors $\omega_2 = \frac{2}{5}\{3\ \ -4\ \ 5\}$, $\omega_1 = \frac{1}{3}\{1\ \ 1\ \ -1\}$. Effect of rotations is depicted on a vector whose initial specification is $p = \{0\ \ -1\ \ 1\}$. *Ox*-axis perpendicular to plane of figure.

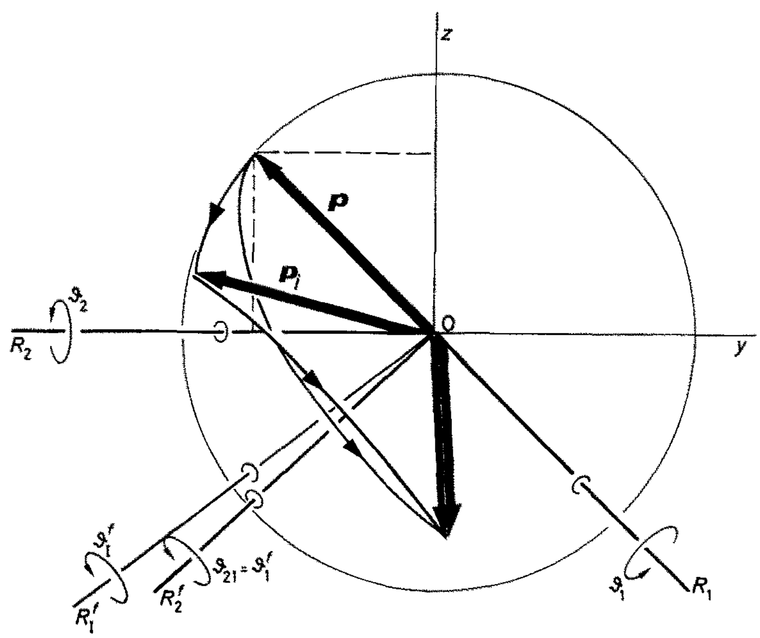

Fig. 23. Compound follower rotation I^f with initial rotational pseudovectors $\omega_1 = \frac{1}{3}\{1\ \ 1\ \ -1\}$, $\omega_2 = \frac{2}{5}\{3\ \ -4\ \ 5\}$. Effect of rotations is depicted on a vector whose initial specification is $p = \{0\ \ -1\ \ 1\}$. *Ox*-axis perpendicular to plane of figure.

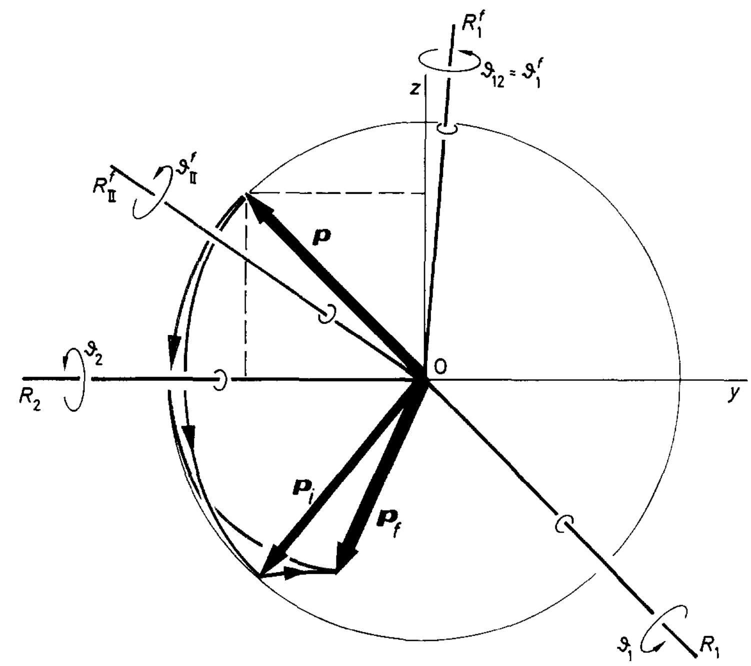

Fig. 24. Compound follower rotation II^f with initial rotational pseudovectors $\boldsymbol{\omega}_2 = \frac{2}{5}\{3 \;\; -4 \;\; 5\}$, $\boldsymbol{\omega}_1 = \frac{1}{3}\{1 \;\; 1 \;\; -1\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0 \;\; -1 \;\; 1\}$. Ox-axis perpendicular to plane of figure.

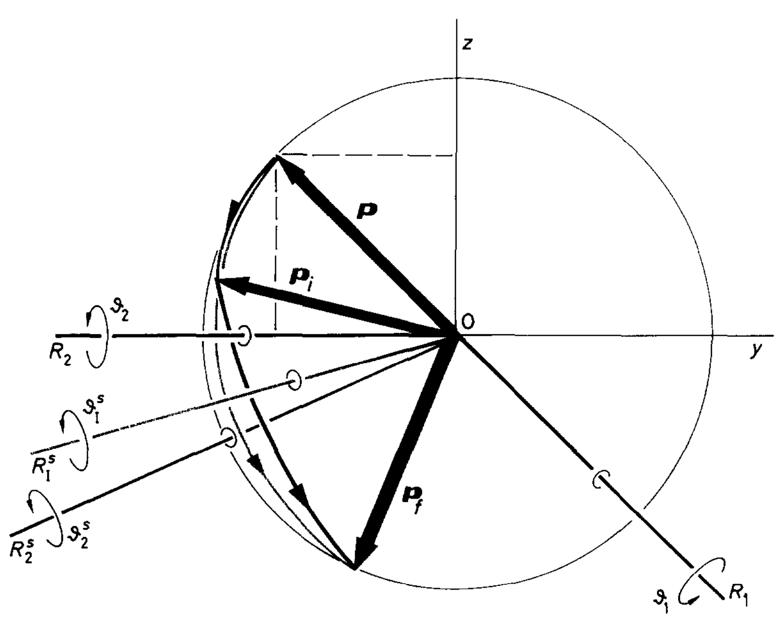

Fig. 25. Compound semitangential rotation I^s with initial rotational pseudovectors $\boldsymbol{\omega}_1 = \frac{1}{3}\{1 \;\; 1 \;\; -1\}$, $\boldsymbol{\omega}_2 = \frac{2}{5}\{3 \;\; -4 \;\; 5\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0 \;\; -1 \;\; 1\}$. Ox-axis perpendicular to plane of figure.

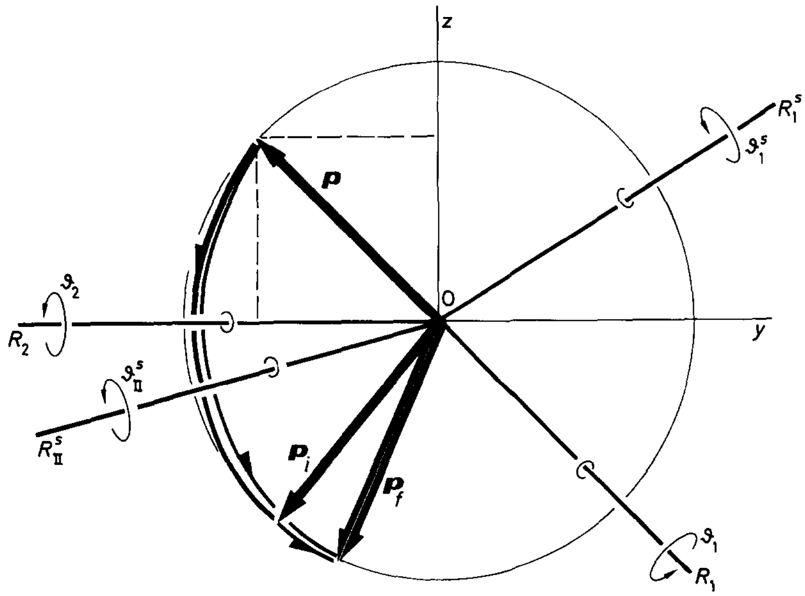

Fig. 26. Compound semitangential rotation II^s with initial rotational pseudovectors $\boldsymbol{\omega}_2 = \frac{2}{5}\{3\ \ -4\ \ 5\}$, $\boldsymbol{\omega}_1 = \frac{1}{3}\{1\ \ 1\ \ -1\}$. Effect of rotations is depicted on a vector whose initial specification is $\boldsymbol{p} = \{0\ \ -1\ \ 1\}$. Ox-axis perpendicular to plane of figure.

$$\boldsymbol{\omega}_1^t\boldsymbol{\omega}_1 = \tfrac{1}{3}, \qquad \boldsymbol{\omega}_2^t\boldsymbol{\omega}_2 = 4, \qquad \boldsymbol{\omega}_1^t\boldsymbol{\omega}_2 = -\tfrac{2}{15} \quad \text{(nonorthogonality)}. \tag{228}$$

The auxiliary matrices $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ are simply

$$\boldsymbol{R}_1 = \tfrac{1}{3}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \qquad \boldsymbol{R}_2 = \tfrac{2}{5}\begin{bmatrix} 0 & 0 & -4 \\ 0 & 0 & -3 \\ 4 & 3 & 0 \end{bmatrix}. \tag{229}$$

Application of the above data in (26) yields for the rotation matrices $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ as applied to fixed axes,

$$\boldsymbol{T}_1 = \tfrac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -1 & 2 & -2 \\ -2 & 1 & 2 \end{bmatrix}, \qquad \boldsymbol{T}_2 = \tfrac{1}{125}\begin{bmatrix} -3 & -96 & -80 \\ -96 & 53 & -60 \\ 80 & 60 & -75 \end{bmatrix}. \tag{230}$$

We possess now all initial data to analyse the different categories of axial arrangements.

6.1. *Compound rotation about fixed and follower axes*

Applying (36), (38), as also their extension to follower axes in (50), (56), we have in the present case

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{T}_2\boldsymbol{T}_1 = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}}, \qquad \boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}} = \boldsymbol{T}_1\boldsymbol{T}_2 = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}. \tag{231}$$

Substitution of (230) furnishes immediately

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \tfrac{1}{375}\begin{bmatrix} 250 & -278 & 29 \\ -125 & -146 & -322 \\ 250 & 205 & -190 \end{bmatrix} = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}} \tag{232}$$

and

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}} = \tfrac{1}{375}\begin{bmatrix} -118 & -26 & -355 \\ -349 & 82 & 110 \\ 70 & 365 & -50 \end{bmatrix} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}. \tag{233}$$

As in Subsections 5.1 and 5.2 the rotation matrices of (232) and (233) may be confirmed by an alternative derivation using the compound pseudovectors

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}} \quad \text{and} \quad \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}} = \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}$$

as defined in (78), (88), (105) and (108). In particular,

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \frac{1}{1 - \boldsymbol{\omega}_1^t\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1\boldsymbol{\omega}_2] = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}}, \qquad \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}} = \frac{1}{1 - \boldsymbol{\omega}_1^t\boldsymbol{\omega}_2}[\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 - \boldsymbol{R}_2\boldsymbol{\omega}_1] = \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}. \tag{234}$$

With the present data we find

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = \tfrac{1}{17}\{31 \quad -13 \quad 9\} = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}}\,, \qquad \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}} = \tfrac{1}{17}\{15 \quad -25 \quad -19\} = \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}\,. \tag{235}$$

We find (see also (86), (86b))

$$(\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = (\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}})^{\mathrm{t}}\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}} = 1211/289\,. \tag{236}$$

The associated auxiliary matrices $\boldsymbol{R}_{\mathrm{I}}^{\mathrm{r}}$, $\boldsymbol{R}_{\mathrm{II}}^{\mathrm{r}}$ are immediately set up and need not be detailed here. Substitution in (26) yields once more the same expressions for the rotation matrices as in (232) and (233).

The results for the fixed and follower sequences I and II are demonstrated in Figs. 21, 22 in which the same colour scheme as in Section 4 is applied.

6.2. *A further confirmation of the procedure for follower axes*

Having established in Subsection 6.1 the $\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}} = \boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}}$ and $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ rotation matrices in (232) and (233), the interested reader may wish to confirm these results for the specific case of the follower axes by the independent procedure set out in Section 2, eqs. (42), (44). Considering first sequence I^{f}, we require the follower rotation $\boldsymbol{\omega}_{21}$ in the form

$$\boldsymbol{\omega}_2^{\mathrm{f}} = \boldsymbol{\omega}_{21} = \boldsymbol{T}_1\boldsymbol{\omega}_2\,. \tag{237}$$

Using (225) and (230) for $\boldsymbol{\omega}_2$ and $\boldsymbol{T}_1$, respectively, we deduce

$$\boldsymbol{\omega}_2^{\mathrm{f}} = \boldsymbol{\omega}_{21} = -\tfrac{2}{15}\{2 \quad 11 \quad 10\} \tag{238}$$

and hence

$$\boldsymbol{R}_{21} = -\tfrac{2}{15}\begin{bmatrix} 0 & -10 & 11 \\ 10 & 0 & -2 \\ -11 & 2 & 0 \end{bmatrix}. \tag{239}$$

Also,

$$\boldsymbol{\omega}_{21}^{\mathrm{t}}\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_2 = 4\,. \tag{240}$$

Applying the information of (239) and (240) in (26) we find

$$\boldsymbol{T}_2^{\mathrm{f}} = \boldsymbol{T}_{21} = \frac{1}{3\cdot 375}\begin{bmatrix} -643 & 776 & -500 \\ -424 & 293 & 1000 \\ 820 & 760 & 125 \end{bmatrix}. \tag{241}$$

Substitution into (42),

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_{21}\boldsymbol{T}_1\,, \tag{42}$$

yields in conjunction with $\boldsymbol{T}_1$ of (230) once more the result of (233).

Turning next our attention to sequence II^{f}, we first determine

$$\boldsymbol{\omega}_1^{\mathrm{f}} = \boldsymbol{\omega}_{12} = \boldsymbol{T}_2\boldsymbol{\omega}_1 . \tag{242}$$

Using (226) and (230) for $\boldsymbol{\omega}_1$ and $\boldsymbol{T}_2$ we find

$$\boldsymbol{\omega}_1^{\mathrm{f}} = \boldsymbol{\omega}_{12} = \tfrac{1}{375}\{-19 \quad 17 \quad 215\} \tag{243}$$

and hence

$$\boldsymbol{R}_{12} = \tfrac{1}{375}\begin{bmatrix} 0 & -215 & 17 \\ 215 & 0 & 19 \\ -17 & -19 & 0 \end{bmatrix}. \tag{244}$$

Also,

$$\boldsymbol{\omega}_{12}^{\mathrm{t}}\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_1 = \tfrac{1}{3} . \tag{245}$$

Applying (244) and (245) in (26) we find

$$\boldsymbol{T}_1^{\mathrm{f}} = \boldsymbol{T}_{12} = \frac{1}{3 \cdot (125)^2}\begin{bmatrix} 23\,618 & -40\,474 & 1145 \\ 40\,151 & 23\,582 & 5390 \\ -5230 & -1735 & 46\,550 \end{bmatrix}. \tag{246}$$

Substitution into (44),

$$\boldsymbol{T}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{T}_{12}\boldsymbol{T}_2 , \tag{44}$$

yields once more the expression (232). The present results are also illustrated in Figs. 23 and 24.

6.3. Application of semitangential rotations

We remind the reader again that in contrast to our previous applications in Subsection 5.3 the current example presupposes nonorthogonal pseudovectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. In order to facilitate the understanding of the reasoning we present in what follows all three possible deductions of the rotation matrix $\boldsymbol{T}^{\mathrm{s}}$. These involve first the direct evaluation of $\boldsymbol{T}^{\mathrm{s}}$ via the compound pseudovector $\boldsymbol{\omega}^{\mathrm{s}}$ of (128). Then we consider separately the sequence I^{s} of the rotations $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2^{\mathrm{s}}$ and the sequence II^{s} of the rotations $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_1^{\mathrm{s}}$, all of which must yield an identical expression for $\boldsymbol{T}^{\mathrm{s}}$.

We start with the construction of the compound pseudovector $\boldsymbol{\omega}^{\mathrm{s}}$. Following (128) we write

$$\boldsymbol{\omega}^{\mathrm{s}} = \frac{1}{1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2}[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2] . \tag{247}$$

Noting (226) and (228) for $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2$, we deduce

$$\boldsymbol{\omega}^{\mathrm{s}} = \tfrac{1}{17}\{23 \quad -19 \quad -5\} \tag{248}$$

and therefore

$$(\boldsymbol{\omega}^{s})^{t}\boldsymbol{\omega}^{s} = 915/289\,. \tag{249}$$

The auxiliary matrix $\boldsymbol{R}^{s}$ reads

$$\boldsymbol{R}^{s} = \tfrac{1}{17}\begin{bmatrix} 0 & 5 & -19 \\ -5 & 0 & -23 \\ 19 & 23 & 0 \end{bmatrix}. \tag{250}$$

Introducing now in (26) the above data, the reader may establish the required rotation matrix in the form

$$\boldsymbol{T}^{s} = \tfrac{1}{301}\begin{bmatrix} 108 & -176 & -219 \\ -261 & 24 & -148 \\ 104 & 243 & -144 \end{bmatrix}. \tag{251}$$

Next we produce the reasoning associated with the I^{s} path. Here it is only necessary to construct the rotation matrix $\boldsymbol{T}_{2}^{s}$ emanating from the semitangential rotation $\boldsymbol{\omega}_{2}^{s}$. To this purpose we use (226), (238), (121) or directly (139) to find

$$\boldsymbol{\omega}_{2}^{s} = \tfrac{1}{2}[\boldsymbol{\omega}_{2}^{i} + \boldsymbol{\omega}_{2}^{f}] = \tfrac{1}{15}\{7 \quad -23 \quad -10\} \tag{252}$$

and hence

$$\boldsymbol{R}_{2}^{s} = \tfrac{1}{15}\begin{bmatrix} 0 & 10 & -23 \\ -10 & 0 & -7 \\ 23 & 7 & 0 \end{bmatrix}. \tag{253}$$

Also, either from (252) or from (143),

$$(\boldsymbol{\omega}_{2}^{s})^{t}\boldsymbol{\omega}_{2}^{s} = 678/225\,. \tag{254}$$

Substituting expressions (253), (254) in (26) we obtain

$$\boldsymbol{T}_{2}^{s} = \frac{1}{3\cdot 301}\begin{bmatrix} -355 & -22 & -830 \\ -622 & 605 & 250 \\ 550 & 670 & -253 \end{bmatrix}. \tag{255}$$

Forming next (144),

$$\boldsymbol{T}_{\mathrm{I}}^{s} = \boldsymbol{T}_{2}^{s}\boldsymbol{T}_{1}\,, \tag{144}$$

we confirm immediately (251).

We conclude our discussion with a brief reference to the path II^s. To this purpose we require as additional information only the rotation matrix $\boldsymbol{T}_{\mathrm{I}}^s$. To set it up we first determine $\boldsymbol{\omega}_{\mathrm{I}}^s$. Using (145) or (146) we find

$$\boldsymbol{\omega}_{\mathrm{I}}^s = \tfrac{1}{375}\{53 \quad 71 \quad 45\} \tag{256}$$

and hence

$$\boldsymbol{R}_{\mathrm{I}}^s = \tfrac{1}{375}\begin{bmatrix} 0 & -45 & 71 \\ 45 & 0 & -53 \\ -71 & 53 & 0 \end{bmatrix}. \tag{257}$$

Also, either from (256) or from (149),

$$(\boldsymbol{\omega}_{\mathrm{I}}^s)^t \boldsymbol{\omega}_{\mathrm{I}}^s = 79(9 \cdot 125)\,. \tag{258}$$

Substituting (257), (258) in (26) we obtain

$$\boldsymbol{T}_{\mathrm{I}}^s = \frac{1}{125 \cdot 301}\begin{bmatrix} 34\,092 & 6556 & 14\,505 \\ 10\,319 & 35\,208 & -8340 \\ 12\,120 & 11\,535 & 33\,700 \end{bmatrix}. \tag{259}$$

Forming next (151),

$$\boldsymbol{T}_{\mathrm{II}}^s = \boldsymbol{T}_{\mathrm{I}}^s \boldsymbol{T}_2\,, \tag{151}$$

we confirm for the third time expression (251). The results of this subsection are illustrated in Figs. 25, 26.

Acknowledgments

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Appendix A. Multiple compound rotations and quarternions

We examined repeatedly in the main body of this paper the construction of the rotation matrix $\boldsymbol{T}$ for any number of arbitrary constituent rotations and alternative specification of axes, i.e., fixed, follower or semitangential. However, with the exception of the semitangential case we restricted the associated problem of an explicit matrix expression for the equivalent single rotation $\boldsymbol{\omega}_{\mathrm{I}}$ or $\boldsymbol{\omega}_{\mathrm{II}}$—which may serve as a fully equipollent substitute representing the combined effect of a prescribed sequence of arbitrary rotations—to the relatively simple case composed of *two* only component rotations. Inevitably our rather extensive discussion on multiple semitangential rotations causes us to pose the question how to deduce in an efficient and elegant manner the equivalent single rotation that reproduces the same overall result as a given sequence of rotations. It is of course always possible, as we mentioned before, to set up this substitute vector in the form of an eigenvector of the compound rotation matrix $\boldsymbol{T}$. Here, however, we seek an explicit matrix expression for this equivalent pseudovector $\boldsymbol{\omega}_{\mathrm{I}}$ or $\boldsymbol{\omega}_{\mathrm{II}}$. The experienced reader will of course concur that this is, in principle, always feasible by a selfevident extension of the method applied in Section 3. However, it is equally clear that this approach inevitably leads to rather complex matrix expressions which demand some manipulative skill in order to reduce them to an ordered scheme displaying a mnemonic pattern. One is at first baffled by this seemingly elementary problem which does not appear to yield an immediate solution of the expected simplicity.

As one ponders over this difficulty one is led nolens volens to guess that the unravelment of this enigma may be achieved by an adaptation of the calculus of quarternions invented by Hamilton. Nevertheless this must not be attempted in the classical formulation of quarternions (see e.g. [9]) but in a modernised guise incorporating vector and matrix methods.

Let us consider to this purpose a dual system of numbers $\mathfrak{q}_1$ composed of a scalar a and a vector $\boldsymbol{b}$ in the form

$$\mathfrak{q}_1 = a + \boldsymbol{b} \tag{A.1}$$

where

$$\boldsymbol{b} = \{b_x \quad b_y \quad b_z\}\,. \tag{A.2}$$

This combination must not surprise the engineer since, in a way, it may be thought of as a generalisation of the field of complex numbers which comprise a real and an imaginary number. In fact, this interpretation is in line with the original concept of Hamilton although not any longer compatible with our way of thinking.

In what follows we reproduce first some definitions concerning elementary operations with quarternions. All laws of addition and subtraction hold as for ordinary numbers with the proviso that book keeping is distinct for scalars and vectors. The most important convention refers to multiplication. To this effect we introduce another quarternion in the form

$$\mathfrak{q}_2 = c + \boldsymbol{d} \tag{A.3}$$

and define the product

$$\mathfrak{q} = \mathfrak{q}_1 \mathfrak{q}_2 = (a + \boldsymbol{b})(c + \boldsymbol{d}) \tag{A.4}$$

as a legitimate and consistent operation if specified as

$$\mathfrak{q} = \mathfrak{q}_1 \mathfrak{q}_2 = ac + c\boldsymbol{b} + a\boldsymbol{d} + \boldsymbol{b} \times \boldsymbol{d} - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{d} \tag{A.5}$$

in which $\mathfrak{q}$ describes again a quarternion. The reader will observe the noncommutativity of the operation (A.5) since the inverse product $\boldsymbol{q}'$ becomes

$$\boldsymbol{q}' = \boldsymbol{q}_2 \boldsymbol{q}_1 = ac + c\boldsymbol{b} + a\boldsymbol{d} - \boldsymbol{b} \times \boldsymbol{d} - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{d}\,. \tag{A.5a}$$

The definitions in (A.5), (A.5a) may appear on first sight odd but are free of any contradictions. We note parenthetically that the distributive and associative rules of multiplication hold as for ordinary numbers and vectors.

We recognise that the scalar and vector components of the product $\boldsymbol{q}_1\boldsymbol{q}_2$ are given respectively by

$$s(\boldsymbol{q}_1 \boldsymbol{q}_2) = ac - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{d} \tag{A.6}$$

and

$$\boldsymbol{v}(\boldsymbol{q}_1 \boldsymbol{q}_2) = c\boldsymbol{b} + a\boldsymbol{d} + \boldsymbol{b} \times \boldsymbol{d}\,. \tag{A.7}$$

When

$$s(\boldsymbol{q}) = 0\,, \tag{A.8}$$

$\boldsymbol{q}$ is denoted a vectorquanternion.

Next we introduce the norm of $\boldsymbol{q}$ which is given by

$$|\boldsymbol{q}| = (a^2 + b_x^2 + b_y^2 + b_z^2)^{1/2}\,. \tag{A.9}$$

We also define as a conjugate quarternion $\bar{\boldsymbol{q}}$ the dual quantity

$$\bar{\boldsymbol{q}} = a - \boldsymbol{b} \quad \text{when } \boldsymbol{q} = a + \boldsymbol{b}\,. \tag{A.10}$$

Following the rules of (A.5) and (A.7), the vector component $\boldsymbol{v}(\boldsymbol{q}\bar{\boldsymbol{q}})$ vanishes and $\boldsymbol{q}\bar{\boldsymbol{q}}$ is reduced to the nonnegative scalar

$$\boldsymbol{q}\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\boldsymbol{q} = a^2 + b_x^2 + b_y^2 + b_z^2\,. \tag{A.11}$$

Hence we obtain for the norm of (A.9),

$$|\boldsymbol{q}| = (\boldsymbol{q}\bar{\boldsymbol{q}})^{1/2} = (\bar{\boldsymbol{q}}\boldsymbol{q})^{1/2}\,. \tag{A.12}$$

We are now in a position to specify an inverse quarternion $\boldsymbol{q}^{-1}$ in the form

$$\boldsymbol{q}^{-1} = \bar{\boldsymbol{q}}/|\boldsymbol{q}|\,, \tag{A.13}$$

which satisfies the required condition

$$\boldsymbol{q}\boldsymbol{q}^{-1} = \boldsymbol{q}^{-1}\boldsymbol{q} = 1\,. \tag{A.14}$$

Of great relevance to our work is the special case of unit quarternions or versors for which $|\boldsymbol{q}| = 1$. For such quarternions we introduce the symbol $\boldsymbol{u}$ and note that (A.13) reduces then to

$$\boldsymbol{u}^{-1} = \bar{\boldsymbol{u}} \quad \text{and}\ \bar{\boldsymbol{u}}\,\boldsymbol{u} = 1, \text{ since } |\boldsymbol{u}| = 1\,. \tag{A.15}$$

Denoting the norm of an arbitrary quarternion $\boldsymbol{q}$ by

$$|\boldsymbol{q}| = r\,, \tag{A.16}$$

we observe that $\boldsymbol{q}$ may always be expressed in the form

$$\boldsymbol{q} = r\boldsymbol{u}\,. \tag{A.17}$$

Now it is evident that any versor $\boldsymbol{u}$ may be written as

$$\boldsymbol{u} = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\,\boldsymbol{e} \tag{A.18}$$

where $\boldsymbol{e}$ is the associated unit vector. The reason for the choice of the angle $\vartheta/2$ will emerge presently.

In the present context we are interested in large rotations that are defined by the pseudovectors $\boldsymbol{\vartheta}$ or $\boldsymbol{\omega}$. Restricting our attention to the latter vector and in particular to its definition in (19) we observe that (A.18) can be written more conveniently as

$$\boldsymbol{u} = \cos\frac{\vartheta}{2}(1 + \boldsymbol{\omega})\,. \tag{A.19}$$

Consider now two versors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ given by

$$\boldsymbol{u}_1 = \cos\frac{\vartheta_1}{2}(1 + \boldsymbol{\omega}_1)\,, \qquad \boldsymbol{u}_2 = \cos\frac{\vartheta_2}{2}(1 + \boldsymbol{\omega}_2)\,. \tag{A.20}$$

Forming the product

$$\boldsymbol{u}_{\mathrm{I}} = \boldsymbol{u}_2\,\boldsymbol{u}_1 \tag{A.21}$$

(note the sequence!), in accordance with (A.5) we obtain the compound unit quarternion

$$\boldsymbol{u}_{\mathrm{I}} = \cos\frac{\vartheta_{\mathrm{I}}}{2}(1 + \boldsymbol{\omega}_{\mathrm{I}}) = \boldsymbol{u}_2\,\boldsymbol{u}_1 = \cos\frac{\vartheta_2}{2}\cos\frac{\vartheta_1}{2}(1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2 + \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_1)\,. \tag{A.22}$$

Thus,

$$\begin{aligned} s(\boldsymbol{u}_{\mathrm{I}}) &= \cos\frac{\vartheta_{\mathrm{I}}}{2} = \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2}\left(1 - \tan\frac{\vartheta_1}{2}\tan\frac{\vartheta_2}{2}\cos\gamma\right) \\ &= \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} - \sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\cos\gamma \end{aligned} \tag{A.23}$$

where γ is the angle enclosed by the vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and

$$\boldsymbol{v}(\boldsymbol{u}_{\mathrm{I}}) = \cos\frac{\vartheta_{\mathrm{I}}}{2}\,\boldsymbol{\omega}_{\mathrm{I}} = \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2}\,[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2]\,. \tag{A.24}$$

We immediately confirm that the operation $\boldsymbol{u}_{\mathrm{I}} = \boldsymbol{u}_2\boldsymbol{u}_1$ yields in its scalar component and in agreement with (87) and (160) the compound angle of rotation $\theta_{\mathrm{I}}^{\mathrm{r}}$ of the sequence $\boldsymbol{\vartheta}_1$, $\boldsymbol{\vartheta}_2$ *and* in its vector component and in agreement with (78) the compound pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ of the sequence $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$.

This compact and elegant result points immediately the way to a generalisation for n rotations described by $\boldsymbol{\omega}_1$ to $\boldsymbol{\omega}_n$. But before proceeding to it we note that (A.22) can in the present case be reformulated in the more convenient matrix expression

$$\boldsymbol{u}_{\mathrm{I}} = \boldsymbol{u}_2\boldsymbol{u}_1 = c_1c_2[(1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2) + \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \boldsymbol{R}_1\boldsymbol{\omega}_2]$$

or

$$\boldsymbol{u}_{\mathrm{I}} = \boldsymbol{u}_2\boldsymbol{u}_1 = c_1c_2[(1 - \boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2) + \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{R}_2\boldsymbol{\omega}_1] \tag{A.25}$$

where $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ are the auxiliary matrices given by (23) and the abbreviation c_i stands for

$$c_i = \cos\vartheta_i/2\,. \tag{A.26}$$

Returning to the case of the sequence I^{r} for n rotations, we immediately deduce from the construction of (A.22) the general result

$$\begin{aligned}\boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}} &= c_{\mathrm{I}}(1 + \boldsymbol{\omega}_{\mathrm{I}}) = \boldsymbol{u}_n\boldsymbol{u}_{n-1}\cdots\boldsymbol{u}_i\cdots\boldsymbol{u}_2\boldsymbol{u}_1\\ &= c(1 + \boldsymbol{\omega}_n)(1 + \boldsymbol{\omega}_{n-1})\cdots(1 + \boldsymbol{\omega}_i)\cdots(1 + \boldsymbol{\omega}_2)(1 + \boldsymbol{\omega}_1)\end{aligned} \tag{A.27}$$

where, in accordance with (A.26),

$$c = c_nc_{n-1}\cdots c_i\cdots c_2c_1\,, \qquad c_{\mathrm{I}} = \cos\vartheta_{\mathrm{I}}/2\,. \tag{A.28}$$

The quarternion expression (A.27) yields in its scalar component $s(\boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}})$ the compound angle of rotation $\vartheta_{\mathrm{I}}^{\mathrm{r}}$ via $c_{\mathrm{I}} = \cos\vartheta_{\mathrm{I}}/2$ and in its vector component $\boldsymbol{v}(\boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}}) = c_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ the associated compound pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$. As the reader will readily admit this is a most gratifying and physically suggestive result. Furthermore, it confirms that quarternions, if applied in a modern guise, prove a powerful tool of analysis not to be disposed off as a historical relic.

We demonstrate now the application of (A.27) on a triple rotation composed of the sequence $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$. Noting that in the present case

$$\boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}} = c_{\mathrm{I}}^{\mathrm{r}}(1 + \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}) = c_1c_2c_3(1 + \boldsymbol{\omega}_3)(1 + \boldsymbol{\omega}_2)(1 + \boldsymbol{\omega}_1) \tag{A.29}$$

and observing the rules of (A.5), (A.6), (A.7) as well as the matrix operations of (24), (72), (81), (82), (115) and (115a) we obtain

$$s(\boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}}) = c_{\mathrm{I}}^{\mathrm{r}} = c_1c_2c_3\{1 - (\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2 + \boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_3 + \boldsymbol{\omega}_3^{\mathrm{t}}\boldsymbol{\omega}_1) - \boldsymbol{\omega}_3^{\mathrm{t}}\boldsymbol{R}_2\boldsymbol{\omega}_1\} \tag{A.30}$$

and

$$v(\boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}}) = c_{\mathrm{I}}^{\mathrm{r}}\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}} = [\boldsymbol{I}_3 - \boldsymbol{R}_1][\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3] - \boldsymbol{R}_2[\boldsymbol{\omega}_2 + \boldsymbol{\omega}_3] - (\boldsymbol{\omega}_2^{\mathrm{t}}\boldsymbol{\omega}_3)\boldsymbol{\omega}_1 + (\boldsymbol{\omega}_3^{\mathrm{t}}\boldsymbol{\omega}_1)\boldsymbol{\omega}_2 - (\boldsymbol{\omega}_1^{\mathrm{t}}\boldsymbol{\omega}_2)\boldsymbol{\omega}_3 . \tag{A.31}$$

Eq. (A.30) yields the angle of rotation $\vartheta_{\mathrm{I}}^{\mathrm{r}}$ via the value of $c_{\mathrm{I}}^{\mathrm{r}} = \cos \vartheta_{\mathrm{I}}^{\mathrm{r}}/2$. The compound pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}$ derives from (A.31), the pattern of which is modelled on (152), and appears to be mnemonically the most appropriate one. On the other hand, we may of course eliminate if we so wish the zero columns $\boldsymbol{R}_i\boldsymbol{\omega}_i = \boldsymbol{o}_3$ in this expression. It is now evident how (A.31) can be generalised in the presence of n rotations $\boldsymbol{\omega}_1$ to $\boldsymbol{\omega}_n$. We refrain, however, from reproducing the formula here. Nevertheless, the attentive reader may like to confirm (A.31) by an extension of the method described in Subsection 3.1 for two component rotations $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$.

Another aspect of compound rotations not in any way connected to quarternions but of some practical relevance, concerns the case of a compound rotation which has subsequently to be modified by the superposition of an additional rotation. Let us assume to this effect that the compound pseudovector $\boldsymbol{\omega}_{(n-1)}$ representing the combined effect of a given sequence $\boldsymbol{\omega}_1$ to $\boldsymbol{\omega}_{n-1}$ is known. What happens at this stage if a supplementary rotation $\boldsymbol{\omega}_n$ is imposed? The answer is simply given by the construction of an overall or dual compound vector $\boldsymbol{\omega}_{\mathrm{I}}$ composed of two pseudovectors only, namely $\boldsymbol{\omega}_{(n-1)}$ and $\boldsymbol{\omega}_n$. This solves the present problem.

An instructive confirmation of (A.27) may be obtained for the limiting case when all rotations $\boldsymbol{\vartheta}_1$ to $\boldsymbol{\vartheta}_n$ take place about parallel axes defined by the same unit vector $\boldsymbol{e}$. Using then preferably the versor $\boldsymbol{u}_i$ in the form of (A.18) we immediately deduce the standard result

$$\vartheta_{\mathrm{I}} = \vartheta_1 + \vartheta_2 + \cdots + \vartheta_i + \cdots + \vartheta_{n-1} + \vartheta_n . \tag{A.32}$$

For the special instance when all angles of rotation ϑ_i are equal, $\vartheta_i = \vartheta$, we find

$$\vartheta_{\mathrm{I}} = n\vartheta \tag{A.33}$$

in agreement with the direct argument via (A.27) and the quarternionian rule of multiplication,

$$\boldsymbol{u}_{\mathrm{I}} = \boldsymbol{u}^n = \left(\cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\boldsymbol{e}\right)^n = \left(\cos\frac{n\vartheta}{2} + \sin\frac{n\vartheta}{2}\boldsymbol{e}\right) \tag{A.34}$$

which accords with the formula of Moivre based on complex numbers.

Before discussing some further aspects of (A.27) we illustrate the applicability of the unit quarternions as regards the formation of the rotation matrix $\boldsymbol{T}$ with which we initiated this paper in Section 1. Consider an arbitrary vector $\boldsymbol{p}$ and an associated similar transformation via the versor $\boldsymbol{u}$ of (A.19) viz.

$$\boldsymbol{u}\boldsymbol{p}\boldsymbol{u}^{-1} = \boldsymbol{u}\boldsymbol{p}\bar{\boldsymbol{u}}. \tag{A.35}$$

Writing this transformation explicitly we obtain, using (A.5),

$$\begin{aligned}\boldsymbol{u}\boldsymbol{p}\bar{\boldsymbol{u}} &= c^2(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{p} - \boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p})(1 - \boldsymbol{\omega}) \\ &= c^2(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{p} - \boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p} + \boldsymbol{R}\boldsymbol{p} + \boldsymbol{R}^2\boldsymbol{p} + (\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p})\boldsymbol{\omega} + \boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p} - \boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{R}\boldsymbol{p}) .\end{aligned} \tag{A.36}$$

In establishing (A.36) we apply the evident relations

$$-\boldsymbol{p}\times\boldsymbol{\omega}=\boldsymbol{\omega}\times\boldsymbol{p}=\boldsymbol{R}\boldsymbol{p} \quad\text{and}\quad -(\boldsymbol{\omega}\times\boldsymbol{p})\times\boldsymbol{\omega}=\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{p})=\boldsymbol{R}^2\boldsymbol{p}\,. \tag{A.37}$$

In order to reduce (A.36) to its final form we observe the evident relations

$$\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{R}\boldsymbol{p}=0 \quad \text{(see (82), (82a))}$$

and

$$\boldsymbol{R}^2\boldsymbol{p}-(\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p})\boldsymbol{\omega}=-(\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{\omega})\boldsymbol{p} \quad\text{or}\quad (\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{p})\boldsymbol{\omega}=\boldsymbol{R}^2\boldsymbol{p}+(\boldsymbol{\omega}^{\mathrm{t}}\boldsymbol{\omega})\boldsymbol{p}\,. \tag{A.38}$$

The interested reader may observe that the second equation in (A.38) is alternatively established by postmultiplication of (84) with $\boldsymbol{p}$. Substitution of (A.38) in (A.37) finally yields

$$\boldsymbol{u}\boldsymbol{p}\,\bar{\boldsymbol{u}}=(c^2+s^2)\boldsymbol{p}+2c^2\boldsymbol{R}\boldsymbol{p}+2c^2\boldsymbol{R}^2\boldsymbol{p}=\boldsymbol{p}+2c^2\boldsymbol{R}\boldsymbol{p}+2c^2\boldsymbol{R}^2\boldsymbol{p} \tag{A.39}$$

where in conformity with (A.26),

$$s=\sin\vartheta/2\,. \tag{A.40}$$

We observe that (A.39) yields the final position $\hat{\boldsymbol{p}}$ of the initial vector $\boldsymbol{p}$ in accordance with (3), (25) and the transformation matrix $\boldsymbol{T}$ of (26). In fact, we obtain

$$\hat{\boldsymbol{p}}=\boldsymbol{u}\boldsymbol{p}\,\bar{\boldsymbol{u}}=\boldsymbol{T}\boldsymbol{p}\,. \tag{A.41}$$

Also,

$$\bar{\boldsymbol{u}}\hat{\boldsymbol{p}}\,\boldsymbol{u}=\boldsymbol{p}=\boldsymbol{T}^{-1}\hat{\boldsymbol{p}}=\boldsymbol{T}^{\mathrm{t}}\hat{\boldsymbol{p}}\,. \tag{A.41a}$$

As the reader will readily observe, (A.41) presents on the left-hand side a quarternionian expression and on the right-hand side a matrix transformation.

It is evident that (A.41) can immediately be generalised to two or more rotations as laid down in (A.21) and (A.27). Thus, for two consecutive rotations $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ the vector $\hat{\boldsymbol{p}}$ becomes

$$\hat{\boldsymbol{p}}=\boldsymbol{u}_2\,\boldsymbol{u}_1\boldsymbol{p}\,\bar{\boldsymbol{u}}_1\bar{\boldsymbol{u}}_2=\boldsymbol{T}_2\boldsymbol{T}_1\boldsymbol{p}=\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}\boldsymbol{p}\,. \tag{A.42}$$

In the general case of n rotations we find

$$\hat{\boldsymbol{p}}=\boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}}\boldsymbol{p}\,\bar{\boldsymbol{u}}_{\mathrm{I}}^{\mathrm{r}}=\boldsymbol{T}_n\boldsymbol{T}_{n-1}\cdots\boldsymbol{T}_2\boldsymbol{T}_1\boldsymbol{p}=\boldsymbol{T}_{\mathrm{I}}^{\mathrm{r}}\boldsymbol{p} \tag{A.43}$$

in accordance with the chain rule of (51).

Let us now consider the sequence II^{r} of the rotations $\boldsymbol{\omega}_n$, $\boldsymbol{\omega}_{n-1}\cdots\boldsymbol{\omega}_i\cdots\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_1$. Without further ado we write

$$\begin{aligned}\boldsymbol{u}_{\mathrm{II}}^{\mathrm{r}}&=c_{\mathrm{II}}(1+\boldsymbol{\omega}_{\mathrm{II}})=\boldsymbol{u}_1\boldsymbol{u}_2\cdots\boldsymbol{u}_i\cdots\boldsymbol{u}_{n-1}\boldsymbol{u}_n\\&=c(1+\boldsymbol{\omega}_1)(1+\boldsymbol{\omega}_2)\cdots(1+\boldsymbol{\omega}_i)\cdots(1+\boldsymbol{\omega}_{n-1})(1+\boldsymbol{\omega}_n)\end{aligned} \tag{A.44}$$

where $c_{\mathrm{II}}=c_{\mathrm{I}}$ and c is given by (A.28).

Following our reasoning in Section 2 we may also assert that our preceding discussion applies equally to the follower sequences II^{f}, I^{f} in place of the rigid sequences I^{r}, II^{r}, respectively. In fact, we have

$$\boldsymbol{u}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{u}_{\mathrm{II}}^{\mathrm{r}} \quad \text{and} \quad \boldsymbol{u}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{u}_{\mathrm{I}}^{\mathrm{r}} \tag{A.45}$$

and

$$c_{\mathrm{I}}^{\mathrm{f}}(1+\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}) = c_{\mathrm{II}}^{\mathrm{r}}(1+\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}}), \qquad c_{\mathrm{II}}^{\mathrm{f}}(1+\boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}}) = c_{\mathrm{I}}^{\mathrm{r}}(1+\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}) \tag{A.46}$$

in which

$$c_{\mathrm{I}}^{\mathrm{f}} = c_{\mathrm{II}}^{\mathrm{f}} = c_{\mathrm{I}}^{\mathrm{r}} = c_{\mathrm{II}}^{\mathrm{r}} \tag{A.47}$$

and

$$\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{r}} \quad \text{and} \quad \boldsymbol{\omega}_{\mathrm{II}}^{\mathrm{f}} = \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{r}}\,. \tag{A.48}$$

Alternatively, we may construct the follower sequence I^{f} of versors emanating from the follower pseudovectors $\boldsymbol{\omega}_i^{\mathrm{f}}$ of Section 2. In fact,

$$\boldsymbol{u}_1 = c_1(1+\boldsymbol{\omega}_1), \quad \boldsymbol{u}_2^{\mathrm{f}} = c_2(1+\boldsymbol{\omega}_2^{\mathrm{f}}), \ldots, \quad \boldsymbol{u}_{n-1}^{\mathrm{f}} = c_{n-1}(1+\boldsymbol{\omega}_{n-1}^{\mathrm{f}}), \quad \boldsymbol{u}_n^{\mathrm{f}} = c_n(1+\boldsymbol{\omega}_n^{\mathrm{f}})\,. \tag{A.49}$$

As a result this follower sequence yields

$$\boldsymbol{u}_{\mathrm{I}}^{\mathrm{f}} = c_{\mathrm{I}}(1+\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{f}}) = c(1+\boldsymbol{\omega}_n^{\mathrm{f}})(1+\boldsymbol{\omega}_{n-1}^{\mathrm{f}}) \cdots (1+\boldsymbol{\omega}_i^{\mathrm{f}}) \cdots (1+\boldsymbol{\omega}_1) \tag{A.50}$$

and must lead to the same $\boldsymbol{u}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{u}_{\mathrm{II}}^{\mathrm{r}}$ as (A.44). The associated rotation matrix

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}} = \boldsymbol{T}_n^{\mathrm{f}} \boldsymbol{T}_{n-1}^{\mathrm{f}} \cdots \boldsymbol{T}_i^{\mathrm{f}} \cdots \boldsymbol{T}_2^{\mathrm{f}} \boldsymbol{T}_1 \tag{A.51}$$

is of course but for the notation identical with (45) and presents an alternative but equivalent expression to the rotation matrix $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{r}} = \boldsymbol{T}_{\mathrm{I}}^{\mathrm{f}}$ of (5a). At this stage of familiarity with the subject it is superfluous to repeat the above account for sequence II^{f}.

However, a brief reference to the case of semitangential rotations is called for. We first introduce the notation

$$c_i^{\mathrm{s}} = \cos \vartheta_i^{\mathrm{s}}/2 \tag{A.52}$$

to define the angle of rotation ϑ_i^{s} of the pseudovector $\boldsymbol{\omega}_i^{\mathrm{s}}$ given by

$$\boldsymbol{\omega}_i^{\mathrm{s}} = (\boldsymbol{\omega}_i^{\mathrm{r}} + \boldsymbol{\omega}_i^{\mathrm{f}})/2\,. \tag{A.53}$$

Then the corresponding versor $\boldsymbol{u}_i^{\mathrm{s}}$ is

$$\boldsymbol{u}_i^{\mathrm{s}} = c_i^{\mathrm{s}}(1+\boldsymbol{\omega}_i^{\mathrm{s}})\,. \tag{A.54}$$

As a result we obtain for the sequence I^{s},

$$\boldsymbol{u}_{\mathrm{I}}^{\mathrm{s}} = c_{\mathrm{I}}^{\mathrm{s}}(1 + \boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}}) = c^{\mathrm{s}}(1 + \boldsymbol{\omega}_{n}^{\mathrm{s}})(1 + \boldsymbol{\omega}_{n-1}^{\mathrm{s}}) \cdots (1 + \boldsymbol{\omega}_{i}^{\mathrm{s}}) \cdots (1 + \boldsymbol{\omega}_{1}) \tag{A.55}$$

from which we deduce $c_{\mathrm{I}}^{\mathrm{s}} = \cos \vartheta_{\mathrm{I}}^{\mathrm{s}}/2$ and the compound pseudovector $\boldsymbol{\omega}_{\mathrm{I}}^{\mathrm{s}}$. The associated rotation matrix becomes clearly

$$\boldsymbol{T}_{\mathrm{I}}^{\mathrm{s}} = \boldsymbol{T}^{\mathrm{s}} = \boldsymbol{T}_{n}^{\mathrm{s}} \boldsymbol{T}_{n-1}^{\mathrm{s}} \cdots \boldsymbol{T}_{i}^{\mathrm{s}} \cdots \boldsymbol{T}_{2}^{\mathrm{s}} \boldsymbol{T}_{1} \,. \tag{A.56}$$

For the sequence II^{s} and $\boldsymbol{T}_{\mathrm{II}}^{\mathrm{s}} = \boldsymbol{T}^{\mathrm{s}}$ the corresponding results are written down by inspection.

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