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Finite Rotations in Space

Space $\equiv \mathbb{R}^n$ $\quad n =$ dimension

General Linear Transformation: Let $T : \mathbb{R}^n \to \mathbb{R}^n$

T is said to be linear if, given any pair of $u, v \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

Component form

In particular, let $\{e_i, i = 1, \dots, n\}$ be a basis of $\mathbb{R}^n$ & express

$$u = u_i e_i \qquad \text{(summation convention)}$$

Then

$$T(u) = T(u_i e_i) = u_i T(e_i)$$

Let

$$\boxed{T(e_i) = T_{ji} e_j} \quad (T_{ji}\ j^{th} \text{ component of } T(e_i))$$

T_{ij} = components of matrix of transformation (w.r.t. e_i)

Say

$$v = T(u)$$

But

$$v = v_i e_i = u_j T(e_j) = u_j T_{ij} e_i$$

$$\Rightarrow \boxed{v_i = T_{ij} u_j} \quad \text{(component form of } T \text{ w.r.t. } \{e_i\})$$

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Def

Let $L(\mathbb{R}^n)$ be set of all L.T.s from $\mathbb{R}^n \to \mathbb{R}^n$

Note: once a basis is chosen $L(\mathbb{R}^n)$ can be identified with the set of n-D square matrices.

Composition of LTs Say $T, S \in L(\mathbb{R}^n)$

Then the operation

$$u :\to S(T(u))$$

defines a LT denoted $S \circ T$. ($\circ \equiv$ composition operator)

Properties of Composition: Matrix of $S \circ T$ is ST (matrix multiplication)

Def A LT is said to be an isomorphism iff
$\boxed{\det T \neq 0}$ (T is invertible or nonsingular)

Def Set of all isomorphisms over $\mathbb{R}^n$ is called $GL(n)$, the general linear group.

note that by props of matrix mult,

(1) $\forall\, R, S \in GL(n)$, $RS \in GL(n)$ (closedness)

(2) $\forall\, R, S, T \in GL(n)$, $R(ST) = (RS)T$ (associativity)

(3) $\forall\, T \in GL(n)$, $\exists\, T^{-1} \in GL(n)$ s.t. $T^{-1}T = TT^{-1} = I \equiv$ Identity matrix

Thus from (1)-(3), $GL(n)$ is a group w.r.t. matrix multiplication

Note: In general $ST \neq TS$ (LTs do not commute)

$\Rightarrow GL(n)$ is not an algebraic group

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Orthogonal Transformations

Def An LT is said to be orthogonal if it preserves the length of vectors, i.e.,

$$|Tu| = |u| \quad \forall\, u \in \mathbb{R}^n \qquad |u| = \langle u, v\rangle^{1/2}$$

$$\langle u, v\rangle = u_i v_i$$

Proposition: T orthogonal $\iff T^T T = I$

proof: $|Tu|^2 = T_{ij} u_j T_{ik} u_k = |u| = \delta_{jk} u_j u_k$

$\iff T_{ij} T_{ik} = \delta_{jk} \iff T^T T = I$

~~Prop~~ Def: Let $O(n)$ be set of all orthogonal transformations

Prop: $S, T \in O(n) \implies ST \in O(n)$

Prop: $T \in O(n) \implies T^{-1} \in O(n)$

Corollary: $O(n)$ is a subgroup of $Gl(n)$ which is called the orthogonal group.

Properties of $O(n)$:

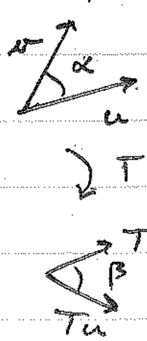
(1) preserve inner products. $T \in O(n)$ &

$u, v \in \mathbb{R}^n \implies \langle T(u), T(v)\rangle = T_{ij} u_j T_{ik} v_k$

$= u_j \delta_{jk} v_k$

$= \langle u, v\rangle$

(2) preserve angles



$$\cos\alpha = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}$$

$$\cos\beta = \frac{\langle Tu, Tv\rangle}{\|Tu\|\,\|Tv\|} \ldots$$

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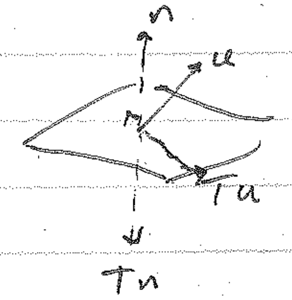
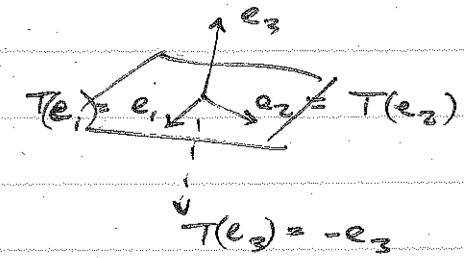
(3) $\det T = \pm 1$: $1 = \det(T^T T) = (\det T)^2$

Are all $T \in O(n)$ rotations? No. Take, e.g. a reflection about a plane. This is orthogonal but not a rotation. To see how apply it to basis w/ $e_3 = n$





$\{e_1, e_2, e_3\}$ right handed
$\{Te_1, Te_2, Te_3\}$ left " $\Big\} \Rightarrow$ T not <u>orientation preserving</u>

Note $\{e_1, e_2\}$ are eigenvectors of T w/ evals = 1
e_3 is evec w/ eval = -1

$\therefore \det(T) = \lambda_1 \lambda_2 \lambda_3 = -1$

Reflections are orientation inverting.

<u>Def</u> A proper orthogonal transformation is one s.t.
$\boxed{\det T = +1}$ (orientation preserving)

<u>Def</u> Set of all proper orthog. trans. is called the special orthogonal group, $SO(n)$

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Properties of $SO(n)$

(1) $\forall R, S \in SO(n)$, $RS \in SO(n)$ ($RS \in O(n)$ & $\det(RS) = \det R \det S = 1$) (~~Proof easy~~)

(2) $\forall R \in SO(n)$ $R^{-1} \in SO(n)$ $(\det R^{-1} = (\det R)^{-1} = 1)$

(3) $SO(n)$ is a subgroup of $O(n)$

(4) Every $R \in SO(2)$ can be expressed as

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ some angle θ (finite rotation)

(5) ~~Every~~ Every $R \in SO(3)$ has an eval $= 1$

Proof Let λ be eval of R. Then

$$Ru = \lambda u, \quad |Ru| = |\lambda||u| \Rightarrow |\lambda| = 1$$

$\Rightarrow$ all λ's are in unit circle in $\mathbb{C}$.

Char eqn $\det(R - \lambda I) = 0$ (cubic

Roots in conjugate pairs $\Rightarrow$ at least one is real, $\lambda_3 = 1$ or $\lambda = -1$

Two possible cases:

(a) all λ_i real $\Rightarrow$ Since $\det R = 1$
λ's must be $(1, 1, 1)$ or $(1, -1, -1)$

(b) Two evals are complex conjugates
$(z, \bar{z}, \lambda_3)$
$\hookrightarrow \det R = |z| \lambda_3 = \lambda_3 = 1$

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6) Eulers Thm Every $R \in SO(3)$ is a finite rotation by some angle θ about some axis e ($|e|=1$)

Proof: Let e be unit evect of R, i.e.,

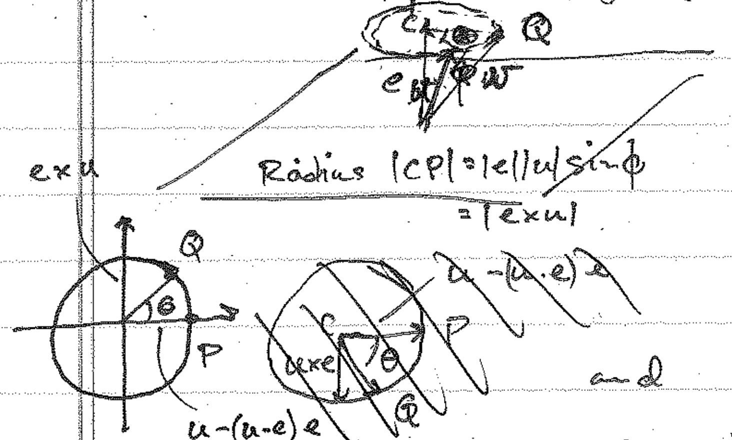
$$\boxed{Re = e}$$

$\therefore$ in any basis $\{e_1, e_2, e\}$

$$R = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

but $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2) \Rightarrow R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(7) Vector expression of finite rotations



Say $v = Ru$

$OC = (u\cdot e)e = (v\cdot e)e$

$CP = OP - OC = u - (u\cdot e)e$

$CQ = OQ - OC = v - (u\cdot e)e$

and $v = Ru = OC + CQ$

$= (u\cdot e)e + [u - (u\cdot e)e]\cos\theta$

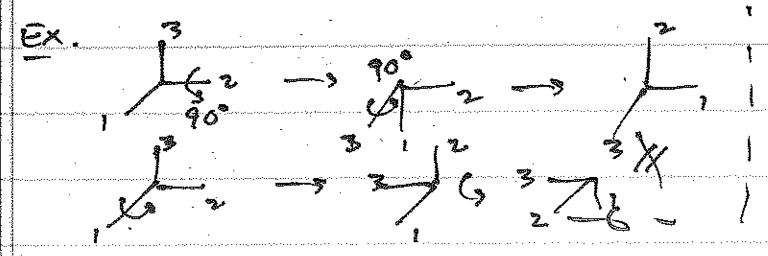
(~~Rodrigues' formula~~ The Canonical Transformation) $+ (\cancel{u\times e}\ e\times u)\sin\theta$

R Determined by a unit vec, e, defining axis of rotation, and angle of rotation θ.

(8) In general $R, S \in SO(n)$ do not commute

(only if coaxial $\Rightarrow \in SO(2) \Rightarrow$ commute)

Ex.



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Infinitesimal Rotations:

Consider time dependent rotation $R(t) \in SO(n)$

At time $t+dt$, $R(t+dt) = R(t) + \dot{R}(t)\,dt + \cdots$

Let $\boxed{\Omega(t) \equiv \dot{R}(t) R(t)^T}$ ($\equiv$ Spin rate tensor)

Then,

$$\boxed{\dot{R}(t) = \Omega(t) R(t)}$$

and

$$R(t+dt) = R(t) + \Omega(t) R(t)\,dt$$
$$= (I + \Omega(t)\,dt)\, R(t) + \cdots$$

∴ The incremental rotation is composition of the initial rotation and $(I + \Omega\,dt) \equiv$ infinitesimal rotation

Properties: (1) $\Omega^T = -\Omega$

proof: $R \in SO(n) \Rightarrow I = R(t+dt)^T R(t+dt) = [I + \Omega\,dt]^T \underbrace{R^T R}_{I} [I + \Omega\,dt]$

$$= I + (\Omega^T + \Omega)\,dt + \text{H.O.T.}$$

$$\Rightarrow \Omega^T + \Omega = 0$$

(2) Can express $\Omega \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

with $\underline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ = axial vector of Ω (angular velocity or vorticity)

Note: $\Omega_{ij} = -\epsilon_{ijk}\,\omega_k$ and $\omega_i = -\frac{1}{2}\epsilon_{ijk}\,\Omega_{jk}$

* Easy to show: $\Omega u = \omega \times u$, so $\boxed{(I + \Omega\,dt)u = u + dt\,\omega \times u}$

Short-hand: $\omega = \hat{\Omega}$ $\quad \Omega = \hat{\omega}$

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Exponential Representation of Rotations

Given square matrix A, define

$$e^A \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

Alternative def'n: Let $\{\lambda_\alpha\}$ be evals & $\{u^\alpha\}$, $\{v^\alpha\}$ be L. & R. evecs of A.

$$A u^\alpha = \lambda_\alpha u^\alpha \quad \text{(no sum on } \alpha) \qquad \text{note: } u^\alpha \cdot v^\beta = \delta^{\alpha\beta}$$

$$A^T v^\alpha = \lambda_\alpha v^\alpha$$

Spectral Thm

$$A = \sum_\alpha \lambda_\alpha u^\alpha \otimes v^\alpha$$

$$A_{ij} = \sum_\alpha \lambda_\alpha u_i^\alpha v_j^\alpha$$

Then define:

$$e^A = \sum_\alpha e^{\lambda_\alpha} u^\alpha \otimes v^\alpha$$

i.e., e^A shares evecs with A, & has evals e^{λ_α}.

Properties:

(1) $\det(e^A) = e^{tr(A)}$ $\qquad tr\, A = A_{ii}$

(2) $(e^A)^T = e^{A^T}$

(3) $e^{A+B} = e^A e^B = e^B e^A$ iff A & B commute $(AB = BA)$ (i.e., share evecs)

Proposition: $R \in SO(n) \iff R = e^W$ for some $W = -W^T$

Proof: $(\Leftarrow)$ Let $W = -W^T$ then show $e^W \in SO(n)$.

$$(e^W)^T(e^W) = e^{W^T} e^W = e^{W^T + W} = e^0 = I$$

and $\det(e^W) = e^{tr(W)} = e^0 = 1$

$\therefore e^W \in SO(n)$ ✓

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($\Rightarrow$) Let $R \in SO(n)$, show $\exists\, W = -W^T$ s.t. $R = e^W$

By Euler's Thm

$$v = Ru = (u\cdot e)e + [u - (u\cdot e)e]\cos\theta + (e\times u)\sin\theta$$

Let $R(s)$ be rotation of same axis but angle $s\theta$

with $s \in [0,1]$

$$v(s) = R(s)u = (u\cdot e)e + [u - (u\cdot e)e]\cos(s\theta) + (e\times u)\sin(s\theta)$$

Rate

$$\frac{d}{ds}v(s) = \left(\frac{d}{ds}R(s)\right)u$$

$$= \theta\left[((u\cdot e)e - u)\sin s\theta + (e\times u)\cos s\theta\right]$$

$\{a\times(b\times c) = b(a\cdot c) - c(a\cdot b)$

$$= (\theta e)\times\left[(e\times u)\sin s\theta + u\cos s\theta\right]$$

$$= (\theta e)\times\left[v - (u\cdot e)e(1-\cos s\theta)\right]$$

$$= \theta e \times v$$

$$= Wv \qquad W = \widehat{(\theta e)}$$

$$\therefore \left(\frac{dR}{ds}\right)u = Wv = WRu \qquad \forall u$$

$$\Rightarrow \frac{dR(s)}{ds} = WR(s) \qquad W = \text{const} \ \& \ R(0) = I$$

Const coeff ODE's, sol'n:

$$\boxed{R(s) = e^{sW}}$$

<u>Note:</u> $R(0) = e^0 = I \quad \& \; R = R(1) = e^W$

Thus a finite rotation is entirely determined by the 3 DOF in W

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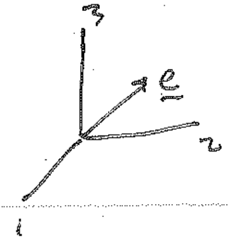
Also note: $W = \theta \hat{e} = \theta \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$



$\therefore \quad R = \exp[\theta \hat{e}]$

Can show also that

$$\underset{\sim}{R} = \exp[\theta \hat{e}] = \underset{\sim}{I} + \sin\theta \, \underset{\sim}{\hat{e}} + (1-\cos\theta) \, \underset{\sim}{\hat{e}}^2$$

$$\begin{aligned} \therefore \quad Ru &= u + \sin\theta \; e \times u + (1-\cos\theta) \; e \times (e \times u) \\ &= u + \sin\theta \; e \times u + (1-\cos\theta) \left[e(e \cdot u) - u \right] \\ &= \cos\theta \; u + \sin\theta \; e \times u + (1-\cos\theta) \; e(e \cdot u) \end{aligned}$$

Same as ~~[illegible]~~ the Canonical Transformation.