Introduction

In this assignment, our goals are to

- implement 3-node and 6-node triangular membrane elements
- implement assembly of force and stiffness matrices for a mesh of these elements
- implement finite element analysis of some example problems using the membrane elements

1 Finite Element Formulation of a Membrane

Membrane theory can be derived from shell theory with the assumption that the shell is very thin and does not offer resistance to bending or transverse shear. We also assume that behaviour of the membrane through its thickness is same as that for the mid-surface and hence it is sufficient to calculate the behaviour of just the mid-surface. The change in the thickness direction can be represented by a single parameter, the thickness stretch, λ.

Let $\mathbf{a}$ and $\mathbf{A}$ denote the curvilinear basis vectors in the current and reference configurations of the mid-surface respectively. If x_{ia} and X_{ia} denote the co-ordinates of the nodes of the triangular membrane element in current and reference configuration respectively then tangent basis vectors in curvilinear frame are given as

$$a_\alpha = \mathbf{x}_{,\alpha} = x_{ia}N_{a,\alpha}$$
$$A_\alpha = \mathbf{X}_{,\alpha} = X_{ia}N_{a,\alpha}$$

where N_a are the shape functions for triangular elements derived in HW 2 and $\alpha = 1, 2$. The third curvilinear tangent basis vector is assumed to always be normal to the other two. This means,

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$$
$$\mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|}$$

For further calculations, we require the dual basis vectors in the refernce configuration of mid-surface, $\mathbf{A}^i$. These can be derived as follows

$$A_{ij} = \mathbf{A}_i \cdot \mathbf{A}_j$$
$$A^{ij} = A_{ij}^{-1}$$
$$\mathbf{A}^i = A^{ij}\mathbf{A}_j$$

where A_{ij} is the metric tensor and A^{ij} is the inverse metric tensor. It should be pointed out that as a consequence of our assumptions in membrane theory we have $\mathbf{A}_3 = \mathbf{A}^3$.

1.1 Deformation Gradient

Using the curvilinear basis vectors derived in the previous section we can write the deformation gradient of the element as

$$\mathbf{F} = \mathbf{a}_\alpha \otimes \mathbf{A}^\alpha + \lambda \mathbf{a}_3 \otimes \mathbf{A}^3$$

where λ is the thickness stretch ratio. We need to provide an initial guess for λ to start with. Then, using the deformation gradient and the Lamé coefficients, we can calculate strain energy density, w, the first Piola-Kirchhoff stress tensor $\mathbf{P}$ and the stiffness modulus C_{iJkL} based on the compressible neo-Hookean constitutive model developed in HW 1. Now, we are in position to implement plane stress condition for the triangular element.

1.2 Plane-stress condition

The component of traction vector in the direction of surface normal in the current configuration can be written as

$$T(\lambda) = \mathbf{a}_3 \cdot (\mathbf{P}(\lambda)\mathbf{A}_3)$$

To enforce planes stress we need to solve for λ such that $T(\lambda) = 0$. We will use Newton-Raphson method to solve for λ. We need Taylor series expansion of $\mathbf{P}(\lambda)$ and $T(\lambda)$.

$$\begin{aligned}
\mathbf{P}(\lambda + d\lambda) &\approx \mathbf{P}(\lambda) + \frac{\partial \mathbf{P}}{\partial \lambda}\mathrm{d}\lambda \\
&= \mathbf{P}(\lambda) + \frac{\partial \mathbf{P}}{\partial \mathbf{F}} : \frac{\partial \mathbf{F}}{\partial \lambda}\mathrm{d}\lambda \\
&= \mathbf{P}(\lambda) + \mathbb{C} : \frac{\partial \mathbf{F}}{\partial \lambda}\mathrm{d}\lambda \\
&= \mathbf{P}(\lambda) + \mathbb{C} : (\mathbf{a}_3 \otimes \mathbf{A}^3)\mathrm{d}\lambda
\end{aligned}$$

Therefore,

$$\begin{aligned}
T(\lambda + \mathrm{d}\lambda) &\approx \mathbf{a}_3 \cdot (\mathbf{P}(\lambda + \mathrm{d}\lambda)\mathbf{A}_3) \\
&= T(\lambda) + \mathbf{a}_3 \cdot \left(\mathbb{C} : (\mathbf{a}_3 \otimes \mathbf{A}^3)\mathbf{A}^3\right)\mathrm{d}\lambda \\
&= T(\lambda) + (\mathbf{a}_3 \otimes \mathbf{A}^3) : \mathbb{C} : (\mathbf{a}_3 \otimes \mathbf{A}^3)\mathrm{d}\lambda
\end{aligned}$$

If $T(\lambda + \mathrm{d}\lambda) = 0$ then

$$\mathrm{d}\lambda = -T(\lambda)\left[(\mathbf{a}_3 \otimes \mathbf{A}^3) : \mathbb{C} : (\mathbf{a}_3 \otimes \mathbf{A}^3)\right]^{-1}$$

This gives the update to λ for the Newton-Raphson interations. The term in the bracket is C_{3333}, component of stiffness modulus in the curvilinear frame. It can be computed in terms of C_{iJkL} which is in the lab-frame as

$$C_{3333} = (\mathbf{a}_3)_i(\mathbf{A}_3)_J C_{iJkL}(\mathbf{a}_3)_k(\mathbf{A}_3)_L$$

1.3 Strain Energy

After solving for λ using Newton-Raphson method for plane stress, we can re-calculate $\mathbf{F}$. Thus, we can calculate strain energy density, w, as per the compressible neo Hookean constitutive model of HW 2. The strain energy of the element, W, can be written as

$$\begin{aligned}
W &= \int_{V^e} w\,\mathrm{d}V^e \\
&= \int_{\Omega^e} w\,\mathrm{d}\Omega^e H
\end{aligned}$$

where H is membrane thickness and Ω is the mid-surface. We need to evaluate the integral using Gauss quadrature for triangular region developed in HW 2. Let n_G be the number of Gauss quadrature points, $\hat{\Omega}$ be the area of standard parametric triangle and $\hat{w}$ be the weight for the Gauss quadrature point. Then we can write

$$\begin{aligned} W &= \int_{\Omega^e} w \, \mathrm{d}\Omega^e H \\ &= \int_{\hat{\Omega}} wH\sqrt{A} \, \mathrm{d}\hat{\Omega} \\ &= \sum_{q=1}^{n_G} w\hat{w}_q H\sqrt{A}\hat{\Omega} \end{aligned}$$

where $A = |A_{ij}|$ is determinant of the metric tensor.

1.4 Internal and External Forces

The internal nodal forces can be written in terms of stress-resultant $\mathbf{n}^\alpha$.

$$\mathbf{n}^\alpha = \mathbf{P} \cdot \mathbf{A}^\alpha H$$

The internal forces are

$$\begin{aligned} f_{ia}^{\text{int}} &= \int_{\hat{\Omega}} n_i^\alpha N_{a,\alpha}\sqrt{A} \, \mathrm{d}\hat{\Omega} \\ &= \sum_{q=1}^{n_G} n_i^\alpha N_{a,\alpha}\Big|_{\theta_q} \hat{w}_q\sqrt{A}\hat{\Omega} \end{aligned}$$

The external forces are obtained from the distributed transverse load on the element, $\mathbf{f}$,

$$\begin{aligned} f_{ia}^{\text{ext}} &= \int_{\hat{\Omega}} f_i N_a\sqrt{A} \, \mathrm{d}\hat{\Omega} \\ &= \sum_{q=1}^{n_G} f_i N_a\Big|_{\theta_q} \hat{w}_q\sqrt{A}\hat{\Omega} \end{aligned}$$

1.5 Stiffness Modulus

To calculate the element stiffness modulus, K_{iakb}, we need to calculate the curvilinear components of Kirchhoff stress tensor, $\boldsymbol{\tau}$,

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{P}\mathbf{F}^T \\ &= \mathbf{F}(\mathbf{F}^{-1}\mathbf{P})\mathbf{F}^T \\ &= \mathbf{F}\mathbf{S}\mathbf{F}^T \end{aligned}$$

where $\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$ is the second Piola-Kirchhoff stress tensor. $\mathbf{S}$ and $\boldsymbol{\tau}$ have the same components but different basis vectors. The relationship can be written as

$$\mathbf{S} = \tau^{\alpha\beta}\mathbf{A}_\alpha \otimes \mathbf{A}_\beta$$

where $\tau^{\alpha\beta}$ are the components of $\boldsymbol{\tau}$ in the curvilinear frame which can be obtained as

$$\tau^{\alpha\beta} = \mathbf{A}^{\alpha} \cdot \left(\mathbf{S}\mathbf{A}^{\beta}\right)$$

We also need the components of stiffness modulus in terms of $\boldsymbol{\tau}$

$$C^{ijkl} = \frac{\partial \tau^{ij}}{\partial a_{kl}}$$

where a_{kl} are components of metric tensor in current configuration. We can obtain this using the stiffness modulus in terms of the second Piola-Kirchoff stress tensor

$$C_{IJKL} = \frac{\partial S_{IJ}}{\partial C_{KL}}$$

where C_{KL} are components of the left Cauchy Green strain tensor. We can calculate C_{IJKL} from C_{iJkL} which is the stiffness modulus in terms of the first Piola-Kirchoff stress tensor.

$$C_{IJKL} = \frac{1}{2} F^{-1}_{Ii} F^{-1}_{Kk} \left(C_{iJkL} - \delta_{ik} S_{JL}\right)$$

Now,

$$C^{ijkl} = C_{IJKL} (\mathbf{A}^{i})_{I} (\mathbf{A}^{j})_{J} (\mathbf{A}^{k})_{K} (\mathbf{A}^{l})_{L}$$

where $(\cdot)_I$ represents the I^{th} component in lab-frame of the vector in the brackets. When we impose the plane-stress condition, we force $\tau^{33} = 0$. This requires modifying $C^{\alpha\beta\gamma\delta}$ such that it is consistent, that is, it can be obtained by differentiating $\tau^{\alpha\beta}$ with respect to $a^{\gamma\delta}$. The modified form is given as

$$\tilde{C}^{\alpha\beta\gamma\delta} = C^{\alpha\beta\gamma\delta} - \frac{C^{\alpha\beta 33}}{C^{3333}} C^{33\gamma\delta}$$

Now we have all components we need to write the element stiffness modulus.

$$\begin{aligned}
K_{iakb} &= \underbrace{\int_{\hat{\Omega}} \left(\tilde{C}^{\alpha\beta\mu\nu} (\mathbf{a}_{\beta} \otimes \mathbf{a}_{\nu})_{ik} N_{a,\alpha} N_{b,\mu}\right) \sqrt{A}\, \mathrm{d}\hat{\Omega}}_{\text{Material Stiffness}} \\
&+ \underbrace{\int_{\hat{\Omega}} \left(\tau^{\alpha\beta} N_{a,\alpha} N_{b,\beta}\right) \sqrt{A}\, \mathrm{d}\hat{\Omega}}_{\text{Geometric Stiffness}} \\
&= \sum_{q=1}^{n_G} \left(\tilde{C}^{\alpha\beta\mu\nu} (\mathbf{a}_{\beta} \otimes \mathbf{a}_{\nu})_{ik} N_{a,\alpha} N_{b,\mu}\right) \Bigg|_{\theta_q} \hat{w}_q \sqrt{A} \hat{\Omega} \\
&+ \sum_{q=1}^{n_G} \left(\tau^{\alpha\beta} N_{a,\alpha} N_{b,\beta}\right) \Bigg|_{\theta_q} \hat{w}_q \sqrt{A} \hat{\Omega}
\end{aligned}$$

It is important to note that in computer implementation the geometric and material stiffness components need to be computed in different loops because they don't have same number of repeated indices.

1.6 Verification Tests

We can verify our computer implementation of the triangular membrane elements using consistency test and by calculating rank of stiffness modulus in zero and finite deformation cases.

1.6.1 Consistency Test

We have implemented the calculation of strain energy, internal force and stiffness modulus for the isoparametric triangular membrane elements. These quantities are interrelated as

$$f_{ia}^{\text{int}} = \frac{\partial W}{\partial x_{ia}} \qquad \text{and}$$

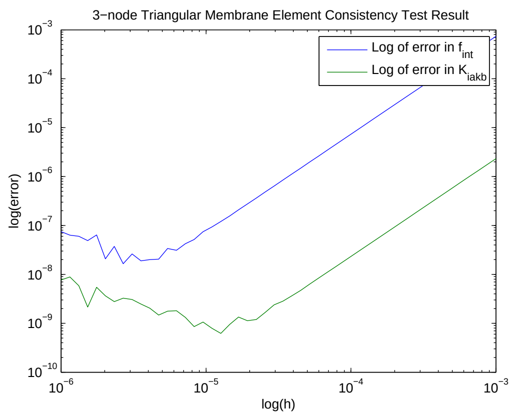
$$K_{iakb} = \frac{\partial f_{ia}^{\text{int}}}{\partial x_{kb}}$$

We can use these two equations to check our implementation for bugs. If there are no bugs, the finite difference approximation of derivatives of W and f_{ia}^{int} with respect to x_{ia} and x_{kb} should match f_{ia}^{int} and K_{iakb}, obtained directly by our code, respectively to acceptable numerical tolerance. The central finite difference approximation of the above equations are given as

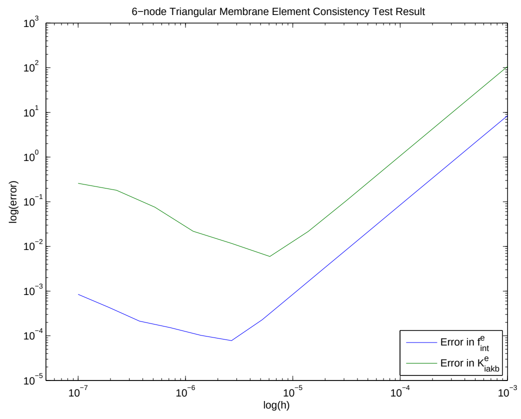
$$f_{ia}^{\text{int}} \approx \frac{W(x_{ia}+h) - W(x_{ia}-h)}{2h}$$

$$K_{iakb} \approx \frac{f_{ia}^{\text{int}}(x_{kb}+h) - f_{ia}^{\text{int}}(x_{kb}-h)}{2h}$$

We chose several values for perturbation $h \in (10^{-7}, 10^{-3})$. The error in the numerical derivatives scaled quadratically with h as can be seen from Figure 1. For the smaller values of h the effect of large rounding errors can be noticed in Figure 1a.



(a) 3-node triangular membrane element



(b) 6-node triangular membrane element

Figure 1: Log-log plot of error versus h shows straight lines with slope 2 indicating error of $\mathcal{O}(h^2)$ as expected for a central finite difference numerical differentiation. Smaller vlues of h introduce large rounding errors as seen in Figure 1a

Table 1: Rank of stiffness matrix for linear and quadratic membrane elements

Deformation	Shape Function	Quadrature	Rank
Zero	Linear	1-point	3
	Quadratic	3-point	9
Finite	Linear	1-point	6
	Quadratic	3-point	15

1.6.2 Stiffness Matrix Rank

The fourth order stiffness modulus K_{iakb} can be *unrolled* into a square 2-D matrix of size $ia \times kb$. For linear 3-node triangular element this will give a 9×9 matrix and for the quadratic 6-node triangular element we get a 18×18 matrix. We calculated the rank of the stiffness matrix for the case of zero deformation i.e. $\mathbf{X} = \mathbf{x}$ and for finite deformation for both linear and quadratic shape functions. The results are presented in Table 1. In case of zero deformation for 3-node element we have only 1 quadrature points. Thus, in 3-D we have 3 rotational and 3 translational degree of freedom for that point. Hence, the stiffness matrix is rank-deficient by 6 which gives rank as $9 - 6 = 3$. For zero-deformation case for 6-node element we have 3 quadrature points which form a triangle. We need to count number of unique rigid body motions such that this triangle remains undeformed. We can have 3 rigid-body translations, 3 rotations about the 3 sides of this triangle and 3 rotations about axis passing through each of the quadrature point and perpendicular to plane of the triangle. Thus, we have 9 degrees of freedom. This we should have rank as $18 - 9 = 9$. In case of finite deformation case, the rotational degrees of freedom are no longer zero-energy modes. So, we expect the rank to increase by 3 and 6 for the 3-node and 6-node elements respectively. Thus, the results presented in Table 1 are consistent with our expectations.

2 Assembly

In finite element analysis, the geometrical domain of the problem is discretized into a mesh of finite elements. Our mesh will be using the finite elements we formulated in the previous section. We need to compute the combined resultant energy, force and stiffness of the whole mesh based on contribution from each element. This is done in the process of assembly in finite element method.

2.1 Meta Arrays

We make use of two "meta" arrays during the assembly process.

- **IEN**: This is the ***E**lement **N**odes* array. It stores the mesh connectivity information. Every node in the mesh is assigned a *global node number*. Each row of **IEN** represents a single triangular element in the mesh. The columns of **IEN** store the global node numbers for the vertices of the triangle represented by that row. In case of 3-node triangular elements **IEN** is of size $n_{el} \times 3$ where n_{el} is the total number of elements in the mesh. For the 6-node triangular membrane element the size of **IEN** is $n_{el} \times 6$.

- **ID**: This is the ***D**estination* array. It stores the *global equation number* corresponding to each *global degree of freedom*. For our elements, each node in the mesh has 3 degrees of freedom associated with it. We call these as the 3 *local degrees of freedom* of each node. Each element has $3 \times n_{nodes\ per\ element}$ local degrees of freedom. So the total number of degrees of freedom in a mesh is given by

$$\text{Total degrees of freedom} = 3 \times \text{total number of nodes}$$

The degree of freedom numbers associated with a global node numbered a can be obtained as

$$GDOF(a) = 3(a-1) + i \qquad \text{where} \quad i = 1, 2, 3$$

In, each row of **ID** array the first column is the global degree of freedom number and the second column is the *global equation number.* If a particular degree of freedom in the mesh is constrained due to a prescribed boundary condition then the second column in **ID** array for that global degree of freedom is set to 0. *Thus, we assemble the force and stiffness matrices only for the unknown degrees of freedom.*

2.2 Potential Energy Assembly

Total *strain energy* of the mesh is the algebraic sum of strain energy of each element. The total *potential energy* of the mesh is the the difference between the total strain energy and the work done by external forces. Since, we know the external force acting on each node of each element and we can calculate the nodal displacements, we can calculate the external work done on the element as

$$W_e^{ext} = \mathbf{f}_e^{ext} \cdot \mathbf{u}_e$$

Therefore, we can write

$$\Pi = \sum_{e=1}^{n_{el}} (W_e^{\text{int}} - W_e^{\text{ext}})$$

2.3 Force Assembly

Using the **IEN** and **ID** arrays, we can get the global equation number for each local degree of freedom for each element. Consider the case of 3-node triangular element. It has 9 local degrees of freedom. The element force vectors will be column vectors with 9 elements, $\mathbf{f}^e$. Suppose the 4^{th} local degree of freedom corresponds to 115^{th} global equation number. Certainly, as the global equation number is non-zero, this is an unknown degree of freedom and thus we have to assemble its contribution to the global force matrix, $\mathbf{f}$. The force (either internal or external) assembly entails that

$$f_{115} \leftarrow f_{115} + f_4^e$$

where $\leftarrow$ indicates assignment operation in a computer program. After assembling both the internal and external forces we can calculate the global residual force vector as

$$\mathbf{r} = \mathbf{f}^{\text{int}} - \mathbf{f}^{\text{ext}}$$

2.4 Stiffness Assembly

Consider the case of 3-node triangular element. It has 9 local degrees of freedom. The stiffness matrix, $\mathbf{K}^e$ will, therefore, be of size 9×9. Suppose the 4^{th} and 6^{th} local degrees of freedom correspond to 115^{th} and 227^{th} global equation numbers respectively. Certainly, as the global equation numbers are non-zero, these are unknown degrees of freedom and thus we have to assemble their contribution to the global stiffness matrix, $\mathbf{K}$. The assembly entails that

$$\begin{aligned}
K_{115,115} &\leftarrow K_{115,115} + K^e_{4,4} \\
K_{115,227} &\leftarrow K_{115,227} + K^e_{4,6} \\
K_{227,115} &\leftarrow K_{227,115} + K^e_{6,4} \\
K_{227,227} &\leftarrow K_{227,227} + K^e_{6,6}
\end{aligned}$$

where $\leftarrow$ indicates assignment operation in a computer program. Since both K^e and K are symmetric, we could have used only three assignments instead of four.

2.5 Verification Tests

We will test our implementation for bugs using consistency test and by checking the rank of stiffness matrix for zero and finite deformations.

2.5.1 Consistency Test

The global potential energy Π, residual force $\mathbf{r}$ and the global stiffness modulus $\mathbf{K}$ are inter-related as

$$\begin{aligned}
r_{ia} &= \frac{\partial \Pi}{\partial x_{ia}} \\
K_{iakb} &= \frac{\partial r_{ia}}{\partial x_{kb}}
\end{aligned}$$

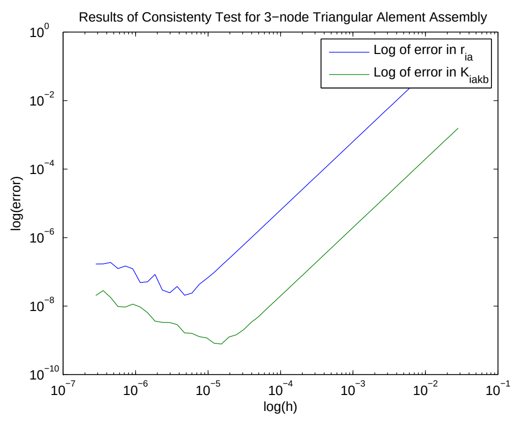
We can use these to test our implementation by comparing the results obtained from our assembly sub-routine with the numerical derivatives calculated using 3-point finite difference scheme. The equations for numerical derivatives are

$$\begin{aligned}
r_{ia} &\approx \frac{W(x_{ia}+h) - W(x_{ia}-h)}{2h} \\
K_{iakb} &\approx \frac{r_{ia}(x_{kb}+h) - r_{ia}(x_{kb}-h)}{2h}
\end{aligned}$$

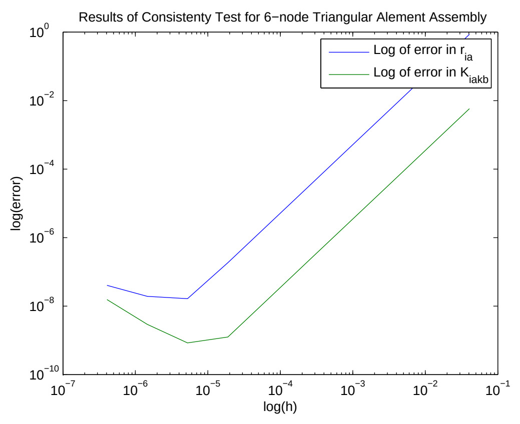
We chose several values for perturbation $h \in (10^{-7}, 10^{-3})$. The error in the numerical derivatives scaled quadratically with h as can be seen from Figure 2.

2.5.2 Rank of Stiffness Matrix

The global stiffness matrix is a square matrix with number of rows and columns equal to the number of unknown global degrees of freedom. We chose a square mesh of two triangular elements, as shown in Figure 3 and constrained one of the corner nodes to have zero displacement in all three directions. Then we prescribed zero and finite displacements for other nodes of the element and



(a) Assembly of 3-node triangular elements



(b) Assembly of 6-node triangular elements

Figure 2: Log-log plot of error versus h shows straight lines with slope 2 indicating error of $\mathcal{O}(h^2)$ as expected for a central finite difference numerical differentiation. Smaller vlues of h introduce large rounding errors.

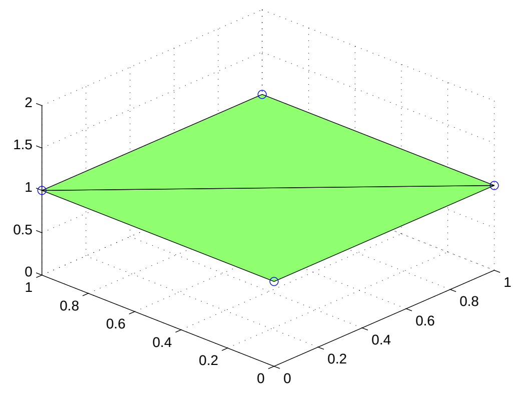
Table 2: Rank of Global Stiffness matrix after assembly for Linear and Quadratic membrane elements

Shape Function	Deformation	Rank
Linear	Zero	9
	Finite	9
Quadratic	Zero	24
	Finite	24

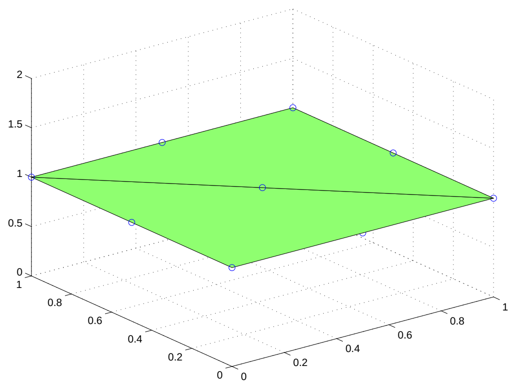
calculated the rank of stiffness matrix in both cases. We did this for both the 3-node and the 6-node element. The results are summarized in the table 2 For the 3-node element mesh, there are 4 global nodes and 12 global degrees of freedom. We have constrained 3 of these degrees of freedom. So we are left with 9 unknown degrees of freedom. Each row and column of the 9×9 assembled stiffness matrix corresponds to one of the unknown degree of freedom. Since, we want our system of equations to give us a unique answer for each unknown degree of freedom we expect that the assembled stiffness matrix has full rank. In our example, it should have rank 9. Similar, analysis for the mesh of 6-node elements shows that we should expect the assembled stiffness matrix to have rank 24. The results shown in Table 2 are thus consistent with our expectations. It is very important to highlight at this point that in the zero deformation case, it was necessary to perturb the unconstrained degrees of freedom in the transverse direction by a small amount. Otherwise, the membrane mesh would be perfectly flat and offer no resistance in the transverse direction giving rise to unwanted rank deficiency in the stiffness matrix.

3 Solving the Equilibrium Problem

The object of our finite element analysis is to calculate nodal displacements that lead to the system being in equilibrium characterized by zero residual force vector. The residual force vector is a non-linear function of the current configuration and we can use Newton-Raphson method to solve for equilibrium displacement as follows



(a) Mesh of 3-node elements



(b) Mesh of 6-node elements

Figure 3: Meshes used to verify the rank of stiffness matrix obtained after assembly. The nodes in the mesh have been circled.

3.1 Newton-Raphson Method for Equilibrium

$$\begin{aligned}\mathbf{r}(\mathbf{x}+\mathbf{u}) &\approx \mathbf{r}(\mathbf{x}) + \frac{\partial \mathbf{r}}{\partial \mathbf{x}}\mathbf{u} \\ &= \mathbf{r}(\mathbf{x}) + \mathbf{K}\mathbf{u}\end{aligned}$$

We want $\mathbf{r}(\mathbf{x}+\mathbf{u}) = 0$, therefore,

$$\mathbf{u} = \mathbf{K}^{-1}\mathbf{r}$$

The $\mathbf{u}$ represents the update required to our current guess for $\mathbf{x}$. We repeat this till the residual is driven to zero to acceptable numerical tolerance.

3.2 Incremental Solution Strategy

When the mesh is subjected to relatively large forces or displacements, the Newton iterations will not converge unless we have a very good initial guess. To resolve this problem we can adopt an incremental solution strategy. We can divide the total prescribed force or displacement in small increments such that for the first increment from zero our Newton iterations for equilibrium converge with the reference configuration itself as our initial guess for the solution. For subsequent increments in load (force or displacement) we can use the solution obtained for the previous increment as the initial guess.

4 Applications

4.1 Planar Isotropic Stretch

We have to solve the problem of a square sheet stretched by a planar isotropic deformation($F_{11} = F_{22} = \lambda, F_{12} = F_{21} = 0$) imposed by prescribing displacements of nodes on the boundaries. Figure 4 shows the reference and deformed configuration mesh, obtained by our finite element

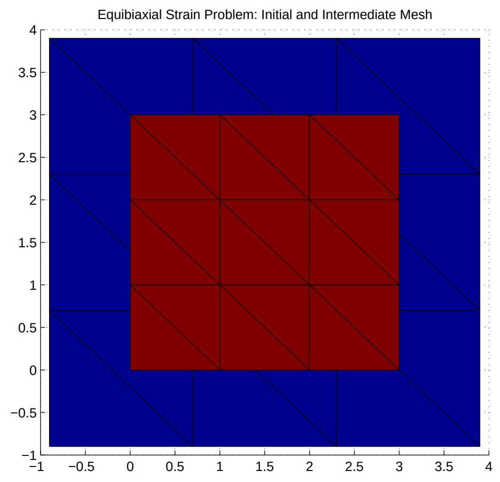


Figure 4: The reference configuration mesh has been superimposed on the deformed mesh. Brown mesh represents the reference configuration and blue mesh represents the deformed configuration obtained by our finite element analysis. The square sheet has stretched isotropically as expected.

analysis, superimposed. Using the incremental solution strategy discussed in previous section, we solved for the deformed configuration for various planar isotropic stretches and compressions of the initial mesh. As a post-processing step for every isotropic stretch, we calculated the mean deformation gradient for all elements and checked if the deformation gradient for any element deviated significantly from the mean. In all cases, the deviation was within numerical tolerance and thus we found that the deformation gradient was uniform across the mesh. It should be noted

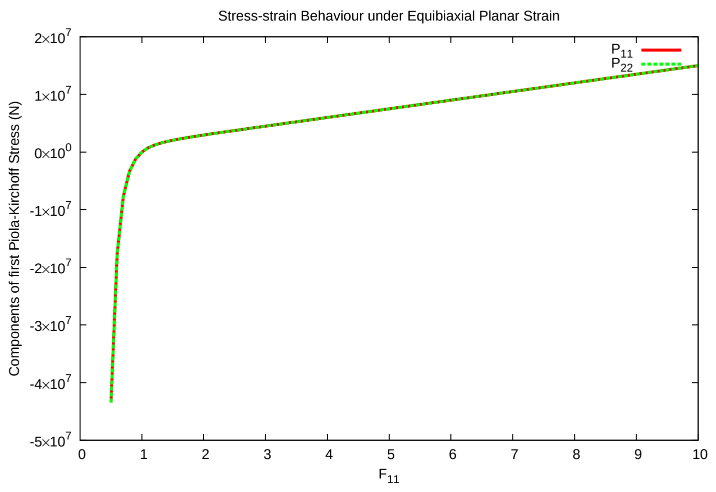
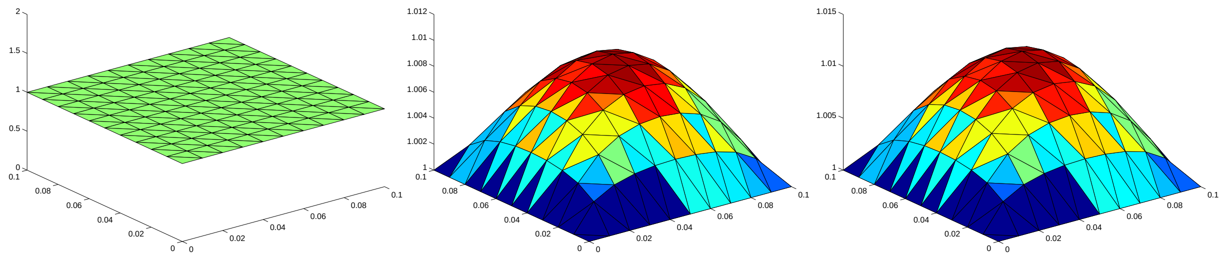


Figure 5: P_{11} and P_{22} components of first Piola-Kirchhoff stress tensor plotted against a component of deformation gradient F_{11}. This result is same as obtained in HW 1 for compressible neo Hookean material.

that we had to prescribe the transverse displacement of all nodes to be zero. Otherwise, as the reference configuration is a perfectly flat mesh, the stiffness matrix has zero stiffness components in the transverse direction which leads to a singular matrix that cannot be inverted as required for our equilibrium Newton iterations. By prescribing the transverse displacement we ensure that those degrees of freedom are not assembled into our global stiffness matrix.

4.2 Simply Supported Membrane under Transverse Load

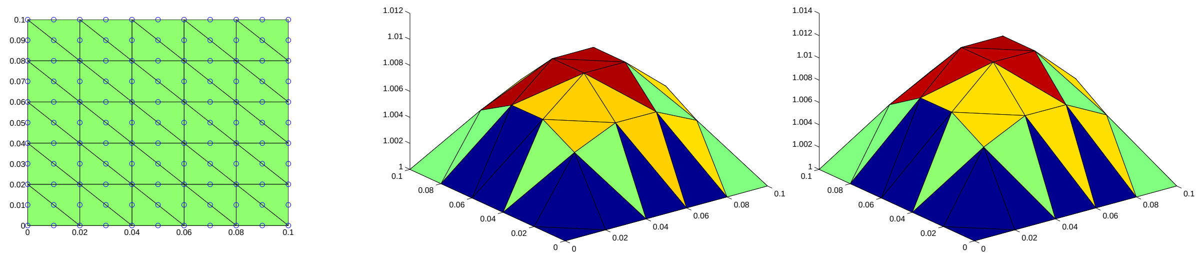
We have to solve the problem of a simply supported square membrane under a uniform transverse load. Side length of the square mesh was chosen to be $L = 10\ cm$. Reference thickness $H = 0.1\ cm$, shear modulus, $\mu_0 = 4 \times 10^5\ N/m^2$ and $\lambda_0 = 10\mu_0$. We will use incremental solution strategy for applied transverse load. We will increase it in increments of 10 N/m^2 from 0 to 1000 N/m^2. Figure 6 shows the mesh at zero, half the maximum and the maximum load full load. We can



(a) Undeformed mesh, 200 elements (b) Deformed mesh under half of maximum load (c) Deformed mesh under maximum load

Figure 6: The reference configuration mesh and the deformed mesh at half load and full load using 3-node triangular membrane elements are shown. The maximum deflection at half load is only slightly less than the deflection at full-load.

observe that the deformed mesh at half-load is almost identical to the deformed mesh at full load. It indicates that the increase in deflection goes on decreasing rapidly as the transverse load increases. This is expected because as the membrane deflects under the transverse load its area increases and it has to stretch. The more it stretches, the stiffer it becomes and thus it takes larger force to stretch it further. This relationshiip between maximum deflection andthe transverse load can also be noticed in Figure 9. Figure 7 shows the mesh of 6-node triangular membrane elements



(a) Top view of undeformed mesh, 50 elements (b) Deformed mesh under half of maximum load (c) Deformed mesh under maximum load

Figure 7: The reference configuration mesh and the deformed mesh at half load and full load using 6-node triangular membrane elements are shown. The deformed mesh plots are misleading because the triangles in the mesh don't show the mid-side nodes.

at zero, half of the maximum and maximum transverse load. Figure 7a is the top-view of the initial mesh. It also shows the mid-side nodes. 6-node triangular elements have mid-side nodes which make the sides of the triangle quadratic functions instead of straight lines. But this detail is missing from Figure 7. A better representation of the displacement field is shown in Figure 8. It is obtained by doing a Delaunay triangulation of the cloud of points formed by the nodes of the original mesh and then plotting the displacement field over it. Like Figure 6, this representation also has the drawback that the triangles in the mesh are not actually the elements but it gives better visualization of the displacement field. 6-node quadratic element mesh gives more accurate solution with lesser number of elements as compared to the mesh of 3-node triangular elements.

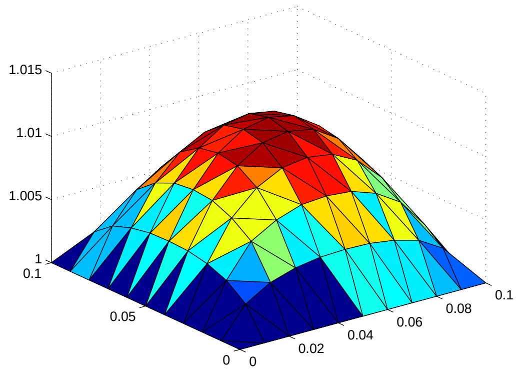
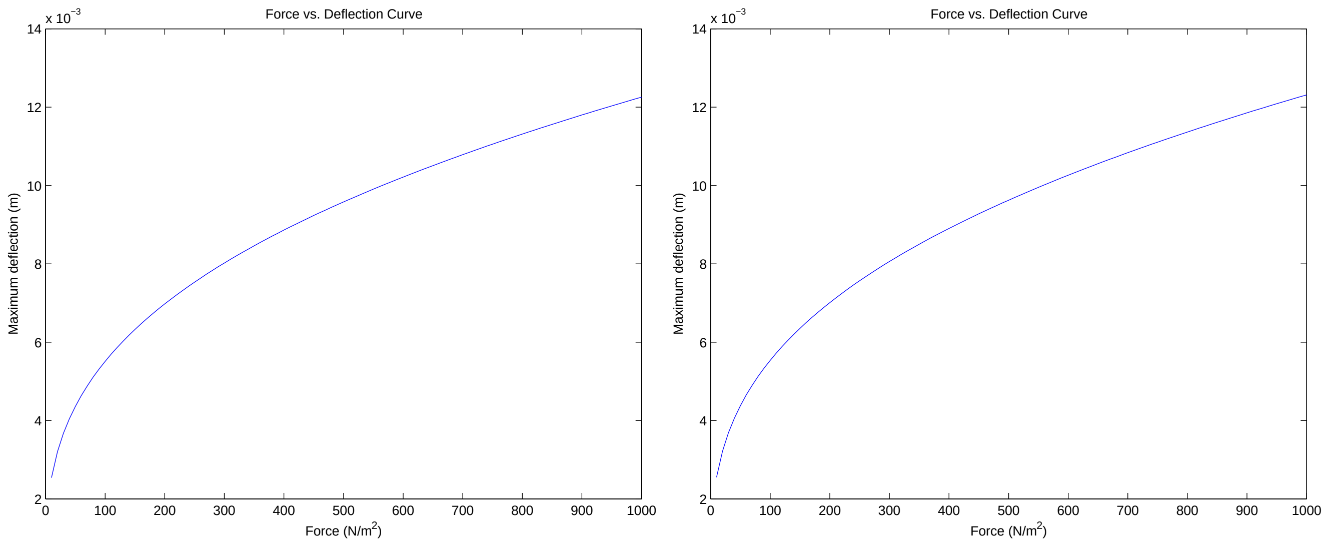


Figure 8: Alternative representation of deformed 6-node triangular element mesh. But the triangles in this figure are not the elements.



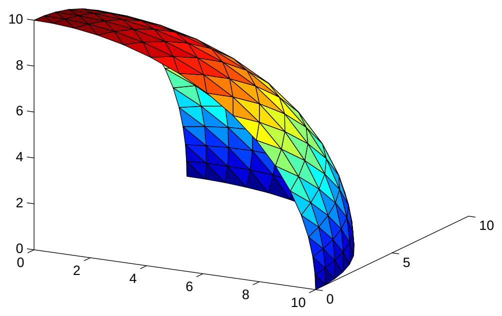
(a) For 3-node element mesh

(b) For 6-node elements mesh

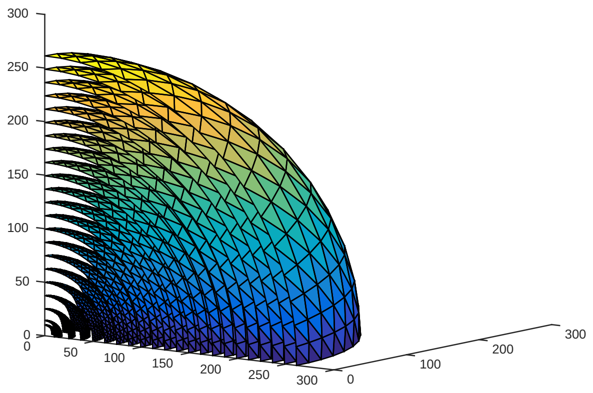
Figure 9: Force-deflection curve for the 3-node and 6-node element meshes. The slope decreases as the maximum deflection increases. This shows that it requires more force to bring about equal increase in maximum deflection as the membrane deforms.

4.3 Spherical balloon

We have to sovle the problem of a spherical balloon of initial radius $R = 10\ cm$ and thickness $H = 0.1\ cm$ under a uniform internal pressure. We used incremental solution strategy as before. The initial mesh and several deformed mesh are shown in Figure 10. Figure 11 shows the variation in stretch-ratio defined as $\lambda = r/R$ where r and R are the current and reference configuration radii. The plot is similar to Figure 5. The Newton-Raphson iterations for enforcing plane-stress at element level could not drive the normal component of traction vector below 1×10^{-6} after a pressure of 60 N/m^2 because the Newton updates became of the order of machine precision. But as the component is still much smaller than the traction vector in the plane of membrane, we continued to solve for equilibrium for larger pressures till 220 N/m^2 when the Newton iterations for the equilibrium stopped converging.



(a) Initial Mesh of radius $R = 10\ cm$



(b) Several deformed mesh of increasing radii

Figure 10: The initial mesh and many deformed meshes of an octant of a spherical membrane subjected to uniform internal pressure.

5 Source Code Listing

- `assemblyT3Lin.m`: This function performs assembly for a mesh of 3-node triangular membrane elements.
- `assemblyT6Quad.m`: This function performs assembly for a mesh of 6-node triangular membrane elements.
- `BalloonProblem.m` This is the driver script from the balloon problem.
- `calcAllCs.m`: This function performs the plane-stress enforcement at element level and calculates various stiffness moduli required to calculate the membrane element stiffness modulus.
- `calcFandP.m`: This function is a post-processing function used to calculate deformation gradient and first Piola-Kirchoff stress tensor. It is used to check for uniform deformation gradient.

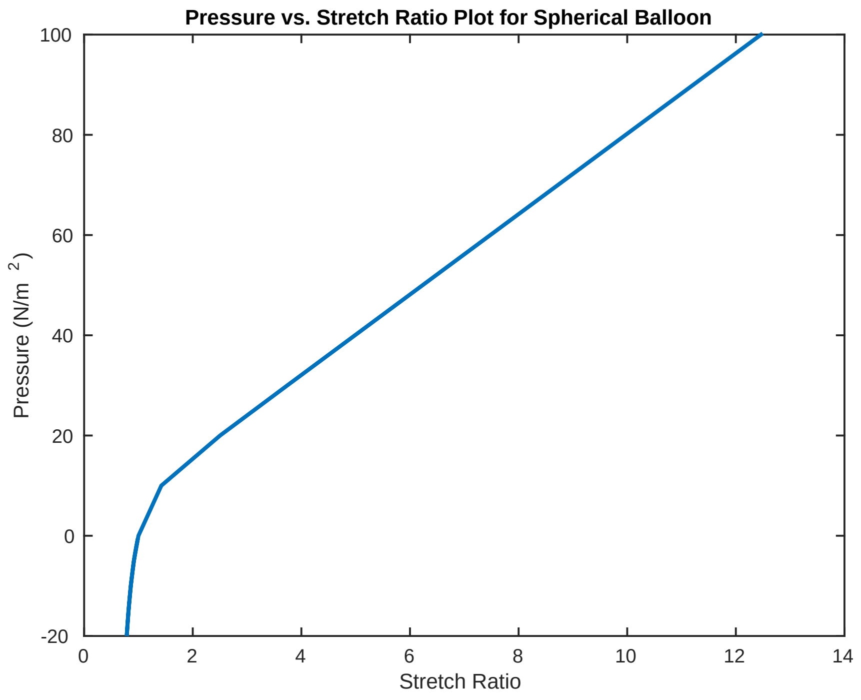


Figure 11: Pressure versus stretch-ratio plot for the balloon problem. It shows similar behavior as the equibiaxial strain problem in Figure 5

- `convertToT6Mesh.m`: This function is used to add mid-side nodes to a 3-node triangular element mesh and regenerate a connectivity matrix to get a 6-node triangular element mesh.
- `EquibiaxialStrain.m`: This is the driver script for the isotropic stretch problem.
- `equiTriMesh.m`: This fucntion creates a mesh of triangles in a large equilateral triangles. It is used to generate the spherical balloon octant mesh.
- `findStiffnessRank.m`: This function calculates rank of a matrix.
- `getBCmatrix.m`: This function accepts a cell-array containing infformation about boundary condition in terms of nodal co-ordinates and local degree of freedom for a mesh and generates a matrix containing the boundary condition specified with respect to global degree of freedom number.
- `neoHookean.m`: This function calculates the strain energy density, first Piola-Kirchhoff stress tensor and the corresponding tangent modulus.
- `plotMesh.m`: This function generates a 3-D plot of a mesh and saves it to file.
- `plotQuiver.m`: This function plots force vector field on a mesh for visualization.
- `Element_Stiffness_rank.m`: This script calculates the rank of stiffness matrix for 3-node and 6-node membrane elements under zero or finite deformations.
- `T3_assy_consistency_check.m`: This script performs consistency check for `assemblyT3Lin.m`

- `T3_assy_Stiffness_Rank.m`: This script calculates the stiffness rank for 3-node triangular element mesh under zero and finite deformation.
- `T3Lin.m`: This function calcuates the 3-node linear triangular shape functions and its derivatives.
- `T3MembraneEle.m`: This function calculates strain energy, nodal force vectors and stiffness modulus for a 3-node triangular membrane element.
- `T3MembraneEle_Verification.m`: This script checks `T3MembraneEle.m` for consistency.
- `T6_assy_consistency_check.m`: This script performs consistency check for `assemblyT6Quad.m`
- `T6_assy_Stiffness_Rank.m`: This script calculates the stiffness rank for 6-node triangular element mesh under zero and finite deformation.
- `T6Quad.m`: This function calcuates the 6-node linear triangular shape functions and its derivatives.
- `T6MembraneEle.m`: This function calculates strain energy, nodal force vectors and stiffness modulus for a 6-node triangular membrane element.
- `T6MembraneEle_Verification.m`: This script checks `T6MembraneEle.m` for consistency.
- `TransverseLoadExample.m`: This is the driver script for transverse load problem.
- `TriGaussQuad.m`: Returns the Gauss-quadrature points and weights for 1-point and 3-point schemes.