MEMBRANES

Membrane theory: For very thin shells assume no stress response to bending or transverse shear.

Recall weak form

$$\int_{\Omega_0} (n^\alpha \cdot \delta a_\alpha + \lambda n^3 \cdot \delta d + \lambda m^\alpha \cdot \delta d_{,\alpha})\, dS - \delta W^{ext} = 0$$

$$\Rightarrow \quad n^\alpha = \frac{\partial w}{\partial a_\alpha} \quad n^3 = \frac{\partial w}{\partial d} \quad m^\alpha = \frac{\partial w}{\partial d_{,\alpha}}$$

(Generalized internal forces)

set $d = a_3 = \dfrac{a_1 \times a_2}{\sqrt{a}}$.

No bending & shear $\Rightarrow$ $n^3 = 0 \quad m^\alpha = 0 \quad n^\alpha \cdot a_3 = 0 \rightarrow n^\alpha = n^{\alpha\beta} a_\beta$

Let resultant of ext. loads be a distributed force $\underset{\sim}{f}$ on the middle surface

$$\Rightarrow \delta W^{ext} = \int_{\Omega_0} f \cdot \delta x \, dS$$

Equilib. in plane $\quad \displaystyle\int_{\Omega_0} n^\alpha \cdot \delta a_\alpha \, dS = \int_{\Omega_0} f \cdot \delta x \, dS \qquad n^\alpha = \int_{-H/2}^{H/2} P \cdot G^\alpha \mu \, d\theta^3 \simeq \underbrace{P \cdot A^\alpha \cancelto{1}{\mu}\Big|_{\theta^3 = 0}}_{\text{Evaluate on middle surface}} H$

across thickness $\quad \displaystyle\int_{\Omega_0}\int_{-H/2}^{H/2} P : [\delta\lambda (d \otimes - \ + \theta d_{,\alpha} \otimes G^\alpha)]\, \mu \, d\theta \, dS = 0$

$$= \int_{\Omega_0}\int_{-H/2}^{H/2} (a_3 \cdot P \cdot A_3 + \underbrace{\theta}_{\ll 1} a_{3,\alpha} \cdot P \cdot G^\alpha)\, \delta\lambda \, \mu \, d\theta^3 dS$$

neglect

$$= \int_{\Omega_0}\int_{-H/2}^{H/2} a_3 \cdot P \cdot A_3 \, \delta\lambda \, \mu \, d\theta^3 dS = 0$$

Enforce this with local Newton iteration. $\quad a_3 \cdot P \cdot A_3 \equiv T(\lambda) = 0$

$$F = a_\alpha \otimes A^\alpha + \lambda a_3 \otimes A_3 \qquad P(\lambda + d\lambda) = P(\lambda) + \frac{\partial P(\lambda)}{\partial F} : (d\lambda \, a_3 \otimes A_3)$$

$\dfrac{\partial P}{\partial F} = \mathbb{C}$ Lagrangian tangent moduli.

$$\Rightarrow T(\lambda + d\lambda) = T(\lambda) + (a_3 \otimes A_3) : \mathbb{C}(\lambda) : (a_3 \otimes A_3)\, d\lambda = 0$$

$$\rightarrow d\lambda = -T(\lambda)\left[(a_3 \otimes A_3) : \mathbb{C} : (a_3 \otimes A_3)\right]^{-1} \quad \text{(Newton Update)}$$

Note: if storing components of all vectors & tensors in the Lab frame $\{E_I\}$ then

$$a_3 \otimes A_3 : \mathbb{C} : a_3 \otimes A_3 = (a_3)_I (A_3)_J \, \mathbb{C}_{IJKL} (a_3)_K (A_3)_L .$$

$\therefore$ we're left with in-plane equilibrium

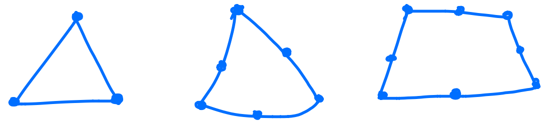
$$D\Pi[x](\delta x) = \int_{\Omega_0} n^\alpha \cdot \delta a_\alpha \, dS - \int_{\Omega_0} f \cdot \delta x \, dS = 0$$

Depends only on middle surface $x = x(\theta^\alpha)$.

FE Interpolation: $\delta a_\alpha = \delta x_{,\alpha}$ & $n^\alpha \to P(a_\alpha \otimes G^\alpha)$ $\Rightarrow$ 1st Derivatives in weak form.

↳ Can use C^0-conforming elements.

E.g., Lagrange elements



$$\underset{\sim}{x}_h(\theta^\alpha) = \sum_{a=1}^{N} \underset{\sim}{x}_a N_a(\theta^\alpha) \qquad \underset{\sim}{x}_a = x_{ia} \underset{\sim}{E}_i \quad \{\underset{\sim}{E}_i\} = \text{Lab Frame}$$

Isoparametric:

$$\underset{\sim}{X}_h(\theta^\alpha) = \sum_{a=1}^{N} \underset{\sim}{X}_a N_a(\theta^\alpha) \qquad \underset{\sim}{X}_a = X_{ia} \underset{\sim}{E}_i$$

Galerkin:

$$\delta x = \sum_{a=1}^{N} \delta x_a N_a(\theta^\alpha) \longrightarrow \delta \underset{\sim}{a}_\alpha = \delta \underset{\sim}{x}_{,\alpha} = \sum_a \delta \underset{\sim}{x}_a N_{a,\alpha}$$

$$\Rightarrow D\Pi[x_h](\delta x_h) = \int_{\Omega_0} \left[n^\alpha \cdot \left(\sum_a \delta x_a N_{a,\alpha} \right) - f \cdot \left(\sum_a \delta x_a N_a \right) \right] dS$$

$$= \sum_a \left\{ \int_{\Omega_0} n^\alpha N_{a,\alpha} \, dS - \int_{\Omega_0} f N_a \, dS \right\} \cdot \delta x_a$$

$$= \sum_a \left(\underset{\sim}{f}_a^{int} - \underset{\sim}{f}_a^{ext} \right) \cdot \delta \underset{\sim}{x}_a = 0$$

Where

$$\underset{\sim}{f}_a^{int} = \int_{\Omega_0} n^\alpha N_{a,\alpha} \sqrt{A} \, d\theta^1 d\theta^2 \equiv \text{Nodal Internal Force}$$

$$\underset{\sim}{f}_a^{ext} = \int_{\Omega_0} \underset{\sim}{f} N_a \sqrt{A} \, d\theta^1 d\theta^2 \equiv \text{Nodal External Force}$$

$$\underset{\sim}{r}_a = \underset{\sim}{f}_a^{int} - \underset{\sim}{f}_a^{ext} \equiv \text{Nodal Residual Force}$$

Comments:

1) Nodal positions are 3-D vectors $\Rightarrow$ membrane can be arbitrarily curved.
2) Internal force array $\underset{\sim}{f}^{int} = \underset{\sim}{f}^{int}(\underset{\sim}{x})$ is in general a NONLINEAR function of nodal position d.o.f.: $n^\alpha = \int P \cdot G^\alpha \mu \, d\theta^3$ $\quad P(F)$ nonlin.

   $\Rightarrow$ Need iterative solution strategy to solve for unknown $\underset{\sim}{x}$:

   – Newton, quasi-Newton, gradient descent methods (cf. Nocedal & Wright)
3) May need to linearize $f^{int}(x) \to$ Tangent Stiffness Matrix

$$f^{int}_{ia} = \frac{\partial \Pi^{int}}{\partial x_{ia}} \longrightarrow k_{iakb} = \frac{\partial f^{int}_{ia}}{\partial x_{kb}} = \frac{\partial^2 \Pi^{int}}{\partial x_{ia} \partial x_{kb}}$$

Linearize the weak form

$$G[x, \eta] \equiv D\Pi[x](\eta) = \frac{d}{d\epsilon} \Pi[x + \epsilon \eta]\Big|_{\epsilon=0} = \int_{\Omega_0} (n^\alpha \cdot \eta_{,\alpha} - f \cdot \eta)\, dS$$

$$D_1 G[x,\eta](\xi) \equiv \frac{d}{d\epsilon} G[x + \epsilon \xi, \eta]_{\epsilon=0} = \int_{\Omega_0} \frac{d}{d\epsilon} n^\alpha(x+\epsilon\xi)\Big|_{\epsilon=0} \cdot \eta_{,\alpha} \, dS$$

$$n^\alpha = P \cdot A^\alpha \mu H \longrightarrow \delta n^\alpha = \left(\frac{\partial P}{\partial F} : \delta F\right) \cdot A^\alpha H$$

$$\frac{\partial P}{\partial F} = \mathbb{C} \quad \delta F = \delta a_\beta \otimes A^\beta \Rightarrow \delta P = \mathbb{C} : \delta a_\beta \otimes A^\beta = (\mathbb{C} \cdot A^\beta) \cdot \delta a_\beta$$

In components rel. to the lab frame $(v_i = \underset{\sim}{v} \cdot \underset{\sim}{E}_i)$

$$\delta n^\alpha_i = \mathbb{C}_{ijkl} (\delta a_\beta)_k (A^\beta)_l (A^\alpha)_j \, H$$

$$\therefore D_1 G[x,\eta](\xi) = \int_{\Omega_0} \mathbb{C}_{ijkl} \, \eta_{i,\alpha} \, \xi_{k,\beta} \, (A^\beta)_l (A^\alpha)_j \, H \sqrt{A} \, d\theta^1 d\theta^2$$

$(\eta_h)_i = \sum_a \eta_{ia} N_a \quad (\xi_h)_k = \sum_b \xi_{kb} N_b$

$$\therefore D_1 G[x,\eta_h](\xi_h) = \sum_a \sum_b \Big[\int_{\Omega_0} \mathbb{C}_{ijkl}\, N_{a,\alpha} N_{b,\beta} (A^\alpha)_j (A^\beta)_l \, H\sqrt{A}\, d\theta^1 d\theta^2\Big] \eta_{ia} \xi_{kb}$$

$$= K_{iakb}\, \eta_{ia} \xi_{kb}$$

$$\Rightarrow K_{iakb} = \int_{\Omega_0} \mathbb{C}_{ijkl}\, N_{a,\alpha} N_{b,\beta} (A^\alpha)_j (A^\beta)_l \, H\sqrt{A}\, d\theta^1 d\theta^2$$

$\mathbb{C}_{ijkl}$ are the Plane Stress Lagrangian Tangent Moduli

Another way, use spatial form of n^α.

Define $\tau = P F^T = P\cdot G^i \otimes g_i$ = Kirchhoff Stress tensor $\quad P = \tau \cdot F^{-T} = \tau \cdot g^i \otimes G_i$

$$\Rightarrow P\cdot A^\alpha = \tau \cdot a^\alpha \quad \rightarrow \quad n^\alpha = \int \tau \cdot a^\alpha \mu\, d\theta^3$$

τ is a spatial tensor : $\tau = \tau^{ij} g_i \otimes g_j$ (on current config.)

Material version is 2nd P-K Stress :

$$S = F^{-1} P = F^{-1} \tau F^{-T} \qquad F = g_i \otimes G^i \rightarrow F^{-1} = G_i \otimes g^i$$

$$= G_i \otimes g^i (\tau^{kl} g_k \otimes g_l)\, g^p \otimes G_p$$

$$= \tau^{ij} G_i \otimes G_j \quad (\text{"pull-back" of } \tau)$$

$$\therefore \tau^{ij} = \tau : (g^i \otimes g^j) = S : (G^i \otimes G^j) = P : (g^i \otimes G^j)$$

Spatial components of τ = Material Components of S.

$$\Rightarrow P\cdot G^\alpha = F S\cdot G^\alpha = (g_i \otimes G^i)\cdot S \cdot G^\alpha = g_i \tau^{i\alpha} = \tau^{\alpha i} g_i \quad (\tau, S \text{ symmetric})$$

$$\rightarrow n^\alpha = \int \tau^{\alpha i} g_i \, \mu\, d\theta^3$$

Let's try to linearize this form.

$$dn^\alpha = \int d\tau^{\alpha i} g_i + \tau^{\alpha i} dg_i \, \mu\, d\theta^3.$$

First note,

$$S = \tau^{ij} G_i \otimes G_j \longrightarrow dS = d\tau^{ij} G_i \otimes G_j.$$

But recall, $P = \frac{\partial w}{\partial F} \Rightarrow dw = P : dF = (F \cdot S) : dF = S : \frac{1}{2} d(F^T F)$

$$\rightarrow S = 2\frac{\partial w}{\partial C} \quad C = F^T F = g_{ij} G^i \otimes G^j$$

(Doyle–Ericksen relation)

$$\therefore 2dw = S : dC = S : (dg_{ij} G^i \otimes G^j) = S : (G^i \otimes G^j) dg_{ij} = \tau^{ij} dg_{ij}$$

$$\rightarrow \tau^{ij} = 2\frac{\partial w}{\partial g_{ij}}$$

Easy to linearize this,

$$d\tau^{ij} = C^{ijkl} dg_{kl} \qquad C^{ijkl} \equiv \frac{\partial \tau^{ij}}{\partial g_{kl}} = 2\frac{\partial^2 w}{\partial g_{ij} \partial g_{kl}}.$$

$$\therefore \quad dn^\alpha = \int g_i \, d\tau^{\alpha i} + \tau^{\alpha i} dg_i \; \mu d\theta^3 = \int g_i C^{\alpha i kl} dg_{kl} + \tau^{\alpha i} dg_i \; \mu d\theta^3$$

But plane stress $\Rightarrow \tau^{\alpha 3} = 0$ & $\tau^{33} = 2\frac{\partial w}{\partial g_{33}} = 0$.

Also, on mid. surf. $\quad g_\alpha = a_\alpha \quad dg_{\alpha\beta} = da_{\alpha\beta} = a_\alpha \cdot da_\beta + a_\beta \cdot da_\alpha$

$$\therefore \quad dn^\alpha = \int a_\beta C^{\alpha\beta\gamma\delta}(a_\gamma \cdot da_\delta + a_\delta \cdot da_\gamma) + \tau^{\alpha\beta} da_\beta$$

$$= \int (a_\beta 2C^{\alpha\beta\gamma\delta} a_\delta \cdot da_\gamma + \tau^{\alpha\beta} \, da_\beta) \mu d\theta^3$$

$$= \left[\int 2C^{\alpha\beta\gamma\delta}(a_\beta \otimes a_\delta) + \tau^{\alpha\gamma} \underset{\sim}{I} \; \mu d\theta^3\right] \cdot da_\gamma$$

Use this to linearize the weak form

$$D_1 G[x, \eta](\xi) = \int_{\Omega_0} \frac{d}{d\epsilon} n^\alpha(x_{,\beta} + \epsilon \xi_{,\beta})\Big|_{\epsilon=0} \eta_{,\alpha} \, dS'$$

$$= \int_{\Omega_0} \left[2C^{\alpha\beta\gamma\delta}(a_\beta \otimes a_\delta) \cdot \xi_{,\gamma} + \tau^{\alpha\beta} \xi_{,\beta}\right] \eta_{,\alpha} \; H\sqrt{A} \, d\theta^1 d\theta^2$$

Now linearize f_a^{int},

$$\delta\Pi^{int} = f_a^{int} \cdot \delta x_a = G[x_h, \delta x_h] \qquad x_h = \sum_a \underset{\sim}{x}_a N_a \quad \delta x_h = \sum_a \delta x_a N_a$$

$$= \left[\int_{\Omega_0} n^\alpha N_{a,\alpha} \, dS'\right] \cdot \delta x_a$$

Now, $\Delta x_h \equiv \sum_a \Delta x_a N_a$

$$\Delta \delta\Pi^{int} = \Delta f_a^{int} \cdot \delta x_a = \left(\frac{\partial f_a^{int}}{\partial x_b} \cdot \Delta x_b\right) \cdot \delta x_a = K_{iakb} \, \delta x_{ia} \Delta x_{kb}$$

$$= D_1 G[x, \delta x_h](\Delta x_h) = \int_{\Omega_0} \left[2C^{\alpha\beta\gamma\delta}(a_\beta \otimes a_\delta) \cdot \Delta x_b N_{b,\gamma} + \tau^{\alpha\beta} \Delta x_b N_{b,\beta}\right] \cdot \delta x_a N_{a,\alpha} H\sqrt{A} \, d\theta^1 d\theta^2$$

$$K_{iakb} = \int_{\Omega_0} \Big[\underbrace{2C^{\alpha\beta\gamma\delta}(a_\beta \otimes a_\delta)_{ik}\, N_{a,\alpha} N_{b,\gamma}}_{\text{Material Stiffness}} + \underbrace{\tau^{\alpha\beta}\, \delta_{ik}\, N_{a,\alpha} N_{b,\beta}}_{\text{Geometric Stiffness}} \Big] H\sqrt{A}\, d\theta^1 d\theta^2$$

$$= \int_{\Omega_0} \big[2C^{\alpha\beta\gamma\delta}(a_\beta \otimes a_\delta)_{ik} + \tau^{\alpha\beta}\, \delta^\gamma_\beta\, \delta_{ik} \big] N_{a,\alpha} N_{b,\gamma}\, H\sqrt{A}\, d\theta^1 d\theta^2$$

Plane Stress Tangent Moduli

Recall $\tau^{\alpha\beta} = \tau^{\alpha\beta}(g_{\gamma\delta}, \lambda)$ $\lambda = g_{33}$ = thickness stretch.

Plane stress $\Rightarrow \frac{1}{2}\tau^{33} = \frac{\partial W}{\partial g_{33}} = \frac{\partial W}{\partial \lambda} = 0 \Rightarrow \lambda = \lambda(g_{\alpha\beta})$

$$\therefore\ d\tau^{\alpha\beta} = \frac{\partial \tau^{\alpha\beta}}{\partial g_{\mu\nu}} dg_{\mu\nu} + \frac{\partial \tau^{\alpha\beta}}{\partial g_{33}} d\lambda \quad \& \quad d\tau^{33} = \frac{\partial \tau^{33}}{\partial g_{\mu\nu}} dg_{\mu\nu} + \frac{\partial \tau^{33}}{\partial g_{33}} d\lambda = 0$$

$$\Rightarrow d\lambda = -(C^{3333})^{-1} C^{33\mu\nu} dg_{\mu\nu} \longrightarrow d\tau^{\alpha\beta} = \left(C^{\alpha\beta\mu\nu} - \frac{C^{\alpha\beta 33}}{C^{3333}} C^{33\mu\nu} \right) dg_{\mu\nu}$$

$\therefore$ Plane Stress moduli are $\tilde{C}^{\alpha\beta\mu\nu} = C^{\alpha\beta\mu\nu} - \frac{1}{C^{3333}} C^{\alpha\beta 33} C^{33\mu\nu}$

Implementation Note: If using a 3-D constitutive routine where components of $\mathbb{C}$ are computed in the lab frame, then we need to compute contravariant components for plane stress correction and for n^α. Here's how that works.

$$S = \tau^{ij} G_i \otimes G_j \rightarrow dS = \frac{\partial S}{\partial C} : dC = \mathbb{C} : dC$$

$$= \frac{\partial \tau^{ij}}{\partial g_{kl}} dg_{kl}\, G_i \otimes G_j$$

$$= C^{ijkl} G_i \otimes G_j \left[dC : (G_k \otimes G_l) \right]$$

$$= C^{ijkl} (G_i \otimes G_j \otimes G_k \otimes G_l) : dC$$

$$\Rightarrow \mathbb{C} = C^{ijkl} G_i \otimes G_j \otimes G_k \otimes G_l \qquad \Rightarrow C^{ijkl} = (G_i \otimes G_j) : \mathbb{C} : (G_k \otimes G_l)$$

Let $C_{IJKL} = \frac{\partial S_{IJ}}{\partial C_{KL}} = 2\frac{\partial^2 W}{\partial C_{IJ} \partial C_{KL}}$ be components of material moduli in lab frame $\{E_I\}$

$$\rightarrow C^{ijkl} = C_{IJKL} (G^i)_I (G^j)_J (G^k)_K (G^l)_L \quad \text{where } (G^i)_I = G^i \cdot E_I, \text{ etc.}$$

Also can show (exercise)

$$C_{iJkL} \equiv \frac{\partial P_{iJ}}{\partial F_{kL}} = \delta_{ik} S_{JL} + 2 F_{iI} F_{kK} C_{IJKL}$$

and

$$C_{IJKL} \equiv \frac{\partial S_{IJ}}{\partial C_{KL}} = \frac{1}{2} F^{-1}_{Ii} F^{-1}_{Kk} \left(C_{iJkL} - \delta_{ik} S_{JL} \right)$$

Use these relations to convert between 1st moduli $C_{iJkL} = \frac{\partial P_{iJ}}{\partial F_{kL}}$ and 2nd moduli $C_{IJKL} = \frac{\partial S_{IJ}}{\partial C_{KL}}$.