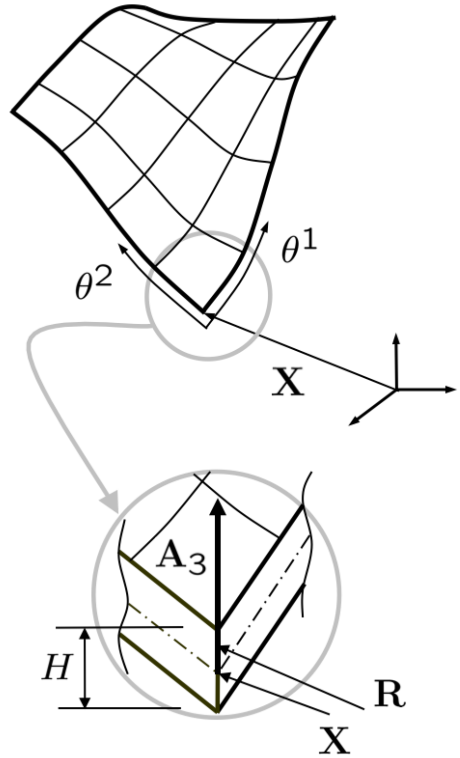
SHELL THEORY



$\mathcal{S}_0 = \Omega_0 \times [-\frac{H}{2}, \frac{H}{2}] \in \mathbb{R}^3$ ≡ Ref. Config

Ω_0 ≡ Reference Middle Surface

H ≡ Reference thickness

Points on the middle surface are defined parametrically by

$$\underset{\sim}{X} = \underset{\sim}{X}(\theta^1, \theta^2)$$

Points off the middle surface are defined

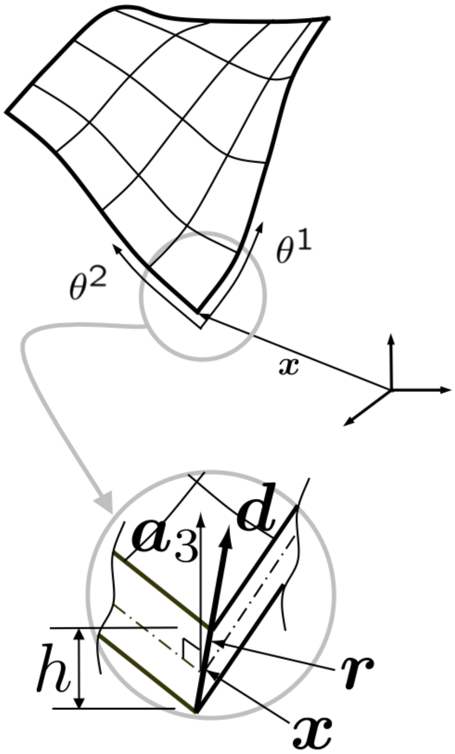
$$\underset{\sim}{R} = \underset{\sim}{\varphi}_0(\theta^1, \theta^2, \theta^3) = \underset{\sim}{X}(\theta^\alpha) + \theta^3 \underset{\sim}{A}_3(\theta^\alpha)$$

where

$A_3(\theta^\alpha)$ ≡ Normal to Ω_0 (unit)

$\{\theta^1, \theta^2\} \in \Omega_0$ ≡ convective curvilinear surface coordinates

$\theta^3 \in [-\frac{H}{2}, \frac{H}{2}]$ ≡ thickness coordinate



$\mathcal{S} \equiv \Omega \times [-\frac{h}{2}, \frac{h}{2}] \in \mathbb{R}^3$ ≡ Current/Deformed Config

Ω ≡ Deformed middle surface

$h = \lambda H$ ≡ deformed thickness

λ ≡ thickness stretch.

Points $\underset{\sim}{r} \in \mathcal{S}$ defined by

$$\underset{\sim}{r} = \underset{\sim}{\varphi}(\theta^\alpha, \theta^3) = \underset{\sim}{x}(\theta^\alpha) + \theta^3 \lambda(\theta^\alpha)\, \underset{\sim}{d}(\theta^\alpha)$$

where

$\underset{\sim}{x}(\theta^\alpha)$ ≡ pt. on the current middle surface

$\underset{\sim}{d}(\theta^\alpha)$ ≡ unit <u>director</u> attached to each pt.

Deformation mapping $\chi = \varphi \circ \varphi_0^{-1}$

To compute the dfm gradient we need basis vectors

$$F = \nabla \chi = g_i \otimes G^i$$

Ref. basis :

$$G_\alpha = \varphi_{0,\alpha} = X_{,\alpha} + \Theta^3 A_{3,\alpha} = \underbrace{A_\alpha - \Theta^3 B_\alpha^\beta A_\beta}_{\text{tangent to } \Omega_0} \qquad \alpha, \beta \in \{1,2\}$$

$$G_3 = \varphi_{0,3} = A_3$$

Curr. basis:

$$g_\alpha = \varphi_{,\alpha} = x_{,\alpha} + \Theta^3 (\lambda d)_{,\alpha} = a_\alpha + \Theta^3 (\lambda d)_{,\alpha}$$

$$g_3 = \varphi_{,3} = \lambda d$$

Two types of theories arise from treatment of λ :

1) "Geometrically Exact" Shell Theory : Set $\lambda = 1$ and deal with plane-stress character of deformation by modeling intrinsically 2-D constitutive laws.
2) "Degenerate Solid Approach": Use 3-D material laws and define λ either
   a) Locally, element by element. $\Rightarrow$ Neglect $\lambda_{,\alpha}$. "Condense out" λ (HW's 1&2)
   b) By interpolation $\Rightarrow$ λ becomes nodal dof. (cf. Bischoff & Ramm)

NOTE: In either 1) or 2a) we can neglect $\lambda_{,\alpha}$.

Metrics and Duals : (Shorthand: $\Theta \equiv \Theta^3$)

$$G_{ij} = G_i \cdot G_j \rightarrow \begin{cases} G_{\alpha\beta} = G_\alpha \cdot G_\beta \\ G_{\alpha 3} = G_\alpha \cdot A_3 = 0 \\ G_{33} = A_3 \cdot A_3 = 1 \end{cases} \Rightarrow [G_{ij}] = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} [G_{\alpha\beta}] & \\ & 1 \end{bmatrix}$$

$$\rightarrow [G^{ij}] = [G_{ij}]^{-1} = \begin{bmatrix} [G^{\alpha\beta}] & \\ & 1 \end{bmatrix} \qquad G^{\alpha\beta} = G_{\alpha\beta}^{-1} \qquad G^i = G^{ij} G_j \rightarrow \begin{cases} G^\alpha = G^{\alpha\beta} G_\beta \\ G^3 = G_3 = A_3 \end{cases}$$

$$G_{\alpha\beta} = (A_\alpha + \Theta A_{3,\alpha})(A_\beta + \Theta A_{3,\beta}) = A_{\alpha\beta} + \Theta (\underbrace{A_{3,\alpha} \cdot A_\beta + A_{3,\beta} \cdot A_\alpha}_{= -2B_{\alpha\beta}}) + (\Theta)^2 A_{3,\alpha} \cdot A_{3,\beta}$$

If we neglect $(B_{\alpha\beta})^2$ terms, $G_{\alpha\beta} \approx A_{\alpha\beta} - 2\Theta^3 B_{\alpha\beta}$

Current:
$$g_{\alpha\beta} = [a_\alpha + \Theta(\lambda d)_{,\alpha}] \cdot [a_\beta + \Theta(\lambda d)_{,\beta}]$$
$$= a_{\alpha\beta} + \Theta\left[(\lambda d)_{,\alpha} \cdot a_\beta + (\lambda d)_{,\beta} \cdot a_\alpha\right] + (\Theta)^2 (\lambda d)_{,\alpha} \cdot (\lambda d)_{,\beta}$$
$$g_{\alpha 3} = a_\alpha \cdot \lambda d + \Theta(\lambda d)_{,\alpha} \cdot (\lambda d) \qquad g_{33} = \lambda^2 \underbrace{d \cdot d}_{=1} = \lambda^2$$

Neglect $\lambda_{,\alpha} \Rightarrow (\lambda d)_{,\alpha} = \lambda d_{,\alpha}$. Also, $(d \cdot d)_{,\alpha} = 0 = 2 d_{,\alpha} \cdot d \Rightarrow d_{,\alpha} \cdot d = 0$
$$\Rightarrow g_{\alpha\beta} = a_{\alpha\beta} + \Theta\lambda\left(d_{,\alpha} \cdot a_\beta + d_{,\beta} \cdot a_\alpha\right) + (\Theta)^2 \lambda^2 d_{,\alpha} \cdot d_{,\beta}$$
$$g_{\alpha 3} = g_{3\alpha} = \lambda a_\alpha \cdot d$$
$$g_{33} = \lambda^2$$

Back to ∇X,
$$F = \nabla X = g_i \otimes G^i = g_\alpha \otimes G^\alpha + g_3 \otimes G^3$$
$$= [a_\alpha + \Theta(\lambda d)_{,\alpha}] \otimes G^\alpha + (\lambda d) \otimes A_3$$
$$= a_\alpha \otimes G^\alpha + \lambda d \otimes A_3 + \Theta \lambda d_{,\alpha} \otimes G^\alpha$$

Strains
$$\underset{\sim}{C} = F^T F = g_{ij}\, G^i \otimes G^j =$$
$$= g_{\alpha\beta} G^\alpha \otimes G^\beta + g_{\alpha 3}(G^\alpha \otimes G^3 + G^3 \otimes G^\alpha) + g_{33} G^3 \otimes G^3$$
$$C = [a_{\alpha\beta} G^\alpha \otimes G^\beta + a_\alpha \cdot \lambda d\,(G^\alpha \otimes G^3 + G^3 \otimes G^\alpha) + \lambda^2 G^3 \otimes G^3$$
$$+ \Theta\lambda(d_{,\alpha} \cdot a_\beta + d_{,\beta} \cdot a_\alpha) G^\alpha \otimes G^\beta + \frac{(\Theta)^2}{2} \lambda^2 d_{,\alpha} \cdot d_{,\beta}\, G^\alpha \otimes G^\beta$$

$$E = \tfrac{1}{2}(C - I) = \tfrac{1}{2}(g_{ij} - G_{ij})\, G^i \otimes G^j$$

$$E_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) + \Theta\left[\lambda(d_{,\alpha} \cdot a_\beta + d_{,\beta} \cdot a_\alpha) + B_{\alpha\beta}\right] + \frac{(\Theta)^2}{2}\left[\lambda^2 d_{,\alpha} \cdot d_{,\beta} - A_{3,\alpha} \cdot A_{3,\beta}\right]$$
$$E_{\alpha 3} = E_{3\alpha} = \lambda a_\alpha \cdot d$$
$$E_{33} = \tfrac{1}{2}(\lambda^2 - 1)$$

Comments:

i) Explicit thickness variation only in $E_{\alpha\beta}$ (in-plane components)

ii) Shear strain is basically constant through the thickness (except for variation in λ, which is allowed in degenerate approach)

iii) Specialized theories:

a) Membrane theory: ignore terms linear and higher in θ.

$$\Rightarrow \quad F(\theta^\alpha, \theta^3) \simeq F(\theta^\alpha, 0) = a_\alpha \otimes G^\alpha$$

$$E_{\alpha\beta} = E^0_{\alpha\beta} \equiv \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}), \quad E_{\alpha 3} = 0, \quad E_{33} = \tfrac{1}{2}(\lambda^2 - 1)$$

b) Kirchhoff-Love theory: $\lambda d = a_3$ (director remains normal)

$$\Rightarrow \quad F = a_\alpha \otimes G^\alpha + a_3 \otimes A_3 + \theta\, a_{3,\alpha} \otimes G^\alpha$$

$$a_{3,\alpha} = \left(\frac{a_1 \times a_2}{\sqrt{a}}\right)_{,\alpha} = -b_{\alpha\beta}\, a^\beta$$

$$\Rightarrow \quad E_{\alpha\beta} = E^\circ_{\alpha\beta} - \theta(b_{\alpha\beta} - B_{\alpha\beta}) + \frac{(\theta)^2}{2}\left(b_{\alpha\mu} b^{\mu\beta} - B_{\alpha\mu} B^{\mu\beta}\right)$$

$$\left(a_{3,\alpha} \cdot a_{3,\beta} = b_{\alpha\mu} b^{\mu\beta} \ldots\right)$$

$$E_{\alpha 3} = 0 \quad \text{(no shear strain)}$$

Weak Forms

$$\text{Energy} \quad \Pi[x, d] = \int_{\Omega_0} \int_{-H/2}^{H/2} w(F)\, dV \quad - \underbrace{W^{ext}}_{\text{Work of Conservative external Loads}}$$

$$dV = \sqrt{G}\, d\theta^1 d\theta^2 d\theta^3 \quad dS = \sqrt{A}\, d\theta^1 d\theta^2 \quad \Rightarrow \quad dV = \mu\, d\theta^3 dS \qquad \mu = \frac{\sqrt{G}}{\sqrt{A}}$$

$$\sqrt{G} = |(G_1 \times G_2) \cdot G_3| \qquad \sqrt{A} = |(A_1 \times A_2) \cdot A_3|$$

Can show (see Wempner, Sec. 8.8) that $\sqrt{G} = \sqrt{A}\left[1 - 2H\theta^3 + K(\theta^3)^2\right]$

where

$$H = \tfrac{1}{2} \operatorname{tr} \underset{\sim}{B} = \tfrac{1}{2} B^\alpha_\alpha \ \text{(mean curvature)} \qquad K = \det(B^\alpha_\beta) \ \text{(Gauss Curvature)}$$

$$\min \Pi \longrightarrow \delta\Pi = 0 = \int_{\Omega_0}\int_{-H/2}^{H/2} P : \delta F \, \mu \, d\theta^3 dS - \delta W^{ext} \qquad P = \frac{\partial W}{\partial F} \text{ (Piola stress)}$$

$$F = a_\alpha \otimes G^\alpha + \lambda d \otimes A_3 + \Theta \lambda \, d_{,\alpha} \otimes G^\alpha$$

$$\Rightarrow \delta F = \delta a_\alpha \otimes G^\alpha + \delta\lambda \left(d \otimes A_3 + \theta^3 d_{,\alpha} \otimes G^\alpha \right) + \lambda \, \delta d \otimes A_3 + \Theta\lambda (\delta d_{,\alpha}) \otimes G^\alpha$$

$$= \delta a_\alpha \otimes G^\alpha + \lambda \, \delta d \otimes A_3 + \delta\lambda \, d \otimes A_3 + \Theta \left[\delta\lambda \, d_{,\alpha} \otimes G^\alpha + \lambda (\delta d_{,\alpha}) \otimes G^\alpha \right]$$

Assume the loads do (virtual) work only through δx and δd

$\Rightarrow$ $\delta\lambda$ contributes nothing to δW^{ext}.

$\therefore$ $\delta\Pi$ decouples into two weak forms

$$\int_{\Omega_0}\int_{-H/2}^{H/2} P : \left[\delta a_\alpha \otimes G^\alpha + \lambda \, \delta d \otimes A_3 + \Theta\lambda (\delta d_{,\alpha}) \otimes G^\alpha \right] \mu \, d\Theta \, dS - \delta W^{ext} = 0 \qquad (1)$$

and

$$\int_{\Omega_0}\int_{-H/2}^{H/2} P : \left[\delta\lambda \left(d \otimes A_3 + \Theta d_{,\alpha} \otimes G^\alpha \right) \right] \mu \, d\Theta \, dS = 0 \qquad (2)$$

Eq. (1) $\rightarrow$ equilib of middle surface position and director.

Eq. (2) $\rightarrow$ equilib through the thickness. Enforce either by

a) interpolating λ or

b) enforcing plane stress constraint locally (at each Gauss point).

For surface & director equilib, define

$$\underset{\sim}{n}^i \equiv \int_{-H/2}^{H/2} \underset{\sim}{P} \cdot \underset{\sim}{G}^i \, \mu \, d\Theta^3 = \text{Stress Resultants}$$

$\underset{\sim}{n}^\alpha$ = In-plane Resultants $\qquad \underset{\sim}{n}^3 \equiv \int_{-H/2}^{H/2} \underset{\sim}{P} \cdot \underset{\sim}{A}_3 \, \mu \, d\Theta^3$ = "Across-the-thickness" Resultant

$$\underset{\sim}{m}^\alpha \equiv \int_{-H/2}^{H/2} \Theta^3 \, \underset{\sim}{P} \cdot \underset{\sim}{G}^\alpha \mu \, d\Theta^3 = \text{Moment Resultants (stress couples)}$$

Then (1) $\longrightarrow$ $\displaystyle\int_{\Omega_0} \left(n^\alpha \cdot \delta a_\alpha + \lambda n^3 \cdot \delta d + \lambda m^\alpha \cdot \delta d_{,\alpha} \right) dS - \delta W^{ext} = 0$