MAE 261B
Computational Mechanics of Solids and Structures

Lecture 18

Subdivision Surface Finite Elements for Kirchhoff-Love Shell Theory

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Outline

- Motivation
- Kirchhoff-Love Thin Shell Theory
  - Kinematics
  - Weak Form
- Introduction to Subdivision Methods in Computer Animation and Geometric Design (CAGD)
  - Cubic B-splines
  - Loop subdivision surfaces
- Subdivision Finite Elements
  - Formulation
  - Examples

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References

Kirchhoff-Love Thin-Shell Theory + Subdivision Surfaces = Subdivision Shell Elements

- F. Cirak, M. Ortiz and P. Schröder, "Subdivision Surfaces: A New Paradigm for Thin-Shell Finite-Element Analysis," *Int. J. Numer. Meth. Engng.* (2000) **47** 2039-2072.

- F. Cirak and M. Ortiz, "Fully C^1-Conforming Subdivision Elements for Finite-Deformation Thin-Shell Analysis," *Int. J. Numer. Meth. Engng.* (2001) **51** 813-833.

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Motivation

- For thin shells, shear deformable theory has challenges:
  - Shear locking, membrane locking
  - Parametrizing shell director $\boldsymbol{d} \in S^2$
- Work arounds are cumbersome
  - Mixed variational methods
  - Extra DOF

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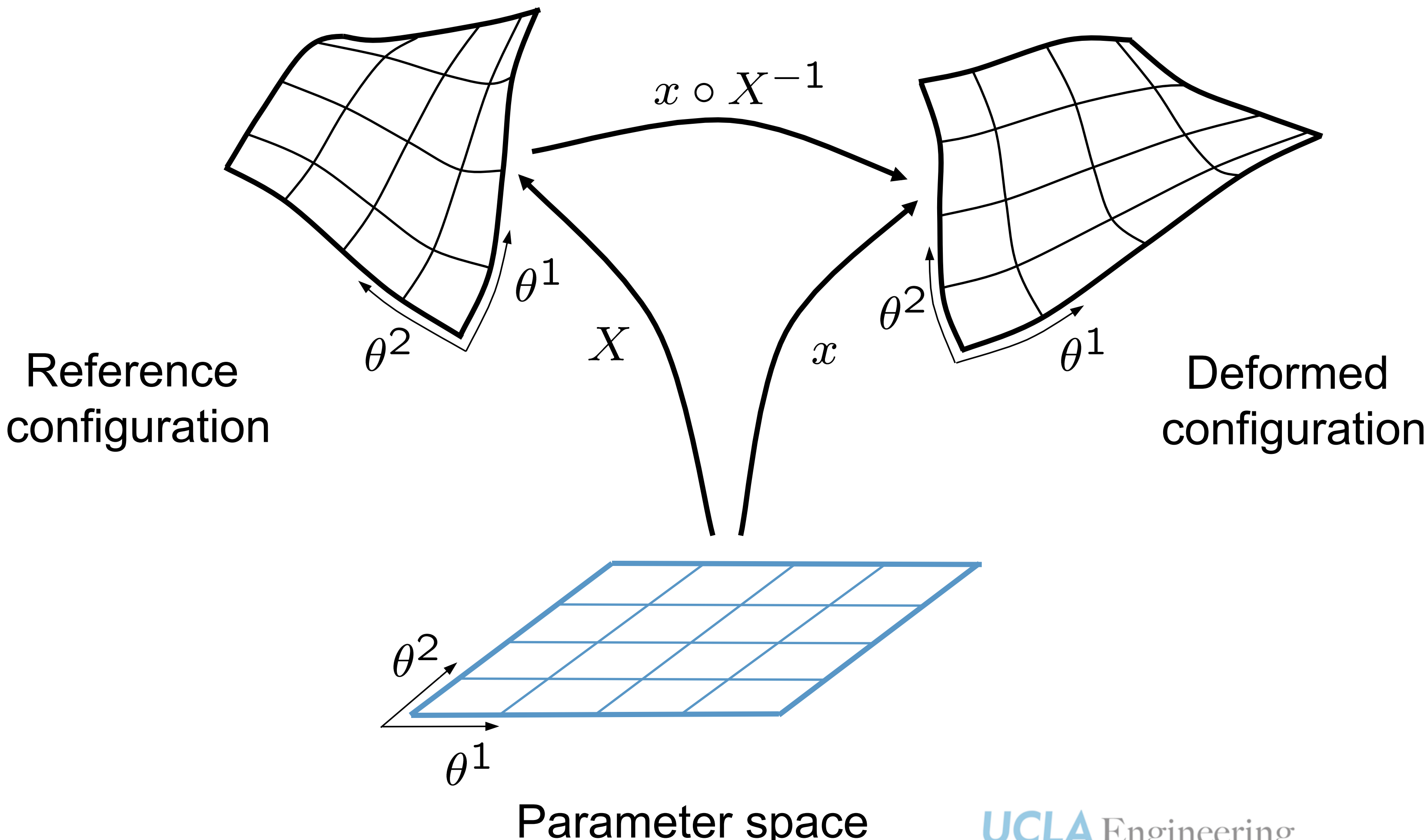
Thin-Shell Kinematics



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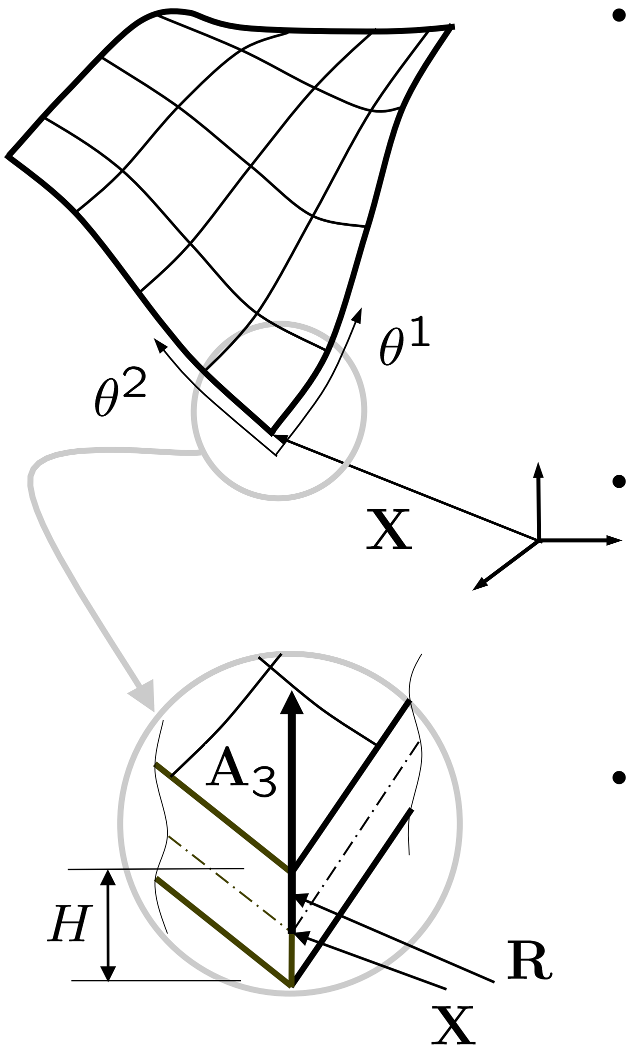
Kirchhoff-Love Kinematics



- Reference configuration:

$$\mathbf{R}(\theta^1, \theta^2, \theta^3) = \mathbf{X}(\theta^1, \theta^2) + \theta^3 \mathbf{A}_3(\theta^1, \theta^2)$$

$$\theta^3 \in \left[-\frac{H}{2}, \frac{H}{2}\right]$$

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{X}}{\partial \theta^\alpha} = \mathbf{X}_{,\alpha} \qquad \mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|}$$

- Deformed configuration

$$\mathbf{r}(\theta^1, \theta^2, \theta^3) = \mathbf{x}(\theta^1, \theta^2) + \lambda \theta^3 \mathbf{a}_3(\theta^1, \theta^2)$$

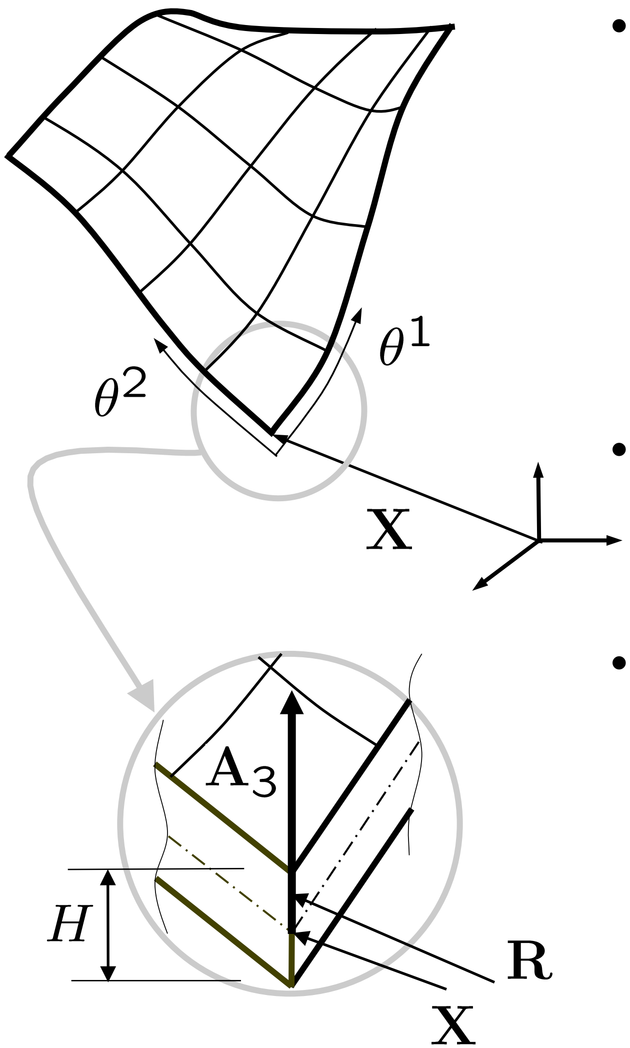
$$|\mathbf{a}_3| = 1 \qquad \lambda = \text{ thickness stretch}$$

- Kirchhoff-Love assumption:

Director $\mathbf{a}_3$ is normal to the deformed middle surface.

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{x}}{\partial \theta^\alpha} = \mathbf{x}_{,\alpha} \qquad \mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$$

Kirchhoff-Love Kinematics



- Basis Vectors:

$$
\begin{aligned}
\boldsymbol{G}_\alpha &= \boldsymbol{R}_{,\alpha} = \boldsymbol{A}_{,\alpha} + \theta^3 \boldsymbol{A}_{3,\alpha} \\
\boldsymbol{G}_3 &= \boldsymbol{R}_{,3} = \boldsymbol{A}_3 \\
\boldsymbol{g}_\alpha &= \boldsymbol{r}_{,\alpha} = \boldsymbol{a}_{,\alpha} + \theta^3 (\lambda \boldsymbol{a}_3)_{,\alpha} \\
\boldsymbol{g}_3 &= \boldsymbol{r}_{,3} = \lambda \boldsymbol{A}_3
\end{aligned}
$$

- Dual Basis Vectors:

$$
\boldsymbol{G}^i \cdot \boldsymbol{G}_j = \delta^i_j, \quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j
$$

- Deformation Gradient

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{g}_i \otimes \boldsymbol{G}^i \\
&= [\boldsymbol{a}_\alpha + \theta^3 (\lambda \boldsymbol{a}_3)_{,\alpha}] \otimes \boldsymbol{G}^\alpha + \lambda \boldsymbol{a}_3 \otimes \boldsymbol{A}_3 \\
&= \underbrace{[\boldsymbol{a}_\alpha \otimes \boldsymbol{G}^\alpha + \lambda \boldsymbol{a}_3 \otimes \boldsymbol{A}_3]}_{\boldsymbol{F}^{(0)}} + \theta^3 \underbrace{[(\lambda \boldsymbol{a}_3)_{,\alpha} \otimes \boldsymbol{G}^\alpha]}_{\boldsymbol{F}^{(1)}}
\end{aligned}
$$

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Shell Equilibrium

- Potential energy:

$$\Pi[\varphi] = \int_{\Sigma_0} \int_{-H/2}^{H/2} \underbrace{W(\boldsymbol{F})}_{\text{strain energy density}} \underbrace{\mu}_{\frac{\sqrt{G}}{\sqrt{A}}} d\theta^3 \sqrt{A} d^2\theta + \Pi^{\text{ext}} \equiv \Pi^{\text{int}} + \Pi^{\text{ext}}$$

- Equilibrium: $\inf_{\boldsymbol{x} \in X} \Pi[x] \quad \Rightarrow \quad \delta\Pi[x] = 0$

$X \equiv$ space of configurations of *finite energy*

$$\begin{aligned} \delta\Pi^{\text{int}} &= \int_{\Sigma_0} \int_{-H/2}^{H/2} \underbrace{\frac{\partial W}{\partial \boldsymbol{F}}}_{\boldsymbol{P}} : \delta \boldsymbol{F} \mu d\theta^3 \sqrt{A} d^2\theta \\ &= \int_{\Sigma_0} [\boldsymbol{n}^\alpha \cdot \delta \boldsymbol{a}_\alpha + \boldsymbol{n}^3 \cdot \delta(\lambda \boldsymbol{a}_3) + \boldsymbol{m}^\alpha \cdot \delta(\lambda \boldsymbol{a}_3)_{,\alpha}] \sqrt{A} d^2\theta \end{aligned}$$

$$\boldsymbol{n}^i \equiv \int_{-H/2}^{H/2} \boldsymbol{P}\boldsymbol{G}^i \mu d\theta^3 \quad \boldsymbol{m}^\alpha \equiv \int_{-H/2}^{H/2} \theta^3 \boldsymbol{P}\boldsymbol{G}^\alpha \mu d\theta^3$$

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Shell Equilibrium

$$\begin{aligned}
\delta\Pi^{\text{int}} &= \int_{\Sigma_0}[\boldsymbol{n}^\alpha\cdot\delta\boldsymbol{a}_\alpha+\boldsymbol{n}^3\cdot\delta(\lambda\boldsymbol{a}_3)+\boldsymbol{m}^\alpha\cdot\delta(\lambda\boldsymbol{a}_3)_{,\alpha}]\sqrt{A}d^2\theta \\
&= \int_{\Sigma_0}[\boldsymbol{n}^\alpha\cdot\delta\boldsymbol{a}_\alpha+\boldsymbol{n}^3\cdot(\lambda\delta\boldsymbol{a}_3)+\boldsymbol{m}^\alpha\cdot(\lambda\delta\boldsymbol{a}_3)_{,\alpha}]\sqrt{A}d^2\theta \\
&+ \int_{\Sigma_0}[\boldsymbol{n}^3\cdot(\boldsymbol{a}_3\delta\lambda)+\boldsymbol{m}^\alpha\cdot(\boldsymbol{a}_3\delta\lambda)_{,\alpha}]\sqrt{A}d^2\theta
\end{aligned}$$

$\delta\boldsymbol{x}$ and $\delta\lambda$ independent,

$$\int_{\Sigma_0}[\boldsymbol{n}^\alpha\cdot\delta\boldsymbol{a}_\alpha+\boldsymbol{n}^3\cdot(\lambda\delta\boldsymbol{a}_3)+\boldsymbol{m}^\alpha\cdot(\lambda\delta\boldsymbol{a}_3)_{,\alpha}]\sqrt{A}d^2\theta+\delta\Pi^{\text{ext}} = 0$$

equilibrium of mid surface

$$\int_{\Sigma_0}[\boldsymbol{n}^3\cdot(\boldsymbol{a}_3\delta\lambda)+\boldsymbol{m}^\alpha\cdot(\boldsymbol{a}_3\delta\lambda)_{,\alpha}]\sqrt{A}d^2\theta = 0$$

equilibrium of thickness stretch

Note: To solve for thickness stretch, two options:
1. discretize λ and solve above equations simultaneously
2. Neglect gradients in λ and solve by static condensation.

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Elastic Constitutive Laws

$$W(\boldsymbol{F}) = \text{ hyperelastic strain energy density} \qquad \boldsymbol{P}(\boldsymbol{F}) = \frac{\partial W}{\partial \boldsymbol{F}} = \text{ first P-K stress}$$

Note: F depends on lambda. Can solve for lambda, enforcing plane-stress assumption locally by condensation.

Plane stress: $\boldsymbol{a}_3 \cdot (\boldsymbol{P}\boldsymbol{A}_3) = 0$

Given initial value for F, set up local Newton iteration at each quadrature point:

$$\boldsymbol{a}_3 \cdot \left[\boldsymbol{P}\Big|_{\boldsymbol{F}^{(n)}} + \underbrace{\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}}}_{\mathbb{C}}\Big|_{\boldsymbol{F}^{(n)}} : \Delta \boldsymbol{F} \right] \boldsymbol{A}_3 = 0 \qquad \Delta \boldsymbol{F} = \Delta\lambda [\boldsymbol{a}_3 \otimes \boldsymbol{A}_3]$$

$$\boldsymbol{a}_3 \cdot \boldsymbol{P}\Big|_{\boldsymbol{F}^{(n)}} \boldsymbol{A}_3 + \Delta\lambda \left([\boldsymbol{a}_3 \otimes \boldsymbol{A}_3] : \mathbb{C}\Big|_{\boldsymbol{F}^{(n)}} : [\boldsymbol{a}_3 \otimes \boldsymbol{A}_3] \right) = 0$$

$$\Rightarrow \Delta\lambda = -\left([\boldsymbol{a}_3 \otimes \boldsymbol{A}_3] : \mathbb{C}\Big|_{\boldsymbol{F}^{(n)}} : [\boldsymbol{a}_3 \otimes \boldsymbol{A}_3] \right)^{-1} \left(\boldsymbol{a}_3 \cdot \boldsymbol{P}\Big|_{\boldsymbol{F}^{(n)}} \boldsymbol{A}_3 \right)$$

$$\boldsymbol{F}^{(n+1)} = \boldsymbol{F}^{(n)} + \Delta \boldsymbol{F}$$

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Shell Equilibrium

If we solve for lambda by condensation, mid-surface equilibrium becomes:

$$\int_{\Sigma_0} [\boldsymbol{n}^\alpha \cdot \delta \boldsymbol{a}_\alpha + \boldsymbol{n}^3 \cdot (\lambda \delta \boldsymbol{a}_3) + \boldsymbol{m}^\alpha \cdot \lambda \delta \underbrace{\boldsymbol{a}_{3,\alpha}}] \sqrt{A} d^2\theta + \delta \Pi^{\mathrm{ext}} = 0$$

$$\begin{aligned} \boldsymbol{a}_{3,\alpha} &= \left(\frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\sqrt{a}} \right)_{,\alpha} \\ &= \frac{1}{\sqrt{a}} [\boldsymbol{I} - \boldsymbol{a}_3 \otimes \boldsymbol{a}_3](\boldsymbol{a}_{1,\alpha} \times \boldsymbol{a}_2 + \boldsymbol{a}_1 \times \boldsymbol{a}_{2,\alpha}) \end{aligned}$$

$$\boldsymbol{a}_{\alpha,\beta} = \boldsymbol{x}_{,\alpha\beta}$$

Need second derivatives of $\boldsymbol{x}$, i.e., C^1-conforming interpolation.

Solution: ***Subdivision Surfaces***

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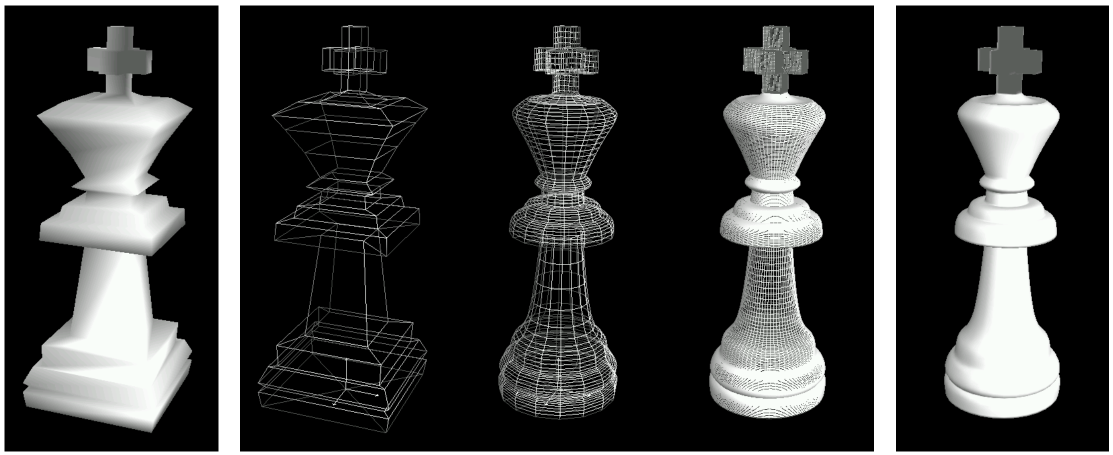
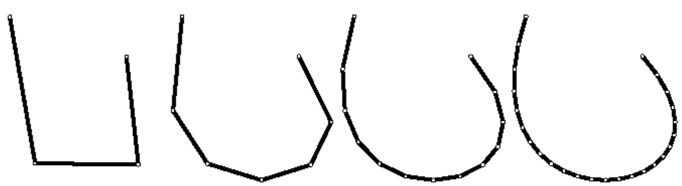
Subdivision in Computer Animation and Geometric Design

- Limit of a sequence of refinements



www.subdivision.org

- Benefits
  - Arbitrary Topology
  - Uniformity of Representation
  - Ease of Implementation



Pixar Studios *Geri's Game*

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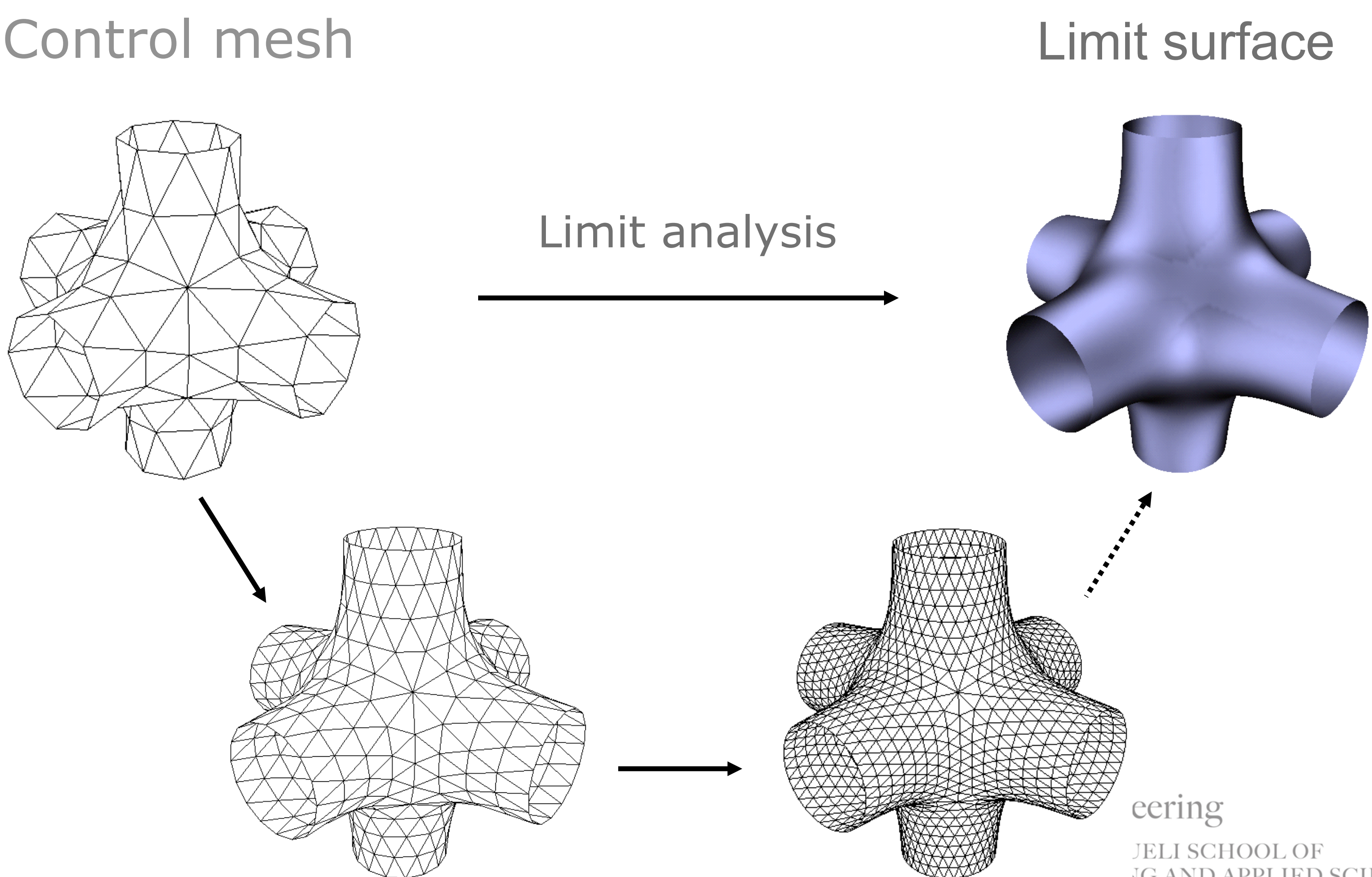
Subdivision Surfaces



Linear Subdivision Analysis

- Subdivision as a linear mapping:

  Let $\mathbf{X} = \{x_0, x_1, \ldots, x_{n-1}\}$, then

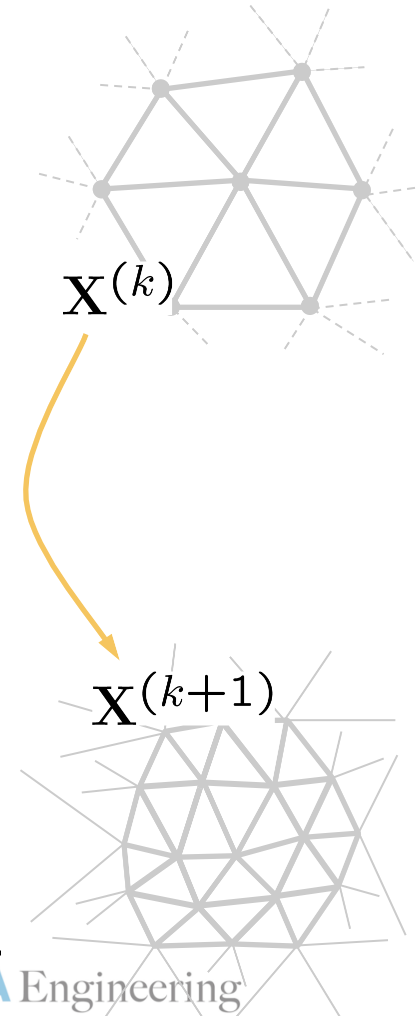
  $$\mathbf{X}^{(k+1)} = \mathbf{S}\mathbf{X}^{(k)} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}\mathbf{X}^{(k)}$$

- Limit curve/surface:

  $$\mathbf{X}^{(\infty)} = \lim_{k\to\infty} \mathbf{S}^k\mathbf{X}^{(0)} = \lim_{k\to\infty} \mathbf{R}\boldsymbol{\Lambda}^k\mathbf{R}^{-1}\mathbf{X}^{(0)}$$

  $\mathbf{X}^{(0)} \equiv$ Control Mesh

  $\longrightarrow$ Tangent vectors, normals, and curvatures



- Smoothness properties:
  - Cubic B-splines: C^2 everywhere.
  - Loop Subdivision Surfaces: C^2 at regular vertices.
    C^1 at irregular vertices.
    L^2 curvatures.

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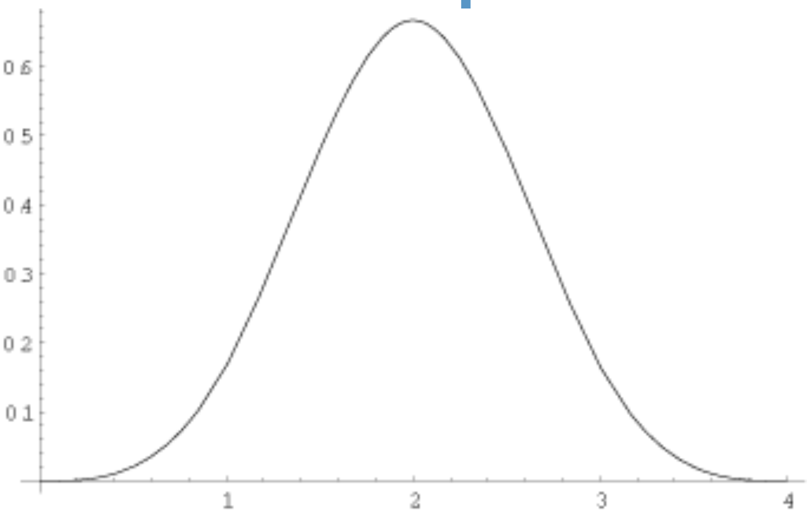
Curve Example: Cubic B-Splines

- Curve segment in terms of Basis Functions:

$$x(t) = \sum_i p_i N(t-i)$$

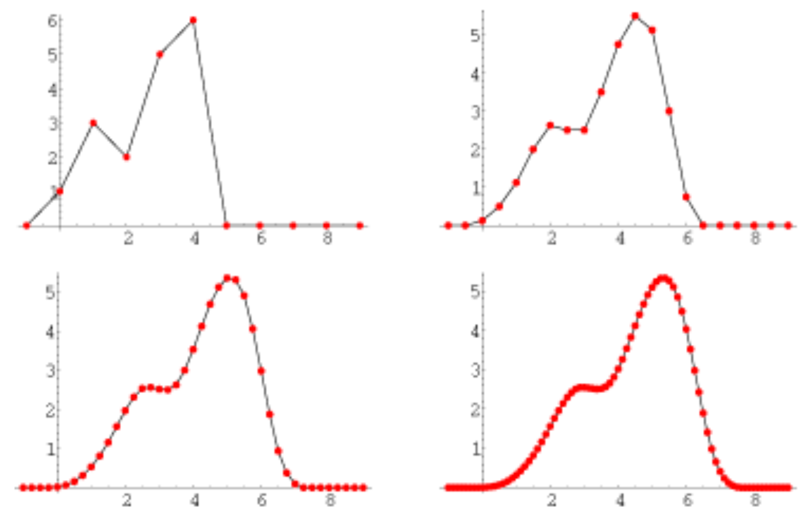


- Subdivision Scheme:
  - Odd:

$$p_{2j+1}^{k+1} = \frac{1}{2}(p_j^k + p_{j+1}^k)$$

  - Even:

$$p_{2j}^{k+1} = \frac{1}{8}p_{j-1}^k + \frac{3}{4}p_j^k + \frac{1}{8}p_{j+1}^k$$



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Curve Example: Cubic B-Splines

$p_{2j+1}^{k+1} = \frac{1}{2}(p_j^k + p_{j+1}^k) \quad p_{2j}^{k+1} = \frac{1}{8}p_{j-1}^k + \frac{3}{4}p_j^k + \frac{1}{8}p_{j+1}^k$

- Linear Mapping:

$$\mathbf{p}^{(k+1)} = \mathbf{S}\mathbf{p}^{(k)} \qquad \mathbf{S} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1/2 & 1/2 & 0 & 0 & 0 & \cdot \\ \cdot & 1/8 & 3/4 & 1/8 & 0 & 0 & \cdot \\ \cdot & 0 & 1/2 & 1/2 & 0 & 0 & \cdot \\ \cdot & 0 & 1/8 & 3/4 & 1/8 & 0 & \cdot \\ \cdot & 0 & 0 & 1/2 & 1/2 & 0 & \cdot \\ \cdot & 0 & 0 & 1/8 & 3/4 & 1/8 & \cdot \\ \cdot & 0 & 0 & 0 & 1/2 & 1/2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

- Smoothing/Filtering Perspective:
  - Linear Interpolation:

$$\begin{aligned} q_{2j}^{k+1} &= p_j^k \\ q_{2j+1}^{k+1} &= \frac{1}{2}(p_j^k + p_{j+1}^k) \end{aligned}$$

  - followed by Laplacian Smoothing:

$$\begin{aligned} \Delta q_h^{k+1} &= \frac{1}{2}(q_{h-1}^{k+1} - 2q_h^{k+1} + q_{h+1}^{k+1}) \\ p_h^{k+1} &= q_h^{k+1} + \frac{1}{2}\Delta q_h^{k+1} \ . \end{aligned}$$

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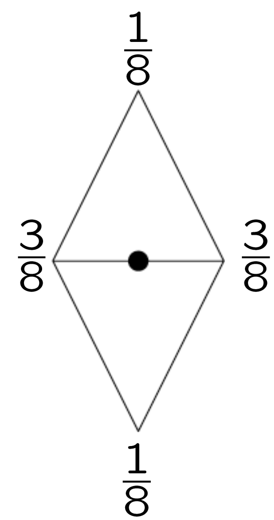
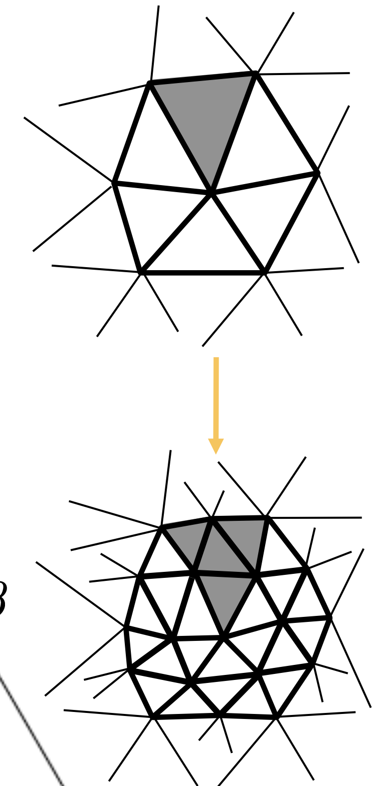
Surface Example: Loop Scheme

- Subdivision Scheme:
  - "Edge" Rule:

$$p^{k+1} = \frac{1}{8}(p_1^k + p_2^k) + \frac{3}{8}(p_3^k + p_4^k)$$



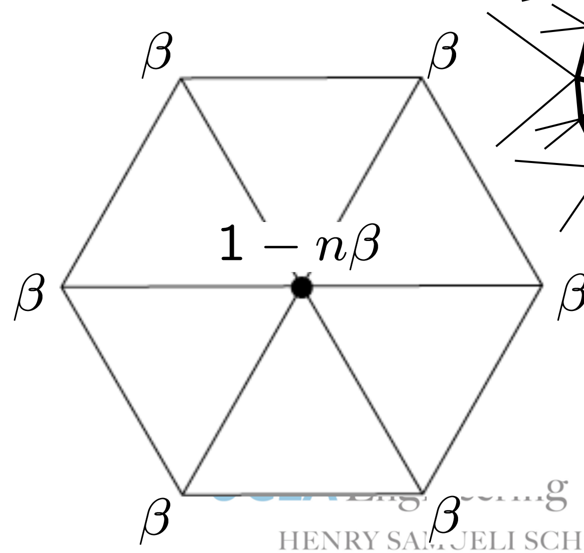


  - "Vertex" Rule:

$$p^{k+1} = (1 - n\beta)p^k + \beta \sum_{i=1}^{n} p_i^k$$

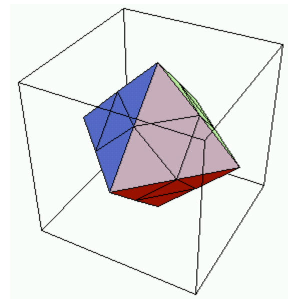
$$\beta = \frac{1}{n}\left[\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2\right]$$

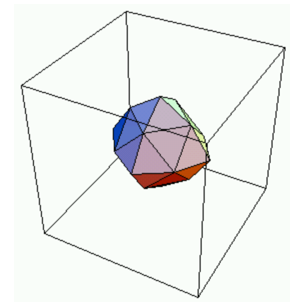
$n \equiv$ vertex valence



(C. Loop, 1987)

Surface Example: Loop Scheme





- Regular case ($n = 6$):
  - Smoothing/Filtering Perspective:
    - Linear interpolation

      - Edges: $q = \frac{1}{2}(p_1^k + p_2^k)$

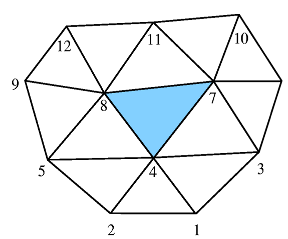
      - Vertices: $q = p^k$

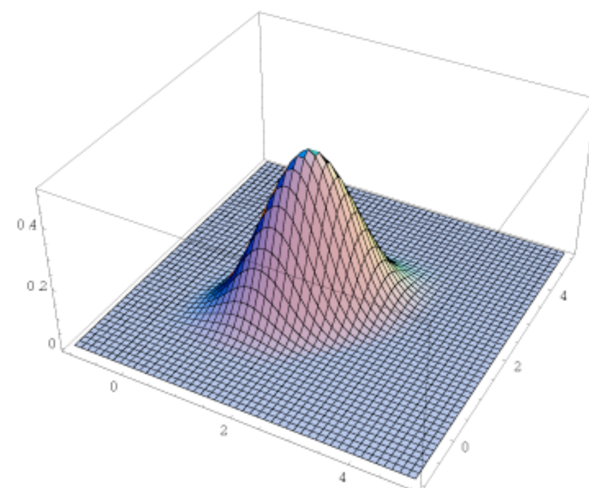
    - followed by Laplacian Smoothing

$$p^{k+1} = q + \frac{3}{4}\sum_{i=1}^{6}\frac{1}{6}(q_i - q)$$

  - Direct Evaluation: Quartic Box Splines

$$p(t, s) = \sum_{i=1}^{12} p_i N_i(t, s)$$





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Subdivision Elements: Loop Scheme

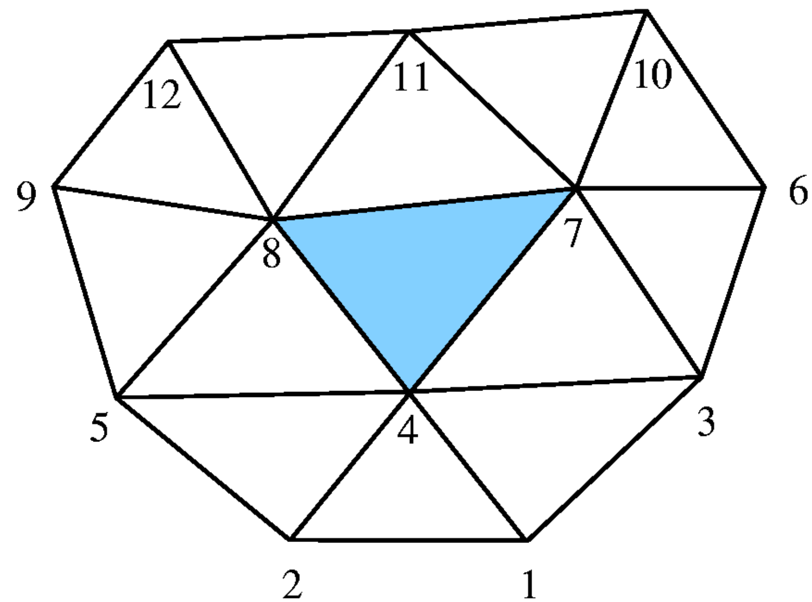
- Interpolation:
  - Reference middle surface:

$$\mathbf{X}(\theta^1, \theta^2) = \sum_{a=1}^{12} \mathbf{X}_a N_a(\theta^1, \theta^2)$$

  - Deformed middle surface:

$$\mathbf{x}(\theta^1, \theta^2) = \sum_{a=1}^{12} \mathbf{x}_a N_a(\theta^1, \theta^2)$$

$N_a \equiv$ Quartic box-splines



A regular patch and its 12 control points

- Sole degrees of freedom: Control-node displacements.
- Interpolation in one element depends on displacements in 1-ring of adjacent elements.

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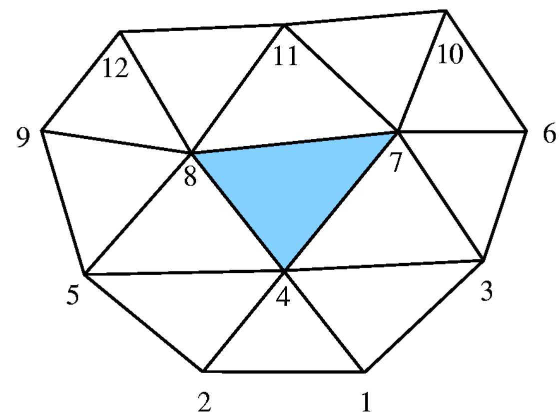
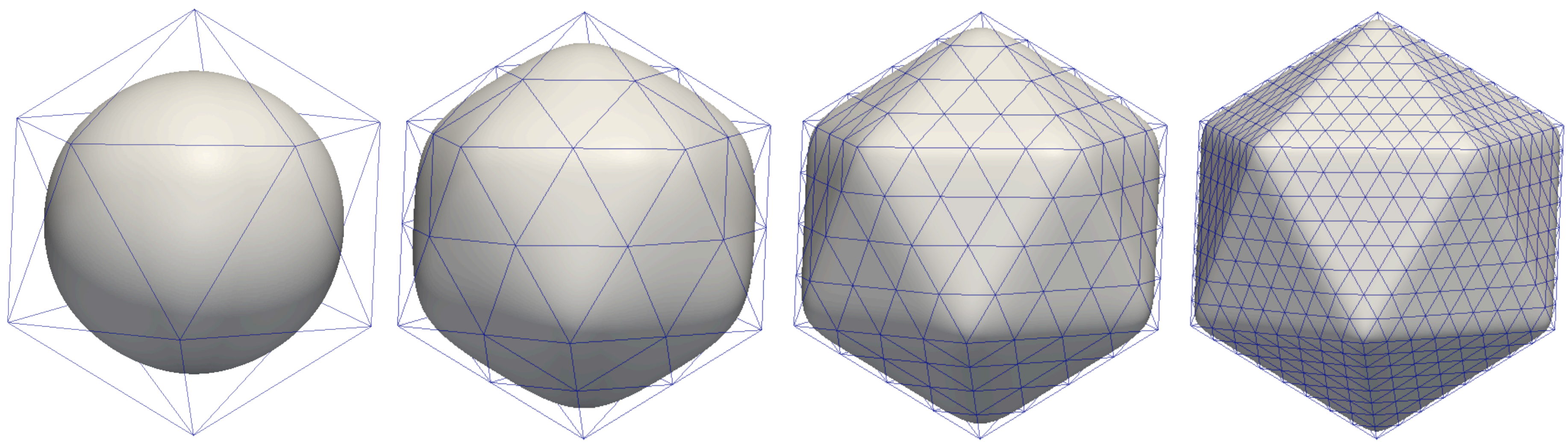
Subdivision Elements: Loop Scheme

- Some properties
  - Shape functions are non-local; support extends over next nearest neighbor elements
  - Shape functions do not satisfy Kronecker-delta property, i.e., limit surface is *non-interpolating!*



A regular patch and its 12 control points



Meshes of an icosahedron at different resolutions, showing non-interpolating limit surfaces

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Finite Element Approximation

Mid-surface equilibrium:

$$\int_{\Sigma_0} [\boldsymbol{n}^\alpha \cdot \delta \boldsymbol{a}_\alpha + \boldsymbol{n}^3 \cdot (\lambda \delta \boldsymbol{a}_3) + \boldsymbol{m}^\alpha \cdot \lambda \delta \boldsymbol{a}_{3,\alpha}] \sqrt{A} d^2\theta + \delta \Pi^{\text{ext}} = 0$$

Subdivision surface approximation:

$$\boldsymbol{X}^h(\theta^\alpha) \equiv \sum_{a=1}^{NP} \boldsymbol{X}_a N_a(\theta^\alpha) \quad \boldsymbol{x}^h(\theta^\alpha) \equiv \sum_{a=1}^{NP} \boldsymbol{x}_a N_a(\theta^\alpha)$$

Discrete equilibrium equations:

$$\boldsymbol{f}_a^{\text{int}} - \boldsymbol{f}_a^{\text{ext}} = 0$$

$$\boldsymbol{f}_a^{\text{int}} = \int_{\Sigma_0} \left(\boldsymbol{n}^\alpha \cdot \frac{\partial \boldsymbol{a}_\alpha}{\partial \boldsymbol{x}_a} + \lambda \boldsymbol{n}^3 \cdot \frac{\partial \boldsymbol{a}_3}{\partial \boldsymbol{x}_a} + \lambda \boldsymbol{m}^\alpha \cdot \frac{\partial \boldsymbol{a}_{3,\alpha}}{\partial \boldsymbol{x}_a} \right) \mu \sqrt{A} d^2\theta$$

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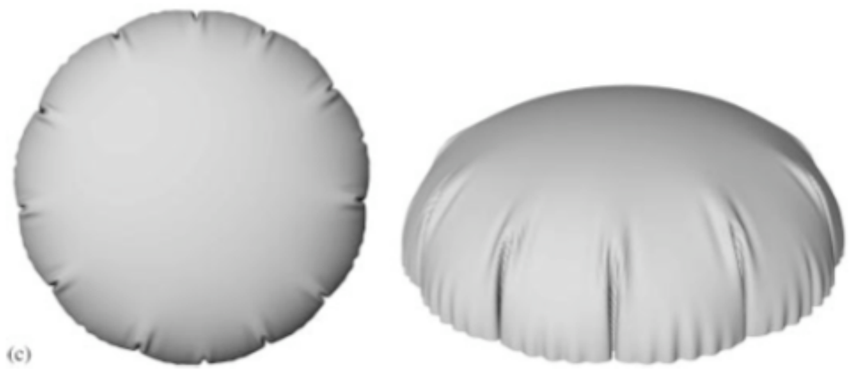
FE Examples



Cirak, Ortiz