Assumed Strain Elements

Reading: Bathe & Dvorkin, IJNME, 1985 → Plates
Dvorkin & Bathe, Eng. Comput., 1984 → Shells

Outline:

1. Review mixed variational principles — Hu Washizu, Hellinger-Reissner
2. Demonstrate idea w/ Timoshenko beam.
3. Extend to Q4 shell element.

Mixed Variational Principles

Recall for lin. elasticity,

M.P.E.
$$\Pi[\underset{\sim}{u}(x)] = \int_B \frac{1}{2}\underset{\sim}{\varepsilon}:\underset{\approx}{\mathbb{C}}:\underset{\sim}{\varepsilon}\, dV - \int_{\partial B_t} t\cdot u\, dS - \int_B b\cdot u\, dV \qquad \underset{\sim}{\varepsilon} \equiv \nabla_s u \qquad \underset{\sim}{\sigma} \equiv \underset{\approx}{\mathbb{C}}\,\underset{\sim}{\varepsilon}$$

$$\min_{u(x)} \Pi[u] \rightarrow \text{equilib } \left(\nabla\cdot\sigma + b = 0 \quad \sigma\cdot n = t \right)$$

Hu-Washizu: Let $\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}, \underset{\sim}{\bar{t}}$ all be independent variables.

$$\Pi_{HW} = \Pi - \int_V \underset{\sim}{\sigma}:(\underset{\sim}{\varepsilon} - \nabla_s \underset{\sim}{u})\, dV - \int_{\partial B_u} \bar{t}\cdot(u - \bar{u})\, dS$$

$$\delta\Pi_{HW} = \int_B \delta\varepsilon:\mathbb{C}:\varepsilon\, dV - \int_B b\cdot\delta u\, dV - \int_{\partial B_t} t\cdot\delta u\, dS$$

$$- \int \delta\sigma:(\varepsilon - \nabla_s u)\, dV - \int \sigma:(\delta\varepsilon - \nabla_s\delta u)\, dV - \int_{\partial B_u} \delta\bar{t}\cdot(u - \bar{u})\, dS - \int_{\partial B_u} \bar{t}\cdot\delta u\, dS$$

$\delta\varepsilon, \delta\sigma, \delta u, \delta\bar{t}$ arbitrary ⟹ E-L eqns:

$$\left.\begin{array}{ll} \text{Hooke:} & \sigma = \mathbb{C}:\varepsilon \\ \text{Equilib:} & \nabla\cdot\sigma + b = 0 \\ \text{Compat.:} & \varepsilon = \nabla_s u \end{array}\right\} \text{ in } B$$

Cauchy: $\sigma\cdot n = t$ on ∂B_t

$\sigma\cdot n = \bar{t}$ on ∂B_u

Hellinger-Reissner:
$$\Pi_{HR} = \Pi_{HW}\Big|_{\substack{\sigma = \mathbb{C}:\varepsilon \\ t = \sigma\cdot n}} = \int_B \left(-\frac{1}{2}\varepsilon:\mathbb{C}:\varepsilon + \varepsilon:\mathbb{C}:\nabla_s u - b\cdot u \right) dV - \int_{\partial B_t} t\cdot u\, dS$$

$\delta\Pi_{HR}$, $\delta u, \delta\varepsilon$ → equilib. + compat.

Timoshenko Beam w/ Mixed Interpolation

Timoshenko Kinematics : $u_x = -z\theta \quad u_z = w \quad \varepsilon_{xx} = u_{x,x} = -z\theta_{,x} = -z\kappa \quad \gamma_{xz} = u_{x,z} + u_{z,x} = w_{,x} - \theta$

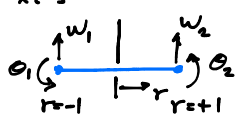
$$\Pi_{HR} = \int_{\mathcal{B}} \left(\tfrac{1}{2}\varepsilon_{xx} E \varepsilon_{xx} - \tfrac{1}{2}\gamma_{xz}^{AS} G \gamma_{xz}^{AS} + \gamma_{xz}^{AS} G \gamma_{xz} - b\cdot u \right) dV + \text{B.T.}$$

$$= \int \left[\tfrac{1}{2} EI\kappa^2 - \tfrac{1}{2} GA (\gamma_{xz}^{AS})^2 + GA\,\gamma_{xz}^{AS}(w_{,x} - \theta) - q w \right] dx + \text{B.T.}$$

Indep. Variables : w = "Displacement" θ = C.S. rotation γ_{xz}^{AS} = "Assumed Shear Strain"

$\Pi_{HR} = \Pi_{HR}[w, \gamma_{xz}^{AS}]$

<u>FE Interp.</u>



$$w(r) = \tfrac{1}{2}(1-r)\,w_1 + \tfrac{1}{2}(1+r)\,w_2 = N_1(r) w_1 + N_2(r) w_2$$

$$\theta(r) = N_1(r)\,\theta_1 + N_2(r)\,\theta_2$$

$$\gamma_{xz}^{AS} = \gamma^{AS} = \text{const}$$

FE arrays : $\underline{u} = (w_1, \theta_1, w_2, \theta_2)^T$

$$\begin{pmatrix} w \\ \theta \end{pmatrix} = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}}_{\underline{N}} \underline{u} \qquad \kappa = \partial_x \theta = \underbrace{\begin{bmatrix} 0 & N_{1,x} & 0 & N_{2,x} \end{bmatrix}}_{\underline{B}_b} \underline{u}$$

$$\gamma_{xz} = w_{,x} - \theta = \underbrace{\begin{bmatrix} N_{1,x} & -N_1 & N_{2,x} & -N_2 \end{bmatrix}}_{\underline{B}_s} \underline{u}$$

$$\gamma_{xz}^{AS} = \gamma^{AS}$$

$$\Rightarrow \Pi_{HR} = \tfrac{1}{2}\underline{u}^T \Big(\underbrace{\int \underline{B}_b^T EI\, \underline{B}_b\, dx}_{\underline{K}_{uu}} \Big) \underline{u} - \tfrac{1}{2}\gamma^{AS} \Big(\underbrace{\int GA\, dx}_{K_{\gamma\gamma}} \Big) \gamma^{AS} + \gamma^{AS} \Big(\underbrace{\int GA\, \underline{B}_s\, dx}_{\underline{K}_{\gamma u}} \Big) \underline{u} - \Big(\underbrace{\int q\, N_w^T\, dx}_{\underline{f}^T} \Big) \underline{u}$$

$$\delta\Pi_{HR} = 0 \Rightarrow \begin{bmatrix} \underline{K}_{uu} & \underline{K}_{\gamma u}^T \\ \underline{K}_{\gamma u} & K_{\gamma\gamma} \end{bmatrix} \begin{pmatrix} \underline{u} \\ \gamma \end{pmatrix} = \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}$$

2 eqns. $\underline{K}_{uu}\underline{u} + \underline{K}_{\gamma u}^T \gamma = \underline{f}$

$\underline{K}_{\gamma u}\underline{u} + K_{\gamma\gamma}\gamma = 0$

"Static Condensation" 4 γ : $\gamma = \frac{1}{K_{\gamma\gamma}} \underline{K}_{\gamma u} \underline{u} = (w_2 - w_1)/L - \tfrac{1}{2}(\theta_1 + \theta_2)$

$$\rightarrow \underbrace{\left[\underline{K}_{uu} + \underline{K}_{\gamma u}^T K_{\gamma\gamma}^{-1} \underline{K}_{\gamma u} \right]}_{\underline{K}} \underline{u} = \underline{f}$$

$\therefore$ $\underline{K}\,\underline{u} = \underline{f}$ Structure of a displacement-only formulation!

Note: Now that we have the final result for the shear $\gamma = \gamma^{AS} = \frac{w_2 - w_1}{L} - \frac{\theta_1 + \theta_2}{2}$

we can try to work backward to reverse engineer a displ. interp s.t. $\gamma = w_{,x} - \theta = \gamma^{AS}$.

$$\left.\begin{aligned} w &= N_1 w_1 + N_2 w_2 + w^* \\ \theta &= N_1 \theta_1 + N_2 \theta_2 \end{aligned}\right\} \rightarrow \gamma = w_{,x} - \theta = N_{1,x} w_1 + N_{2,x} w_2 + w^*_{,x} - N_1\theta_1 - N_2\theta_2$$

$$\Rightarrow \quad w^*_{,x} - N_1\theta_1 - N_2\theta_2 = \text{const}$$

$$\ldots \quad w^* = \tfrac{1}{8} L_e (1 - r^2)(\theta_1 - \theta_2)$$

Another way to see it...

Pure-displ. beam $\quad w = N_1 w_1 + N_2 w_2 \quad \theta = N_1\theta_1 + N_2\theta_2$

$$\rightarrow \gamma^{DI} = w_{,x} - \theta = \frac{w_2 - w_1}{L} + \frac{\theta_1 + \theta_2}{2} + \frac{r}{2}(\theta_2 - \theta_1)$$

Apply torque loading s.t. beam should bend 

$$w_1 = \theta_1 = 0 \longrightarrow \gamma = \frac{w_2}{L} + \frac{\theta_2}{2} + \frac{r}{2}\theta_2$$

Pure bending $\Rightarrow \gamma = 0 \rightarrow$ only possible if $w_2 = \theta_2 = 0 \longrightarrow$ LOCKING.

$\Rightarrow$ Motivation for constant strain. "Tie" the shear strain interpolation to the displ. interp by

$$\gamma^{AS}(r) = \gamma^{DI}\big|_{r=0} \quad \text{Quadrature Pt.}$$

More generally, define $\quad \gamma^{AS} = \sum_{A=1}^{NAS} N^*_A \, \gamma^{DI}\big|_{P_A}$

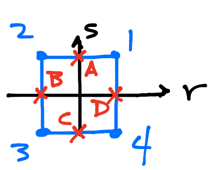
N^*_A: shear shape fns $\qquad P_A$: Assumed Strain Points

PLATE/SHELL ELEMENTS

Use same idea: tie the assumed strain interpolation to the displacement interpolation at selected pts.

Reading: Ch. 5 of Bathe, <u>Finite Element Procedures</u> (sec. 5.4)

<u>Ex</u> Q4 element ("MITC4") $\qquad$ Bilinear Shp fns



$$w(r,s) = \sum_{a=1}^{4} w_a N_a(r,s) \qquad N_1 = \tfrac{1}{4}(1+r)(1+s) \qquad N_2 = \tfrac{1}{4}(1-r)(1+s)$$

$$\vec{\beta}(r,s) = \sum_{a=1}^{4} \vec{\beta}_a N_a(r,s) \qquad N_3 = \tfrac{1}{4}(1-r)(1-s) \qquad N_4 = \tfrac{1}{4}(1+r)(1-s)$$

Curvilinear Strains

$$\gamma_{rz}^{DI} = w_{,r} - \beta_r$$

$$= w_1 \tfrac{1}{4}(1+s) - \beta_{r1} \tfrac{1}{4}(1+r)(1+s)$$

$$- w_2 \tfrac{1}{4}(1+s) - \beta_{r2} \tfrac{1}{4}(1-r)(1+s)$$

$$- w_3 \tfrac{1}{4}(1-s) - \beta_{r3} \tfrac{1}{4}(1-r)(1-s)$$

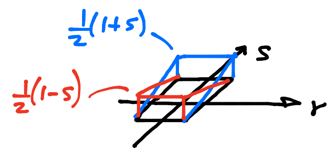
$$+ w_4 \tfrac{1}{4}(1-s) - \beta_{r4} \tfrac{1}{4}(1+r)(1-s)$$

No hope of cancelling w & β terms unless we get rid of r-dependence. Try setting $r \to 0$

$$\gamma_{rz}^{AS} = \gamma_{rz}^{DI}\Big|_{r=0} = \tfrac{1}{2}(1+s)\underbrace{\left(\frac{w_1-w_2}{2} - \frac{\beta_{r1}+\beta_{r2}}{2}\right)}_{=\gamma_{rz}^{DI}|_{r=0,s=1}} + \tfrac{1}{2}(1-s)\underbrace{\left(\frac{w_4-w_3}{2} - \frac{\beta_{r3}+\beta_{r4}}{2}\right)}_{=\gamma_{rz}^{DI}|_{r=0,s=-1}}$$

Or,

$$\gamma_{rz}^{AS} = \tfrac{1}{2}(1+s)\,\gamma_{rz}^{DI}\Big|_A + \tfrac{1}{2}(1-s)\,\gamma_{rz}^{DI}\Big|_C$$



In the same spirit, define sz shear interp.

$$\gamma_{sz}^{AS} = \tfrac{1}{2}(1+r)\,\gamma_{sz}^{DI}\Big|_D + \tfrac{1}{2}(1-s)\,\gamma_{sz}^{DI}\Big|_B$$

Comments:

1) Straightforward to extend to curvilinear strains $E_{ij} = \tfrac{1}{2}(g_{ij} - G_{ij})$ → NONLINEAR

2) Can use for both shear- and membrane-locking.

3) Satisfies inf-sup condition → will converge.

4) Extensions to higher order quads and triangles