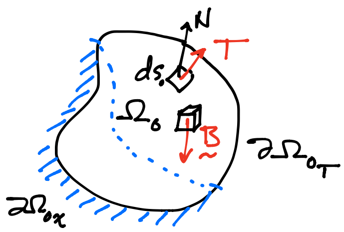
DYNAMICS

BVP of continuum dynamics



Cons. lin. momentum

$$\hookrightarrow \nabla \cdot \underset{\sim}{P} + \rho_0 \underset{\sim}{B} = \rho_0 \ddot{\underset{\sim}{\chi}} \quad \text{in } \Omega_0 \qquad (*)$$

$\ddot{\underset{\sim}{\chi}}(X,t)$ = material acceleration

$$\underset{\sim}{P} \cdot \underset{\sim}{N} = \underset{\sim}{T} \quad \text{on } \partial\Omega_{0T}$$

$$\underset{\sim}{\chi}(X,t) = \bar{\chi}(X,t) \quad \text{on } \partial\Omega_{0x}$$

Initial conditions:

$$\chi(X,0) = \chi_0(X) \quad \text{init def. mapping}$$

$$\dot{\chi}(X,0) = V_0(X) \quad \text{init. material velocity}$$

Hyperelastic Constitutive law:

$$\underset{\sim}{P} = \frac{\partial W}{\partial \underset{\sim}{F}} \qquad P_{iJ} = \frac{\partial W}{\partial F_{iJ}}$$

Q/ How do we approximate the dynamical IBVP? Is there a variational principle?

Hamilton's Principle:

Action: $$I[\chi] = \int_{t_1}^{t_2} \underbrace{\mathcal{L}[\chi, \dot{\chi}]}_{\text{Lagrangian}} \, dt$$

$$\mathcal{L}[\chi, \dot{\chi}] = \int_{\Omega_0} \left\{ \tfrac{1}{2} \rho_0 |\dot{\chi}|^2 - W(\nabla \chi) \right\} dV_0$$

Hamilton: $\delta I = 0$ for trajectory of system and all admissible variations

E-L equations $\Leftrightarrow$ equations of motion (*)

This is not a minimum principle $\Rightarrow$ Cannot apply Rayleigh-Ritz

But we can formulate minimum principles for dynamics by means of Time Discretization.

Two Strategies for dynamics:

- Traditional: Discretize eqns of motion directly
- Variational: Discretize I in time

$\Rightarrow$ action sum + stationarity $\rightarrow$ discrete eqns. of motion

Discretize time: $t_0, t_1, t_2, \ldots, t_n, t_{n+1} = t_n + \Delta t, \ldots$

Newmark's algorithm (standard for struct. dynamics)

Denote: $\underset{\sim}{V}(\underline{X},t) = \dot{\underset{\sim}{\chi}}(\underline{X},t)$ = material velocity

$\underset{\sim}{A}(\underline{X},t) = \ddot{\underset{\sim}{\chi}}(\underline{X},t)$ = " acceleration

Problem: given $\{\chi_n, V_n\}$ @ time t_n $\left(\Rightarrow \rho_0 \overset{\text{Also known}}{A_n} = \nabla \cdot \underline{P}(\nabla\chi_n) + \rho_0 \underset{\sim}{B}_n\right)$

want to compute $\{\chi_{n+1}, V_{n+1}, A_{n+1}\}$ @ t_{n+1}

Newmark eqns.
$$\left.\begin{aligned} \chi_{n+1} &= \chi_n + \Delta t\, V_n + \Delta t^2 \left[(\tfrac{1}{2}-\beta) A_n + \beta A_{n+1} \right] \\ V_{n+1} &= V_n + \Delta t \left[(1-\gamma) A_n + \gamma A_{n+1} \right] \\ \nabla \cdot P(\nabla \chi_{n+1}) &+ \rho_0 B_{n+1} = \rho_0 A_{n+1} \end{aligned}\right\} \quad (**)$$

(System of nonlinear PDEs in unknowns $\{\chi_{n+1}, V_{n+1}, A_{n+1}\}$.

$(**) \rightarrow$ continuum incremental (time-discretized) problem. Apply FEM to this.

Want: a minimum principle for χ_{n+1}.

Let $\chi_{n+1}^{pre} = \chi_n + \Delta t\, V_n + \Delta t^2 (\tfrac{1}{2}-\beta) A_n$ = Newmark Predictor for χ_{n+1}.
Then
$$\chi_{n+1} = \chi_{n+1}^{pre} + \beta \Delta t^2 A_{n+1}$$
Newmark corrector for χ_{n+1}

Notice $\dfrac{\chi_{n+1} - \chi_{n+1}^{pre}}{\beta \Delta t^2} = A_{n+1}$ looks like a sort of discrete time derivative

Insert this into $(**.3)$
$$\nabla \cdot P(\chi_{n+1}) + \rho_0 B_0 = \rho_0 \frac{\chi_{n+1} - \chi_{n+1}^{pre}}{\beta \Delta t^2}$$

The A_{n+1} term appears as a sort of body force that is linear in χ_{n+1}!
Can we derive that force from a potential?

Potential Energy extended to include inertia terms:

$$I[\chi_{n+1}] = \int_{\Omega_0} \left\{ \frac{1}{2} \rho_0 \left| \frac{\chi_{n+1} - \chi_{n+1}^{pre}}{\beta \Delta t^2} \right|^2 + w(\nabla \chi) \right\} dV_0$$

$$- \int_{\Omega_0} \rho_0 B_{n+1} \cdot \chi_{n+1} \, dV_0 - \int_{\partial \Omega_{0T}} T \cdot \chi_{n+1} \, dA$$

$$\min I \longrightarrow \delta I = \int_{\Omega_0} \rho_0 A_{n+1} \cdot \delta \chi_{n+1} + \frac{\partial w}{\partial F} : \nabla \delta \chi_{n+1} - \rho_0 B_{n+1} \cdot \delta \chi_{n+1} \, dV_0$$
$$- \int_{\partial \Omega_{0T}} T \cdot \delta \chi_{n+1} \, dA$$

Divergence theorem,

$$\delta I = \int_{\Omega_0} \left(- \nabla \cdot P_{n+1} - \rho_0 B_{n+1} + \rho_0 A_{n+1} \right) \cdot \delta \chi_{n+1} \, dV_0$$
$$+ \int_{\partial \Omega_{0T}} \left(P_{n+1} N - T_{n+1} \right) \delta \chi_{n+1} \, dA$$

E-L eqns. $\quad \nabla \cdot P_{n+1} + \rho_0 B_{n+1} = \rho_0 A_{n+1}$ in Ω_0 $\qquad$ "Equivalent" static problem

NBCs $\quad P_{n+1} N = T_{n+1}$ on $\partial \Omega_{0T}$

i) E-L eqns of I_{n+1} are the time-discretized eqns. of motion for χ_{n+1}.

ii) Assumption: minimizers define the "stable" configurations at t_{n+1}

iii) Inertia stabilizes the functional to some extent. We could stabilize more by adding viscosity. (Just wait.)

$$\boxed{\text{IBVP}} \xrightarrow[\text{discretization}]{\text{time}} \boxed{\begin{array}{c}\text{(Incremental)} \\ \text{BVP}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{minimum} \\ \text{principle}\end{array}} \xrightarrow[\text{discretization}]{\text{spatial (FE)}} \boxed{\begin{array}{c}\text{(Incremental)} \\ \text{FE eqns.}\end{array}}$$

Finite Element Discretization

Apply Rayleigh-Ritz to (*)

$$\boxed{\chi(\mathbf{X},t) = \sum_{a=1}^{N} x_a(t)\, \underbrace{N_a(\mathbf{X})}_{\text{independant of time}}}$$

Discrete time:

$$\chi_n(\mathbf{X}) = \sum_a x_a^n N_a(\mathbf{X})$$

$$\chi_{n+1}(\mathbf{X}) = \sum_a \underbrace{x_a^{n+1}}_{\underline{x}_{n+1} \text{ (array of position DoF @ } t_{n+1})} N_a(\mathbf{X})$$

Potential energy:

$$I_{n+1}(\underline{x}_{n+1}) = \int_{\Omega_0} \frac{\rho_0}{2\beta\Delta t^2} \left| \sum_a \left(x_a^{n+1} - x_a^{pre,n+1}\right) N_a \right|^2 dV_0 + \int_{\Omega_0} W(F_{n+1})\, dV_0 + \text{F.T.}$$

$$= \sum_{a,b} \frac{1}{2\beta\Delta t^2} \underbrace{\left(\int_{\Omega_0} \rho_0 \delta_{ik} N_a N_b\, dV_0\right)}_{\text{mass matrix}} \left(x_{ia}^{n+1} - x_{ia}^{pre,n+1}\right)\left(x_{kb}^{n+1} - x_{kb}^{pre,n+1}\right)$$

$$+ \dots$$

$$M_{iakb} = \int_{\Omega_0} \rho_0 \delta_{ik} N_a N_b\, dV_0$$

R-R: $\min I_{n+1} \rightarrow \delta I_{n+1} = 0$

$$\hookrightarrow \underline{M}\, \underbrace{\frac{\underline{x}_{n+1} - \underline{x}_{n+1}^{pre}}{2\beta\Delta t^2}}_{\underline{a}_{n+1}} + \underline{f}^{int}(x_{n+1}) - \underline{f}^{ext} = 0$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta t \underline{v}_n + \Delta t^2\left[\left(\tfrac{1}{2} - \beta\right)\underline{a}_n + \beta \underline{a}_{n+1}\right]$$

$$\underline{v}_{n+1} = \underline{v}_n + \Delta t\left[(1-\gamma)\underline{a}_n + \gamma \underline{a}_{n+1}\right]$$

$$\underline{M}\,\underline{a}_{n+1} + \underline{f}^{int}(x_{n+1}) = \underline{f}^{ext}_{n+1}$$

SPECIAL CASES

Explicit Dynamics: $\beta = 0$

$$\underline{x}_{n+1} = \underline{x}_{n+1}^{pre}$$
$$\underline{a}_{n+1} = \underline{M}^{-1}\left[f_{n+1}^{ext} - f^{int}(x_{n+1})\right]$$
$$\underline{v}_{n+1} = \underline{v}_n + \Delta t\left[(1-\gamma)\underline{a}_n + \gamma \underline{a}_{n+1}\right]$$

Comments:

i) Only efficient if $\underline{M}$ is diagonal. Then $\underline{M}^{-1}$ is trivial.

Example: Row-sum lumping

$$M^L_{iaia} = \int_{\Omega_0} \rho_0 N_a \, dV_0 = \sum_b \int \rho_0 N_a \underline{N_b} \, dV_0 = \sum_b M_{iaib}$$

(no sum on i or a) $\qquad \sum_b N_b = 1$

See Hughes or Zienkiewicz & Taylor for more on this.

ii) Explicit Newmark is only <u>conditionally</u> stable.

Courant-Friedrichs-Levy condition: An elastic wave should travel a distance smaller than the size of an element, h, in a time step, Δt.

$$\Delta x = c\Delta t < h$$
$$c = \sqrt{E/\rho} = \text{speed of sound in material}$$

Use this as a rough guide.

If you exceed stability threshold ($\Delta t \gtrsim h/c$) oscillations will grow without bound.

Implicit Dynamics : $\beta \neq 0$

Newton-Raphson, "traditional" approach.

i) Initialize $k=0$

$$x_{n+1}^{(0)} = x_{n+1}^{pre} = x_n + \Delta t\, v_n + \left(\tfrac{1}{2}-\beta\right)\Delta t^2 a_n$$

$$v_{n+1}^{(0)} = v_{n+1}^{pre} = v_n + (1-\gamma)\,\Delta t\, a_n$$

$$a_{n+1}^{(0)} = 0$$

ii) Linearize equs about $x_{n+1}^{(k)}$, $v_{n+1}^{(k)}$, $a_{n+1}^{(k)}$

$$x_{n+1}^{(k+1)} = x_{n+1}^{(k)} + \Delta x$$

$$v_{n+1}^{(k+1)} = v_{n+1}^{(k)} + \Delta v \qquad \Delta v = \gamma \Delta t\, \Delta a$$

$$a_{n+1}^{(k+1)} = a_{n+1}^{(k)} + \Delta a \qquad \Delta a = \Delta x / \beta \Delta t^2$$

$$\left.\begin{matrix}\Delta v = \gamma \Delta t\, \Delta a \\ \Delta a = \Delta x/\beta\Delta t^2\end{matrix}\right\} \Delta v = \frac{\gamma}{\beta \Delta t}\Delta x$$

$$M\,\Delta a + \underbrace{\frac{\partial f^{int}}{\partial x}(x_{n+1}^{(k)})}_{\text{tangent stiffness}}\,\Delta x = f_{n+1}^{ext} - f^{int}(x_{n+1}^{(k)}) = -r(x_{n+1}^{(k)})$$

iii) Solve $\left(\frac{1}{\beta \Delta t^2} M + K\right)\Delta x = -r(x_{n+1}^{(k)})$

iv) $x_{n+1}^{(k+1)} = x_{n+1}^{(k)} + \Delta x \quad v_{n+1}^{(k+1)} = \ldots \quad a_{n+1}^{(k+1)} = \ldots$

v) If $|r| > tol$ $k \leftarrow k+1$. Go to ii)

Else $x_{n+1} = x_{n+1}^{(k)}$.

Unconditional Stability for $\gamma \geq \frac{1}{2}$ $\beta \geq \frac{1}{4}\left(\gamma+\frac{1}{2}\right)^2$. (See Hughes)