MAE 261B – Computational Mechanics of Solids and Structures

Lecture 16: Shells

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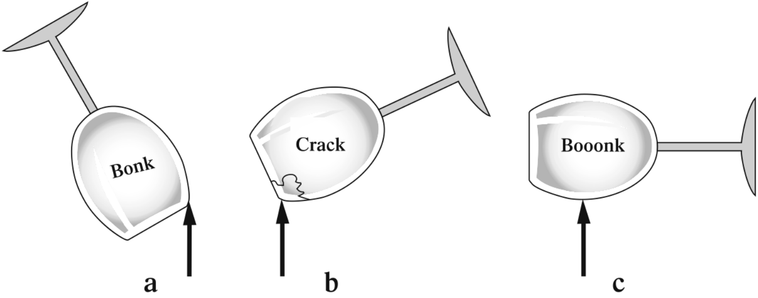
Consequences of initial curvature

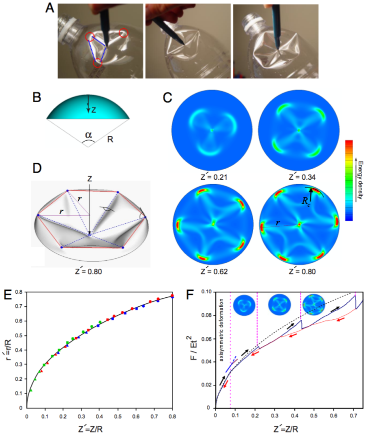
Stretching and bending are difficult to separate for curved shells.

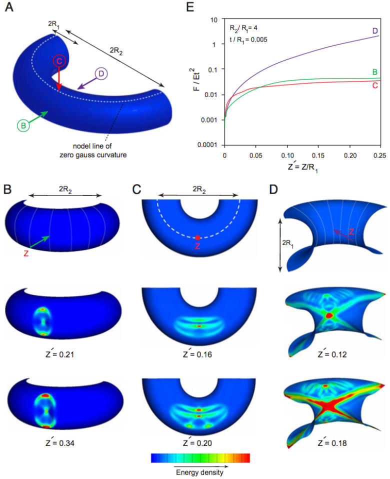


Wempner and Talaslidis (2003)

Example: Role of Gaussian Curvature

Sign of Gauss Curvature matters. $K = b_1 b_2$.





Vaziri and Mahadevan (2008)

Finite Element Approaches to Shells

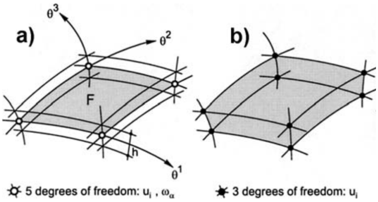
2 basic categories:

A. Classical (degenerate solid) Shell

- Track midsurface explicitly
- Flavors:
  1. Co-rotational: local frame rotates with element; small-strain/large-rotation approximations.
  2. Geometrically exact: no geometric approximations; resultants only; no 3-D constitutive laws
  3. Continuum-based: shell kinematics input into 3-D constitutive laws

B. Solid Shell

- Track multiple surfaces explicitly
- Multiple nodes across thickness
- Displacement DOF only
- Different shape functions in lateral and thickness directions.



Kratzig and Jun (2003)

Shell Theory

Shell Kinematics

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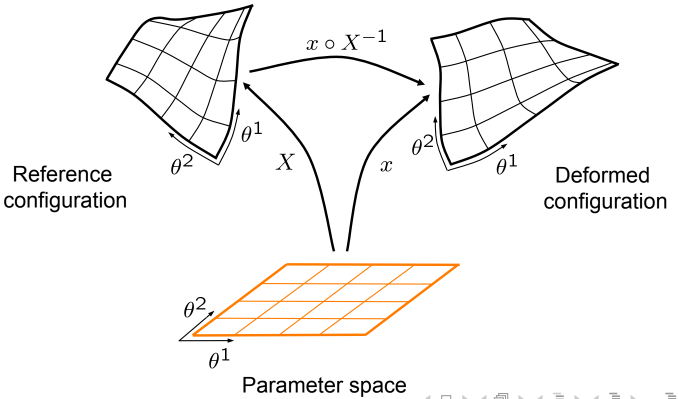
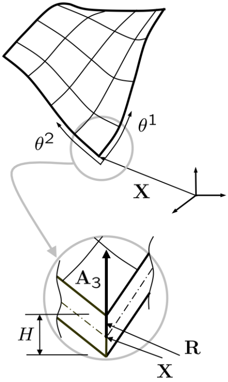
Explicitly track the mid surface of the deforming shell:

$$\boldsymbol{X}(\theta^1, \theta^2) \equiv \textit{Reference} \text{ position of mid surface}$$
$$\boldsymbol{x}(\theta^1, \theta^2) \equiv \textit{Deformed} \text{ position of mid surface}$$



Reference Configuration

Material Points are located by distance from the mid surface along the reference *director* $\boldsymbol{A}_3$.



- Reference position:

$$\boldsymbol{R}(\theta^1, \theta^2, \theta^3) \equiv \boldsymbol{X}(\theta^1, \theta^2) + \theta^3 \boldsymbol{A}_3(\theta^1, \theta^2)$$

- Thickness coordinate:

$$\theta^3 \in [-H/2, H/2]$$

- Surface basis vectors:

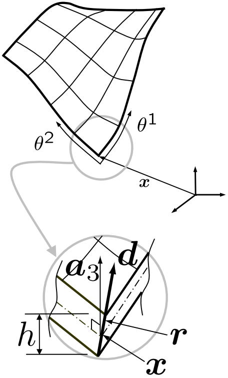
$$\boldsymbol{A}_\alpha \equiv \boldsymbol{X}_{,\alpha} \equiv \partial \boldsymbol{X} / \partial \theta^\alpha$$

- Reference Director:

$$\boldsymbol{A}_3 \equiv \frac{\boldsymbol{A}_1 \times \boldsymbol{A}_2}{|\boldsymbol{A}_1 \times \boldsymbol{A}_2|}$$

Deformed Configuration

Classical Shell theory assumption: Deformed *director* $\boldsymbol{d}$ is inextensible and simply*rotates* during defomation. Generally it is *not* equal to the deformed surface normal.



- Deformed position:

$$\boldsymbol{r}(\theta^1, \theta^2, \theta^3) \equiv \boldsymbol{x}(\theta^1, \theta^2) + \lambda(\theta^1, \theta^2)\theta^3 \boldsymbol{d}(\theta^1, \theta^2)$$

- Inextensible director: $|\boldsymbol{d}(\theta^1, \theta^2)| = 1$
  We say the director is on the **unit sphere**, denoted $\boldsymbol{d} \in \mathrm{S}^2$.
- Scalar thickness stretch: $\lambda(\theta^1, \theta^2)$
- Surface basis vectors:

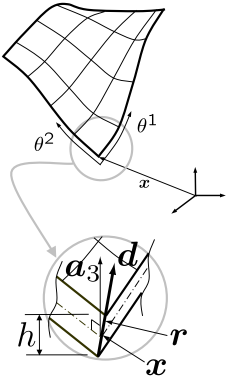
$$\boldsymbol{a}_\alpha \equiv \boldsymbol{x}_{,\alpha} \equiv, \quad \boldsymbol{a}_3 \equiv \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}$$

Notes:

- The more general case including thickness and transverse shear deformation is a sort of extension of Reissner-Mindlin of plate theory.
- Common approximation: neglect spatial variation of thickness stretch, determine λ by condensation.

Kirchhoff-Love (K-L) Theory

Two additional "simplifying" assumptions:



1. Neglect thickness stretch:

$$\lambda = 1 \quad \Rightarrow \quad \boldsymbol{r} = \boldsymbol{x} + \theta^3 \boldsymbol{d}$$

2. No shear deformation; director remains normal to deformed surface:

$$\boldsymbol{d}(\theta^1, \theta^2) = \boldsymbol{a}_3 \equiv \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}$$

$$\Rightarrow \quad \boldsymbol{r}(\theta^1, \theta^2, \theta^3) = \boldsymbol{x}(\theta^1, \theta^2) + \theta^3 \boldsymbol{a}_3(\theta^1, \theta^2).$$

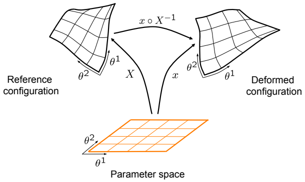
Therefore, in K-L theory, the deformation of the shell is completely determined by the deformation of the mid surface.

Deformation

$$\boldsymbol{R}(\theta^1,\theta^2,\theta^3) \equiv \boldsymbol{X}(\theta^1,\theta^2) + \theta^3 \boldsymbol{A}_3(\theta^1,\theta^2)$$
$$\boldsymbol{r}(\theta^1,\theta^2,\theta^3) \equiv \boldsymbol{x}(\theta^1,\theta^2) + \lambda(\theta^1,\theta^2)\theta^3 \boldsymbol{d}(\theta^1,\theta^2)$$

Deformation gradient:

$$\boldsymbol{F} = \boldsymbol{g}_i \otimes \boldsymbol{G}^i$$



Basis vectors:

$$\boldsymbol{G}_\alpha \equiv \boldsymbol{R}_{,\alpha} = \boldsymbol{A}_\alpha + \theta^3 \boldsymbol{A}_{3,\alpha}$$
$$\boldsymbol{G}_3 \equiv \boldsymbol{R}_{,3} = \boldsymbol{A}_3$$
$$\boldsymbol{g}_\alpha \equiv \boldsymbol{r}_{,\alpha} = \boldsymbol{a}_\alpha + \lambda\theta^3 \boldsymbol{d}_{,\alpha}$$
$$\boldsymbol{g}_3 \equiv \boldsymbol{r}_{,3} = \lambda \boldsymbol{d}$$

Dual Basis vectors:

$$\boldsymbol{G}^i \cdot \boldsymbol{G}_j = \delta^i_j \quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$$

References

G. Wempner and D. Talaslidis. *Mechanics of Solids and Shells, Theory and Approxmation*. CRC Press, 2003.

Ashkan Vaziri and L. Mahadevan. Localized and extended deformations of elastic shells. *PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE UNITED STATES OF AMERICA*, 105(23): 7913–7918, 2008.

WB Kratzig and D Jun. On 'best' shell models - From classical shells, degenerated and multi-layered concepts to 3D. *ARCHIVE OF APPLIED MECHANICS*, 73(1-2):1–25, 2003.