FINITE ELEMENT APPROXIMATION THEORY

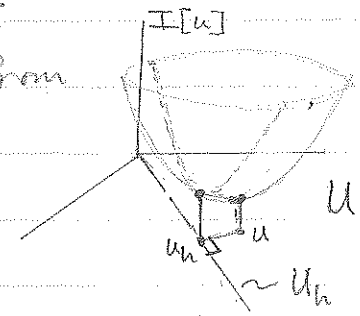
Rayleigh-Ritz (Constrained minimization)

$\mathcal{U}_h$ finite dimensional subspaces of $\mathcal{U}$

$\mathcal{U}_h = \text{span}\{ N_a : B \to \mathbb{R},\ a = 1, \dots, N \}$

Approximate solution(s) u_h follow from

$$\inf_{u \in \mathcal{U}_h} I[u]$$



Questions: (1) How do we set up $\mathcal{U}_h$?

(2) $\lim_{h \to 0} u_h$ a minimizer of I over $\mathcal{U}$? (convergence)

(3) Mesh adaption? (best $\mathcal{U}_h$ for fixed N)

Interpolation scheme: $\underset{\sim}{u}_h(\underset{\sim}{X}) = \sum_{a=1}^{N} \underset{\sim}{u}_a N_a(\underset{\sim}{X})$

$$\boxed{u_i(\underset{\sim}{X}) = \sum_{a=1}^{N} u_{ia} N_a(\underset{\sim}{X})}$$

Array of DOF: $\underline{u}_h = \{ \underset{\sim}{u}_1, \dots, \underset{\sim}{u}_N \} \in \mathbb{R}^{3N}$

Rayleigh-Ritz: $\boxed{\inf_{\underset{\sim}{u} \in \mathcal{U}_h} I[\underset{\sim}{u}]}$

Restriction of I to $\mathcal{U}_h \to I_h$

$$I_h(\underline{u}_h) = I[\underset{\sim}{u}_h], \quad \forall \underline{u}_h \in \mathcal{U}_h$$

RR': $\inf_{\underset{\sim}{u}_h \in \mathcal{U}_h} I[\underset{\sim}{u}_h] \iff \boxed{\inf_{\underline{u}_h \in \mathbb{R}^{3N}} I_h(\underline{u}_h)}$

Discritized Equations of Equilibrium

$$\underset{\sim}{\epsilon}_h = \nabla^S \underset{\sim}{u}_h$$

$$\epsilon_{ij} = \sum_{a=1}^{N} \frac{1}{2}\left[u_{ia} N_{a,j}(\underset{\sim}{x}) + u_{ja} N_{a,i}\right]$$

$$I_h \equiv \int_B W(\underset{\sim}{\epsilon}_h)\, dV - \int_B f_i u_i \, dV - \int_{S_2} \bar{t}_i u_i \, dS$$

$$= \int_B W(\underset{\sim}{\epsilon}_h)\, dV - \int_B f_i\Big(\sum_{a=1}^{N} u_{ia} N_a\Big) dV - \int_{S_2} \bar{t}_i\Big(\sum_{a=1}^{N} u_{ia} N_a\Big) dS$$

$$I_h = \int_B W(\underset{\sim}{\epsilon}_h)\, dV - \sum_{a=1}^{N} f_{ia}^{ext} u_{ia}$$

$$f_{ia}^{ext} = \int_B f_i N_a \, dV + \int_{S_2} \bar{t}_i N_a \, dS \quad \text{External forces}$$

Euler-Lagrange Eqns of RR′

$$DI_h(u_h) = 0 \qquad \frac{\partial I_h}{\partial u_{ia}} = 0$$

$$\Rightarrow \int_B \frac{\partial W(\underset{\sim}{\epsilon}_h)}{\partial u_{ia}} dV - f_{ia}^{ext} = 0$$

$$\frac{\partial W(\underset{\sim}{\epsilon}_h)}{\partial u_{ia}} = \frac{\partial W}{\partial \epsilon_{kl}} \frac{\partial \epsilon_{kl}}{\partial u_{ia}} = \sigma_{kl} \frac{\partial}{\partial u_{ia}} \Big\{ \sum_{b=1}^{N} \frac{1}{2}\left[u_{kb} N_{b,l} + u_{lb} N_{b,k}\right] \Big\}$$

$$= \sigma_{kl} \Big\{ \frac{1}{2} \sum_{b=1}^{N} \left(\delta_{ik} \delta_{ab} N_{b,l} + \delta_{il} \delta_{ab} N_{b,k} \right) \Big\}$$

$$= \sigma_{kl} \Big\{ \frac{1}{2} \left(\delta_{ik} N_{a,l} + \delta_{il} N_{a,k} \right) \Big\}$$

$$= \frac{1}{2}\left(\sigma_{il} N_{a,l} + \sigma_{ki} N_{a,k} \right)$$

$$= \frac{1}{2}\left(\sigma_{ij} N_{a,j} + \sigma_{ji} N_{a,j} \right) \qquad \sigma_{ij} = \sigma_{ji}$$

$$= \sigma_{ij} N_{a,j}$$

$$\therefore \Rightarrow \underbrace{\int_B \sigma_{ij}(\nabla^S u_h) N_{a,j} \, dV}_{} - \int_B f_i N_a \, dV - \int_{S_2} \bar{t}_i N_a \, dS = 0$$

$$f_{ia}^{int}(u_h) - f_{ia}^{ext} = 0$$

Discrete equilibrium equations (may be nonlinear)

Hookean material

$$\frac{\partial W}{\partial \epsilon_{ij}} = \sigma_{ij} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}} \epsilon_{kl} = c_{ijkl} \epsilon_{kl}$$
$$= \sum_{b=1}^{N} c_{ijkl} u_{kb} N_{b,l}(x)$$

$$\Rightarrow \quad f_{ia}^{int} = \int_B \sum_{b=1}^{N} c_{ijkl} u_{kb} N_{b,l} N_{a,j} \, dV$$

$$= \sum_{b=1}^{N} \left(\int_B c_{ijkl} N_{a,j} N_{b,l} \, dV \right) u_{kb}$$

stiffness components

$$\boxed{f_{ia}^{int} = \sum_{b=1}^{N} K_{iakb} u_{kb}}$$

$$\therefore \quad \sum_{b=1}^{N} K_{iakb} u_{kb} - f_{ia}^{ext} = 0 \qquad i = 1,2,3 \quad a = 1, \dots, N$$

$$\boxed{\underline{K}_h \underline{u}_h - \underline{f}_h = \underline{0}}$$ linear algebraic system

Discrete FE Equilibrium Equations for Linear Elasticity

$$f_{ia}^{ext} = \int_B f_i(\underline{x}) N_a(\underline{x}) \, dV + \int_{S_2} \bar{t}_i(\underline{x}) N_a(\underline{x}) \, dS$$

$$K_{iakb} = \int_B c_{ijkl}(\underline{x}) N_{a,j}(\underline{x}) N_{b,l}(\underline{x}) \, dV$$

FINITE ELEMENT INTERPOLATION

Method for generating convenient u_h's. Based on discretization of domain B (meshing / triangulation).

Partition $B = \bigcup_{e=1}^{E} \Omega^e$, $\{\Omega^e, e = 1, \dots, E\}$ collection of Finite Elements.

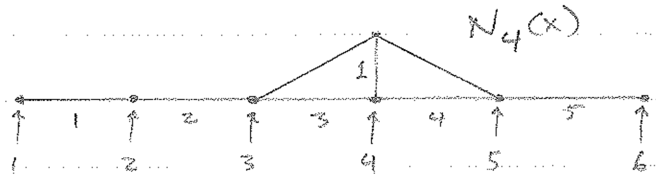
Global Finite Element Interpolent

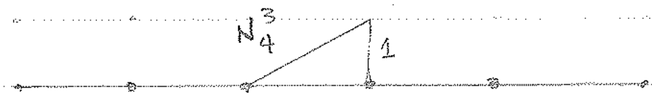
$$\boxed{\underset{\sim}{u}_h = \sum_{a=1}^{N} \underset{\sim}{u}_a N_a(\underset{\sim}{x})} \qquad (1)$$

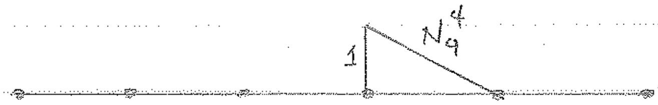
$\underset{\sim}{u}_a \equiv$ value of unknown field (displacements) at node a in the mesh

$N_a(x) \equiv$ Global shape functions defined by "piecing together" local shape functions $N_a^e(\underset{\sim}{x})$, i.e., $N_a(\underset{\sim}{x}) = N_a^e(\underset{\sim}{x})$ if $\underset{\sim}{x} \in \Omega^e$

Ex: 1-D $u(x)$ w/ linear shape fns.

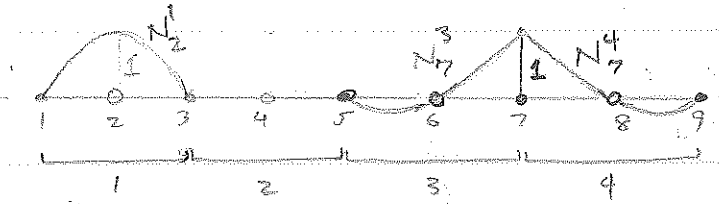




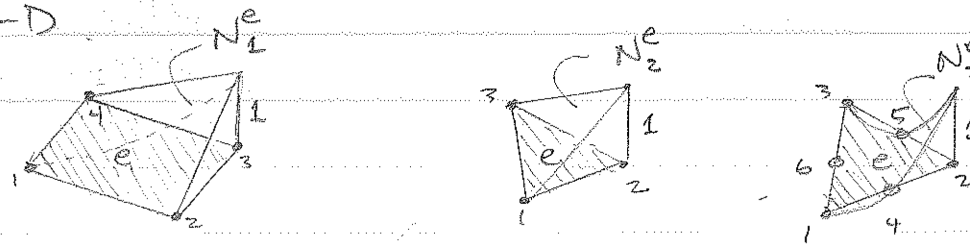


Examples of simplicial interpolation

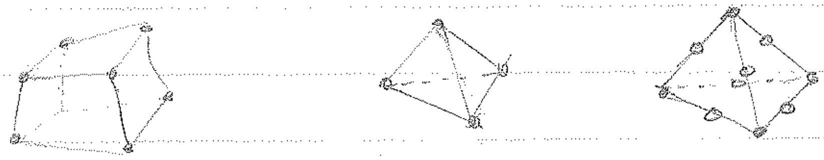
Higher order interpolents?



2-D



3-D



Local Shape Functions $\Rightarrow$ compute element contributions to $\underline{f}_h$, $\underline{K}_h$ separately, then assemble

$$f_{ia}^{ext} = \sum_{e=1}^{E} \left\{ \int_{\Omega^e} f_i N_a^e(x)\, dV + \int_{\partial\Omega^e \cap S_2} \bar{t}_i N_a^e(x)\, dS \right.$$

$$K_{iakb} = \sum_{e=1}^{E} \int_{\Omega} c_{ijkl} N_{a,j}^e N_{b,l}\, dV$$

$$f_{ia}^e = \int_{\Omega^e} f_i N_a^e(x)\, dV + \int_{\partial\Omega^e \cap S_2} \bar{t}_i N_a^e(x)\, dS \qquad K_{iakb}^e = \int_{\Omega^e} c_{ijkl} N_{a,j}^e N_{b,l}^e\, dV$$

$$f_{ia} = \sum_{e=1}^{E} f_{ia}^e \qquad K_{iakb} = \sum_{e=1}^{E} K_{iakb}^e$$

Here we will focus on SIMPLICIAL elements (lines, triangles & tets) because of their well developed mathematical framework. Useful for

(1) error analysis

(2) automatic meshing (triangulation)

Simplices

(i) Basic building blocks of polyhedra

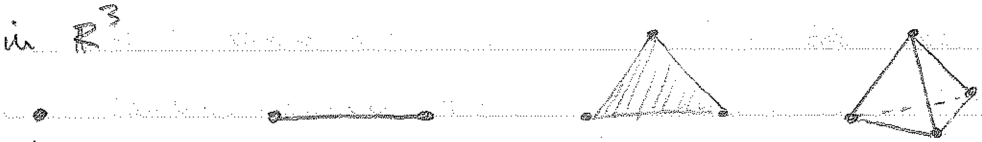
(ii) Natural basis for FE interpolation over solid bodies

Def A K-simplex of vertices $x_a \in \mathbb{R}^d$, $a = 1, \dots, k+1$, $k \le d$ is the set of all convex combinations of x_a's

$$K = \left\{ x \in \mathbb{R}^d : x = \sum_{a=1}^{k+1} \lambda_a x_a \, , \; 0 \le \lambda_a \le 1 \, , \; \sum_{a=1}^{k+1} \lambda_a = 1 \right\}$$

= "all points in between the x_a's"

Ex in $\mathbb{R}^3$



0-simplex (vertex) 1-simplex (edge) 2-simplex (triangle) 3-simplex (tetrahedron)

Barycentric Coordinates K is a d-simplex in $\mathbb{R}^d$ and let $\underline{x} \in K$. Then

$$\left. \begin{aligned} \underline{x} &= \sum_{a=1}^{d+1} \lambda_a \underline{x}_a \\ 1 &= \sum_{a=1}^{d+1} \lambda_a \end{aligned} \right\} \quad \underbrace{\begin{pmatrix} & x_{ia} & \\ & & \\ 1 & \cdots & 1 \end{pmatrix}}_{\mathbb{M}} \underbrace{\begin{Bmatrix} \lambda_1 \\ \vdots \\ \lambda_{d+1} \end{Bmatrix}}_{\boldsymbol{\lambda}} = \underbrace{\begin{Bmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{Bmatrix}}_{\mathbb{X}}$$

Def K is nondegenerate (degenerate) if

$\det \mathbb{M} \neq 0 \quad (\det \mathbb{M} = 0)$

Volume of $K \equiv |K| = \frac{1}{d!} |\det \mathbb{M}|$

If K nondegenerate

$$\boldsymbol{\lambda} = \mathbb{M}^{-1} \mathbb{X} \Rightarrow \lambda_a = \lambda_a(\underline{x})$$

affine functions of $\underline{x}$

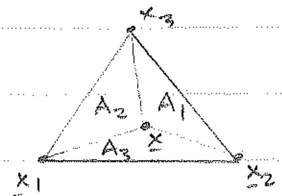
$\lambda_a(x)$ $a = 1, \ldots, d+1$ $\equiv$ barycentric cooordinates of $\underline{x}$ w.r.t. K

Barycenter: $\lambda_a = \dfrac{1}{d+1}$

Also note: $\lambda_a(\underline{x}_b) = \delta_{ab}$ $a, b = 1, \ldots, d+1$

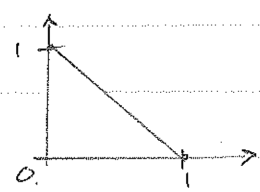
<u>Geometric Interpretation</u>

EX1 2-simplex in $\mathbb{R}^2$



$$\lambda_a = \frac{A_a}{A} \qquad A = \sum_{a=1}^{3} A_a$$

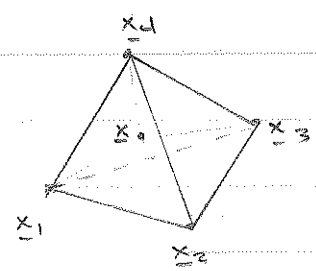
$a = 1, 2, 3$



$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 1 \end{pmatrix} \qquad \det M = 1 \quad d! = 2$$

$$\frac{1}{d!} \det M = \frac{1}{2} = A$$

EX2 3-simplex in $\mathbb{R}^3$



$$\lambda_a = \frac{V_a}{V} \qquad a = 1, \ldots, 4$$

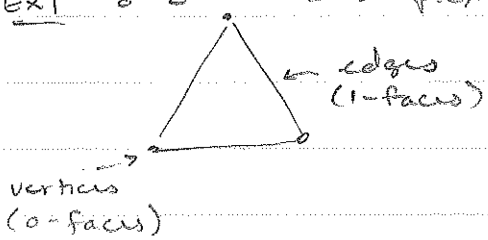
In General for a d-simplex in $\mathbb{R}^d$

$$\lambda_a = \frac{\mathrm{vol}(x_1, \ldots, x_{a-1}, x, x_{a+1}, \ldots, x_{d+1})}{\mathrm{vol}(x_1, \ldots, x_{a-1}, x_a, x_{a+1}, \ldots, x_{d+1})}$$

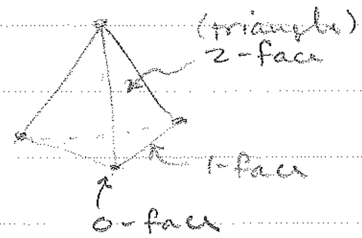
Simplicial Complexes

<u>Def</u> Let K be a d-simplex in $\mathbb{R}^n$. A <u>k-face</u> $S \subset K$ is a k-simplex obtained by setting to zero $d-k$ λ's

<u>EX1</u> $d=2 \rightarrow$ 2-simplex in $\mathbb{R}^2$ (Tri)



<u>EX2</u> $d=3$ 3-simplex in $\mathbb{R}^3$ (Tet)



In general a d-simplex has $\underbrace{\binom{d+1}{k+1}}_{\text{binomial coefficient}} = \dfrac{(d+1)!}{(k+1)!\,(d+1-k-1)!}$ k-faces

<u>EX</u> $d=2$ $k=0$ $\binom{3}{1} = 3$ vertices

$d=2$ $k=1$ $\binom{3}{2} = 3$ edges

$d=2$ $k=2$ $\binom{3}{3} = 1$ triangle

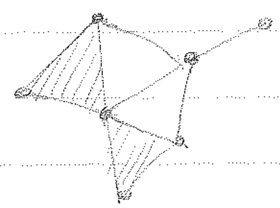
<u>Def</u> A finite collection $\mathcal{C}$ of simplices defines a <u>simplicial complex</u> $\Leftrightarrow$:

(i) If $K \in \mathcal{C}$ and S is a face of $K \Rightarrow S \in \mathcal{C}$

(ii) If $K_1, K_2 \in \mathcal{C}$ then $K_1 \cap K_2 \in \mathcal{C}$ or is empty

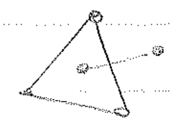
Simplicial Complex = "Mesh"

<u>Ex 1</u>



Simplicial complex

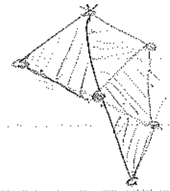
<u>Ex 2</u>



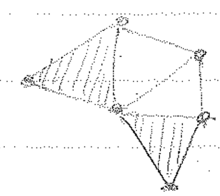
Not a simplicial complex

<u>Def</u> A subset Ω of $\mathbb{R}^n$ is <u>triangulable</u> if it can be represented as a simplicial complex.

<u>Def</u> A simplicial complex $\mathcal{C}$ is a <u>polyhedral solid</u> in $\mathbb{R}^d$ if all simplices $S \in \mathcal{C}$ of dimension $d-1$ are contained in at least a d-simplex.



Polyhedral Solid in $\mathbb{R}^2$



Not a polyhedral solid (But is a simplicial complex)

FE Discretization: Represent body B as a polyhedral solid. Interpolate fields on simplices.

Polynomial Interpolation over Simplices

1^{st} discuss Polynomials in more than one dimension.

$\mathbb{N} \equiv$ "natural numbers" (non-negative integers)

Def A multiindex α of dimension d is a member of $\mathbb{N}^d$. (i.e., $\alpha = \{\alpha_1, \ldots, \alpha_d\}$ $\alpha_i \in \mathbb{N}$)

Def The degree of multiindex $\alpha \in \mathbb{N}^d$ is

$$|\alpha| = \alpha_1 + \cdots + \alpha_d$$

Notation: $\underline{x} \in \mathbb{R}^d$, $\alpha \in \mathbb{N}^d \Rightarrow x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ (monomial)

Notation: Let K be a d-simplex in $\mathbb{R}^d$

$P_k(K) = \{$ Polynomials of degree less than or equal to k over $K \}$

Notation: if $p \in P_k(K)$, $K \subset \mathbb{R}^d$

$$p = \sum_{|\alpha| \le k} a_\alpha x^\alpha$$

Ex (i) Linear Polynomials in $\mathbb{R}^d$: If $K \subset \mathbb{R}^d$ d-simplex and $p \in P_1(K)$ then

$$p = \sum_{|\alpha| \le 1} a_\alpha x^\alpha = a_{\{0,\ldots,0\}} \cdot 1 + a_{\{1,0,\ldots,0\}} x_1 + \cdots + a_{\{0,\ldots,0,1\}} x_d$$

(ii) $k=2$, $d=2$ $\quad p = \sum_{|\alpha| \le 2} a_\alpha x^\alpha \quad x = (x_1, x_2) \in \mathbb{R}^2$

$$p = a_{00} \cdot 1 + a_{10} x_1 + a_{01} x_2 + a_{20} x_1^2 + a_{11} x_1 x_2 + a_{02} x_2^2$$

$\#$ of monomials $= \dim P_k(\mathbb{R}^d) = \binom{d+k}{d} = \frac{(d+k)!}{d!\,k!}$ $\quad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$

Linear Interpolation over simplices

$K \subset \mathbb{R}^d$, a d-simplex, non-degenerate

Proposition: Let $f: K \to \mathbb{R}$. Then there is a unique linear polynomial which interpolates f at vertices of K.

<u>Proof</u>. Let $p(x) = \sum_{a=1}^{d+1} f(x_a)\lambda_a(x) \in P_1(K)$. Then

$p(x_b) = \sum_{a=1}^{d+1} f(x_a)\lambda_a(x_b) \Rightarrow p(x_b) = \sum_{a=1}^{d+1} f(x_a)\delta_{ab} = f(x_b)$

$\Rightarrow p(x)$ interpolates $f(x)$ at all vertices of K.

<u>Uniqueness</u>. Let p_1, p_2 be interpolating polynomials. Then $p_1 - p_2 \in P_1(K)$ and $(p_1 - p_2)(x_a) = 0 \quad a = 1, \ldots, d+1$

$\therefore (p_1 - p_2)(x) = \sum_{|\alpha| \le 1} a_\alpha x^\alpha \to (p_1 - p_2)(x_b) = \sum_{|\alpha|\le 1} a_\alpha x_b^\alpha$

in matrix form

$$\mathbb{p} = \begin{Bmatrix} (p_1 - p_2)(x_1) \\ \vdots \\ (p_1 - p_2)(x_{d+1}) \end{Bmatrix} \quad a = \begin{Bmatrix} a_{\{0,\ldots,0\}} \\ a_{\{1,0,\ldots,0\}} \\ \vdots \\ a_{\{0,\ldots,0,1\}} \end{Bmatrix} \quad M = \begin{bmatrix} & x_{ia} & \\ & & \\ 1 & \cdots & 1 \end{bmatrix}$$

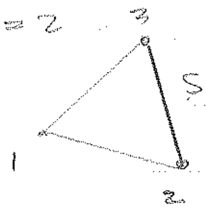
$d+1 \qquad\qquad d+1 \qquad\qquad d+1$ by $d+1$

$$\mathbb{p} = M^T a = 0 \Rightarrow a = 0 \Rightarrow a_\alpha = 0 \Rightarrow p_1 - p_2 = 0$$

$\therefore \boxed{p(x) = \sum_{a=1}^{d+1} f(x_a)\lambda_a(x)}$ is unique linear interpolant of $f(x)$ over K

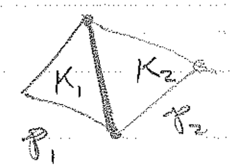
<u>Corollary</u>. Restriction of a linear polynomial $p \in P_1(K)$ to a face $S \subset K$ is completely determined by values at vertices of S.

Ex $d = 2$



$\lambda_1 = 0$ on $S \Rightarrow p|_S = f(x_2)\lambda_2(x) + f(x_3)\lambda_3(x)$
$\uparrow \qquad \uparrow$

This corollary ensures C^0 interpolation over solids



$p_1|_S = p_2|_S$

p_1, p_2 interpolating polynomials in $P_1(K_1)$, $P_1(K_2)$

Higher-Order Interpolation over Simplices

$K \subset \mathbb{R}^d$, d-simplex
$\dim P_k(K) = \binom{d+k}{k} \equiv n$

Seek a basis $\{N_a, a = 1, \dots, n\}$ of $P_k(K)$ such that
(i) (FE normalization)
$N_a(x_b) = \delta_{ab} \quad \forall x_b \in X_k(K) \equiv$ nodal set of order k over K

(ii) Let S be a face of K. Want
$\{N_a|_S, a \in X_k(K) \cap S\}$ to be a basis for $P_k(S)$
($p|_S$ depends only on nodes of $S \Rightarrow C^0$-continuity)

(iii) (Symmetry): If $x \in X_k(K)$ and
$$x = \sum_{a=1}^{d+1} \lambda_a x_a$$
then
$$y = \sum_{a=1}^{d+1} \lambda_{\sigma(a)} x_a \in X_k(K)$$
$\forall$ permutations σ of $\{1, \dots, d+1\}$

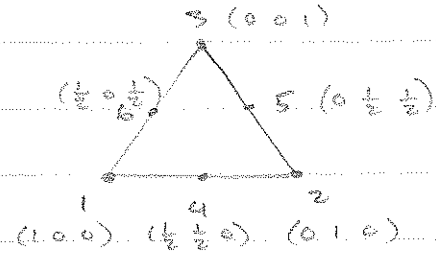
The nodal set X_k which satisfies (iii) above

$$X_k = \{ x = \sum_{a=1}^{d+1} \lambda_a x_a ,\ \lambda_a \in \{0, \tfrac{1}{k}, \dots, \tfrac{k-1}{k}, 1\},\ a = 1, \dots, d+1 \}$$

Ex (1) $k=1$ $\quad \lambda_a \in \{0, 1\} \Rightarrow$ Nodes = Vertices

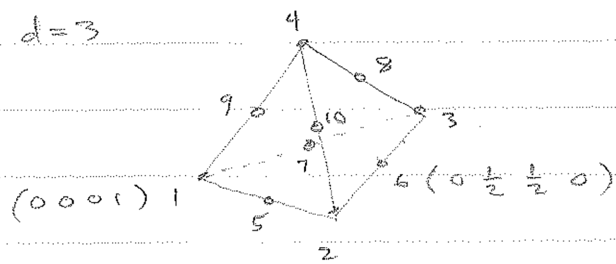
(2) $k=2$ $\quad \lambda_a \in \{0, \tfrac{1}{2}, 1\} \Rightarrow$ Vertices + Edge midpoints

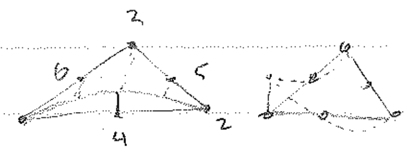
$d=2$



Quadratic Interpolation over triangles

$d=3$





Quadratic Shape Functions satisfying (i)–(iii) above

$d=2$

$$N_4 = 4\lambda_1\lambda_2 \qquad N_1 = \lambda_1 - \tfrac{1}{2}N_6 - \tfrac{1}{2}N_4 = \lambda_1(2\lambda_1 - 1)$$

$$N_5 = 4\lambda_2\lambda_3 \qquad N_2 = \lambda_2 - \tfrac{1}{2}N_4 - \tfrac{1}{2}N_5 = \lambda_2(2\lambda_2 - 1)$$

$$N_6 = 4\lambda_3\lambda_1 \qquad N_3 = \lambda_3 - \tfrac{1}{2}N_5 - \tfrac{1}{2}N_6 = \lambda_3(2\lambda_3 - 1)$$

$d=3$

$$N_5 = 4\lambda_1\lambda_2 \qquad N_1 = \lambda_1 - \tfrac{1}{2}N_5 - \tfrac{1}{2}N_7 - \tfrac{1}{2}N_9 = \lambda_1(2\lambda_1 - 1)$$

$$N_6 = 4\lambda_2\lambda_3$$

$$N_7 = 4\lambda_3\lambda_1 \qquad N_2 = \lambda_2 - \tfrac{1}{2}N_5 - \tfrac{1}{2}N_6 - \tfrac{1}{2}N_{10} = \lambda_2(2\lambda_2 - 1)$$

$$N_8 = 4\lambda_3\lambda_4$$

$$N_9 = 4\lambda_4\lambda_1 \qquad N_3 = \lambda_3 - \tfrac{1}{2}N_6 - \tfrac{1}{2}N_7 - \tfrac{1}{2}N_8 = \lambda_3(2\lambda_3 - 1)$$

$$N_{10} = 4\lambda_2\lambda_4 \qquad N_4 = \lambda_4 - \tfrac{1}{2}N_8 - \tfrac{1}{2}N_9 - \tfrac{1}{2}N_{10} = \lambda_4(2\lambda_4 - 1)$$

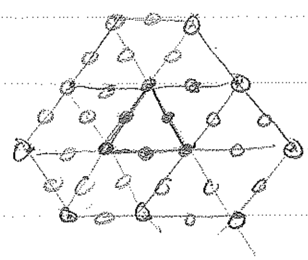
Global Interpolation

Let: $\mathcal{C}$ = a triangulation of solid body $\mathcal{B}$

$\mathcal{N}$ = nodal set (vertices of $\mathcal{C}$ + higher-order nodes)

$K \subset \mathcal{C}$ = a "finite element"

$\mathcal{N}^K = \mathcal{N} \cap K$ (nodal set of K)



$\{\circ\} = \mathcal{N}$

$\{\bullet\} = \mathcal{N}^K$

Let $u: \mathcal{B} \to \mathbb{R}$

$u^K \equiv u|_K$ = restriction of u to $K \subset \mathcal{C}$

Let u_I^K be polynomial of order $\leq k$ which interpolates u over K @ $\mathcal{N}^K$

Let $\{N_a^K, a = 1, \ldots, n\}$ = local shape functions

Then $$u_I^K(x) = \sum_{a \in \mathcal{N}^K} u(x_a) N_a^K(x), \quad x \in K$$

The Global FE interpolant of u is a function

$u_I : \mathcal{B} \to \mathbb{R}$ s.t. $u_I|_K = u_I^K$

Global Shape functions: $N_a : \mathcal{B} \to \mathbb{R}$ s.t. $N_a|_K = N_a^K \quad \forall K \subset \mathcal{C}$

$$u_I(x) = \sum_{a \in \mathcal{N}} u(x_a) N_a(x)$$

Properties ...

Properties (by construction):

(i) (FE normalization) $N_a(x_b) = \delta_{ab}, \ \forall a, b \in \mathcal{N}$

(ii) (Piecewise polynomial interpolation)

$\{N_a|_K, \ a \in \mathcal{N}^K\}$ basis for $P_k(K)$

(iii) (C^0-continuous) N_a continuous over B $\forall a \in \mathcal{N}$

(span of $\{N_a\}$)

$\therefore$ Let $U_h \equiv \{ u_I = \sum_{a \in \mathcal{N}} u_a N_a \}_X$ finite dimensional subspace of U

R-R: $\inf_{u \in U_h} I[u]$

$u_h = \{u_a, \ a \in \mathcal{N}\}$

$\inf_{u_h \in \mathbb{R}^{dN}} I_h(u_h)$

$$I_h(u_h) = \int_B W(\epsilon_h)\, dV - f_h^{ext} \cdot u_h$$

$$= \sum_{K \in \mathcal{C}} \int_K W(\epsilon_h)\, dV - f_h^{ext} \cdot u_h$$

$$\frac{\partial I_h}{\partial u_{ia}} = 0 \Rightarrow f_{ia}^{int} = f_{ia}^{ext}$$

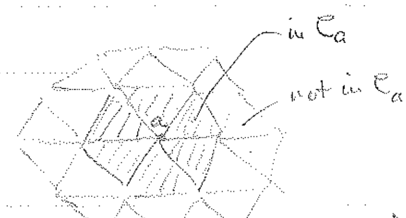
$$f_{ia}^{ext} = \int_B f_i N_a \, dV + \int_{S_2} \bar{t}_i N_a \, dS$$

$$= \sum_{K \in \mathcal{C}} \left(\int_K f_i N_a^K \, dV + \int_{\partial K \cap S_2} \bar{t}_i N_a^K \, dV \right)$$

$$f_{ia}^{int} = \frac{\partial}{\partial u_{ia}} \int_B W(\epsilon_h)\, dV = K_{iakb} u_{kb}$$

$$K_{iakb} = \int_B c_{ijkl} N_{a,j} N_{b,l} \, dV$$

$$= \sum_{K \subset \mathcal{C}} \int_K c_{ijkl} N^K_{a,j} N^K_{b,l} \, dV$$



Let $\mathcal{C}_a = \{ K \subset \mathcal{C} \text{ s.t. } a \in K \}$ (simplices to which a is connected)

Then

$$f^{ext}_{ia} = \sum_{K \subset \mathcal{C}_a} \int_K f_i N^K_a \, dV + \int_{\partial K \cap S_2} t_i N^K_a \, dS$$

and

$$K_{iakb} = \sum_{K \subset \mathcal{C}_a \cap \mathcal{C}_b} \int_K c_{ijkl} N^K_{a,j} N^K_{b,l} \, dV$$

Compute local arrays for each $\underline{K}$

$$f^{\underline{K}}_{ia} = \int_K f_i N_a \, dV + \int_{\partial K \cap S_2} t_i N_a \, dS \quad , \; a \in \underline{K}$$

$$K^{\underline{K}}_{iakb} = \int_{\underline{K}} c_{ijkl} N^{\underline{K}}_{a,j} N^{\underline{K}}_{b,l} \, dV \quad , \; a, b \in \underline{K}$$

Assembly operation: $f^{ext}_{ia} = \sum_{\underline{K} \subset \mathcal{C}_a} f^{\underline{K}}_{ia} \qquad a \in \mathcal{N}$

$$K_{iakb} = \sum_{\underline{K} \subset \mathcal{C}_a \cap \mathcal{C}_b} K^{\underline{K}}_{iakb} \qquad a, b \in \mathcal{N}$$

$f^{\underline{K}}_{ia}$, $K^{\underline{K}}_{iakb}$ involve integrals over $\underline{K}$.

Generic problem: $\int_{\underline{K}} f \, dV$

To compute local array integrals, choose some standard domain & evaluate numerically

$$\left.\begin{array}{l} \text{– isoparametric elements} \\ \text{– numerical quadrature} \end{array}\right\} \longrightarrow \int_{\underline{K}} f \, dV$$