Riks/Wempner Arc-Length Method

$$f(x,\lambda) = f^{int}(x) - \lambda f^{ext} = 0$$

$$ds^2 = dx^2 + d\lambda^2 \psi^2 |f^{ext}|^2$$

Incremental form

$$\Delta l^2 = |\Delta x|^2 + \Delta\lambda^2 \psi^2 |f^{ext}|^2$$

Coupled eqns

$$f(x_{n+1},\lambda_{n+1}) = f^{int}(x_{n+1}) - \lambda_{n+1} f^{ext} = 0$$

$$a(x_{n+1},\lambda_{n+1}) = |x_{n+1}-x_n|^2 + (\lambda_{n+1}-\lambda_n)^2 \psi^2 |f^{ext}|^2 - \Delta l^2 = 0$$

Solve for x_{n+1} λ_{n+1}

$$\lambda^{(k+1)} = \delta\lambda + \Delta\lambda^{(k)} + \lambda_n$$

Linearize

$$f^{int}(x^{(k)}) + K(x^{(k)})(x^{(k+1)} - x^{(k)}) - \lambda^{(k+1)} f^{ext} = 0$$

$$|x^{(k)} - x_n|^2 + 2(x^{(k)} - x_n)\cdot(x^{(k+1)} - x^{(k)}) + 2(\lambda^{(k)} - \lambda_n)(\lambda^{(k+1)} - \lambda^{(k)})\psi^2|f^{ext}|^2 + (\lambda^{(k)} - \lambda_n)^2\psi^2|f^{ext}|^2 - (\Delta l)^2 = 0$$

Define $\Delta x^{(k)} = x^{(k)} - x_n$ $\quad \Delta\lambda^{(k)} = \lambda^{(k)} - \lambda_n$

$\delta x = x^{(k+1)} - x^{(k)}$ $\quad \delta\lambda = \lambda^{(k+1)} - \lambda^{(k)}$

$$f^{int}(x^{(k)}) + K(x^{(k)})\,\delta x - (\delta\lambda + \lambda^{(k)}) f^{ext} = 0$$

$$|\Delta x^{(k)}|^2 + 2\Delta x^{(k)}\cdot\delta x + 2\Delta\lambda^{(k)}\delta\lambda\,\psi^2|f^{ext}|^2 + (\lambda^{k} - \lambda_n)^2\psi^2|f^{ext}|^2 - \Delta l^2 = 0$$

OR

$$K(x^{(k)})\,\delta x - \delta\lambda\, f^{ext} = \lambda^{(k)} f^{ext} - f^{int}(x^{(k)}) = -f(x^{(k)},\lambda^{(k)})$$

$$2\Delta x^{(k)}\cdot\delta x + 2\Delta\lambda^{(k)}\psi^2|f^{ext}|^2\,\delta\lambda = \Delta l^2 - |\Delta x^{(k)}|^2 - (\Delta\lambda^{(k)})^2\psi^2|f^{ext}|^2 = -a(x^{(k)},\lambda^{(k)})$$

$$\begin{bmatrix} K & -f^{ext} \\ 2\Delta x^{(k)T} & 2\Delta\lambda^{(k)}\psi^2|f^{ext}|^2 \end{bmatrix}\begin{bmatrix}\delta x \\ \delta\lambda\end{bmatrix} = \begin{bmatrix} -f^{(k)} \\ -a^{(k)} \end{bmatrix}$$

"Spherical Arc-Length" Method.

$$f = f^{int}(x) - \lambda f^{ext} = 0$$

$$a = |x - x_n|^2 + (\lambda - \lambda_n)^2 \psi^2 |f^{ext}| - (\Delta l)^2 = 0$$

$$f(x_n^{(k)} + \delta x, \lambda_n^{(k)} + \delta\lambda) = f(x^{(k)}, \lambda^{(k)}) + \left.\frac{\partial f}{\partial x}\right|_{x^{(k)}, \lambda^{(k)}} \delta x + \left.\frac{\partial f}{\partial \lambda}\right|_{x^{(k)}, \lambda^{(k)}} \delta\lambda$$

$$= f(x^{(k)}, \lambda^{(k)}) + K(x^{(k)})\delta x \mp f^{ext}\delta\lambda = 0$$

$$\delta x = -K^{-1} f^{(k)} + \delta\lambda K^{-1} f^{ext}$$

$$= \delta\bar{x} + \delta x_t \qquad \delta\bar{x} = -K^{-1} f^{(k)} \qquad \delta x_t = +K^{-1} f^{ext}$$

$$\Delta x^{(k+1)} = \Delta x^{(k)} + \delta x = \Delta x^{(k)} + \delta\bar{x} + \delta x_t \delta\lambda$$

$$|\Delta x^{(k+1)}|^2 + (\Delta\lambda^{(k+1)})^2 \psi^2 |f^{ext}|^2 = \Delta l^2$$

$$\Delta\lambda^{(k+1)} = \Delta\lambda^{(k)} + \delta\lambda$$

$$|\Delta x^{(k)} + \delta\bar{x} + \delta x_t \delta\lambda|^2 + (\Delta\lambda^{(k)} + \delta\lambda)^2 \psi^2 |f^{ext}|^2 = \Delta l^2$$

$$|\Delta x^k + \delta\bar{x}|^2 + 2(\Delta x^k + \delta\bar{x}) \cdot (\delta x_t)\delta\lambda + |\delta x_t|^2 \delta\lambda^2 + \left[(\Delta\lambda^{(k)})^2 + 2\Delta\lambda^{(k)}\delta\lambda + \delta\lambda^2\right]\psi^2 |f^{ext}|^2 = \Delta l^2$$

$$\left[(\Delta x^{(k)} + \delta\bar{x})^2 - \Delta l^2 + (\Delta\lambda^{(k)})^2 \psi^2 |f^{ext}|^2\right] + \left[2(\Delta x^{(k)} + \delta\bar{x}) \cdot \delta x_t + 2\Delta\lambda^{(k)} \psi^2 |f^{ext}|^2\right]\delta\lambda$$
$$+ \left[|\delta x_t|^2 + \psi^2 |f^{ext}|^2\right]\delta\lambda^2 = 0$$

$$a_0 + a_1 \delta\lambda + a_2 \delta\lambda^2 = 0$$

$a_0 = \ldots$

Linearized Arc-Length Method

Linearize $f \longrightarrow \quad f(x^k, \lambda^k) + K(x^k)\,\delta x - f^{ext}\,\delta\lambda = 0$

$$\hookrightarrow \delta x = \delta \bar{x} + \delta x_t\, \delta\lambda \qquad (1)$$

$$\delta\bar{x} = -K^{-1} f^{(k)} \quad \delta x_t = K^{-1} f^{ext}$$

Linearize $a \longrightarrow \quad a(x^k, \lambda^k) + 2\, \Delta x^k \cdot \delta x + 2 \Delta\lambda^k\, \delta\lambda\, \psi^2 / f^{ext\,2} = 0$

$$(1) \longrightarrow \quad a^k + 2\Delta x^k \cdot (\delta\bar{x} + \delta x_t\, \delta\lambda) + 2\Delta\lambda^k \delta\lambda\, \psi^2 / f^{ext\,2} = 0$$

$$\hookrightarrow \left(a^k + 2\Delta x^k \cdot \delta\bar{x}\right) + \left[2\Delta x^k \cdot \delta x_t + 2\Delta\lambda^k \psi^2 / f^{ext\,2}\right]\delta\lambda = 0$$

$$\delta\lambda = -\frac{a^{(k)}/2 + \Delta x^{(k)} \cdot \delta\bar{x}}{\Delta x^{(k)} \cdot \delta x_t + \Delta\lambda^{(k)} \psi^2 / f^{ext\,2}}$$

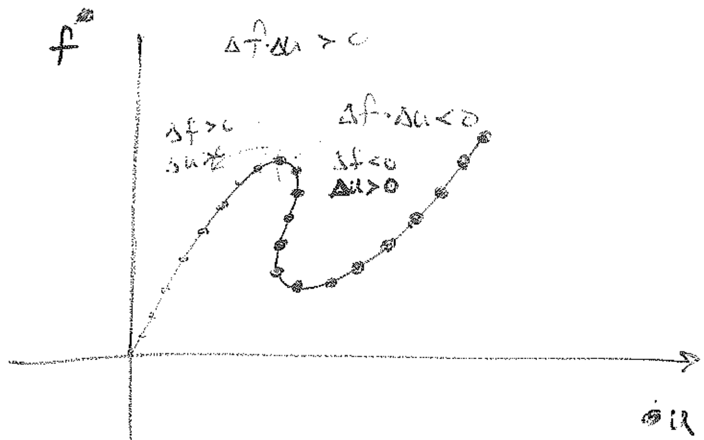
Algorithm

Solve for $\delta\lambda^{(k)}$ (either spherical ~~cubic eq.~~ or linearized) $\longrightarrow \delta x^{(k)} = \delta\bar{x} + \delta x_t\, \delta\lambda^{(k)}$

$\lambda^{(k+1)} = \lambda^{(k)} + \delta\lambda^{(k)}$

$x^{(k+1)} = x^{(k)} + \delta x^{(k)}$

$k \leftarrow k+1$



<u>Predictor step for Spherical Arc-Length Method</u>

Recall, linearizing $f \rightarrow$

$$\delta x = \delta \bar{x} + \delta x_t \, \delta \lambda$$

$$\delta \bar{x} = -K^{-1} f^{(0)}_{n+1} \qquad \delta x_t = K^{-1} f^{ext}$$

~~Constraint~~
At beginning of an increment $f^{(0)}_{n+1} = f_n = 0$ (sol'n of last increment)

$\Rightarrow$ Predictor step is

$$\Delta x_p = \delta x_t \, \Delta \lambda_p$$

predictor load increment

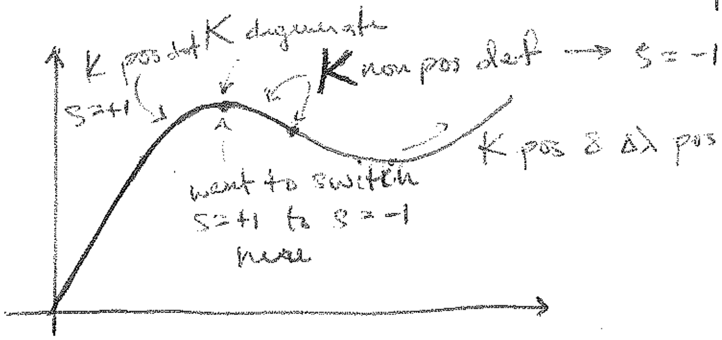
Constraint

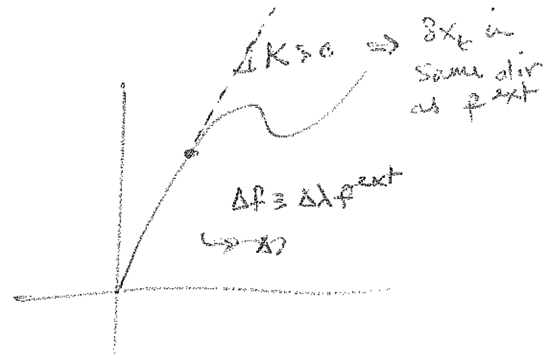
$$a = |\Delta x_p|^2 + \Delta \lambda_p^2 \, \psi^2 |f^{ext}|^2 - \Delta l^2 = 0$$

$$= \left(|\delta x_t|^2 + \psi^2 |f^{ext}|^2 \right) (\Delta \lambda)^2 - \Delta l^2$$

$$\hookrightarrow \quad \Delta \lambda = \pm \frac{\Delta l}{\left(|\delta x_t|^2 + \psi^2 |f^{ext}|^2 \right)^{1/2}} = s \frac{\Delta l}{\left(|\delta x_t|^2 + \psi^2 |f^{ext}|^2 \right)^{1/2}} \qquad s = \pm 1$$

Which sign do we choose? Depends



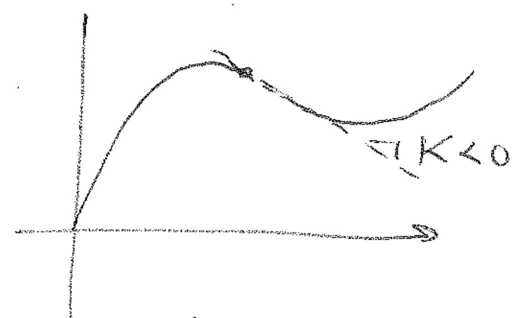


$$\Delta f_p = \Delta\lambda_p f^{ext} \qquad \Delta x_p = K^{-1} f^{ext} \Delta\lambda_p$$
$$= K^{-1} \Delta f_p$$
$$\text{or } \Delta f_p = K \Delta x_p$$

$$\therefore \Delta W_p = \Delta x_p \cdot \Delta f_p = \Delta x_p K \Delta x_p = \Delta f_p \cdot K^{-1} \Delta f_p$$

if K pos def $\Delta W_p > 0$

if K not " " ΔW_p maybe < 0



Alternative:
Switch s when ΔW_p switches