MAE 261B – Computational Mechanics of Solids and Structures

Lecture 17: Shell Theory

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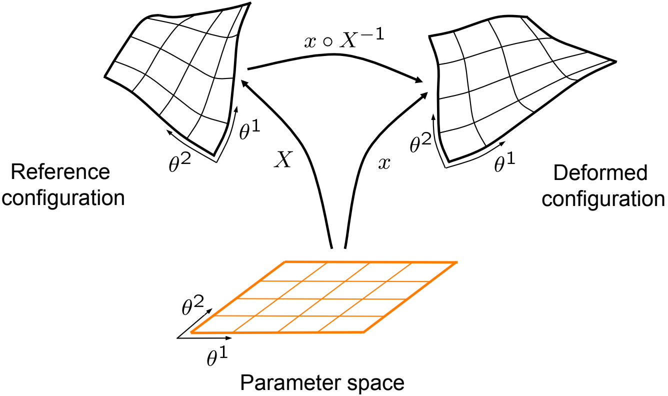
Shell Theory

Shell Kinematics

Explicitly track the mid surface of the deforming shell:

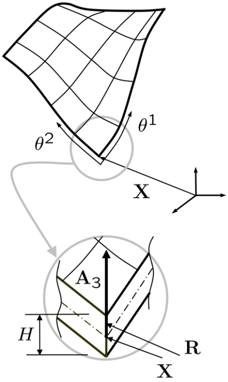
$$\boldsymbol{X}(\theta^1, \theta^2) \equiv \textit{Reference} \text{ position of mid surface}$$

$$\boldsymbol{x}(\theta^1, \theta^2) \equiv \textit{Deformed} \text{ position of mid surface}$$



Reference Configuration

Material Points are located by distance from the mid surface along the reference *director* $\boldsymbol{A}_3$.



- Reference position:

$$\boldsymbol{R}(\theta^1, \theta^2, \theta^3) \equiv \boldsymbol{X}(\theta^1, \theta^2) + \theta^3 \boldsymbol{A}_3(\theta^1, \theta^2)$$

- Thickness coordinate:

$$\theta^3 \in [-H/2, H/2]$$

- Surface basis vectors:

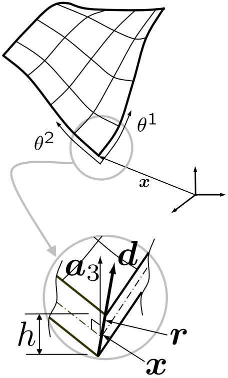
$$\boldsymbol{A}_\alpha \equiv \boldsymbol{X}_{,\alpha} \equiv \partial \boldsymbol{X} / \partial \theta^\alpha$$

- Reference Director:

$$\boldsymbol{A}_3 \equiv \frac{\boldsymbol{A}_1 \times \boldsymbol{A}_2}{|\boldsymbol{A}_1 \times \boldsymbol{A}_2|}$$

Deformed Configuration

Classical Shell theory assumption: Deformed *director* $\boldsymbol{d}$ is inextensible and simply *rotates* during defomation. Generally it is *not* equal to the deformed surface normal.



- Deformed position:

$$\boldsymbol{r}(\theta^1, \theta^2, \theta^3) \equiv \boldsymbol{x}(\theta^1, \theta^2) + \lambda(\theta^1, \theta^2)\theta^3 \boldsymbol{d}(\theta^1, \theta^2)$$

- Inextensible director: $|\boldsymbol{d}(\theta^1, \theta^2)| = 1$
  We say the director is on the **unit sphere**, denoted $\boldsymbol{d} \in \mathrm{S}^2$.
- Scalar thickness stretch: $\lambda(\theta^1, \theta^2)$
- Surface basis vectors:

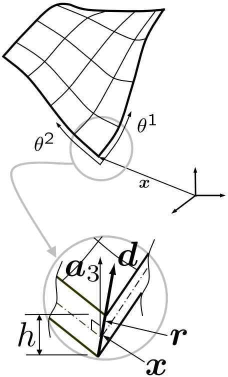
$$\boldsymbol{a}_\alpha \equiv \boldsymbol{x}_{,\alpha}, \quad \boldsymbol{a}_3 \equiv \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}$$

Notes:

- The more general case including thickness and transverse shear deformation is a sort of extension of Reissner-Mindlin of plate theory.
- Common approximation: neglect spatial variation of thickness stretch, determine λ by condensation.

Kirchhoff-Love (K-L) Theory

Two additional “simplifying” assumptions:



1. Neglect thickness stretch:

$$\lambda = 1 \quad \Rightarrow \quad \boldsymbol{r} = \boldsymbol{x} + \theta^3 \boldsymbol{d}$$

2. No shear deformation; director remains normal to deformed surface:

$$\boldsymbol{d}(\theta^1, \theta^2) = \boldsymbol{a}_3 \equiv \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}$$

$$\Rightarrow \quad \boldsymbol{r}(\theta^1, \theta^2, \theta^3) = \boldsymbol{x}(\theta^1, \theta^2) + \theta^3 \boldsymbol{a}_3(\theta^1, \theta^2).$$

Therefore, in K-L theory, the deformation of the shell is completely determined by the deformation of the mid surface.

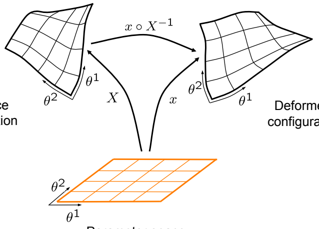
Deformation

Position mappings

$$\boldsymbol{R}(\theta^1, \theta^2, \theta^3) \equiv \boldsymbol{X}(\theta^1, \theta^2) + \theta^3 \boldsymbol{A}_3(\theta^1, \theta^2)$$
$$\boldsymbol{r}(\theta^1, \theta^2, \theta^3) \equiv \boldsymbol{x}(\theta^1, \theta^2) + \lambda(\theta^1, \theta^2)\theta^3 \boldsymbol{d}(\theta^1, \theta^2)$$

Basis vectors:

$$\boldsymbol{G}_\alpha \equiv \boldsymbol{R}_{,\alpha} = \boldsymbol{A}_\alpha + \theta^3 \boldsymbol{A}_{3,\alpha}$$
$$\boldsymbol{G}_3 \equiv \boldsymbol{R}_{,3} = \boldsymbol{A}_3$$
$$\boldsymbol{g}_\alpha \equiv \boldsymbol{r}_{,\alpha} = \boldsymbol{a}_\alpha + \theta^3 (\lambda \boldsymbol{d})_{,\alpha}$$
$$\boldsymbol{g}_3 \equiv \boldsymbol{r}_{,3} = \lambda \boldsymbol{d}$$



Dual Basis vectors

$$\boldsymbol{G}^i \cdot \boldsymbol{G}_j = \delta^i_j \quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$$

Deformation gradient:

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{g}_i \otimes \boldsymbol{G}^i \\ &= [\boldsymbol{a}_\alpha + \theta^3 (\lambda \boldsymbol{d})_{,\alpha}] \otimes \boldsymbol{G}^\alpha + \lambda \boldsymbol{d} \otimes \boldsymbol{A}_3 \\ &= [\boldsymbol{a}_\alpha \otimes \boldsymbol{G}^\alpha + \lambda \boldsymbol{d} \otimes \boldsymbol{A}_3] + \theta^3 [(\lambda \boldsymbol{d})_{,\alpha} \otimes \boldsymbol{G}^\alpha] \end{aligned}$$

Virtual Work

- Internal Virtual Work:

$$\delta\Pi^{\text{int}} = \int_{\mathcal{B}_0} \boldsymbol{P} : \delta\boldsymbol{F} dV_0$$

$\boldsymbol{P} =$ 1st Piola-Kirchhoff Stress Reference volume element

$$dV_0 = (\boldsymbol{G}_1 \times \boldsymbol{G}_2) \cdot \boldsymbol{G}_3 d\theta^1 d\theta^2 d\theta^3 \equiv \sqrt{G} d^3\theta.$$

(See Wempner and Talaslidis (2003), Ch 2)

- Split the volume element into thickness and surface elements

$$dV_0 = \mu d\theta^3 \sqrt{A} d^2\theta$$

where $\mu \equiv \frac{\sqrt{G}}{\sqrt{A}}$, and

$$\sqrt{A} d^2\theta \equiv (\boldsymbol{A}_1 \times \boldsymbol{A}_2) \cdot \boldsymbol{A}_3 d\theta^1 d\theta^2$$

Virtual Work

Recall the deformation gradient

$$\boldsymbol{F} = [\boldsymbol{a}_\alpha \otimes \boldsymbol{G}^\alpha + \lambda \boldsymbol{d} \otimes \boldsymbol{A}_3] + \theta^3[(\lambda \boldsymbol{d})_{,\alpha} \otimes \boldsymbol{G}^\alpha]$$

It's variation is then

$$\delta \boldsymbol{F} = \delta \boldsymbol{a}_\alpha \otimes \boldsymbol{G}^\alpha \delta(\lambda \boldsymbol{d}) \otimes \boldsymbol{G}^3 + \theta^3[\delta(\lambda \boldsymbol{d})_{,\alpha} \otimes \boldsymbol{G}^\alpha]$$

Such that

$$\boldsymbol{P} : \delta \boldsymbol{F} = \delta \boldsymbol{a}_\alpha \cdot (\boldsymbol{P}\boldsymbol{G}^\alpha) + \delta(\lambda \boldsymbol{d}) \cdot (\boldsymbol{P}\boldsymbol{G}^3) + \theta^3[\delta(\lambda \boldsymbol{d})_{,\alpha} \cdot (\boldsymbol{P}\boldsymbol{G}^\alpha)]$$

And

$$\delta \Pi^{\text{int}} = \int_{\Sigma_0} \int_{-H/2}^{H/2} \{\delta \boldsymbol{a}_\alpha \cdot (\boldsymbol{P}\boldsymbol{G}^\alpha) + \delta(\lambda \boldsymbol{d}) \cdot (\boldsymbol{P}\boldsymbol{G}^3) + \theta^3[\delta(\lambda \boldsymbol{d})_{,\alpha} \cdot (\boldsymbol{P}\boldsymbol{G}^\alpha)]\} \mu d\theta^3 \sqrt{A} d^2\theta$$

Resultants

$$\delta\Pi^{\text{int}} = \int_{\Sigma_0}\int_{-H/2}^{H/2}\{\delta\boldsymbol{a}_\alpha\cdot(\boldsymbol{P}\boldsymbol{G}^\alpha) + \delta(\lambda\boldsymbol{d})\cdot(\boldsymbol{P}\boldsymbol{G}^3) + \theta^3[\delta(\lambda\boldsymbol{d})_{,\alpha}\cdot(\boldsymbol{P}\boldsymbol{G}^\alpha)]\}\mu d\theta^3\sqrt{A}d^2\theta$$

Define

$$\boldsymbol{n}^i \equiv \int_{-H/2}^{H/2}\boldsymbol{P}\boldsymbol{G}^i\mu d\theta^3 \quad = \quad \text{Stress Resultants}$$

$$\boldsymbol{m}^\alpha \equiv \int_{-H/2}^{H/2}\theta^3\boldsymbol{P}\boldsymbol{G}^\alpha\mu d\theta^3 \quad = \quad \text{Moment Resultants}$$

Such that,

$$\boxed{\delta\Pi^{\text{int}} = \int_{\Sigma_0}[\boldsymbol{n}^\alpha\cdot\delta\boldsymbol{a}_\alpha + \boldsymbol{n}^3\cdot\delta(\lambda\boldsymbol{d}) + \boldsymbol{m}^\alpha\cdot\delta(\lambda\boldsymbol{d})_{,\alpha}]\sqrt{A}d^2\theta}$$

External Loads

Assume existence of distributed loads and moments

$$\bar{\boldsymbol{n}} = \text{force per unit reference } \textit{area} \text{ on } \Sigma_0$$

$$\bar{\boldsymbol{m}} = \text{moment per unit reference } \textit{area} \text{ on } \Sigma_0$$

along with edge loads and moments

$$\bar{\bar{\boldsymbol{n}}} = \text{force per unit reference } \textit{length} \text{ on } \partial\Sigma_0$$

$$\bar{\bar{\boldsymbol{m}}} = \text{moment per unit reference } \textit{length} \text{ on } \partial\Sigma_0$$

External virtual work is then

$$\delta\Pi^{\text{ext}} = \int_{\Sigma_0} [\bar{\boldsymbol{n}} \cdot \delta\boldsymbol{x} + \bar{\boldsymbol{m}} \cdot \delta\boldsymbol{d}]\sqrt{A}d^2\theta + \int_{\partial\Sigma_0} [\bar{\bar{\boldsymbol{n}}} \cdot \delta\boldsymbol{x} + \bar{\bar{\boldsymbol{m}}} \cdot \delta\boldsymbol{d}]dS.$$

Variation of Shell Director

Recall, $\boldsymbol{d} \in S^2 \Rightarrow |\boldsymbol{d}| = 1$. Thus,

$$\delta(\boldsymbol{d} \cdot \boldsymbol{d}) = 2\boldsymbol{d} \cdot \delta\boldsymbol{d} = 0.$$

Meaning that $\delta\boldsymbol{d}$ must be normal to $\boldsymbol{d}$, i.e., it's in the director's tangent space $\delta\boldsymbol{d} \in T_{\boldsymbol{d}}S^2$.

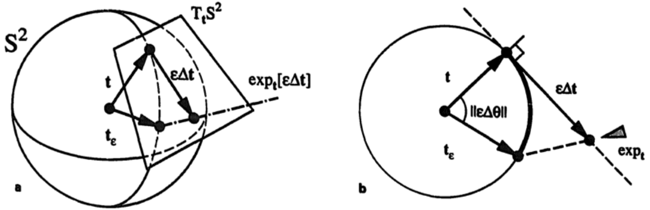


Fig. 2.2.(a) The unique geodesic starting at $t \in S^2$ and tangent to $\Delta t \in T_t S^2$. This curve (an arc of great circle) is the image of the straight line $t + \varepsilon\Delta t$. (b) Illustration of (2.7) for the exponential map.

Simo et al. (1990)

Exponential Map on The Unit Sphere

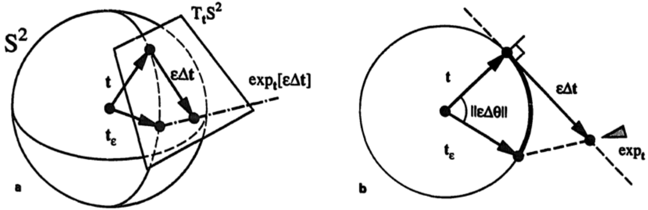


Fig. 2.2.(a) The unique geodesic starting at $t \in S^2$ and tangent to $\Delta t \in T_t S^2$. This curve (an arc of great circle) is the image of the straight line $t + \varepsilon \Delta t$. (b) Illustration of (2.7) for the exponential map.

- Identify $\delta \boldsymbol{d} \in S^2$ as an *infinitesimal director displacement*. Note, it has the effect of taking the director off the unit sphere.
- Define the **exponential mapping** in S^2 as mapping the straight line tangent to $\delta \boldsymbol{d}$ onto the geodesic passing through $\boldsymbol{d}$ tangent to $\delta \boldsymbol{d}$

$$\boldsymbol{d}_\epsilon = \exp[\epsilon \delta \boldsymbol{d}] \equiv \cos(\epsilon |\delta \boldsymbol{d}|) \boldsymbol{d} + \frac{\sin(\epsilon |\delta \boldsymbol{d}|)}{|\delta \boldsymbol{d}|} \delta \boldsymbol{d}$$

- $\exp[\epsilon \delta \boldsymbol{d}]$ maps the displacement back on to the unit sphere.

Director Rotation

Connect S^2 to SO(3),

$$\boldsymbol{d}_\epsilon = \exp_{\boldsymbol{d}}[\epsilon \delta \boldsymbol{d}] = \exp[\epsilon \widehat{\delta \boldsymbol{\theta}}] \boldsymbol{d}$$

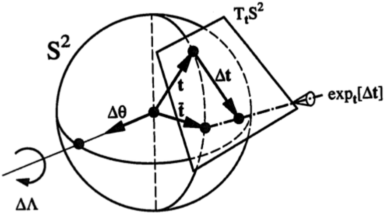
Then

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \boldsymbol{d}_\epsilon = \widehat{\delta \boldsymbol{\theta}} \boldsymbol{d} = \delta \boldsymbol{\theta} \times \boldsymbol{d}$$

$$\Rightarrow \quad \boxed{\delta \boldsymbol{d} = \delta \boldsymbol{\theta} \times \boldsymbol{d}}.$$

Or,

$$\delta \boldsymbol{\theta} = \boldsymbol{d} \times \delta \boldsymbol{d}.$$



Simo et al. (1990)

Parameterizing Director Rotation

Choose some reference orientation $\boldsymbol{E} \in \mathbb{R}^3$, and define rotation $\boldsymbol{\Lambda}$ $\in$SO(3) such that

$$\boldsymbol{d} = \boldsymbol{\Lambda}\boldsymbol{E}$$

Then *pull back* $\delta\boldsymbol{\theta}$,

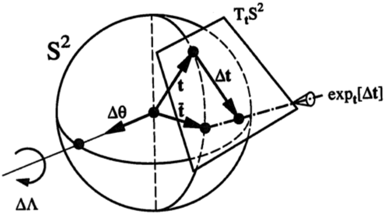
$$\delta\boldsymbol{\Theta} \equiv \boldsymbol{\Lambda}^T \delta\boldsymbol{\theta}$$

Such that,

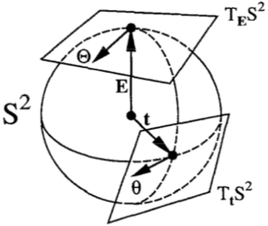
$$\delta\boldsymbol{d} = \boldsymbol{\Lambda}\delta\boldsymbol{\theta} \times \boldsymbol{\Lambda}\boldsymbol{E}.$$

or,

$$\delta\boldsymbol{d} = \boldsymbol{\Lambda}\delta\boldsymbol{D}, \quad \delta\boldsymbol{D} \equiv \delta\boldsymbol{\Theta} \times \boldsymbol{E}.$$



Simo et al. (1990)



Simo and Fox (1989)

References

G. Wempner and D. Talaslidis. *Mechanics of Solids and Shells, Theory and Approxmation*. CRC Press, 2003.

J.C. Simo, D.D. Fox, and M.S. Rifai. On a stress restultant geometrically exact shell model. Part III: Computational aspects of the nonlinear theory. *CMAME*, 79:21–70, 1990.

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