Lecture 8

FEA for Solids in 2-D & 3-D

William S. Klug

Department of Mechanical and Aerospace Engineering

UCLA

Introduction to Finite Element Methods

MAE M168/CEE 135C

Section 8.1

Elasticity BVP – Strong and Weak Forms

Equilibrium BVP – The Strong Form

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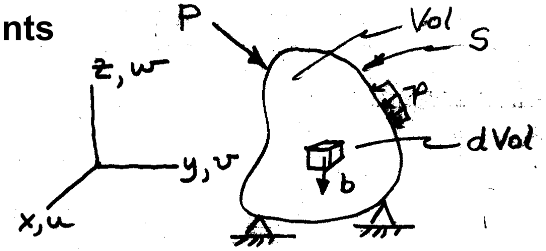
Displacements



$$\mathbf{u}(x,y)=\begin{Bmatrix} u \\ v \end{Bmatrix} \quad or \quad \mathbf{u}(x,y,z)=\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

2-D 3-D

Strains (2-D)

$$\varepsilon=\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}=\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix}=\underbrace{\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}}_{\partial}\underbrace{\begin{Bmatrix} u \\ v \end{Bmatrix}}_{\mathbf{u}}=\partial\mathbf{u}$$

$$\varepsilon=\partial\mathbf{u}$$

Differential operator

Hooke's Law (2-D)

$$\sigma=\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}=\underbrace{\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & E_{11} \end{bmatrix}}_{\mathbf{E}}\underbrace{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}}_{\varepsilon}=\mathbf{E}\varepsilon$$

Isotropic:

Plane Stress

$$\sigma_z=\tau_{xz}=\tau_{yz}=0$$

$$\mathbf{E}=\frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix}$$

Plane Strain

$$\varepsilon_z=\gamma_{xz}=\gamma_{yz}=0$$

$$\mathbf{E}=\frac{E}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix} 1-\upsilon & \upsilon & 0 \\ \upsilon & 1-\upsilon & 0 \\ 0 & 0 & 1/2-\upsilon \end{bmatrix}$$

Equilibrium (2-D)

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+b_x=0$$

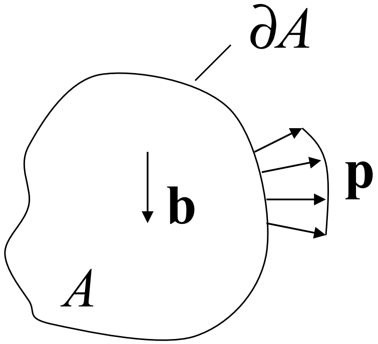
$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+b_y=0$$

$$\mathbf{b}=\begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$$

Body Force

$$\underbrace{\begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}}_{\partial^T}\underbrace{\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}}_{\sigma}+\underbrace{\begin{Bmatrix} b_x \\ b_y \end{Bmatrix}}_{\mathbf{b}}=0 \qquad \partial^T\sigma+\mathbf{b}=0$$

Weak Form



Strong Form: $\varepsilon = \partial \mathbf{u} \quad \sigma = \mathbf{E}\varepsilon \quad \partial^T(\sigma) + \mathbf{b} = \mathbf{0}$

$$\partial^T(\mathbf{E}\partial\mathbf{u}) + \mathbf{b} = \mathbf{0} \quad \text{+ BCs}$$

Weighted Residual: $\int_A \mathbf{w}^T \left[\partial^T(\mathbf{E}\partial\mathbf{u}) + \mathbf{b}\right] dA = 0$

Weight function (virtual displ.)

(Integrate by parts, insert NBCs, …)

Weak Form (virtual work):

$$0 = \int_A \left[-(\partial\mathbf{w})^T \mathbf{E}\,\partial\mathbf{u} + \mathbf{b}\right] dA + \int_{\partial A} \left[\mathbf{w}^T\mathbf{p}\right] dl$$

Admissibility Conditions:

1) Strain energy should be finite (square integrable)
   - ➢ No jumps in displacement (continuous, no fractures)
2) Enforce Displacement Boundary Conditions

Section 8.2

2-D Interpolation: Basic Elements

FE Interpolation of Displacements

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$$u(x,y) = c_{00} + c_{10}x + c_{01}y + c_{11}xy + c_{20}x^2 + \ldots$$

- Minimal polynomial for finite strains: <u>linear</u>

- 2-D planar surface fits $u(x, y)$
  - Need at least 1st 3 – terms

$$u(x,y) \to c_{00} + c_{10}x + c_{01}y$$

- In 3-D $$u(x,y,z) \to c_{000} + c_{100}x + c_{010}y + c_{001}z$$

- This is minimal requirement for convergence.

Minimal Elements in 2-D & 3-D

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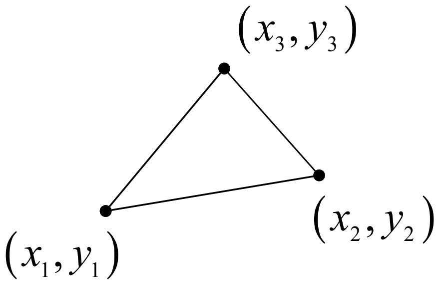
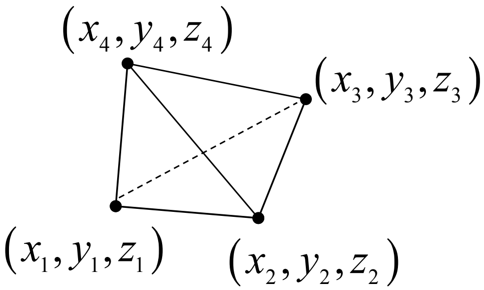
2-D: Triangle



$$u(x,y) = u_1 N_1(x,y) + u_2 N_2(x,y) + u_3 N_3(x,y)$$

3-D: Tetrahedron



$$u(x,y,z) = u_1 N_1 + u_2 N_2 + u_3 N_3 + u_4 N_4$$

Simplest Element in 2-D: Triangle

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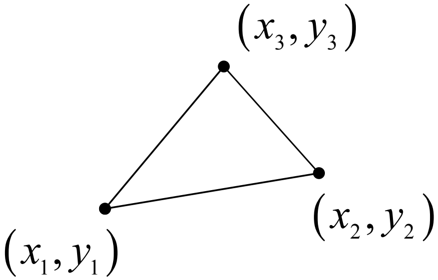
Interpolate a function $T(x, y)$ among nodal values



$$T(x,y) = a + bx + cy = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$T_1 = T(x_1, y_1)$$
$$T_2 = T(x_2, y_2)$$
$$T_3 = T(x_3, y_3)$$

$$\begin{aligned} T_1 &= a + bx_1 + cy_1 \\ T_2 &= a + bx_2 + cy_2 \\ T_3 &= a + bx_3 + cy_3 \end{aligned} \Rightarrow \underbrace{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}}_{\mathbf{T}^e} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{Bmatrix} a \\ b \\ c \end{Bmatrix}}_{\alpha}$$

$$\alpha = \mathbf{C}^{-1}\mathbf{T}^e \Rightarrow T(x,y) = \underbrace{\begin{bmatrix} 1 & x & y \end{bmatrix} \mathbf{C}^{-1}}_{\mathbf{N}(x,y)} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

Tri3 Shape Functions

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$$\mathbf{N}(x,y)=\begin{bmatrix}1 & x & y\end{bmatrix}\mathbf{C}^{-1}=\begin{bmatrix}N_1(x,y) & N_2(x,y) & N_3(x,y)\end{bmatrix}$$

$$N_1(x,y)=\frac{1}{2A}\left[(x_2y_3-x_3y_2)+(y_2-y_3)x+(x_3-x_2)y\right]$$

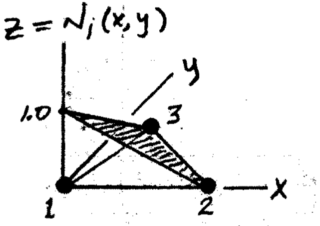
$$N_2(x,y)=\frac{1}{2A}\left[(x_3y_1-x_1y_3)+(y_3-y_1)x+(x_1-x_3)y\right]$$

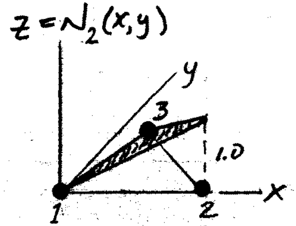
$$N_3(x,y)=\frac{1}{2A}\left[(x_1y_2-x_2y_1)+(y_1-y_2)x+(x_2-x_1)y\right]$$

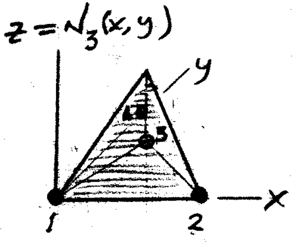
$$A=\frac{1}{2}\det\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}=\text{ area of triangle}$$

Kronecker Delta: $$N_i(x_j,y_j)=\delta_{ij}=\begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j\end{cases}$$

Partition of Unity: $$N_1(x,y)+N_2(x,y)+N_3(x,y)=1$$







4-Node Bilinear Rectangular Element (Quad4)

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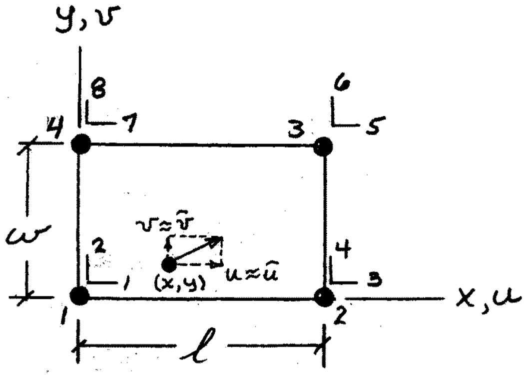
- Interpolate x- and y- displacement components

$$\hat{u}(x,y) \text{ and } \hat{v}(x,y)$$

- Choose bilinear polynomial functions

$$u = \hat{u} = (a+bx)(c+dy)$$
$$= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$



$$u_1, u_2, u_3, u_4$$
$$\text{and}$$
$$v_1, v_2, v_3, v_4$$

Why Bilinear?

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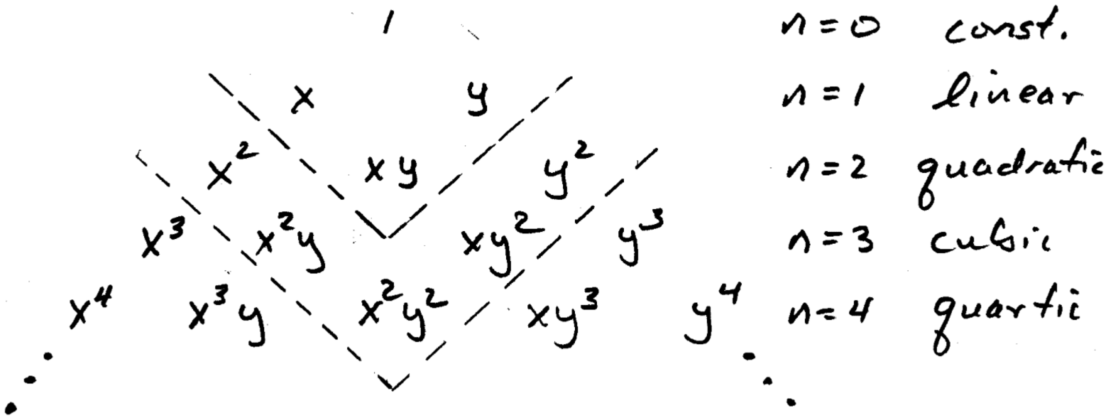
$$\widehat{u} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$\widehat{v} = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$$

- Why not use quadratic terms?

$$\widehat{u} = \alpha_1 + \alpha_2 x + \alpha_3 y + \boxed{\alpha_4 x^2}$$
$$\widehat{v} = \alpha_5 + \alpha_6 x + \alpha_7 y + \boxed{\alpha_8 y^2}$$

Pascal's Triangle

$$\hat{u}(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$



- Linear combination of all monomials above any horizontal line constitutes polynomial of order n.
- The bilinear term $\alpha_4 xy$ is *surplus*.

Q4 Shape Functions

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$$u = \widehat{u} = (a+bx)(c+dy)$$
$$= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$\widehat{u}(x,y) = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

- Four interpolation conditions at nodes 1-4

$$\widehat{u}(x_1,y_1) = u_1 \Rightarrow \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1 = u_1$$
$$\widehat{u}(x_2,y_2) = u_2 \Rightarrow \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_2 y_2 = u_2$$
$$\widehat{u}(x_3,y_3) = u_3 \Rightarrow \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_3 y_3 = u_3$$
$$\widehat{u}(x_4,y_4) = u_4 \Rightarrow \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4 = u_4$$

$$\underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}}_{\mathbf{d}} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}}_{\alpha}$$

$$\alpha = \mathbf{C}^{-1}\mathbf{d} \Rightarrow \widehat{u}(x,y) = \underbrace{\begin{bmatrix} 1 & x & y & xy \end{bmatrix} \mathbf{C}^{-1}}_{\mathbf{N}(x,y)} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\mathbf{N}(x,y) = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \mathbf{C}^{-1} = \begin{bmatrix} N_1(x,y) & N_2(x,y) & N_3(x,y) & N_4(x,y) \end{bmatrix}$$

Q4 Shape Functions

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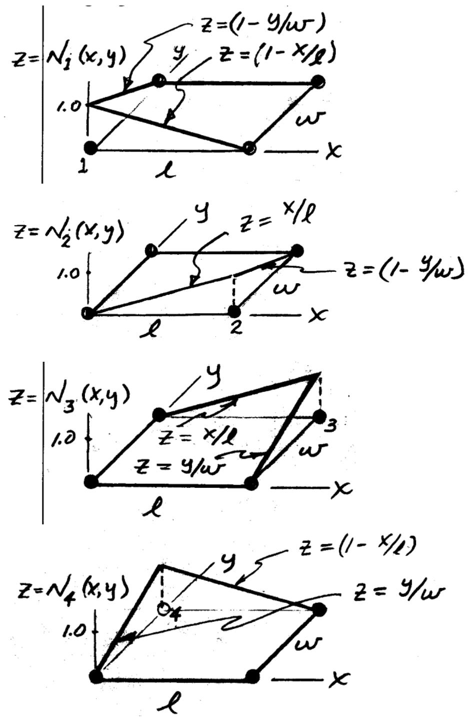
$$\left.\begin{aligned}
N_1(x,y) &= (1-x/l)(1-y/w)\\
N_2(x,y) &= (x/l)(1-y/w)\\
N_3(x,y) &= (x/l)(y/w)\\
N_4(x,y) &= (1-x/l)(y/w)
\end{aligned}\right\}\quad \begin{array}{c}(0\le x\le l)\\ (0\le y\le w)\end{array}$$



Kronecker Delta:

$$N_i(x,y)=\begin{cases}1 & \text{at node } i\\ 0 & \text{at all other nodes}\end{cases}$$

Partition of Unity:

$$N_1(x,y)+N_2(x,y)+N_3(x,y)+N_4(x,y)=1$$

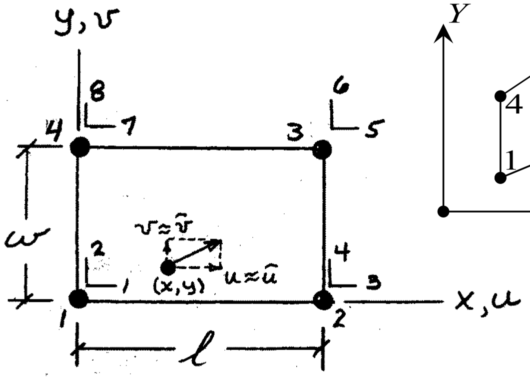
Note: Shape functions are *not flat*. Quadratic along diagonal.

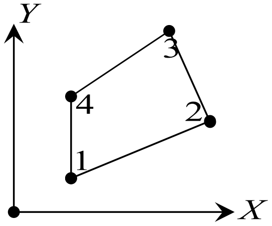
Section 8.3

2-D Isoparametric Elements

Motivation

- Deal with distorted shapes.





- Compute integrals over standard domain.

$$0=\int_A\left[-\left(\partial\mathbf{w}\right)^T\mathbf{E}\partial\mathbf{u}+\mathbf{b}\right]dA+\int_{\partial A}\left[\mathbf{w}^T\mathbf{p}\right]dl$$

Isoparametric Mapping in 1-D

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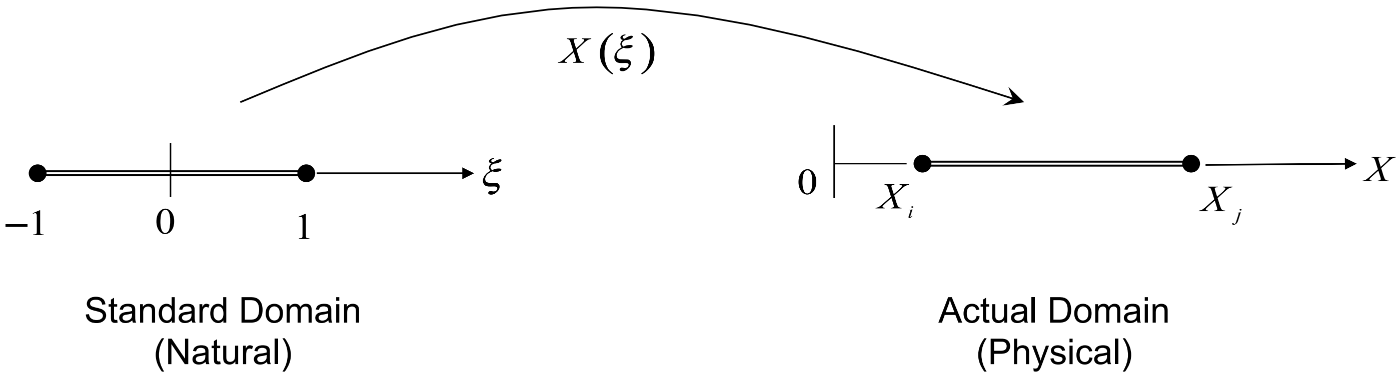

$$X(\xi) = N_1(\xi)X_1 + N_2(\xi)X_2 = \mathbf{N}(\xi)\mathbf{X}$$

Transformation *interpolates position* between nodal positions in actual physical domain

Isoparametric Mapping in 2-D

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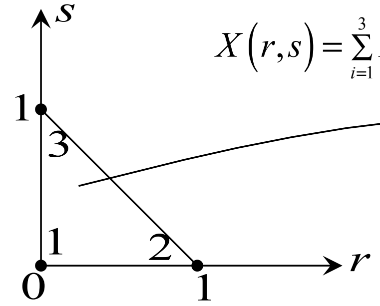
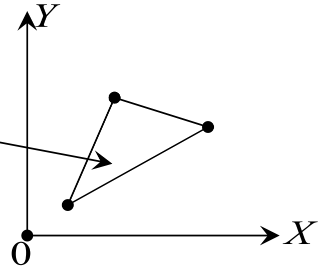
$$u(r,s)=\sum_{i=1}^{3}N_i(r,s)u_i \qquad v(r,s)=\sum_{i=1}^{3}N_i(r,s)v_i$$

$$X(r,s)=\sum_{i=1}^{3}N_i(r,s)X_i \qquad Y(r,s)=\sum_{i=1}^{3}N_i(r,s)Y_i$$



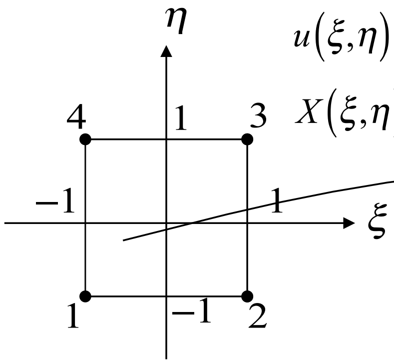
Standard Domain
(Natural)

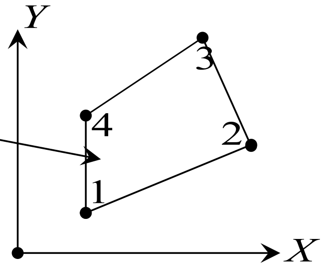


Actual Domain
(Physical)

$$u(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)u_i \qquad v(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)v_i$$

$$X(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)X_i \qquad Y(\xi,\eta)=\sum_{i=1}^{4}N_i(\xi,\eta)Y_i$$





The Isoparametric Recipe

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- Define shape functions in standard domain.
- Create mapping between standard and physical coordinate systems by *interpolation with shape functions*.
- Build shape functions and do integrals in *natural coordinates*.

Section 8.4

3-Node Linear Triangle

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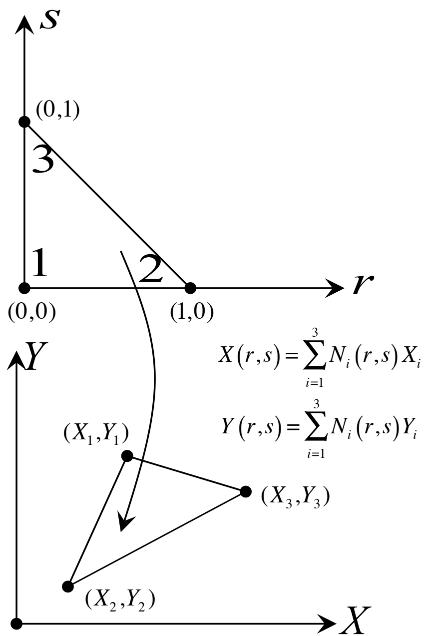
- Building shape functions in (r ,s)
  - Linear in r & s
  -

$$N_i\left(r_j,s_j\right)=\delta_{ij}$$

$$N_1\left(r,s\right)=1-r-s$$

$$N_2\left(r,s\right)=r$$

$$N_3\left(r,s\right)=s$$



Isoparametric Element
(Field & Position use same shape functions)

$$X(r,s)=\sum_{i=1}^{3}N_i(r,s)X_i$$

$$T(r,s)=\sum_{i=1}^{3}N_i(r,s)T_i$$

$$Y(r,s)=\sum_{i=1}^{3}N_i(r,s)Y_i$$

$$T(r,s)=\underbrace{\begin{bmatrix} N_1(r,s) & N_2(r,s) & N_3(r,s)\end{bmatrix}}_{\mathbf{N}(r,s)}\underbrace{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}}_{\mathbf{T}}=\mathbf{N}(r,s)\mathbf{T}$$

$$\begin{Bmatrix} X(r,s) \\ Y(r,s) \end{Bmatrix}=\begin{bmatrix} N_1(r,s) & 0 & N_2(r,s) & 0 & N_3(r,s) & 0 \\ 0 & N_1(r,s) & 0 & N_2(r,s) & 0 & N_3(r,s) \end{bmatrix}\begin{Bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \end{Bmatrix}$$

Computing Gradients

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$$T(r,s) = \underbrace{\begin{bmatrix} N_1(r,s) & N_2(r,s) & N_3(r,s) \end{bmatrix}}_{\mathbf{N}(r,s)} \underbrace{\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}}_{\mathbf{T}} = \mathbf{N}(r,s)\mathbf{T}$$

$$\nabla T = \nabla \mathbf{N}(r,s)\mathbf{T} = \begin{bmatrix} N_{1,X} & N_{2,X} & N_{3,X} \\ N_{1,Y} & N_{2,Y} & N_{3,Y} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} \qquad (\bullet)_{,X} \equiv \frac{\partial(\bullet)}{\partial X} \qquad (\bullet)_{,Y} \equiv \frac{\partial(\bullet)}{\partial Y}$$

- Need derivatives with respect to physical coordinates (X,Y)
- N_i's are functions of natural coordinates (r,s).
  - Use Chain rule

Transforming Derivatives

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$$\left.\begin{aligned}\frac{\partial N_i}{\partial r} &= \frac{\partial N_i}{\partial X}\frac{\partial X}{\partial r} + \frac{\partial N_i}{\partial Y}\frac{\partial Y}{\partial r} \\ \frac{\partial N_i}{\partial s} &= \frac{\partial N_i}{\partial X}\frac{\partial X}{\partial s} + \frac{\partial N_i}{\partial Y}\frac{\partial Y}{\partial s}\end{aligned}\right\} \quad \begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial X}{\partial r} & \dfrac{\partial Y}{\partial r} \\ \dfrac{\partial X}{\partial s} & \dfrac{\partial Y}{\partial s} \end{bmatrix}}_{\mathbf{J}\text{ = Jacobian matrix}} \begin{Bmatrix} \dfrac{\partial N_i}{\partial X} \\ \dfrac{\partial N_i}{\partial Y} \end{Bmatrix}$$

$$\frac{\partial X}{\partial r} = \frac{\partial}{\partial r}\left(\sum_i X_i N_i(r,s)\right) = \sum_i X_i \frac{\partial N_i}{\partial r} \qquad \frac{\partial Y}{\partial r} = \sum_i Y_i \frac{\partial N_i}{\partial r}$$

$$\frac{\partial X}{\partial s} = \sum_i X_i \frac{\partial N_i}{\partial s} \qquad \frac{\partial Y}{\partial s} = \sum_i Y_i \frac{\partial N_i}{\partial s}$$

$$N_1(r,s) = 1 - r - s \rightarrow \frac{\partial N_1}{\partial r} = -1 \quad \frac{\partial N_1}{\partial s} = -1$$

$$N_2(r,s) = r \rightarrow \frac{\partial N_2}{\partial r} = 1 \quad \frac{\partial N_2}{\partial s} = 0$$

$$N_3(r,s) = s \rightarrow \frac{\partial N_3}{\partial r} = 0 \quad \frac{\partial N_3}{\partial s} = 1$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial X} \\ \dfrac{\partial N_i}{\partial Y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{Bmatrix}$$

Exercise: Compute $\mathbf{J}$ and shp. fun. gradients. Then compare to (x,y) formulation.

Section 8.5

4-Node Bi-linear Quadrilateral

4-Node Quadrilateral

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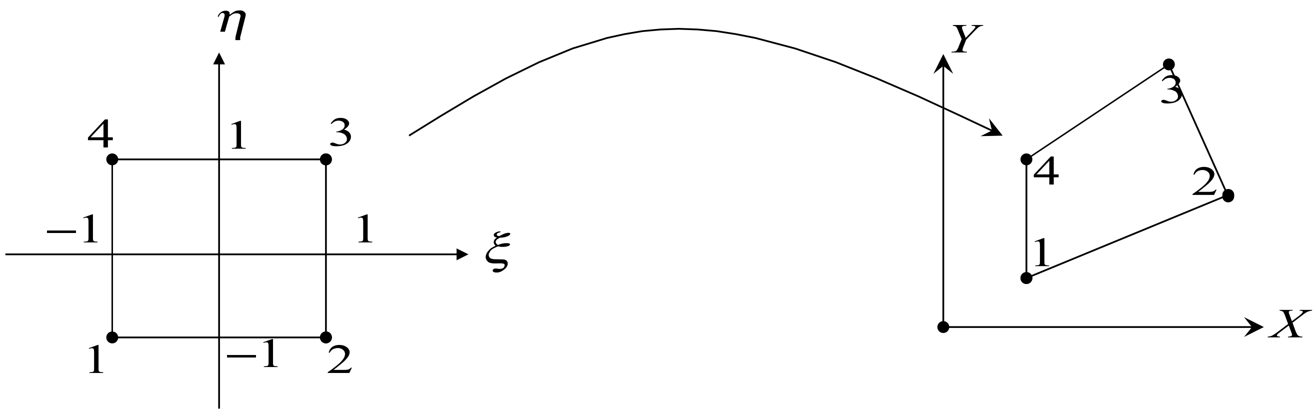

Natural

Actual

Consider a vector field:

$$\mathbf{u}(\xi,\eta) = \begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{Bmatrix}$$

Isoparametric Element
(Field & Position use same shape functions)

$$\underbrace{\begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{Bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix}}_{\mathbf{d}}$$

$$\underbrace{\begin{Bmatrix} X(\xi,\eta) \\ Y(\xi,\eta) \end{Bmatrix}}_{\mathbf{X}} = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{Bmatrix} X_1 \\ Y_1 \\ \vdots \\ \vdots \\ X_4 \\ Y_4 \end{Bmatrix}}_{\mathbf{C}}$$

Shape Functions in Natural Coordinate

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- Polynomial interpolation

$$X(\xi,\eta) = X_1 N_1(\xi,\eta) + X_2 N_2 + X_3 N_3 + X_4 N_4$$

$$X_i = X(\xi_i,\eta_i)$$

- 4 nodes ➔ 4 interpolation conditions ➔ 4 monomials

$$X(\xi,\eta) = a_{00} + a_{10}\xi + a_{01}\eta + a_{11}\xi\eta$$

Pascal's Triangle

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$$X(\xi,\eta) = a_{00} + a_{10}\xi + a_{01}\eta + a_{11}\xi\eta$$

$$1$$

$$\xi \quad \eta$$

Need for Linear Completeness

$$\xi^2 \quad \xi\eta \quad \eta^2$$

Least biased of terms from the row

$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$

$$\therefore X(\xi,\eta) \text{ will be linear in both } \xi \;\&\; \eta$$

Also want Kronecker:

$$N_i(\xi_j,\eta_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Shape Functions

UCLA

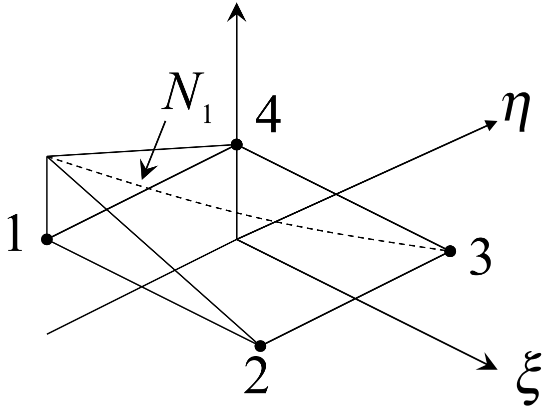
"Tensor Products" of 1-D shape functions $N_1 = \frac{1}{2}(1-\xi)$ $N_2 = \frac{1}{2}(1+\xi)$

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

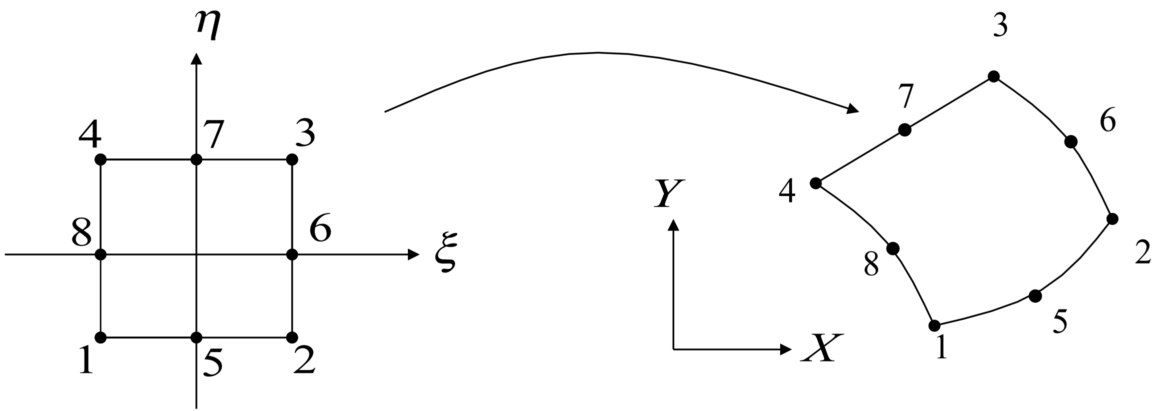


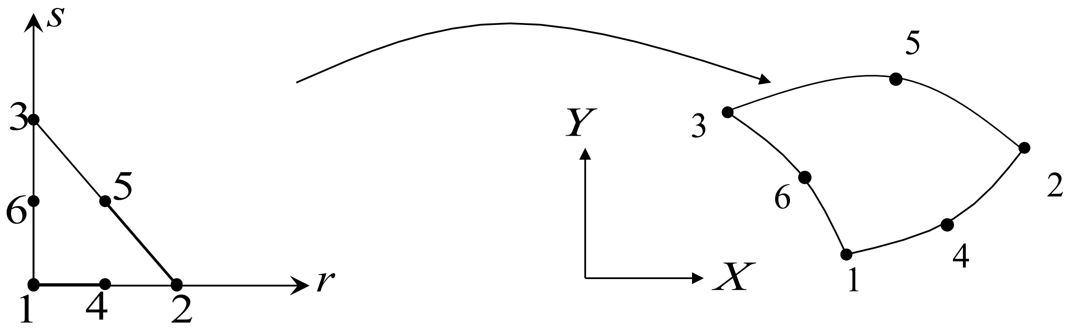
Section 8.6

Quadratic Triangles and Quads

Higher Order Isoparametric Elements

UCLA



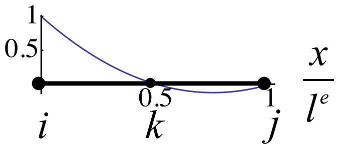


9-Node Bi-quadratic Quadrilateral

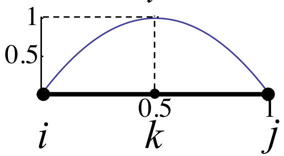
UCLA

Tensor products of 1-D shape functions (just like for Q4)

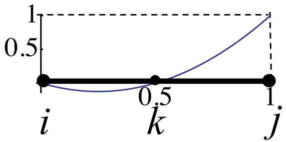
$$N_i(x) = \frac{2}{l^2}\left(x - \frac{l}{2}\right)(x - l)$$

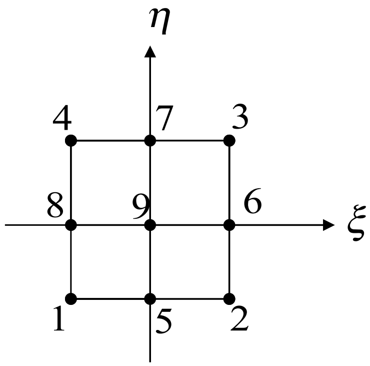


$$N_k(x) = -\frac{4}{l^2}x(x - l)$$



$$N_j(x) = \frac{2}{l^2}x\left(x - \frac{l}{2}\right)$$





9-node Quad
"Q9"

$$N_1 = \frac{1}{2}\xi(\xi - 1) \times \frac{1}{2}\eta(\eta - 1)$$

$$N_2 = \frac{1}{2}\xi(\xi + 1) \times \frac{1}{2}\eta(\eta - 1)$$

$$\ldots$$

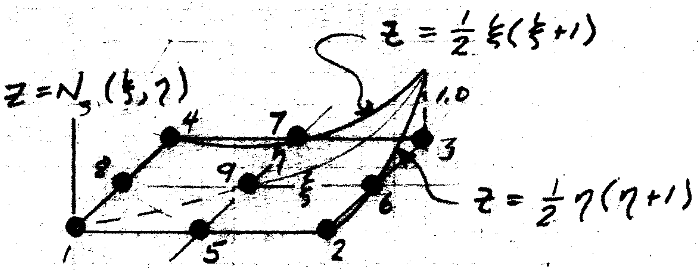
$$N_5 = (1+\xi)(1-\xi) \times \frac{1}{2}\eta(\eta - 1)$$

$$\ldots$$

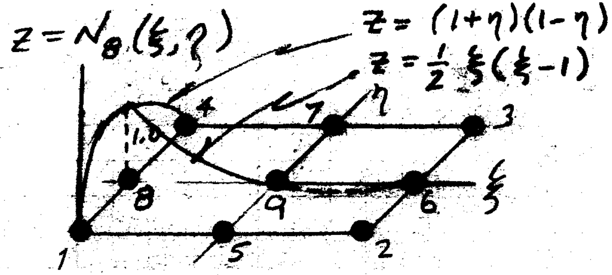
$$N_9 = (1+\xi)(1-\xi) \times (1+\eta)(1-\eta)$$

Bi-quadratic Shape Functions

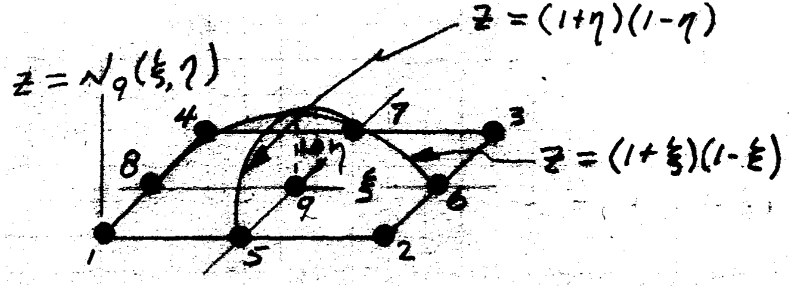
UCLA



$$N_3 = \frac{1}{4}\xi\eta(\xi+1)(\eta+1)$$

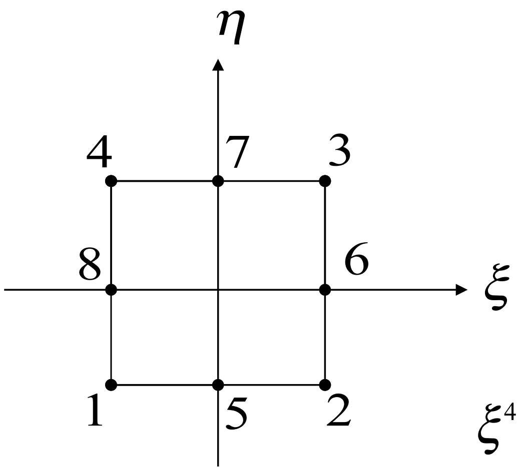


$$N_8 = \frac{1}{2}\xi(\xi-1)(1+\eta)(1-\eta)$$

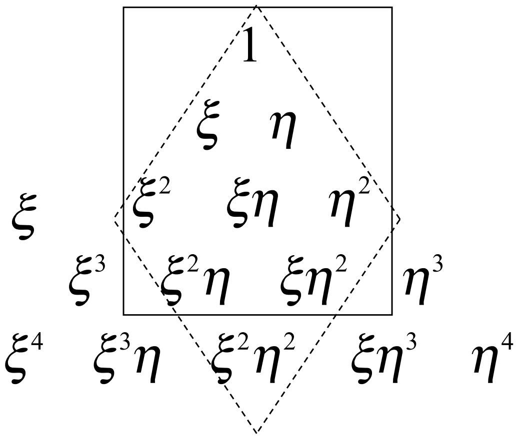


$$N_9 = (1+\xi)(1-\xi)(1+\eta)(1-\eta)$$

8-Node “Serendipity” Quad



8-node Quad “Q8”



- - - - - “Q9” element

——— “Q8” element

$$N_1 = \frac{1}{4}(\xi - 1)(1-\eta)(1+\xi+\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta)$$

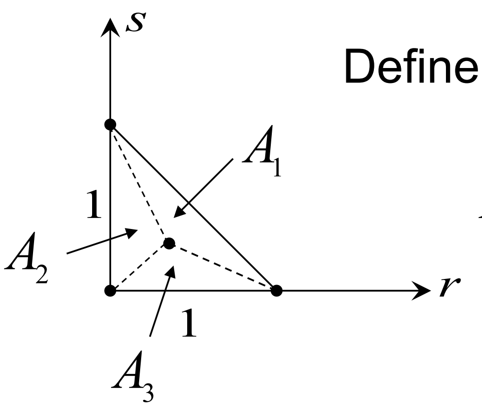
$$\dots$$

$$N_5 = \frac{1}{2}(1-\eta)(1-\xi^2)$$

$$\dots$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Triangles: Area Coordinates



Define area coordinates: $L_i = \dfrac{A_i}{A}$

$$A = A_1 + A_2 + A_3 = 1/2 \qquad 1 = L_1 + L_2 + L_3$$

$$A_2 = \frac{1}{2}(1)(r) = \frac{r}{2} \qquad A_3 = \frac{1}{2}(1)(s) = \frac{s}{2}$$

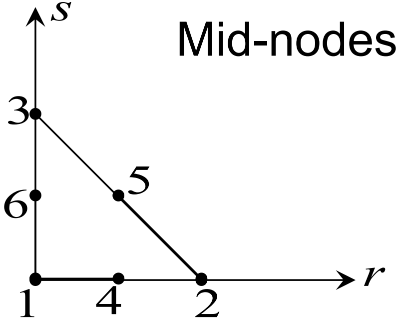
$$A_1 = A - A_2 - A_3 = \frac{1}{2} - \frac{r}{2} - \frac{s}{2}$$

$$L_1 = 1 - r - s$$
$$L_2 = r$$
$$L_3 = s$$

$$N_i\left(L_1, L_2, L_3\right) = L_i$$

Quadratic Triangles

UCLA



Mid-nodes at $(r,s) = \frac{1}{2} \quad \Rightarrow \quad L_i = \frac{1}{2}$

Choose $N_1(r,s),\ldots,N_6(r,s)$

to satisfy $N_i(r_j,s_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$N_1(r,s) = L_1(2L_1 - 1) = (1-r-s)(1-2r-2s)$$
$$N_2(r,s) = L_2(2L_2 - 1) = r(2r-1)$$
$$N_3(r,s) = L_3(2L_3 - 1) = s(2s-1)$$
$$N_4(r,s) = 4L_1L_2 = 4(1-r-s)r$$
$$N_5(r,s) = 4L_2L_3 = 4rs$$
$$N_6(r,s) = 4L_3L_1 = 4s(1-r-s)$$

Section 8.7

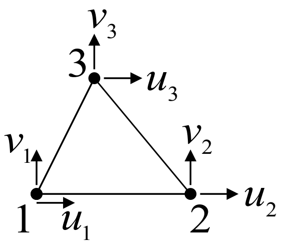
Galerkin FE in 2-D

Galerkin FEA

Weak Form

$$0 = \int_A \left[-(\partial \mathbf{w})^T \mathbf{E} \partial \mathbf{u} + \mathbf{b} \right] dA + \int_{\partial A} \left[\mathbf{w}^T \mathbf{p} \right] dl \qquad \mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Ex: 3-Node Triangle



$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$v = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\underbrace{\begin{Bmatrix} u \\ v \end{Bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}}_{\mathbf{d}}$$

$$\Rightarrow \mathbf{u} = \mathbf{N}\mathbf{d}$$

Strain-Displacement Operator

UCLA

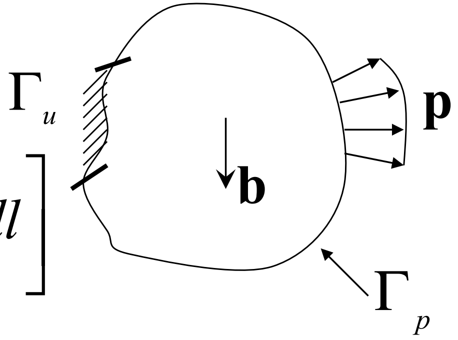
$$\mathbf{u} = \mathbf{N}\mathbf{d}$$

$$\varepsilon = \partial\mathbf{u} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \Rightarrow \varepsilon = (\partial\mathbf{N})\mathbf{d} = \mathbf{B}\mathbf{d}$$

$$\mathbf{B} = \partial\mathbf{N} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$= \left[\begin{array}{cc:cc:cc} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{array}\right] = \left[\begin{array}{c|c|c} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \end{array}\right]$$

Galerkin FE Equations



$$0 = \sum_e \left[\int_{A^e} (\partial \mathbf{w})^T \mathbf{E} \partial \mathbf{u} dA - \int_{A^e} \mathbf{w}^T \mathbf{b} dA - \int_{\Gamma_p^e} \mathbf{w}^T \mathbf{p} dl \right]$$

$$\mathbf{u}(x, y) = \mathbf{N}(x, y)\mathbf{d} \qquad \partial \mathbf{u} = (\partial \mathbf{N})\mathbf{d} = \mathbf{B}\mathbf{d}$$

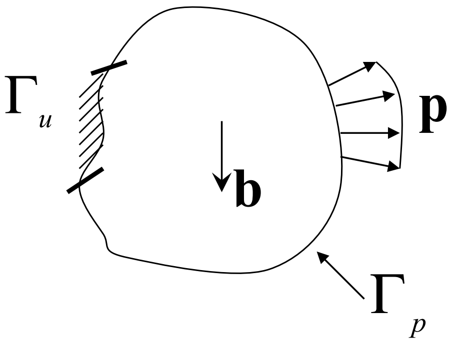
$$\mathbf{w} \to \mathbf{N} \qquad \Rightarrow \partial \mathbf{w} \to \partial \mathbf{N} = \mathbf{B}$$

$$0 = \sum_e \left[\left(\int_{A^e} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dA \right) \mathbf{d}^e - \int_{A^e} \mathbf{N}^T \mathbf{b} \, dA - \int_{\Gamma_p^e} \mathbf{N}^T \mathbf{p} \, dl \right]$$

Galerkin FE Equations

UCLA

$$0=\sum_e\left[\left(\int_{A^e}\mathbf{B}^T\mathbf{E}\mathbf{B}\,dA\right)\mathbf{d}^e-\int_{A^e}\mathbf{N}^T\mathbf{b}\,dA-\int_{\Gamma_p^e}\mathbf{N}^T\mathbf{p}\,dl\right]$$



$$0=\sum_e\left[\mathbf{k}^e\mathbf{d}^e-\mathbf{q}^e\right]$$

$$\mathbf{k}^e=\int_{A^e}\mathbf{B}^T\mathbf{E}\mathbf{B}dA \qquad \mathbf{q}^e=\int_{A^e}\mathbf{N}^T\mathbf{b}dA+\int_{\Gamma_p^e}\mathbf{N}^T\mathbf{p}dl$$

Isoparametric
(Displacement & Position use same shape functions)

$$\mathbf{k}^e = \int_{A^e} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dA \qquad \mathbf{q}^e = \int_{A^e} \mathbf{N}^T \mathbf{b} \, dA + \int_{\Gamma_p^e} \mathbf{N}^T \mathbf{p} \, dl$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & \cdots & \cdots & \cdots \\ 0 & N_1 & 0 & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix} \qquad \begin{Bmatrix} X(r,s) \\ Y(r,s) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & \cdots & \cdots & \cdots \\ 0 & N_1 & 0 & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} X_1 \\ Y_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\ \dfrac{\partial N_i}{\partial s} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial X}{\partial r} & \dfrac{\partial Y}{\partial r} \\ \dfrac{\partial X}{\partial s} & \dfrac{\partial Y}{\partial s} \end{bmatrix}}_{\mathbf{J}} \begin{Bmatrix} \dfrac{\partial N_i}{\partial X} \\ \dfrac{\partial N_i}{\partial Y} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial N_i}{\partial X} \\ \dfrac{\partial N_i}{\partial Y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

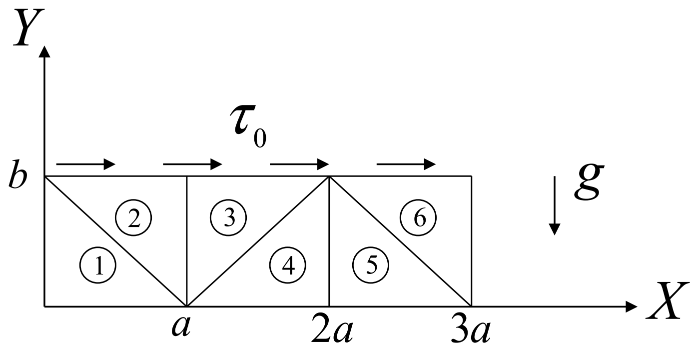
$$dA = dX\, dY = (\det \mathbf{J})\, d\xi\, d\eta$$

Section 8.8

Example Calculations: Linear Triangle

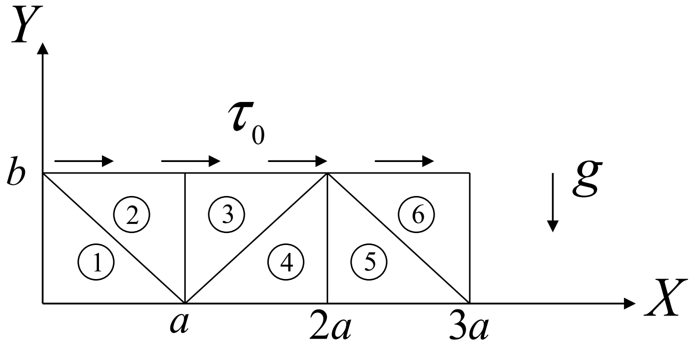
Example: Linear Triangle

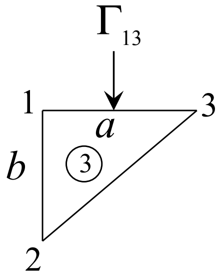
UCLA



Compute: $\mathbf{k}^{(3)}$, $\mathbf{q}^{(3)}$

Element Stiffness





$$(x_1, y_1) = (a, b)$$
$$(x_2, y_2) = (a, 0)$$
$$(x_3, y_3) = (2a, b)$$

$$\mathbf{k} = \int_{A^e} \mathbf{B}^T \mathbf{E} \mathbf{B} \, dA = \mathbf{B}^T \mathbf{E} \mathbf{B} A$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \end{bmatrix}$$

Element Stiffness

UCLA

$$\mathbf{k} = \int_{A^e} \mathbf{B}^T \mathbf{E} \mathbf{B} dA = \mathbf{B}^T \mathbf{E} \mathbf{B} A$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \end{bmatrix}$$

$$(x_1, y_1) = (a, b)$$
$$(x_2, y_2) = (a, 0)$$
$$(x_3, y_3) = (2a, b)$$

$$N_1(x,y) = \frac{1}{2A}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$
$$N_2(x,y) = \frac{1}{2A}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$
$$N_3(x,y) = \frac{1}{2A}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

$$\mathbf{B}_1 = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} \end{bmatrix} = \begin{bmatrix} y_2 - y_3 & 0 \\ 0 & x_3 - x_2 \\ x_3 - x_2 & y_2 - y_3 \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 0 & a \\ a & -b \end{bmatrix}$$

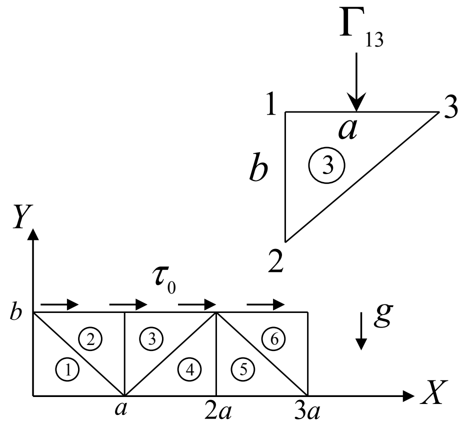
Element Nodal Forces

UCLA

$$\mathbf{q}^{(3)} = \int_{A^{(3)}} \mathbf{N}^T \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix} dA + \int_{\Gamma_{13}} \mathbf{N}^T \begin{Bmatrix} \tau_0 \\ 0 \end{Bmatrix} dx$$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

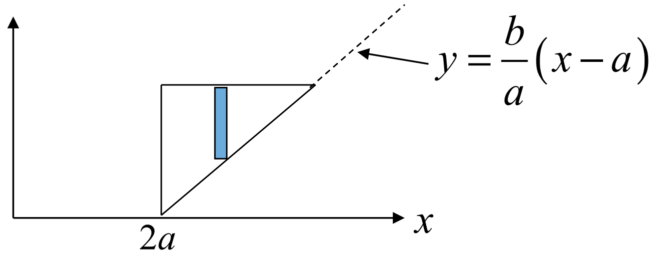
$$\mathbf{q}^{(3)} = \int_{A^{(3)}} -\rho g \begin{Bmatrix} 0 \\ N_1 \\ 0 \\ N_2 \\ 0 \\ N_3 \end{Bmatrix} dA + \int_{\Gamma_{13}} \tau_0 \begin{Bmatrix} N_1 \\ 0 \\ N_2 \\ 0 \\ N_3 \\ 0 \end{Bmatrix} dx$$



Two types of integrals

$$\int_A N_i dA \quad and \quad \int_{x_1}^{x_3} N_i \Big|_{y=b} dx$$

Area Integral



$$\int_A N_i dA = \int_a^{2a} \int_{\frac{b}{a}(x-a)}^{b} N_i dx dy$$

$$N_1(x,y) = \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$N_2(x,y) = \frac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

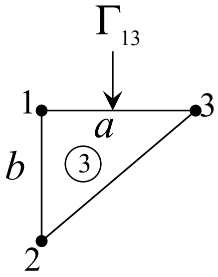
$$N_3(x,y) = \frac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

$$\int N_1 dA = \frac{1}{6} ab = \frac{1}{3} A^{(3)}$$

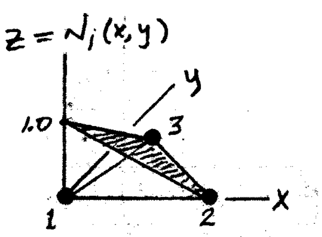
$$\int N_2 dA = \frac{1}{6} ab$$

$$\int N_3 dA = \frac{1}{6} ab$$

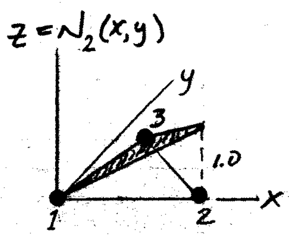
Line Integral



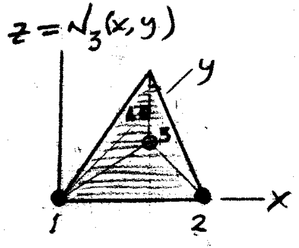
$$\int_{x_1}^{x_3} N_i\Big|_{y=b}\, dx = \int_a^{2a} N_i\left(x,b\right)dx$$



$$\Rightarrow \int_a^{2a} N_1\left(x,b\right)dx = \frac{1}{2}a$$



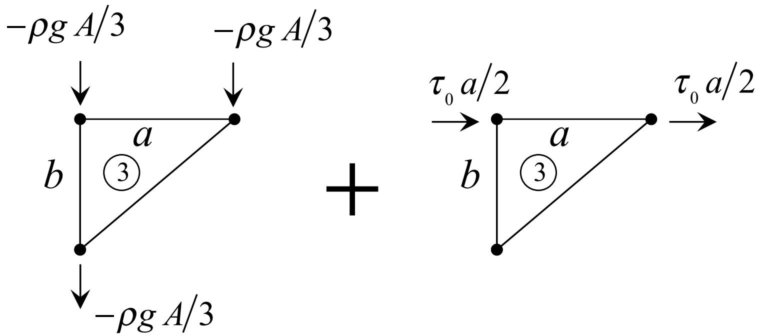
$$\Rightarrow \int_a^{2a} N_2\left(x,b\right)dx = 0$$



$$\Rightarrow \int_a^{2a} N_3\left(x,b\right)dx = \frac{1}{2}a$$

Nodal Forces

$$\mathbf{q}^{(3)} = -\frac{\rho A^{(3)}}{3} g \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} + \frac{\tau_0 a}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}$$



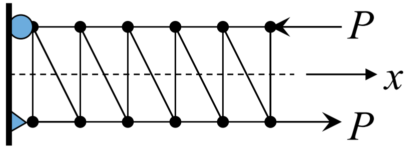
Section 8.9

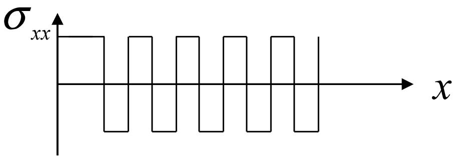
Comments on Element Behavior

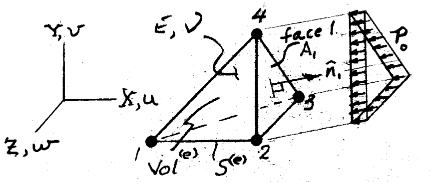
Properties of Linear Triangle

UCLA

- Displacement fields u & v are linear in x, y in each element.
- Strain fields are constant/uniform in each element (nickname: CST or Constant Strain Triangle).
- Element is undesirably stiff.
  - (Stiff for elasticity, but OK for thermal)
- Slow convergence.
- Cannot represent pure bending.
  - Constant stress instead of linear with y.
  - Non-zero stress on x-axis.
- Spurious (parasitic) shear stress.
- In 3-D, 4-node Tetrahedron has analogous properties

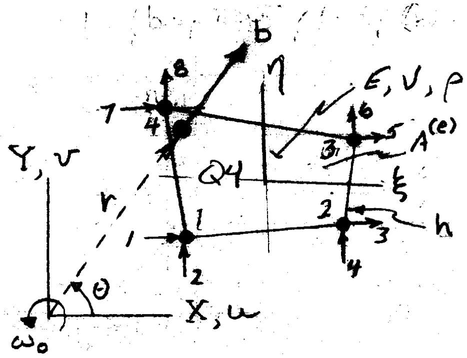


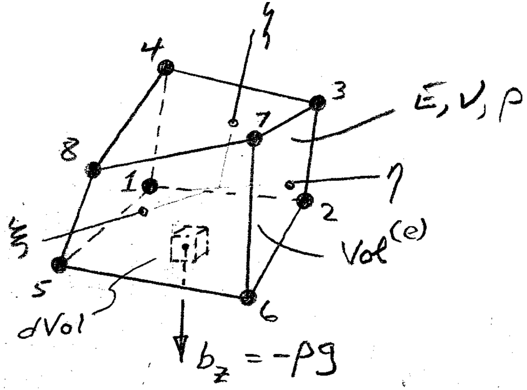




Q4 and Hex 8 Properties

- Displacement bilinear/trilinear.
- Strains partially linear (e.g., ε_{xx} linear in y constant in x)
- Less stiff than CST/Tet4, but still undesirably stiff (when no “tricks” implemented).
- Slow convergence.





Quadratic Elements

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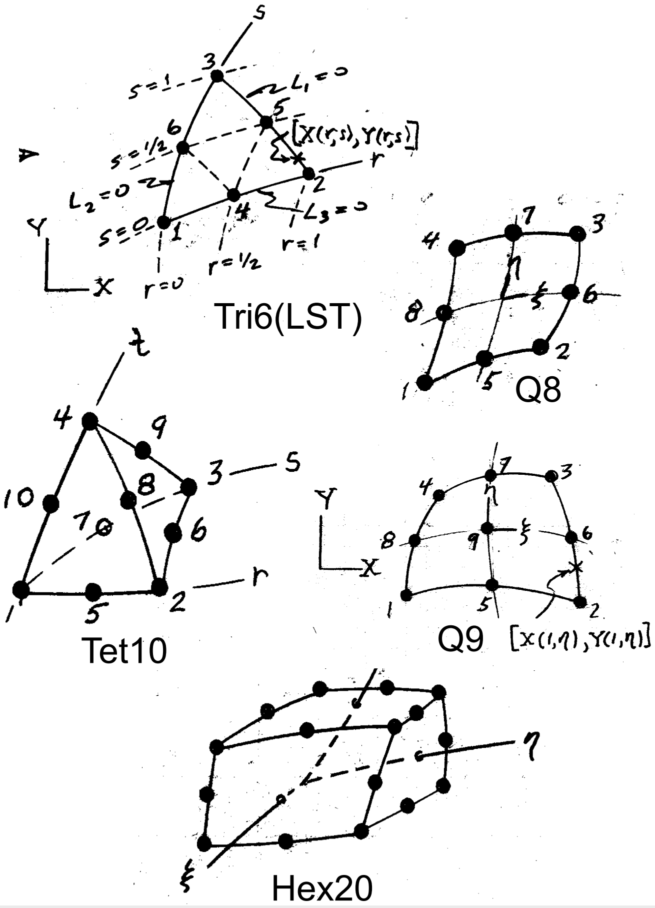
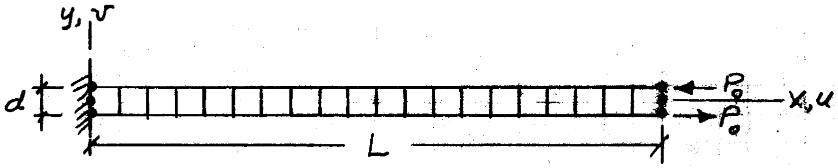
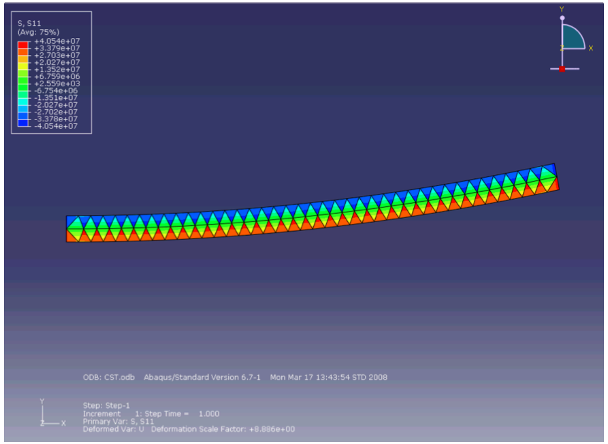
- Displacement quadratic.
- Strains fully linear
- Very accurate with no tricks.
- Fast convergence.
- “Expensive”
  - Need higher-order quadrature rules (many more points)
  - Analysis runs slower



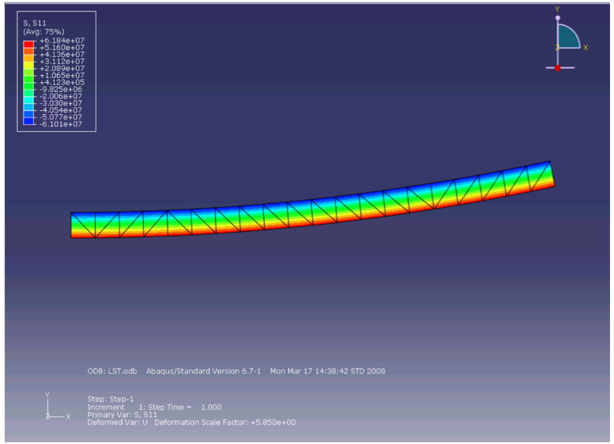
Example Problem: Bending a Beam in 2-D

UCLA





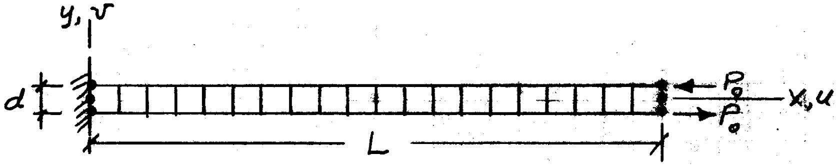
Bending Stress with Tri3 (CST)

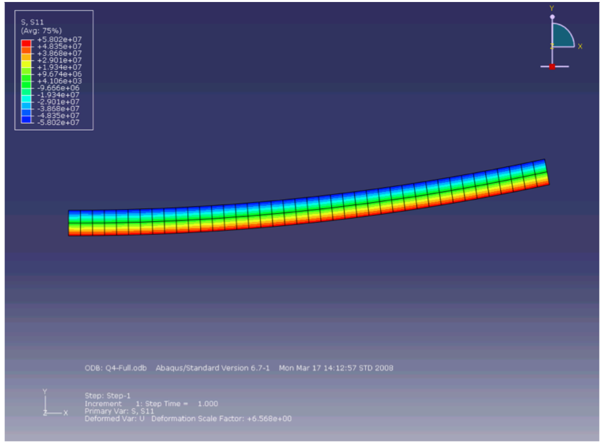


Bending Stress with Tri6 (LST)

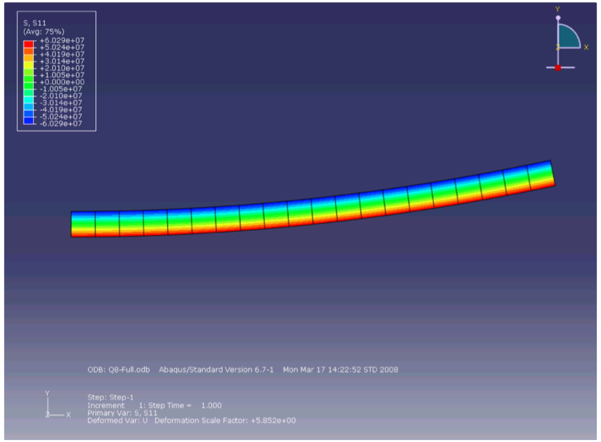
Example Problem: Bending a Beam in 2-D

UCLA





Bending Stress with Q4



Bending Stress with Q8

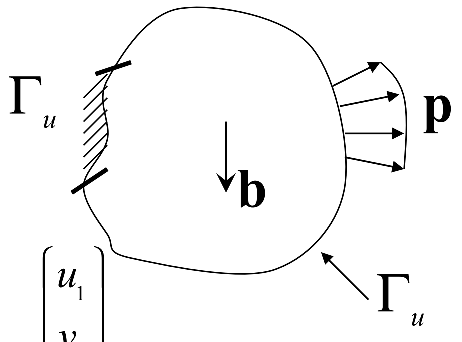
Section 8.10

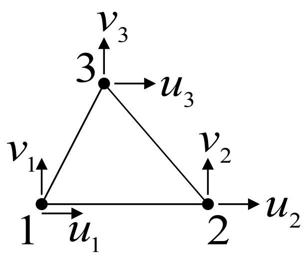
Numerical Integration: 2-D Gauss Quadrature

Summary and Review

$$\sum_e \left[\mathbf{k}^e \mathbf{d}^e - \mathbf{q}^e\right] = 0$$

$$\mathbf{k}^e = \int_{A^e} \mathbf{B}^T \mathbf{E} \mathbf{B}\, dA \qquad \mathbf{q}^e = \int_{A^e} \mathbf{N}^T \mathbf{f}\, dA + \int_{\Gamma_p^e} \mathbf{N}^T \mathbf{p}\, dl$$





$$\underbrace{\begin{Bmatrix} u \\ v \end{Bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}}_{\mathbf{d}}$$

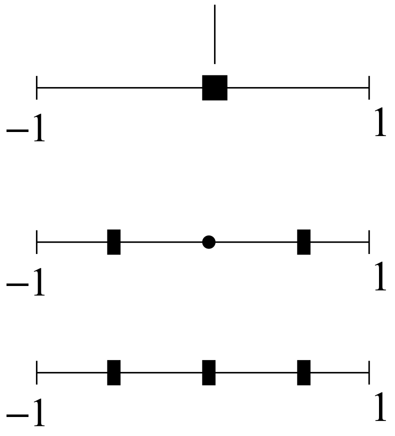
$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial X} & 0 & \frac{\partial N_2}{\partial X} & 0 & \frac{\partial N_3}{\partial X} & 0 \\ 0 & \frac{\partial N_1}{\partial Y} & 0 & \frac{\partial N_2}{\partial Y} & 0 & \frac{\partial N_3}{\partial Y} \\ \frac{\partial N_1}{\partial Y} & \frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial Y} & \frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial Y} & \frac{\partial N_3}{\partial X} \end{bmatrix}$$

Quadrature in 1-D

$$I = \int_{-1}^{1} \phi \, d\xi \approx w_1 \phi(\xi_1) + w_2 \phi(\xi_2) + \ldots + w_n \phi(\xi_n) = \sum_{i=1}^{n} w_i \phi(\xi_i)$$

ξ_i = quadrature points

w_i = quadrature weights



n	ξ_p	w_p	$2n-1$
2	-0.57735027 0.57735027	1.0 1.0	3
3	-0.77459667 0.0 0.77459667	0.55555555 0.88888889 0.55555555	5
4	-0.86113631 -0.33998104 0.33998104 0.86113631	0.34785485 0.65214515 0.65214515 0.34785485	7
5	-0.90617975 -0.53846931 0.0 0.53846931 0.90617975	0.23692689 0.47862867 0.56888889 0.47862867 0.23692689	9

Quadrature in 2-D

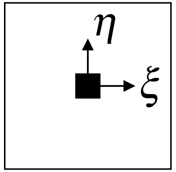
UCLA

$$I = \int_{-1}^{1}\int_{-1}^{1}\phi(\xi,\eta)\,d\xi\,d\eta \approx \int_{-1}^{1}\left[\sum_{i=1} w_i\phi(\xi_i)\right]d\eta$$

$$\approx \sum_i \sum_j w_i w_j \phi(\xi_i,\eta_j)$$

$$= \sum_p w_p \phi(\xi_p,\eta_p)$$

Example: 1 pt.



$$w_1 = 4$$

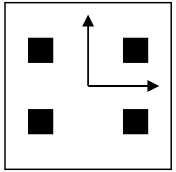
$$(\xi_1,\eta_1) = 0$$

Can integrate exactly

$$\phi = a_0 + a_1\xi + a_2\eta + a_3\xi\eta$$

Quadrature in 2-D

Example: 4 pts.



$$w_i w_j = 1$$

$$(\xi_i, \eta_j) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}} \right)$$

Can integrate exactly

$$\xi^k \eta^l \qquad k, l \le 3$$

$$1$$

$$\xi \quad \eta$$

$$\xi^2 \quad \xi\eta \quad \eta^2$$

$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$

$$\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$$

$$\xi^5 \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \eta^5$$

$$\xi^3\eta^3$$

One pt.

2 by 2

Quadrature for Triangles

$$I = \int_A \phi(r,s)\,dA \approx \sum_{i=1}^{n} \phi(r_i,s_i)\,J(r_i,s_i)\,w_i$$

$$J = \frac{1}{2}\det \mathbf{J}$$

Area of natural geometry is 1/2

k = 1 k = 2 k = 2 k = 3

$$r^l s^m \text{ integrated exactly if } l + m \le k$$

Choices in Quadrature

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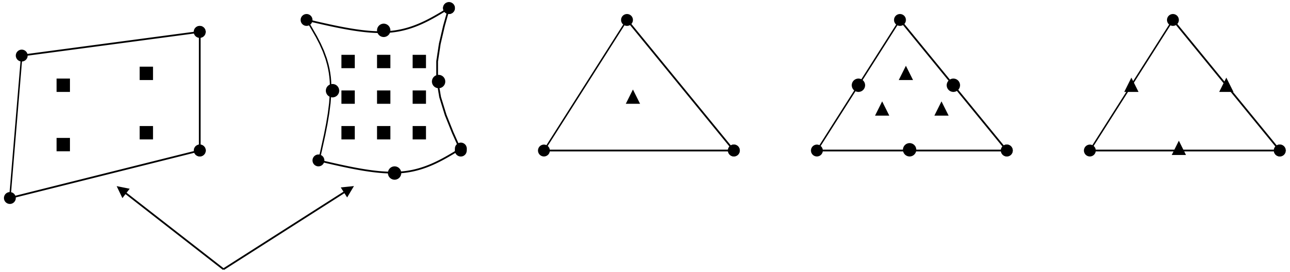
- Because of Jacobians quadrature is almost always inexact or approximate.
- Does increasing number of quad points increase accuracy?
  - Not always.
  - Mostly, yes.
  - Sometimes reducing quad points increases accuracy.
  - If number of quad points is too few can suffer *spurious modes*.

Basic Quadrature Choices

1) Full integration/quadrature:
   - Enough points that components of **k** are exact for undistorted elements.
     - Straight sides, mid-nodes centered, rectangular if quad.



Distorted, so full
quadrature is not exact

Basic Quadrature Choices

2) Reduced integration/quadrature:
   - Reduce computational time/expense.
   - May improve accuracy if element is artificially stiff (softening side effect).

What's the catch?

- Spurious modes
  - "Hourglass", "singular", "zero-energy", mechanism, etc.
- Reduced quadrature can miss/fail to pick up certain deformations → rank-deficient **k**
  - Zero-energy motions besides rigid body motions.

Modes of Q4 elements

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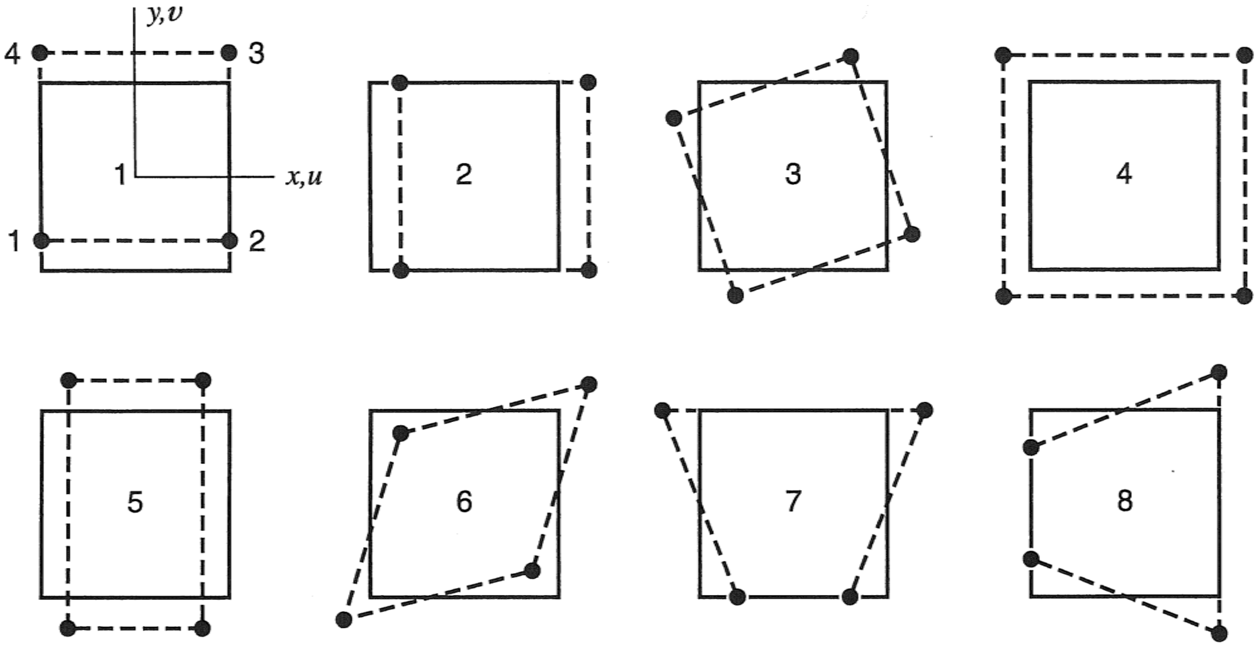

Modes of Q8 & Q9 elements

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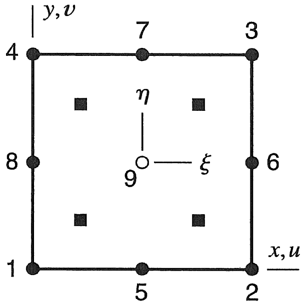
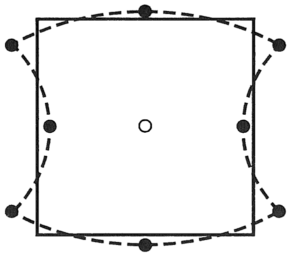
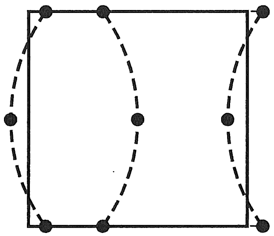


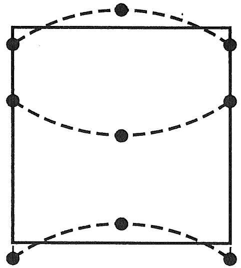
Eight- or nine-node elements



Eight- and nine-node elements

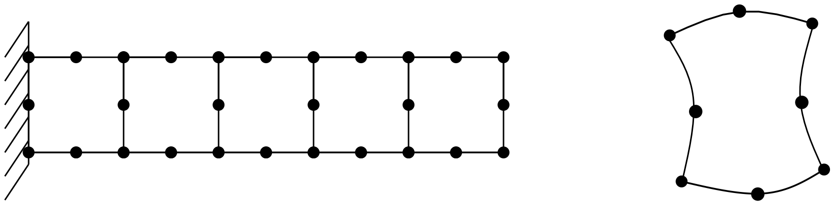


Nine-node element only



Nine-node element only

These mode can sometimes be suppressed by the mesh



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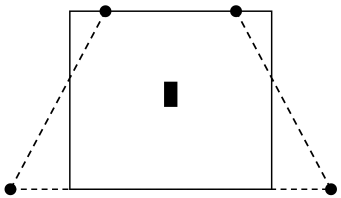
3) “Reduced” with stabilization (hourglass control)

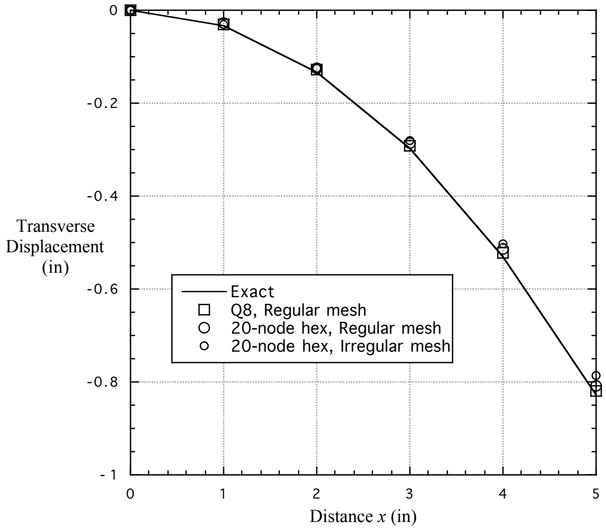
- Artificial stiffness is added specifically to spurious modes.

4) “Selective Reduced” integration

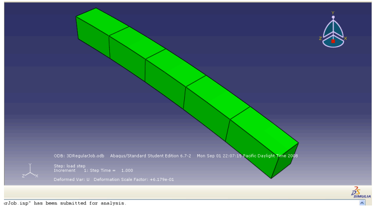
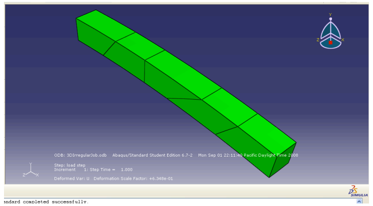
- Motivation:
  - No shear for bending
    - Good !
  - No stress at all for bending
    - Bad !
  - Compromise
    - Full for normal stresses/strains
    - Reduced for shear stresses/ strains



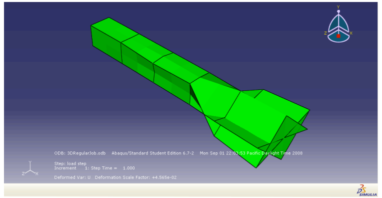
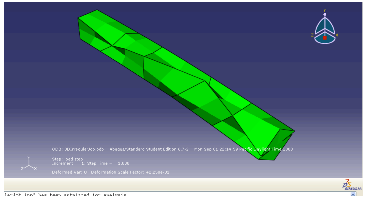
Example: “Hourglassing”



Regular | Irregular

Full Integration

Reduced Integration

“Selective Reduced” integration

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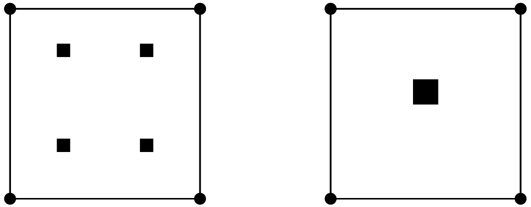
Rules:



Terms: ε_{xx} ε_{yy} γ_{xy}

Must be careful of frame invariance. Tricks involved.

E.g. Plane Stress:

$$E = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{E}{1-\upsilon^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{(1-\upsilon)}{2} \end{bmatrix}$$

Note: cannot do this with CST.

Example Problem: Bending a Beam in 2-D

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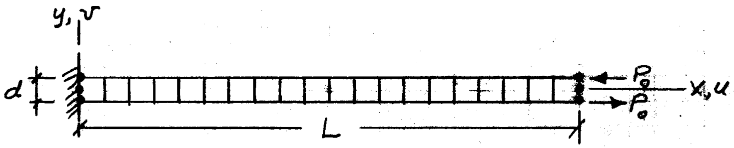
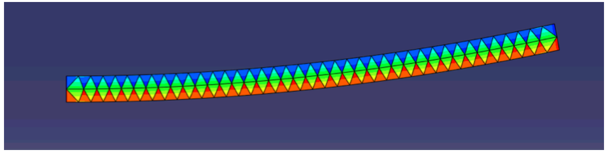




Element	$\tilde{v}[mm]$	$Error\left(\tilde{v}-v_{E-B}[mm]\right)$	$\%Error\left(\frac{\tilde{v}-v_{E-B}}{v_{E-B}}\right)$	Comment
CST	2.251	-1.17757	-34.35	Way to stiff
Q4 full	3.045	-0.38357	-11.19	Too stiff
Q4 reduced	4.569	1.14043	33.26	Way to flexible
LST	3.419	-0.00957	-0.28	Ok
Q8 full	3.418	-0.01057	-0.31	Ok
Q8 reduced	3.418	-0.01057	-0.31	Ok

Comments About Quadrature

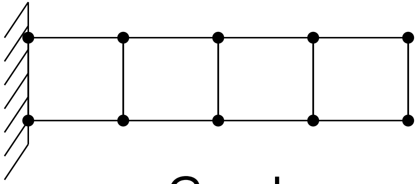
UCLA

- Distortion of elements generally increases error.
- Can trust the solver to choose the right integration order to some extent, but **beware!**
- Watch for spurious modes with reduced integration.
- Solver calculates stresses at Gauss points, not at nodes. Keep that in mind when interpreting results.

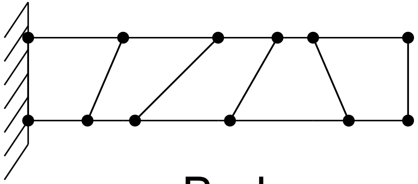
$$\mathbf{u} = \mathbf{N}\mathbf{d} \qquad \varepsilon = \partial \mathbf{u}$$

$$\Rightarrow \varepsilon = \left(\partial \mathbf{N}\right)\mathbf{d} = \mathbf{B}\mathbf{d} \qquad \sigma = \mathbf{E}\varepsilon = \mathbf{E}\mathbf{B}\mathbf{d}$$

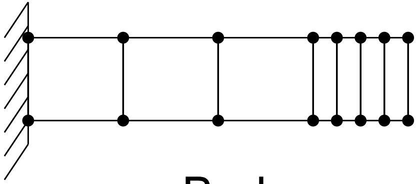
Element Distortion



Good

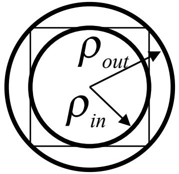


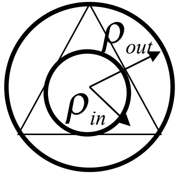
Bad



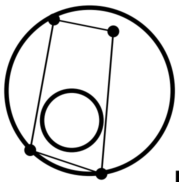
Bad

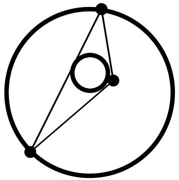
Keep aspect ratios small $\dfrac{\rho_{out}}{\rho_{in}}$





Good



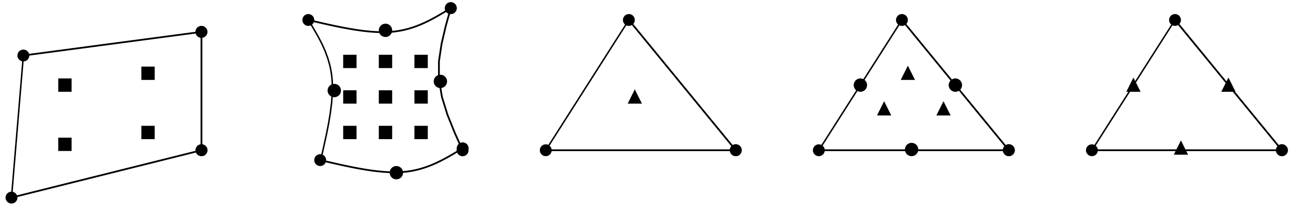


Bad

Summary and Review

UCLA

$$I = \int_{-1}^{1}\int_{-1}^{1}\phi\left(\xi,\eta\right)d\xi d\eta \approx \sum_{i=1} w_i\phi\left(\xi_i,\eta_i\right)$$



- Quadrature Choices:
  - Full: exact for undistorted element
  - Reduced: less than full, beware spurious modes
  - Stabilized reduced: artificial stiffness for spurious modes
  - Selective reduced: reduced for shear, full for rest

Section 8.11

Error and Convergence

Sources of Error

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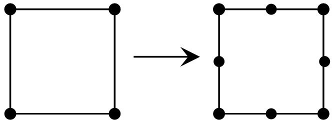
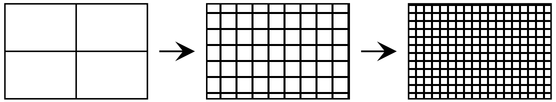
1. User Error:
   - Mistakes you make with software.
     - E.g., typos, wrong element, bad mesh, …
   - ***Prevent***: always double-check results, compare to prelim. analysis, refine mesh, etc.
2. Numerical Error:
   - Finite precision of computer arithmetic.
   - Don’t except accuracy to 20 significant digits.
   - Single precision: 8 digits
   - Double precision: 16 digits
   - Usually not a problem.
3. Modeling Error:
   - Difference between physical system and model.
     - ❑ Geometry, loads, BS’s, etc.
     - ❑ E.g., airplane wing with rivets.
   - **Quantify & Reduce:**
     - ❑ Can build multiple models at various resolutions and compare.
     - ❑ Also compare to testing. Requires engineering intuition.

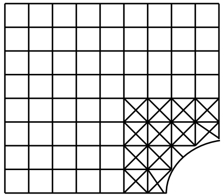
4. Discretization Error:

- Due to interpolation
- ***Reduce by REFINEMENT***:
  - p-refinement: use higher order polynomial elements.
  - h-refinement: use more, smaller elements





  - Uniform: refine everywhere.
  - Non-uniform: refine only where derivatives of stress and strain are large (stress-concentration !)



Discretization Error

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- How does error reduce with p- & h-refinement ?
  - Requires some cool math … not here.
  - End results:
    - h = "element size", p = degree of polynomial interpolation.
    - Error for primary field (displacement): $C(\sigma)h^{p+1}$
    - Error for r^{th} derivative of field: $C(\sigma)h^{p+1-r}$
  - Ex: Elasticity with Q4

$$p = 1 \Rightarrow \ \text{error}(\mathbf{u}) \sim h^2$$

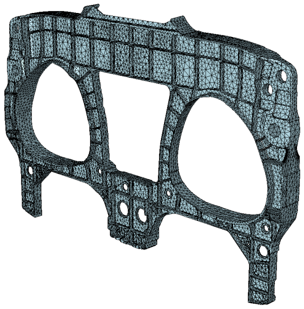
$$\sigma, \varepsilon \text{ involves } 1^{st} \text{ derivtives of } \mathbf{u} \Rightarrow r = 1$$

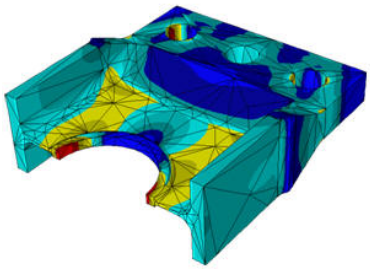
$$\text{error}(\sigma, \varepsilon) \sim h^1$$

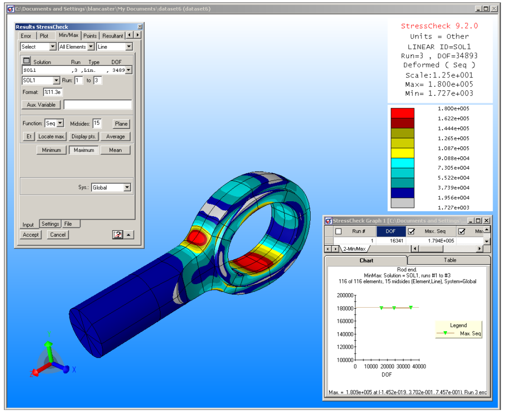
Displacements converge faster than stresses and strains!

Discretization Error

- *p*- vs. *h*-refinement
  - Most “standard” FEA codes focus on h-refinement:
    - Linear & quadratic elements
  - *p*-method Codes (e.g., StressCheck)
    - Refine with *p*: Keep mesh fixed, add nodes & use higher-order shape f’ns.
    - Isoparametric: high-order mapping for geometry
  - *p* controls *rate* of convergence $err(u) = Ch^{p+1}$ $err(\sigma) = Ch^{p}$
    - Sequential increase in *p* reduces error more quickly than decrease in *h.*
    - Exponential convergence vs. polynomial convergence.







StressCheck: http://www.esrd.com/

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5. Quadrature Error:

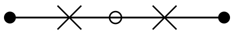
- Full vs. reduced quadrature
  - Full integration: Components of $\mathbf{k}^e$ are exact for “undistorted” element with evenly spaced nodes.
  - Reduced integration: Fewer points than full adds error to $\mathbf{k}^e$.
- Examples

$\mathbf{N}$ linear

$\mathbf{B}$ constant

$\mathbf{B}^T\mathbf{B}$ constant

$\Rightarrow$ 1 pt. rule full



$\mathbf{B}$ linear

$\mathbf{B}^T\mathbf{B}$ quadratic

$\Rightarrow$ 2 pt. rule full

$\mathbf{B}$ quadratic

$\mathbf{B}^T\mathbf{B}$ quadratic

$4 \leq 2n-1 \Rightarrow n=3$

$\Rightarrow$ 3 pt. rule full

Summary and Review

UCLA

- Sources of Error:
  1. User Error: Mistakes you make with software.
  2. Numerical Error: Usually not a problem.
  3. Modeling Error: Model is not reality.
  4. Discretization Error: Refine the mesh.
  5. Quadrature Error
- Quadrature Choices:
  1. Full integration (exact for undistorted elements)
  2. Reduced (spurious modes)
  3. Stabilized-reduced
  4. Selective reduced