ELASTICITY

Elastic Materials:

1. Reversible, path independent, no hysteresis
2. No internal processes, no viscosity
3. No dissipation

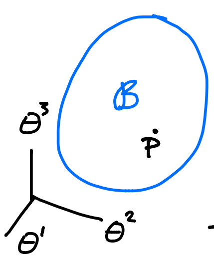
ELASTICITY $\neq$ LINEARITY

1. Elastic materials can be nonlinear: LARGE deformations or nonlinear material behavior.
2. Linear materials can be inelastic, e.g., linear viscoelasticity.

Outline:

1. Finite kinematics
2. Governing principles (Conservation, MFI)
3. Constitutive theory

KINEMATICS



<u>Defn</u> $\mathcal{B} \subset \mathbb{R}^3 \equiv$ A collection of material particles (the BODY)

$P = (\theta^1, \theta^2, \theta^3) \in \mathcal{B}$ a material point identified by coordinates

<u>Defn</u> Position map $\varphi : \mathcal{B} \times [t_1, t_2] \to \mathbb{R}^3$ gives the current/deformed position of material points in $\mathcal{B}$ at time t.

$$\Omega \equiv \{ \underset{\sim}{x} = x_i \underset{\sim}{E}_i \mid \underset{\sim}{x} = \underset{\sim}{\varphi}(\theta^1, \theta^2, \theta^3, t), (\theta^1, \theta^2, \theta^3) \in \mathcal{B} \} \equiv \text{Current Configuration of } \mathcal{B}$$

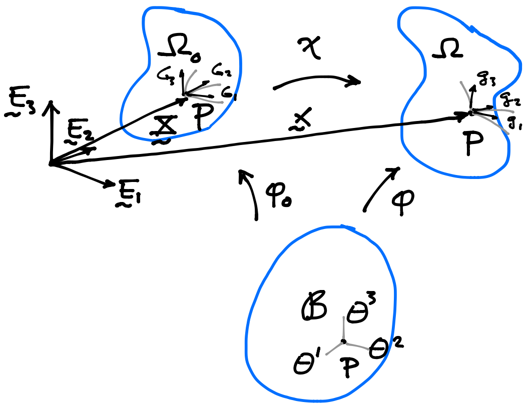
$\{ \underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3 \} \equiv$ Cartesian (orthonormal) frame (LAB FRAME)

<u>Defn</u> $\underset{\sim}{\varphi}_0 : \mathcal{B} \to \mathbb{R}^3$, $\underset{\sim}{\varphi}_0(\cdot) \equiv \underset{\sim}{\varphi}(\cdot, 0)$ $\equiv$ Initial/Reference position map.

$$\Omega_0 \equiv \{ \underset{\sim}{X} : X_I \underset{\sim}{E}_I \mid \underset{\sim}{X} = \underset{\sim}{\varphi}_0(\theta^1, \theta^2, \theta^3), (\theta^1, \theta^2, \theta^3) \in \mathcal{B} \} \equiv \text{Reference Configuration of } \mathcal{B}.$$

$\underline{\text{Defn}}$ $\underset{\sim}{\chi} : \Omega_0 \to \Omega$, $\chi \equiv \underset{\sim}{\varphi} \circ \underset{\sim}{\varphi}_0^{-1}$ = The Deformation Mapping (Motion)

$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t) = \underset{\sim}{\varphi}(\theta^1, \theta^2, \theta^3, t), \quad (\theta^1, \theta^2, \theta^3) = \underset{\sim}{\varphi}_0^{-1}(\underset{\sim}{X})$$



LOCAL COORDINATE BASES

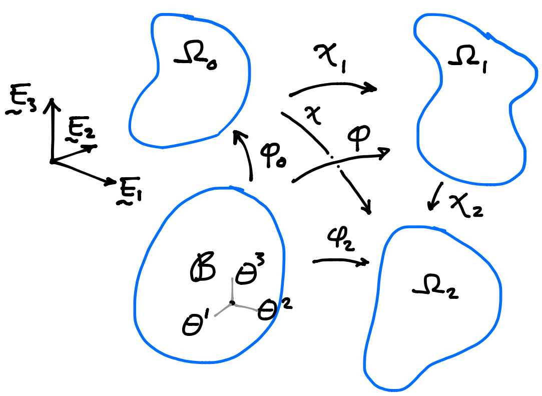
$$\underset{\sim}{G}_i \equiv \varphi_{0,i} = \frac{\partial \varphi_0}{\partial \theta^i} \qquad g_i = \varphi_{,i} = \frac{\partial \varphi}{\partial \theta^i}$$

$$\underset{\sim}{G}^i \cdot \underset{\sim}{G}_j = \delta^i_j \qquad \underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j$$

$$G_{ij} = \underset{\sim}{G}_i \cdot \underset{\sim}{G}_j \quad G^{ij} = \underset{\sim}{G}^i \cdot \underset{\sim}{G}^j \qquad g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j \quad g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j$$

$$G = \det G_{ij} \qquad g = \det g_{ij}$$

COMPOSITION OF MAPPINGS



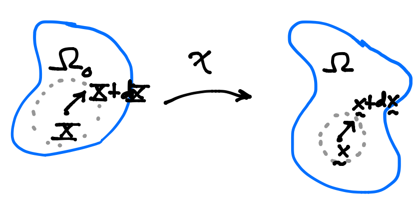
$$\chi(X) = \chi_2(\chi_1(X)) = (\chi_2 \circ \chi_1)(X)$$

$$\boxed{\chi = \chi_2 \circ \chi_2}$$

Deformations are combined by composition of mappings $\Rightarrow$ Multiplicative group structure. (Lie)

ANALYSIS OF LOCAL DEFORMATIONS

Principle of Local Action: assume thermodynamic state (internal free energy density) at a material pt. depends only on (local) state of an infinitesimal material neighborhood containing the pt.



$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t)$$

$$d\underset{\sim}{x} = \frac{\partial \underset{\sim}{\chi}}{\partial \underset{\sim}{X}} \cdot d\underset{\sim}{X} = \nabla \underset{\sim}{\chi} \cdot d\underset{\sim}{X}$$

But $\underset{\sim}{x} = \varphi(\underset{\sim}{\theta}, t)$ & $\underset{\sim}{X} = \varphi_0(\underset{\sim}{\theta}) \to d\underset{\sim}{x} = \underset{\sim}{\varphi}_{,i} d\theta^i = \underset{\sim}{g}_i d\theta^i$ $\quad d\underset{\sim}{X} = \varphi_{0,i} d\theta^i = \underset{\sim}{G}_i d\theta^i$

$$\Rightarrow \quad d\underset{\sim}{x} = \underset{\sim}{g}_i d\theta^i = \nabla \underset{\sim}{\chi}\, \underset{\sim}{G}_i d\theta^i \Rightarrow \underset{\sim}{g}_i = \nabla \underset{\sim}{\chi}\, \underset{\sim}{G}_i$$

Denote $\mathbf{F} \equiv \nabla \boldsymbol{\chi}$ = Deformation Gradient. $g_i = F G_i$

$\Rightarrow$ F is a 2-tensor that maps vectors on the ref config to "deformed versions" on the current configuration. Provides full description of deformation in neighborhood of X

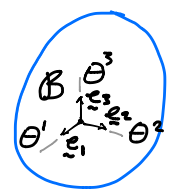
Express $\mathbf{F}$ in mixed bases, $\mathbf{F} = F^{i}_{\cdot j}\, g_i \otimes \mathbf{G}^j \rightarrow$ What are $F^{i}_{\cdot j}$?

$$g_k = F G_k = (F^{i}_{\cdot j}\, g_i \otimes G^j)\, G_k = F^{i}_{\cdot j}\, g_i \underbrace{(G^j \cdot G_k)}_{\delta^j_k} = F^{i}_{\cdot k}\, \mathbf{g}_i$$

$$\rightarrow \quad g_k = F^{i}_{\cdot k}\, g_i \Rightarrow F^{i}_{\cdot k} = \delta^i_k$$

$$\therefore \boxed{\mathbf{F} = \delta^i_k\, \mathbf{g}_i \otimes \mathbf{G}^k = \mathbf{g}_i \otimes \mathbf{G}^i}$$

Another way to see it: Composition of mappings



$\{\mathbf{e}_i\} \equiv$ Orthonormal Basis in parametric space.

$$\chi = \varphi \circ \varphi_0^{-1} \qquad \nabla\chi = \nabla\varphi\, \nabla\varphi_0^{-1}$$

$$\nabla\varphi = \frac{\partial\varphi}{\partial\theta^i} \otimes \mathbf{e}^i = \mathbf{g}_i \otimes \mathbf{e}^i \qquad \nabla\varphi_0 = \frac{\partial\varphi_0}{\partial\theta^i} \otimes \mathbf{e}^i = \mathbf{G}_i \otimes \mathbf{e}^i \rightarrow \nabla\varphi_0^{-1} = [G_i \otimes e^i]^{-1} = \mathbf{e}^i \otimes \mathbf{G}^i$$

$$\therefore \nabla\chi = \nabla\varphi\, \nabla\varphi_0^{-1} = (\mathbf{g}_i \otimes \mathbf{e}^i)(\mathbf{e}^i \otimes \mathbf{G}^i) = \mathbf{g}_i \otimes \mathbf{G}^i \checkmark$$

Strain & Deformation

Ref. & deformed arclengths: $dS^2 = d\mathbf{X} \cdot d\mathbf{X} \qquad ds^2 = d\mathbf{x} \cdot d\mathbf{x} = (\mathbf{F}\, d\mathbf{X})(\mathbf{F}\, d\mathbf{X}) = d\mathbf{X} \cdot (\mathbf{F}^T \mathbf{F})\, d\mathbf{X}$

$$\rightarrow ds^2 - dS^2 = d\mathbf{X} \cdot (F^T F)\, d\mathbf{X} - d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{X} \cdot (F^T F - I)\, d\mathbf{X}$$

Defn $\mathbf{C} = \mathbf{F}^T \mathbf{F} \equiv$ Right Cauchy-Green Deformation tensor.

Defn $\mathbf{E} = \frac{1}{2}(C - I) \equiv$ Green Strain (Green-Lagrange)

$$ds^2 = d\mathbf{X} \cdot \mathbf{C}\, d\mathbf{X} \qquad ds^2 - dS^2 = 2\, d\mathbf{X} \cdot \mathbf{E}\, d\mathbf{X}$$

In the curvilinear bases,

$$F = g_i \otimes G^i \rightarrow F^T = G^i \otimes g_i$$

$$\therefore \quad C = F^T F = (G^j \otimes g_j)(g_i \otimes G^i) = G^j \otimes (g_j \cdot g_i)\, G^i = g_{ji}\, G^j \otimes G^i$$

$\longrightarrow C = g_{ij}\, G^i \otimes G^j$ represents the deformed metric.

Also, $I = G_i \otimes G^j = G_{ij}\, G^i \otimes G^j$.

$$\Rightarrow 2E = F^T F - I = g_{ij}\, G^i \otimes G^j - G_{ij}\, G^i \otimes G^j$$

$$\boxed{E = \tfrac{1}{2}(g_{ij} - G_{ij})\, G^i \otimes G^j}$$

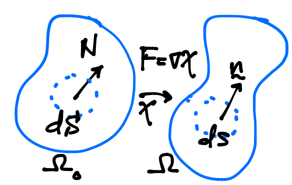
E represents the difference in metrics.

Summary: $F = g_i \otimes G^i$ (mixed tensor) $\quad C = g_{ij}\, G^i \otimes G^j$ (Ref. tensor) $\quad E = \tfrac{1}{2}(g_{ij} - G_{ij})\, G^i \otimes G^j$ (Ref. tensor)

Changes in Volume & Area:

Volume: $\quad dv = \sqrt{g}\, \underbrace{d\Theta^1 d\Theta^2 d\Theta^3}_{\equiv d^3\Theta} \qquad dV = \sqrt{G}\, d^3\Theta$

$$dv = \det F\, dV \to \det F = \frac{dv}{dV} = \frac{\sqrt{g}}{\sqrt{G}} \equiv J > 0$$



Area (Nanson's formula): $\quad \underset{\sim}{n}\, ds = J \underset{\sim}{N} \cdot \underset{\sim}{F}^{-1} dS = J \underset{\sim}{F}^{-T} \underset{\sim}{N}\, dS$

$F = g_i \otimes G^i \to F^{-1} = G_i \otimes g^i$ (check this) $\to F^{-T} = g^i \otimes G_i$

$$\Rightarrow \underset{\sim}{n}\, ds = \frac{\sqrt{g}}{\sqrt{G}} (g^i \otimes G_i) N\, dS$$

$$\boxed{\underset{\sim}{n}\, ds = \sqrt{\tfrac{g}{G}}\, N_i\, \underset{\sim}{g}^i\, dS}$$

$\underset{\sim}{n} = n_i g^i \quad \underset{\sim}{N} = N_i G^i \Rightarrow \boxed{n_i\, ds = N_i \sqrt{\tfrac{g}{G}}\, dS}$

Polar Decomposition

Can decompose $F = \nabla\chi$ into pure stretch & pure rotation

$$F = RU = VR \qquad R^T R = R R^T = I \quad \det R = +1 \quad (R \in SO(3))$$

$$U^T = U \quad V^T = V \quad \text{(unique)}$$

$U \equiv$ Right (material) stretch

$V \equiv$ Left (spatial) stretch

$R \in SO(3) \equiv$ (Proper) Rotation

Exercise: Show: $U = \sqrt{C} \quad U^2 = C$

$V = \sqrt{FF^T} \quad V^2 = B = FF^T$ ("Finger tensor")

EXERCISE: Consider a right circular cylinder initially of inner radius a_0 and outer radius b_0, and thickness h_0. Using as curvilinear coords (R, Φ, Z) in the ref. config. consider the mappings

$$\mathbf{X} = \boldsymbol{\varphi}_0(R, \Phi, Z) = R\,\mathbf{e}_R + Z\,\mathbf{e}_Z$$

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\varphi}(R, \Phi, Z) = r(R)\,\mathbf{e}_R + z(R, Z)\,\mathbf{e}_Z$$

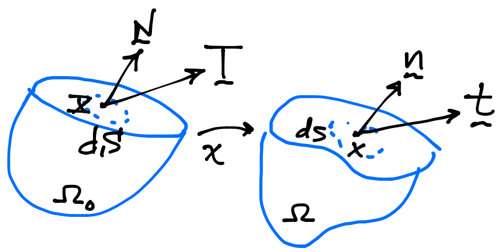
Compute the basis on the deformed config $\mathbf{g}_i = \mathbf{x}_{,i}$ and ref. config. $\mathbf{G}_i = \mathbf{X}_{,i}$ and

a) the deformation gradient $F = \nabla\chi$

b) Right Cauchy-Green Deformation tensor $C = F^T F$

c) Green Strain $E = \frac{1}{2}(C - I)$.

TRACTIONS & STRESS



(Actual) Resultant force on ds: $d\mathbf{f} = \mathbf{t}\,ds = \mathbf{T}\,dS$

$\mathbf{t}(\mathbf{x}, \mathbf{n}) \equiv$ Cauchy (true) traction vector (true force per unit deformed area)

$\mathbf{T}(\mathbf{X}, \mathbf{N}) \equiv$ 1st Piola-Kirchhoff (nominal) traction vector (true force per unit reference area)

$\mathbf{t}$ & $\mathbf{T}$ point in <u>same</u> direction!

CAUCHY'S STRESS THEOREM

$\exists$ unique 2-tensors $\boldsymbol{\sigma}$ & $\mathbf{P}$ such that

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\sigma}(\mathbf{x})\,\mathbf{n} \qquad \mathbf{T}(\mathbf{X}, \mathbf{N}) = \mathbf{P}(\mathbf{X})\,\mathbf{N}$$

$\Rightarrow$ Traction is a LINEAR function of the normal.

$\boldsymbol{\sigma} \equiv$ Cauchy (true) stress tensor. Spatial tensor (on Ω)

$\mathbf{P} \equiv$ 1st Piola-Kirchhoff (nominal) stress tensor. Material tensor (on Ω_0)

In components,

$$\mathbf{n} = n_i \mathbf{g}^i \quad \mathbf{t} = t^i \mathbf{g}_i \quad \boldsymbol{\sigma} = \sigma^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \quad \rightarrow \quad \sigma^{ij} n_j = t^i$$

$$\mathbf{N} = N_i \mathbf{G}^i \quad \mathbf{T} = T^i \mathbf{g}_i \quad \mathbf{P} = P^{ij} \mathbf{g}_i \otimes \mathbf{G}_j \quad \rightarrow \quad P^{ij} N_j = T^i$$

Recall Nanson: $\boldsymbol{n}\,ds = J\boldsymbol{F}^{-T}\boldsymbol{N}\,dS$

$$\Rightarrow \quad \boldsymbol{\sigma}\boldsymbol{n}\,ds = \boldsymbol{t}\,ds = d\boldsymbol{f} = \boldsymbol{T}\,dS = \boldsymbol{P}\boldsymbol{N}\,dS$$

$$\Rightarrow \quad \boldsymbol{\sigma}(J\boldsymbol{F}^{-T}\boldsymbol{N})\,dS = \boldsymbol{P}\boldsymbol{N}\,dS$$

$$\Rightarrow \quad \boxed{\boldsymbol{P} = J\boldsymbol{F}^{-T}\boldsymbol{\sigma}} \rightarrow \boxed{\boldsymbol{\sigma} = J^{-1}\boldsymbol{P}\boldsymbol{F}^{T}}$$

Also define

$$\boxed{\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}} \equiv 2^{nd} \text{ P-K stress.}$$

Conservation Laws

(i) Cons. of Mass

$\rho \equiv$ mass density per unit def. volume

$\rho_0 \equiv$ " " " " ref. "

$$\rightarrow \quad dm = \rho\,dv = \rho_0\,dV \quad \rightarrow \quad \rho = \rho_0 \frac{dV}{dv}$$

But recall $J = \det \boldsymbol{F} = \dfrac{dv}{dV} = \dfrac{\sqrt{g}}{\sqrt{G}}$. $\Rightarrow \boxed{\rho = \rho_0/J = \sqrt{G/g}\,\rho_0}$

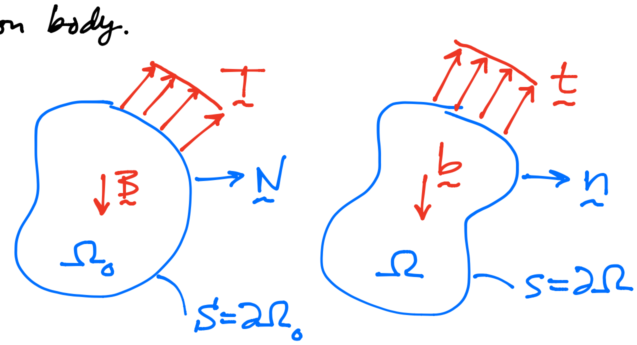
(ii) Cons. of lin. momentum/Equilibrium

$\boldsymbol{b} \equiv$ spatial body force/unit mass

$\boldsymbol{B} \equiv$ Material " " " " $\qquad \boldsymbol{B}(\boldsymbol{X}) = \boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{b}(\chi(\boldsymbol{X})) \qquad B = b\circ\chi$

$\qquad B : \Omega_0 \rightarrow \mathbb{R}^3 \quad b : \Omega \rightarrow \mathbb{R}^3$

$\boldsymbol{R} \equiv$ Resultant of all forces acting on body.



Spatial form:

$$\boldsymbol{R} = \int_\Omega \rho\boldsymbol{b}\,dv + \int_s \boldsymbol{t}\,ds = \int_\Omega \rho\boldsymbol{b}\,dv + \int_{\partial\Omega} \boldsymbol{\sigma}\boldsymbol{n}\,ds$$

$$= \int_\Omega (\rho\boldsymbol{b} + \nabla\cdot\boldsymbol{\sigma})\,dv \quad \text{(Div. Thm.)}$$

Material form:

$$\boldsymbol{R} = \int_{\Omega_0} \rho(b\circ\chi)J\,dV + \int_{\partial\Omega_0} (\sigma\circ\chi)JF^{-T}N\,dS = \int_{\Omega_0} \rho_0 B\,dV + \int_{\partial\Omega_0} PN\,dS = \int_{\Omega_0} (\rho_0 B + \nabla\cdot P)\,dV$$

Set $\underset{\sim}{R}=0$ and enforce for any part of the body. Shrinking Ω_0 to a point,

Local Equilibrium: $\quad \operatorname{div}\underset{\sim}{\sigma}+\rho\underset{\sim}{b}=0 \quad$ (Spatial)

$$\operatorname{Div}\underset{\sim}{P}+\rho_0\underset{\sim}{B}=0 \quad \text{(Material)}$$

(iii) Cons. of Angular momentum.

$$\underset{\sim}{\sigma}^T=\underset{\sim}{\sigma} \quad \text{Cauchy Stress is symmetric.}$$

What about 1st P-K & 2nd P-K?

$$\underset{\sim}{\sigma}=J^{-1}PF^T \rightarrow \sigma^T=J^{-1}(PF^T)^T=J^{-1}FP^T$$

$$\Rightarrow \boxed{PF^T=FP^T}$$

$$S=F^{-1}P=JF^{-1}\sigma F^{-T} \rightarrow S^T=J(F^{-1}\sigma F^{-T})^T=JF^{-1}\sigma^T F^{-T}=S$$

$$\boxed{\underset{\sim}{S}^T=\underset{\sim}{S}}$$

$\underset{\sim}{\sigma}$ & $\underset{\sim}{S}$ are symmetric. $\underset{\sim}{P}$ is not (in general).

ELASTIC MATERIALS

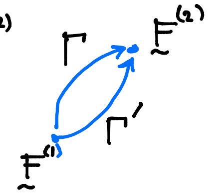
Compute work done for differential deformation $d\underset{\sim}{x}$,

$$
\begin{aligned}
dW &= \int_{\partial\Omega_0} T_i\, dx_i\, dS + \int_{\Omega_0} \rho_0 B_i\, dx_i\, dV \\
&= \int_{\partial\Omega_0} P_{iJ}N_J\, dx_i\, dS + \cdots \\
&= \int_{\Omega_0} (P_{iJ}\, dx_i)_{,J}\, dV + \cdots \\
&= \int_{\Omega_0} \overbrace{(P_{iJ,J}+\rho_0 B_i)}^{0}\, dx_i\, dV + \int_{\Omega_0} P_{iJ}\, dx_{i,J}\, dV \\
&= \int_{\Omega_0} P_{iJ}\, dF_{iJ}\, dV
\end{aligned}
$$

$\Rightarrow$ differential work/energy stored per volume is $dw=P_{iJ}\,dF_{iJ}=\underset{\sim}{P}:d\underset{\sim}{F}$

Work done over a deformation path Γ from $\underset{\sim}{F}^{(1)}$ to $\underset{\sim}{F}^{(2)}$

$$w_{1\to 2} = \int_\Gamma P_{iJ}\, dF_{iJ}$$



<u>Defn</u> The material is ELASTIC if $w_{1\to 2}$ depends only on the end points $F^{(1)}$ & $F^{(2)}$ $\Rightarrow$ $\int_{\Gamma'} P:dF = \int_\Gamma P:dF \quad \forall\, \Gamma, \Gamma'$ between 1 & 2.

$$\Rightarrow \oint P:dF = \int_\Gamma P:dF + \int_{-\Gamma'} P:dF = 0 \;\rightarrow\; \text{Reversible!}$$

Then $\exists\; w(F)$ s.t. $P_{iJ} = \dfrac{\partial w}{\partial F_{iJ}}$ ($P_{iJ}\, dF_{iJ}$ is a perfect differential)

$w(F)$ is called the strain energy density.

<u>VIRTUAL WORK & ENERGY</u>

Weighted residual of equilib. (material)

$$
\begin{aligned}
0 &= \int_{\Omega_0} (\mathrm{Div}\, \underset{\sim}{P} + \underset{\sim}{B}) \cdot \underset{\sim}{v}\, dV \qquad \forall\, \underset{\sim}{v} \in \mathbb{R}^3 \\
&= \int_{\Omega_0} [\nabla\cdot(P^T v) - P : \nabla v + \underset{\sim}{B}\cdot \underset{\sim}{v}]\, dV \\
&= \int_{\partial\Omega_0} N\cdot(P^T v)\, dS - \int_{\Omega_0} P : \nabla \underset{\sim}{v}\, dV + \int_{\Omega_0} B\cdot v\, dV \\
&= \int_{\partial\Omega_0} v\cdot(\underbrace{PN}_{T})\, dS - \cdots \\
&= \underbrace{\int_{\partial\Omega_0} \underset{\sim}{T}\cdot v\, dS}_{\text{Virt. Work of Tractions}} - \underbrace{\int_{\Omega_0} P : \nabla v\, dV}_{\text{Internal Virt. Work of Stress}} + \underbrace{\int_{\Omega_0} \underset{\sim}{B}\cdot v\, dV}_{\text{Virt. Work of body forces}} \equiv G[\chi, v]
\end{aligned}
$$

$G[\chi, v] = 0$ is the WEAK FORM of the equilib problem.

Question: Does a functional $I[\chi]$ exist s.t.

$$DI[\chi](v) \equiv \frac{d}{d\epsilon} I[\chi + \epsilon v]\Big|_{\epsilon=0} = G[\chi, v] \;?$$

(1^{st} variation of $I \rightarrow G$?)

Recall answer: Vainberg's thm.

$$I \text{ exists} \Leftrightarrow D_1 G[\chi, v](\xi) = D_1 G[\chi, \xi](v).$$

Here: $D_1 G[\chi, v](\xi) = \frac{d}{d\epsilon} \int_{\Omega_0} P(\nabla\chi + \epsilon\nabla\xi) : \nabla v \, dV \Big|_{\epsilon=0}$

$$= \int_{\Omega_0} \nabla v : \frac{\partial P}{\partial F} : \nabla \xi \, dV$$

$$D_1 G[\chi, \xi](v) = \int_{\Omega_0} \nabla\xi : \frac{\partial P}{\partial F} : \nabla v \, dV$$

$\therefore$ I exists $\Leftrightarrow$ $\frac{\partial P}{\partial F}$ is symmetric.

In cartesian components (in some local orthonormal frame $\{E_i\}_{i\in\{1,2,3\}}$)

$$P_{iJ} = \underset{\sim}{E}_i \cdot (\underset{\sim}{P}\underset{\sim}{E}_J) \qquad F_{iJ} = \underset{\sim}{E}_i \cdot (\underset{\sim}{F}\underset{\sim}{E}_J)$$

$$\underset{\sim}{v} = v_i \underset{\sim}{E}_i \quad \underset{\sim}{\xi} = \xi_i \underset{\sim}{E}_i \rightarrow (\nabla v)_{iJ} = \underset{\sim}{E}_i \cdot (\nabla \underset{\sim}{v} \underset{\sim}{E}_J) = \frac{\partial v_i}{\partial X_J} = v_{i,J} \qquad (\nabla\xi)_{iJ} = \xi_{i,J}$$

$$\nabla v : \frac{\partial P}{\partial F} : \nabla\xi = v_{i,J} \frac{\partial P_{iJ}}{\partial F_{kL}} \xi_{k,L} \qquad \therefore I \text{ exists} \Leftrightarrow \frac{\partial P_{iJ}}{\partial F_{kL}} = \frac{\partial P_{kL}}{\partial F_{iJ}}.$$

This is only possible if $\exists\ w(\underset{\sim}{F})$ s.t. $P_{iJ} = \frac{\partial w}{\partial F_{iJ}}$ $P = \frac{\partial w}{\partial F}$

$W(\nabla\chi) \equiv$ Strain Energy Density.

Material is ELASTIC! we can define a variational principle:

minimum potential energy

$$I[\chi] = \int w(\nabla\chi)\, dV - \int_{\Omega_0} \rho_0 \underset{\sim}{B} \cdot \underset{\sim}{\chi}\, dV - \int_{\partial\Omega_0} \underset{\sim}{T} \cdot \chi \, dS$$

$$\min I[\nabla\chi] \Rightarrow DI[\chi](v) = 0 \Rightarrow \begin{cases} \nabla \cdot P + \rho_0 B = 0 & \text{in } \Omega_0 \\ PN = T & \text{on } \partial\Omega_0 \end{cases}$$