2)

i) $H = \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$ $\quad HQH^T = I$

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}^T = \frac{1}{4}\begin{bmatrix} 4 & & & \\ & 4 & & \\ & & 4 & \\ & & & 4 \end{bmatrix} = I$$

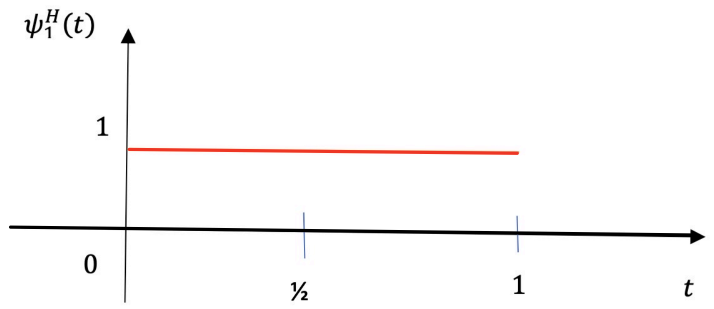
$$\frac{1}{4}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}^T\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & & & 0 \\ & 4 & & \\ & & 4 & \\ 0 & & & 4 \end{bmatrix} = I$$

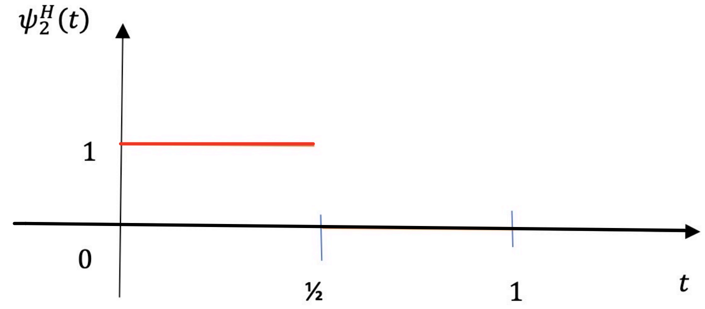
ii) $$\left(\varphi_1^H(t),\ \varphi_2^H(t),\ \varphi_3^H(t),\ \varphi_4^H(t)\right) = \overline{\varphi}(t)$$

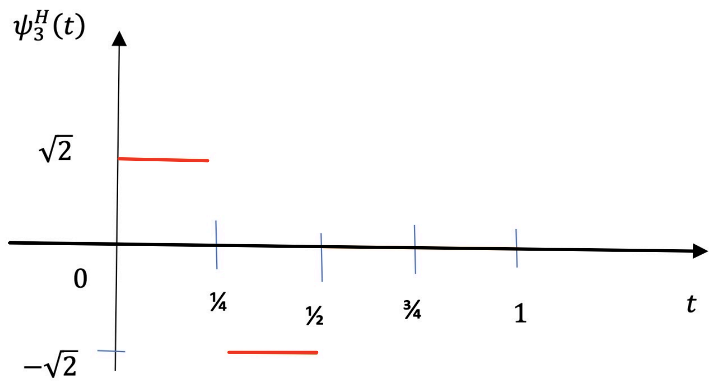
$$= \sqrt{4}\,\overline{\varphi}(t)\,(H_4) = \overline{\varphi}(t)\,(\sqrt{4})(H_4) =$$

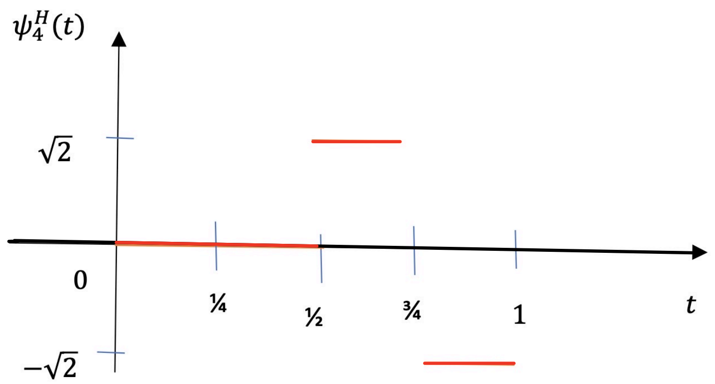
normalize

$$\begin{pmatrix} 1_{[0,1]}(t) \\ 1_{[0,\frac{1}{2}]}(t) + \left(-1_{[\frac{1}{2},1]}(t)\right) \\ \sqrt{2}_{(0,\frac{1}{4})}(t) + \left(-\sqrt{2}_{(\frac{1}{4},\frac{1}{2})}(t)\right) + 0_{(0,\frac{1}{2})}(t) \\ 1(t) + \sqrt{2}_{(\frac{1}{2},\frac{3}{4})}(t) - \sqrt{2}_{[\frac{3}{4},1]}(t) \end{pmatrix}$$




iii)

$$\int_0^1 (\phi(t) - \tilde{\phi}(t))^2 dt = \int_0^1 \phi^2(t)\, dt - 2\int_0^1 \phi(t)\tilde{\phi}(t)\, dt + \int_0^1 \tilde{\phi}^2(t)\, dt$$

$$\overset{①}{=} \int_0^1 \phi^2(t)\, dt - \sum_{i=1}^{4} (\phi_i^{o(t)})\, dt =$$

$$\overset{②}{=} \int_0^1 (a + b\cos(2\pi t) + c\cdot\cos^2(\pi t))^2 dt - \sum_{i=1}^{4} (\langle \phi(t), \psi_i^H(t)\rangle)^2 dt$$

$$= \int_0^1 a^2 + 2ab\cos(2\pi t) + 2ac\cdot\cos^2(\pi t) + b^2\cos^2(2\pi t) + 2bc\cdot\cos(2\pi t)\cos^2(\pi t) + c^2\cos^4(\pi t)\, dt$$

$$= a^2\int_0^1 dt + 2ab\int_0^1 \cos(2\pi t) + 2ac\int_0^1 \cos^2(\pi t)\, dt + b^2\int_0^1 \cos^2(2\pi t)\, dt$$

$$+ 2bc\int_0^1 \cos(2\pi t)\cos^2(\pi t)\, dt + c^2\int_0^1 \cos^4(\pi t)\, dt$$

$$= \frac{3c^2}{8} + \frac{b^2 + bc}{2} + ac + 0 + a^2$$

$$= \left(a + \frac{c}{2}\right)^2 \pm \left(\frac{\sqrt{2}}{\pi}\left(b + \frac{c}{2}\right)\right)^2 =$$

$$= a^2 + ac + c\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4b^2}{\pi^2} + \frac{4bc}{\pi^2}$$

Finally:

$$MSE = a^2 + ac + \frac{b^2 + bc}{2} + \frac{3c^2}{8}\left(a^2 + ac + c^2\right.$$

$$a^2 + ac + \frac{b^2 + bc}{2} + \frac{3c^2}{8}\left(a^2 + ac + c^2\left(\frac{1}{4} + \frac{1}{\pi^2}\right) + \frac{4b^2}{\pi^2} + \frac{4bc}{\pi^2}\right)$$

$$= \left(b\left(\frac{1}{2} - \frac{4}{\pi^2}\right)\right) \cdot (b + c) + c^2\left(\frac{1}{8} - \frac{1}{\pi^2}\right)$$

that baise on what we saw in the

eyeorial of

$$\phi(t) = \sum_{i=1}^{4} \langle \phi(t), \psi_i^H(t) \rangle \psi_i^H(t)$$

1) $\langle \phi(t), \psi_4^H(t) \rangle = a + c\left(\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_0^1 = a + \frac{c}{2}$

2) $\langle \phi(t), \psi_2^H(t) \rangle = \left(\frac{a}{2} + c\left[\frac{\sin(2\pi t)}{4\pi} + \frac{t}{2}\right]_0^{\frac{1}{2}}\right) - \left(\frac{a}{2} + c\left[\frac{\sin(2\pi t)}{4} + \frac{t}{2}\right]_{\frac{1}{2}}^1\right) = \frac{a-a}{2} + \frac{c-c}{4} = 0$

iv)

$$(\phi_i^{opt})^2 = (\langle \phi(t), \psi_{i(t)}^{w} \rangle)^2 \text{ in}$$

$$(\phi_i^{opt})^2 = (\langle \phi(t), \psi_{i(t)}^{w} \rangle)^2$$

First we will sort the squared coefficients into

$$|\phi_1^{opt}| = a + \frac{c}{2} > |\phi_3^{opt}| = \frac{2}{\pi}\left(b + \frac{c}{2}\right) > |\phi_4^{opt}| = 0 = |\phi_2^{opt}|$$

$$a \geq b \geq 0 \quad \text{and} \quad c \geq 0$$

for the k-term approximation MSE we will choose the first k functions with the biggest squared projections

1) best 1-term:

$$\tilde{\phi}_1(t) = (\phi_1^{opt} \cdot \psi_1(t)) = \left(a + \frac{c}{2}\right) 1_{[0,1]}(t) = 1.114\,[1]_0^{\frac{1}{4}} + 1.114\,[1]_{\frac{1}{4}}^{3}$$

2) the best 2-term approximation is the same as 3 and 4 because the basis is 4 dim so:

$$\tilde{\phi}_2(t) = \phi_1^{opt} \cdot \psi_1^{w}(t) + \phi_3^{opt} \cdot \psi_3^{w}(t)$$

$$= \left(a + \frac{c}{2} + \frac{2\left(b + \frac{c}{2}\right)}{\pi}\right)_{[0,\frac{1}{4})}(t) + \left(a + \frac{c}{2} - \frac{2\left(b + \frac{c}{2}\right)}{\pi}\right)_{(\frac{1}{4},\frac{3}{4})}$$

$$+ \left(a + \frac{c}{2} + \frac{2\left(b + \frac{c}{2}\right)}{\pi}\right)_{[\frac{3}{4},1]}$$

V)

Assuming $a = \frac{1}{\pi}$, $b=1$ and $c=\frac{3}{2}$

now the best k approximations will be:

first lets sort $(\phi_i^{opt})^2$

$$|\phi_3^{opt}| = \frac{2}{\pi}\left(b+\frac{c}{2}\right) = \frac{7}{2\pi} > \phi_1^{opt} = a+\frac{c}{2} = \frac{1}{\pi}+\frac{3}{4} > |\phi_4^{opt}| = 0 = |\phi_2^{opt}|$$

1) best 1- term

$$\tilde{\phi}_1(t) = \phi_3^{opt} \circ \psi_3^w(t) = 1.114\left(\frac{1}{4}+\frac{2}{4}+\frac{1}{4}\right) = 1.114$$

2) $\tilde{\phi}_2(t) = \phi_3^{opt} \circ \psi_3^w(t) + \phi_1^{opt} \cdot \psi_3^w(t) = 2.182\left(\frac{1}{4}+\frac{1}{4}\right) \cdot \frac{0.4578}{2}$