Introduction to Data Processing and representation

236201

HW3

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<u>Q1:</u>

a. Let's check the behavior for $k = 2$:

$$J^2 = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 \end{pmatrix}$$

As we can see, we got a matrix with one $(k-1)$ cyclic shift of the rows downward (or $n-1$ cyclic shifts of the rows upwards). Therefore, for $k \in \mathbb{N}$ we get a matrix J^k with $k-1$ cyclic shifts of the rows downward. For $k = n$ we get a matrix J^n with $n-1$ cyclic shifts of the rows downward (or one cyclic shifts of the rows upwards) which means we get:

$$J^n = \begin{pmatrix} 1 & \dots & \dots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} = I_{n \times n}$$

b. $\det(J - \lambda I) = \det \begin{pmatrix} -\lambda & \dots & \dots & 0 & 1 \\ 1 & -\lambda & \ddots & \ddots & 0 \\ 0 & 1 & -\lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -\lambda \end{pmatrix} =$

$$= -\lambda \cdot \det \begin{pmatrix} -\lambda & \dots & \dots & 0 \\ 1 & -\lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -\lambda \end{pmatrix} - 0 + 0 - 0 + \cdots +$$

$$(-1)^{n+1} \det \begin{pmatrix} 1 & -\lambda & \dots & 0 \\ 0 & 1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & -\lambda \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

By applying the determinant of upper or lower triangular matrix is the product of all the diagonal elements of the matrix we get:

$$\det(J - \lambda I) = \det \begin{pmatrix} -\lambda & \dots & \dots & 0 & 1 \\ 1 & -\lambda & \ddots & \ddots & 0 \\ 0 & 1 & -\lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -\lambda \end{pmatrix} =$$

$$= -\lambda \cdot \det\begin{pmatrix} -\lambda & \dots & \dots & 0 \\ 1 & -\lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -\lambda \end{pmatrix} + (-1)^{n+1} \det\begin{pmatrix} 1 & -\lambda & \dots & 0 \\ 0 & 1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & -\lambda \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

$$= -\lambda \cdot (-\lambda)^{n-1} + (-1)^{n+1} 1^{n-1} = (-\lambda)^n + (-1)^{n+1}$$

By solving $\det(J - \lambda I) = (-\lambda)^n + (-1)^{n+1} = 0$ we get that $\lambda^n = 1$, and by using the polar form $\lambda = re^{i\theta}$ we get:

$$r^n e^{in\theta} = 1e^{i\theta}$$

$$r^n = 1 \Rightarrow r = 1$$

$$n\theta = 0 = 2\pi k \ for\ k = 0,1,2, \dots, n-1$$

$$\Rightarrow \theta = \frac{2\pi k}{n} \ for\ k = 0,1,2, \dots, n-1$$

Since every $n \times n$ matrix has exactly n complex eigenvalue, we get that the n eigenvalues of J are given by:

$$\lambda_k = e^{i\frac{2\pi k}{n}} \ for\ k = 0,1,2, \dots, n-1$$

c. To complete the eigendecomposition of J we need to find the corresponding eigenvector of each eigenvalue found in the last section:

$$Jv_k = \lambda_k v_k \Rightarrow (J - \lambda_k I)v_k = 0$$

$$\begin{pmatrix} -\lambda_k & \dots & \dots & 0 & 1 \\ 1 & -\lambda_k & \ddots & \ddots & 0 \\ 0 & 1 & -\lambda_k & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -\lambda_k \end{pmatrix} \begin{pmatrix} v_{k_1} \\ v_{k_2} \\ v_{k_3} \\ \vdots \\ v_{k_n} \end{pmatrix} = 0$$

$$\begin{pmatrix} -\lambda_k v_{k_1} + v_{k_n} \\ -\lambda_k v_{k_2} + v_{k_1} \\ -\lambda_k v_{k_3} + v_{k_2} \\ \vdots \\ -\lambda_k v_{k_n} + v_{k_{n-1}} \end{pmatrix} = 0$$

For $v_{k_1} = 1 = \lambda_k^0$ we get:

$$\begin{pmatrix} v_{k_1} \\ v_{k_2} \\ v_{k_3} \\ \vdots \\ v_{k_n} \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda_k \end{pmatrix}$$

Thus, we get:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ e^{-i\frac{2\pi}{n}} \\ e^{-i\frac{2\pi \cdot 2}{n}} \\ \vdots \\ e^{-i\frac{2\pi \cdot (n-1)}{n}} \end{pmatrix}, v_n = \begin{pmatrix} 1 \\ e^{-i\frac{2\pi(n-1)}{n}} \\ e^{-i\frac{2\pi(n-1)\cdot 2}{n}} \\ \vdots \\ e^{-i\frac{2\pi(n-1)\cdot(n-1)}{n}} \end{pmatrix}$$

Therefore, J is diagonalizable and can be decomposed to the form

$$J = PDP^{-1}$$

Where P is a matrix which contains the eigenvectors of J in its columns and D is a diagonal matrix constructed from the corresponding eigenvalues of J.

We can diagonalize J in a unitary basis, by normalizing the matrix P such that $U=\frac{1}{\sqrt{n}}P$ (eigenvectors are up to multiplication by scalar), U is symmetric (because P is symmetric) and unitary since its rows and columns provide:

$$\sum_{l=0}^{n-1} U^{kl}(U^{rl})^* = \sum_{l=0}^{n-1} U^{(k-r)l} = \frac{1}{n}\sum_{l=0}^{n-1} e^{-\frac{i2\pi}{n}(k-r)l} = \begin{cases}1 & k=r\\ 0 & k\neq r\end{cases}$$

Thus, we get:

$$U^*JU = \begin{pmatrix} \lambda_0 & \cdots & \cdots & 0 & 0 \\ 0 & \lambda_1 & \ddots & \ddots & 0 \\ 0 & 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \lambda_{n-1} \end{pmatrix}$$

d. Using the results from section a we get:

$$H = h_1J + h_2J^2 + \cdots + h_{n-1}J^{n-1} + h_0J^n = h_0I_{n\times n} + h_1J + h_2J^2 + \cdots + h_{n-1}J^{n-1}$$
$$\Rightarrow P(J) = h_0I_{n\times n} + h_1J + h_2J^2 + \cdots + h_{n-1}J^{n-1}$$

e. Claim: if λ is an eigenvalue of a matrix A with a corresponding eigenvector v, then λ^k is an eigenvalue of matrix A^k with the same eigenvector v.
Proof: let A be a matrix with eigenvalue λ and a corresponding eigenvector v.
$A^kv = A^{k-1}(Av) = A^{k-1}\lambda v = \lambda A^{k-1}v = \lambda A^{k-2}(Av) = \lambda^2A^{k-2}v = \cdots = \lambda^k v$
Let's compute the eigenvalues of $H = P(J)$:

$$Hv_k = \left(\sum_{m=0}^{n-1} h_mJ^m\right)v_k = \sum_{m=0}^{n-1} h_m(J^mv_k) \underbrace{=}_{claim} \sum_{m=0}^{n-1} h_m(\lambda_k^m v_k)$$
$$= \left(\sum_{m=0}^{n-1} h_m\lambda_k^m\right)v_k = P(\lambda_k)v_k = \lambda_{H_k}v_k$$

Therefore, the eigenvalues λ_{H_k} of H with the corresponding eigenvector v_k are:

$$\lambda_{H_k} = P(\lambda_k) = P\left(e^{i\frac{2\pi k(k-1)}{n}}\right)$$
$$= h_1e^{i\frac{2\pi k}{n}} + h_2e^{i\frac{2\pi k\cdot 2}{n}} + \cdots + h_{n-1}e^{i\frac{2\pi k(n-1)}{n}} + h_0$$
$$for\ k = 0,1,2,\dots,n-1$$

H is diagonalizable by the same unitary matrix U we found on section c, with the eigenvalues computed above, which means it is diagonalizable in a unitary basis.

f. In the last section we saw that H is diagonalizable in unitary basis using the basis matrix U. Since U is symmetric and each one of its elements is $W^{*kt} = e^{-\frac{2\pi}{n}ikt} for\ k,r = 0,1,2,\dots,n-1$, its conjugate is the conjugate of the DFT matrix. Given that H is diagonalizable using $U = DFT$, we get:

$$H = B\Lambda B^*$$

$$H \underbrace{=}_{H\ is\ real} \overline{H} = \overline{DFT^*}\Lambda\overline{DFT} \underbrace{=}_{DFT\ is\ symmetric} \overline{\overline{DFT^T}}\Lambda\overline{DFT} \underbrace{=}_{DFT\ is\ symmetric} DFT\overline{\Lambda}DFT^*$$

That is, both the DFT and its complex conjugate can be chosen as the diagonalization basis matrix – we can choose the eigenvalues to be $\lambda_k = e^{-i\frac{2\pi k}{n}}$ for $k = 0,1,2,\dots,n-1$ or $\lambda_k = e^{i\frac{2\pi k}{n}}$ for $k = 0,1,2,\dots,n-1$).

g. Let B be the diagonalize basis matrix and let's denote $\sqrt{n}B = U$, then we get:

$$U^*HU = \Lambda$$
$$\Rightarrow (U^*HU)^T = \Lambda^{\mathrm{T}}$$
$$\xRightarrow[\substack{U,U^*\ are\\ symmetic}]{} UH^TU^* = \Lambda$$
$$\xRightarrow[\substack{multiplying\\ by\ U\ from\\ the\ right}]{} UH^T = \Lambda U$$

$$U\begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_{n-1} & h_0 & h_1 & \cdots & h_{n-2} \\ h_{n-2} & h_{n-1} & h_0 & \cdots & h_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & h_3 & \cdots & h_0 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ 1 & \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$
$$\xRightarrow[\sqrt{n}B=U]{} \sqrt{n}B\begin{pmatrix} h_0 \\ h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \end{pmatrix} = \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}$$

Thus, we get:

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n}B\begin{pmatrix} h_0 \\ h_{n-1} \\ h_{n-2} \\ \vdots \\ h_1 \end{pmatrix}$$

h. In order to show the two circulant matrices $H_1, H_2 \in \mathbb{R}^{n\times n}$ commute, we need to show that $H_1H_2 = H_2H_1$.
Let B be the diagonalize basis matrix and let's denote $\sqrt{n}B = U$, then we get:

$$U^*HU = \Lambda$$

Assuming $H_1 = U\Lambda_1U^*$, $H_2 = U\Lambda_2U^*$ we get:

$$H_1H_2 = U\Lambda_1U^*U\Lambda_2U^* \underbrace{=}_{\substack{U\ is\\ unitary}} U\Lambda_1\Lambda_2U^* \underbrace{=}_{\substack{diagonal\\ matrices\\ commute}} U\Lambda_2\Lambda_1U^* = U\Lambda_1U^*U\Lambda_2U^*$$
$$= H_2H_1$$

In section d we saw that H_1 and H_2 are given by polynomial expression of the matrix J, let's denote $H_1 = P_1(J)$, $H_2 = P_2(J)$.

Therefore,

$$H_1H_2 = P_1(J)P_2(J) = \sum_{m=0}^{n-1} h_{1_m}J^m \sum_{l=0}^{n-1} h_{2_l}J^l = \sum_{m=0}^{n-1}\sum_{l=0}^{n-1} h_{1_m}h_{2_l}J^{m+l}$$

As we saw earlier J^{m+l} is a circulant matrix, therefore $h_1h_2J^{m+l}$ is also a circulant matrix. Since a sum of two circulant matrix is also a circulant matrix, we get that H_1H_2 is also a circulant matrix.

i. Let's check the behavior for $k = 2$:

$$DFT^2 = \frac{1}{\sqrt{n}}\begin{pmatrix} w^{*0\cdot 0} & w^{*1\cdot 0} & w^{*2\cdot 0} & \cdots & w^{*(n-1)\cdot 0} \\ w^{*0\cdot 1} & w^{*1\cdot 1} & w^{*2\cdot 1} & \cdots & w^{*(n-1)\cdot 1} \\ w^{*0\cdot 2} & w^{*2\cdot 1} & w^{*2\cdot 2} & \cdots & w^{*(n-1)\cdot 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w^{*0\cdot(n-1)} & w^{*1\cdot(n-1)} & w^{*2\cdot(n-1)} & \cdots & w^{*(n-1)\cdot(n-1)} \end{pmatrix}$$

$$\cdot\frac{1}{\sqrt{n}}\begin{pmatrix} w^{*0\cdot 0} & w^{*1\cdot 0} & w^{*2\cdot 0} & \cdots & w^{*(n-1)\cdot 0} \\ w^{*0\cdot 1} & w^{*1\cdot 1} & w^{*2\cdot 1} & \cdots & w^{*(n-1)\cdot 1} \\ w^{*0\cdot 2} & w^{*2\cdot 1} & w^{*2\cdot 2} & \cdots & w^{*(n-1)\cdot 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w^{*0\cdot(n-1)} & w^{*1\cdot(n-1)} & w^{*2\cdot(n-1)} & \cdots & w^{*(n-1)\cdot(n-1)} \end{pmatrix}$$

$$\Rightarrow DFT^2_{kr} = \frac{1}{n}\sum_{l=0}^{n-1} w^{*lk}w^{*rl} = \frac{1}{n}\sum_{l=0}^{n-1} e^{-\frac{i2\pi lk}{n}}e^{-\frac{i2\pi lr}{n}} = \frac{1}{n}\sum_{l=0}^{n-1} e^{-\frac{i2\pi l(k+r)}{n}}$$

$$= \begin{cases} 1 & (k+r)\ mod\ n = 0 \\ 0 & else \end{cases}$$

$$\Rightarrow DFT^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & 1 \\ 0 & \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 1 & \ddots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

As we can see, this is the vertically flipped form of J. Let's denote it by J_v.

For $k = 4$ we get:

$$DFT^4 = DFT^2 \cdot DFT^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & 1 \\ 0 & \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 1 & \ddots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & 1 \\ 0 & \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 1 & \ddots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

To calculate DFT^3 we use the fact that $DFT \cdot DFT^* = I$, thus, we get:

$$DFT^3 = DFT^3 \cdot I = DFT^3 \cdot DFT \cdot DFT^* = DFT^4 \cdot DFT^* = DFT^*$$

In conclusion we get:

$$DFT^k = \begin{cases} I_{n\times n} & k\ mod\ 4 = 0 \\ DFT & k\ mod\ 4 = 1 \\ J_v & k\ mod\ 4 = 2 \\ DFT^* & k\ mod\ 4 = 3 \end{cases}$$

j. The convolution of x and y can be computed by using a circulant matrix H_x (same as H we saw earlier but with the elements of x) multiplied by y:

$$z = x \otimes y = H_x y$$

$$\Rightarrow (DFT)z = (DFT)H_x y \underbrace{=}_{DFT^*DFT=I} (DFT)H_x(DFT^*)(DFT)y$$

As we saw earlier, $(DFT)H_x(DFT^*)$ is a diagonal matrix and hence symmetric.

$$\Rightarrow (DFT)z = (DFT)H_x(DFT^*)(DFT)y = (DFT^*)H_x(DFT) \cdot (DFT)y$$

Using the results from section g, we get:

$$\sqrt{n}(DFT)y = \begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix} \Rightarrow (DFT)y = \frac{1}{\sqrt{n}} \begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix}$$

$$\Rightarrow (DFT)z = (DFT^*)H_x(DFT)(DFT)y$$

$$= \begin{pmatrix} \lambda_0^x & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1^x & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2^x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1}^x \end{pmatrix} \cdot \frac{1}{\sqrt{n}} \begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_0^x \\ \lambda_1^x \\ \lambda_2^x \\ \vdots \\ \lambda_{n-1}^x \end{pmatrix} \odot \frac{1}{\sqrt{n}} \begin{pmatrix} \lambda_0^y \\ \lambda_1^y \\ \lambda_2^y \\ \vdots \\ \lambda_{n-1}^y \end{pmatrix} = \sqrt{n}(DFT)x \odot (DFT)y$$

Q2:

a. $$\phi^F = \sqrt{2N} \begin{pmatrix} \sum_{k=0}^{2N-1} W^{*k\cdot 0} \cdot \phi(k) \\ \sum_{k=0}^{2N-1} W^{*k\cdot 1} \cdot \phi(k) \\ \sum_{k=0}^{2N-1} W^{*k\cdot 2} \cdot \phi(k) \\ \vdots \\ \sum_{k=0}^{2N-1} W^{*k\cdot(2N-1)} \cdot \phi(k) \end{pmatrix} =$$

$$\sqrt{2N}\begin{pmatrix} W^{*0\cdot 0}+\frac{W^{*0}}{2}+\frac{W^{*(2N-1)\cdot 0}}{2} \\ W^{*0\cdot 1}+\frac{W^{*1}}{2}+\frac{W^{*(2N-1)\cdot 1}}{2} \\ W^{*0\cdot 2}+\frac{W^{*2}}{2}+\frac{W^{*(2N-1)\cdot 2}}{2} \\ \vdots \\ W^{*0\cdot(2N-1)}+\frac{W^{*(2N-1)}}{2}+\frac{W^{*(2N-1)\cdot(2N-1)}}{2} \end{pmatrix} =$$

$$\sqrt{2N}\begin{pmatrix} 1+\frac{W^{*0}}{2}+\frac{W^{0}}{2} \\ 1+\frac{W^{*1}}{2}+\frac{W^{1}}{2} \\ 1+\frac{W^{*2}}{2}+\frac{W^{2}}{2} \\ \vdots \\ 1+\frac{W^{*(2N-1)}}{2}+\frac{W^{(2N-1)}}{2} \end{pmatrix}$$

b. For $k \in [0,\dots,2N-1]$:

$$\gamma^F(k) = \frac{1}{\sqrt{2N}}\cdot\sum_{n=0}^{2N-1} W_{2N}^{*n\cdot k}\cdot\gamma_n = \frac{1}{\sqrt{2N}}\cdot\sum_{n=0}^{N-1} W_{2N}^{*(2n)\cdot k}\cdot\psi_n =$$

$$\frac{1}{\sqrt{2N}}\cdot\sum_{n=0}^{N-1} W_{N}^{*n\cdot k}\cdot\psi_n = \frac{1}{\sqrt{2}}\cdot\psi^F\big(k\ (mod\ N)\big)$$

$$\Rightarrow \gamma^F = \frac{1}{\sqrt{2}}\cdot\left[\psi_0^F,\psi_1^F,\dots,\psi_{N-1}^F,\psi_0^F,\psi_1^F,\dots,\psi_{N-1}^F\right]^T$$

c. For $k \in [0,\dots,2N-1]$:

$$(\gamma*\phi)(k) = \sum_{m=0}^{2N-1}\phi(m)\cdot\gamma(k-m) =$$

$$\phi(0)\cdot\gamma(k-0)+\phi(1)\cdot\gamma(k-1)+\phi(2N-1)\cdot\gamma(k-2N+1) =$$

$$\gamma(k)+\frac{\gamma(k-1)}{2}+\frac{\gamma(k-2N+1)}{2} =$$

$$\gamma(k)+\frac{\gamma(k-1)}{2}+\frac{\gamma(k+1)}{2} = \begin{cases} \psi\left(\frac{k}{2}\right), & k\ is\ even \\ \frac{\psi\left(\frac{k-1}{2}\right)}{2}+\frac{\psi\left(\frac{k+1}{2}\right)}{2}, & k\ is\ odd \end{cases}$$

d. For $k \in [0,\dots,2N-1]$:

$$h^F(k) = \frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{2N-1} W_{2N}^{*n\cdot k} \cdot h_n =$$

$$\frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{N-1} W_{2N}^{*(2n)\cdot k} \cdot \psi_n + \frac{1}{\sqrt{2N}} \cdot \sum_{n=0}^{N-1} W_{2N}^{*(2n+1)\cdot k} \cdot \frac{\psi_n + \psi_{n+1}}{2} =$$

$$\frac{1}{\sqrt{2}} \cdot \psi^F\big(k\ (mod\ N)\big) + \frac{1}{\sqrt{8N}} \cdot \left(\sum_{n=0}^{N-1} W_{2N}^{*(2n+1)\cdot k} \cdot \psi_n + \cdot \sum_{n=0}^{N-1} W_{2N}^{*(2n+1)\cdot k} \cdot \psi_{n+1}\right) =$$

$$\frac{1}{\sqrt{2}} \cdot \psi^F\big(k\ (mod\ N)\big) + \frac{1}{\sqrt{8N}} \cdot \left(W_{2N}^{*k} \sum_{n=0}^{N-1} W_{2N}^{*(2n)\cdot k} \cdot \psi_n + \frac{1}{W_{2N}^{*k}} \sum_{n=0}^{N-1} W_{2N}^{*(2n)\cdot k} \cdot \psi_n\right) =$$

$$\frac{1}{\sqrt{2}} \cdot \psi^F\big(k\ (mod\ N)\big) + \frac{W_{2N}^{*k} + \frac{1}{W_{2N}^{*k}}}{\sqrt{8}} \cdot \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} W_{2N}^{*(2n)\cdot k} \cdot \psi_n\right) =$$

$$\frac{W_{2N}^{*k} + W_{2N}^{k} + 2}{\sqrt{8}} \cdot \psi^F\big(k\ (mod\ N)\big)$$