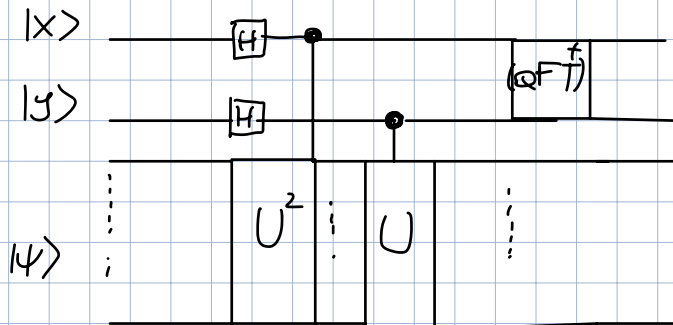


1) a)

$$\text{size}(S) = \underset{| \psi \rangle}{2^n} \cdot \underset{| \chi \rangle}{2} \cdot \underset{| \chi \rangle}{2} = 2^{n+2}$$

$$S = 2^{2+n} \times 2^{2+n}$$



b) $(H^{\otimes 2} \otimes I^{\otimes n}) (|0\rangle \otimes |0\rangle \otimes \psi) = H|0\rangle \otimes H|0\rangle \otimes \psi$

$$= \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |\psi\rangle$$

$$U^x \rightarrow \frac{1}{2} (|00\rangle \psi + |01\rangle \psi) + e^{4\pi i \varphi} |10\rangle \psi + e^{6\pi i \varphi} |11\rangle \psi$$

$$U^y \rightarrow \frac{1}{2} (|00\rangle \psi) + e^{2\pi i \varphi} |01\rangle \psi + e^{4\pi i \varphi} |10\rangle \psi + e^{6\pi i \varphi} |11\rangle \psi$$

c)

$$QFT |\psi_1 \psi_0\rangle =$$

$$\begin{aligned} & + \frac{1}{2} (|0\rangle \otimes |0\rangle \otimes e^{2\pi i (2\psi_1 + \psi_0)}) \\ & + \frac{1}{2} (|0\rangle \otimes |1\rangle \otimes e^{2\pi i (2\psi_1 + \psi_0 + 2)}) \\ & + \frac{1}{2} (|1\rangle \otimes |0\rangle \otimes e^{2\pi i (2\psi_1 + \psi_0 + 2)}) \\ & + \frac{1}{2} (|1\rangle \otimes |1\rangle \otimes e^{2\pi i (2\psi_1 + \psi_0 + 3)}) \end{aligned} = \frac{1}{2} \sum_{y_1, y_0 \in \{0,1\}} e^{\frac{2\pi i}{4} (2\psi_1 + \psi_0) (2y_1 + y_0)} |y_1 y_0\rangle \otimes |\psi\rangle = QFT |\psi_1 \psi_0\rangle \otimes |\psi\rangle$$

$$\Rightarrow |\psi_1\rangle \otimes |\psi_0\rangle \otimes |\psi\rangle$$

$$\downarrow$$

$$(QFT)^\dagger (QFT) = I$$

d) The output of the first two qubits
is $|\psi_1 \psi_0\rangle$, we can measure them and
then calculate $\psi = \frac{\psi_0}{2} + \frac{\psi_1}{2}$ by got the phase.

$$2) \quad a) \quad M \in U \Leftrightarrow M M^\dagger = I = M^\dagger M$$

$$\underline{V_f}: \quad \langle y | V_f | x \rangle = e^{-\frac{2\pi i}{M} f(x)} \langle y | x \rangle = \begin{cases} e^{-\frac{2\pi i}{M} f(x)} & y=x \\ 0 & x \neq y \end{cases} \quad (*)$$

$$\langle y | QFT | x \rangle = \frac{1}{\sqrt{M}} \sum_{y'=0}^{M-1} e^{\frac{2\pi i}{M} y' x} \langle y | y' \rangle \quad \text{According to } (*) \quad \int_{\mathbb{Z}_M} y' = I$$

QFT:

$$\langle y | QFT | x \rangle = \frac{1}{\sqrt{M}} \sum_{y'=0}^{M-1} e^{\frac{2\pi i}{M} (x y - x' y)} = \begin{cases} 1 & x = x' \\ 0 & \text{else} \end{cases}$$

$$\text{Then for } (QFT)(QFT)^\dagger = (QFT)^\dagger(QFT) = I$$

b)

$$\text{define: } |x_1 x_2 \dots x_n| \xrightarrow{1 \leq n \in \mathbb{N}} \forall m \in \mathbb{N}, \exists x_1 \dots x_n \in \{0,1\}: m = x_1 2^0 + \dots + x_n 2^{n-1}$$

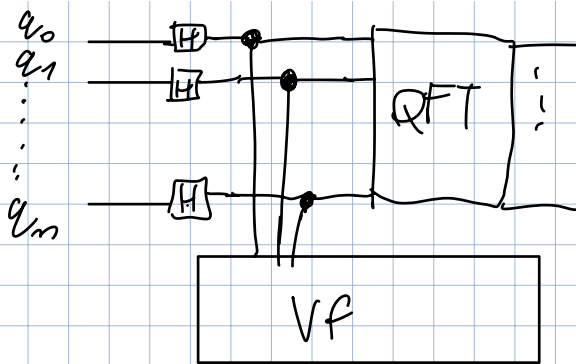
Present natural number as vector of bits

$$\text{for example } x=3 \rightarrow x = |0\rangle \dots \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle$$

$$(1)2^1 + (1)2^0 = 3$$

else for to represent the identity $I_M = (QFT)(V_f)H \otimes M|0\rangle$

We need n qubits since $M = 2^n$



$$c) H \otimes m |0\rangle \xrightarrow{H} \frac{1}{\sqrt{m}} \sum_x^{m-1} |x\rangle$$

$$\xrightarrow{Vf} \frac{1}{\sqrt{m}} \sum_x^{m-1} e^{(-2\pi i f(x) \frac{1}{m})} |x\rangle$$

$$\xrightarrow{QFT} \frac{1}{m} \sum_x^{m-1} \sum_y^{m-1} e^{(-2\pi i f(x) \frac{1}{m})} e^{(2\pi i x y) \frac{1}{m}} |y\rangle = |\psi\rangle$$

d)

$f(x) = A + B \bmod m$, in computational base we will get:

$$\frac{1}{\sqrt{m}} \sum_{x=0}^{m-1} e^{\frac{-2\pi i}{m} Ax} e^{\frac{-2\pi i}{m} B} e^{\frac{2\pi i}{m} xy} = \frac{1}{\sqrt{m}} \sum_{x=0}^{m-1} e^{\frac{2\pi i}{m} (y-A)x}$$

The probability to measure $|y\rangle$ is:

$$P(y) = \frac{1}{m} \left| \sum_{x=0}^{m-1} e^{\frac{2\pi i}{m} (y-A)x} \right|^2$$

So one measurement is sufficient to find A

because $P(y \neq A) = 0$ and $P(y = A) = 1$

3) a)

$$H \rightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle (-1)^x) \otimes |y\rangle \xrightarrow{CU} \frac{1}{\sqrt{2}} (|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y\rangle)$$

$$V \rightarrow \frac{1}{\sqrt{2}} (|0\rangle \otimes V|y\rangle + (-1)^x |1\rangle \otimes VU|y\rangle)$$

b)

$$\begin{aligned} P_{\text{prob}}(c_1, c_2) &= \frac{1}{2} | \langle c_1 c_2 | 1 \rangle (-1)^x \otimes VU|y\rangle + \langle c_1 c_2 | 0 \rangle \otimes V|y\rangle |^2 \\ &= \frac{1}{2} | \langle c_1 1 | \times \langle c_2 | VU|y\rangle (-1)^x + \langle c_1 0 | \times \langle c_2 | V|y\rangle |^2 \\ &= \begin{cases} \frac{1}{2} | \langle c_2 | V|y\rangle |^2 & c_1 = 0 \\ \frac{1}{2} | \langle c_2 | VU|y\rangle |^2 & c_1 = 1 \end{cases} \end{aligned}$$

c)

$$P(c_1=0) = \frac{1}{2} = P(c_1=1)$$

$$P_{c_1 c_2}(0,0) = |\langle 0|0\rangle|^2 \cdot |\langle 0|V|y\rangle|^2 = |\langle 0|V|y\rangle|^2$$

$$P(0,1) = |\langle 0|0\rangle|^2 \cdot |\langle 1|V|y\rangle|^2 = |\langle 1|V|y\rangle|^2$$

$$P(1,0) = |\langle 0|VU|y\rangle|^2$$

$$P(1,1) = |\langle 1|VU|y\rangle|^2$$

$$P(c_1=0, c_2) = \frac{1}{2} | \langle c_2 | V|y\rangle |^2$$

$$P(c_1=1, c_2) = \frac{1}{2} | \langle c_2 | VU|y\rangle |^2$$

$$d) |0\rangle \xrightarrow{H} (|0\rangle + |1\rangle) \frac{1}{\sqrt{2}} \xrightarrow{W} (|1\rangle - |0\rangle) \frac{1}{\sqrt{2}} \xrightarrow{H} (|0\rangle + |1\rangle) \frac{1}{\sqrt{2}}$$

so the state before measure is:

$$(|1\rangle - |0\rangle) \otimes (|0\rangle + |1\rangle) \left(\frac{1}{\sqrt{2}}\right)^2 = (|10\rangle - |00\rangle + |11\rangle - |01\rangle) \frac{1}{2}$$

$$P_{(0,0)}^{C_1 C_2} = |\langle 00 | 10 \rangle \frac{1}{2} - \langle 00 | 00 \rangle + \langle 00 | 11 \rangle - \langle 00 | 01 \rangle|^2 = \frac{1}{4}$$

$$P_{(0,1)} = |\langle 01 | 10 \rangle \frac{1}{2} - \langle 01 | 00 \rangle + \langle 01 | 11 \rangle - \langle 01 | 01 \rangle|^2 = \frac{1}{4}$$

$$P_{(1,0)} = |\langle 10 | 10 \rangle \frac{1}{2} - \langle 10 | 00 \rangle + \langle 10 | 11 \rangle - \langle 10 | 01 \rangle|^2 = \frac{1}{4}$$

$$P_{(1,1)} = |\langle 11 | 10 \rangle \frac{1}{2} - \langle 11 | 00 \rangle + \langle 11 | 11 \rangle - \langle 11 | 01 \rangle|^2 = \frac{1}{4}$$

3)e)