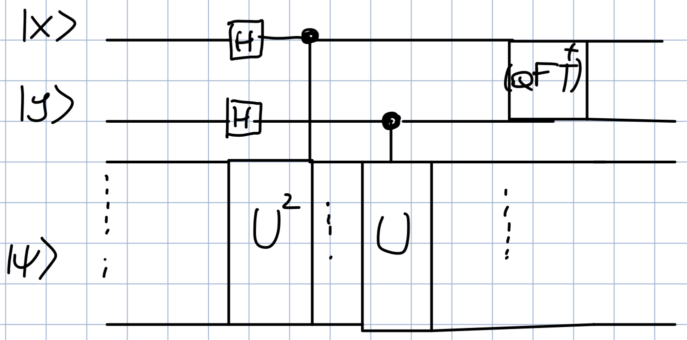
1) a)

$$\text{size}(S) = \underset{|\psi\rangle}{2^n} \cdot \underset{|y\rangle}{2} \cdot \underset{|x\rangle}{2} = 2^{n+2}$$

$$S = 2^{2+n} \times 2^{2+n}$$



b)

$$(H^{\times 2} \otimes I^{\times n})(|0\rangle \otimes |0\rangle \otimes \psi) = H|0\rangle \otimes H|0\rangle \otimes \psi$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |\psi\rangle$$

$$\xrightarrow{U^{2x}} \frac{1}{2}(|00\psi\rangle + |01\psi\rangle + e^{4\pi i\varphi}|10\psi\rangle + e^{6\pi i\varphi}|11\psi\rangle)$$

$$U^y \rightarrow \frac{1}{2}(|00\psi\rangle + e^{2\pi i\varphi}|01\psi\rangle + e^{4\pi i\varphi}|10\psi\rangle + e^{6\pi i\varphi}|11\psi\rangle)$$

c)

$$QFT|\psi_1\psi_0\rangle =$$

$$+\frac{1}{2}(|0\rangle \otimes |0\rangle \otimes e^{2\pi i(2\psi_1+\psi_0)})$$
$$+\frac{1}{2}(|0\rangle \otimes |1\rangle \otimes e^{2\pi i(2\psi_1+\psi_0+2)}) = \frac{1}{2}\sum_{y_1 y_0 \in \{0,1\}} e^{\frac{2\pi i}{4}(2\psi_1+\psi_0)(2y_1+y_0)} |y_1 y_0\rangle \otimes |\psi\rangle = QFT|\psi_1\psi_0\rangle \otimes |\psi\rangle$$
$$+\frac{1}{2}|1\rangle \otimes |0\rangle \otimes e^{2\pi i(2\psi_1+\psi_0+2)})$$
$$+\frac{1}{2}(|1\rangle \otimes |1\rangle) \otimes e^{2\pi i(2\psi_1+\psi_0+3)})$$

$$\Rightarrow |\psi_1\rangle \otimes |\psi_0\rangle \otimes |\psi$$

$$\downarrow$$

$$(QFT)^{\dagger}(QFT) = I$$

d) The output of the first two qubits is $|\varphi_1 \varphi_0\rangle$, we can measure them and then calculate $\varphi = \frac{\varphi_0}{4} + \frac{\varphi_1}{2}$ by got the phase.

2) a) $M \in U \leftrightarrow MM^\dagger = I = M^\dagger M$

<u>V_f</u>:

$$\langle y|V_f|x\rangle = e^{-\frac{2\pi i}{M}f(x)}\langle y|x\rangle = \begin{cases} e^{-\frac{2\pi i}{M}f(x)} & y=x \\ 0 & x\neq y \end{cases} \quad (*)$$

$$\langle y|QFT|x\rangle = \frac{1}{\sqrt{M}}\sum_{y'=0}^{M-1} e^{\frac{2\pi i}{M}y'x}\langle y|y'\rangle \underset{\text{According } (*)}{=} \delta_{yy'} = I$$

<u>QFT</u>:

$$\langle y|QFT|x\rangle = \frac{1}{\sqrt{M}}\sum_{y'=0}^{M-1} e^{\frac{2\pi i}{M}(xy-x'y)} = \begin{cases} 1 & x=x' \\ 0 & \text{else} \end{cases}$$

Ther for $(QFT)(QFT)^\dagger = (QFT)^\dagger(QFT) = I$

b)

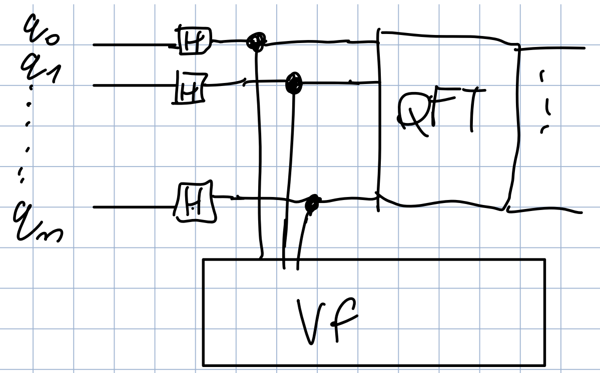
define: $\underset{2^0\,2^1\,\dots\,2^n}{|x_1 x_2 \dots x_n|} \longrightarrow \forall m \in \mathbb{N}, \exists x_1 .. x_n \overset{\text{linear}}{\in} \{0,1\}: m = x_1 2^0 \dots + x_n 2^{n-1}$

Present natural number as vector of bits

for example $x=3 \rightarrow x = |0\rangle \dots \otimes |0\rangle \otimes \underset{(1)2^1+(1)2^0=3}{|1\rangle \otimes |1\rangle}$

there for to represent the identity $|\psi\rangle = (QFT)(V_f)H^{\otimes m}|0\rangle$

we need m qubits since $M = 2^m$



c) $H^{\otimes n}|0\rangle \xrightarrow{H} \frac{1}{\sqrt{n}} \sum_{x}^{M-1} |x\rangle$

$$\xrightarrow{Vf} \frac{1}{\sqrt{n}} \sum_{x}^{M-1} e^{(-2\pi i\, f(x))\frac{1}{M}} |x\rangle$$

$$\xrightarrow{QFT} \frac{1}{M} \sum_{x}^{M-1} \sum_{y}^{M-1} e^{(-2\pi i\, f(x))\frac{1}{M})}\, e^{(2\pi i\, xy)\frac{1}{M}} |y\rangle = |\psi\rangle$$

d)

$f(x) = A + B \bmod M$, in compitational base we will get:

$$\frac{1}{\sqrt{m}} \sum_{x=0}^{M-1} e^{\frac{-2\pi}{M} Ax}\, e^{-\frac{2\pi}{M} B}\, e^{\frac{2\pi i}{M} xy} = \frac{1}{\sqrt{m}} \sum_{x=0}^{M-1} e^{\frac{2\pi i}{M}(y-A)x}$$

the probability to measere $|y\rangle$ is:

$$P(y) = \frac{1}{M} \left| \sum_{x=0}^{M-1} e^{\frac{2\pi i}{M}(y-A)} \right|^2$$

so one measurement is sutesfy to find A

because $P(y \neq A) = 0$ and $P(y = A) = 1$

3) a)

$$\xrightarrow{H} \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle(-1)^x\right) \otimes |y\rangle \xrightarrow{CU} \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle U |y\rangle\right)$$

$$\xrightarrow{V} \frac{1}{\sqrt{2}}\left(|0\rangle \otimes V|y\rangle + (-1)^x |1\rangle VU|y\rangle\right)$$

b)

$$P_{rob}(c_1, c_2) = \frac{1}{2} \left| \langle c_1 c_2 | 1\rangle(-1)^x \otimes VU|y\rangle + \langle c_1 c_2 |0\rangle \otimes V|y\rangle \right|^2$$

$$= \frac{1}{2}\left|\langle c_1|1\rangle\langle c_2|VU|y\rangle(-1)^x + \langle c_1|0\rangle\langle c_2|V|y\rangle\right|^2$$

$$= \begin{cases} \frac{1}{2}\,|\langle c_2|V|y\rangle|^2 & c_1 = 0 \\ \frac{1}{2}\,|\langle c_2|VU|y\rangle|^2 & c_1 = 1 \end{cases}$$

c)

$$P(c_1 = 0) = \frac{1}{2} = P(c_1 = 1)$$

$$\overset{c_1 c_2}{P(0,0)} = |\langle 0|0\rangle|^2 \cdot |\langle 0|V|y\rangle|^2 = |\langle 0|V|y\rangle|^2$$

$$P(0\,1) = |\langle 0|0\rangle|^2 \cdot |\langle 1|V|y\rangle|^2 = |\langle 1|V|y\rangle|^2$$

$$P(1\,0) = |\langle 0|VU|y\rangle|^2$$

$$P(1\,1) = |\langle 1|VU|y\rangle|^2$$

$$P(c_1 = 0, c_2) = \frac{1}{2}\,|\langle c_2|V|y\rangle|^2$$

$$P(c_1 = 1, c_2) = \frac{1}{2}\,|\langle c_2|VU|y\rangle|^2$$

d) $|0\rangle \xrightarrow{H} (|0\rangle + |1\rangle)\frac{1}{\sqrt{2}} \xrightarrow{W} (|1\rangle - |0\rangle)\frac{1}{\sqrt{2}} \xrightarrow{H} (|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}$

so the state befor measure is:

$$((|1\rangle - |0\rangle) \otimes (|0\rangle + |1\rangle))\left(\frac{1}{\sqrt{2}}\right)^2 = (|10\rangle - |00\rangle + |11\rangle - |01\rangle)\frac{1}{2}$$

$p\overset{c_1\ c_2}{(0,0)} = |\langle 00|10\rangle \frac{1}{2} - |00\rangle + |11\rangle - |01\rangle|^2 = \frac{1}{4}$

$p(0,1) = |\langle 01|10\rangle \frac{1}{2} - |00\rangle + |11\rangle - |01\rangle|^2 = \frac{1}{4}$

$p(1,0) = |\langle 10|10\rangle \frac{1}{2} - |00\rangle + |11\rangle - |01\rangle|^2 = \frac{1}{4}$

$p(1,1) = |\langle 11|10\rangle \frac{1}{2} - |00\rangle + |11\rangle - |01\rangle|^2 = \frac{1}{4}$

3)e)