The area between functions $x^{\frac{1}{n}}$ and x^n in the domain $0 \leq x \leq 1$ is equal to $\frac{n-1}{n+1}$

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1 Abstract

When the graphs of functions $y = x^{\frac{1}{n}}$ and $y = x^n$ are plotted [in the domain $0 \leq x \leq 1$,range/co-domain $0 \leq y \leq 1$] then an area is carved between the two functions by the virtue of intersection of the two graphs (at 0 and 1). If we consider the area formed by the unit square [$0 \leq x \leq 1$ and $0 \leq y \leq 1$] to be 100%, then the above mentioned 'carved out area due to intersection' is a function of 'n' and grows as the value of 'n' grows, asymptotically reaching 100% for n=∞. The relationship between the 'carved area' and 'n' is simply: $\Delta = (\frac{n-1}{n+1} \times 100)\%$ of the total area between the unit square formed.

2 Derivation

Let us see three plots to understand the behavior of the area carved out by the intersection of two curves. Below are the curves for n=2, 3 and 10 (Figure 1). It is easy to calculate the area formed by the intersection between

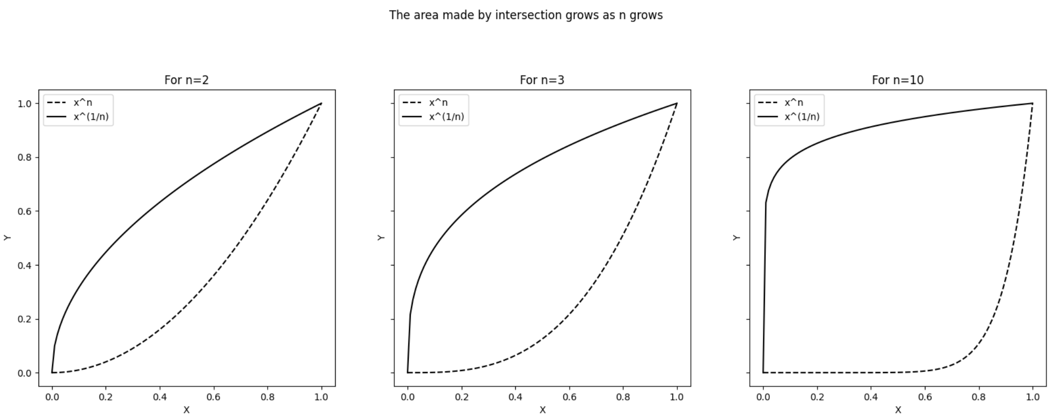

Figure 1: The area made by intersection is an increasing function of n

the two functions by going through these 3 steps.

1. Calculate the area under the function $x^{\frac{1}{n}}$.
2. Calculate the area under the function $y = x^n$.
3. Subtract the second step from the first step.

Thus we can calculate the area as below.

$$\begin{aligned}
\Delta &= \int_0^1 x^{\frac{1}{n}}\,dx - \int_0^1 x^n\,dx \\
&= \frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}\Big|_0^1 - \frac{x^{n+1}}{n+1}\Big|_0^1 \\
&= \left[\frac{1^{\frac{1}{n}+1}}{\frac{1}{n}+1} - \frac{0^{\frac{1}{n}+1}}{\frac{1}{n}+1}\right] - \left[\frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}\right] \\
&= \frac{1}{\frac{1}{n}+1} - \frac{1}{n+1} \\
&= \frac{n}{n+1} - \frac{1}{n+1} \\
&= \frac{n-1}{n+1}
\end{aligned}$$

For $n = 1$ there should be no area while for $n = \infty$ the area should cover the whole square intuitively. As a sanity check we can calculate the area for $n = 1$ and $n = \infty$ and we find the results to be 0 and 1 respectively, as expected.

3 Discussion

Below is a table for different values of n (from 1 to 100) for the equation $y = \frac{n-1}{n+1}$.

n	**fraction**
1	0.0000
2	0.3333
3	0.5000
4	0.6000
5	0.6667
6	0.7143
10	0.8182
25	0.9231
50	0.9608
100	0.9802

This observation can be turned into a plot as below (Figure 2).

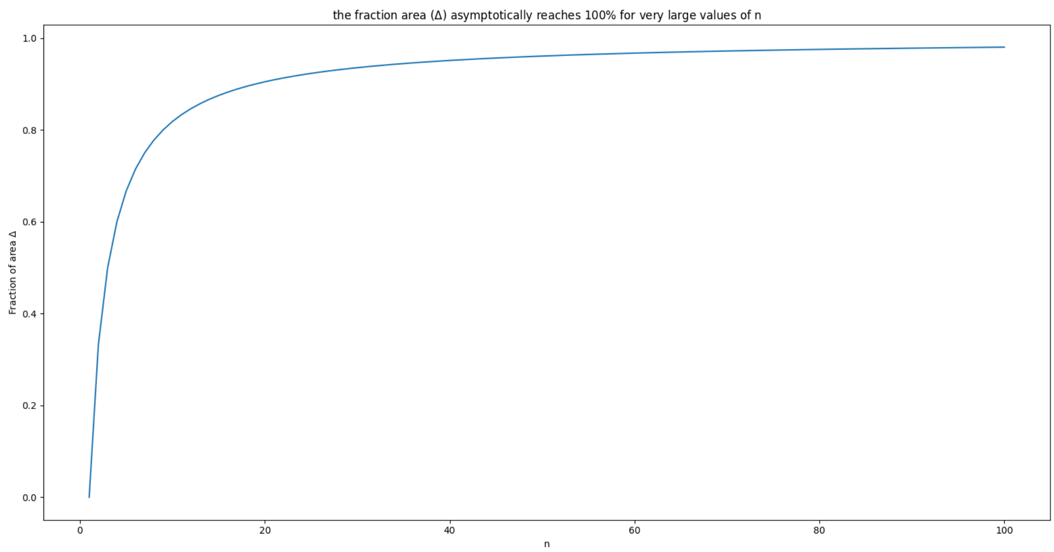

Figure 2: The fraction asymptotically reaches 100% for large values of n

There are a few interesting properties of this equation.

1. As we can see this equation $\left(y = \frac{n-1}{n+1}\right)$ is a special case of Möbius-Transformation. A Möbius-Transformation is defined as $f(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$ and z is a complex number. In this context this function is special since z is replaced by x and $a = c = d = 1 \neq b = -1$. Please see `https://math.stackexchange.com/questions/5059632/does-the-function-y-fracx-1x1-have-a-name` for some very helpful comments on this.

2. The n^{th} derivative of this function is defined as: $f'^{n}\left(\frac{x-1}{x+1}\right) = \frac{-1^{(n+1)} * 2n!}{(x+1)^{(n+1)}}$, where $1 \leq n$.

3. The fifth composition of the function with itself is the original function again [see `https://math.stackexchange.com/q/5059632/357360`].

$$
\begin{aligned}
f(n) &= \frac{n-1}{n+1} \\
f(f(n)) &= \frac{-1}{n} \\
f(f(f(n))) &= \frac{n+1}{-n+1} \\
f(f(f(f(n)))) &= n \\
f(f(f(f(f(n))))) &= \frac{n-1}{n+1}
\end{aligned}
$$