

The area between functions  $x^{\frac{1}{n}}$  and  $x^n$  in the domain  $0 \leq x \leq 1$  is equal to  $\frac{x-1}{x+1}$

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## 1 Abstract

When the graphs of functions  $y = x^{\frac{1}{n}}$  and  $y = x^n$  are plotted [in the domain  $0 \leq x \leq 1$ , range/co-domain  $0 \leq y \leq 1$ ] then an area is carved between the two functions by the virtue of intersection of the two graphs (at 0 and 1). If we consider the area formed by the unit square [ $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ ] to be 100%, then the above mentioned 'carved out area due to intersection' is a function of 'n' and grows as the value of 'n' grows, asymptotically reaching 100% for  $n=\infty$ . The relationship between the 'carved area' and 'n' is simply:  $\Delta = (\frac{x-1}{x+1} \times 100)\%$  of the total area between the unit square formed.

## 2 Derivation

Let us see three plots to understand the behavior of the area carved out by the intersection of two curves. Below are the curves for  $n=2, 3$  and  $10$  (Figure 1). It is easy to calculate the area formed by the intersection between

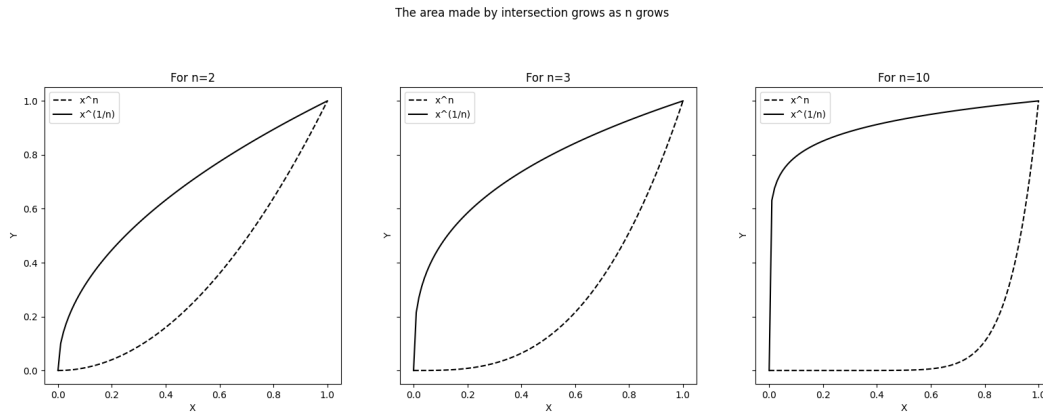


Figure 1: The area made by intersection is an increasing function of n

the two functions by going through these 3 steps.

1. Calculate the area under the function  $x^{\frac{1}{n}}$ .
2. Calculate the area under the function  $y = x^n$ .
3. Subtract the second step from the first step.

Thus we can calculate the area as below.

$$\begin{aligned}
\Delta &= \int_0^1 x^{\frac{1}{n}} dx - \int_0^1 x^n dx \\
&= \left. \frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} \right|_0^1 - \left. \frac{x^{n+1}}{n+1} \right|_0^1 \\
&= \left[ \frac{1^{\frac{1}{n}+1}}{\frac{1}{n}+1} - \frac{0^{\frac{1}{n}+1}}{\frac{1}{n}+1} \right] - \left[ \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} \right] \\
&= \frac{1}{\frac{1}{n}+1} - \frac{1}{n+1} \\
&= \frac{n}{n+1} - \frac{1}{n+1} \\
&= \frac{n-1}{n+1}
\end{aligned}$$

For  $n = 1$  there should be no area while for  $n = \infty$  the area should cover the whole square intuitively. As a sanity check we can calculate the area for  $n = 1$  and  $n = \infty$  and we find the results to be 0 and 1 respectively, as expected.

### 3 Discussion

Below is a table for different values of  $n$  (from 1 to 100).

<b>n</b>	<b>fraction</b>
1	0.0000
2	0.3333
3	0.5000
4	0.6000
5	0.6667
6	0.7143
10	0.8182
25	0.9231
50	0.9608
100	0.9802

This observation can be turned into a plot as below (Figure 2).

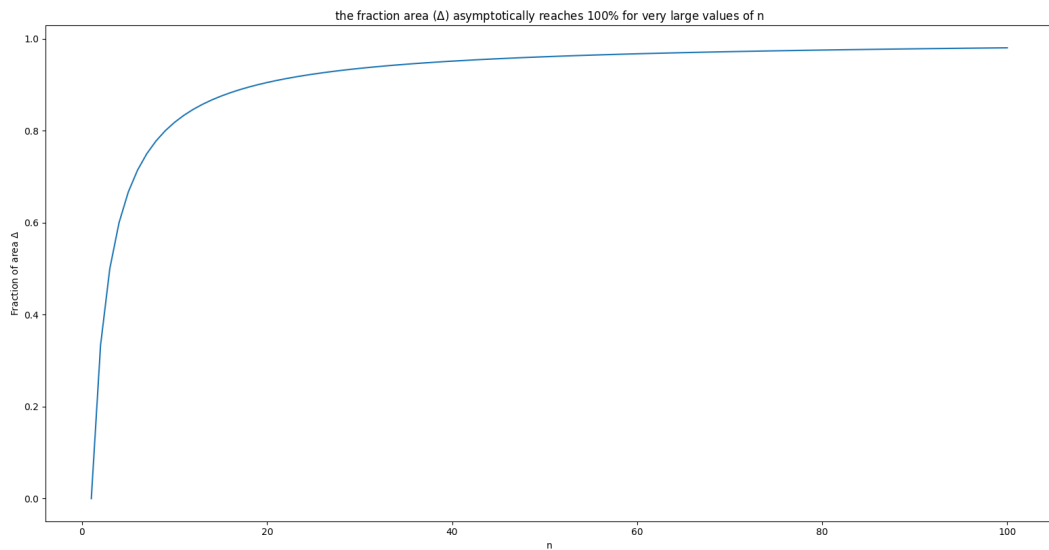


Figure 2: The fraction asymptotically reaches 100% for large values of  $n$

As we can see this equation is a special case of Möbius-Transformation. A Möbius-Transformation transformation is defined as  $f(z) = \frac{az+b}{cz+d}$ , where  $ad - bc \neq 0$ . In this context this function is special since  $z$  is replaced by  $x$  and  $a = c = d = 1 \neq b = -1$ . Please see <https://math.stackexchange.com/questions/5059632/does-the-function-y-fracx-1x1-have-a-name> for some very helpful comments on this.