Digital Signal Processing(BEC-42)

Unit-2

Lecture-8

(Frequency Transformation)

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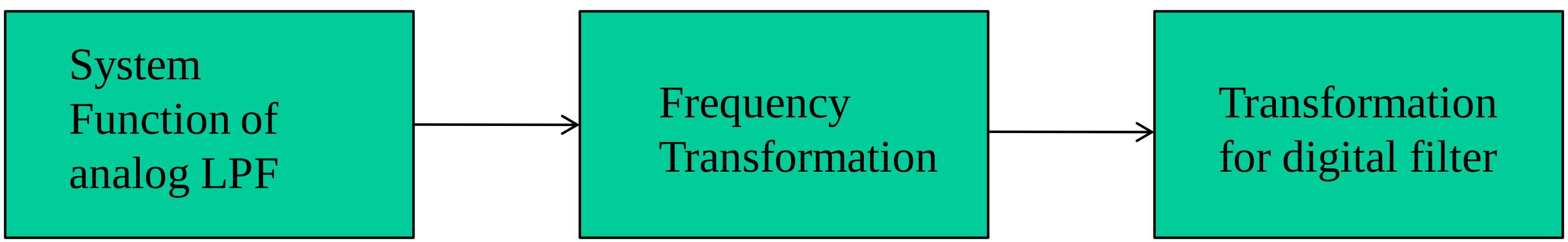
Frequency Transformation

Types of frequency selective filters:

- Low-pass
- High-pass
- Band-pass
- Stop-band



Analog Frequency Transformation

(i) Low-pass with cut-off frequency Ω_c to low-pass with a new cut-off frequency Ω_c^*

$$s \to \frac{\Omega_c}{\Omega_c^*} s$$

Thus, if the system response of the prototype filter is $H_p(s)$, the system response of the new low-pass filter will be

$$H(s) = H_p\left(\frac{\Omega_c}{\Omega_c^*} s\right)$$

(ii) Low-pass with cut-off frequency Ω_c to high-pass with cut-off frequency Ω_c^*

$$s \to \frac{\Omega_c \Omega_c^*}{s}$$

The system function of the high-pass filter is then,

$$H(s) = H_p\left(\frac{\Omega_c \Omega_c^*}{s}\right)$$

(iii) Low-pass with cut-off frequency Ω_c to band-pass with lower cut-off frequency Ω_1 and higher cut-off frequency Ω_2

$$s \rightarrow \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$$

The system function of the high-pass filter is then

$$H(s) = H_p \left(\Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)} \right)$$

(iv) Low-pass with cut-off frequency Ω_c to bandstop with lower cut-off frequency Ω_1 and higher cut-off frequency Ω_2

$$s \rightarrow \Omega_c \frac{s^2(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$$

The system function of the bandstop filter is then,

$$H(s) = H_p \left(\Omega_c \frac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2} \right)$$

Type	*Transformation*
Low-pass	$s \rightarrow \frac{\Omega_c}{\Omega_c^*} s$
High-pass	$s \rightarrow \frac{\Omega_c \Omega_c^*}{s}$
Bandpass	$s \rightarrow \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Bandstop	$s \rightarrow \Omega_c \frac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

Digital Frequency Transformation

Transformation is obtained by replacing z^{-1} by a function of z^{-1},i.e. $f(z^{-1})$

This mapping must take into account the stability criterion.

Type	*Transformation*	*Design Parameter*
Low–pass	$z^{-1} \rightarrow \dfrac{z^{-1} - a}{1 - a z^{-1}}$	$a = \dfrac{\sin[(\omega_c - \omega_c^*)/2]}{\sin[(\omega_c + \omega_c^*)/2]}$
High-pass	$z^{-1} \rightarrow -\dfrac{z^{-1} + a}{1 + a z^{-1}}$	$a = \dfrac{\cos[(\omega_c - \omega_c^*)/2]}{\cos[(\omega_c + \omega_c^*)/2]}$

Type	*Transformation*	*Design Parameter*
Bandpass	$z^{-1} \rightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a^2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$a_1 = -2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \dfrac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$ $K = \cot\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$
Bandstop	$z^{-1} \rightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	$a_1 = -2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \dfrac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$ $K = \tan\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$

Problem: Transform the prototype LPF with system function

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Into a high-pass filter with cut-off frequency Ω_c^* ..

Solution: Low-pass with cut-off frequency Ω_c to high-pass with cut-off frequency Ω_c^*

$$s \rightarrow \frac{\Omega_c \Omega_c^*}{s}$$

Thus we have,

$$H_{hpf}(s) = \frac{\Omega_c}{\left(\frac{\Omega_c \Omega_c^*}{s}\right) + \Omega_c} = \frac{s}{s + \Omega_c^*}$$

Problem: Design an BPF to satisfy the following specifications:

i. 3 dB upper and lower cut-off frequencies are 100 Hz and 3.8 kHz.
ii. Stop-band attenuation of 20 dB at 20 Hz and 8 kHz.
iii. No ripple with both pass-band and stop-band.

Solution:

Design of Butterworth filter (bandpass filter)

Given $\Omega_1 = 2\pi \times 20$ rad/sec, $\Omega_2 = 2\pi \times 8000$ rad/sec, $\Omega_l = 2\pi \times 100$ rad/sec,

$\Omega_u = 2\pi \times 3800$ rad/sec,

$\delta_1 = 3$ dB and $\delta_2 = 20$ dB

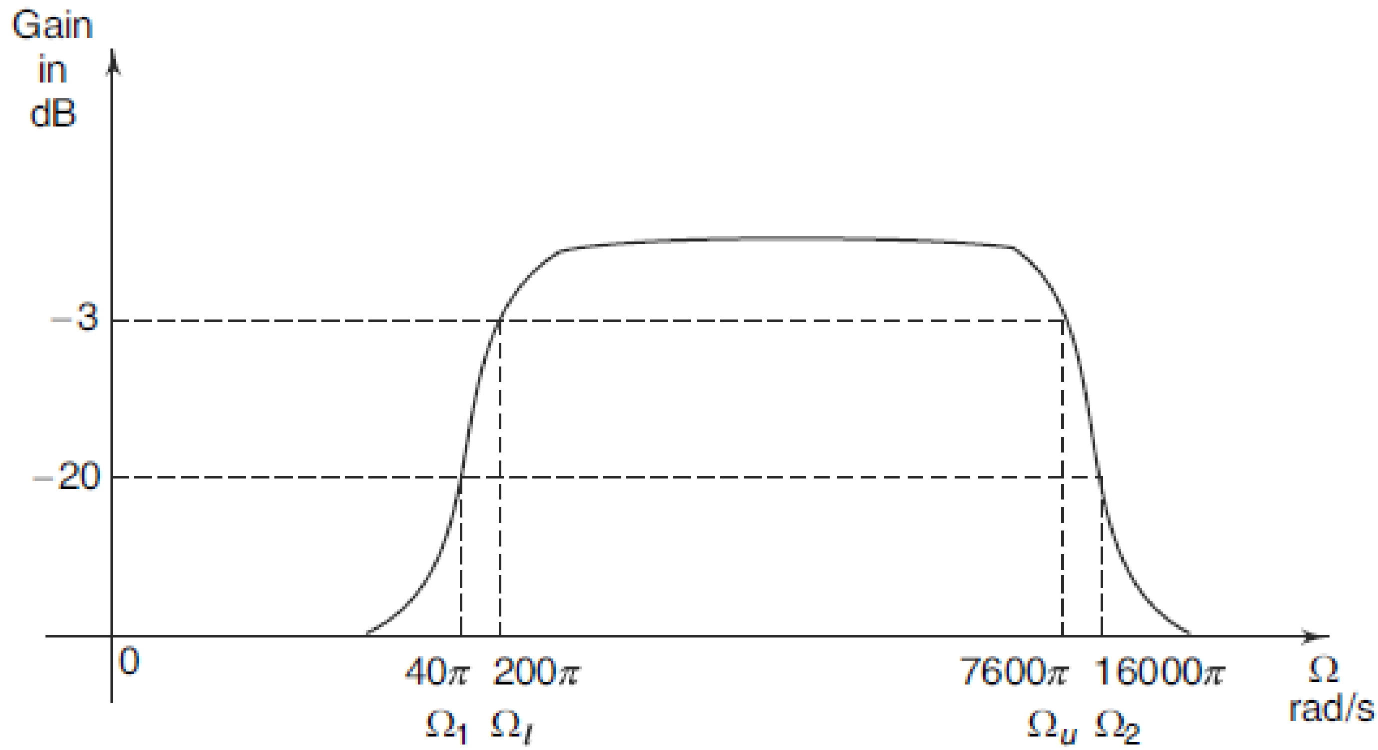
To find Ha (s)

Step I: To find the ratio Ω_s/Ω_p

We know that $$\frac{\Omega_s}{\Omega_p} = \min\left(|A|, |B|\right)$$

where $$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)} = 5.129$$

$$B = \frac{-\Omega_2^2 + \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)} = 2.149$$



Butterworth BPF

Therefore, $$\frac{\Omega_s}{\Omega_p} = 2.149$$

Step II: To find the order of filter (N):

$$N = \frac{\log_{10}\left(\dfrac{10^{0.1\delta_2}-1}{10^{0.1\delta_1}-1}\right)}{2\log_{10}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\dfrac{10^2-1}{10^{0.3}-1}\right)}{2\log_{10}(2.149)} = 3.006$$

Therefore, the order of the filter is selected as $N = 4$.

Step III: Cut-off frequency Ω_c:

$$\Omega_c = 7600\pi \text{ rad/s}$$

Step IV: To find poles:

$$s_k = \Omega_c e^{j(\pi/2+(2k+1)\pi/2N)} \quad \text{where } k = 0, 1, \ldots, N-1$$

Here, $s_k = 7600 \quad e^{j(\pi/2+(2k+1)\pi/8)}, k = 0, 1, 2$

$$s_0 = -9136.99 + j22058.6$$
$$s_1 = -22058.6 + j9136.99$$
$$s_2 = -22058.6 - j9136.99$$
$$s_3 = -9136.99 - j22058.6$$

Step V: To find $H_a(s)$:

$$H_a(s) = \frac{1}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)}$$

Here, $s_2 = s_1^*$ and $s_3 = s_0^*$

Therefore, $$H_a(s) = \frac{1}{(s-s_0)(s-s_0^*)(s-s_1)(s-s_1^*)}$$

Let $s_0 = a + jb$ and $s_1 = c + jd$

Therefore, $$H_a(s) = \frac{1}{(s^2 - 2as + a^2 + b^2)(s^2 - 2cs + c^2 + d^2)}$$

Here, $a = -9136.99$, $b = 22058.6$, $c = -22058.6$ and $d = 9136.99$

$$H_a(s) = \frac{1}{(s^2 - 2(-9136.99)s + 5.7\times 10^8)(s^2 - 2(-22058.6)s + 5.7\times 10^8)}$$

$$\frac{1}{(s^2 + 1.83\times 10^4 s) + 5.7\times 10^8)(s^2 + 4.41\times 10^4 s + 5.7\times 10^8)}$$

We know that

$$(s^2+as+b)(s^2+cs+b)=s^4+s^3(a+c)+s^2(2b+ac)+sb(a+c)+b^2$$

Therefore,

$$H_a(s)=\frac{1}{s^4+6.2\times10^4 s^3+1.947\times10^9 s^2+3.56\times10^{13} s+3.25\times10^{17}}$$

Step VI: To transform LPF to BPF:

$$s \Rightarrow \frac{s^2+\Omega_l\Omega_u}{s(\Omega_u-\Omega_l)}$$

Here, $\Omega_2 = 200\,\pi$ rad/s

$\Omega_u = 7600\,\pi$ rad/s

$$s \Rightarrow \frac{s^2+1.5\times10^7}{s(2.32\times10^4)}$$

Let $x = 1.5 \times 10^7$ and $y = 2.32 \times 10^4$

Therefore, $$s \rightarrow \frac{s^2+x}{sy}$$

$$H_a(s)=\frac{1}{s^4+ms^3+ns^2+os+p}$$

where $m = 6.24 \times 10^4$
$n = 1.947 \times 10^9$
$o = 3.56 \times 10^{13}$
$p = 3.25 \times 10^{17}$

Therefore,

$$H_a(s) = \frac{1}{\left(\dfrac{s^2+x}{sy}\right)^4 + m\left(\dfrac{s^2+x}{sy}\right)^3 + n\left(\dfrac{s^2+x}{sy}\right)^2 + o\left(\dfrac{s^2+x}{sy}\right) + p}$$

$$H_a(s) = \frac{(sy)^4}{(s^2+x)^4 + m(s^2+x)^3 sy + ns^2y^2(s^2+x)^2 + os^3y^3(s^2+x) + ps^4y^4}$$

$$= \frac{s^4y^4}{(s^8+4s^6x+4s^2x^3+6s^4x^2+x^4) + msy(s^6+3s^4x+3s^2x^2+x^3) + ns^2y^2(s^4+2s^2x+x^2) + os^5y^3 + os^3y^3x + ps^4y^4}$$

$$= \frac{s^4y^4}{(s^8+4s^6x+4s^2x^3+6s^4x^2+x^4) + ms^7y + 3ms^5xy + 3ms^3yx^2 + msyx^3 + ns^6y^2 + 2ns^4xy^2 + ns^2y^2x^2 + os^3y^3x + ps^4y^4 + os^5y^3}$$

$$== \frac{s^4 y^4}{s^8 + ms^7 y + 4s^6 x + ns^6 y^2 + 3ms^5 xy + os^5 y^3 + 6s^4 x^2 + 2ns^4 xy^2 + ps^4 y^4 + 3ms^3 yx^2 + os^3 y^3 x + 4s^2 x^3 + ns^2 y^2 x^2 + msyx^3 + x^4}$$

$$= \frac{s^4 y^4}{s^8 + (my)s^7 + (4x + ny^2)s^6 + (3mxy + oy^3)s^5 + (6x^2 + 2nxy^2 + py^4)s^4 + (3myx^2 + oy^3 x)s^3 + (4x^3 + ny^2 x^2)s^2 + myx^3 s + x^4}$$

$$H_a(s) = \frac{2.9 \times 10^{17} s^4}{s^8 + 1.45 \times 10^9 s^7 + 1.048 \times 10^{18} s^6 + 4.45 \times 10^{26} s^5 + 9.492 \times 10^{34} s^4 \; 6.67 \times 10^{33} s^3 + 2.358 \times 10^{32} s^2 + 4.89 \times 10^{30} s + 5.06 \times 10^{28}}$$