State-Space Analysis

\# **State:** It represents present operating condition of a system in terms of stored values within the system.

eg: L, C, sequential circuit.

\# **State-variable:** Variables that are used to represent state of a system are called as state-variables.

eg: i_L, v_c, Q (seq. ckt.)

state-vars
Name → $Q_3\ Q_2\ Q_1\ Q_0$
value → 1 1 0 1 → state

\# **State-space:** Set of all possible combination of state variable. State of a system is an element of state space.

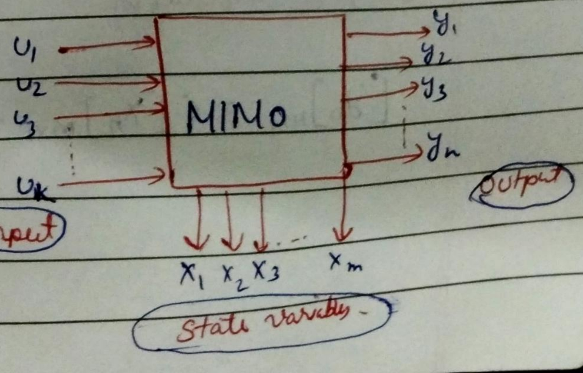
Advantages.

→ Applicable for LTI & LTV system.
→ Consider initial conditions.
→ All internal state variables can be considered.
→ Applicable for MIMO
→ Controllability & observability can be determined

\# **Representation:**

Input: $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}_{k\times 1}$; output: $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$

State-variables = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m\times 1}$



State equation: $\dot{x} = Ax + By$

$$\boxed{\dot{x} = Ax + Bu} \Rightarrow [\dot{x}]_{m \times 1} = [A][x]_{m \times 1} + [B][u]_{k \times 1}$$

$$\dot{x} = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_m/dt \end{bmatrix}_{m \times 1} \qquad [A]_{m \times m} \qquad [B]_{m \times k}$$

Output Equation: $\boxed{y = Cx + Du}$

$$[y]_{n \times 1} = [C][x]_{m \times 1} + [D][u]_{k \times 1}$$

$$[C]_{n \times m} \qquad [D]_{n \times k}$$

Discrete-Time-System.

→ State equation:

$$[x_{n+1}]_{m \times 1} = [A][x_n]_{m \times 1} + [B][u]_{k \times 1}$$

x_n : n^{th} sample of x

x_{n+1} : $(n+1)^{th}$ sample of x

Output equation:

$$[y_n]_{n \times 1} = C[x_n]_{m \times 1} + D[u_n]_{k \times 1}$$

Differential equation to state model.

$$\frac{d^3y}{dt^3} + \frac{6\,d^2y}{dt^2} + \frac{3\,dy}{dt} + 4y = 7u$$

State variable. $x_1 = y$

$x_2 = dy/dt = \dot{x}_1$

$x_3 = d^2y/dt^2 = \dot{x}_2$

$\dot{x}_3 = d^3y/dt^3$

$\dot{x}_3 = -4y - 3\,dy/dt - 6\,d^2y/dt^2 + 7u.$

$\dot{x}_3 = -4x_1 - 3\,dy/dt - 6\,d^2y/dt^2 + 7u$

$$\boxed{\dot{x}_3 = -4x_1 - 3x_2 - 6x_3 + 7u}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} u$$

o/p –

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

$$\frac{d^5y}{dt^5} + 0 \cdot \frac{d^4y}{dt^4} + 6\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 9y = 11u$$

Date
Page

Q $$\frac{d^5y}{dt^5} + 6\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 9y = 11u$$

$$\frac{d^5y}{dt^5} + 6d^4$$

State equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -9 & 0 & -7 & -6 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 11 \end{bmatrix}$$

Output eq^n:

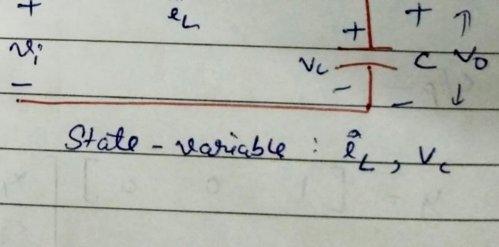
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + [0]u$$

Electrical Network to state model.

$$\begin{cases} v_L = L\frac{di}{dt} \\ i_L = C\frac{dv_c}{dt} \end{cases}$$

KVL

$v_L = v_s - i_L R - v_c$



~~KVL~~ —

$$\frac{di_L}{dt} = \frac{-R}{L} i_L - \frac{1}{L} v_c + \frac{v_s}{L}$$

State-variable : i_L, v_c

$i_c = i_L$

KCL —

$$\frac{dv_c}{dt} = \frac{i_L}{c} \; ; \quad V_0 = v_c$$

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/c & 0 \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} V_s$$

$$V_0 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + [0] V_s$$

Q1 A system is represented by $3\frac{dy}{dt} + 2y = u$. What is the transfer fun^n of ~~th~~ to the system $\frac{Y(s)}{U(s)}$

$$\Rightarrow \quad 3\frac{dy}{dt} + 2y = u$$

L.T of eq^n ①

$$3sY(s) + 2Y(s) = U(s)$$

$$Y(s)[3s+2] = U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{1}{3s+2}}$$

Q2 The dynamic model of a pendulum is given by -
$\frac{d^2\theta}{dt^2} + 400\theta = 100T$.

where - θ = displacement
T = Torque

It representation in time ~~sp~~ scale - state variable from $\dot{x} = \alpha x + \beta u$ can have constant ? $\{\alpha = ?\ \ \beta = ?$

$$\frac{d^2\theta}{dt^2} + 400\theta = 100T \quad \Rightarrow \frac{d^2\theta}{dt^2} - 0\frac{d\theta}{dt} + 4.00\,\theta = 1.00\,T$$

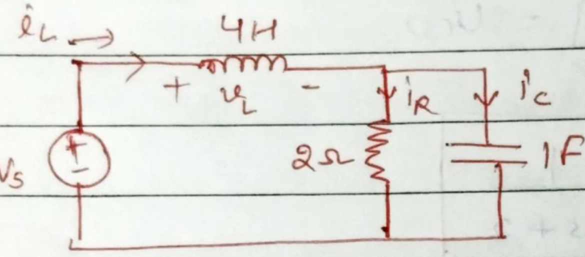
$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4.00 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} T$$

Q3 With reference to state space representation of eqⁿ

$\frac{d^2y}{dt^2} + 2\frac{dy(t)}{dt} + 3y(t) = u(t)$, the matrix A & B are ?

$$\Rightarrow \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Q Consider the network shown below -



Find the state space representation for the above network -

KVL :- $V_L = V_S - V_C$ $\quad \{ V_S = V_L + V_C$

$$\frac{di_L}{dt} = \cancel{+V_S/4}$$

$$\frac{di_L}{dt} = \frac{V_S}{L} - \frac{V_C}{L} \Rightarrow \boxed{\frac{di_L}{dt} = +\frac{V_S}{4} - \frac{V_C}{4}}$$

KCL -

$i_C = i_L - i_R$ $\quad \{ i_L = i_R + i_C$

$$\boxed{\frac{dV_C}{dt} = i_L - \frac{V_C}{R} = i_L - \frac{V_C}{2}}$$

$$\boxed{\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & -1/4 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} V_s}$$

CANONICAL FORM

Transfer function of a system is unique but there may be multiple state model associated with it & such models are called canonical form.

(a) Transfer function $= \dfrac{b\,(s^3 + c_2 s^2 + c_1 s + c_0)}{(s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)}$

with -ve sign

→ Controllable canonical form - (CCF)

$\dot{X} = AX + BU$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b \end{bmatrix} u$$

→ O/P -

$Y = CX + DU$

$$y = \begin{bmatrix} c_0 & c_1 & c_2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b) $T(s) = \dfrac{b\,(s^3 + c_2 s^2 + c_1 s + c_0)}{(s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)}$

→ Observable canonical form (OCF)

$$A = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}$$

$$B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 & b \end{bmatrix}$$

Q $G(s) = \frac{Y(s)}{U(s)} = \frac{20(10s+1)}{s^3+3s^2+2s+1}$ (Transfer function)

Find the state equation & output equation.

$$\frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)} \cdot \frac{Y(s)}{X_1(s)} = \frac{20}{s^2+3s^2+2s+1} \cdot (10s+1)$$

Let $\frac{X_1(s)}{U(s)} = \frac{20}{s^3+3s^2+2s+1}$ — ① & $\frac{Y(s)}{X_1(s)} = (10s+1)$ — ②

eqⁿ ① can be re-written as —

$X_1(s)\,[s^3+3s^2+2s+1] = 20\,U(s)$

$X_1(s).s^3 = \dot{x}_3$
$X_1(s).s^2 = \dot{x}_2 = x_3$
$X_1(s).s^1 = \dot{x}_1 = x_2$
$X_1(s) = x_1$

on taking inverse laplace —

$\dot{x}_3 + 3x_3 + 2x_2 + x_1 = 20u(t)$

$\dot{x}_3 = -x_1 - 2x_2 - 3x_3 + 20u(t)$

State equation
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 20\end{bmatrix}[u]$$

Now- eqⁿ ② can be re-written as —

$\frac{Y(s)}{X_1(s)} = (10s+1)$

$Y(s) = (10s+1)X_1 = 10s\,X_1(s) + X_1(s)$

On taking inverse Laplace —

$y(t) = 10x_2 + x_1$

Output equation
$$y = \begin{bmatrix}1 & 10 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Q $\dfrac{Y(s)}{U(s)} = \dfrac{K}{s^0 (s+1)(s+2)(s^2+1)}$

$\dfrac{Y(s)}{U(s)} = \dfrac{X_1(s)}{U(s)} \cdot \dfrac{Y(s)}{X_1(s)} = \dfrac{K}{(s+1)(s+2)(s^2+1)} \cdot 1$

so — $\dfrac{X_1(s)}{U(s)} = \dfrac{K}{(s+1)(s+2)(s^2+1)} \Rightarrow \dfrac{K}{(s^4+3s^3+3s^2+3s+2)}$

$X_1(s)\,[s^4+3s^3+3s^2+3s+2] = K\,U(s)$

On taking inverse laplace —

$\dot{x}_4 + 3x_4 + 3x_3 + 3x_2 + 2x_1 = K\,u(t)$

$\dot{x}_4 = -3x_4 - 3x_3 - 3x_2 - 2x_1 + K\,u(t)$

$X_1(s) = X_1$

$\dot{x}_1 = x_2$

$\ddot{x}_1 = \dot{x}_2 = x_3$

$\dddot{x}_1 = \dot{x}_3 = x_4$

$\ddddot{x}_1 = \dot{x}_4 = x_5$ —

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} [u]$$

& — $\dfrac{Y(s)}{X_1(s)} = 1 \Rightarrow Y(s) = X_1(s)$

Taking inverse laplace transform —

$y(t) = x_1$

$$y = [1\ 0\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

CONTROLLABILITY

A control system is said to be controllable if the initial states of the control system are changed to some other desired states by controlled input in finite duration of time.

KALAM'AN TEST

Step 1 Write the matrix $Q_C = [B \quad AB \quad A^2B \ldots A^{n-1}B]$

Find determinant of Q_C, if it is not equal to 0

Then control system is controllable.

Q Verify the controllability of a control system which is represented by state equation.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}}_{2\times2 \,\hookrightarrow A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\hookrightarrow x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hookrightarrow B} \underbrace{[U]}_{\hookrightarrow U}$$

Sol^n

$$\dot{x} = Ax + BU$$

Kalam's test: $Q_C = [B \quad AB \quad A^2B \ldots A^{n-1}B]$

Given - $A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad n = 2$

So -

$$Q_C = [B \quad AB] = \begin{bmatrix} \underset{\downarrow B}{1} & \overset{}{-1} \\ 0 & \underset{\uparrow AB}{1} \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1+0 \\ 1+0 \end{bmatrix}$$

$$A.B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step 2 $|Q_c| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$

$|Q_c| = 1$

$\neq 0 \rightarrow$ controllable

Q The state eqⁿ of a system is given as –

$\dot{x}_1 = 2x_1 + x_2 + 4$

$\dot{x}_2 = -2x_2$ check for controllability.

$\Rightarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [4]$$

$A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Q_c = [B \quad AB]$

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$Q_c = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$|Q_c| = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$ Not controllable.

Q $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Check the controllability of the system.

OBSERVABILITY

A control system is said to be observable if it is able to determine the initial state of the control system by observing the o/p in finite duration of time.

Kalaman's test

Step1 From the matrix $Q_o = [C^T \; A^T C^T \; (A^T)^2 C^T \dots (A^T)^{n-1} C^T]$

Step2 Take determinant of Q_o, if it is not equal to zero then the control system is observable.

Q Verify the observability of a control system which is represented in the state space model.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} [u]$$

$$y = \underbrace{[1 \quad 1]}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solⁿ Given -

$A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $C = [1 \quad 1]$; $D = 0$

Step 1 $Q_o = [C^T \; A^T C^T \; (A^T)^2 C^T \dots (A^T)^{n-1} C^T]$

$C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $A^T = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$ $\boxed{n = 2}$

$\boxed{Q_o = [C^T \quad A^T C^T]}$

$A^T . C^T = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+1 \\ -2+0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$\boxed{Q_o = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}}$

Step 2 $|Q_o| = \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = (-2 - (-1)) = -1$ $|Q_o| \neq 0$ OBSERVABLE

Transfer function decomposition

The process of obtaining state model from the transfer function is called as transfer function decomposition.

It is of three types—

(i) Direct decomposition—

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^2+5s+6}$$

$$Y(s)\left[s^2+5s+6\right] = 10\,U(s)$$

Let— $y(t) = x_1$

I.L.T.

$$\frac{d^2y}{dt^2} + \frac{5\,dy}{dt} + 6\,y(t) = 10u(t)$$

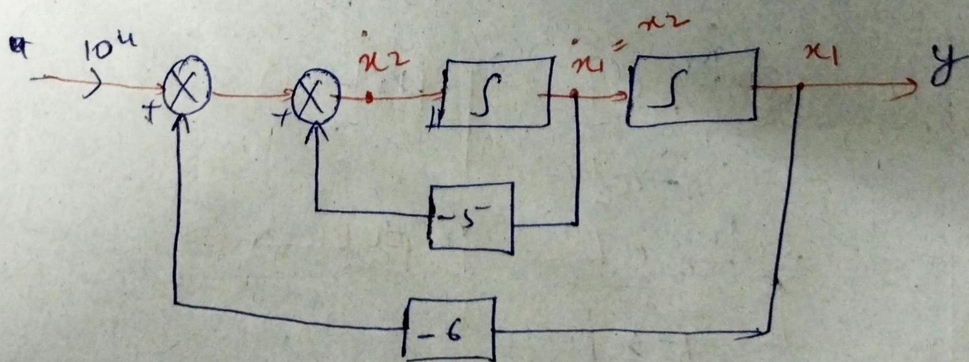
or— $\dot{x}_2 + 5x_2 + 6x_1 = 10u$

$$\dot{x}_2 = -5x_2 - 6x_1 + 10u$$

or $\boxed{\dot{x}_2 = -6x_1 - 5x_2 + 10u}$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 10\end{bmatrix}[u]$$

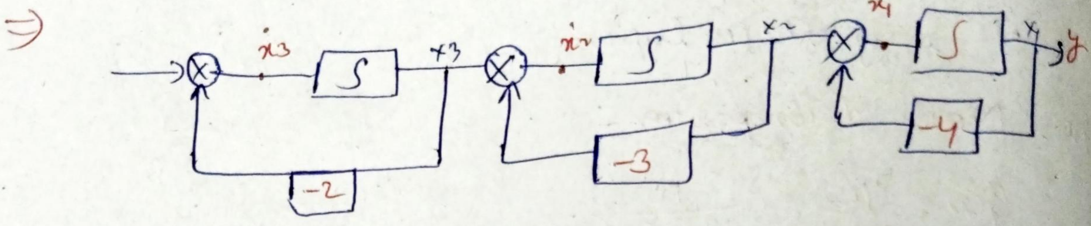
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$



② Cascade decomposition

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+2)(s+3)(s+4)}$$

$\Rightarrow$



$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + u$$

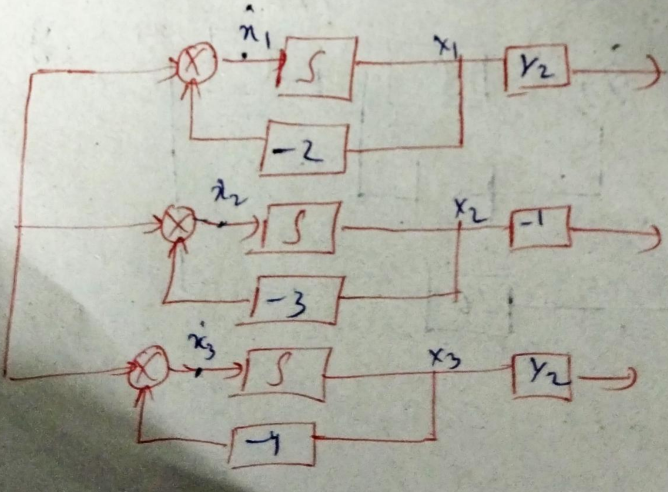
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

upper triangular form

③ Parallel decomposition

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+2)(s+3)(s+4)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+4)}$$

$$\frac{Y(s)}{U(s)} = \frac{1/2}{(s+2)} + \frac{(-1)}{(s+3)} + \frac{(1/2)}{(s+4)}$$



$$y = \frac{1}{2}x_1 - x_2 + \frac{1}{2}x_3$$

$$y = \begin{bmatrix} +1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{x}_1 = -2x_1 + 4$$
$$\dot{x}_2 = -3x_2 + 4$$
$$\dot{x}_3 = -4x_3 + 4$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [4]$$

All poles are in diagonal.