Digital Signal Processing(BEC-42)

Unit-2

Lecture-7

(Chebyshev Filter Design)

Date:08/10/2020

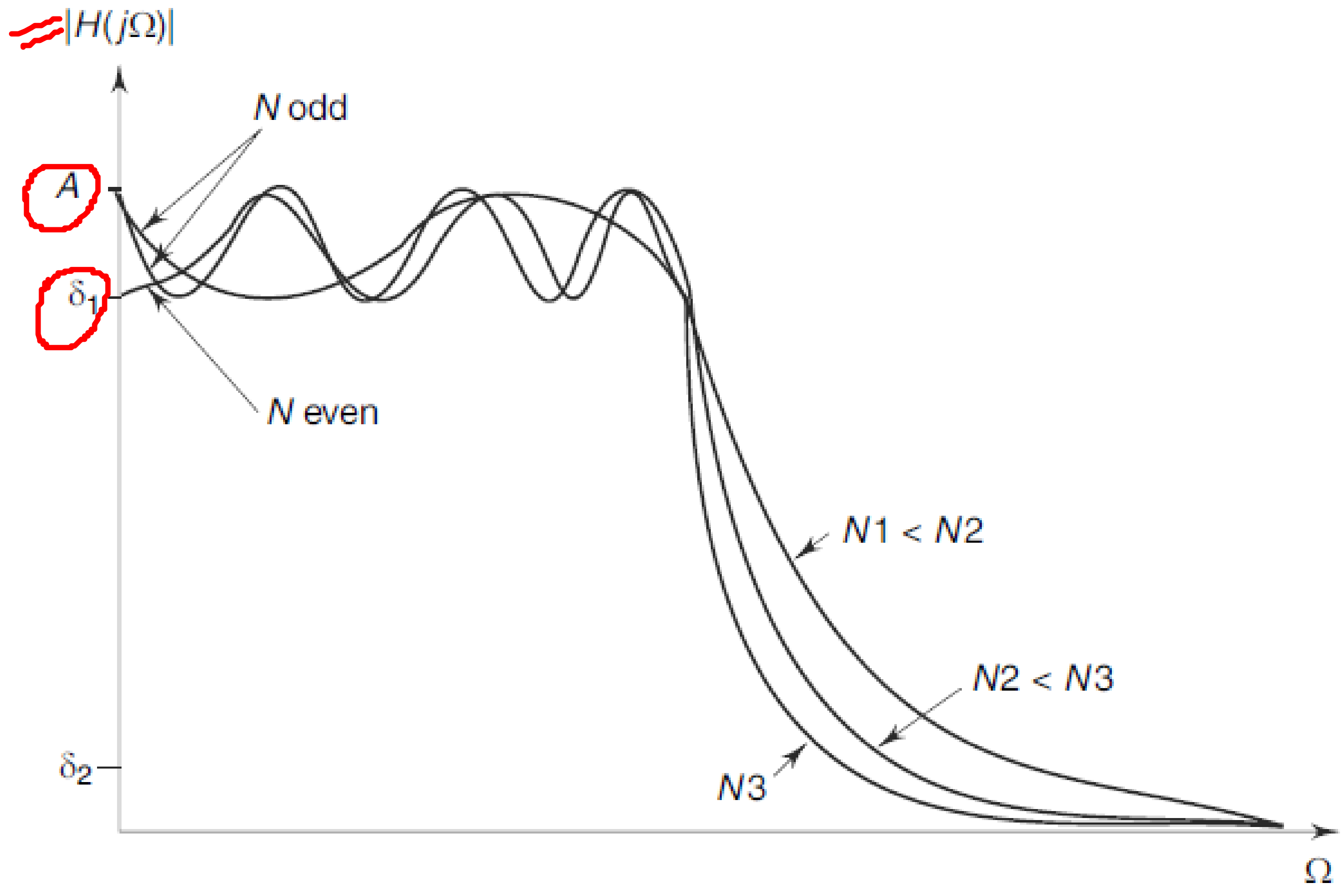
Chebyshev Filter

The Chebyshev low-pass filter has a magnitude response given by

$$\left|H(j\Omega)\right| = \frac{A}{[1+\varepsilon^2 C_N^2(\Omega/\Omega_c)]^{0.5}} \quad \mathbf{1}$$

where A is the filter gain, ε is a constant and Ω_c is the 3 dB cut-off frequency. The Chebyshev polynomial of the I kind of Nth order, $C_N(x)$ is given by

$$C_N(x) = \begin{cases} \cos(N\cos^{-1} x), & \text{for } |x| \le 1 \\ \cos(N\cosh^{-1} x), & \text{for } |x| \ge 1 \end{cases}$$



Magnitude Response of a Low-pass Chebyshev Filter

Now we can obtain the design parameters of chebyshev filter by considering the low pass filter with specifications mentioned as

$$\begin{aligned} \delta_1 \le |H(e^{j\omega})| \le 1 \qquad & 0 \le \omega \le \omega_1 \\ |H(e^{j\omega})| \le \delta_2 \qquad & \omega_2 \le \omega \le \pi \end{aligned} \qquad 2$$

Ω_1

Using equation 1and 2 and putting A=1, we write as

N Ω_2

Ω_c

$$\delta_1^2 \le \frac{1}{1+\varepsilon^2 C_N^2(\Omega_1/\Omega_c)} \le 1$$

$$\frac{1}{1+\varepsilon^2 C_N^2(\Omega_1/\Omega_c)} \le \delta_2^2 \qquad 3$$

Assuming $\Omega_c = \Omega_1$, we will have $C_N(\Omega_c/\Omega_c) = C_N(1) = 1$.

$$\delta_1^2 \le \frac{1}{1+\varepsilon^2}$$

Assuming equality in the above equation, the expression for ε is

$$\varepsilon = \left[\frac{1}{\delta_1^2} - 1\right]^{0.5}$$

Now the order of analog filter N can be determined using equation 3. assuming $\Omega_c = \Omega_1$

$$C_N(\Omega_2 / \Omega_1) \geq \frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}$$

Since $\Omega_2 > \Omega_1$,

$$\cosh[N \cosh^{-1}(\Omega_2 / \Omega_1)] \geq \frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}$$

$$\boxed{N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2 / \Omega_1)}}$$

The transfer function of Chebyshev filter are usually mentioned in the factored form as

$$H(s)=\prod_{k=1}^{N/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}\quad N=2,4,6,\ldots$$ even

or

$$H(s)=\frac{B_0\Omega_c}{s+c_0\Omega_c}\prod_{k=1}^{(N-1)/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}\quad N=3,5,7,\ldots.$$ odd

The coefficients b_k and c_k are given by

$$b_k = 2y_N \sin[(2k-1)\pi/2N]$$

$$c_k = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N}$$

$$c_0 = y_N$$

The parameter y_N is given by

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5}+\frac{1}{\varepsilon}\right]^{\frac{1}{N}}-\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5}+\frac{1}{\varepsilon}\right]^{-\frac{1}{N}}\right\}$$

The parameter B_k can be obtained from

$$\frac{A}{(1+\varepsilon^2)^{0.5}} = \prod_{k=1}^{N/2}\frac{B_k}{c_k}, \quad \text{for } N \text{ even}$$

and

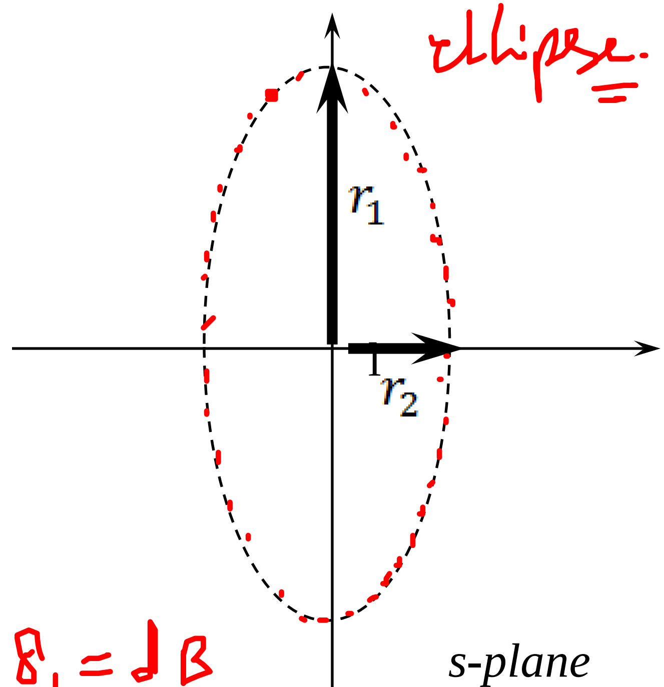
$$A = \prod_{k=0}^{\frac{N-1}{2}}\frac{B_k}{c_k} \text{ for } N \text{ odd}$$

Now the system function of equivalent digital filter can be obtained using BLT or impulse invariant techniques.

$\Omega_c = 1$ rad/sec

Poles of Normalized Chebyshev Filter

Pole of Chebyshev filters lies on a ellipse.



The semi-major and minor axes of ellipse are

$$r_1 = \cosh y \qquad r_2 = \sinh y$$

$$r_1 = \Omega_p \left[\frac{\beta^2 + 1}{2\beta}\right] \qquad r_2 = \Omega_p \left[\frac{\beta^2 - 1}{2\beta}\right]$$

$$\beta = \left(\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right)^{\frac{1}{N}} \qquad \varepsilon = \sqrt{10^{0.1\delta_1} - 1}$$

$\delta_1 = dB$

order of filter

$$\theta_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N}, k = 0, \ldots, N-1$$

$\varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1}$

$\delta_1 \neq dB$

The poles of filters are given as

$$s_k = r_2 cos\theta_k + j r_1 sin\theta_k$$

$$H(s) = \frac{K}{(s-s_0)(s-s_1)(s-s_2)\cdots(s-s_k)}$$

Constant $K = \begin{cases} b_0 & N = \text{odd} \\ \dfrac{b_0}{\sqrt{1+\epsilon^2}} & N = \text{even} \end{cases}$

Problem: Design a Chebyshev filter with a maximum pass-band attenuation of 2.5 dB at $\boldsymbol{\Omega_p = 20\, rad/s}$ and the stop-band attenuation of 30 dB at $\boldsymbol{\Omega_s = 50\, rad/s}$.

$N = ?$

$\delta_1 =$

$\delta_2 =$

$\Omega_p =$

$\Omega_s =$

Solution:

Given $\Omega_1 = \Omega_p = 20$ rad/s and $\Omega_2 = \Omega_s = 50$ rad/s

$$\alpha_p = 2.5 = -20 \log \delta_1$$

Therefore, $\delta_1 = 0.7499$

$$\alpha_s = 30 = -20 \log \delta_2$$

Therefore, $\delta_2 = 0.0316$

The order of the filter is

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

where $$\varepsilon = \left[\frac{1}{\delta_1^2} - 1\right]^{0.5} = \left[\frac{1}{(0.7499)^2} - 1\right]^{0.5} = 0.882$$

Therefore, $$N \geq \frac{\cos h^{-1}\left\{\left(\frac{1}{0.882}\right)\left(\frac{1}{(0.0316)^2}-1\right)^{0.5}\right\}}{\cos h^{-1}\left(\frac{50}{20}\right)} \geq \frac{\cos h^{-1}(35.8614)}{\cos h^{-1}(2.5)} \geq 2.727$$

Hence, $$N = 3$$

$$\beta = \left[\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right]^{\frac{1}{N}} = \left[\frac{\sqrt{1+(0.882)^2}+1}{0.882}\right]^{\frac{1}{3}} = 1.3826$$

$$r_1 = \Omega_p\left[\frac{\beta^2+1}{2\beta}\right] = 21.06$$

$$r_2 = \Omega_p\left[\frac{\beta^2-1}{2\beta}\right] = 6.60$$

$N = 3$

$$\theta_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N},\ k = 0,1,\ldots,(N-1)$$

$$= \frac{\pi}{2} + (2k+1)\frac{\pi}{6},\ k = 0,1,2 = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$$

$$\left[s_k = r_2 \cos\theta_k + j r_1 \sin\theta_k \right]$$

$$s_1 = -3.3 - j\,18.23$$

$$s_2 = -6.6$$

$$s_3 = -3.3 + j\,18.23$$

$[s - (-3.3 + j\,18.23)]$

$[s - (-3.3 + j\,18.23)]$

Denominator of $H(s) = (s + 6.6)(s^2 + 6.6s + 343.2)$

Numerator of $H(s) = 2265.27$

Hence, the transfer function $H(s) = \dfrac{2265.27}{(s+6.6)(s^2+6.6s+343.2)}$.

Problem: The specification of the desired low-pass digital filter is

$$0.9 \le H(\omega) \le 1.0 \qquad\qquad 0 \le \omega \le 0.25\pi$$

$$|H(\omega)| \le 0.24, \qquad\qquad 0.5\pi \le \omega \le \pi$$

Design a Chebyshev digital filter using impulse invariant transformation.

Solution:

Given $\delta_1 = 0.9, \quad \delta_2 = 0.24, \quad \omega_p = 0.25\pi$ and $\omega_s = 0.5\pi$

Therefore, $\dfrac{\Omega_2}{\Omega_1} = \dfrac{\omega_s T}{\omega_p T} = \dfrac{0.5\pi T}{0.25\pi T} = 2$

where, $\varepsilon = \left[\dfrac{1}{\delta_1^2} - 1\right]^{\frac{1}{2}} = \left[\dfrac{1}{0.9^2} - 1\right]^{\frac{1}{2}} = 0.484$

The order of the filter is

$$N \geq \frac{\cos h^{-1}\left\{\dfrac{1}{\varepsilon}\left[\dfrac{1}{\delta_2^2} - 1\right]^{0.5}\right\}}{\cos h^{-1}\left(\dfrac{\Omega_2}{\Omega_1}\right)} \geq \frac{\cos h^{-1}\left\{\dfrac{1}{0.484}\left[\dfrac{1}{(0.24)^2} - 1\right]^{0.5}\right\}}{\cos h^{-1}(2)} \geq 2.136$$

Therefore, $N = 3$

$$\beta = \left[\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right]^{\frac{1}{N}} = \left[\frac{\sqrt{1+(0.484)^2}+1}{0.484}\right]^{\frac{1}{3}} = 1.6337$$

$$r_1 = \Omega_p\left[\frac{\beta^2+1}{2\beta}\right] = 0.25\pi\left[\frac{(1.6337)^2+1}{2(1.6337)}\right] = 0.882$$

$$r_2 = \Omega_p\left[\frac{\beta^2-1}{2\beta}\right] = 0.25\pi\left[\frac{(1.6337)^2-1}{2(1.6337)}\right] = 04012$$

The poles are

$$s_k = r_2\cos\theta_k + jr_1\sin\theta_k$$

$$\theta_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N}, \qquad k = 0,1,\ldots.,(N-1)$$

$$= \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$$

$$s_1 = (0.4012)\cos\left(\frac{2\pi}{3}\right) + j0.882\sin\left(\frac{2\pi}{3}\right) = -0.2 + j0.764$$

N=3

$$s_2 = -0.4012$$

$$s_3 = -0.2 - j0.764$$

Denominator of the transfer function $H(s)$ is

$$(s+0.4012)(s+0.2-j0.764)(s+0.2+j0.764)$$

Numerator of $H(s) = 0.25$

Hence, the transfer function $H(s)$ is

$$H(s) = \frac{0.25}{(s+0.4012)(s+0.2-j0.764)(s+0.2+j0.764)}$$

$$= \frac{A_1}{s+0.4012} + \frac{A_2}{s+0.2-j0.764} + \frac{A_3}{s+0.2+j0.764}$$

IIM

H(z)

$$A_1 = H(s)\times(s+0.4012)\big|_{s=0.4012} = 0.4$$

$$A_2 = H(s)\times(s+0.2-j0.764)\big|_{s=-0.2+j0.764}$$

$$= \frac{0.25}{\left(-0.2 + j0.764 + 0.4012\right)\left(-0.2 + j0.764 + 0.2 + j0.764\right)} = -0.138 + j0.5242$$

$$A_3 = A_2^* = -0.138 - j0.5242$$

$$H(s) = \frac{0.4}{s + 0.4012} + \frac{-0.138 + j0.5242}{s + 0.2 - j0.764} + \frac{-0.138 - j0.5242}{s + 0.2 + j0.764}$$

Using the transformation,

$$\frac{1}{s - p_i} \to \frac{1}{1 - e^{p_i T} z^{-1}}$$

$$H\left(z\right) = \frac{0.4}{1 - e^{-0.4012} z^{-1}} + \frac{-0.138 + j0.5242}{1 - e^{-0.2} e^{+j0.764} z^{-1}} - \frac{0.138 + j0.5242}{1 - e^{-0.2} e^{-j0.764} z^{-1}}$$

Hence, $H(z) = \dfrac{0.4}{1 - 0.6695 z^{-1}} - \dfrac{0.138 - j0.5242}{1 - \left(0.5912 + j0.5664\right) z^{-1}} - \dfrac{0.138 + j0.5242}{1 - \left(0.5912 - j0.5664\right) z^{-1}}.$