

S. Palani

# Principles of Digital Signal Processing

*Second Edition*



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Second Edition



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Pvt. Ltd.



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**PINGALA** (200BC) in India invented that any number can merely be represented by binary system using **1s** and **0s**. A spiritual significance of this is that **1** represented a symbol for **GOD** while **0** represented nothingness. God created the universe out of nothing. This book is dedicated to the father of the Universe, **THE GOD**.

## Preface to the Second Edition

We are very happy to bring out the second edition of the book *Digital Signal Processing* with the new title *Principles of Digital Signal Processing* to meet the requirements of the students of Electronics and Communication Engineering of undergraduate level. First of all, the authors would like to thank the members of the teaching faculty and student community for the wide patronage extended to the first edition. A thorough revision of all the chapters in the previous edition has been undertaken, and few numerical problems have been worked out and included for the questions that appeared in recent university examinations. A new chapter has been included which describes the representation of discrete time signal and systems. From the first edition, two chapters, namely DSP architecture and power spectrum density, have been removed since they are not in the syllabus content. Elaborate treatment on the topics such as filtering methods based on DSP, noise power spectrum and adaptive filters has been given in the new edition.

The authors would like to thank Shri Sunil Sexana, Managing Director, ANE Books Pvt. Ltd., India. Shri A. Rathinam, General Manager (South), for the encouragement given to us. We would also like to thank Mr. V. Ashok for the at most care he took to key the voluminous technical book like this.

Pudukkottai, India

S. Palani

# Preface to the First Edition

Signal processing is all about taking a signal, applying some changes to it and then getting a new signal out. The change might be amplification or filtration or something else, but nearly all electronic circuits can be considered as signal processors. Thus, the signal processor might be composed of discrete components like capacitors and resistors, or it could be a complex integrated circuits, or it could be a digital system which accepts a signal on its input and outputs the changed signal. Digital signal processing (DSP) is the processing of signals by digital means. The term “digital” comes from “digit,” meaning a number and so “digital” literally means numerical. A signal carries a stream of information representing anything from stock prices to data from a remote sensing satellite. If they are represented in the form of stream of numbers, they are called digital signals. The processing of a digital signal is done by digital signal processor (DSP) by performing numerical calculations. Digital signal processors require several things to work properly. The processor should be fast enough with enough precision to support the required mathematics it needs to implement. It requires memory to store programming, samples, intermediate results and final results. It also requires A/D and D/A converters to bring real signals into and out of the digital domain. Further, it requires programming to do the job.

The main applications of DSP are audio signal processing, audio compression, digital image processing, video compression, speech processing, speech recognition, digital communications, radar, sonar, seismology and biomedicine. Specific examples include speech compression and transmission in digital mobile phones, room matching equalization of sound, analysis and control of industrial process, seismic data processing, medical imaging such as CAT scans and MRI, MP3 compression, image manipulation, computer-generated animations in movies, high fidelity loud speaker crossovers and equalization, audio effects, etc. Digital signal processing is often implemented using specialized microprocessors such as the DSP56000, the TMS320 or the SHARC. Multi-core implementations of DSPs have started to emerge from companies including Free scale and stream processors.

The book is divided into eight chapters. Chapter 1 presents an introduction to the field of the signal processing and provides overview of the development of DSP,

analog and digital signals, DSP domains, different types of filters used to eliminate unwanted noise from the signal, DSP applications and its implementation.

The discrete time sequence  $x[n]$  can be transformed as  $X(j\omega)$  by Fourier transform and can be analyzed using digital computer. However,  $X(j\omega)$  is a continuous function of frequency  $\omega$  and computationally difficulties are encountered while analyzing  $X(j\omega)$  using DSP. In Chapter 2, we convert  $X(j\omega)$  into equally spaced samples. Such a sequence is called discrete Fourier transform (DFT) which is a powerful computational tool for the frequency analysis of discrete time signals. Several methods are available for computing DFT. However, fast Fourier transform (FFT) algorithms eliminate redundant calculations and offer rapid frequency domain analysis. In Chapter 2, the properties of DFT, FFT algorithms, decimation in time and decimation in frequency, linear filtering and correlation are discussed.

The digital filters are classified as infinite impulse response (IIR) and finite impulse response (FIR) filters. Digital IIR filter design procedure is the extension of analog filter design. In Chapter 3, designs of IIR filter using impulse invariant method and bilinear transformation are described. IIR filter is also designed using system function  $H(s)$ . Low pass IIR Butterworth Chebyshev digital filter designs are also described in this chapter.

Design techniques for finite impulse response digital filter are discussed in Chapter 4. These filters are designed using windows such as Rectangular window, Hamming window, Kaiser window and Hanning window. FIR filters are designed using frequency sampling method.

The linear time invariant discrete time system is described by linear differential equation with constant coefficients. These coefficients and the signal variables are assumed, when implemented in digital hardware to take a specified range and stored in finite length in a digital machine. During discretization and quantization processes, errors occur in different form. They are discussed in Chapter 5. Quantization noise, over flow error, limit cycle oscillations, signal scaling and sampling and hold operations are also discussed in this chapter.

In Chapter 6, introduction to multi-rate digital signal processing is given. The necessity to use MDSP and its application are also described. The concepts of decimation and interpolations are explained. Polyphase implementation of FIR filters for interpolators and decimators are thoroughly discussed and presented. Multi-rate implementation of sampling rate conversion and design of narrow band filters are explained with necessary examples. Finally applications of MDSP are explained.

Digital signal processing is the processing of signals by digital means. The digital signal processor which processes the signal should be fast, with enough precision, and should have supporting memory to store programming, samples, intermediate and final results. In Chapter 7, we describe different types of DSP architecture, advanced addressing modes, pipe lining and overview of instructions set of TMS320C5X and C54x.

One of the important applications of digital signal processing is the spectral analysis of the signals. The signal processing methods that characterize the frequency content of a signal are termed as spectral analysis. The distribution of power with frequency is called power density spectrum. Similarly, the distribution of energy

with frequency is called energy density spectrum. In Chapter 8, different methods of estimating power spectrum density and energy spectrum density are presented and their merits and demerits discussed.

The notable features of this book include the following:

1. The syllabus content of digital signal processing of undergraduate level of most of the Indian Universities has been well covered.
2. The organization of the chapters is sequential in nature.
3. Large number of numerical examples have been worked out.
4. Learning objectives and summary are given in each chapter.
5. For the students to practice, short and long questions with answers are given at the end of each chapter.

The authors take this opportunity to thank Shri Sunil Saxena, Managing Director, Ane Books Pvt. Ltd., India, for coming forward to publish this book. We would like to express our sincere thanks to Shri A. Rathinam, General Manager (South), Ane Books Pvt. Ltd., who took the initiatives to publish the book in a short span of time. We would like to express our sincere thanks to Mr. V. Ashok who has done a wonderful job to key the voluminous book like this in a very short time and beautifully too. Suggestions and constructive criticisms are welcome from staff and students.

Pudukkottai, India

S. Palani

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## About the Author



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He has published more than a hundred research papers in reputed international journals and has won many cash awards. Under his guidance, 17 research scholars were awarded Ph.D. He has carried out several research projects worth about several lakhs rupees funded by the Government of India and AICTE. As Theme Leader of the Indo—UK, REC Project on energy, he has visited many universities and industries in the UK. He is the author of the books titled *Control Systems Engineering, Signals and Systems, Digital Signal Processing, Linear System Analysis, and Automatic Control Systems*.

# Chapter 1

## Representation of Discrete Signals and Systems



### Learning Objectives

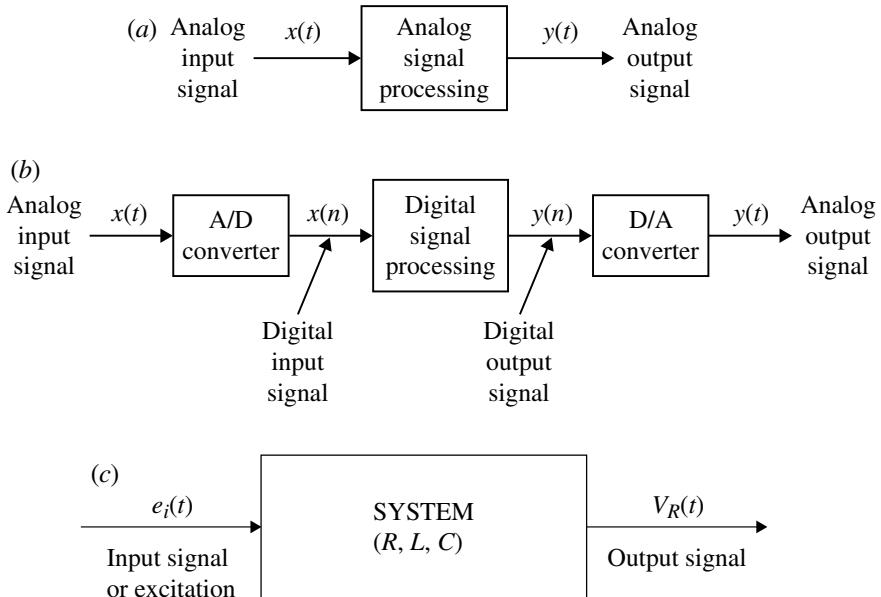
After completing this chapter, you should be able to:

- ✖ define various terminologies related to signals and systems.
- ✖ classify signals and systems.
- ✖ give mathematical description and representation of signals and systems.
- ✖ perform basic operations on DT signals.
- ✖ classify DT signals as periodic and non-periodic, odd and even and power and energy signals.
- ✖ classify DT systems.

### 1.1 Introduction

Most of the signals encountered in science and engineering are analog in nature. That is, the signals are functions of a continuous variable, such as time or space, and usually take on values in a continuous range. Such signals may be processed directly by appropriate analog systems (such as filter or frequency analyzers or frequency multipliers) for the purpose of changing their characteristics or extracting some desired information. In such a case, the signal has been processed directly in its analog form. Both the input signal and the output signal are in analog form and are shown in Fig. 1.1a.

Digital Signal Processing provides an alternative method for processing the analog signal and is shown in Fig. 1.1b. To perform processing digitally, there is a need for an interface between the analog signal and the digital processor. This interface is called an analog to digital (A/D) converter. The output of A/D converter is a digital signal that is applied as an input to the digital processor.



**Fig. 1.1** **a** Analog signal processing. **b** Digital signal processing. **c** Block diagram representation of signals and systems

The digital signal processor may be a large programmable digital computer or a small microprocessor programmed to perform the desired operations on the input signal. Programmable machines provide the flexibility to change the signal processing operation through a change in the software, whereas hardwired machines are difficult to reconfigure. In application where the digital output from the digital signal processor is to be given to the user in analog form, we must provide interface, and this is called digital to analog (D/A) converter. Thus the signal is provided to the user in analog form.

Digital Signal Processing has developed very rapidly over the past five decades mainly due to the advances in digital computer technology and very large-scale integrated electronic circuits. These inexpensive smaller but faster and more powerful digital computers are capable of performing very complex signal processing functions which are usually too difficult to perform by analog circuitry. The following are the advantages of Digital Signal Processing (DSP) over analog processing.

1	Flexibility	Digital programmable systems allow flexibility in reconfiguring the DSP operations by simply changing the program
2	Accuracy	DSP provides better control of accuracy requirements, while tolerance limits have to be met in the analog counterpart
3	Easy storage	Digital signals can be easily stored in magnetic media without deterioration or loss of signal fidelity. They can also be easily transportable and can be processed off-time in remote laboratories
4	Processing	DSP allows for the implementation of more sophisticated signal processing than its analog counterpart
5	Cost effective	With advancement in VLSI technology, digital implementation of the signal process system is cheaper

The limitation of DSP is that the conversion speed of ADC and the process speed of signal processors should be very high to perform real-time processing. Signals of high bandwidth require fast sampling rate ADCs and fast processors.

Some of the applications of digital signal processor are as follows: speech processing, signal transmission on telephone channels, image processing, biomedical, seismology and consumer electronics.

Speech processing	Speech compression and decompression for voice storage system and for transmission and reception of voice signals
Communication	Elimination of noise by filtering and echo cancelation by adaptive filtering in transmission channels
Biomedical	Spectrum analysis of ECG signals to identify various disorders in heart. Spectrum analysis of EEG signals to study the malfunction or disorders in the brain
Consumer electronics	Music synthesis, digital audio and video
Seismology	Spectrum analysis of seismic signals can be used to predict the earthquake, nuclear explosions and earth movement
Image processing	Two-dimensional filtering on images for image enhancement, finger-print matching, identifying hidden images in the signals received by radars, etc.,

The concepts of signals and systems play a very important role in many areas of science and technology. These concepts are very extensively applied in the field of circuit analysis and design, long-distance communication, power system generation and distribution, electron devices, electrical machines, biomedical engineering, aeronautics, process control, speech and image processing to mention a few. **Signals represent some independent variables which contain some information about the behavior of some natural phenomenon.** Voltages and currents in electrical and electronic circuits, electromagnetic radio waves, human speech and sounds produced by animals are some of the examples of signals. **When these signals are operated on some objects, they give out signals in the same or modified form. These objects are called systems.** A system is, therefore, defined as the interconnection of objects with a definite relationship between objects and attributes. Signals appearing at various stages of the system are attributes.  $R$ ,  $L$ ,  $C$  components, spring, dash-pots, mass, etc., are the objects. The electrical and electronic circuits comprising of

$R$ ,  $L$ ,  $C$  components and amplifiers, the transmitter and receiver in a communication system, the petrol and diesel engines in an automobile, chemical plants, nuclear reactor, human beings, animals, a government establishment, etc., are all examples of systems. **In this book, we deal with only discrete signals and systems.**

## 1.2 Terminologies Related to Signals and Systems

Before we give mathematical descriptions and representations of various terminologies related to signals and systems, the following terminologies which are very frequently used are defined as follows.

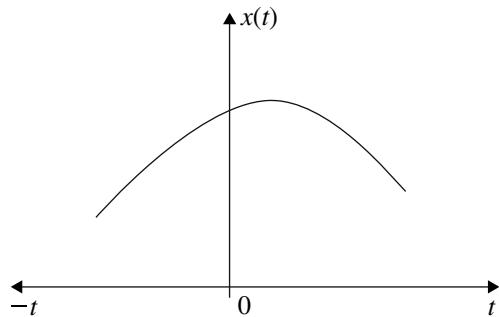
### 1.2.1 Signal

A signal is defined as a physical phenomenon which carries some information or data. The signals are usually functions of independent variable time. There are some cases where the signals are not functions of time. The electrical charge distributed in a body is a signal which is a function of space and not time.

### 1.2.2 System

A system is defined as the set of interconnected objects with a definite relationship between objects and attributes. The interconnected components provide desired function.

Objects are parts or components of a system. For example, switches, springs, masses, dashpots, etc., in a mechanical system and inductors, capacitors, resistors in an electrical system are the objects. The displacement of mass, spring and dashpot and the current flow and the voltage across the inductor, capacitor and resistor are the attributes. There is a definite relationship between the objects and attributes. The voltages across  $R$ ,  $L$ ,  $C$  series components can be expressed as  $v_R = iR$ ;  $V_L = L \frac{di}{dt}$  and  $V_C = \frac{1}{C} \int idt$ . If this series circuit is excited by the voltage source  $e_i(t)$ , the  $e_i(t)$  is the input attribute or the input signal. If the voltage across any of the objects  $R$ ,  $L$  and  $C$  is taken, then such an attribute is called the output signal. The block diagram representation of input and output (voltage across the resistor) signals and the system is shown in Fig. 1.1c.

**Fig. 1.2** CT signal

### 1.3 Continuous and Discrete Time Signals

Signals are broadly classified as follows:

1. Continuous time signal (CT signal).
2. Discrete time signal (DT signal).
3. Digital signal.

The signal that is specified for every value of time  $t$  is called continuous time signal and is denoted by  $x(t)$ . On the other hand, the signal that is specified at discrete value of time is called discrete time signal. The discrete time signal is represented as a sequence of numbers and is denoted by  $x[n]$ , where  $n$  is an integer. Here, time  $t$  is divided into  $n$  discrete time intervals. The continuous time signal (CT) and discrete time signal (DT) are represented in Figs. 1.2 and 1.3, respectively.

It is to be noted that in continuous time signal representation the independent variable  $t$  which has unit as sec. is put in the parenthesis  $(\cdot)$ , and in discrete time signal, the independent variable  $n$  which is an integer is put inside the square parenthesis  $[\cdot]$ . Accordingly, the dependent variables of the continuous time signal/system are denoted as  $x(t)$ ,  $g(t)$ ,  $u(t)$ , etc. Similarly the dependent variables of discrete time signals/systems are denoted as  $x[n]$ ,  $g[n]$ ,  $u[n]$ , etc.

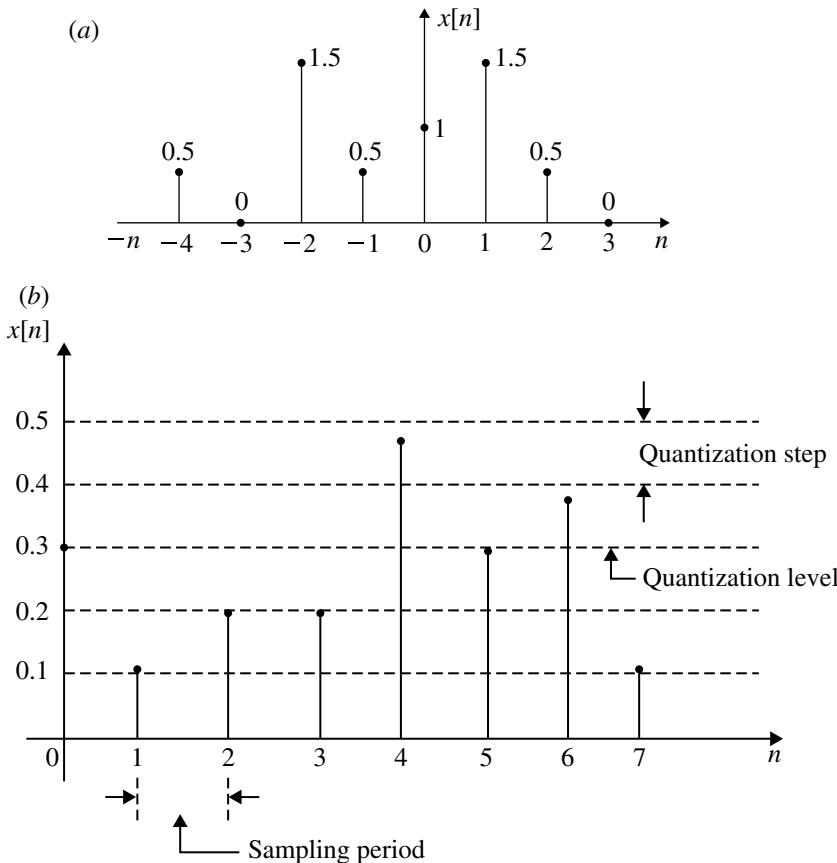
A discrete time signal  $x[n]$  is represented by the following two methods:

1.

$$x[n] = \begin{cases} \left(\frac{1}{a}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (1.1)$$

Substituting various values of  $n$  where  $n \geq 0$  in Eq. (1.1), the sequence for  $x[n]$  which is denoted by  $x\{n\}$  is written as follows:

$$x\{n\} = \left\{ 1, \frac{1}{a}, \frac{1}{a^2}, \dots, \frac{1}{a^n} \right\}$$



**Fig. 1.3** **a** Discrete time (DT) signal. **b** Digital signal

2. The sequence is also represented as given below.

$$x\{n\} = \{3, 2, 5, 4, 6, 8, 2\}$$

↑

The arrow indicates the value of  $x[n]$  at  $n = 0$  which is 5 in this case. The numbers to the left of the arrow indicate to the negative sequence  $n = -1, -2$ , etc. The numbers to the right of the arrow correspond to  $n = 1, 2, 3, 4$ , etc. Thus, for the above sequence  $x[-1] = 2$ ;  $x[-2] = 3$ ;  $x[0] = 5$ ;  $x[1] = 4$ ;  $x[2] = 6$ ;

$x[3] = 8$  and  $x[4] = 2$ . If no arrow is marked for a sequence, the sequence starts from the first term in the extreme left. Consider the sequence

$$x\{n\} = \{5, 3, 4, 2\}.$$

Here,  $x[0] = 5$ ;  $x[1] = 3$ ;  $x[2] = 4$  and  $x[3] = 2$ . There is no negative sequence here.

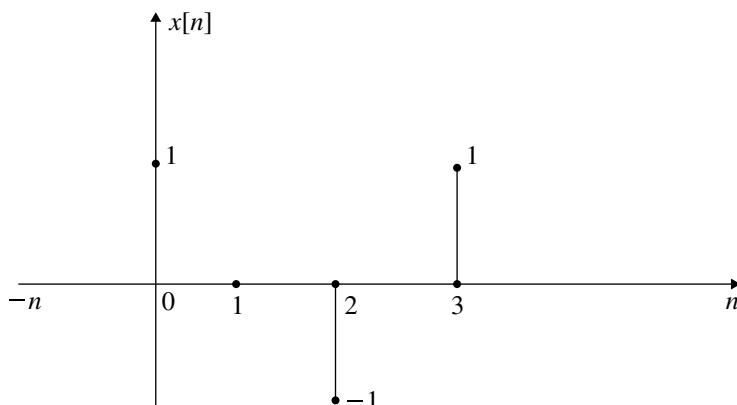
A digital signal is not very much different from a discrete signal except that a digital signal amplitude is quantized at certain specific level. This is because a digital computer can accept only sequences of numbers which are expressed in terms of bits. The representation of digital signal is shown in Fig. 1.3b. The amplitude at any discrete time interval can be only one of quantization levels. If the amplitude is chosen from a finite set of numbers, the amplitude is said to be digitized. The quantization level is equidistance and is called quantization step. An analog signal can be converted into a digital signal by means of sampling and quantizing. It is then digitized by rounding off its value to the closest permissible level. Thus, error exists when an analog signal is digitized and quantized. This is called quantization error. The digital signal is, therefore, digitized in time and quantized in amplitude.

### Example 1.1

Graphically represent the following sequence:

$$x\{n\} = \{1, 0, -1, 1\}$$

**Solution** The graphical representation  $x\{n\} = \{1, 0, -1, 1\}$  is shown in Fig. 1.4.



**Fig. 1.4** Graphical representation of  $x[n]$

**Example 1.2**

Graphically represent the following sequence:

$$x\{n\} = \{-2, 1, 0, 1, 2, 0, 1\}$$

↑

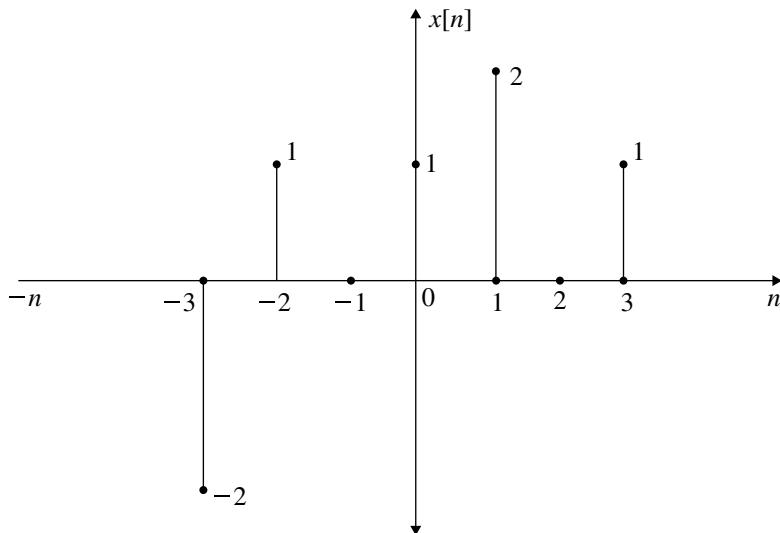
---

**Solution** The sequence

$$x\{n\} = \{-2, 1, 0, 1, 2, 0, 1\}$$

↑

is represented in Fig. 1.5.



**Fig. 1.5** Graphical representation of  $x[n]$

## 1.4 Basic Discrete Time Signals

Similar to continuous time signals, basic discrete signals are available. However, these signals are represented at discrete interval of time “ $n$ ” where  $n$  is an integer. Representations of basic discrete time signals are discussed below.

### 1.4.1 The Unit Impulse Sequence

The basic impulse sequence is shown in Fig. 1.6. The unit impulse sequence also called sample is defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \quad (1.2)$$

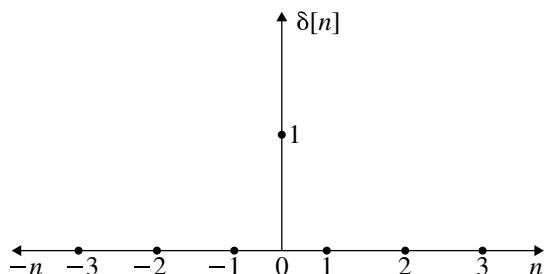
### 1.4.2 The Basic Unit Step Sequence

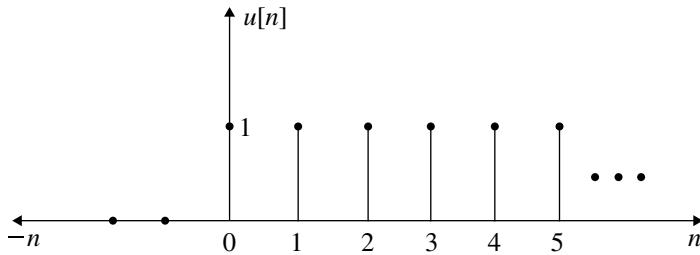
The basic unit step sequence is represented in Fig. 1.7. It is denoted by  $u(n)$ . It is defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (1.3)$$

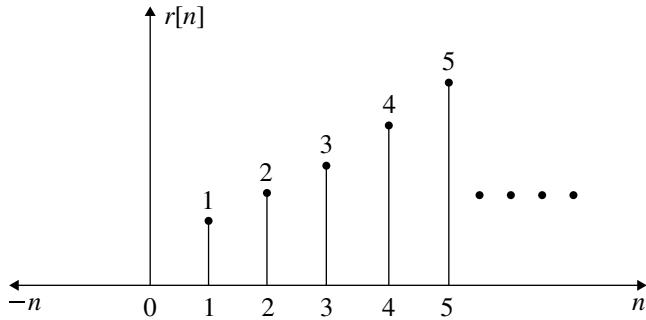
Any discrete sequence  $x[n]$  for  $n \geq 0$  is expressed as  $x[n]u[n]$ . For  $n < 0$ , it is expressed as  $x[n]u[-n]$ . It is to be noted that at  $n = 0$ , the value of  $u[n] = 1$ .

**Fig. 1.6** Basic unit impulse sequence





**Fig. 1.7** Basic unit step sequence



**Fig. 1.8** Basic unit ramp sequence

### 1.4.3 The Basic Unit Ramp Sequence

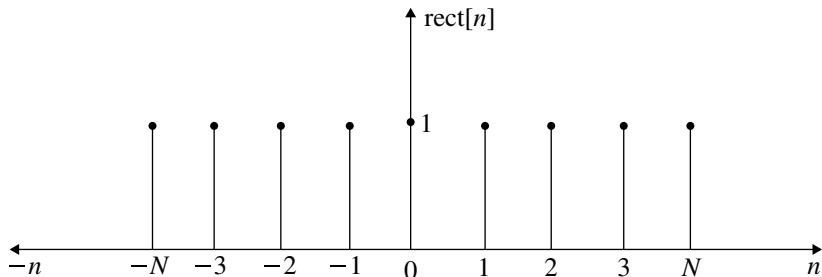
The basic unit ramp sequence which is denoted by  $r[n]$  is represented in Fig. 1.8. It is defined as

$$r[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (1.4)$$

### 1.4.4 Unit Rectangular Sequence

The discrete time unit Rectangular sequence is shown in Fig. 1.9. It is defined as

$$\text{rect}[n] = \begin{cases} 1 & |n| \leq N \\ 0 & |n| > N \end{cases} \quad (1.5)$$



**Fig. 1.9** Unit Rectangular sequence

The above equation can also be expressed as

$$\text{rect}[n] = 1 \quad -N \leq n \leq N.$$

### 1.4.5 Sinusoidal Sequence

The discrete time sinusoidal signal is defined by the following mathematical expression:

$$x[n] = A e^{-\alpha n} \sin(\omega_0 n + \phi) \quad (1.6)$$

where  $A$  and  $\alpha$  are real numbers and  $\phi$  is the phase shift. Depending on the value of  $\alpha$ , the sinusoidal sequence is divided into the following categories:

- A purely sinusoidal sequence ( $\alpha = 0$ ).
- Decaying sinusoidal sequence ( $\alpha > 0$ ).
- Growing sinusoidal sequence ( $\alpha < 0$ ).

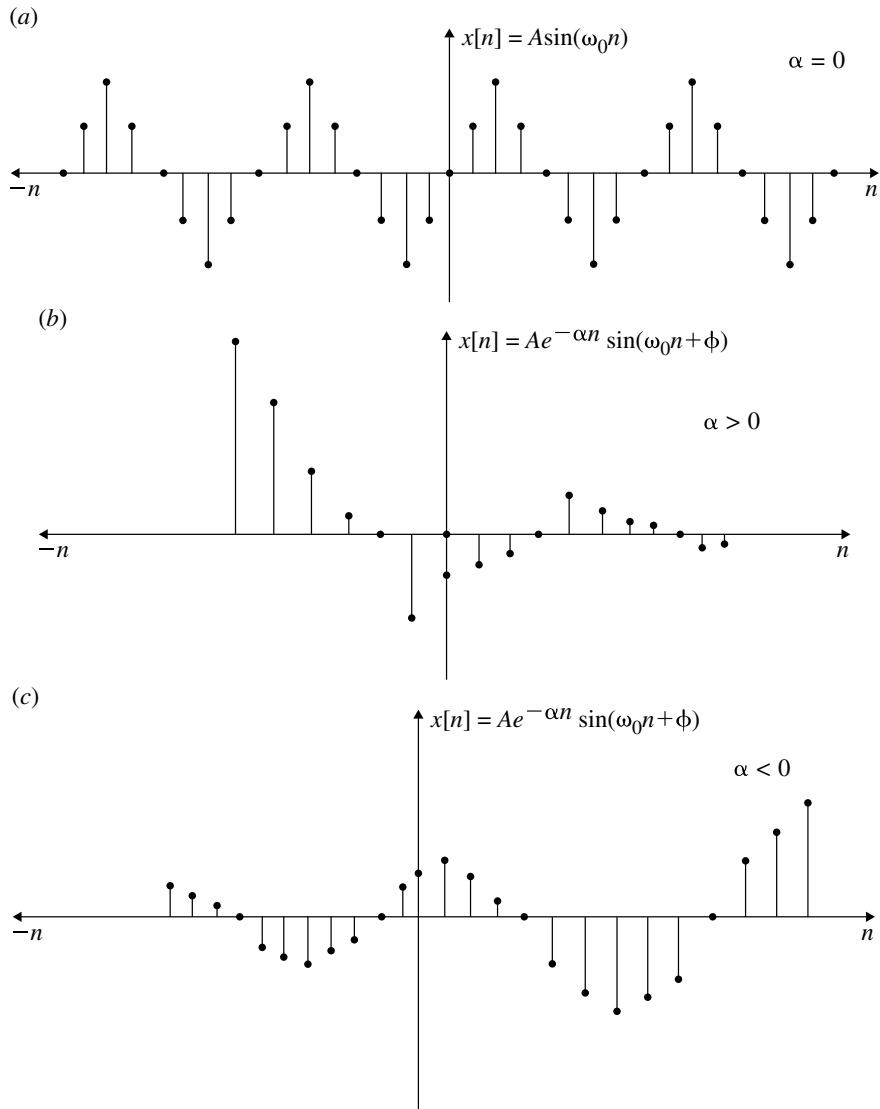
The above sinusoidal sequences are illustrated in Fig. 1.10a–c, respectively.

### 1.4.6 Discrete Time Real Exponential Sequence

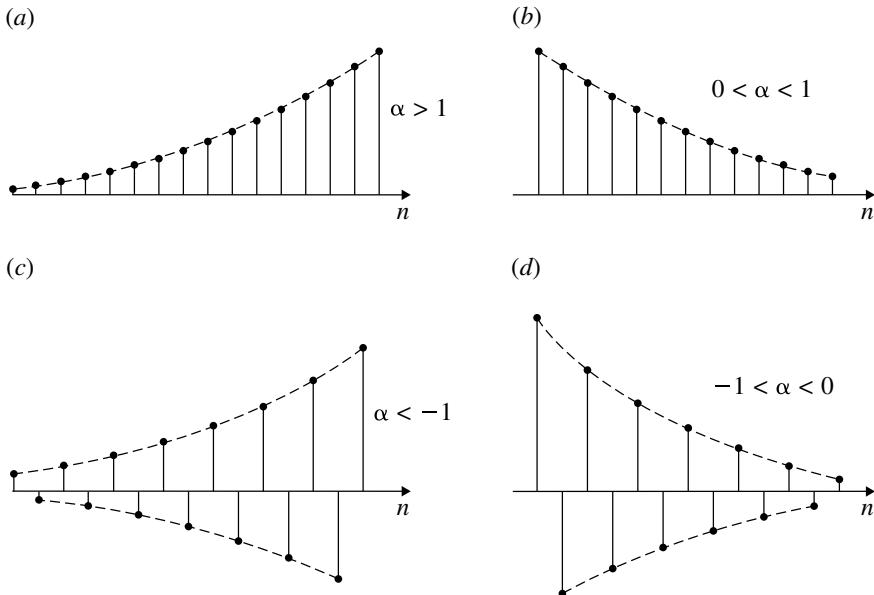
The general complex exponential sequence is defined as

$$x[n] = A \alpha^n \quad (1.7)$$

where  $A$  and  $\alpha$  are in general complex numbers.



**Fig. 1.10** Discrete time sinusoidal signal. **a** Purely sinusoidal. **b** Decaying sinusoidal. **c** Growing sinusoidal



**Fig. 1.11** Discrete time real exponential sequences. **a**  $\alpha > 1$ . **b**  $0 < \alpha < 1$ . **c**  $\alpha < -1$ . **d**  $-1 < \alpha < 0$

In Eq. (1.7) if  $A$  and  $\alpha$  are real, the sequence is called real exponential. These sequences for various values of  $\alpha$  are shown in Fig. 1.11. Depending on the value of  $\alpha$ , the sequence is classified as follows:

1. Exponentially growing signal ( $\alpha > 1$ , Fig. 1.11a).
2. Exponentially decaying signal ( $0 < \alpha < 1$ , Fig. 1.11b).
3. Exponentially growing for alternate value of  $n$  ( $\alpha < -1$ , Fig. 1.11c).
4. Exponentially decaying for alternate value of  $n$  ( $-1 < \alpha < 0$ , Fig. 1.11d).

## 1.5 Basic Operations on Discrete Time Signals

The basic operations that are applied to continuous time signals are also applicable to discrete time signals. The time  $t$  in CT signal is replaced by  $n$  in DT signals. The basic operations as applied to DT signals are explained below.

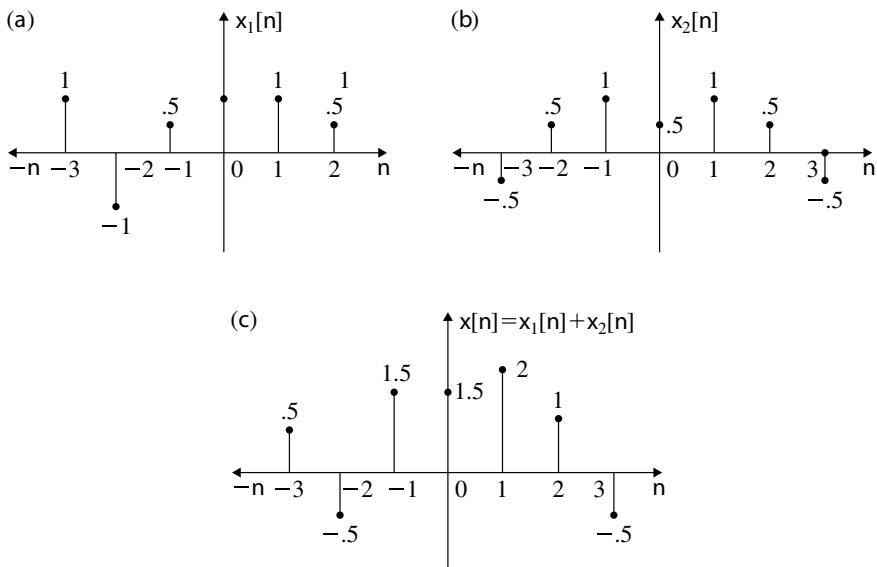


Fig. 1.12 Addition of DT signals

### 1.5.1 Addition of Discrete Time Sequence

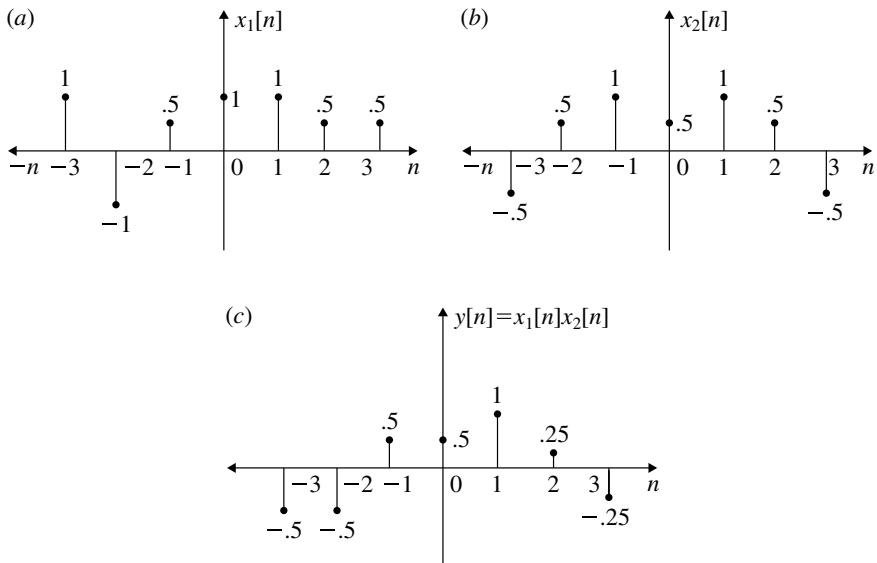
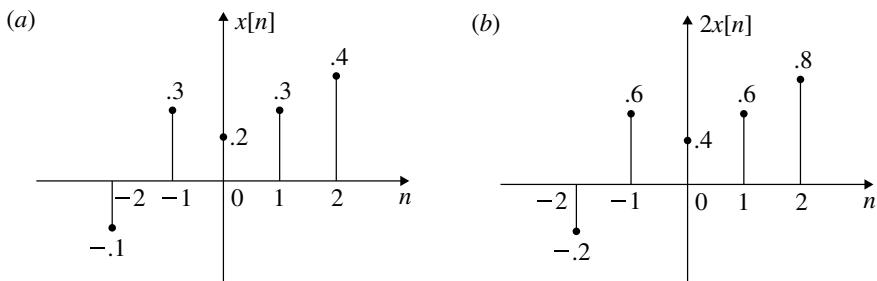
Addition of discrete time sequence is done by adding the signals at every instant of time. Consider the signals  $x_1[n]$  and  $x_2[n]$  shown in Fig. 1.12a, b, respectively. The addition of these signals at every  $n$  is done and represented as  $y[n] = x_1[n] + x_2[n]$ . This is shown in Fig. 1.12c.

### 1.5.2 Multiplication of DT Signals

The multiplication of two DT signals  $x_1[n]$  and  $x_2[n]$  is obtained by multiplying the signal values at each instant of time  $n$ . Consider the signals  $x_1[n]$  and  $x_2[n]$  represented in Fig. 1.13a, b. At each instant of time  $n$ , the samples of  $x_1[n]$  and  $x_2[n]$  are multiplied and represented as shown in Fig. 1.13c.

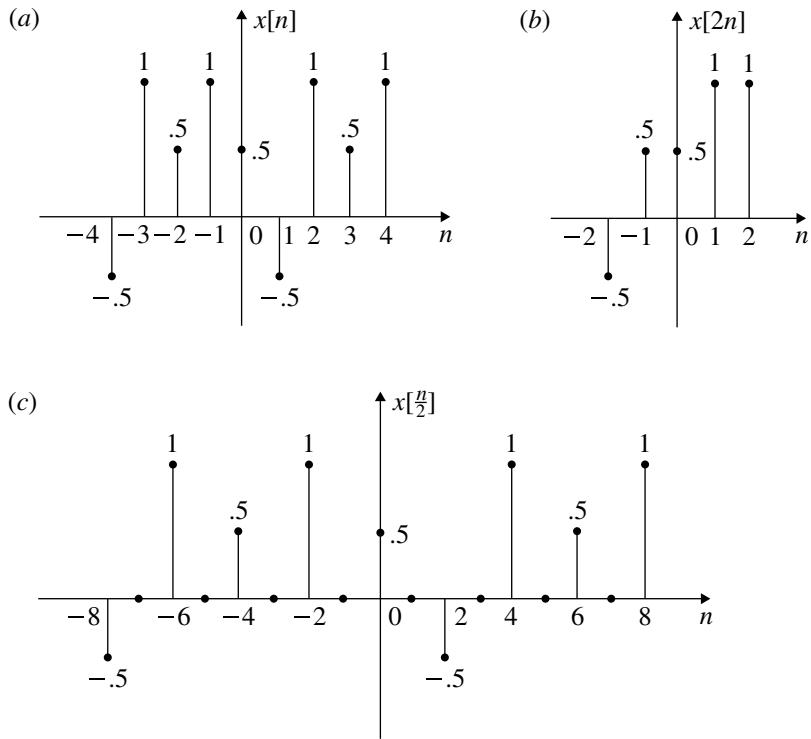
### 1.5.3 Amplitude Scaling of DT Signal

Let  $x[n]$  be a discrete time signal. The signal  $Ax[n]$  is represented by multiplying the amplitude of the sequence by  $A$  at each instant of time  $n$ . Consider the signal  $x[n]$  shown in Fig. 1.14a. The signal  $2x[n]$  is represented and shown in Fig. 1.14b.

**Fig. 1.13** Multiplications of two DT signals**Fig. 1.14** Amplitude scaling of DT signals

#### 1.5.4 Time Scaling of DT Signal

The time compression or expansion of a DT signal in time is known as time scaling. Consider the signal  $x[n]$  shown in Fig. 1.15a. The time compressed signal  $x[2n]$  and time expanded signal  $x[\frac{n}{2}]$  are shown in Fig. 1.15b, c respectively. One should note that while doing compression and expansion of DT signal, **only for integer value of  $n$ , the samples exist. For non-integer value of  $n$ , the samples do not exist.**



**Fig. 1.15** Time scaling of DT signal

### Time Compression

Let

$$\begin{aligned} y[n] &= x[2n] \\ y[-2] &= x[-4] = -0.5 \\ y[-1] &= x[-2] = 0.5 \\ y[0] &= x[0] = 0.5 \end{aligned}$$

$$\begin{aligned} y[1] &= x[2] = 1 \\ y[2] &= x[4] = 1. \end{aligned}$$

The plot of  $x[2n]$  is shown in Fig. 1.15b.

## Time Expansion

Let

$$\begin{aligned}
 y[n] &= x\left[\frac{n}{2}\right] \\
 y[-8] &= x[-4] = -0.5 \\
 y[-6] &= x[-3] = 1 \\
 y[-4] &= x[-2] = 0.5 \\
 y[-2] &= x[-1] = 1 \\
 y[0] &= x[0] = 0.5 \\
 y[2] &= x[1] = -0.5 \\
 y[4] &= x[2] = 1 \\
 y[6] &= x[3] = 0.5 \\
 y[8] &= x[4] = 1.
 \end{aligned}$$

The plot of  $x[\frac{n}{2}]$  is shown in Fig. 1.15c.

### 1.5.5 Time Shifting of DT Signal

As in the case of CT signal, time shifting property is applied to DT signal also. Let  $x[n]$  be the DT signal. Let  $n_0$  be the time by which  $x[n]$  is time shifted. Since  $n$  is an integer,  $n_0$  is also an integer. The following points are applicable while DT signal is time shifted.

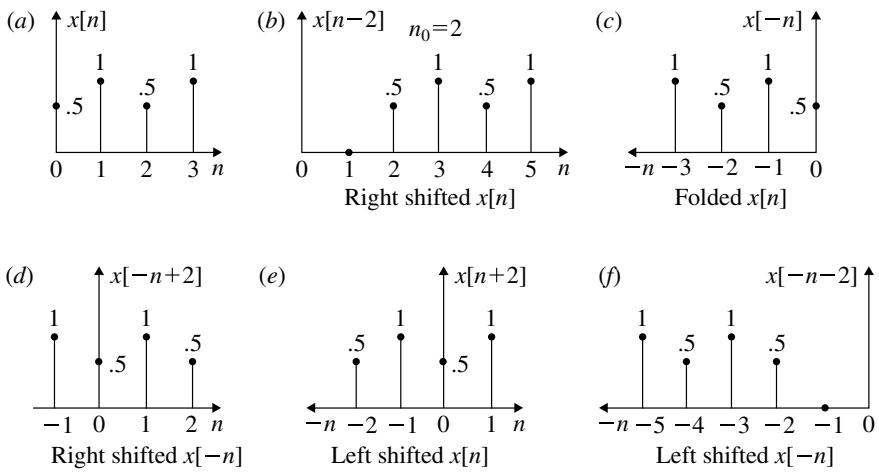
- For the DT signals  $x[-n - n_0]$  and  $x[n + n_0]$ , the signals  $x[-n]$  and  $x[n]$  are to be left shifted by  $n_0$ .
- For the DT signals  $x[n - n_0]$  and  $x[-n + n_0]$ , the signals  $x[n]$  and  $x[-n]$  are to be right shifted by  $n_0$ .

Figure 1.16 shows time shifting of DT signal.

In Fig. 1.16a the sequence  $x[n]$  is shown. The sequence  $x[n - 2]$  which is right shifted by two samples is shown in Fig. 1.16b.  $x[-n]$  which is the folded signal is shown in Fig. 1.16c.  $x[-n + 2]$  which is left shifted of  $x[-n]$  is shown in Fig. 1.16d.  $x[n + 2]$  which is right shifted of  $x[n]$  is shown in Fig. 1.16e.  $x[-n - 2]$  which is left shifted of  $x[-n]$  is shown in Fig. 1.16f.

### 1.5.6 Multiple Transformation

The transformations, namely amplitude scaling, time reversal, time shifting, time scaling, etc., are applied to represent DT sequence. The sequence of operation of these transformations is important and followed as described below.



**Fig. 1.16** Time shifting of DT signal

Consider the following DT signal:

$$y[n] = Ax \left[ -\frac{n}{a} + n_0 \right]$$

1. Plot  $x[n]$  sequence and obtain  $Ax[n]$  by amplitude scaling.
2. Using time reversal (folding), plot  $Ax[-n]$ .
3. Using time shifting, plot  $Ax[-n + n_0]$  where  $n_0 > 0$ . The time shift is to be right of  $x[-n]$  by  $n_0$  samples.
4. Using time scaling, plot  $Ax[-\frac{n}{a} + n_0]$  where  $a$  is in integer. In the above case, keeping amplitude constant, time is expanded by  $a$ .

The following examples illustrate the above operations:

### Example 1.3

Let  $x[n]$  and  $y[n]$  be as given in Fig. 1.17a, b, respectively. Plot

- (a)  $x[2n]$
- (b)  $x[3n - 1]$
- (c)  $x[n - 2] + y[n - 2]$
- (d)  $y[1 - n]$

**Solution**

- (a) **To plot  $x[2n]$**  Here, the DT sequence is time compressed by a factor 2. Hence, the samples only with even numbers are divided by a factor 2 and the corresponding amplitudes marked and shown in Fig. 1.17c. When odd values of  $n$  are divided by the factor 2, it becomes a fraction, and they are skipped.
- (b) **To plot  $x[3n - 1]$**  The plot of  $x[n - 1]$  is obtained by right shifting of  $x[n]$  by  $n_0 = 1$ . This is shown in Fig. 1.17d. When  $x[n - 1]$  is time compressed by a factor 3,  $x[3n - 1]$  is obtained. Only integers which are divisible by 3 in the sequence  $x[n - 1]$  are to be taken to plot  $x[3n - 1]$ . Thus samples for  $n = 0$  and  $n = 3$  will be plotted as shown in Fig. 1.17e.
- (c) **To plot  $x[n - 2] + y[n - 2]$**  The sequence  $x[n - 2]$  is obtained by right shifting of  $x[n]$  by 2 and is shown in Fig. 1.17f. Similarly, the sequence  $y[n - 2]$  is obtained by right shifting of  $y[n]$  by 2 and is shown in Fig. 1.17g. The sequence  $x[n - 2] + y[n - 2]$  is obtained by summing up the sequences in Fig. 1.17g, for all  $n$  and is shown in Fig. 1.17h.
- (d) **To plot  $y[1 - n]$**  The sequence  $y[-n]$  is obtained by folding  $y[n]$  and is shown in Fig. 1.17i.  $y[-n]$  is right shifted by 1 sample to get the sequence  $y[1 - n]$ . This is shown in Fig. 1.17j.

**Example 1.4**

Consider the sequence shown in Fig. 1.18a. Express the sequence in terms of step function.

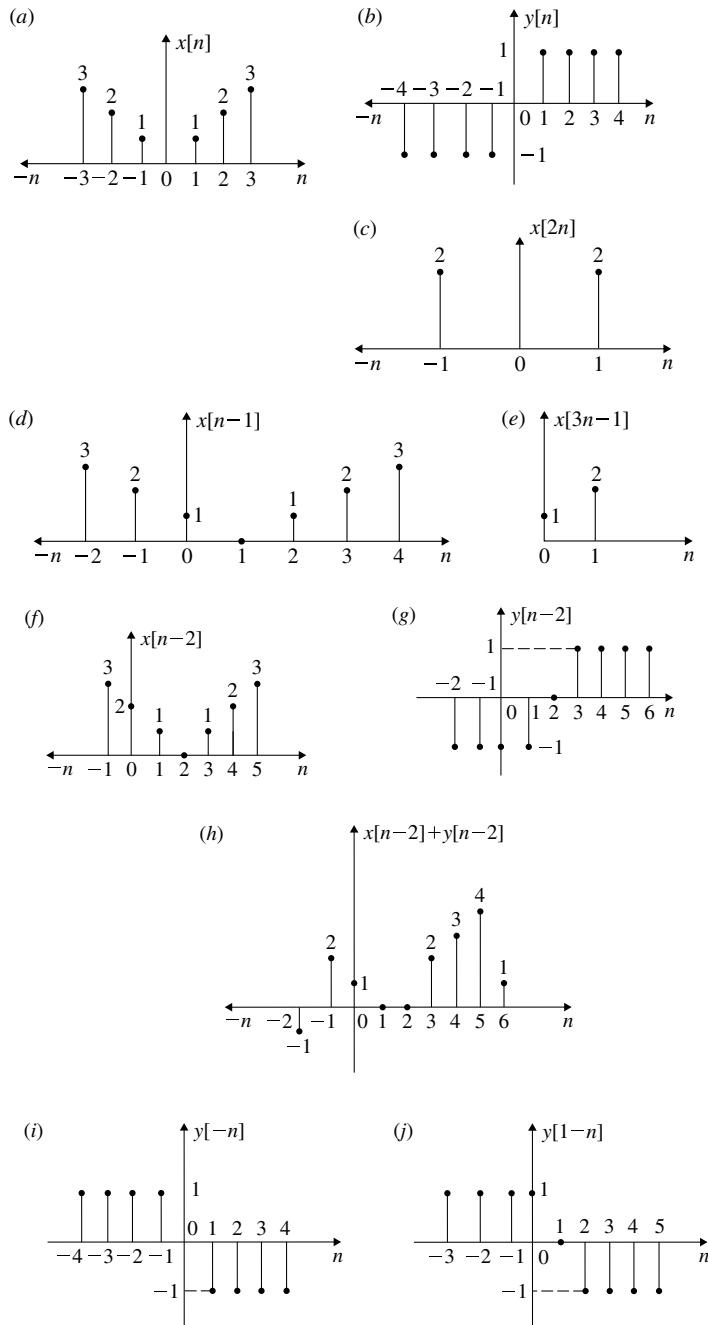
**Solution** The unit step sequence  $u[n]$  is shown in Fig. 1.18b. The unit negative step sequence with a time delay of  $n_0 = 4$  is shown in Fig. 1.18b. It is evident from Fig. 1.18 that  $\{u[n] - u[n - 4]\}$  gives the required  $x[n]$  sequence which is represented in Fig. 1.18a. Thus,  $x[n] = \{u[n] - u[n - 4]\}$ .

**Example 1.5**

Consider the sequence shown in Fig. 1.19a. Express the sequence in terms of step function.

**Solution**

- Figure 1.19a represents the sequence  $x[n]$  in the interval  $-3 \leq n \leq 4$ .
- Consider  $u[n + 3]$  which is represented in Fig. 1.19b. The sequence interval is  $-3 \leq n < \infty$ .
- Consider the step sequence with a time delay of  $n_0 = 5$  and inverted. This can be written as  $-u[n - 5]$  for the interval  $5 \leq n < \infty$ . This is represented in Fig. 1.19c.

**Fig. 1.17** Two discrete sequences

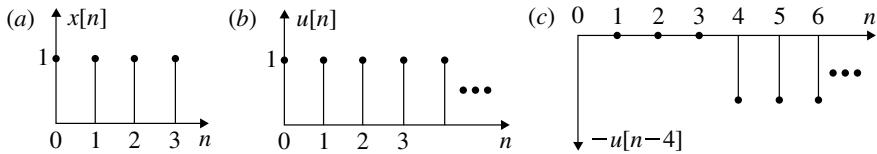


Fig. 1.18 Sequences expressed in terms of step sequences

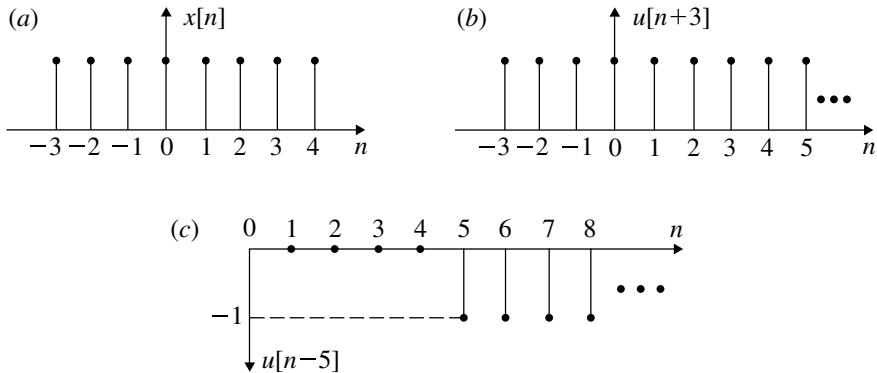


Fig. 1.19 DT sequences expressed in terms of step sequences

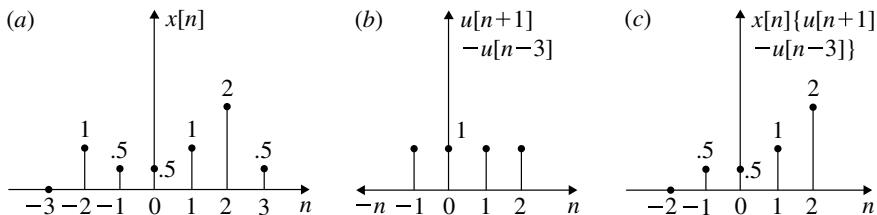


Fig. 1.20 Multiplication of DT sequences

4. Now consider the sum of the sequences  $u[n + 3]$  and  $-u[n - 5]$ . This is nothing but  $x[n]$ . Thus

$$x[n] = u[n + 3] - u[n - 5]$$

### Example 1.6

A discrete time sequence  $x[n]$  is shown in Fig. 1.20a. Find

$$x[n]\{u[n + 1] - u[n - 3]\}$$

**Solution**

1.  $x[n]$  sequence is represented in Fig. 1.20a.
  2.  $\{u[n+1] - u[n-3]\}$  sequence is nothing but the time delayed unit step sequence with  $n_0 = 3$ , being subtracted from the time advanced unit step sequence with  $n_0 = 1$ . This sequence is represented in Fig. 1.20b.
  3. Multiplying, sample wise of Fig. 1.20a, b, the required sequence  $x[n]\{u[n+1] - u[n-3]\}$  is obtained and represented in Fig. 1.20c.
- 

**Example 1.7**

Sketch  $x[n] = a^n$  where  $-2 \leq n \leq 2$  for the two cases shown below:

$$\begin{aligned} (1) \quad a &= \left(-\frac{1}{4}\right) \\ (2) \quad a &= -4 \end{aligned}$$


---

(Anna University, May, 2007)

**Solution** For  $x[n] = (-\frac{1}{4})^n$  and  $x[n] = (-4)^n$  where  $-2 \leq n \leq 2$ ,  $x[n]$  is found and tabulated below:

$n$	-2	-1	0	1	2
$x[n] = (-\frac{1}{4})^n$	16	-4	1	$-\frac{1}{4}$	$\frac{1}{16}$
$x[n] = (-4)^n$	$\frac{1}{16}$	$-\frac{1}{4}$	1	-4	16

The samples of  $x[n]$  are plotted and shown in Fig. 1.21.  $x[n] = (-\frac{1}{4})^n$  is represented in Fig. 1.21a, and  $x[n] = (-4)^n$  is represented in Fig. 1.21b.

---

**Example 1.8**

Express

$$x[n] = (-1)^n \quad -2 \leq n \leq 2$$

as a sum of scaled and shifted step function.

(Anna University, May, 2007)

**Solution**

- (1)  $x[n] = (-1)^n$  is tabulated for  $-2 \leq n \leq 2$ . The samples corresponding to the above table are sketched and shown as  $x[n]$  in Fig. 1.22a.

$n$	-2	-1	0	1	2
$x[n]$	1	-1	1	-1	1

- (2) Consider the step sequence  $u[n+2]$  for  $-2 \leq n \leq 2$ . The samples are shown in Fig. 1.22b.

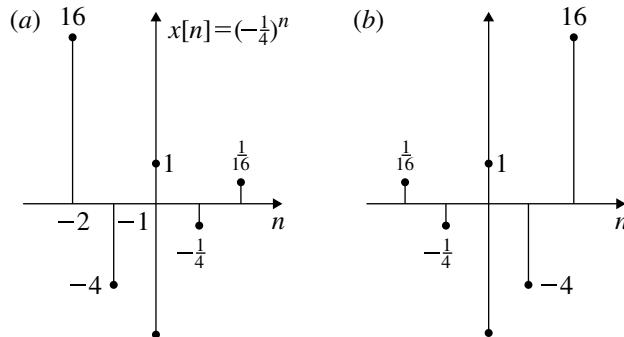


Fig. 1.21 DT sequences of Example 1.7

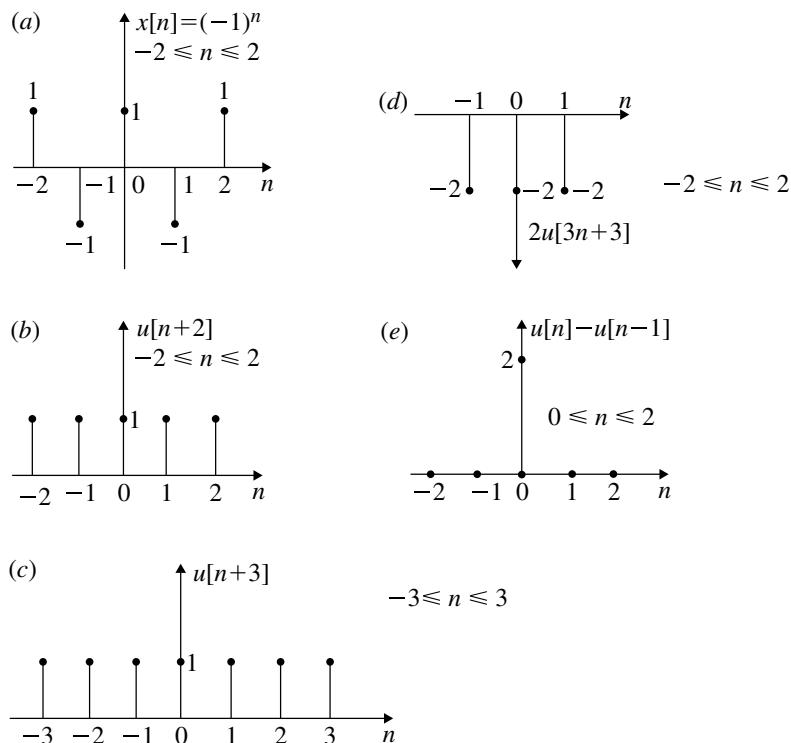


Fig. 1.22 DT sequences of Example 1.8

- (3) Consider unit step sequence  $u[n + 3]$  for  $-3 \leq n \leq 3$ . This is represented in Fig. 1.22c. From Fig. 1.22c,  $-2u[3n + 3]$  is obtained by amplitude inversion and multiplication and time scaling (compression). This is represented in Fig. 1.22d for  $-2 \leq n \leq 2$ .
- (4) Consider the step sequence  $2\{u[n] - u[n - 1]\}$  for  $n \geq 0$ . This is represented in Fig. 1.22e. This is nothing but the sample of strength 2 at  $n = 0$ .
- (5) Now, by adding the samples in Fig. 1.22b, d, e, it can be easily verified that

$$\boxed{x[n] = u[n + 2] - 2u[3n + 3] + 2[u[n] - u[n - 1]]} \quad -2 \leq n \leq 2$$


---

### Example 1.9

Given

$$x[n] = \{1, 2, 3, -4, 6\}$$

↑

Plot the signal  $x[-n - 1]$ .

*(Anna University, May, 2007)*

#### Solution

1. The sequence  $x[n]$  is represented in Fig. 1.23a.
2. By folding  $x[n]$ ,  $x[-n]$  is obtained and represented in Fig. 1.23b.
3.  $x[-n]$  is shifted to the left by one sample, and  $x[-n - 1]$  is obtained. This is represented in Fig. 1.23c.

## 1.6 Classification of Discrete Time Signals

Discrete time signals are classified as follows:

1. Periodic and non-periodic signals.
2. Odd and even signals.
3. Power and energy signals.

They are discussed below with suitable examples.

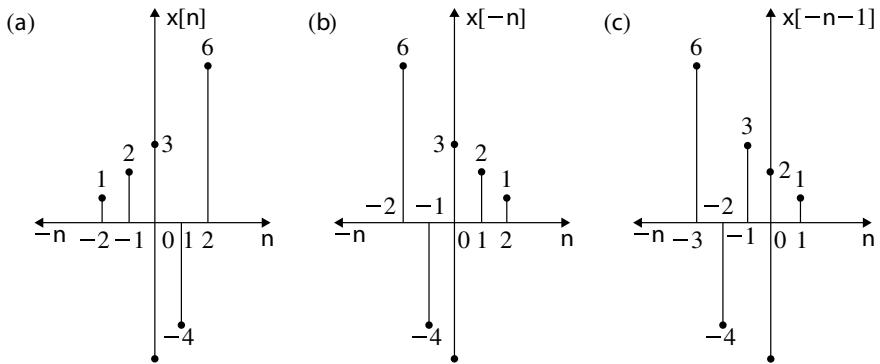


Fig. 1.23 DT sequences of Example 1.9

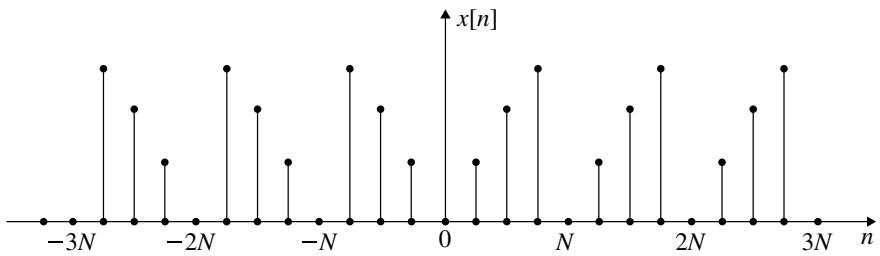


Fig. 1.24 Periodic sequence

### 1.6.1 Periodic and Non-periodic DT Signals

A discrete time signal (sequence)  $x[n]$  is said to be periodic with period  $N$  which is a positive integer if

$$x[n + N] = x[n] \quad \text{for all } n \quad (1.8)$$

Consider the DT sequence shown in Fig. 1.24. The signal gets repeated for every  $N$ . For Fig. 1.24, the following equation is written:

$$x[n + mN] = x[n] \quad \text{for all } n \quad (1.9)$$

where  $m$  is any integer. The smallest positive integer  $N$  in Eq. (1.9) is called the fundamental period  $N_0$ . Any sequence which is not periodic is said to be non-periodic or aperiodic.

**Example 1.10**

Show that complex exponential sequence  $x[n] = e^{j\omega_0 n}$  is periodic and find the fundamental period.

---

**Solution**

$$\begin{aligned}x[n] &= e^{j\omega_0 n} \\x[n+N] &= e^{j\omega_0(n+N)} \\&= e^{j\omega_0 n}e^{j\omega_0 N} \\&= e^{j\omega_0 n} \quad \text{if } e^{j\omega_0 N} = 1 \\&\omega_0 N = m2\pi \quad \text{where } m \text{ is any integer.}\end{aligned}$$

$$N = m \frac{2\pi}{\omega_0}$$

or

$$\frac{\omega_0}{2\pi} = \frac{m}{N} = \text{ rational number.}$$

Thus,  $e^{j\omega_0 n}$  is periodic if  $\frac{m}{N}$  is rational. For  $m = 1$ ,  $N = N_0$ . The corresponding frequency  $F_0 = \frac{1}{N_0}$  is the fundamental frequency.  $F_0$  is expressed in cycles and not Hz. Similarly  $\omega_0$  is expressed in radians and not in radians per second.

---

**Example 1.11**

Consider the following DT signal.

$$x[n] = \sin(\omega_0 n + \phi)$$

Under what condition, the above signal is periodic?

---

**Solution**

$$\begin{aligned}x[n] &= \sin(\omega_0 n + \phi) \\x[n+N] &= \sin(\omega_0(n+N) + \phi) \\&= \sin(\omega_0 n + \omega_0 N + \phi) \\&= \sin(\omega_0 n + \phi) \quad \text{if } \omega_0 N = 2\pi m \quad \text{where } m \text{ is an integer} \\&= x[n]\end{aligned}$$

$$\frac{\omega_0}{2\pi} = \frac{m}{N} = \text{ rational}$$

**Example 1.12**

If  $x_1[n]$  and  $x_2[n]$  are periodic, then show that the sum of the composite signal  $x[n] = x_1[n] + x_2[n]$  is also periodic with the least common multiple (LCM) of the fundamental period of individual signal.

**Solution** Let  $N_1$  and  $N_2$  be the fundamental periods of  $x_1[n]$  and  $x_2[n]$ , respectively. Since both  $x_1[n]$  and  $x_2[n]$  are periodic,

$$\begin{aligned}x_1[n] &= x_1[n + mN_1] \\x_2[n] &= x_2[n + kN_2] \\x[n] &= x_1[n] + x_2[n] \\&= x_1[n + mN_1] + x_2[n + kN_2]\end{aligned}$$

For  $x[n]$  to be periodic with period  $N$ ,

$$\begin{aligned}x[n + N] &= x_1[n + N] + x_2[n + N] \\x[n] &= x[n + N] \\x_1[n + mN_1] + x_2[n + kN_2] &= x_1[n + N] + x_2[n + N]\end{aligned}$$

The above equation is satisfied if

$$mN_1 = kN_2 = N$$

$m$  and  $k$  which are integers are chosen to satisfy the above equation. It implies that  $N$  is the LCM of  $N_1$  and  $N_2$ .

On similar line it can be proved that if  $x_1[n]$  and  $x_2[n]$  are periodic signals with fundamental period  $N_1$  and  $N_2$ , respectively, then  $x[n] = x_1[n]x_2[n]$  is periodic if

$$mN_1 = kN_2 = N$$

**Example 1.13**

Find whether the following signals are periodic. If periodic, determine the fundamental period

- (a)  $x[n] = e^{j\pi n}$
- (b)  $x[n] = \cos\left[\frac{n}{8} - \pi\right]$
- (c)  $x[n] = \sin^2 \frac{\pi}{4} n$

**Solution**

(a)  $x[n] = e^{j\pi n}$

$$\begin{aligned}\omega_0 &= \pi \\ N &= \frac{2\pi}{\omega_0}m\end{aligned}$$

$$N = \frac{2\pi}{\pi} = 2 \quad \text{if } m = 1$$

$x[n]$  is periodic with fundamental period 2.

(b)  $x[n] = \cos\left[\frac{n}{8} - \pi\right]$

$$\begin{aligned}\omega_0 &= \frac{1}{8} \\ N &= \frac{2\pi}{\omega_0}m = 16\pi m\end{aligned}$$

For any integer value of  $m$ ,  $N$  is not integer. Hence,  $x[n]$  is not periodic.

$x[n]$  is not periodic

(c)  $x[n] = \sin^2 \frac{\pi}{4}n$

$$\begin{aligned}x[n] &= \sin^2 \frac{\pi}{4}n \\ &= \frac{1}{2} - \frac{1}{2} \cos \frac{2\pi}{4}n \\ &= x_1[n] + x_2[n] \\ x_1[n] &= \frac{1}{2} = \frac{1}{2}(1)^n \text{ is periodic with } N_1 = 1 \\ x_2[n] &= -\frac{1}{2} \cos \frac{\pi}{2}n \\ \omega_0 &= \frac{\pi}{2} \\ N_2 &= \frac{2\pi}{\omega_0}m = 4m = 4 \quad \text{for } m = 1 \\ \frac{N_1}{N_2} &= \frac{1}{4} \\ \text{or } 4N_1 &= N_2 = N\end{aligned}$$

$N = 4$

**Example 1.14**

Find the periodicity of the DT signal

$$x[n] = \sin \frac{2\pi}{3}n + \cos \frac{\pi}{2}n$$

(Anna University, December, 2007)

**Solution**

$$\begin{aligned} x[n] &= \sin \frac{2\pi}{3}n + \cos \frac{\pi}{2}n \\ &= x_1[n] + x_2[n] \\ x_1[n] &= \sin \frac{2}{3}\pi n \\ \omega_1 &= \frac{2}{3}\pi \\ N_1 &= \frac{2\pi}{\omega_1}m_1 = \frac{2\pi}{2\pi}3m_1 = 3 \quad \text{for } m_1 = 1 \\ x_2[n] &= \cos \frac{\pi}{2}n \\ \omega_2 &= \frac{\pi}{2} \\ N_2 &= \frac{2\pi}{\omega_2}m_2 = \frac{2\pi}{\pi}2m_2 = 4 \quad \text{for } m_2 = 1 \\ \frac{N_1}{N_2} &= \frac{3}{4} \text{ or } 4N_1 = 3N_2 = N \end{aligned}$$

$N = 12$

**Example 1.15**

Determine whether the following signal is periodic. If periodic, find its fundamental period.

$$x[n] = \cos\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{4}\right)$$

(Anna University, December, 2006)

**Solution**

$$\begin{aligned}
 x[n] &= \cos\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{4}\right) \\
 &= x_1[n]x_2[n] \\
 x_1[n] &= \cos \frac{n\pi}{2} \\
 \omega_1 &= \frac{\pi}{2} \\
 N_1 &= \frac{2\pi}{\omega_1} m_1 = \frac{2\pi}{\pi} 2m_1 = 4 \quad \text{for } m_1 = 1
 \end{aligned}$$

$$\begin{aligned}
 x_2[n] &= \cos \frac{n\pi}{4} \\
 \omega_2 &= \frac{\pi}{4} \\
 N_2 &= \frac{2\pi}{\omega_2} m_2 = \frac{2\pi}{\pi} 4m_2 = 8 \quad \text{for } m_2 = 1 \\
 \frac{N_1}{N_2} &= \frac{4}{8} = \frac{1}{2} \quad \text{or} \\
 2N_1 &= N_2 = N
 \end{aligned}$$

$N = 8$

The signal is periodic, and the fundamental period  $N = 8$ .

**Example 1.16**

Test whether the following signals are periodic or not and if periodic, calculate the fundamental period.

- (a)  $x[n] = \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{8}n\right) + 3 \cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$   
 (b)  $x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$

(Anna University, December, 2007)

**Solution**

(a)

$$x[n] = \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{8}n\right) + 3\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$$

$$= x_1[n] + x_2[n] + x_3[n]$$

$$x_1[n] = \cos\frac{\pi}{2}n$$

$$\omega_1 = \frac{\pi}{2}; \quad N_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\pi} \quad \text{for } m_1 = 1$$

$$N_1 = 4$$

$$x_2[n] = \sin\left(\frac{\pi}{8}n\right)$$

$$\omega_2 = \frac{\pi}{8}; \quad N_2 = \frac{2\pi}{\omega_2}m_2 = \frac{2\pi}{\pi}8 \quad \text{for } m_2 = 1$$

$$N_2 = 16$$

$$x_3[n] = 3\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$$

$$\omega_3 = \frac{\pi}{4}; \quad N_3 = \frac{2\pi}{\omega_3}m_3 = \frac{2\pi}{\pi}4 \quad \text{for } m_3 = 1$$

$$N_3 = 8$$

To find the LCM of  $N_1$ ,  $N_2$  and  $N_3$ ,

4	4,	8,	16
2	1,	2,	4
	1,	1,	2

$$\text{LCM} = 4 \times 2 \times 2 = 16$$

$$N = 16$$

The signal is periodic.

(b)

$$x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$$

$$= x_1[n] + x_2[n]$$

$$x_1[n] = e^{j\frac{2\pi}{3}n}$$

$$\begin{aligned}\omega_1 &= \frac{2\pi}{3}; N_1 = \frac{2\pi}{\omega_1}m_1 = \frac{2\pi}{2\pi}3 \quad \text{for } m_1 = 1 \\ N_1 &= 3 \\ x_2[n] &= e^{j\frac{3\pi}{4}n} \\ \omega_2 &= \frac{3\pi}{4}; N_2 = \frac{2\pi}{\omega_2}m_2 = \frac{2\pi}{3\pi}4m_2\end{aligned}$$

$$\begin{aligned}N_2 &= 8 \quad \text{for } m_2 = 3 \\ \frac{N_1}{N_2} &= \frac{3}{8} \\ 8N_1 &= 3N_2 = N = 24\end{aligned}$$

$N = 24$

The signal is periodic with fundamental period  $N = 24$ .

### 1.6.2 Odd and Even DT Signals

DT signals are classified as odd and even signals. The relationships are analogous to CT signals.

A discrete time signal  $x[n]$  is said to be an even signal if

$$x[-n] = x[n] \quad (1.10)$$

A discrete time signal  $x[n]$  is said to be an odd signal if

$$x[-n] = -x[n] \quad (1.11)$$

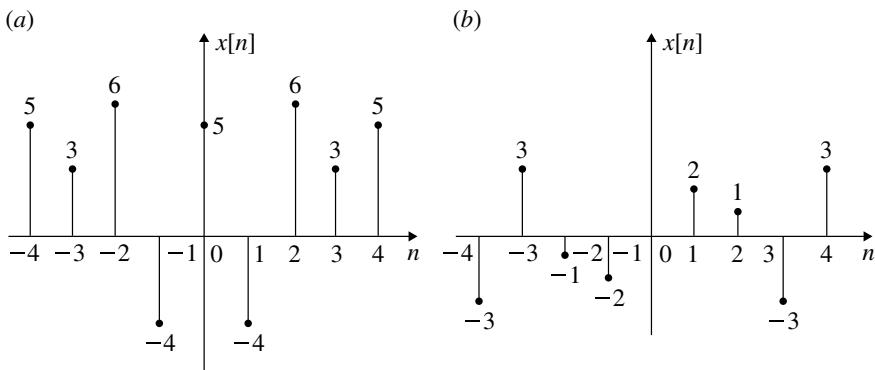
The signal  $x[n]$  can be expressed as the sum of odd and even signals as

$$x[n] = x_e[n] + x_0[n] \quad (1.12)$$

The even and odd components of  $x[n]$  can be expressed as

$$x_e[n] = \frac{1}{2}[x[n] + x[-n]] \quad (1.13)$$

$$x_0[n] = \frac{1}{2}[x[n] - x[-n]] \quad (1.14)$$



**Fig. 1.25** **a** Even function and **b** odd function

It is to be noted that

- An even function has an odd part which is zero.
- An odd function has an even part which is zero.
- The product of two even signals or of two odd signals is an even signal.
- The product of an odd and an even signal is an odd signal.
- At  $n = 0$ , the odd signal is zero.

The even and odd signals are represented in Fig. 1.25a, b, respectively.

### Example 1.17

Determine whether the following functions are odd or even:

$$\begin{array}{ll} \text{(a)} & x[n] = \sin 2\pi n \\ \text{(b)} & x[n] = \cos 2\pi n \end{array}$$

### Solution

$$\text{(a) } x[n] = \sin 2\pi n$$

$$\begin{aligned} x[-n] &= \sin(-2\pi n) = -\sin 2\pi n \\ &= -x[n] \end{aligned}$$

This is an odd signal.

(b)  $x[n] = \cos 2\pi n$

$$\begin{aligned}x[-n] &= \cos(-2\pi n) = \cos 2\pi n \\&= x[n]\end{aligned}$$

This is an even signal.

### Example 1.18

Find the even and odd components of DT signal given below. Verify the same by graphical method.

$$x[n] = \{-2, 1, 3, -5, 4\}$$

↑

**Solution**  $x[-n]$  is obtained by folding  $x[n]$ . Thus

$$\begin{aligned}x[-n] &= \{4, -5, 3, 1, -2\} \\&\quad \uparrow \\-x[-n] &= \{-4, 5, -3, -1, 2\} \\&\quad \uparrow\end{aligned}$$

$$\begin{aligned}x_e[n] &= \frac{1}{2} [x[n] + x[-n]] \\&= \frac{1}{2} [\{-2, 1, 3, -5, 4\} + \{4, -5, 3, 1, -2\}] \\&= \frac{1}{2} [(-2+4), (1-5), (3+3), (-5+1), (4-2)] \\&\quad \uparrow\end{aligned}$$

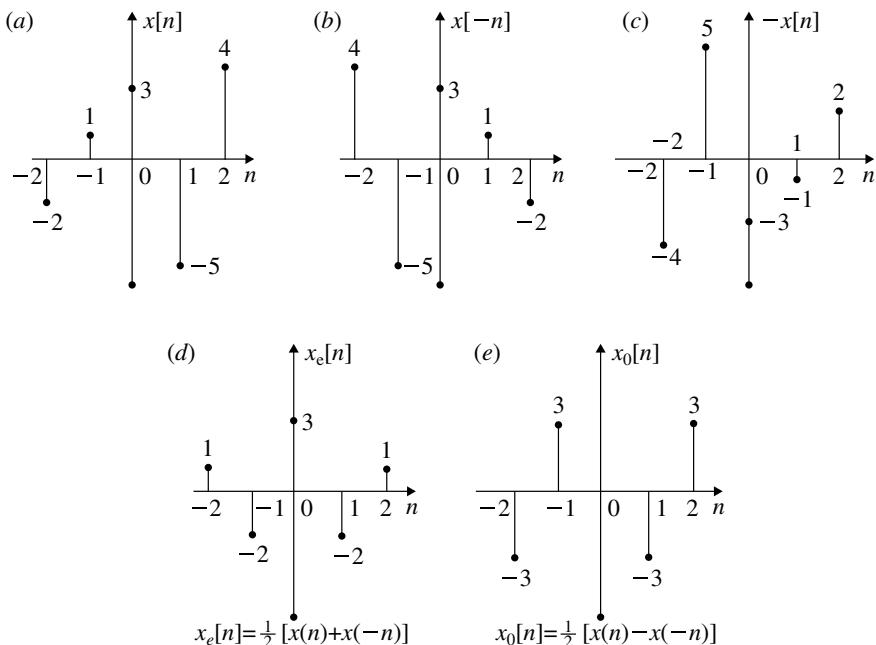
$x_e[n] = \{1, -2, 3, -2, 1\}$ 

↑

$$\begin{aligned}
 x_0[n] &= \frac{1}{2} [x[n] - x[-n]] \\
 &= \frac{1}{2} \left\{ \{-2, 1, 3, -5, 4\} + \{-4, 5, -3, -1, 2\} \right\} \\
 &= \frac{1}{2} [(-2 - 4), (1 + 5), (3 - 3), (-5 - 1), (4 + 2)] \\
 &= \boxed{x_0[n] = \{-3, 3, 0, -3, 3\}}
 \end{aligned}$$

**Odd and even components by graphical method.****Solution**

1.  $x[n]$  is represented in Fig. 1.26a.
2.  $x[-n]$  is obtained by folding  $x[n]$  which is represented in Fig. 1.26b.
3.  $-x[n]$  is obtained by inverting  $x[-n]$  of Fig. 1.26b. This is represented in Fig. 1.26c.

**Fig. 1.26** Graphical determination of even and odd function from  $x[n]$

4.  $x_e[n] = \frac{1}{2} [x[n] + x[-n]]$ . Figure 1.26a, b sample wise are added, and their amplitudes are divided by the factor 2. This gives  $x_e[n]$  and is represented in Fig. 1.26d.
  5.  $x_0[n] = \frac{1}{2} [x[n] - x[-n]]$ . Figure 1.26a, c sample wise are added, and their amplitudes are divided by a factor 2 to get  $x_0[n]$ . This is represented in Fig. 1.26e.
- 

### Example 1.19

Find the even and odd components of the following DT signal and sketch the same.

$$x[n] = \{-2, 1, 2, -1, 3\}$$

(Anna University, December, 2007)

#### Solution

$$\begin{aligned}
 x[n] &= \{-2, 1, 2, -1, 3\} \\
 x[-n] &= \{3, -1, 2, 1, -2\} \\
 x_e[n] &= \frac{1}{2} \{x[n] + x[-n]\} \\
 &= \frac{1}{2} [\{-2, 1, 2, -1, 3\} + \{3, -1, 2, 1, -2\}] \\
 &= \{1.5, -0.5, 1, 0.5, -2, 0.5, 1, -0.5, 1.5\} \\
 x_0[n] &= \frac{1}{2} [x[n] - x[-n]] \\
 &= \frac{1}{2} [\{-2, 1, 2, -1, 3\} - \{3, -1, 2, 1, -2\}] \\
 x_0[n] &= \{-1.5, 0.5, -1, -0.5, 0, 0.5, 1, -0.5, 1.5\}
 \end{aligned}$$

Even and odd components of  $x[n]$  are represented in Fig. 1.27a, b, respectively.

### 1.6.3 Energy and Power of DT Signals

For a discrete time signal  $x[n]$ , the total energy is defined as

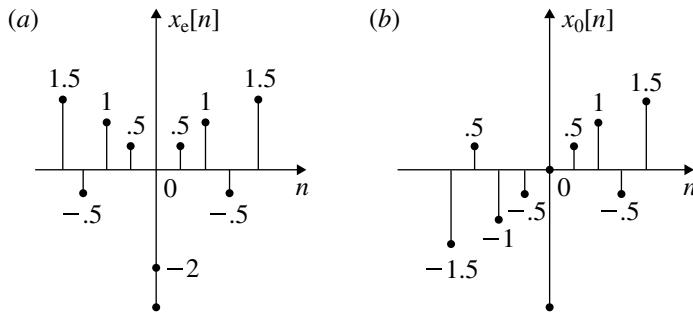


Fig. 1.27 **a** Even function and **b** odd function

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad (1.15)$$

The average power is defined as

$$P = \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2 \quad (1.16)$$

From the definitions of energy and power, the following inferences are derived:

1.  $x[n]$  is an energy sequence if  $0 < E < \infty$ . For finite energy signal, the average power  $P = 0$ .
2.  $x[n]$  is a power sequence if  $0 < P < \infty$ . For a sequence with average power  $P$  being finite, the total energy  $E = \infty$ .
3. Periodic signal is a power signal, and *vice versa* is not true. Here, the energy of the signal per period is finite.
4. Signals which do not satisfy the definitions of total energy and average power are neither termed as power signal nor energy signal. The following summation formulae are very often used while evaluating the average power and total energy of DT sequence.

1.

$$\begin{aligned} \sum_{n=0}^{N-1} a^n &= \frac{(1-a^N)}{(1-a)} & a \neq 1 \\ &= N & a = 1 \end{aligned} \quad (1.17)$$

2.

$$\sum_{n=0}^{\infty} a^n = \frac{1}{(1-a)} \quad a < 1 \quad (1.18)$$

3.

$$\sum_{n=m}^{\infty} a^n = \frac{a^m}{(1-a)} \quad a < 1 \quad (1.19)$$

4.

$$\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2} \quad a < 1 \quad (1.20)$$

**Example 1.20**

Determine whether the following signals are energy signals or power signals:

- (a)  $x[n] = A\delta[n]$
- (b)  $x[n] = u[n]$
- (c)  $x[n] = \text{ramp } n$
- (d)  $x[n] = A$
- (e)  $x[n] = 2e^{j(\pi n + \theta)}$
- (f)  $x[n] = \cos \frac{\pi}{2} n$

**Solution**

- (a)  $x[n] = A\delta[n]$

$$\begin{aligned} x[n] &= A\delta[n] \\ &= A \quad n = 0 \\ &= 0 \quad n \neq 0 \end{aligned}$$

$$\text{Energy } E = \sum_{n=0}^0 (A)^2$$

$E = A^2$

For unit impulse,  $A = 1$  and  $E = 1$ .

(b)  $x[n] = u[n]; n \geq 0$

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=0}^N |x(n)|^2 \\ &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=0}^N 1 \end{aligned}$$

But  $\sum_{n=0}^N 1 = (N+1)$

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \frac{(N+1)}{(2N+1)} \\ &= \lim_{N \rightarrow \infty} \frac{N(1 + \frac{1}{N})}{N(2 + \frac{1}{N})} = \frac{1}{2} \end{aligned}$$

$P = \frac{1}{2}$
$E = \infty$

(c)  $x[n] = \text{ramp } n; n \geq 0$

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=0}^N |x[n]|^2 \\ P &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=0}^N n^2 \end{aligned}$$

But  $\sum_{n=0}^N n^2 = \frac{N(N+1)(2N+1)}{6}$

$$P = \lim_{N \rightarrow \infty} \frac{N(N+1)(2N+1)}{(2N+1)6}$$

$P = \infty$
--------------

$$\begin{aligned} E &= \lim_{N \rightarrow \infty} \sum_{n=0}^N n^2 \\ &= \lim_{N \rightarrow \infty} \frac{N(N+1)(2N+1)}{6} = \infty \end{aligned}$$

$$\boxed{E = \infty}$$

The signal  $x[n] = n$  is neither power signal nor energy signal.  
 (d)  $x[n] = A$

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=-\infty}^{\infty} A^2 \\ &= \lim_{N \rightarrow \infty} \frac{A^2}{(2N+1)} (2N+1) \quad \left[ \sum_{n=-\infty}^{\infty} 1 = (2N+1) \right] \end{aligned}$$

$$\boxed{\begin{array}{l} P = A^2 \\ E = \infty \end{array}}$$

(e)  $x[n] = 2e^{j(\pi n + \theta)}$

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{-N}^N |2e^{j(n\pi + \theta)}|^2 \\ &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} 4 \sum_{-N}^N |e^{j(n\pi + \theta)}|^2 \end{aligned}$$

But  $|e^{j(n\pi + \theta)}| = 1$  and  $\sum_{-N}^N 1 = (2N+1)$

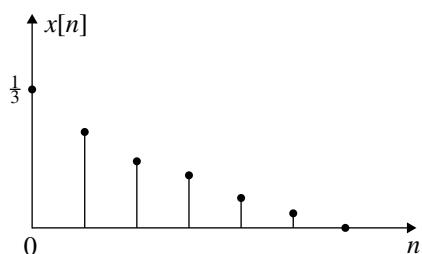
$$P = \lim_{N \rightarrow \infty} 4 \frac{(2N+1)}{(2N+1)} = 4$$

$$\boxed{\begin{array}{l} P = 4 \\ E = \infty \end{array}}$$

(f)  $x[n] = \cos \frac{\pi}{2} n$

$$P = \frac{1}{(2N+1)} \sum_{-N}^N \cos^2 \frac{\pi}{2} n$$

Since  $\sum_{-N}^N \cos \pi n = 0$ ,

**Fig. 1.28**  $x[n] = \left(\frac{1}{3}\right)^2 u[n]$ 

$$\begin{aligned}
 P &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=-N}^N \frac{(1 + \cos \pi n)}{2} \\
 &= \frac{1}{2} \lim_{N \rightarrow \infty} \frac{(2N+1)}{(2N+1)} \\
 &= \frac{1}{2}
 \end{aligned}$$

$P = \frac{1}{2}$   
 $E = \infty$

**Example 1.21**

Determine the energy of the signal shown in Fig. 1.28 whose

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

(Anna University, December, 2007)

**Solution**

$$\begin{aligned}
 E &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\frac{1}{3}\right)^{2n} \\
 &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\frac{1}{9}\right)^n \\
 &= 1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \dots \\
 &= \frac{1}{1 - \frac{1}{9}}
 \end{aligned}$$

$$\boxed{E = \frac{9}{8}} \\ P = 0$$

**Example 1.22**

Find the energy of the following sequence shown below:

$$x[n] = n \quad 0 \leq n \leq 4$$

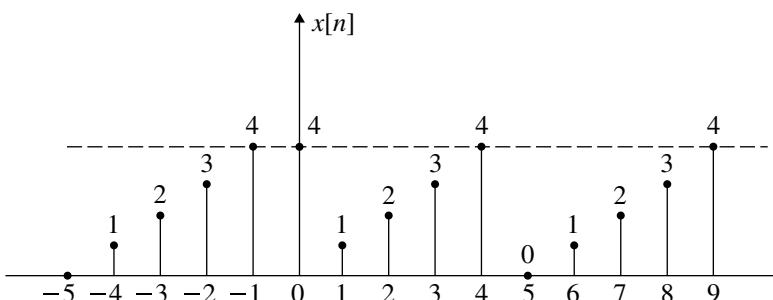
**Solution**

$$\begin{aligned} x[n] &= n \\ &= \{0, 1, 2, 3, 4\} \\ E &= \sum_{n=0}^4 n^2 \\ &= 0 + 1 + 4 + 9 + 16 \end{aligned}$$

$$\boxed{E = 30}$$

**Example 1.23**

Determine the average power and the energy per period of the sequence shown in Fig. 1.29.



**Fig. 1.29**  $x[n]$  of Example 1.23

**Solution** The fundamental period  $N$  of the signal is 5. Hence, the average power per period is

$$P = \frac{1}{5} \sum_{n=0}^4 n^2 = \frac{1}{5}[0 + 1 + 4 + 9 + 16]$$

$$\boxed{P = 6}$$

Average energy per period is

$$\begin{aligned} E &= \sum_{n=0}^4 n^2 \\ &= [0 + 1 + 4 + 9 + 16] \end{aligned}$$

$$\boxed{E = 30}$$

### Example 1.24

Find the energy and power of the following signal:

$$x[n] = a^n u[n]$$

for the following cases:

- (a)  $|a| < 1$
- (b)  $|a| = 1$
- (c)  $|a| > 1$

### Solution

(a)  $x[n] = a^n u[n]$  where  $|a| < 1$  and  $n \geq 0$

$$\begin{aligned} E &= \sum_{n=0}^{\infty} (a^n)^2 \\ &= 1 + a^2 + a^4 + \dots \end{aligned}$$

$$E = \frac{1}{1 - |a|^2}$$

$$P = 0$$

(b)  $x[n] = a^n u[n]$  where  $|a| = 1$

$$E = \underset{N \rightarrow \infty}{Lt} \sum_0^N 1^n = \underset{N \rightarrow \infty}{Lt} (N + 1)$$

$$E = \infty$$

$$P = \underset{N \rightarrow \infty}{Lt} \frac{1}{(2N + 1)} \sum_0^N (1)^n$$

$$\begin{aligned} P &= \underset{N \rightarrow \infty}{Lt} \frac{(N + 1)}{(2N + 1)} \\ &= \underset{N \rightarrow \infty}{Lt} \frac{N(1 + \frac{1}{N})}{N(2 + \frac{1}{N})} \end{aligned}$$

$$P = \frac{1}{2}$$

(c)  $x[n] = a^n u[n]$  where  $|a| > 1$

$$\begin{aligned} E &= \underset{N \rightarrow \infty}{Lt} \sum_0^N a^n \\ &= 1 + a + a^2 + \cdots + a^N \end{aligned}$$

$$E = \infty$$

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=0}^N a^n \\ &= \lim_{N \rightarrow \infty} \frac{1}{(N+1)} \frac{(1-a^{N+1})}{(1-a)} \end{aligned}$$

$$P = \infty$$

The signal is neither energy nor power signal.

### Example 1.25

Find the energy of the following signal:

$$x[n] = \text{ramp}[n] - 2 \text{ramp}[n-4] + \text{ramp}[n-8]$$

### Solution

$$\begin{aligned} x[n] &= \text{ramp}[n] - 2 \text{ramp}[n-4] + \text{ramp}[n-8] \\ &= x_1[n] + x_2[n] + x_3[n] \end{aligned}$$

$x_1[n], x_2[n]$  and  $x_3[n]$  are shown in Fig. 1.30a–c ,respectively. Figure 1.30d represents  $x[n]$ . From Fig. 1.30d, the energy of the signal  $x[n]$  is obtained as

$$E = 1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 2^2 + 1^2$$

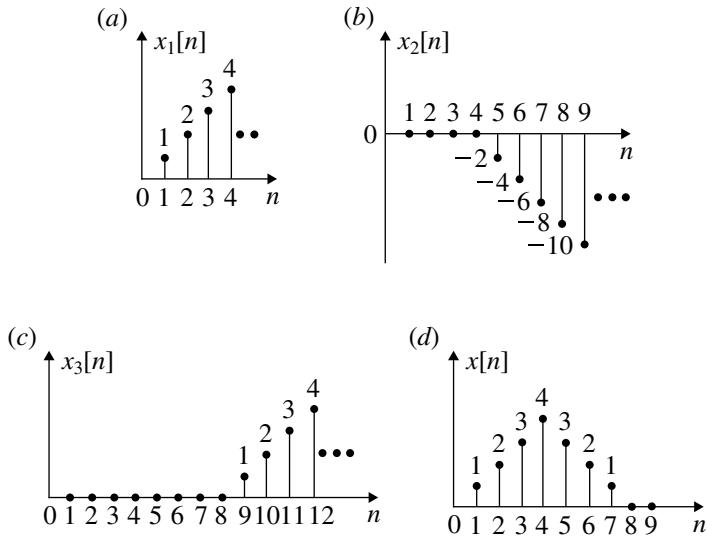
$$E = 44$$

### Example 1.26

Determine the value of power and energy of each of the following signals:

$$(a) \quad x[n] = e^{j(\frac{\pi n}{2} + \frac{\pi}{8})}$$

$$(b) \quad x[n] = \left(\frac{1}{2}\right)^n u[n]$$



**Fig. 1.30** DT energy signal of Example 1.25.

### Solution

$$(a) \quad x[n] = e^{j(\frac{\pi n}{2} + \frac{\pi}{8})}$$

$$\begin{aligned} P &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{-N}^N |e^{j(\frac{\pi n}{2} + \frac{\pi}{8})}|^2 \\ &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{-N}^N 1 \end{aligned}$$

$$P = \frac{(2N+1)}{(2N+1)} = 1$$

$P = 1 \text{ and } E = \infty$

(b)  $x[n] = \left[\frac{1}{2}\right]^n u[n]$

$$\begin{aligned} E &= \lim_{N \rightarrow \infty} \sum_0^N \left(\frac{1}{2}\right)^{2n} \\ &= \lim_{N \rightarrow \infty} \sum_0^N \left(\frac{1}{4}\right)^n = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \end{aligned}$$

$E = \frac{4}{3} \text{ and } P = 0$

### Example 1.27

Find the energy of the following DT signal

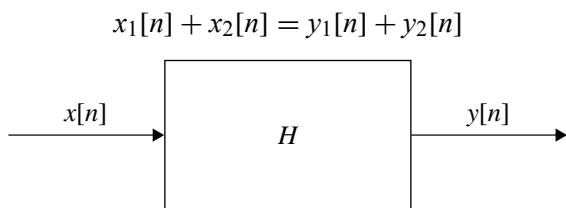
$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^n & n \geq 0 \\ &= 3^n & n < 0 \end{aligned}$$

(Anna University, April, 2005)

### Solution

$$\begin{aligned} E &= \left[ \sum_{-\infty}^{-1} (3)^{2n} + \sum_0^{\infty} \left(\frac{1}{2}\right)^{2n} \right] \\ &= \left[ \sum_{-\infty}^{-1} (9)^n + \sum_0^{\infty} \left(\frac{1}{4}\right)^n \right] \\ &= \left[ \sum_1^{\infty} (9)^{-n} + \frac{1}{(1 - \frac{1}{4})} \right] \\ &= \left[ \sum_1^{\infty} \left(\frac{1}{9}\right)^n + \frac{4}{3} \right] \\ &= \left[ \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \dots \right] + \frac{4}{3} \\ &= \frac{1}{9} \left[ 1 + \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \dots \right] + \frac{4}{3} \end{aligned}$$

**Fig. 1.31** Block diagram representation of discrete time system



$$\begin{aligned} &= \frac{1}{9} \frac{1}{[1 - \frac{1}{9}]} + \frac{4}{3} \\ &= \frac{1}{8} + \frac{4}{3} \end{aligned}$$

$$E = \frac{35}{24}$$

## 1.7 Discrete Time System

The block diagram of a discrete time system is shown in Fig. 1.31.  $x[n]$  is the excitation (input) signal, and  $y[n]$  is the response (output) signal of the DT system.  $H$  represents the functional relationship between the input and output which is described by difference equation. The input-output signals appear at discrete interval of time  $n$ , where  $n = 0, 1, 2 \dots$  which is an integer.  $n$  can also take negative value of an integer.

## 1.8 Properties of Discrete Time System

Like CT systems, DT systems also possess similar properties which are given below:

1. Linearity and nonlinearity
2. Time varying and time invariant
3. Causal and non-causal
4. Stable and unstable
5. Static (instantaneous) and dynamic (system without and with memory)
6. Invertibility and inverse.

### 1.8.1 Linear and Nonlinear Systems

A linear discrete time system obeys the property of superposition. As discussed for CT system, the superposition property is composed of homogeneity and additivity. Let  $x_1[n]$  excitation produce  $y_1[n]$  response and  $x_2[n]$  produce  $y_2[n]$  response. According to additivity property of superposition theorem, if both  $x_1[n]$  and  $x_2[n]$  are applied simultaneously, then

$$x_1[n] + x_2[n] = y_1[n] + y_2[n]$$

Let  $a_1x_1[n]$  and  $a_2x_2[n]$  be the inputs. According the homogeneity (scaling) property, when these signals are separately applied,

$$\begin{aligned} a_1x_1[n] &= a_1y_1[n] \\ a_2x_2[n] &= a_2y_2[n] \end{aligned}$$

If  $a_1x_1[n] + a_2x_2[n]$  are simultaneously applied, the output is obtained by applying superposition theorem as,

$$a_1x_1[n] + a_2x_2[n] = a_1y_1[n] + a_2y_2[n]$$

In the above equation,  $a_1x_1[n] + a_2x_2[n]$  is called the weighted sum of input, and  $a_1y_1[n] + a_2y_2[n]$  is called the weighted sum of the output. Therefore, the following procedure is followed to test the linearity of a DT system.

1. Express

$$\begin{aligned} y_1[n] &= f(x_1[n]) \\ y_2[n] &= f(x_2[n]) \end{aligned}$$

2. Find the weighted sum of the output as

$$y_3[n] = a_1y_1[n] + a_2y_2[n]$$

3. Find the output  $y_4[n]$  due to the weighted sum of input as

$$y_4[n] = f(a_1x_1[n] + a_2x_2[n])$$

4. If  $y_3[n] = y_4[n]$ , then given DT system is linear. Otherwise it is nonlinear.

The following examples illustrate the method of testing a DT system for its linearity.

**Example 1.28**

Test whether the following DT systems are linear or not:

- (a)  $y[n] = x^2[n]$
- (b)  $y[n] = x[4n + 1]$
- (c)  $y[n] = x[n] + \frac{1}{x[n + 1]}$
- (d)  $y[n] = x[n^2]$
- (e)  $y[n] = x[n] + nx[n + 1]$

**Solution**

(a)  $y[n] = x^2[n]$

$$\begin{aligned}y_1[n] &= x_1^2[n] \\y_2[n] &= x_2^2[n]\end{aligned}$$

1. The weighted sum of the output  $y_3[n]$  is

$$\begin{aligned}y_3[n] &= a_1 y_1[n] + a_2 y_2[n] \\&= a_1 x_1^2[n] + a_2 x_2^2[n]\end{aligned}$$

2. The output due to the weighted sum of the input  $y_4[n]$  is

$$\begin{aligned}y_4[n] &= [a_1 x_1[n] + a_2 x_2[n]]^2 \\&= a_1^2 x_1^2[n] + a_2^2 x_2^2[n] + 2a_1 a_2 x_1[n] x_2[n]\end{aligned}$$

3.

$$y_3[n] \neq y_4[n]$$

The system is nonlinear.

(b)  $y[n] = x[4n + 1]$

$$\begin{aligned}a_1 y_1[n] &= a_1 x_1[4n + 1] \\a_2 y_2[n] &= a_2 x_2[4n + 1] \\y_3[n] &= a_1 y_1[n] + a_2 y_2[n]\end{aligned}$$

1. The weighted sum of the output is

$$\begin{aligned}y_3[n] &= a_1y_1[n] + a_2y_2[n] \\&= a_1x_1[4n+1] + a_2x_2[4n+1]\end{aligned}$$

2. The output due to the weighted sum of the input is

$$y_4[n] = a_1x_1[4n+1] + a_2x_2[4n+1]$$

3.

$$y_3[n] = y_4[n]$$

The system is linear.

(c)  $y[n] = x[n] + \frac{1}{x(n+1)}$

$$\begin{aligned}a_1y_1[n] &= a_1 \left[ x_1[n] + \frac{1}{x_1(n+1)} \right] \\a_2y_2[n] &= a_2 \left[ x_2[n] + \frac{1}{x_2(n+1)} \right]\end{aligned}$$

1. The weighted sum of the output  $y_3[n]$  is

$$\begin{aligned}y_3[n] &= a_1y_1[n] + a_2y_2[n] \\&= a_1 \left[ x_1[n] + \frac{1}{x_1(n+1)} \right] + a_2 \left[ x_2[n] + \frac{1}{x_2(n+1)} \right]\end{aligned}$$

2. The output due to the weighted sum of the input is

$$\begin{aligned}y_4[n] &= f[a_1x_1[n] + a_2x_2[n]] \\&= a_1[x_1[n] + a_2x_2[n]] + \left[ \frac{1}{a_1x_1[n+1] + a_2x_2[n+1]} \right]\end{aligned}$$

3.

$$y_3[n] \neq y_4[n]$$

The system is nonlinear.

(d)  $y[n] = x[n^2]$

$$\begin{aligned}a_1y_1[n] &= a_1x_1[n^2] \\a_2y_2[n] &= a_2x_2[n^2]\end{aligned}$$

1. The weighted sum of the output  $y_3[n]$  is

$$\begin{aligned}y_3[n] &= a_1y_1[n] + a_2y_2[n] \\&= a_1x_1[n^2] + a_2x_2[n^2]\end{aligned}$$

2. The output  $y_4[n]$  due to the weighted sum of input is

$$y_4[n] = a_1x_1[n^2] + a_2x_2[n^2]$$

3.

$$y_3[n] = y_4[n]$$

The system is linear.

(e)  $y[n] = x[n] + nx[n+1]$

$$\begin{aligned}a_1y_1[n] &= a_1[x_1[n] + nx_1[n+1]] \\a_2y_2[n] &= a_2[x_2[n] + nx_2[n+1]]\end{aligned}$$

1. The weighted sum of the output is

$$\begin{aligned}y_3[n] &= a_1y_1[n] + a_2y_2[n] \\&= a_1[x_1[n] + nx_1[n+1]] + a_2[x_2[n] + nx_2[n+1]]\end{aligned}$$

2. The output due to the weighted sum of the input is

$$y_4[n] = a_1x_1[n] + a_2x_2[n] + a_1nx_1[n+1] + a_2nx_2[n+1]$$

3.

$$y_3[n] = y_4[n]$$

The system is linear.

### 1.8.2 Time Invariant and Time Varying DT Systems

Consider the discrete time system represented in block diagram of Fig. 1.32a. If the input is  $x[n]$ , then the output is  $y[n]$ . If the input is time delayed by  $n_0$ , which becomes  $x[n - n_0]$ , the output becomes  $y[n - n_0]$ . The signal representation and the delayed signals are shown in Fig. 1.32b, c, respectively. Such systems are called time invariant.

**If an arbitrary excitation  $x[n]$  of a system causes a response  $y[n]$  and the delayed excitation  $x[n - n_0]$  where  $n_0$  is any arbitrary integer causes  $y[n - n_0]$ , then the system is said to be time invariant.**

#### Procedure to Check Time Invariance of DT Systems

1. For the delayed input  $x[n - n_0]$ , find the output  $y[n, n_0]$ .
2. Obtain the delayed output  $y[n - n_0]$  by substituting  $n = n - n_0$  in  $y[n]$ .
3. If  $y[n, n_0] = y[n - n_0]$ , the system is time invariant. Otherwise the system is time varying.

The following examples illustrate the method of testing the time invariancy of DT systems.

#### Example 1.29

Determine whether the following systems are time invariant or not:

- (a)  $y[n] = nx[n]$
- (b)  $y[n] = x[2n]$
- (c)  $y[n] = x[-n]$
- (d)  $y[n] = \sin(x[n])$
- (e)  $y[n] = x[n]x[n - 1]$

#### Solution

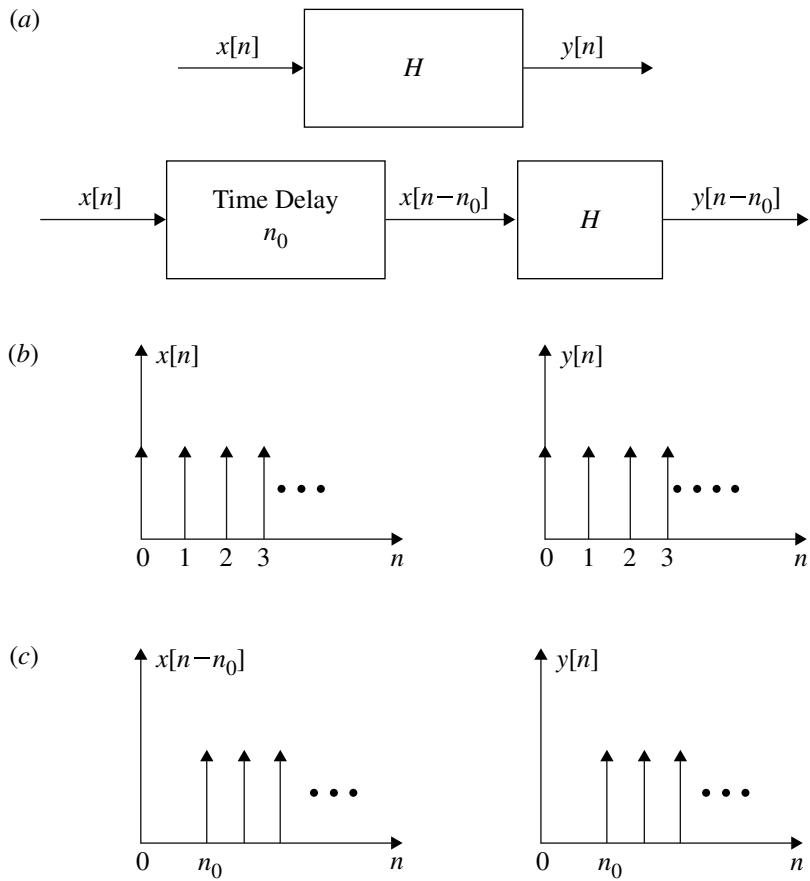
(a)  $y[n] = nx[n]$

1. The output for the delayed input  $x[n - n_0]$  is

$$y[n, n_0] = nx[n - n_0]$$

2. The delayed output for the input  $x[n]$  is

$$y[n - n_0] = (n - n_0)x[n - n_0]$$



**Fig. 1.32** Block diagram and signal representation to illustrate time invariancy of DT system

3.

$$y[n, n_0] \neq y[n - n_0]$$

The system is time variant.

(b)  $y[n] = x[2n]$

$$\begin{aligned} y[n, n_0] &= x[2n - n_0] \\ y[n - n_0] &= x[2(n - n_0)] \\ &= x[2n - 2n_0] \\ y[n, n_0] &\neq y[n - n_0] \end{aligned}$$

The system is time varying.

(c)  $y[n] = x[-n]$

$$\begin{aligned}y[n, n_0] &= x[-n - n_0] \\y[n - n_0] &= x[-(n - n_0)] \\&= x[-n + n_0] \\y[n, n_0] &\neq y[n - n_0]\end{aligned}$$

The system is time varying.

(d)  $y[n] = \sin(x[n])$

$$\begin{aligned}y[n, n_0] &= \sin(x[n - n_0]) \\y[n - n_0] &= \sin(x[n - n_0]) \\y[n, n_0] &= y[n - n_0]\end{aligned}$$

The system is time invariant.

(e)  $y[n] = x[n]x[n - 1]$

$$\begin{aligned}y[n, n_0] &= x[n - n_0]x[n - n_0 - 1] \\y[n - n_0] &= x[n - n_0]x[n - n_0 - 1] \\y[n, n_0] &= y[n - n_0]\end{aligned}$$

The system is time invariant.

### 1.8.3 Causal and Non-causal DT Systems

A discrete time system is said to be causal if the response of the system depends on the present or the past inputs applied. The systems is non-causal if the output depends on the future input.

The following examples illustrate the method of identifying causal and non-causal systems.

---

### Example 1.30

Determine whether the following systems are causal or not:

- (a)  $y[n] = x[n - 1]$
- (b)  $y[n] = x[n] + x[n - 1]$
- (c)  $y[n - 1] = x[n]$
- (d)  $y[n] = \sin(x[n])$
- (e)  $y[n] = \sum_{k=-\infty}^{n+4} x(k)$
- (f)  $y[n] = \sum_{k=0}^{-3} x(k)$

---

### Solution

(a)  $y[n] = x[n - 1]$

$$\begin{aligned}y[0] &= x[-1] \\y[1] &= x[0]\end{aligned}$$

The output depends on the past value of  $x[n]$ . Hence

The system is causal.

(b)  $y[n] = x[n] + x[n - 1]$

$$\begin{aligned}y[0] &= x[0] + x[-1] \\y[1] &= x[1] + x[0]\end{aligned}$$

here  $x[1]$  is present value and  $x[0]$  is past value. The output depends on the present and past inputs. Hence

The system is causal.

(c)  $y[n - 1] = x[n]$ 

$$\begin{aligned}y[-1] &= x[0] && \text{(Future input)} \\y[0] &= x[1] && \text{(Future input)}\end{aligned}$$

The output depends on the future inputs. Hence

The system is non-causal.

(d)  $y[n] = \sin x[n]$ 

$$\begin{aligned}y[0] &= \sin x[0] \\y[-1] &= \sin x[-1]\end{aligned}$$

The output depends on the present input. Hence

The system is causal.

(e)  $y[n] = \sum_{k=-\infty}^{n+4} x[k]$ 

$$\begin{aligned}y[0] &= \sum_{-\infty}^4 x[k] \\&= x[-\infty] + x[-\infty + 1] + \dots + x[-1] + x[0] + x[1] + x[2] + x[3] + x[4]\end{aligned}$$

$$\begin{aligned}x[-\infty] + x[-\infty + 1], \dots, x[-1] &= \text{Past input} \\x[0] &= \text{Present input} \\x[0], x[1], x[2], x[3] \text{ and } x[4] &= \text{Future input}\end{aligned}$$

The output depends on past, present and future input. Hence

The system is non-causal.

$$(f) \quad y[n] = \sum_{k=0}^{n-3} x[k]$$

$$\begin{aligned} y[0] &= \sum_{k=0}^{-3} x[k] \\ &= x[0] + x[-1] + x[-2] + x[-3] \\ x[0] &= \text{Present input} \\ x[-1] &= \text{Past input} \end{aligned}$$

The output depends on the present and past input. Hence

The system is causal.

#### 1.8.4 Stable and Unstable Systems

A discrete time system is said to be stable if for any bounded input, it produces a bounded output. This implies that the impulse response

$$y[n] = \sum_{-\infty}^{\infty} |h[n]| < \infty$$

is absolutely summable.

For a bounded input,

$$|x[n]| \leq M_x < \infty$$

the output

$$|y[n]| \leq M_y < \infty$$

From the above two conditions, it can be obtained

$$y[n] = \sum_{-\infty}^{\infty} |h[n]| < \infty$$

The following examples illustrate the above procedure.

**Example 1.31**

Check whether the DT systems described by the following equations are stable or not.

- (a)  $y[n] = \sin x[n]$
- (b)  $y[n] = \sum_{k=0}^{n+1} x[k]$
- (c)  $y[n] = e^{x[n]}$
- (d)  $h[n] = 3^n u[n + 3]$
- (e)  $y[n] = x[-n - 3]$
- (f)  $y[n] = x[n - 1] + x[n] + x[n + 1]$
- (g)  $h[n] = e^{-|n|}$
- (h)  $h[n] = n u[n]$
- (i)  $h[n] = 3^n u[n - 3]$
- (j)  $h[n] = 2^n u[-n]$

**Solution**

(a)  $y[n] = \sin x[n]$

If  $x[n]$  is bounded, then  $\sin x[n]$  is also bounded and so  $y[n]$  is also bounded

The system is stable.

(b)  $y[n] = \sum_{k=0}^{n+1} x[k]$

Here, as  $n \rightarrow \infty$ ,  $y[n] \rightarrow \infty$ , and the output is unbounded. For bounded input,  $n$  should be a finite number. In that case  $y[n]$  is bounded, and the system is stable.

The system is stable.	for $n = \text{finite}$
The system is unstable.	for $n = \infty$

(c)  $e^{x[n]}$

For  $|x[n]|$  bounded,  $e^{|x[n]|}$  is bounded, and the system is stable.

The system is stable.

(d)  $h[n] = 3^n u[n + 3]$

$$\begin{aligned}|y[n]| &= \sum_{n=-3}^{\infty} 3^n \\&= (3)^{-3} + (3)^{-2} + (3)^{-1} + (3)^0 + (3)^1 + \cdots + (3)^{\infty} \\&= \infty\end{aligned}$$

The output is unbounded.

The system is unstable.

(e)  $y[n] = x[-n - 3]$

$$\begin{aligned}y[n] &= x[-n - 3] \\&= 1 \quad n = -3 \\&= 0 \quad \text{otherwise}\end{aligned}$$

The system is stable.

(f)  $y[n] = x[n - 1] + x[n] + x[n + 1]$

$$y[0] = \delta[-1] + \delta[0] + \delta[1] = 0 + 1 + 0 = 1$$

$$y[1] = \delta[0] + \delta[1] + \delta[2] = 1 + 0 + 0 = 1$$

$$y[-1] = \delta[-2] + \delta[-1] + \delta[0] = 0 + 0 + 1 = 1$$

$$y[-2] = \delta[1] + \delta[2] + \delta[3] = 0 + 0 + 0 = 0$$

$$y[2] = \delta[1] + \delta[2] + \delta[3] = 0 + 0 + 0 = 0$$

$$y[n] = \sum_{-\infty}^{\infty} |h[k]| = 1 + 1 + 1 = 3 < \infty$$

The system is stable.

(g)  $h[n] = e^{-|n|}$ 

$$\begin{aligned}
 y[n] &= \sum_{-\infty}^{\infty} e^{|n|} = \sum_{-\infty}^{-1} e^{|n|} + \sum_0^{\infty} e^{-|n|} \\
 &= \sum_1^{\infty} e^{-n} + \sum_0^{\infty} e^{-n} \\
 &= e^{-1} + e^{-2} + \cdots + 1 + e^{-1} + e^{-2} + \cdots \\
 &= e^{-1}[1 + e^{-1} + e^{-2} + \cdots] + 1 + e^{-1} + e^{-2} + \cdots \\
 &= e^{-1} \frac{1}{[1 - e^{-1}]} + \frac{1}{[1 - e^{-1}]} \\
 &= \frac{e^{-1}}{(1 - e^{-1})} + \frac{1}{(1 - e^{-1})} \\
 &= \frac{[1 + e^{-1}]}{[1 - e^{-1}]} < \infty
 \end{aligned}$$

The system is stable.

(h)  $h[n] = n u[n]$ 

$$y[n] = \sum_0^{\infty} n = 1 + 2 + \cdots + \infty = \infty$$

The system is unstable.

(i)  $h[n] = 3^n u[n - 3]$ 

$$y[n] = \sum_3^{\infty} 3^n = 3^3 + 3^2 + \cdots + \infty = \infty$$

The system is unstable.

(j)  $h[n] = 2^n u[-n]$

$$\begin{aligned}y[n] &= \sum_{-\infty}^{-1} 2^n = \sum_{1}^{\infty} \left(\frac{1}{2}\right)^n \\&= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots \\&= \frac{1}{2} \left[ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots \right] \\&= \frac{1}{2} \left[ \frac{1}{1 - \frac{1}{2}} \right] = 1 < \infty\end{aligned}$$

The system is stable.

### 1.8.5 Static and Dynamic Systems

A discrete time system is said to be static (memoryless or instantaneous) if the output response depends on the present value only and not on the past and future values of excitation. Discrete systems described by difference equations require memory, and hence they are dynamic systems.

The following examples illustrate the method identifying static and dynamic discrete systems.

#### Example 1.32

Identify whether the following systems are static or dynamic:

- (a)  $y[n] = x[3n]$
- (b)  $y[n] = \sin(x[n])$
- (c)  $y[n - 1] + y[n] = x[n]$
- (d)  $y[n] = \text{sgn}|x[n]|$

**Solution**

(a)  $y[n] = x[3n]$

$$\begin{aligned}y[1] &= x[3] \\y[-1] &= x[-3]\end{aligned}$$

The output  $y[1]$  and  $y[-1]$  depend on the future value  $x[3]$  and the past input  $x[-3]$ , respectively. Hence

The system is dynamic.

(b)  $y[n] = \sin(x[n])$

$$\begin{aligned}y[0] &= \sin(x[0]) \\y[1] &= \sin(x[1])\end{aligned}$$

The output depends on the present input at all time. Hence

The system is static.

(c)  $y[n-1] + y[n] = x[n]$

The system is described by first-order difference equation which require memory. Hence

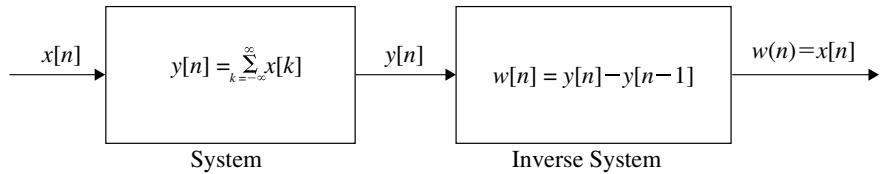
The system is dynamic.

(d)  $y[n] = \text{sgn}|x[n]|$

$$\begin{aligned}\text{sgn}|x[n]| &= 1 && \text{for } n > 0 \\&= -1 && \text{for } n < 0 \\y[1] &= x[1] = 1 \\y[-1] &= x[-1] = -1\end{aligned}$$

The output depends on the present value of the input. Hence

The system is static.



**Fig. 1.33** Inverse discrete time system

### 1.8.6 Invertible and Inverse Discrete Time Systems

A discrete time system is said to be invertible if distinct input leads to distinct output. If a system is invertible, then an inverse system exists.

Consider the system shown in Fig. 1.33. The input  $x[n]$  produces the output  $y[n]$ . This system is in cascade with its inverse system. The output of this system is nothing but the difference of the two successive inputs  $y[n] - y[n - 1]$ . This is the input to the original system. Thus, by connecting an inverse system in cascade with the original system, the excitation signal  $x[n]$  is re-established provided the original system is invertible. The concept of invertibility is very widely used in communications.

### Example 1.33

Determine whether the following systems are static, causal, time invariant, linear and stable.

- (a)  $y[n] = x[4n + 1]$
- (b)  $y[n] = x[n] + n x[n + 1]$
- (c)  $y[n] = x[n]u[n]$
- (d)  $y[n] = \log_{10} x[n]$
- (e)  $y[n] = x^2[n]$

(Anna University, 2007)

#### Solution

(a)  $y[n] = x[4n + 1]$

1.

$$y[0] = x[1]$$

The output depends on future input. Hence

The system is dynamic and non-causal.

2. The output due to the delayed input is

$$y[n, n_0] = x[4n - n_0 + 1]$$

The delayed output due to the input is

$$\begin{aligned} y[n - n_0] &= x[4(n - n_0) + 1] \\ &= x[4n - 4n_0 + 1] \\ y[n, n_0] &\neq y[n - n_0] \end{aligned}$$

The system is time variant.

- 3.

$$\begin{aligned} a_1 y_1[n] &= a_1 x_1[4n + 1] \\ a_2 y_2[n] &= a_2 x_2[4n + 1] \\ y_3[n] &= a_1 y_1[n] + a_2 y_2[n] \\ &= a_1 x_1[4n + 1] + a_2 x_2[4n + 1] \\ y_4[n] &= a_1 x_1[4n + 1] + a_2 x_2[4n + 1] \\ y_3[n] &= y_4[n] \end{aligned}$$

The system is linear.

4. The input is time shifted and time compressed signal. As long as the input is bounded, the output is also bounded.

The system is stable.

The system is

(1) Dynamic, (2) non-causal, (3) time variant, (4) linear and (5) stable.

(b)  $y[n] = x[n] + n x[n + 1]$

1.

$$y[0] = x[0] + 0 \times x[1]$$

The output depends on present and future inputs.

The system is dynamic and non-causal.

2. The output due to the delayed input is

$$y[n, n_0] = x[n - n_0] + nx[n - n_0 + 1]$$

The delayed output due to the input is

$$\begin{aligned} y[n - n_0] &= x[n - n_0] + (n - n_0)x[n - n_0 + 1] \\ y[n, n_0] &\neq y[n - n_0] \end{aligned}$$

The system is time variant.

3. The weighted sum of the output due to the input is

$$y_3[n] = a_1 y_1[n] + a_2 y_2[n] = a_1 x_1[n] + a_1 n x_1[n + 1] + a_2 x_2[n] + a_2 n x_2[n + 1]$$

The output due to the weighted sum of the input is

$$\begin{aligned} y_4[n] &= a_1 \{x_1[n] + nx_1[n + 1]\} + a_2 \{x_2[n] + nx_2[n + 1]\} \\ y_3[n] &= y_4[n] \end{aligned}$$

The system is linear.

4. As long as  $x[n]$  is bounded,  $y[n]$  is also bounded for  $n = \text{finite}$ .

The system is stable.

The system is

(1) Dynamic, (2) non-causal, (3) time variant, (4) linear and (5) stable.

(c)  $y[n] = x[n]u[n]$

1.

$$\begin{aligned}y[0] &= x[0]u[0] \\y[1] &= x[1]u[1]\end{aligned}$$

The output depends on present input only.

The system is static and causal.

2. For a causal signal  $n \geq 0$ . The weighted sum of the output due to input is

$$\begin{aligned}y_3[n] &= a_1y_1[n] + a_2y_2[n] \\&= \{a_1x_1[n] + a_2x_2[n]\}u[n]\end{aligned}$$

The output due to the weighted sum of input is

$$\begin{aligned}y_4[n] &= \{a_1x_1[n] + a_2x_2[n]\}u[n] \\y_3[n] &= y_4[n]\end{aligned}$$

The system is linear.

3. The output due to the delayed input is

$$y[n, n_0] = x_1[n - n_0]u[n]$$

The delayed output due to the input is

$$\begin{aligned}y[n - n_0] &= x_1[n - n_0]u[n - n_0] \\y[n, n_0] &\neq y[n - n_0]\end{aligned}$$

The system is time variant.

4. As long as  $x[n]$  is bounded,  $y[n]$  is also bounded.

The system is stable.

The system is

(1) Static, (2) causal, (3) linear, (4) time variant and (5) stable.

(d)  $y[n] = \log_{10} x[n]$

1.

$$\begin{aligned}y[0] &= \log_{10} x[0] \\y[1] &= \log_{10} x[1] \\y[-1] &= \log_{10} x[-1]\end{aligned}$$

The output depends on present input only.

The system is static and causal.

2. The weighted sum of the output due to input is

$$\begin{aligned}y_3[n] &= a_1 y_1[n] + a_2 y_2[n] \\&= a_1 \log_{10} x_1[n] + a_2 \log_{10} x_2[n]\end{aligned}$$

The output due to the weighted sum of input is

$$\begin{aligned}y_4[n] &= \log_{10}(a_1 x_1[n] + a_2 x_2[n]) \\y_3[n] &\neq y_4[n]\end{aligned}$$

The system is nonlinear.

3. The output due to the delayed input is

$$y[n, n_0] = \log_{10}[n - n_0]$$

The delayed output due to input is

$$\begin{aligned}y[n - n_0] &= \log_{10}[n - n_0] \\y[n, n_0] &= y[n - n_0]\end{aligned}$$

The system is time invariant.

4. As long as  $x[n]$  is bounded,  $\log_{10} x[n]$  is bounded and  $y[n]$  is also bounded.

The system is stable.

The system is

(1) Static, (2) causal, (3) nonlinear, (4) time invariant and (5) stable.

(e)  $y[n] = x^2[n]$

1.

$$\begin{aligned}y[0] &= x^2[0] \\y[1] &= x^2[1]\end{aligned}$$

The output depends on present input only.

The system is static and causal.

2. The weighted sum of the output due to input is

$$\begin{aligned}y_3[n] &= a_1 y_1[n] + a_2 y_2[n] \\&= a_1 x_1^2[n] + a_2 x_2^2[n]\end{aligned}$$

The output due to weighted sum of input is

$$\begin{aligned}y_4[n] &= \{a_1 x_1[n] + a_2 x_2[n]\}^2 \\&= a_1^2 x_1^2[n] + a_2^2 x_2^2[n] + 2a_1 a_2 x_1[n] x_2[n] \\y_3[n] &\neq y_4[n]\end{aligned}$$

The system is nonlinear.

3. The output due to the delayed input is

$$y[n, n_0] = x^2[n - n_0]$$

The delayed output due to the input is

$$\begin{aligned} y[n - n_0] &= x^2[n - n_0] \\ y[n, n_0] &= y[n - n_0] \end{aligned}$$

The system is time invariant.

4. If  $x[n]$  is bounded,  $x^2[n]$  is bounded, and  $y[n]$  is also bounded.

The system is stable.

The system is

(1) Static, (2) causal, (3) nonlinear, (4) time invariant and (5) stable.

## Summary

- Signals are broadly classified as continuous time (CT) and discrete time (DT) signals. They are further classified as deterministic and stochastic, periodic and non-periodic, odd and even and energy and power signals.
- Basic DT signals include impulse, step, ramp, parabolic, Rectangular pulse, triangular pulse, signum function, sinc function, sinusoid, real and complex exponentials.
- Basic operations on DT signals include addition, multiplication, amplitude scaling, time scaling , time shifting, reflection or folding and amplitude inverted signals.
- In time shifting of DT signal, for  $x(n + n_0)$  and  $x(-n - n_0)$  the time shift is made to the left of  $x(n)$  and  $x(-n)$ , respectively, by  $n_0$ . For  $x(n - n_0)$  and  $x(-n + n_0)$  the time shift is made to the right of the  $x(n)$  and  $x(-n)$ , respectively, by  $n_0$ .
- To plot DT signals, the operation performed is in the following sequence. The signal is folded (if necessary), time shifted, time scaled, amplitude scaled and inverted.
- Signals are classified as even signals and odd signals. Even signals are **symmetric** about the vertical axis, whereas odd signals are **anti-symmetric** about the time origin. Odd signals pass through the origin. The product of two even signals or two odd signals is an even signal. The product of an even and an odd signal is an odd signal.

- A DT signal which repeats itself every  $N$  sequence is called a periodic signal. If the signal is not periodic, it is called an aperiodic or non-periodic signal. The necessary condition for the composite of two or more signals to be periodic is that the individual signal should be periodic.
- A signal is an energy signal if the total energy of the signal satisfies the condition  $0 < E < \infty$ . A signal is called a power signal if the average power of the signal satisfies the condition  $0 < P < \infty$ . If the energy of a signal is finite, the average power is zero. If the power of the signal is finite, the signal has infinite energy. All periodic signals are power signals. However all power signals need not be periodic. Signals which are deterministic and non-periodic are usually energy signals. Some signals are neither energy signal nor power signal.
- The system is broadly classified as continuous and discrete time system.
- The DT systems are further classified based on the property of causality, linearity, time invariancy, invertibility, memory and stability.
- A discrete time system is said to be causal if the impulse response  $h(n) = 0$  for  $n < 0$ .
- If the impulse response of a discrete time system is absolutely summable, then the system is said to be BIBO stable.

### Short Questions and Answers

**1. How are signals classified?**

Signals are generally classified as CT and DT signals. They are further classified as deterministic and non-deterministic, odd and even, periodic and non-periodic and power and energy signals.

**2. What are odd and even signals?**

A continuous CT signal is said to be an even signal if it satisfies the condition  $x(-t) = x(t)$  for all  $t$ . It is said to be an odd signal if  $x(-t) = -x(t)$  for all  $t$ . For a DT signal if  $x[-n] = x[n]$  condition is satisfied, it is an even sequence (signal). If  $x[-n] = -x[n]$ , the sequence is called odd sequence.

**3. How even and odd components of a signal are mathematically expressed?**

$$x_e[n] = \frac{1}{2}\{x[n] + x[-n]\}$$

$$x_0[n] = \frac{1}{2}\{x[n] - x[-n]\}$$

**4. What are periodic and non-periodic signals?**

A discrete time signal is said to be a period signal if it satisfies the condition  $x[n] = x[n + N]$  for all  $n$ . A signal which is not periodic is said to be non-periodic.

**5. What is the fundamental period of a periodic signal? What is fundamental frequency?**

A DT signal is said to be periodic if it satisfies the condition  $x(n) = x(n + N)$ . If

this condition is satisfied for  $N = N_0$ , it is also satisfied for  $N = 2N_0, 3N_0, \dots$ . The smallest value of  $N$  that satisfies the above condition is called fundamental period. The fundamental frequency  $f_0 = \frac{1}{N_0}$  Hz. It is also expressed as  $\omega_0 = \frac{2\pi}{N_0}$  rad.

### 6. What are power and energy signals?

For a DT signal  $x[n]$ , the total energy is defined as

$$E = \sum_{n=-\infty}^{\infty} x^2[n]$$

The average power is defined as

$$P = \lim_{T \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x^2[n]$$

### 7. Determine whether the signal $x[n] = \cos[0.1\pi n]$ is periodic.

The signal  $x[n]$  is periodic with fundamental period  $N_0 = 20$ .

### 8. Find whether the signal $x[n] = 5 \cos[6\pi n]$ is periodic.

The signal is periodic with fundamental period  $N_0 = 1$ .

### 9. Find the average power of the signal.

$$x[n] = u[n] - u[n - N]$$

The average power  $P = 1$ .

### 10. Find the total energy of

$$x[n] = \{1, 1, 1\}$$

↑

The total energy  $E = 3$ .

### 11. If the discrete time signal $x[n] = \{0, 0, 0, 3, 2, 1, -1, -7, 6\}$ then find $y[n] = x[2n - 3]\?$

$$y[n] = \{0, 0, 0, 3, 1, -7\}$$

### 12. What is the energy of the signal $x[n] = u[n] - u[n - 6]\?$

$$E = 6$$

**13. What are the properties of systems?**

Systems are generally classified as continuous and discrete time systems. Further classifications of these systems are done based on their properties which include (a) linear and nonlinear, (b) time invariant and time variant, (c) static and dynamic, (d) causal and non-causal, (e) stable and unstable and (f) invertible and non-invertible.

**14. Define system. What is linear system?**

A system is defined as the interconnection of objects with a definite relationship between objects and attributes.

A system is said to be linear if the weighted sum of several inputs produces weighted sum of outputs. In other words, the system should satisfy the homogeneity and additivity of superposition theorem if it is to be linear. Otherwise it is a nonlinear system.

**15. What is time invariant and time varying system?**

A system is said to be time invariant if the output due to the delayed input is same as the delayed output due to the input. If the continuous time system is described by the differential equation, its coefficients should be time independent for the system to be time invariant. In the case of discrete time system, the coefficients of the difference equation describing the system should be time independent (constant) for the system to be time invariants. If the above conditions are not satisfied, the system (CT as well as DT) is said to be time variant.

**16. What are static and dynamic systems?**

If the output of the system depends only on the present input, the system is said to be static or instantaneous. If the output of the system depends on the past and future inputs, the system is not static, and it is called dynamic system. Static system does not require memory, and so it is called memoryless system. Dynamic system requires memory, and hence, it is called system with memory. Systems which are described by differential and difference equations are dynamic systems.

**17. What are causal and non-causal systems?**

If the system output depends on present and past inputs, it is called causal system. If the system output depends on future input, it is called non-causal system.

**18. What are stable and unstable systems?**

If the input is bounded and output is also bounded, the system is called BIBO stable system. If the input is bounded and the output is unbounded, the system is unstable. System whose impulse response curve has finite area is also called stable systems.

**19. What are invertible and non-invertible systems?**

A system is said to be invertible if the distinct inputs give distinct outputs.

### Long Answer Type Questions

- 1. For the following DT signal find even and odd components**

$$x[n] = \{1, -3, 2, 5, 4\}$$

$$x_e[n] = \{2, 2.5, 1, -1.5, 1, -1.5, 1, 2.5, 2\}$$



$$x_0[n] = \{-2, -2.5, -1, 1.5, 0, -1.5, 1, 2.5, 2\}$$



- 2. Find whether the following signals are periodic. If periodic, find the fundamental period.** (a)  $x[n] = \cos(\frac{n}{8} - \pi)$ ; (b)  $x[n] = \cos(\frac{\pi}{8} + \frac{\pi}{2}) + \cos(\frac{\pi}{6} - \frac{\pi}{2})$ ; (c)  $x[n] = \cos(\frac{5\pi N}{12} + \frac{\pi}{2}) + \sin \frac{10\pi n}{8}$ ; (d)  $x[n] = e^{j3n} + e^{j4\pi n}$ .  
 (a) Not periodic. (b) Periodic with fundamental period  $N_0 = 48$  samples/s. (c) Periodic with fundamental period  $N_0 = 24$  samples/s. (d) Non-periodic.

- 3. Given  $x[n]$  and  $y[n]$**

$$x[-1] = 2$$

$$x[n] = 1 \quad 1 \leq n \leq 5$$

$$x[6] = \frac{1}{2}$$

$$= 0 \quad \text{for other } n$$

Plot (a)  $x[\frac{n}{2}]$  and (b)  $E_v x[n]$ .

(Anna University, 2007).

(a)

$$x[n] = \{2, 1, 1, 1, 1, 1, 1, 0.5\}$$



$$x\left[\frac{n}{2}\right] = \{2, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0.5\}$$



(b)

$$x_e[n] = \{0.25, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 1, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.25\}$$



4. Find whether the following signal is periodic. If periodic, find the fundamental period.

$$x[n] = \cos\left(2\pi n + \frac{\pi}{2}\right) \sin\left(5\pi n - \frac{\pi}{4}\right) \sin\left(8\pi n + \frac{\pi}{2}\right)$$

The signal is periodic. Their fundamental period  $N_0 = 2$  samples/s.

5. For the systems given below determine whether each of them is (a) static, (b) causal, (c) time invariant, (d) linear and (e) stable.

$$y[n] = x[5n]$$

- (a) The system response depends on present, past and future inputs. Hence, it is dynamic.
- (b) Non-causal.
- (c) The output due to the delayed input is not same as the delayed output. Hence, it is time variant.
- (d) The weighted sum of the output is same as output due to the weighted sum of the input. The system is linear.
- (e) If the input  $x[5n]$  is bounded, the output  $y[n]$  is also bounded. The system is stable.

6.

$$y[k+2] + 3y[k+1] + 4y[k] = x[k]$$

- (a) The system is dynamic.
- (b) The system is causal.
- (c) The system is time invariant.
- (d) The system is linear.
- (e) The system is stable.

7.

$$y[n] = 5x[3^n]$$

- (a) The system is dynamic.
- (b) The system is non-causal.
- (c) The system is time invariant.
- (d) The system is linear.
- (e) The system is stable.

8.

$$y[n] = \sin(2\pi x[n]) + x[n+1]$$

- (a) The system is dynamic.
- (b) The system is non-causal.
- (c) The system is time invariant.
- (d) The system is nonlinear.
- (e) The system is stable.

9.  $x[n] = u[n-4] - u[n-10]$  and  $h[n] = u[n-5] - u[n-16]$ .  
Find  $y[n] = x[n] * h[n]$ .

$$\begin{aligned} y[n] &= (n-8) & 9 \leq n \leq 13 \\ &= 6 & 10 \leq n \leq 19 \\ &= (25-n) & 20 \leq n \leq 24 \\ &= 0 & n > 24 \end{aligned}$$

10.  $x[n] = 4^n u[-n-2]$  and  $h[n] = u[n-2]$ . Find  $y[n] = x[n] * h[n]$ .

$$\begin{aligned} y[n] &= \frac{1}{3} \left[ \frac{1}{4} \right]^{n-1} & n < 0 \\ &= \frac{1}{12} & n > 0 \end{aligned}$$

11. Determine whether the following LTID time systems whose impulse response given below are stable. (a)  $h[n] = n \sin 2\pi n u[n]$ , (b)  $h[n] = 5^n u[-n]$  and (c)  $h[n] = 2^{-n} u[n-5]$ .

- (a)  $y[n] = \infty$  B.I.B.O. unstable.
- (b)  $y[n] = \frac{1}{4} < \infty$  B.I.B.O. stable.
- (c)  $y[n] = \frac{1}{6} < \infty$  B.I.B.O. stable.

12. Determine the power and energy of the following signals.

- (a)  $x[n] = \left(\frac{1}{3}\right)^n$
- (b)  $x[n] = e^{j(\frac{n\pi}{2} + \frac{\pi}{4})}$
- (c)  $x[n] = \sin\left(\frac{n\pi}{4}\right)$
- (d)  $x[n] = e^{2n} u(n)$

Ans:

- (a)  $E = \frac{9}{8}J; P = 0$
- (b)  $E = \infty; P = 1W$
- (c)  $E = \infty; P = \frac{1}{2}W$
- (d)  $E = \infty; P = \infty$  (Neither power nor energy)

13. Determine whether each of the following signal is periodic. Find the fundamental period.

- (a)  $x[n] = e^{j6\pi n}$
- (b)  $x[n] = e^{j\frac{3}{5}(n+\frac{1}{2})}$
- (c)  $x[n] = \cos\left(\frac{2\pi}{3}\right)n$
- (d)  $x[n] = \cos\frac{n\pi}{3} + \cos\frac{3n\pi}{4}$

Ans:

- (a) Periodic.  $N = 1$  sample
- (b) Not Periodic.
- (c) Periodic.  $N = 3$  samples
- (d) Periodic.  $N = 24$  samples

14. Find the odd and even components of the following sequence.

$$x(n) = \{2, 3, 4, 5, 6\}$$

↑

Ans:

$$x_e(n) = \{4, 4, 4, 4, 4\}$$

↑

$$x_0(n) = \{-2, -1, 0, 1, 2\}$$

**15. Test whether the following systems are time variant or time invariant.**

- (a)  $y(n) = x(n) - x(n - 1)$
- (b)  $y(n) = nx(n)$
- (c)  $y(n) = x(-n)$
- (d)  $y(n) = x(n) \cos \omega_0 n$
- (e)  $y(n) = x(n) - bx(n - 1)$
- (f)  $y(n) = x(n) + c$
- (g)  $y(n) = e^{x(n)}$
- (h)  $y(n) = nx^2(n)$
- (i)  $y(n) = \sum_{k=0}^m a_k(n - k) - \sum_{k=1}^m b_k y(n - k)$

Ans:

- (a) Time invariant.
- (b) Time varying.
- (c) Time varying.
- (d) Time varying.
- (e) Time varying.
- (f) Time invariant.
- (g) Time invariant.
- (h) Time varying.
- (i) Time invariant.

**16. Test the linearity of the following systems.**

- (a)  $y(n) = nx(n)$
- (b)  $y(n) = x(n^2)$
- (c)  $y(n) = x^2(n)$
- (d)  $y(n) = Ax(n) + B$
- (e)  $y(n) = e^{x(n)}$

Ans: (a) Linear; (b) linear; (c) nonlinear; (d) nonlinear and (e) nonlinear

**17. Test the causality of the following systems.**

- (a)  $y(n) = x(n) - x(n - 1)$
- (b)  $y(n) = \sum_{k=\infty}^n x(k)$
- (c)  $y(n) = nx(n)$

Ans: (a) Causal; (b) causal; and (c) causal

18. Test whether the following systems are stable or not.

- (a)  $y(n) = x(-n - 2)$
- (b)  $y(n) = x(n) + by(n - 1)$
- (c)  $y(n) = \begin{cases} a^n & n \leq 0, \\ b^n & n < 0. \end{cases}$

Ans: (a) Stable; (b) stable if  $|b| < 1$ ; and (c) stable if  $|a| < 1$  and  $|b| > 1$

# Chapter 2

## Discrete and Fast Fourier Transforms (DFT and FFT)



### Learning Objectives

After completing this chapter, you should be able to:

- ✖ define Discrete Fourier transform (DFT) and inverse discrete Fourier transform (IDFT).
- ✖ establish the properties of DFT.
- ✖ study different methods of circular convolution and solve numerical problems.
- ✖ study different fast Fourier transform (FFT) algorithms and their applications.
- ✖ study the use of FFT algorithms in linear filtering and correlation.

### 2.1 Introduction

In the study of signals and systems, the discrete time periodic signals are represented by discrete time Fourier series (DFS) using a parallel development of continuous system. The Fourier series representation in these cases was applicable only if the signals are periodic. If the signal is non-periodic, then applying a limiting process, the aperiodic discrete time signal  $x[n]$  can be expressed as a sum of everlasting exponentials or sinusoids. The spectrum of  $x(\omega)$  so obtained is called discrete time Fourier transform (DTFT). If the spectrum obtained by DTFT is sampled for one period of the Fourier transform, such a transformation is called discrete Fourier transform (DFT) which is a very powerful computational tool for the evaluation of Fourier transform. DFT finds wide applications in linear filtering, correlation analysis and spectrum analysis. Some special algorithms are developed for the easy implementation of DFT which result in saving of considerable computation time.

Such algorithms are called fast Fourier transform (FFT). By divide-and-conquer approach, the DFT which has a size  $N$ , where  $N$  is a composite number is reduced to the smaller DFTs and computation is performed. The computational algorithms are developed when the size of  $N$  is power of 2 and power of 4.

## 2.2 Discrete Fourier Transform (DFT)

By Fourier transform, the sequence  $x[n]$  is transformed as  $X(\omega)$  in the frequency domain. If the time sequence  $x[n]$  is continuous and periodic, the transformation is called discrete time Fourier transform (DTFT). For non-periodic signal  $x[n]$ , a slightly modified transform technique which is known as discrete Fourier transform (DFT) is used which transforms  $x[n]$  to  $X(\omega)$ . DFT is a very powerful tool for the analysis and synthesis of discrete signals and systems. The method is ideally suited for use in digital computer or specially designed digital hardware. The DFT is obtained by sampling one period of DTFT only at a finite number of frequency points. It has the following features:

1. The original finite duration signal can be easily recovered from its DFT since there exists one-to-one correspondence between  $x[n]$  and  $X(\omega)$ .
2. For the calculation of the DFT of finite duration sequences, a very efficient and fast technique called FFT has been developed.
3. As far as realization in digital computer is concerned, DFT is the appropriate representation since it is discrete and of finite length in both the time and frequency domains.
4. DFT is closely related to discrete Fourier series, the Fourier transform, convolution, correlation and filtering.

### 2.2.1 The Discrete Fourier Transform Pairs

Consider the sequence  $x[n]$  of length  $N$ . The Fourier transform of  $x[n]$  is given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \quad (2.1)$$

In Eq. (2.1),  $X(\Omega)$  is the continuous function of  $\Omega$ . The range of  $\Omega$  is from  $-\pi$  to  $\pi$  or 0 to  $2\pi$ . Hence, calculating  $X(\Omega)$  on digital computer or DSP is impossible. It is, therefore, necessary to compute  $X(\Omega)$  at discrete values of  $\Omega$ . When Fourier transform  $X(\Omega)$  is calculated at only discrete points  $k$ , it is called discrete Fourier transform (DFT). The DFT is denoted by  $X(k)$ . For finite discrete points  $N$ , Eq. (2.1) is written as

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \quad (2.2)$$

where  $k = 0, 1, 2, \dots, (N - 1)$ .  $X(k)$  is computed at  $k = 0, 1, 2, \dots, (N - 1)$  discrete points.  $X(k)$  is the sequence of  $N$  samples. The sequence  $x[n]$  is obtained back form

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad (2.3)$$

Let us define  $W_N = e^{-j2\pi/N}$ , where  $W_N$  is called twiddle factor. Equations (2.2) and (2.3) are called DFT and IDFT or simply discrete Fourier transform pair. They can be represented in terms of twiddle factor as given below

$$X(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad (2.4)$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad (2.5)$$

Let the sequence  $x[n]$  be resentenced as a vector  $x_N$  of  $N$  samples as

$$x_N = \begin{bmatrix} n = 0 & x(0) \\ n = 1 & x(1) \\ \vdots & \vdots \\ n = N - 1 & x(N - 1) \end{bmatrix}_{N \times 1} \quad (2.6)$$

and  $X(k)$  be represented as a vector  $X_N$  of  $N$  samples as

$$X_N = \begin{bmatrix} k = 0 & X(0) \\ k = 1 & X(1) \\ \vdots & \vdots \\ k = N - 1 & X(N - 1) \end{bmatrix}_{N \times 1} \quad (2.7)$$

The twiddle factor  $W_N^{kn}$  is represented as a matrix with  $k$  rows and  $N$  column as

$$W_N = \begin{bmatrix} k = 0 & \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \\ \vdots \\ k = N - 1 & \end{bmatrix}_{N \times N} \quad (2.8)$$

Thus, Eqs. (2.4) and (2.5) can be written with matrix form as

$$X_N = [W_N]x_N \quad (2.9)$$

$$x_N = \frac{1}{N}[W_N^*]X_N \quad (2.10)$$

where  $W_N^* = W_N^{-kn}$

$$\begin{aligned} W_N &= e^{-j\frac{2\pi}{N}} \\ &= 1 \angle -2\pi/N \end{aligned} \quad (2.11)$$

From Eq. (2.11), the magnitude of the twiddle factor is 1 and the phase angle is  $-\frac{2\pi}{N}$ . It lies on the unit circle in the complex plane from 0 to  $2\pi$  angle, and it gets repeated for every cycle.

## 2.2.2 Four-Point, Six-Point and Eight-Point Twiddle Factors

As in Eq. (2.11), the magnitude of the twiddle factor is 1 and the angle  $-2\pi$  is equally divided in the interval  $N$ . The most commonly used intervals are  $N = 4$  and  $N = 8$ . For  $N = 4$ , the angle between any  $N = 0$  and  $N = 1$  is  $\frac{\pi}{2}$ .

### 2.2.2.1 Four-Point Twiddle Factor

For  $N = 4$

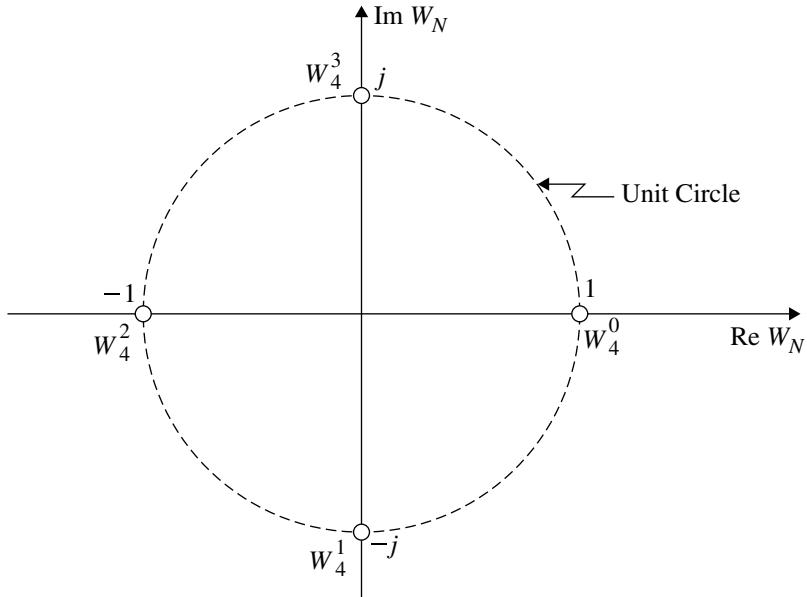
$$W_N = \begin{bmatrix} n=0 & 1 & 2 & 3 \\ k=0 & W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ 1 & W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ 2 & W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ 3 & W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \quad (2.12)$$

**Note:**  $W_4^4 = W_4^0$ ;  $W_4^6 = W_4^2$  and  $W_4^9 = W_4^1$ . From Eq. (2.11)

$$W_4^1 = 1 \angle -\pi/2$$

For  $N = 4$ , the unit circle is divided into four equal segments in the clockwise sequence and labelled as  $W_4^0$ ,  $W_4^1$ ,  $W_4^2$  and  $W_4^3$ . From Fig. 2.1, the twiddle factors are obtained as

$$W_4^0 = 1; \quad W_4^1 = -j; \quad W_4^2 = -1; \quad W_4^3 = -j$$



**Fig. 2.1** Representation of  $W_4^{-nk}$

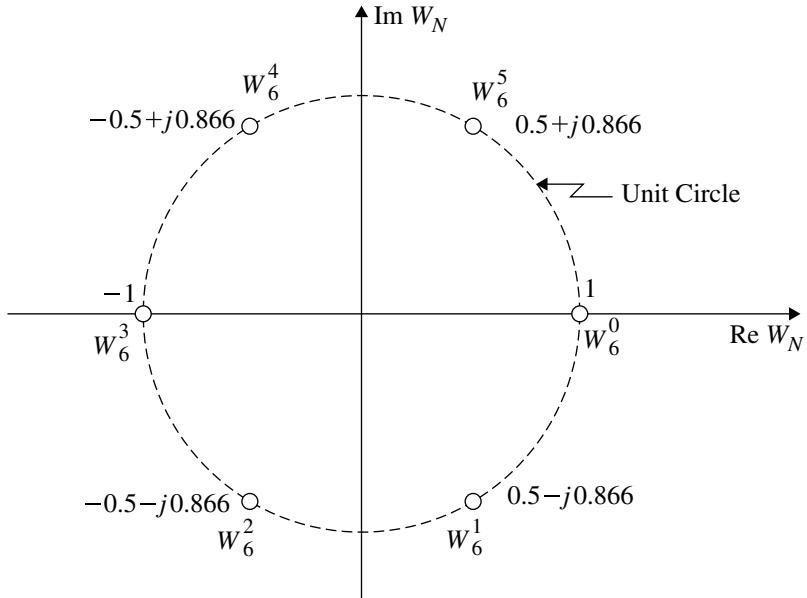
Equation (2.12) is written as

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad (2.13)$$

Equation (2.13) represents the twiddle factor to express DFT of any sequence  $x[n]$ . Twiddle factors for six-point DFT and eight-point DFT are derived below.

### 2.2.2.2 Six-Point Twiddle Factor

For  $N = 6$ , the unit circle is divided into six equal segments and in the clockwise sequence labelled as  $W_6^0$ ,  $W_6^1$ ,  $W_6^2$ ,  $W_6^3$ ,  $W_6^4$  and  $W_6^5$  noting that  $W_6^6 = W_6^0$ ,  $W_6^7 = W_6^1$  and so on. This is shown in Fig. 2.2. Each segment is shifted by  $-60^\circ$  on the unit circle. For  $N = 6$ ,  $W_6$  is obtained by multiplying the rows and columns of  $W_6$  and is given below.



**Fig. 2.2** Representation of  $W_6^{-nk}$

$$W_N = \begin{bmatrix} W_6^0 & W_6^0 & W_6^0 & W_6^0 & W_6^0 \\ W_6^0 & W_6^1 & W_6^2 & W_6^3 & W_6^4 \\ W_6^0 & W_6^2 & W_6^4 & W_6^6 & W_6^8 \\ W_6^0 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} \\ W_6^0 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} \\ W_6^0 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} \end{bmatrix} \quad (2.14)$$

$$W_6^0 = W_6^6 = W_6^{12} = W_6^{18} = W_6^{24} = 1$$

$$W_6^1 = W_6^7 = W_6^{13} = W_6^{19} = W_6^{25} = 1e^{-j\frac{\pi}{3}} = 0.5 - j0.866$$

$$W_6^2 = W_6^8 = W_6^{14} = W_6^{20} = W_6^{26} = 1e^{-j\frac{2\pi}{3}} = -0.5 - j0.866$$

$$W_6^3 = W_6^9 = W_6^{15} = W_6^{21} = W_6^{27} = -1$$

$$W_6^4 = W_6^{10} = W_6^{16} = W_6^{22} = W_6^{28} = 1e^{j\frac{2\pi}{3}} = -0.5 + j0.866$$

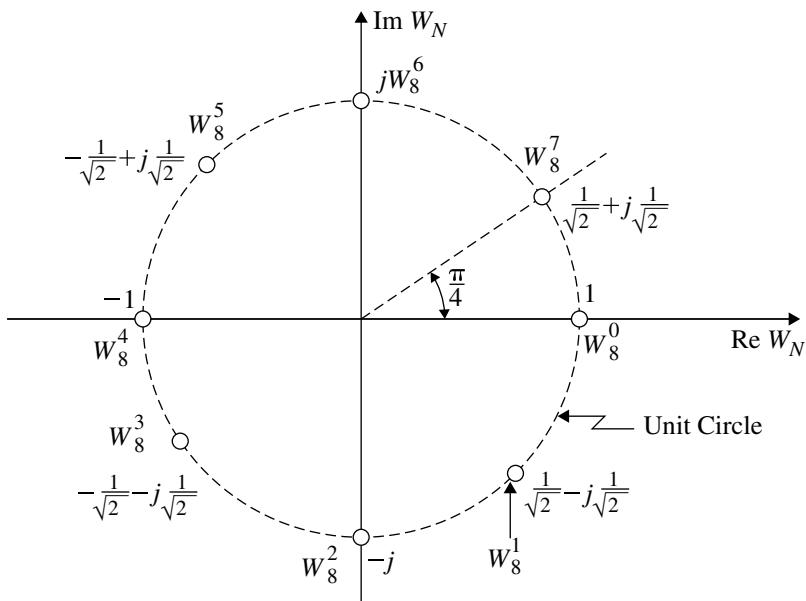
$$W_6^5 = W_6^{11} = W_6^{17} = W_6^{23} = W_6^{29} = 1e^{j\frac{\pi}{3}} = 0.5 + j0.866$$

Substituting the values of the elements of the matrix  $W_6$ , we get

$$W_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.5 - j0.866 & -0.5 - j0.866 & -1 & -0.5 + j0.866 & 0.5 + j0.866 \\ 1 & -0.5 - j0.866 & -0.5 + j0.866 & 1 & -0.5 - j0.866 & -0.5 + j0.866 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.5 + j0.866 & -0.5 - j0.866 & 1 & -0.5 + j0.866 & -0.5 - j0.866 \\ 1 & 0.5 + j0.866 & -0.5 + j0.866 & -1 & -0.5 - j0.866 & 0.5 - j0.866 \end{bmatrix} \quad (2.15)$$

### 2.2.2.3 Eight-Point Twiddle Factor

The eight-point twiddle factor is represented in Fig. 2.3. The twiddle factor  $W_8$  is obtained from Fig. 2.3 and is shown below.



**Fig. 2.3** Representation of  $W_8^{-kn}$

$$W_8 = \begin{bmatrix} W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \quad (2.16)$$

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & j & -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} & -j & \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \end{bmatrix} \quad (2.17)$$

### Example 2.1

Compute the DFT of the sequence  $x[n] = \{1, j, -1, -j\}$  for  $N = 4$ .

(Anna University, November, 2006)

#### Solution

##### Method 1

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}; \quad X_4 = \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$$

From Eq. (2.9)

$$X_4 = W_4 x_4$$

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$$

$$= \begin{bmatrix} 1+j-1-j \\ 1+1+1+1 \\ 1-j-1+j \\ 1-1+1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$X(0) = 0$
$X(1) = 4$
$X(2) = 0$
$X(3) = 0$

**Method 2**

$$X(k) = \sum_{n=0}^3 x[n] e^{-j \frac{2\pi kn}{4}}; \quad k = 0, 1, 2, 3, \dots$$

For  $k = 0$

$$\begin{aligned} X(0) &= \sum_{n=0}^3 x[n] \\ &= x[0] + x[1] + x[2] + x[3] \\ &= 1 + j - 1 - j = 0 \end{aligned}$$

For  $k = 1$

$$\begin{aligned} X(1) &= \sum_{n=0}^3 x[n] e^{-j \frac{\pi n}{2}} \\ &= x[0] + x[1] e^{-j \frac{\pi}{2}} + x[2] e^{-j\pi} + x[3] e^{-j \frac{3\pi}{2}} \\ &= 1 + j(-j) + (-1)(-1) + (-j)(j) \\ &= 1 + 1 + 1 + 1 = 4 \end{aligned}$$

For  $k = 2$

$$\begin{aligned} X(2) &= \sum_{n=0}^3 x[n]e^{-j\pi n} \\ &= x[0] + x[1]e^{-j\pi} + x[2]e^{-j2\pi} + x[3]e^{-j3\pi} \\ &= 1 + j(-1) + (-1)(1) + (-j)(-1) \\ &= 1 - j - 1 + j = 0 \end{aligned}$$

For  $k = 3$

$$\begin{aligned} X(3) &= \sum_{n=0}^3 x[n]e^{-j\frac{3\pi n}{2}} \\ &= x[0] + x[1]e^{-\frac{3\pi}{2}} + x[2]e^{-j3\pi} + x[3]e^{-j\frac{9\pi}{2}} \\ &= 1 + j(j) + (-1)(-1) + (-j)(-j) \\ &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

$X(0) = 0$
$X(1) = 4$
$X(2) = 0$
$X(3) = 0$

Method 1 is simpler and quicker.

### Example 2.2

Find eight-point DFT of  $x[n] = \{1, -1, 1, -1, 1, -1, 1, -1\}$ .

(Anna University, April, 2004)

#### Solution

$$\begin{aligned} X_N &= W_N x_N \\ &= W_8 x_8 \end{aligned} \tag{2.18}$$

$W_8$  is given in Eq. (2.17).

$$x_8 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$X(0)$  is obtained by multiplying  $x_8$  with the first row of  $W_8$ . Thus

$$X(0) = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 = 0$$

$X(1) = 2\text{nd row of } W_8 \text{ to multiply } x_8$

$$= 1 - \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} - j + \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} + j = 0$$

$X(2) = 3\text{rd row of } W_8 \text{ to multiply } x_8$

$$= 1 + j - 1 - j + 1 + j - 1 - j = 0$$

$$\begin{aligned} X(3) &= 1 + \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} + j - \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} - 1 - \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} - j + \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

$X(4) = 5\text{th row of } W_8 \text{ to multiply } x_8$

$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8$$

$X(5) = 6\text{th row of } W_8 \text{ to multiply } x_8$

$$\begin{aligned} &= 1 + \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} - j - \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} - 1 - \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} + j + \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

$X(6) = 1 - j - 1 + j + 1 - j - 1 + j = 0$

$$\begin{aligned} X(7) &= 1 - \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} + j + \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} - j - \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

$$X_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

### 2.2.3 Zero Padding

In evaluating the DFT we assumed that the length of the DFT which is  $N$  is equal to the length  $L$  of the sequence  $x[n]$ . If  $N < L$ , time domain aliasing occurs due to under sampling and in the process we could miss out some important details and get misleading information. To avoid this  $N$ , the number of samples of  $x[n]$  is increased by adding some dummy sample of 0 value. This addition of dummy samples is known as zero padding. The zero padding not only increases the number of samples but also helps in getting a better idea of the frequency spectrum of  $X(\Omega)$ .

---

#### Example 2.3

Compute the four-point DFT of the sequence

$$x[n] = 1 \quad 0 \leq n < 2$$

---

**Solution** For the given sequence  $L = 3$  and  $N = 4$ . By adding a dummy samples of 0 values (zero padding), the given sequence becomes

$$x[n] = \{1, 1, 1, 0\}$$

$$x_N = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$W_4$  is given in Eq. (2.13).

$$\begin{aligned} X_4 &= W_4 x_4 \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$X(0) = [1 + 1 + 1 + 0] = 3$$

$$X(1) = [1 - j - 1 + 0] = -j$$

$$X(2) = [1 - 1 + 1 + 0] = 1$$

$$X(3) = [1 + j - 1 + 0] = j$$

$$X_4 = \begin{bmatrix} 3 \\ -j \\ 1 \\ j \end{bmatrix}$$

**Example 2.4**

Compute the four-point DFT of the following sequences:

- (1)  $x[n] = \{1, 1, 1, 1\}$
- (2)  $x[n] = \{1, 1, 0, 0\}$
- (3)  $x[n] = \cos \pi n$
- (4)  $x[n] = \sin \frac{n\pi}{2}$

*(Anna University, April, 2004; November, 2007)*

**Solution**

$$(1) \quad x[n] = \{1, 1, 1, 1\}$$

$$\begin{aligned}x_4 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\X_4 &= W_4 x_4 \\&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}X(0) &= 1 + 1 + 1 + 1 \\&= 4\end{aligned}$$

$$\begin{aligned}X(1) &= 1 - j - 1 + j \\&= 0\end{aligned}$$

$$\begin{aligned}X(2) &= 1 - 1 + 1 - 1 \\&= 0\end{aligned}$$

$$\begin{aligned}X(3) &= 1 + j - 1 - j \\&= 0 \\X_4 &= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}\end{aligned}$$

(2)  $x[n] = \{1, 1, 0, 0\}$

$$\begin{aligned} X_4 &= W_4 x_4 \\ X_4 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ X(0) &= 1 + 1 + 0 + 0 \\ &= 2 \\ X(1) &= 1 - j + 0 + 0 \\ &= (1 - j) \end{aligned}$$

$$\begin{aligned} X(2) &= 1 - 1 + 0 + 0 \\ &= 0 \\ X(3) &= 1 + j + 0 + 0 \\ &= (1 + j) \\ X_4 &= \begin{bmatrix} 2 \\ 1 - j \\ 0 \\ 1 + j \end{bmatrix} \end{aligned}$$

(3)  $x[n] = \cos \pi n$ ; where  $n = 0, 1, 2, 3, \dots$

$$\begin{aligned} x[n] &= \{1, -1, 1, -1\} \\ X_4 &= W_4 x_4 \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \\ X(0) &= 1 - 1 + 1 - 1 \\ &= 0 \\ X(1) &= 1 + j - 1 - j \\ &= 0 \\ X(2) &= 1 + 1 + 1 + 1 \\ &= 4 \\ X(3) &= 1 - j - 1 + j \\ &= 0 \\ X_4 &= \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \end{aligned}$$

$$(4) \quad x[n] = \sin \frac{n\pi}{2}; \text{ where } n = 0, 1, 2, 3, \dots$$

$$x[n] = \{0, 1, 0, -1\}$$

$$X_4 = W_4 x_4$$

$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned} X(0) &= 0 + 1 + 0 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} X(1) &= 0 - j + 0 - j \\ &= -j2 \end{aligned}$$

$$\begin{aligned} X(2) &= 0 - 1 + 0 + 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} X(3) &= 0 + j + 0 + j \\ &= j2 \end{aligned}$$

$$X_4 = \begin{bmatrix} 0 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

### Example 2.5

Find the  $N$ -point DFT of the following sequences for  $0 \leq n \leq N - 1$ .

$$x[n] = \delta[n]$$

### Solution

$$x[n] = \delta[n]$$

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi k n}{N}} \\ x[n] &= \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \end{aligned}$$

$X(k) = 1$

**Example 2.6**

Find the IDFT of the following functions with  $N = 4$ .

- (1)  $X(k) = \{1, 0, 1, 0\}$
- (2)  $X(k) = \{6, (-2 + j2), -2, (-2 - j2)\}$

**Solution**

(1)  $X(k) = \{1, 0, 1, 0\}$

From Eq. (2.10)

$$x_N = \frac{1}{N} [W_N^*] X_N$$

$$X_N = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$W_N^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

For  $N = 4$

$$x_N = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x[0] = \frac{1}{4}[1 + 0 + 1 + 0] = 0.5$$

$$x[1] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

$$x[2] = \frac{1}{4}[1 + 0 + 1 + 0] = 0.5$$

$$x[3] = \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

$x[n] = \{0.5, 0, 0.5, 0\}$

$$(2) \quad x[n] = \{6, (-2 + j2), -2, (-2 - j2)\}$$

$$x_N = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ -2 + j2 \\ -2 \\ -2 - j2 \end{bmatrix}$$

$$x[0] = \frac{1}{4}[6 - 2 + j2 - 2 - 2 - j2] = 0$$

$$x[1] = \frac{1}{4}[6 - j2 - 2 + 2 + j2 - 2] = 1$$

$$x[2] = \frac{1}{4}[6 + 2 - j2 - 2 + 2 + j2] = 2$$

$$x[3] = \frac{1}{4}[6 + j2 + 2 + 2 - j2 + 2] = 3$$

$x[n] = \{0, 1, 2, 3\}$

### Example 2.7

Compute four-point DFT of causal three sample sequence given by

$$x(n) = \begin{cases} \frac{1}{3}, & 0 \leq n \leq 2 \\ 0, & \text{else.} \end{cases}$$

Plot the magnitude and phase spectrum.

**Solution**  $N$ -point DFT of  $x(n)$  is

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}; \quad k = 0, \dots, N-1$$

Here  $N = 4$

$$X(k) = \sum_{n=0}^3 x(n)e^{-j\frac{2\pi kn}{4}} = \sum_{n=0}^3 x(n)e^{-j\frac{\pi kn}{2}}; \quad k = 0, 1, 2, 3, \dots$$

$$X(k) = x(0)e^0 + x(1)e^{-j\frac{\pi k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}}$$

$$= \frac{1}{3} + \frac{1}{3}e^{-j\frac{\pi k}{2}} + \frac{1}{3}e^{-j\pi k} + 0$$

$$= \frac{1}{3} \left[ 1 + e^{-j\frac{\pi k}{2}} + e^{-j\pi k} \right]$$

$$X(k) = \frac{1}{3} \left[ 1 + \cos \frac{\pi k}{2} - j \sin \frac{\pi k}{2} + \cos \pi k - j \sin \pi k \right]; \quad k = 0, 1, 2, 3, \dots$$

For  $k = 0$

$$\begin{aligned} X(0) &= \frac{1}{3} [1 + \cos 0 - j \sin 0 + \cos 0 - j \sin 0] \\ &= \frac{1}{3}[1 + 1 + 1] = 1\angle 0 \end{aligned}$$

For  $k = 1$

$$\begin{aligned} X(1) &= \frac{1}{3} \left[ 1 + \cos \frac{\pi}{2} - j \sin \frac{\pi}{2} + \cos \pi - j \sin \pi \right] \\ &= \frac{1}{3}[1 - j - 1] = -\frac{1}{3}j = \frac{1}{3}\angle -\pi/2 \end{aligned}$$

For  $k = 2$

$$\begin{aligned} X(2) &= \frac{1}{3} [1 + \cos \pi - j \sin \pi + \cos 2\pi - j \sin 2\pi] \\ &= \frac{1}{3}[1 - 1 + 1] = \frac{1}{3}\angle 0 \end{aligned}$$

For  $k = 3$

$$\begin{aligned} X(3) &= \frac{1}{3} \left[ 1 + \cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2} + \cos 3\pi - j \sin 3\pi \right] \\ &= \frac{1}{3}[1 + j - 1] = \frac{j}{3} = \frac{1}{3}\angle \pi/2 \end{aligned}$$

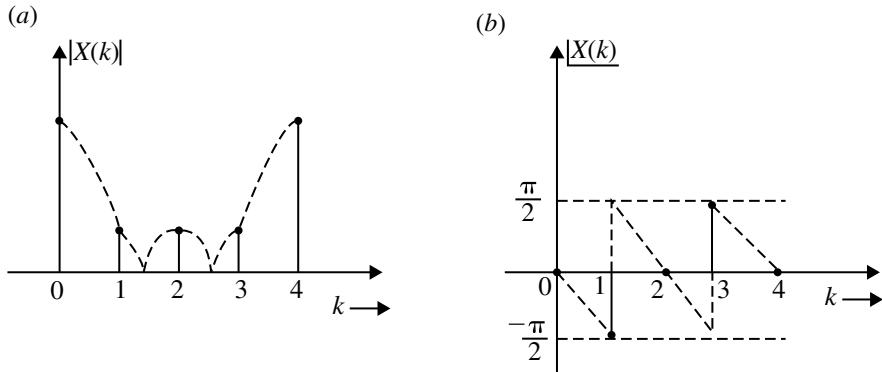
Therefore

$$X(k) = \left\{ 1\angle 0, \frac{1}{3}\angle -\pi/2, \frac{1}{3}\angle 0, \frac{1}{3}\angle \pi/2 \right\}$$

$$\text{Magnitude function } |X(k)| = \left\{ 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\}$$

$$\text{Phase function } \angle X(k) = \left\{ 0, -\frac{\pi}{2}, 0, \frac{\pi}{2} \right\}$$

The magnitude and phase spectrum shown in Fig. 2.4a, b respectively.



**Fig. 2.4** Magnitude and phase spectrum of Example 2.7

### Example 2.8

Compute the four-point DFT of the sequence

$$x(n) = \{0, 1, 2, 3\}$$

Sketch the magnitude and phase spectrum.

**Solution** Given

$$x(n) = \{0, 1, 2, 3\}$$

Here the length of the sequence  $L = 4$ . Hence, we can compute four-point DFT since  $N \geq L$ . The four-point DFT of the sequence  $x(n)$  is given by

$$\begin{aligned} X(k) &= \sum_{n=0}^3 x(n)e^{-j\frac{2\pi kn}{4}} = \sum_{n=0}^3 x(n)e^{-j\frac{\pi kn}{2}} \\ &= x(0)e^0 + x(1)e^{-j\frac{\pi k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}} \\ &= 0 + e^{-j\frac{\pi k}{2}} + 2e^{-j\pi k} + 3e^{-j\frac{3\pi k}{2}} \\ X(4) &= \left( \cos \frac{\pi k}{2} - j \sin \frac{\pi k}{2} \right) + 2 \left( \cos \pi k - j \sin \pi k \right) + 3 \left( \cos \frac{3\pi k}{2} - j \sin \frac{3\pi k}{2} \right) \end{aligned}$$

For  $k = 0$

$$\begin{aligned} X(0) &= (\cos 0 - j \sin 0) + 2(\cos 0 - j \sin 0) + 3(\cos 0 - j \sin 0) \\ &= 1 + 2 + 3 = 6 = 6\angle 0 \end{aligned}$$

For  $k = 1$

$$\begin{aligned} X(1) &= \left(\cos \frac{\pi}{2} - j \sin \frac{\pi}{2}\right) + 2(\cos \pi - j \sin \pi) + 3 \left(\cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2}\right) \\ &= -j - 2 + 3j = -2 + 2j = 2.8 \angle 135^\circ = 2.8 \angle 135 \times \frac{\pi}{180} \\ &= 2.8 \angle 0.75\pi \end{aligned}$$

For  $k = 2$

$$\begin{aligned} X(2) &= (\cos \pi - j \sin \pi) + 2(\cos 2\pi - j \sin 2\pi) + 3(\cos 3\pi - j \sin 3\pi) \\ &= -1 + 2 - 3 = -2 = 2 \angle 180 = 2 \angle 180 \times \frac{\pi}{180} \\ &= 2 \angle \pi \end{aligned}$$

For  $k = 3$

$$\begin{aligned} X(3) &= \left(\cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2}\right) + 2(\cos 3\pi - j \sin 3\pi) + 3 \left(\cos \frac{9\pi}{2} - j \sin \frac{9\pi}{2}\right) \\ &= j - 2 - 3j = -2 - 2j = 2.8 \angle -135^\circ = 2.8 \angle -135 \times \frac{\pi}{180} \\ &= 2.8 \angle -0.75\pi \end{aligned}$$

Therefore,

$$X(k) = \{6 \angle 0, 2.8 \angle 0.75\pi, 2 \angle \pi, 2.8 \angle -0.75\pi\}$$

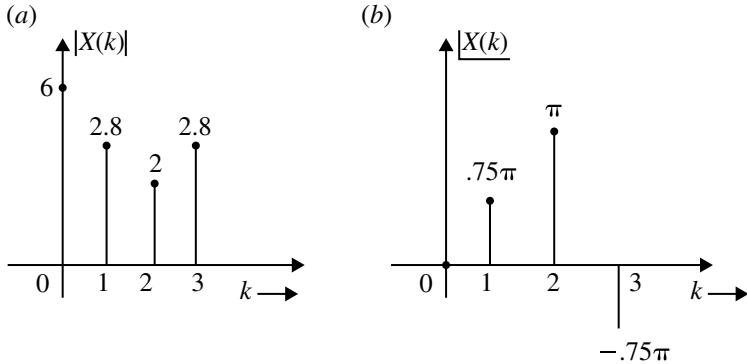
$$\text{Magnitude function } |X(k)| = \{6, 2.8, 2, 2.8\}$$

$$\text{Phase function } \angle X(k) = \{0, 0.75\pi, \pi, -0.75\pi\}$$

The magnitude and phase spectrum are shown in Fig. 2.5a, b respectively.

### Example 2.9

Find the DFT of a sequence  $x(n) = \{1, 1, 0, 0\}$  and find the IDFT of  $Y(k) = \{1, 0, 1, 0\}$



**Fig. 2.5** Magnitude and phase spectrum of Example 2.8

**Solution** Let us assume  $L = N = 4$ . The four-point DFT of  $x(n)$  is

$$\begin{aligned}
 X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \dots, N-1 \\
 X(0) &= \sum_{n=0}^3 x(n)e^{-j\frac{2\pi kn}{4}} = \sum_{n=0}^3 x(n)e^{-j\frac{\pi kn}{2}} \\
 X(0) &= \sum_{n=0}^3 x(n) = x(0) + x(1) + x(2) + x(3) \\
 &= 1 + 1 + 0 + 0 = 2 \\
 X(1) &= \sum_{n=0}^3 x(n)e^{-j\frac{\pi n}{2}} = x(0) + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}} \\
 &= 1 + \cos \frac{\pi}{2} - j \sin \frac{\pi}{2} = 1 - j \\
 X(2) &= \sum_{n=0}^3 x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} \\
 &= 1 + \cos \pi - j \sin \pi = 1 - 1 = 0 \\
 X(3) &= \sum_{n=0}^3 x(n)e^{-j\frac{3\pi n}{2}} = x(0) + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}} \\
 &= 1 + \cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2} = 1 + j
 \end{aligned}$$

Therefore

$$X(k) = \{2, 1-j, 0, 1+j\}$$

Given  $Y(k) = \{1, 0, 1, 0\}$ .

The four-point IDFT of  $Y(k)$  is

$$\begin{aligned}y(n) &= \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j \frac{2\pi k n}{N}}, \quad n = 0, 1, \dots, N-1 \\y(n) &= \frac{1}{4} \sum_{k=0}^3 Y(k) e^{j \frac{2\pi k n}{3}} \\y(0) &= \frac{1}{4} \sum_{k=0}^3 Y(k) \\&= \frac{1}{4} [Y(0) + Y(1) + Y(2) + Y(3)]\end{aligned}$$

$$\begin{aligned}y(0) &= \frac{1}{4} [1 + 0 + 1 + 0] = 0.5 \\y(1) &= \frac{1}{4} \sum_{k=0}^3 Y(k) e^{j \frac{\pi k}{2}} \\&= \frac{1}{4} \left[ Y(0) + Y(1) e^{j \frac{\pi}{2}} + Y(2) e^{j \pi} + Y(3) e^{j \frac{3\pi}{2}} \right] \\&= \frac{1}{4} \left[ Y(0) + Y(1) j \sin \frac{\pi}{2} + Y(2) \cos \pi + Y(3) j \sin \frac{3\pi}{2} \right] \\&= \frac{1}{4} [1 + 0 - 1 + 0] = 0 \\y(2) &= \frac{1}{4} \sum_{k=0}^3 Y(k) e^{j \frac{4\pi k}{4}} \\&= \frac{1}{4} [Y(0) + Y(1) e^{j \pi} + Y(2) e^{j 2\pi} + Y(3) e^{j 3\pi}] \\&= \frac{1}{4} [1 + 0 + \cos 2\pi + 0] \\&= \frac{1}{4} [1 + 0 + 1 + 0] = 0.5 \\y(3) &= \frac{1}{4} \sum_{k=0}^3 Y(k) e^{j \frac{3\pi k}{2}} \\&= \frac{1}{4} \left[ Y(0) + Y(1) e^{j \frac{3\pi}{2}} + Y(2) e^{j 3\pi} + Y(3) e^{j \frac{9\pi}{2}} \right] \\&= \frac{1}{4} [1 + 0 + \cos 3\pi + 0] \\&= \frac{1}{4} [1 + 0 - 1 + 0] = 0\end{aligned}$$

Therefore,

$$y(n) = \{0.5, 0, 0.5, 0\}$$

### Example 2.10

Find the DFT of a sequence

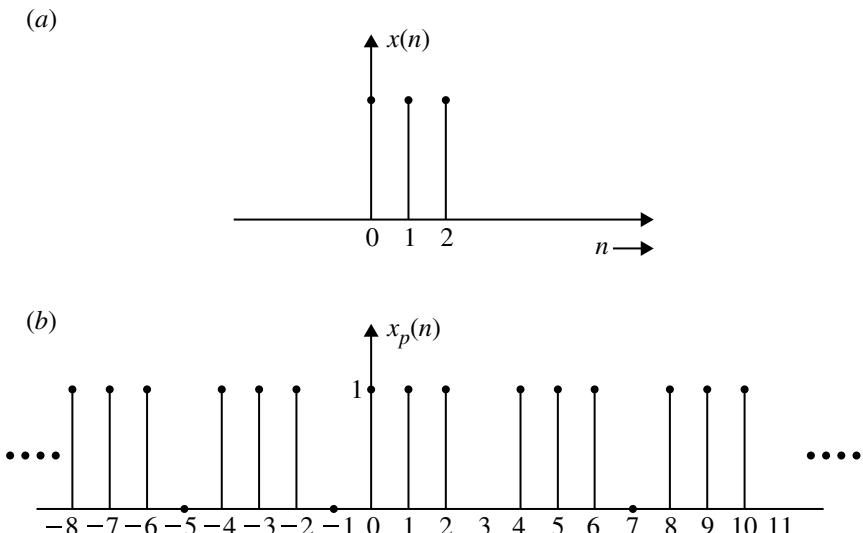
$$x(n) = \begin{cases} 1, & \text{for } 0 \leq n \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

for (i)  $N = 4$  and (ii)  $N = 8$ . Plot  $|X(k)|$  and  $\angle X(k)$ . Comment on the result.

**Solution** Given length of the sequence  $L = 3$ .  $x[n]$  is represented in Fig. 2.6a.

(i) For  $N = 4$  the periodic extension of the sequence is shown in Fig. 2.6b. Consider,

$$x[n] = \{1, 1, 1, 0\}$$



**Fig. 2.6** DT signal and its periodic extension

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^3 x(n)e^{-j\frac{2\pi kn}{4}} \\
X(0) &= \sum_{n=0}^3 x(n) = x(0) + x(1) + x(2) + x(3) \\
&= 3 = 3\angle 0 \\
X(1) &= \sum_{n=0}^3 x(n)e^{-j\frac{\pi n}{2}} \\
&= x(0) + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}} \\
&= 1 + (-j) + (-1) \\
&= -j \\
&= 1\angle\pi/2
\end{aligned}$$

$$\begin{aligned}
X(2) &= \sum_{n=0}^3 x(n)e^{-j\pi n} \\
&= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} \\
&= 1 + (-1) + 1 = 1 = 1\angle 0 \\
X(3) &= \sum_{n=0}^3 x(n)e^{-j\frac{3\pi n}{2}} \\
&= x(0) + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}} \\
&= 1 + j - 1 = j = 1\angle\pi/2
\end{aligned}$$

Therefore,

$$\begin{aligned}
X(k) &= \{3\angle 0, 1\angle\pi/2, 1\angle 0, 1\angle\pi/2\} \\
|X(k)| &= \{3, 1, 1, 1\} \\
\angle X(k) &= \left\{0, -\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}
\end{aligned}$$

Magnitude and phase spectrum are shown in Fig. 2.7a, b respectively.

(ii) **For  $N = 8$**

$$x(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$$

$$X(k) = \sum_{n=0}^7 x(n)e^{-j\frac{2\pi kn}{8}}, \quad k = 0, \dots, 7$$

$$X(0) = \sum_{n=i}^7 x(n) = 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0 = 3$$

$$= 3\angle 0$$

$$X(1) = \sum_{n=0}^7 x(n)e^{-j\frac{\pi n}{4}}$$

$$= x(0) + x(1)e^{-j\frac{\pi}{4}} + x(2)e^{-j\frac{2\pi}{4}} + 0 + 0$$

$$= 1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - j = 1.707 - j1.707$$

$$X(1) = 2.414\angle -\pi/4$$

$$X(2) = \sum_{n=0}^7 x(n)e^{-j\frac{\pi n}{2}}$$

$$= x(0) + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + 0 + 0$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi$$

$$= 1 - j - 1 = -j$$

$$= 1\angle -\pi/2$$

$$X(3) = \sum_{n=0}^7 x(n)e^{-j\frac{3\pi n}{4}}$$

$$= x(0) + x(1)e^{-j\frac{3\pi}{4}} + x(2)e^{-j\frac{3\pi}{2}} + 0 + 0$$

$$= 1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 - 0.707 - j0.707 + j = 0.293 + j0.293$$

$$= 0.414\angle\pi/3$$

$$X(4) = \sum_{n=0}^7 x(n)e^{-jn\pi}$$

$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + 0 + 0$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$$

$$= 1 - 1 + 1 = 1\angle 0$$

$$\begin{aligned}
X(5) &= \sum_{n=0}^7 x(n)e^{-j\frac{5\pi n}{4}} \\
&= x(0) + x(1)e^{-j\frac{5\pi}{4}} + x(2)e^{-j\frac{5\pi}{2}} \\
&= 1 + \cos \frac{5\pi}{4} - j \sin \frac{5\pi}{7} + \cos \frac{5\pi}{2} - j \sin \frac{5\pi}{2} \\
&= 1 - 0.707 + j0.707 - j \\
&= 0.293 - j0.293 = 0.414\angle-\pi/4
\end{aligned}$$

$$\begin{aligned}
X(6) &= \sum_{n=0}^7 x(n)e^{-j\frac{3\pi n}{2}} \\
&= x(0) + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + 0 \\
&= 1 + \cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2} + \cos 3\pi - j \sin 3\pi \\
&= 1 + j - 1 = j \\
&= 1\angle\pi/2
\end{aligned}$$

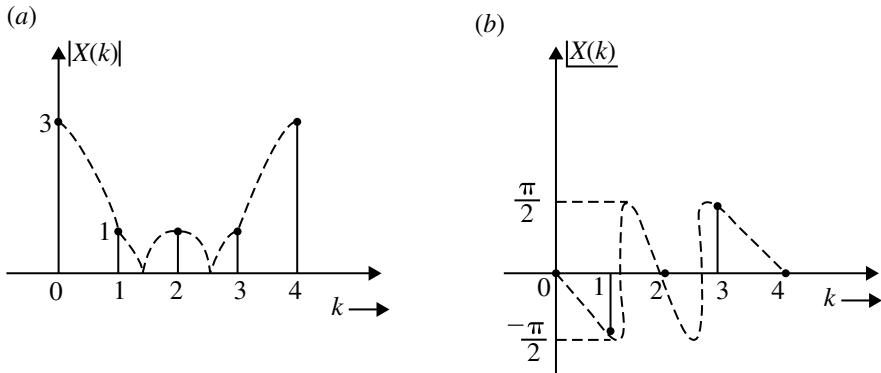
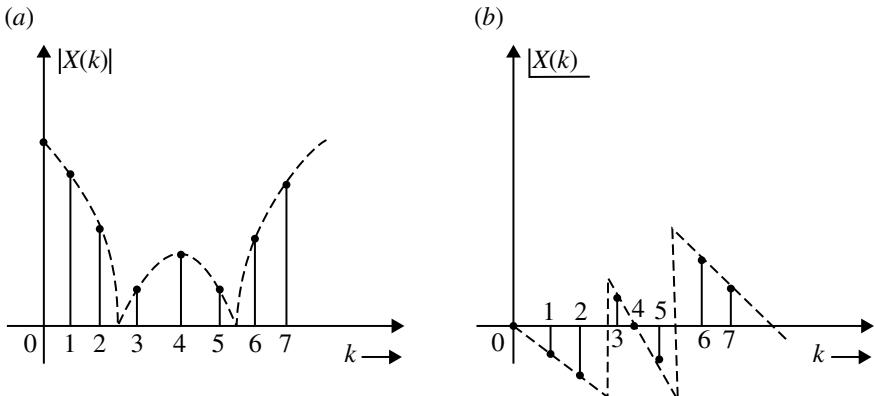
$$\begin{aligned}
X(7) &= \sum_{n=0}^7 x(n)e^{-j\frac{7\pi nk}{4}} \\
&= 1 + e^{-j\frac{7\pi}{4}} + e^{-j\frac{7\pi}{2}} + 0 \\
&= 1 + \cos \frac{7\pi}{4} - j \sin \frac{7\pi}{4} + \cos \frac{7\pi}{2} - j \sin \frac{7\pi}{2} + 0 \\
&= 1 + 0.707 + j0.707 + j \\
&= 1.707 + j1.707 \\
&= 2.414\angle\pi/4
\end{aligned}$$

Therefore

$$\begin{aligned}
X(k) &= \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\} \\
\angle X(k) &= \left\{0, -\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, 0, -\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\right\}
\end{aligned}$$

The magnitude and phase spectrum are shown in Fig. 2.8a, b respectively.

*Comments:* It is very difficult to extrapolate the entire frequency spectrum with  $N = 4$ , i.e., the resolution of the spectrum is very poor. In order to increase the resolution, we must increase  $N$ . It is possible to extrapolate the frequency spectrum with  $N = 8$ . Thus, the zero padding gives us a high density spectrum and provides a better displayed magnitude and phase plots.

**Fig. 2.7** Magnitude and phase spectrum for  $N = 4$ **Fig. 2.8** Magnitude and phase spectrum for  $N = 8$ **Example 2.11**

Find IDFT of the sequence  $X(k) = \{5, 0, 1 - j, 0, 1, 0, 1 + j, 0\}$

(Anna University, December, 2003)

**Solution** Given  $N = 8$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j \frac{2\pi k n}{N}}, \quad n = 0, \dots, N-1$$

$$x(n) = \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \frac{\pi k n}{4}}, \quad n = 0, \dots, 7$$

$$\begin{aligned}
x(0) &= \frac{1}{8} \sum_{k=0}^7 X(k) \\
&= \frac{1}{8}[5 + 0 + 1 - j + 0 + 1 + 0 + 1 + j + 0] = 1 \\
x(1) &= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \frac{\pi k}{4}} \\
&= \frac{1}{8}[5 + (1-j)j + 1(-1) + (1+j)(-j)] \\
&= \frac{1}{8}[6] = 0.75 \\
x(2) &= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \frac{\pi k}{2}} \\
&= \frac{1}{8}[5 + (1-j)(-1) + 1(1) + (1+j)(-1)] \\
&= \frac{1}{8}[4] = 0.5 \\
x(3) &= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \frac{3\pi k}{4}} \\
&= \frac{1}{8}[5 + (1-j)(-j) + 1(-1) + (1+j)(j)] \\
&= \frac{1}{8}[2] = 0.25 \\
x(4) &= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \pi k} \\
&= \frac{1}{8}[5 + (1-j)(1) + 1(1) + (1+j)(1)] = 1 \\
x(5) &= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \frac{5\pi k}{4}} \\
&= \frac{1}{8}[5 + (1-j)(j) + 1(1) + (1+j)(-j)] \\
&= \frac{1}{8}[6] = 0.75 \\
x(6) &= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \frac{3\pi k}{2}} \\
&= \frac{1}{8}[5 + (1-j)(-1) + 1(1) + (1+j)(-1)] \\
&= \frac{1}{8}[4] = 0.5
\end{aligned}$$

$$\begin{aligned}
 x(7) &= \frac{1}{8} \sum_{k=0}^7 X(k) e^{j \frac{7\pi k}{4}} \\
 &= \frac{1}{8} [5 + (1-j)(-j) + 1(-1) + (1+j)(j)] \\
 &= \frac{1}{8} [2] = 0.25
 \end{aligned}$$

Therefore,

$$x(n) = \{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$$


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### Example 2.12

Two finite duration sequences are given by

$$\begin{aligned}
 x[n] &= \sin\left(\frac{n\pi}{2}\right) && \text{for } n = 0, 1, 2, 3 \\
 h[n] &= 2^n && \text{for } n = 0, 1, 2, 3
 \end{aligned}$$

- (a) Calculate the four-point DFT  $X(k)$ . (b) Calculate the four-point DFT  $H(k)$  and  
 (c) If  $Y(k) = X(k)H(k)$ , determine the inverse DFT  $y(n)$  of  $Y(k)$  and sketch it.

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(Anna University, December, 2007)

#### Solution

- (a) To calculate the four-point DFT  $X(k)$

Given

$$\begin{aligned}
 x[n] &= \sin\left(\frac{n\pi}{2}\right) && \text{for } n = 0, 1, 2, 3 \\
 x[n] &= \{0, 1, 0, -1\}
 \end{aligned}$$

$$\begin{aligned}
 X_4 &= W_4 x_4 \\
 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}
 \end{aligned}$$

$$X(0) = 0 + 1 + 0 - 1 = 0$$

$$X(1) = 0 - j + 0 - j = -j2$$

$$X(2) = 0 - 1 + 0 + 1 = 0$$

$$X(3) = 0 + j + 0 + j = j2$$

$$X_4 = \begin{bmatrix} 0 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$$X(k) = \{0, -2j, 0, 2j\}$$

(b) Calculate the four-point DFT  $H(k)$

Given

$$h[n] = 2^n \quad \text{for } n = 0, 1, 2, 3$$

$$h[n] = \{1, 2, 4, 8\}$$

$$H_4 = W_4 h_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}$$

$$H(0) = 1 + 2 + 4 + 8 = 15$$

$$H(1) = 1 - 2j - 4 + 8j = -3 + 6j$$

$$H(2) = 1 - 2 + 4 - 8 = -5$$

$$H(3) = 1 + 2j - 4 - 8j = -3 - 6j$$

$$H_4 = \begin{bmatrix} 15 \\ -3 + 6j \\ -5 \\ -3 - 6j \end{bmatrix}$$

$$H(k) = \{15, -3 + 6j, -5, -3 - 6j\}$$

(c)

$$Y(k) = X(k)H(k)$$

$$= \{0, -j2, 0, j2\}\{15, -3 + j6, -5, -3 - j6\}$$

$$= \{0 \times 15, (-j2)(-3 + j6), 0 \times (-5), j2(-3 - j6)\}$$

$$= \{0, 6j + 12, 0, -6j + 12\}$$

$$Y(k) = \{0, 12 + 6j, 0, 12 - 6j\}$$

From Eq. (2.10)

$$y_n = \frac{1}{N} [W_N^*] Y_N$$

$$Y_n = \begin{bmatrix} 0 \\ 12 + 6j \\ 0 \\ 12 - 6j \end{bmatrix}$$

$$W_N^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

For  $N = 4$

$$y_N = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 0 \\ 12 + 6j \\ 0 \\ 12 - 6j \end{bmatrix}$$

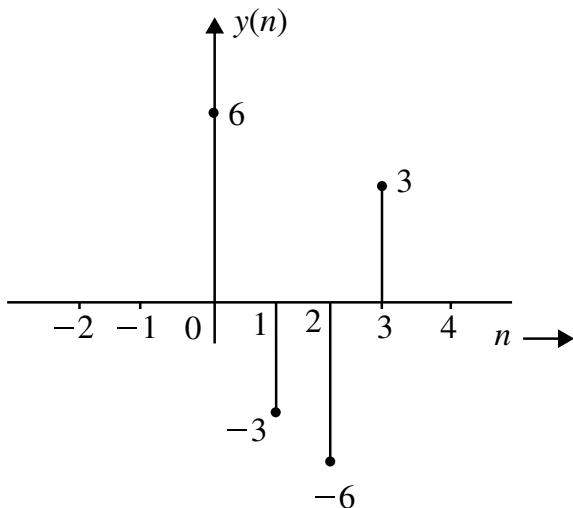
$$y[0] = \frac{1}{4} [0 + 12 + 6j + 0 + 12 - 6j] = 6$$

$$y[1] = \frac{1}{4} [0 + 12j - 6 + 0 - 12j - 6] = -3$$

$$y[2] = \frac{1}{4} [0 - 12 - 6j + 0 - 12 + 6j] = -6$$

$$y[3] = \frac{1}{4} [0 - 12j + 6 + 0 + 12j + 6] = 3$$

$$\boxed{y[n] = \{6, -3, -6, 3\}}$$



**Example 2.13**

Compute the DFT of

$$x[n] = e^{-0.5n}, \quad 0 \leq n \leq 5$$

(Anna University, June, 2007)

**Solution** For the given sequence  $L = 5$ . In general  $N \geq L$ .

Let  $N = 8$

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi kn}{8}} \\ X(k) &= \sum_{n=0}^5 e^{-0.5n}e^{\frac{-j2\pi kn}{8}} \\ &= \sum_{n=0}^5 e^{-0.5n}e^{\frac{-j\pi kn}{4}} \\ &= \sum_{n=0}^5 \left(e^{-0.5}e^{\frac{-j\pi k}{4}}\right)^n \\ &= \sum_{n=0}^5 \left(0.607e^{\frac{-j\pi k}{4}}\right)^n \\ &= \frac{1 - \left(0.607e^{\frac{-j\pi k}{4}}\right)^6}{1 - 0.607e^{\frac{-j\pi k}{4}}} \end{aligned}$$

$$X(k) = \frac{1 - 0.05e^{\frac{-j3\pi k}{2}}}{1 - 0.607e^{\frac{-j\pi k}{4}}}$$

The following summation formula is used above.

$$\sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x}$$

**Example 2.14**

Calculate the DFT of the sequence

$$x[n] = \{1, 1, -2, -2\}$$

(Anna University, November, 2006)

**Solution** Given

$$x[n] = \{1, 1, -2, -2\}$$

$$x_4 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}$$

$W_4$  is given in Eq. (2.13)

$$\begin{aligned} X_4 &= W_4 x_4 \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} \end{aligned}$$

$$X(1) = 1 + 1 - 2 + 2 = 2$$

$$X(2) = 1 - j + 2 + 2j = 3 + j$$

$$X(3) = 1 - 1 - 2 - 2 = -4$$

$$X(4) = 1 + j + 2 - 2j = 3 - j$$

$$X_4 = \begin{bmatrix} 2 \\ 3 + j \\ -4 \\ 3 - j \end{bmatrix}$$

Four-point DFT of  $x(n)$  is

$$X(k) = \{2, 3 + j, -4, 3 - j\}$$

**Example 2.15**

A finite duration sequence of length  $L$  is given as

$$x(n) = \begin{cases} 1, & 0 \leq n \leq L-1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the  $N$ -point DFT of the sequence for  $N = L$ .

*(Anna University, June, 2007)*

**Solution** For  $N = L$ . A finite duration  $x(n)$  is

$$x(n) = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$$

$N$ -point DFT of  $x(n)$  is

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} 1 \cdot e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} \left( e^{-j\frac{2\pi k}{N}} \right)^n \\ &= \frac{1 - e^{-j\frac{2\pi kN}{N}}}{1 - e^{-j\frac{2\pi k}{N}}} \quad \because e^{-j\frac{2\pi kN}{N}} = e^{-j2\pi k} = 1 \\ &= 0 \end{aligned}$$

**Example 2.16**

Compute DFT of  $x[n]$

$$x(n) = \begin{cases} 0, & 0 \leq n \leq 4, \\ 1, & 5 \leq n < 7. \end{cases}$$

*(Anna University, December, 2005)*

**Solution** Given

$$x(n) = \begin{cases} 0, & 0 \leq n \leq 4, \\ 1, & 5 \leq n < 7. \end{cases}$$

$$x[n] = \{0, 0, 0, 0, 0, 1, 1\}$$

Let us assume  $N = L = 8$ . Therefore,  $x[n] = \{0, 0, 0, 0, 0, 0, 1, 1, 0\}$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi nk}{N}}, \quad k = 0, \dots, N-1$$

$$X(k) = \sum_{n=0}^7 x(n)e^{\frac{-j2\pi nk}{8}}, \quad k = 0, 1, \dots, 7$$

$$X(k) = \sum_{n=0}^7 x(n)e^{\frac{-j\pi nk}{4}}, \quad k = 0, 1, \dots, 7$$

For  $k = 0$

$$X(0) = \sum_{n=0}^7 x(n)$$

$$X(0) = 0 + 0 + 0 + 0 + 0 + 1 + 1 + 0 = 2$$

For  $k = 1$

$$X(1) = \sum_{n=0}^7 x(n)e^{\frac{-j\pi n}{4}}$$

$$= x(0) + x(1)e^{\frac{-j\pi}{4}} + x(2)e^{\frac{-j\pi}{2}} + x(3)e^{\frac{-j3\pi}{4}} + x(4)e^{-j\pi}$$

$$+ x(5)e^{\frac{-j5\pi}{4}} + x(6)e^{\frac{-j3\pi}{2}} + x(7)e^{\frac{-j7\pi}{4}}$$

$$= 0 + 0 + 0 + 0 + 0 + e^{\frac{-j5\pi}{4}} + e^{\frac{-j3\pi}{2}} + 0$$

$$X(1) = -0.707 + 0.707j + j$$

$$= -0.707 + 1.707j$$

For  $k = 2$

$$X(2) = \sum_{n=0}^7 x(n)e^{\frac{-j\pi n}{2}}$$

$$= x(5)e^{\frac{-j5\pi}{2}} + x(6)e^{-j3\pi}$$

$$= -j - 1 = -1 - j$$

For  $k = 3$

$$\begin{aligned} X(3) &= \sum_{n=0}^7 x(n)e^{-j\frac{3\pi n}{4}} \\ &= x(5)e^{-j\frac{15\pi}{4}} + x(6)e^{-j\frac{9\pi}{2}} \\ &= 0.707 + 0.707j - j \\ &= 0.707 - 0.293j \end{aligned}$$

For  $k = 4$

$$\begin{aligned} X(4) &= \sum_{n=0}^7 x(n)e^{-j\pi n} \\ &= x(5)e^{-j5\pi} + x(6)e^{-j6\pi} \\ &= -1 + 1 = 0 \end{aligned}$$

For  $k = 5$

$$\begin{aligned} X(5) &= \sum_{n=0}^7 x(n)e^{-j\frac{5\pi n}{4}} \\ &= x(5)e^{-j\frac{25\pi}{4}} + x(6)e^{-j\frac{15\pi}{2}} \\ &= 0.707 - 0.707j + j \\ &= 0.707 + 0.293j \end{aligned}$$

For  $k = 6$

$$\begin{aligned} X(6) &= \sum_{n=0}^7 x(n)e^{-j\frac{3\pi n}{2}} \\ &= x(5)e^{-j\frac{15\pi}{2}} + x(6)e^{-j9\pi} \\ &= j - 1 = -1 + j \end{aligned}$$

For  $k = 7$

$$\begin{aligned} X(7) &= \sum_{n=0}^7 x(n)e^{-j\frac{7\pi n}{4}} \\ &= x(5)e^{-j\frac{35\pi}{2}} + x(6)e^{-j\frac{21\pi}{2}} \\ &= -0.707 - 0.707j - j \\ &= -0.707 - 1.707j \end{aligned}$$

$$X(k) = \left\{ 2, -0.707 + 1.707j, -1 - j, 0.707 - 0.293j, 0, \right. \\ \left. 0.707 + 0.293j, -1 + j, -0.707 - 1.707j \right\}$$

**Example 2.17**

Compute the DFT of the sequence  $x[n] = e^{-n}$  where  $0 \leq n \leq 4$ .

(Anna University, December, 2004)

**Solution** Let  $N = 8$ , therefore

$$x(n) = \begin{cases} e^n, & 0 \leq n \leq 4, \\ 0, & 5 \leq n \leq 7. \end{cases}$$

$$\begin{aligned} X(k) &= \sum_{n=0}^7 x(n)e^{-j\frac{2\pi nk}{8}} \\ &= \sum_{n=0}^4 x(n)e^{-j\frac{2\pi nk}{8}} \\ &= \sum_{n=0}^4 e^{-n}e^{-j\frac{2\pi nk}{8}} \\ &= \sum_{n=0}^4 \left( e^{-1} \cdot e^{-j\frac{2\pi k}{8}} \right)^n \\ &= \sum_{n=0}^4 \left( 0.3679e^{-j\frac{\pi k}{4}} \right)^n \end{aligned}$$

$$X(k) = \frac{1 - \left( 0.3679e^{-j\frac{\pi k}{4}} \right)^5}{1 - 0.3679e^{-j\frac{\pi k}{4}}}, \quad k = 0, \dots, N-1$$

because

$$\sum_{n=0}^N x^n = \frac{1 - x^{N+1}}{1 - x}$$

**Example 2.18**

Determine the eight-point DFT of the sequence

$$x[n] = \{0, 0, 1, 1, 1, 0, 0, 0\}$$

(Anna University, May, 2004)

**Solution** Given

$$X_N = W_N x_N$$

$$X_8 = W_8 x_8$$

$W_8$  is given Eq. (2.17)

$$x_8 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$X(0)$  is obtained by multiplying  $x_8$  with first row of  $W_8$ . Thus

$$X(0) = 0 + 0 + 1 + 1 + 1 + 0 + 0 + 0 = 3$$

$X(1) = 2$ nd row of  $W_8$  to multiply  $x_8$

$$= -j - \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} - 1 = -1.707 - j1.707$$

$X(2) = 3$ rd row of  $W_8$  to multiply  $x_8$

$$= -1 + j + 1 = j$$

$X(3) = 4$ th row of  $W_8$  to multiply  $x_8$

$$= j + \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} - 1 = -0.293 + j0.293$$

$X(4) = 5$ th row of  $W_8$  to multiply  $x_8$

$$= 1 - 1 + 1 = 1$$

$X(5) = 6$ th row of  $W_8$  to multiply  $x_8$

$$= -j + \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} - 1 = -0.293 - j0.293$$

$X(6) = 7$ th row of  $W_8$  to multiply  $x_8$

$$\begin{aligned}
 &= -1 - j + 1 = -j \\
 X(7) &= \text{8th row of } W_8 \text{ to multiply } x_8 \\
 &= j - \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} - 1 = -1.707 + j1.707
 \end{aligned}$$

Therefore,

$$\boxed{X(k) = \left\{ 3, -1.707 - j1.707, j, -0.293 + j0.293, \right.} \\
 \left. 1, -0.293 - j0.293, -j, -1.707 + j1.707 \right\}$$

### Example 2.19

Find the  $N$ -point DFT of the following signals

$$\begin{aligned}
 (a) \quad x[n] &= \delta(n - n_0) \\
 (b) \quad x[n] &= a^n
 \end{aligned}$$

### Solution

$$(a) \quad x[n] = \delta(n - n_0)$$

$$\begin{aligned}
 X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn} \\
 x[n] &= \begin{cases} 1, & n = n_0, \\ 0, & n \neq n_0. \end{cases}
 \end{aligned}$$

$$\boxed{X(k) = e^{-j2\pi kn_0}}$$

$$(b) \quad x[n] = a^n$$

$$\begin{aligned}
 X(k) &= \sum_{n=0}^{N-1} a^n e^{-j2\pi kn} \\
 &= \sum_{n=0}^{N-1} \left( ae^{-j2\pi k} \right)^n
 \end{aligned}$$

using summation formula

$$\sum_{n=0}^N x^n = \frac{1 - x^{N+1}}{1 - x}$$

we get

$$X(k) = \frac{1 - \left(a e^{-j\frac{2\pi k}{N}}\right)^N}{1 - a e^{-j\frac{2\pi k}{N}}}$$

$$X(k) = \frac{1 - a^N}{1 - a e^{-j\frac{2\pi k}{N}}}$$

## 2.3 Relationship of the DFT to Other Transforms

### 2.3.1 Relationship to the Fourier Series Coefficients of a Periodic Sequence

Fourier series of the discrete time signal is written as

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j \frac{2\pi n k}{N}} \quad -\infty < n < \infty \quad (2.19)$$

where the Fourier series coefficients are represented as

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi n k}{N}} \quad k = 0, \dots, N-1 \quad (2.20)$$

$$\text{DFT} \implies X(k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi n k}{N}} \quad (2.21)$$

Comparing Eqs. (2.20) and (2.21), we get

$$C_k = \frac{1}{N} X(k)$$

$$X(k) = N C_k \quad (2.22)$$

### 2.3.2 Relationship to the Fourier Transform of an Aperiodic Sequence

Fourier transform of  $x(n)$  is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad (2.23)$$

$$X(k) = X(\omega) \Big|_{\omega=\frac{2\pi k}{N}}, \quad k = 0, \dots, N-1 \quad (2.24)$$

The finite duration sequence

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}$$

### 2.3.3 Relationship to the $z$ -Transform

The  $z$ -transform of  $N$ -point sequence  $x(n)$  is given by,

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n} \quad (2.25)$$

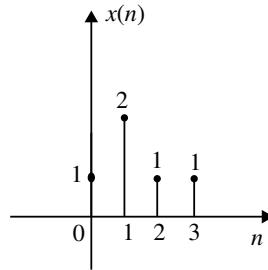
Let us evaluate  $X(z)$  at  $N$  equally spaced points on unit circle that is at  $z = e^{j\frac{2\pi k}{N}}$

$$\begin{aligned} X(z) \Big|_{z=e^{j\frac{2\pi k}{N}}} &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \\ &= X(k) \\ X(k) &= X(z) \Big|_{z=e^{j\frac{2\pi k}{N}}} \end{aligned} \quad (2.26)$$

From Eq. (2.26), we can conclude that the  $N$ -point DFT of a finite duration sequence can be obtained from the  $z$ -transform of the sequence at  $N$  equally spaced points around the unit circle.

**Example 2.20**

Consider the finite length sequence  $x(n)$  as represented below:



Let  $X(z)$  be the  $z$ -transform of  $x(n)$ . If we sample  $X(z)$  at

$$z = e^{j(\frac{2\pi}{4})k}, \quad k = 0, 1, 2, 3$$

Using the relation

$$X_1(k) = X(z) \Big|_{z=e^{j(\frac{2\pi}{4})k}}$$

find the sequence  $x_1(n)$ .

**Solution**

$$\begin{aligned} X_1(k) &= X(z) \Big|_{z=e^{j(\frac{2\pi}{4})k}} \\ X(z) &= \sum_{n=0}^3 x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} \\ &= 1 + 2z^{-1} + z^{-2} + z^{-3} \end{aligned}$$

$$\begin{aligned} X_1(k) &= (1 + 2z^{-1} + z^{-2} + z^{-3}) \Big|_{z=e^{j(\frac{2\pi}{4})k}} \\ X_1(k) &= 1 + 2e^{-j\frac{2\pi k}{4}} + e^{-j\frac{2\pi k \cdot 2}{4}} + e^{-j\frac{2\pi k \cdot 3}{4}} \\ X_1(k) &= x_1(0) + x_1(1)e^{-j\frac{2\pi k}{4}} + x_1(2)e^{-j\frac{2\pi k \cdot 2}{4}} + x_1(3)e^{-j\frac{2\pi k \cdot 3}{4}} \end{aligned}$$

Therefore,

$$x_1(n) = \boxed{\{1, 2, 1, 1\}}$$

## 2.4 Properties of DFT

### 2.4.1 Periodicity

If a sequence  $x(n)$  is periodic with periodicity of  $N$  samples, then  $N$ -point DFT of the sequence is also periodic with periodicity of  $N$  samples.

Hence, if  $x(n)$  and  $X(k)$  are an  $N$  point DFT pair, then

$$\begin{aligned} x(n+N) &= x(n) \quad \text{for all } n \\ X(k+N) &= X(k) \quad \text{for all } k \end{aligned} \tag{2.27}$$

#### Proof

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi kn}{N}} \\ X(k+N) &= \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi(k+N)n}{N}} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi kn}{N}} e^{-j 2\pi n} \\ e^{-j 2\pi n} &= 1 \quad \text{for all } n, (\text{Here } n \text{ is an integer}) \\ X(k+N) &= \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi kn}{N}} \end{aligned}$$

$$X(k+N) = X(k)$$

### 2.4.2 Linearity

If

$$x_1(n) \xrightarrow[N]{\text{DFT}} X_1(k) \quad \text{and} \quad x_2(n) \xrightarrow[N]{\text{DFT}} X_2(k)$$

then for any real-valued or complex-valued constants  $a_1$  and  $a_2$

$$a_1 x_1(n) + a_2 x_2(n) \xrightarrow[N]{\text{DFT}} a_1 X_1(k) + a_2 X_2(k)$$

(2.28)

**Proof**

$$\begin{aligned}
 \text{DFT}\{a_1x_1(n) + a_2x_2(n)\} &= \sum_{n=0}^{N-1} (a_1x_1(n) + a_2x_2(n))e^{-j\frac{2\pi kn}{N}} \\
 &= a_1 \sum_{n=0}^{N-1} x_1(n)e^{-j\frac{2\pi kn}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n)e^{-j\frac{2\pi kn}{N}} \\
 &= a_1 X_1(k) + a_2 X_2(k)
 \end{aligned}$$

### 2.4.3 Circular Shift and Circular Symmetric of a Sequence

Consider a finite duration sequence  $x(n)$  and its periodic extension  $x_p(n)$ . The periodic extension of  $x(n)$  can be expressed as  $x_p(n) = x(n + N)$  where  $N$  is the periodicity. For example, let

$$x(n) = \{1, 2, 3, 4\} \quad \text{and} \quad N = 4$$

↑

The sequence and its periodic extension are shown in Fig. 2.9a, b respectively. Let us shift the periodic sequence  $x_p(n)$  by two units of time to the right.

Let us denote one period of this shifted sequence by  $x'(n)$ . The sequence  $x'(n)$  can be represented by  $x(n - 2, (\text{mod}4))$  where mod 4 indicates that the sequence repeats after four samples.

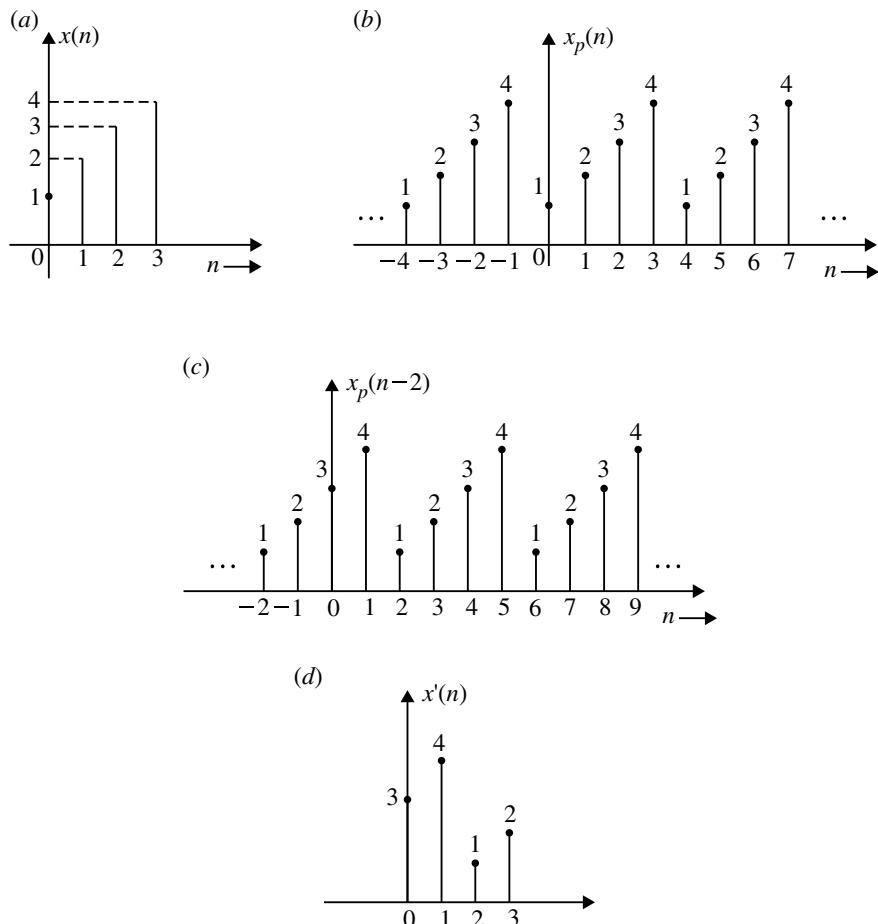
$$\begin{aligned}
 x'(n) &= x(n - 2, \text{ mod } 4) \\
 x'(0) &= x(-2, \text{ mod } 4) = x(2) = 3 \\
 x'(1) &= x(-1, \text{ mod } 4) = x(3) = 4 \\
 x'(2) &= x(0, \text{ mod } 4) = x(0) = 1 \\
 x'(3) &= x(1, \text{ mod } 4) = x(1) = 2
 \end{aligned}$$

Circular representation of  $x(n)$  and  $x'(n)$  are shown in Fig. 2.10.

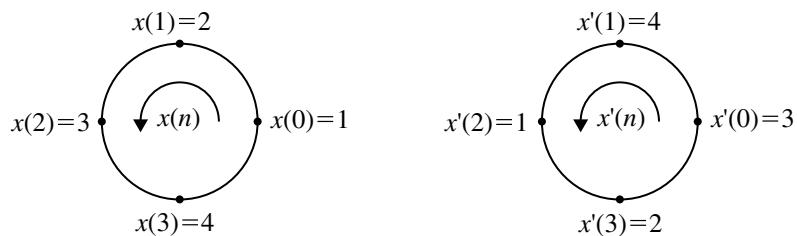
From this,  $x'(n)$  is simply  $x(n)$  shifted circularly by two units in time where the counterclockwise direction has been arbitrarily selected as positive direction. From this we conclude that a circular shift of an  $N$ -point sequence is equivalent to linear shift of its periodic extension.

Let  $x(n)$  be a  $N$ -point sequence represented on a circle and  $x'(n)$  be its shifted sequence by  $k$  units of time

$$x'(n) = x(n - k, \text{ mod } 4) = x(n - k)_N \quad (2.29)$$



**Fig. 2.9** **a** Discrete time signal  $x(n)$ , **b** periodic extension of  $x(n)$ , **c** shifted periodic extension of  $x(n)$  and **d** discrete time signal  $x'(n)$



**Fig. 2.10** Circular representation

### Even Symmetry

An  $N$ -point sequence is called even, if it is symmetric about zero on the circle

$$x(N - n) = x(n) \quad \text{for } 0 \leq n \leq N - 1 \quad (2.30)$$

### Odd Symmetry

An  $N$ -point sequence is called odd, if it is asymmetric about zero on the circle

$$x(N - n) = -x(n) \quad \text{for } 0 \leq n \leq N - 1 \quad (2.31)$$

### Time Reversal

Time reversal of a  $N$ -point sequence is obtained as

$$x(-n, \bmod N) = x(N - n) \quad (2.32)$$

$$x(-n)_N = x(N - n), \quad 0 \leq n \leq N - 1 \quad (2.33)$$

Time reversal is equivalent to plotting  $x(n)$  in a clockwise direction on a circle.

Even and odd sequences for a periodic sequence  $x_p(n)$  are given as

$$\text{Even: } x_p(n) = x_p(-n) = x_p(N - n) \quad (2.34)$$

$$\text{Odd: } x_p(n) = -x_p(-n) = -x_p(N - n) \quad (2.35)$$

If the periodic sequence is complex valued, then

$$\text{Conjugate Even: } x_p(n) = x_p^*(N - n) \quad (2.36)$$

$$\text{Conjugate Odd: } x_p(n) = -x_p^*(N - n) \quad (2.37)$$

$$x_p(n) = x_{pe}(n) + x_{po}(n) \quad (2.38)$$

$$x_{pe}(n) = \frac{1}{2}[x_p(n) + x_p^*(N - n)] \quad (2.39)$$

$$x_{po}(n) = \frac{1}{2}[x_p(n) - x_p^*(N - n)] \quad (2.40)$$

#### 2.4.4 Symmetry Properties of the DFT

Let us assume that  $N$ -point sequence  $x(n)$  and its DFT are complex valued then,

$$\begin{aligned} x(n) &= x_R(n) + jx_I(n) \quad 0 \leq n \leq N - 1 \\ X(k) &= X_R(k) + jX_I(k) \quad 0 \leq k \leq N - 1 \end{aligned} \quad (2.41)$$

$$\begin{aligned}
&= \sum_{n=0}^{N-1} (x_R(n) + jx_I(n)) e^{-j \frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{N-1} (x_R(n) + jx_I(n)) \left( \cos \frac{2\pi kn}{N} - j \sin \frac{2\pi kn}{N} \right) \\
X(k) &= \sum_{n=0}^{N-1} \left[ x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right] \\
&\quad - j \sum_{n=0}^{N-1} \left[ x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right] \\
&= X_R(k) + jX_I(k)
\end{aligned}$$

where

$$\begin{aligned}
X_R(k) &= \sum_{n=0}^{N-1} \left[ x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right] \\
X_I(k) &= - \sum_{n=0}^{N-1} \left[ x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right]
\end{aligned}$$

Similarly

$$x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[ X_R(k) \cos \frac{2\pi kn}{N} - X_I(k) \sin \frac{2\pi kn}{N} \right] \quad (2.42)$$

$$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[ X_R(k) \sin \frac{2\pi kn}{N} + X_I(k) \cos \frac{2\pi kn}{N} \right] \quad (2.43)$$

### Case (i) Real Value Sequence

If  $x(n)$  is real,

$$X(N-k) = X^*(k) = X(-k)$$

$X(k) = X^*(N-k)$ ,  $X(k)$  has complex conjugate symmetry

$$\begin{aligned}
|X(N-k)| &= |X(k)| \\
\angle X(N-k) &= -\angle X(k)
\end{aligned} \quad (2.44)$$

**Case (ii) Real and Even Value Sequence** If  $x(n)$  is real and even that is,

$$\begin{aligned} x(n) &= x(N-n), \quad 0 \leq n \leq N-1 \\ X_I(k) &= 0, \quad x_I(n) = 0 \\ \text{DFT} \implies X(k) &= X_R(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi kn}{N}; \quad 0 \leq n \leq N-1 \quad (2.45) \\ \text{IDFT} \implies x(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos \frac{2\pi kn}{N}; \quad 0 \leq n \leq N-1 \quad (2.46) \end{aligned}$$

**Case (iii) Real and Odd Sequence**

If  $x(n)$  is real and odd that is,

$$\begin{aligned} x(n) &= -x(N-n) \\ X_R(k) &= 0, \quad x_I(n) = 0 \\ \text{DFT} \implies X(k) &= jX_I(k) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi kn}{N}; \quad 0 \leq n \leq N-1 \quad (2.47) \\ \text{IDFT} \implies x(n) &= j \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sin \frac{2\pi kn}{N}; \quad 0 \leq k \leq N-1 \quad (2.48) \end{aligned}$$

**Case (iv) Purely Imaginary Sequence**

$$x(n) = jx_I(n)$$

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi kn}{N} \quad (2.49)$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi kn}{N} \quad (2.50)$$

$$X(k) = X_R(k) + X_I(k) \quad (2.51)$$

### Example 2.21

The first five points of the eight-point DFT of a real-valued sequences are

$$\{28, -4 + j9.565, -4 + j4, -4 + j1.656, -4\}$$

Determine the remaining three points.

**Solution** Give  $N = 8$ ,

$$\begin{aligned} X(0) &= 28 & X(1) &= -4 + j9.565 \\ X(2) &= -4 + j4 & X(3) &= -4 + j1.656 \\ X(4) &= -4 \end{aligned}$$

We have

$$X(k) = X^*(N - k)$$

The remaining three DFT points are,

$$\begin{aligned} X(5) &= X^*(8 - 5) = X^*(3) = -4 - j1.656 \\ X(6) &= X^*(8 - 6) = X^*(2) = -4 - j4 \\ X(7) &= X^*(8 - 7) = X^*(1) = -4 - j9.565 \end{aligned}$$

### 2.4.5 Multiplication of Two DFTs and Circular Convolution

Let  $x_1(n)$  and  $x_2(n)$  are finite duration sequence of length  $N$ . Their respective  $N$ -point DFTs are

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j \frac{2\pi n k}{N}}; \quad k = 0, 1, 2, \dots, N-1 \quad (2.52)$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j \frac{2\pi n k}{N}}; \quad k = 0, 1, 2, \dots, N-1 \quad (2.53)$$

If we multiply the two DFTs together, the result is a DFT, say  $X_3(k)$  of a sequence  $x_3(n)$  of length  $N$

$$X_3(k) = X_1(k)X_2(k); \quad k = 0, 1, 2, \dots, N-1 \quad (2.54)$$

$$\text{IDFT}\{X_3(k)\} = x_3(m)$$

By the definition of IDFT

$$\begin{aligned} x_3(m) &= \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j \frac{2\pi k m}{N}} \quad m = 0, \dots, (N-1) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{j \frac{2\pi k m}{N}} \end{aligned} \quad (2.55)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[ \sum_{n=0}^{N-1} x_1(n) e^{-j \frac{2\pi n k}{N}} \right] \left[ \sum_{l=0}^{N-1} x_2(l) e^{-j \frac{2\pi k l}{N}} \right] e^{j \frac{2\pi k m}{N}}$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{j \frac{2\pi k(m-n-l)}{N}}$$

Let  $(m - n - l) = PN$ , where  $P$  is an integer.

$$\begin{aligned} e^{\frac{j2\pi k(m-n-l)}{N}} &= e^{\frac{j2\pi kPN}{N}} \\ &= e^{j2\pi kp} = (e^{j2\pi p})^k \end{aligned}$$

From finite geometric series sum formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & \text{for } a = 1 \\ \frac{1-a^N}{1-a}, & \text{for } a \neq 1. \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}} &= \sum_{k=0}^{N-1} (e^{j2\pi p})^k = \sum_{k=0}^{N-1} 1^k = N \\ \therefore x_3(m) &= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) N \\ &= \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \end{aligned}$$

If  $x_2(l)$  is a periodic sequence with periodicity of  $N$  samples then  $x_2(l \pm PN) = x_2(l)$ . Here  $m - n - l = PN$ , and therefore  $l = m - n - PN$

$$\begin{aligned} x_2(l) &= x_2(m - n - PN) = x_2(m - n)_N \\ &= x_2((m - n), \mod N) \\ \therefore x_3(n) &= \sum_{n=0}^{N-1} x_1(n) \sum_{n=0}^{N-1} x_2(m - n)_N \\ &= \sum_{n=0}^{N-1} x_1(n) x_2(m - n)_N \\ x_3(n) &= x_1(n) \circledast x_2(n) \end{aligned}$$

$$\begin{aligned}\text{IDFT } [X_1(k)X_2(k)] &= x_1(n) \circledast x_2(n) \\ X_1(k)X_2(k) &= \text{DFT } [x_1(n) \circledast x_2(n)]\end{aligned}\quad (2.56)$$

Therefore,

$$x_1(n) \circledast x_2(n) \xrightarrow[N]{\text{DFT}} X_1(k)X_2(k) \quad (2.57)$$

Thus, we conclude that multiplication of the DFTs of two sequence is equivalent to the DFT of the circular convolution of the two sequences.

#### 2.4.6 Time Reversal of a Sequence

If

$$x(n) \xrightarrow[N]{\text{DFT}} X(k)$$

then

$$x((-n))_N = x(N - n) \xleftarrow[N]{\text{DFT}} X((-k))_N = X(N - k) \quad (2.58)$$

Hence, reversing the  $N$ -point sequence in time is equivalent to reversing the DFT values (enter the sequence is clockwise direction in the circle).

**Proof**

$$\text{DFT } \{x(n - N)\} = \sum_{n=0}^{N-1} x(N - m) e^{-j \frac{2\pi kn}{N}}$$

Let  $m = N - n$

$$\begin{aligned}\text{DFT } \{x(n - N)\} &= \sum_{n=0}^{N-1} x(m) e^{-j \frac{2\pi k(N-m)}{N}} \\ &= \sum_{m=0}^{N-1} x(m) e^{j \frac{2\pi km}{N}} \cdot e^{-j \frac{2\pi kN}{N}} \\ &= \sum_{m=0}^{N-1} x(m) e^{-j \frac{2\pi (N-k)m}{N}} = X(N - k)\end{aligned}$$

$$X(N - k) = X((-k))_N, \quad 0 \leq n \leq N - 1$$

### 2.4.7 Circular Time Shift of a Sequence

If

$$x(n) \xrightarrow[N]{\text{DFT}} X(k)$$

then

$$x(n-l)_N \xleftrightarrow[N]{\text{DFT}} X(k)e^{-j\frac{2\pi kl}{N}} \quad (2.59)$$

**Proof**

$$\text{DFT } \{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N e^{-j\frac{2\pi kn}{N}}$$

$$\text{DFT } \{x((n-l))_N\} = \sum_{n=0}^{l-1} x((n-l))_N e^{-j\frac{2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n-l)_N e^{-j\frac{2\pi kn}{N}}$$

let

$$\begin{aligned} x((n-l))_N &= x(N + (n-l)) \\ &= x(N + n - l) \\ m &= N + n - l \end{aligned}$$

$$\begin{aligned} \therefore \sum_{n=0}^{l-1} x((n-l))_N e^{-j\frac{2\pi kn}{N}} &= \sum_{m=N-l}^{N-1} x(N + n - l) e^{-j\frac{2\pi km}{N}} \\ &= \sum_{m=N-l}^{N-1} x(m) e^{-j\frac{2\pi k(m+l-N)}{N}} \\ &= \sum_{m=N-l}^{N-1} x(m) e^{-j\frac{2\pi km}{N}} \cdot e^{-j\frac{2\pi kl}{N}} \\ \sum_{n=l}^{N-1} x(n-l) e^{-j\frac{2\pi kn}{N}} &= \sum_{m=0}^{N-1-l} x(m) e^{-j\frac{2\pi k(m+l)}{N}} \end{aligned}$$

Therefore,

$$\begin{aligned}
 \text{DFT } \{x(n-l)_N\} &= \sum_{m=N-l}^{N-1} x(m) e^{-j \frac{2\pi k(m+l)}{N}} + \sum_{m=0}^{N-1-l} x(m) e^{-j \frac{2\pi k(m+l)}{N}} \\
 &= \sum_{m=0}^{N-1} x(m) e^{-j \frac{2\pi k(m+l)}{N}} \\
 &= e^{-j \frac{2\pi kl}{N}} \sum_{m=0}^{N-1} x(m) e^{-j \frac{2\pi km}{N}} \\
 &= e^{-j \frac{2\pi kl}{N}} \cdot X(k)
 \end{aligned}$$

#### 2.4.8 Circular Frequency Shift

If

$$x(n) \xrightarrow[N]{\text{DFT}} X(k)$$

then

$$x(n) e^{j \frac{2\pi ln}{N}} \xleftarrow[N]{\text{DFT}} X((k-l))_N \quad (2.60)$$

**Proof**

$$\begin{aligned}
 \text{DFT } \{X((k-l))_N\} &= \frac{1}{N} \sum_{k=0}^{N-1} X((k-l))_N e^{j \frac{2\pi kn}{N}} \\
 &= \frac{1}{N} \sum_{k=0}^{l-1} X((k-l))_N e^{j \frac{2\pi kn}{N}} + \frac{1}{N} \sum_{k=l}^{N-1} X((k-l))_N e^{j \frac{2\pi kn}{N}}
 \end{aligned}$$

Let  $X((k-l))_N = X(N+k-l)$

$$\begin{aligned}
 N+k-l &= m \\
 \text{when } k=0 &\implies m=N-l \\
 \text{when } k=l-1 &\implies m=N-1
 \end{aligned}$$

$$\begin{aligned}
\frac{1}{N} \sum_{k=0}^{l-1} X((k-l))_N e^{j \frac{2\pi k n}{N}} &= \frac{1}{N} \sum_{m=N-l}^{N-1} x(N+k-l) e^{j \frac{2\pi k n(m+l-N)}{N}} \\
&= \frac{1}{N} \sum_{m=N-l}^{N-1} X(m) e^{j \frac{2\pi(m+l)n}{N}} \\
\frac{1}{N} \sum_{k=l}^{N-1} X((k-l))_N e^{j \frac{2\pi k n}{N}} &= \frac{1}{N} \sum_{m=0}^{N-1-l} X(m) e^{j \frac{2\pi(m+l)n}{N}} \\
\therefore \text{DFT } \{X((k-l))_N\} &= \frac{1}{N} \sum_{m=0}^{N-1} X(m) e^{j \frac{2\pi(m+l)n}{N}} \\
&= e^{j \frac{2\pi l n}{N}} \frac{1}{N} \sum_{m=0}^{N-1} X(m) e^{j \frac{2\pi m n}{N}} \\
&= e^{j \frac{2\pi l n}{N}} x(n) \\
X((k-l))_N &= \text{DFT } \left\{ x(n) e^{j \frac{2\pi l n}{N}} \right\}
\end{aligned}$$

### 2.4.9 Complex-Conjugate Properties

If

$$x(n) \xrightarrow[N]{\text{DFT}} X(k)$$

then

$$x^*(n) \xleftarrow[N]{\text{DFT}} X^*((-k))_N = X^*(n-k) \quad (2.61)$$

**Proof**

$$\begin{aligned}
\text{DFT } \{X^*(n)\} &= \sum_{n=0}^{N-1} x^*(n) e^{-j \frac{2\pi k n}{N}} \\
&= \left[ \sum_{n=0}^{N-1} x(n) e^{j \frac{2\pi k n}{N}} \right]^* \\
\text{DFT } \{X^*(n)\} &= \left[ \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi(N-k)n}{N}} \right]^* \\
&= X^*(n-k)
\end{aligned}$$

### 2.4.10 Circular Correlation

#### 2.4.10.1 Circular Cross-Correlation

In general, for complex-valued sequence  $x(n)$  and  $y(n)$  if

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k) \quad \text{and} \quad y(n) \xleftrightarrow[N]{\text{DFT}} Y(k)$$

then

$$\gamma_{xy}(l) \xleftrightarrow[N]{\text{DFT}} R_{xy}(k) = X(k)Y^*(k) \quad (2.62)$$

#### 2.4.10.2 Circular Auto Correlation

If  $x(n) = y(n)$

$$\gamma_{xx} \xleftrightarrow[N]{\text{DFT}} R_{xx}(k) = |X(k)|^2 \quad (2.63)$$

### 2.4.11 Multiplication of Two Sequences

If

$$x_1(n) \xleftrightarrow[N]{\text{DFT}} X_1(k) \quad \text{and} \quad x_2(n) \xleftrightarrow[N]{\text{DFT}} X_2(k)$$

then

$$x_1(n)x_2(n) \xleftrightarrow[N]{\text{DFT}} \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \circledast X_2(k) \quad (2.64)$$

### 2.4.12 Parseval's Theorem

For complex-valued sequences  $x(n)$  and  $y(n)$ . If

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k) \quad \text{and} \quad y(n) \xleftrightarrow[N]{\text{DFT}} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k) \quad (2.65)$$

If  $x(n) = y(n)$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

**Proof**

$$\begin{aligned} \text{IDFT } \{R_{xy}(k)\} &= \gamma_{xy}(l) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k) e^{j \frac{2\pi k l}{N}} \end{aligned}$$

using cross correlation

$$\begin{aligned} \sum_{n=0}^{N-1} x(n)y^*(n-l) &= \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k) e^{j \frac{2\pi k l}{N}} \\ \sum_{n=0}^{N-1} x(n)y^*(n) &= \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k) \quad [\text{when } l=0] \end{aligned}$$

If  $x(n) = y(n)$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k)$$

Now

$$\begin{aligned} R_{xy}(k) &= X(k)Y^*(k) \\ X(k) &= Y(k) \\ R_{xy}(k) &= X(k)X^*(k) = |X(k)|^2 \end{aligned}$$

Therefore,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

The properties of DFT are given in Table 2.1.

**Table 2.1** Summary of properties of the discrete Fourier transform

No.	Finite-length sequence (length $N$ )	$N$ -point DFT (length $N$ )
1.	$x(n)$	$X(k)$
2.	$x_1(n), x_2(n)$	$X_1(k), X_2(k)$
3.	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n - m))_N]$	$e^{\frac{-j2\pi kn}{N}} X[k]$
6.	$e^{\frac{-j2\pi ln}{N}} Nx(n)$	$X[((k - l))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m)x_2((n - m))_N$	$X_1[k]X_2[k]$
8.	$x_1(n)x_2(n)$	$\frac{1}{2} \sum_{l=0}^{N-1} X_1(l)X_2((k - l))_N$
9.	$x^*(n)$	$X^*((-k))_N$
10.	$x^*((-n))_N$	$X^*(k)$
11.	$\text{Re}\{x(n)\}$	$X_e(k) = \frac{1}{2}\{X((k))_N + X^*((-k))_N\}$
12.	$j\text{Im}\{x(n)\}$	$X_o(k) = \frac{1}{2}\{X((k))_N - X^*((-k))_N\}$
13.	$x_e = \frac{1}{2}\{x(n) + x^*((-n))_N\}$	$\text{Re}\{X(k)\}$
14.	$x_o = \frac{1}{2}\{x(n) - x^*((-n))_N\}$	$j\text{Im}\{X(k)\}$
15.	Symmetric properties	$X(k) = X^*((-k))_N$ $\text{Re}\{X(k)\} = \text{Re}\{X((-k))_N\}$ $\text{Im}\{X(k)\} = -\text{Im}\{X((-k))_N\}$ $ X(k)  =  X((-k))_N $ $\angle X(k) = -\angle X((-k))_N$
16.	$x_e(n) = \frac{1}{2}\{x(n) + x((-n))_N\}$	$\text{Re}\{X(k)\}$
17.	$x_o(n) = \frac{1}{2}\{x(n) - x((-n))_N\}$	$j\text{Im}\{X(k)\}$

## 2.5 Circular Convolution

The circular convolution of two sequences requires that, one of the sequences should be periodic. If both the sequences are non-periodic, then periodically extend one of the sequences and then perform circular convolution.

The circular convolution can be performed only if both the sequences consist of same number of samples. If the sequences have different number of samples, then convert the smaller size sequences to the size of larger size sequence by appending zero padding. The circular convolution has length same as the length of input sequences, whereas in linear convolution the length is  $N_1 + N_2 - 1$ . That is,

$$\text{Linear convolution} \Rightarrow \text{output length} = N_1 + N_2 - 1 \quad (2.66)$$

$$\text{Circular convolution} \Rightarrow \text{output length} = \max(N_1, N_2) \quad (2.67)$$

### 2.5.1 Method of Performing Circular Convolution

#### 2.5.1.1 Graphical Method (Concentric Circle Method) (Method 1)

Let  $x_3(m)$  be the sequence obtained by circular convolution of  $x_1(n)$  and  $x_2(n)$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2(m-n) = x_1(n) \circledast x_2(n) \quad (2.68)$$

#### Procedure

1. Graph  $N$  samples of  $x_1(n)$  at equally spaced points around the outer circle in counterclockwise direction.
2. Start at the same point as was done for  $x_1(n)$ . Graph  $N$  samples of  $x_2(n)$  at equally spaced points around an inner circle in clockwise direction.
3. Multiply corresponding samples on the two circles and sum the products to produce the output.
4. Rotate the inner circle keeping the outer circle fixed by one sample in the counterclockwise direction and repeat step 3 to obtain the next value of output.
5. Repeat step 4 until the inner circle first sample coincides with the first sample of the exterior circle once again. This completes one complete rotation.

#### Example 2.22

Perform circular convolution of two sequences

$$x_1(n) = \{2, 1, 2, 1\} \quad \text{and} \quad x_2(n) = \{1, 2, 3, 4\}$$

↑    ↑

#### Solution Given

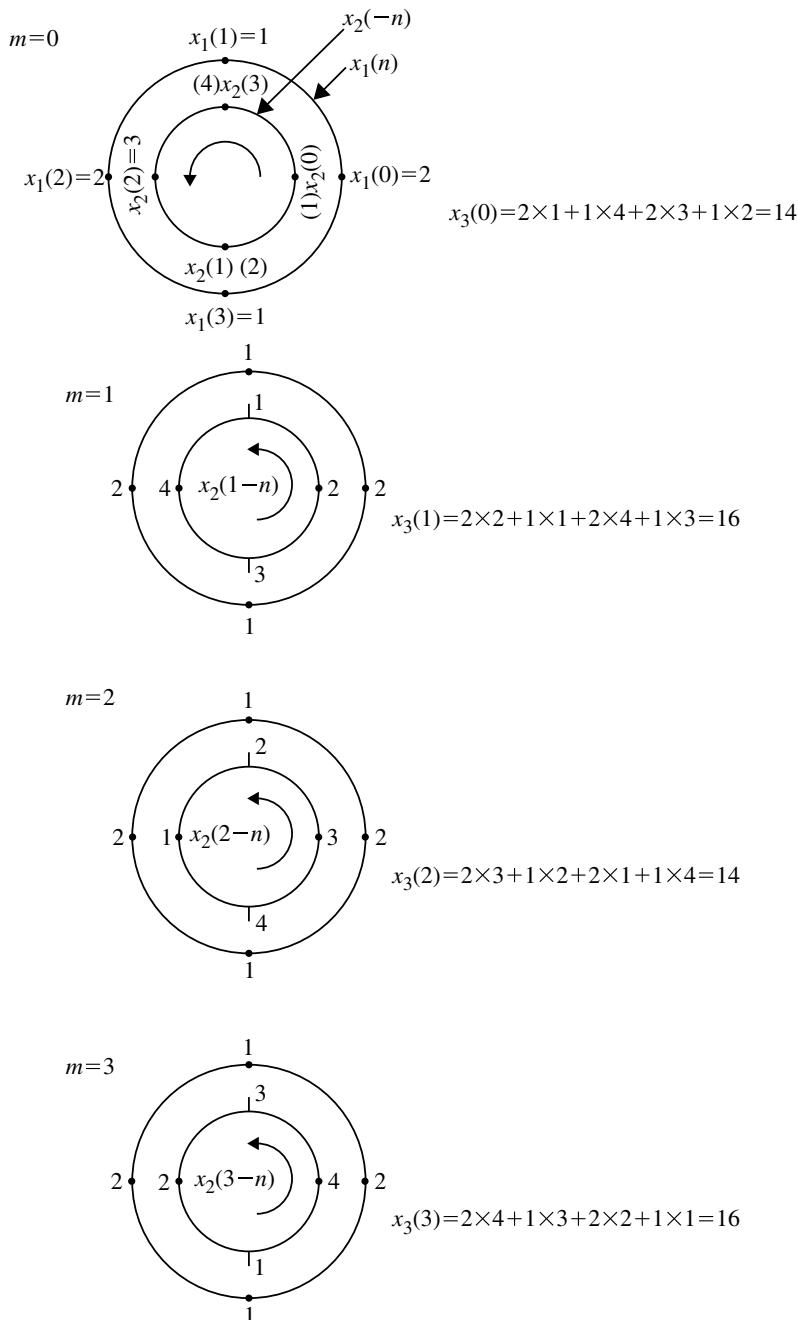
$$x_1(n) = \{2, 1, 2, 1\} \quad \text{and} \quad x_2(n) = \{1, 2, 3, 4\}$$

↑    ↑

$$\therefore x_3(m) = \{14, 16, 14, 16\}$$

↑

The circular convolution is shown in Fig. 2.11.

**Fig. 2.11** Circular convolution of Example 2.22

### 2.5.1.2 Filtering Methods Based on DFT

DFT method gives discrete frequency representation, and this is successfully used as a computational tool for linear filtering. It has been established that the product of two DFTs is equivalent to circular convolution in the time domain. However, to find the output of a linear filter, the circular convolution cannot be used. Let  $x[n]$ ,  $h[n]$  and  $y[n]$  be the input, impulse response and output respectively of the linear filter. In the frequency domain these variables are expressed as  $X(\omega)$ ,  $H(\omega)$  and  $Y(\omega)$  and the output spectrum of the linear filter is given by

$$Y(\omega) = X(\omega)H(\omega)$$

The output sequence  $y[n]$  of the filter is obtained by taking inverse Fourier transform. Since  $Y(\omega)$ ,  $X(\omega)$  and  $H(\omega)$  are functions of the continuous variable  $\omega$ , the computation cannot be done on a digital computer since digital computer performs computation of quantities which occur at discrete frequency interval. However, using DFT, the spectrum  $Y(\omega)$  is represented uniquely by samples in the frequency domain. Since the  $N$ -point DFT of the output sequence  $y[n]$  is sufficient to represent it in the frequency domain as  $Y(\omega)$ , then it is possible to compute the  $N$ -point DFTs of  $X(\omega)$  and  $H(\omega)$  and hence  $Y(\omega)$ . By taking IDFT of  $X(\omega)H(\omega)$ ,  $y[n]$  of the linear filter is obtained. This methodology is illustrated by the following examples with the well-defined procedure.

#### Procedure

1. Take  $N$ -point DFT  $x_1(n)$  and  $x_2(n)$ .

$$X_1(k) = \text{DFT}(x_1(n)) \quad \text{and} \quad X_2(k) = \text{DFT}[x_2(n)]$$

2. Determine product of  $X_1(k)$  and  $X_2(k)$ . Let  $X_3(k) = X_1(k)X_2(k)$ .
- 3.

$$x_3(m) = \text{IDFT}[X_3(k)] = \text{IDFT}[X_1(k)X_2(k)]$$

#### Example 2.23

Find  $x_3(m)$  using DFT and IDFT method.

$$x_1(n) = \{2, 1, 2, 1\} \quad \text{and} \quad x_2(n) = \{1, 2, 3, 4\}$$

**Solution**

$$X_1(k) = \sum_{n=0}^3 x_1(n)e^{-j\frac{2\pi kn}{4}}, \quad k = 0, 1, 2, 3 \dots$$

$$X_1(k) = x_1(0) + x_1(1)e^{-j\frac{\pi k n}{2}} + x_1(2)e^{-j\pi k} + x_1(3)e^{-j\frac{3\pi k n}{2}}$$

$$\text{For } k = 0 \Rightarrow X_1(0) = 2 + 1 + 2 + 1 = 6$$

$$\text{For } k = 1 \Rightarrow X_1(1) = 2 + e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}} = 2 - j - 2 + j = 0$$

$$\text{For } k = 2 \Rightarrow X_1(2) = 2 + e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi} = 2 - 1 + 2 - 1 = 2$$

$$\text{For } k = 3 \Rightarrow X_1(3) = 2 + e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}} = 2 + j - 2 - j = 0$$

$$X_1(k) = \{6, 0, 2, 0\}$$

↑

$$X_2(k) = \sum_{n=0}^3 X_2(n)e^{-j\frac{2\pi nk}{4}}, \quad k = 0, 1, 2, 3 \dots$$

$$X_2(k) = x_2(0) + x_2(1)e^{-j\frac{\pi k}{2}} + x_2(2)e^{-j\pi k} + x_2(3)e^{-j\frac{3\pi k}{2}}$$

$$\text{For } k = 0$$

$$X_2(0) = 1 + 2 + 3 + 4 = 10$$

$$\text{For } k = 1$$

$$X_2(1) = 1 + 2e^{-j\frac{\pi}{2}} + 3e^{-j\pi} + 4e^{-j\frac{3\pi}{2}} = 1 - 2j - 3 + 4j = -2 + j2$$

$$\text{For } k = 2$$

$$X_2(2) = 1 + 2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi} = 1 - 2 + 3 - 4 = -2$$

$$\text{For } k = 3$$

$$X_2(3) = 1 + 2e^{-j\frac{3\pi}{2}} + 3e^{-j3\pi} + 4e^{-j\frac{9\pi}{2}} = 1 + 2j - 3 - 4j = -2 - j2$$

$$\therefore X_2(k) = \{10, -2 + j2, -2, -2 - j2\}$$

↑

$$X_3(k) = X_1(k)X_2(k)$$

$$= [6, 0, 2, 0][10, -2 + j2, -2, -2 - j2]$$

$$= [6 \times 10, 0 \times (-2 + j2), 2 \times (-2), 0(-2 - j2)]$$

$$= \{60, 0, -4, 0\}$$

↑

$$x_3(m) = \text{IDFT}[X_3(k)]$$

$$\begin{aligned}
&= \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j \frac{2\pi m k}{N}}, \quad m = 0, 1, 2, 3, \dots \\
&= \frac{1}{4} \sum_{k=0}^3 X_3(k) e^{j \frac{2\pi m k}{4}} \\
&= \frac{1}{4} [X_3(0) + X_3(2) e^{j\pi m}] \\
x_3(m) &= \frac{1}{4} [60 - 4e^{j\pi m}]
\end{aligned}$$

For  $m = 0$

$$X_3(0) = \frac{1}{4} [60 - 4] = \frac{56}{4} = 14$$

For  $m = 1$

$$X_3(1) = \frac{1}{4} [60 - 4e^{j\pi}] = \frac{64}{4} = 16$$

For  $m = 2$

$$X_3(2) = \frac{1}{4} [60 - 4e^{2j\pi}] = \frac{56}{4} = 14$$

For  $m = 3$

$$X_3(3) = \frac{1}{4} [60 - 4e^{3j\pi}] = \frac{64}{4} = 16$$

$$\boxed{x_3(m) = x_1(n) \circledast x_2(n) = \{14, 16, 14, 16\}}$$

↑

### 2.5.1.3 Circular Convolution Using Matrices (Method 3)

Circular convolution of  $x_1(n)$  and  $x_2(n)$  is

$$\begin{aligned}
x(m) &= x_1(n) \circledast x_2(n) \\
&= \left[ \begin{array}{cccc|c} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x(1) & x_1(0) \\ x_2(1) & x(0) & x_2(N-1) & \cdots & x(2) & x_1(1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x(0) & x_1(N-1) \end{array} \right] \\
&= \left[ \begin{array}{c} x_3(0) \\ x_3(1) \\ \vdots \\ x_3(m-1) \end{array} \right]
\end{aligned}$$

### Example 2.24

Perform the circular convolution using matrices method

$$x_1(n) = \{2, 1, 2, 1\} \quad \text{and} \quad x_2(n) = \{1, 2, 3, 4\}$$

(Anna University, May, 2007)

## Solution

$$\begin{bmatrix} x_2(n-k)_N \\ x_2(0) \ x_2(3) \ x_2(2) \ x_2(1) \\ x_2(1) \ x_2(0) \ x_2(3) \ x_2(2) \\ x_2(2) \ x_2(1) \ x_2(0) \ x_2(3) \\ x_2(3) \ x_2(2) \ x_2(1) \ x_2(0) \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} x_3(m) \\ x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

Therefore,

$$x_3(m) = x_1(n) \text{ } \textcircled{N} \text{ } x_2(n) = \{14, 16, 14, 16\}$$

### 2.5.2 Performing Linear Convolution Using DFT

To perform linear convolution using DFT both the sequences should be converted into  $N_1 + N_2 - 1$ -point sequences by padding with zeros. Then take  $N_1 + N_2 - 1$ -point DFT of both the sequences and determine the product of their DFTs. The resultant sequence is obtained by taking inverse DFT of the product of the DFTs.

### Example 2.25

Find the response (linear convolution) of the system whose impulse response and input sequence are

using DFT-IDFT.

**Solution** Here  $N_1 = 2$  and  $N_2 = 2$

$$N = N_1 + N_2 - 1 = 2 + 2 - 1 = 3$$

$$\therefore x(n) = \begin{cases} 1, 0.5, 0 \end{cases} \text{ and } h(n) = \begin{cases} 0.5, 1, 0 \end{cases}$$

↑   ↑

$$\begin{aligned} X(k) &= \sum_{n=0}^2 x(n)e^{-j\frac{2\pi kn}{3}}, \quad k = 0, \dots, 2 \\ X(0) &= 1 + 0.5 = 1.5 \\ X(1) &= \sum_{n=0}^2 x(n)e^{-j\frac{2\pi n}{3}} = 1 + 0.5e^{-j\frac{2\pi}{3}} = 0.75 - j0.433 \\ X(2) &= \sum_{n=0}^3 x(n)e^{-j\frac{2\pi n2}{3}} = \sum_{n=0}^3 x(n)e^{-j\frac{4\pi n}{3}} \\ &= 1 + 0.5e^{-j\frac{4\pi}{3}} + 0 = 0.75 + j0.433 \end{aligned}$$

Therefore,

$$\begin{aligned} X(k) &= \{1.5, 0.75 - j0.433, 0.75 + j0.433\} \\ H(k) &= \sum_{n=0}^2 h(n)e^{-j\frac{2\pi kn}{3}}, \quad k = 0, \dots, 2 \\ H(0) &= h(0) + h(1) + h(3) = 0.5 + 1 + 0 = 1.5 \\ H(1) &= \sum_{n=0}^2 h(n)e^{-j\frac{2\pi n}{3}} \\ &= 0.5 + e^{-j\frac{2\pi}{3}} + 0 = 0.5 - 0.5 - j0.866 \\ &= -j0.866 \\ H(2) &= \sum_{n=0}^2 h(n)e^{-j\frac{4\pi n}{3}} \\ &= 0.5 + e^{-j\frac{4\pi}{3}} + 0 = j0.866 \\ H(k) &= \{1.5, -j0.866, j0.866\} \end{aligned}$$

Let

$$\begin{aligned} Y(k) &= X(k)H(k) \\ &= \{2.25, -0.375 - j0.6495, -0.375 + j0.6495\} \end{aligned}$$

The sequence  $y(n)$  is obtained by taking IDFT of  $Y(k)$

$$\begin{aligned} y(n) &= \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\frac{j2\pi kn}{N}}, \quad n = 0, \dots, N-1 \\ y(n) &= \frac{1}{3} \sum_{k=0}^2 Y(k) e^{\frac{j2\pi kn}{3}}, \quad n = 0, \dots, 2 \end{aligned}$$

For  $n = 0$ ,

$$y(0) = \frac{1}{3}[y(0) + y(1) + y(2)] = \frac{1}{3}[1.5] = 0.5$$

For  $n = 1$ ,

$$\begin{aligned} y(1) &= \frac{1}{3} \sum_{k=0}^2 Y(k) e^{\frac{j2\pi k}{3}} \\ &= \frac{1}{3} \left[ y(0) + y(1)e^{\frac{j2\pi}{3}} + y(2)e^{\frac{j4\pi}{3}} \right] \\ &= \frac{1}{3} [2.25 + 0.75 + j0 + 0.75 + j0] = 1.25 \end{aligned}$$

For  $n = 2$ ,

$$\begin{aligned} y(2) &= \frac{1}{3} \left[ y(0) + y(1)e^{\frac{j4\pi}{3}} + y(2)e^{\frac{j8\pi}{3}} \right] \\ &= \frac{1}{3} [2.25 - 0.375 + j0.6495 - 0.375 - j0.6495] \\ &= 0.5 \end{aligned}$$

$$\therefore \boxed{y(n) = \{0.5, 1.25, 0.5\}}$$

**Example 2.26**

Determine the output response  $y(n)$  if

$$h(n) = \{1, 1, 1\} \quad \text{and} \quad x(n) = \{1, 2, 3, 1\}$$

by using (i) Linear convolution and (ii) Circular convolution.

**Solution****(i) Linear Convolution**

$$N_1 = 4 \quad \text{and} \quad N_2 = 3$$

$$N = N_1 + N_2 - 1 = 4 + 3 - 1 = 6$$

The linear convolution obtained by graphical method is shown in Fig. 2.12.

**(ii) Circular Convolution With Zero Padding**

To get the result of linear convolution with circular convolution we have to add appropriate number of zeros to both sequence.

$$N = N_1 + N_2 - 1 = 4 + 3 - 1 = 6$$

$$\therefore x(n) = \{1, 2, 3, 1, 0, 0\}$$

$$h(n) = \{1, 1, 1, 0, 0, 0\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$\therefore y(n) = \{1, 3, 6, 6, 4, 1\}$$



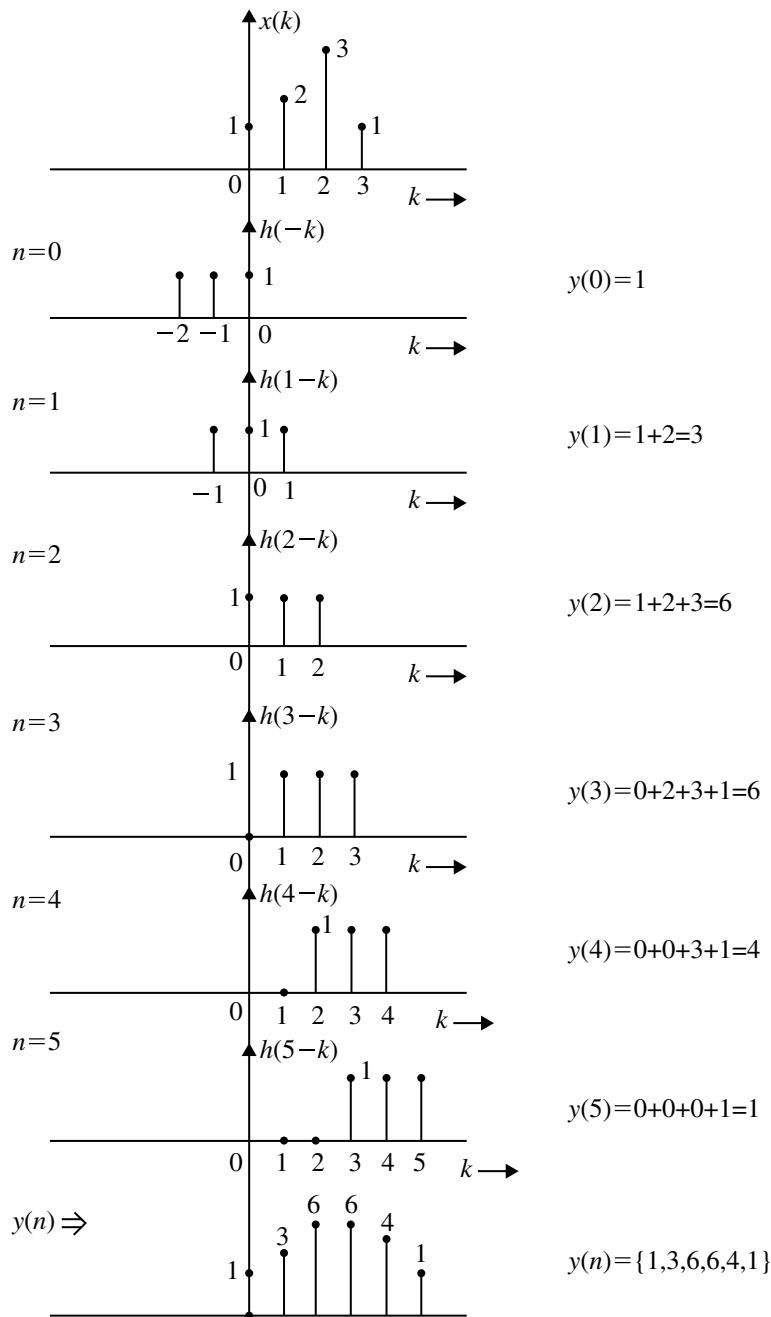


Fig. 2.12 Linear convolution of two sequences for Example 2.26

**Example 2.27**

For the  $x_1(n)$ ,  $x_2(n)$  and  $N$  given compute  $x_1(n) \circledast x_2(n)$

$$(a) \quad \begin{aligned} x_1(n) &= \delta(n) + \delta(n - 1) + \delta(n - 2) \\ x_2(n) &= 2\delta(n) - \delta(n - 1) + 2\delta(n - 2) \quad (N = 3) \end{aligned}$$

$$(b) \quad \begin{aligned} x_1(n) &= \delta(n) + \delta(n - 1) - \delta(n - 2) + \delta(n - 3) \\ x_2(n) &= \delta(n) - \delta(n - 2) + \delta(n - 4) \quad (N = 5) \end{aligned}$$

**Solution**

(a) Given

$$\begin{aligned} x_1(n) &= \{1, 1, 1\} \quad \text{and} \quad x_2(n) = \{2, -1, 2\} \\ &\qquad \uparrow \qquad \qquad \qquad \uparrow \\ N &= 3 \end{aligned}$$

$$\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$y(n) = x_1(n) \circledast x_2(n) = \{ \underset{\uparrow}{3}, 3, 3 \}$$

(b) Given

$$\begin{aligned} x_1(n) &= \{1, 1, -1, 1\} \quad \text{and} \quad x_2(n) = \{1, 0, -1, 0, 1\} \\ &\qquad \uparrow \qquad \qquad \qquad \uparrow \\ N &= 5 \end{aligned}$$

Add one zero to the sequence  $x_1(n)$

$$\begin{aligned} x_1(n) &= \{1, 1, -1, 1, 0\} \quad \text{and} \quad x_2(n) = \{1, 0, -1, 0, 1\} \\ &\qquad \uparrow \qquad \qquad \qquad \uparrow \end{aligned}$$

Using matrix method, we get,

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$y(n) = \{3, 0, -1, 0, 2\}$$

↑

### Example 2.28

Find the circular convolution of  $x(n) \circledast h(n)$

$$x(n) = 1, \quad 0 \leq n \leq 10$$

$$h(n) = \left(\frac{1}{2}\right)^n, \quad 0 \leq n \leq 10$$

(Anna University, December, 2005)

#### Solution

$$x(n) = 1, \quad 0 \leq n \leq 10$$

$$x(n) = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$$

$$h(n) = \left(\frac{1}{2}\right)^n, \quad 0 \leq n \leq 10$$

$$= \{1, 0.5, 0.25, 0.125, 0.0625, 0.03125, 0.0156, 0.008, 0.004, 0.002, 0.001\}$$

$$h((N-k)_N)$$

$$\begin{bmatrix} 1 & 0.001 & 0.002 & 0.004 & 0.008 & 0.0156 & 0.03125 & 0.0625 & 0.125 & 0.25 & 0.5 \\ 0.5 & 1 & 0.001 & 0.002 & 0.004 & 0.008 & 0.0156 & 0.03125 & 0.0625 & 0.125 & 0.25 \\ 0.25 & 0.5 & 1 & 0.001 & 0.002 & 0.004 & 0.008 & 0.0156 & 0.03125 & 0.0625 & 0.125 \\ 0.125 & 0.25 & 0.5 & 1 & 0.001 & 0.002 & 0.004 & 0.008 & 0.0156 & 0.03125 & 0.0625 \\ 0.0625 & 0.125 & 0.25 & 0.5 & 1 & 0.001 & 0.002 & 0.004 & 0.008 & 0.0156 & 0.03125 \\ 0.03125 & 0.0625 & 0.125 & 0.25 & 0.5 & 1 & 0.001 & 0.002 & 0.004 & 0.008 & 0.0156 \\ 0.0156 & 0.03125 & 0.0625 & 0.125 & 0.25 & 0.5 & 1 & 0.001 & 0.002 & 0.004 & 0.008 \\ 0.008 & 0.0156 & 0.03125 & 0.0625 & 0.125 & 0.25 & 0.5 & 1 & 0.001 & 0.002 & 0.004 \\ 0.004 & 0.008 & 0.0156 & 0.03125 & 0.0625 & 0.125 & 0.25 & 0.5 & 1 & 0.001 & 0.002 \\ 0.002 & 0.004 & 0.008 & 0.0156 & 0.03125 & 0.0625 & 0.125 & 0.25 & 0.5 & 1 & 0.001 \\ 0.001 & 0.002 & 0.004 & 0.008 & 0.0156 & 0.03125 & 0.0625 & 0.125 & 0.25 & 0.5 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} x(n) \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x(n) \circledast h(n) \\ 1.999 \approx 2 \end{bmatrix}$$

$$x(n) \circledast h(n) = \{2, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$$

**Example 2.29**

Find the output sequence  $y(n)$  if

$$h(n) = \{1, 1, 1\} \text{ and } x(n) = \{1, 2, 3, 1\}$$

using circular convolution.

(Anna University, May, 2004)

**Solution**

$$\begin{aligned} \text{Length of output sequence} &= N_1 + N_2 - 1 \\ &= 3 + 4 - 1 = 6 \end{aligned}$$

Therefore

$$h(n) = \{1, 1, 1, 0, 0, 0\} \text{ and } x(n) = \{1, 2, 3, 1, 0, 0\}$$

using circular convolution

$$y(n) = \begin{bmatrix} h((N-k)_N) \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0+0+0+0+0 \\ 1+2+0+0+0+0 \\ 1+2+3+0+0+0 \\ 0+2+3+1+0+0 \\ 0+0+3+1+0+0 \\ 0+0+0+1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

### Example 2.30

Find the convolution sum of

$$x(n) = \begin{cases} 1, & n = -2, 0, 1 \\ 2, & n = -1 \\ 0, & \text{otherwise.} \end{cases}$$

(Anna University, December, 2003)

**Solution**  $x(k)$  and  $h(n - k)$  are represented as shown in Fig. 2.13.  $x(n - k)$  is moved such that it overlaps with  $x(k)$  and  $y(n)$  is obtained as shown in Fig. 2.13.

convolution sum

$$y(n) = x(n) * h(n)$$

$$= \sum_{n=0}^{N-1} x(k)h(n-k)$$

Output sequence starts at

$$n = n_1 + n_2 = -2 + 0 = -2$$

Length of output (convolution sum) sequences

$$N = N_1 + N_2 - 1 = 4 + 4 - 1 = 7$$

$$y(n) = \{1, 1, 0, 1, -2, 0, -1\}$$

↑

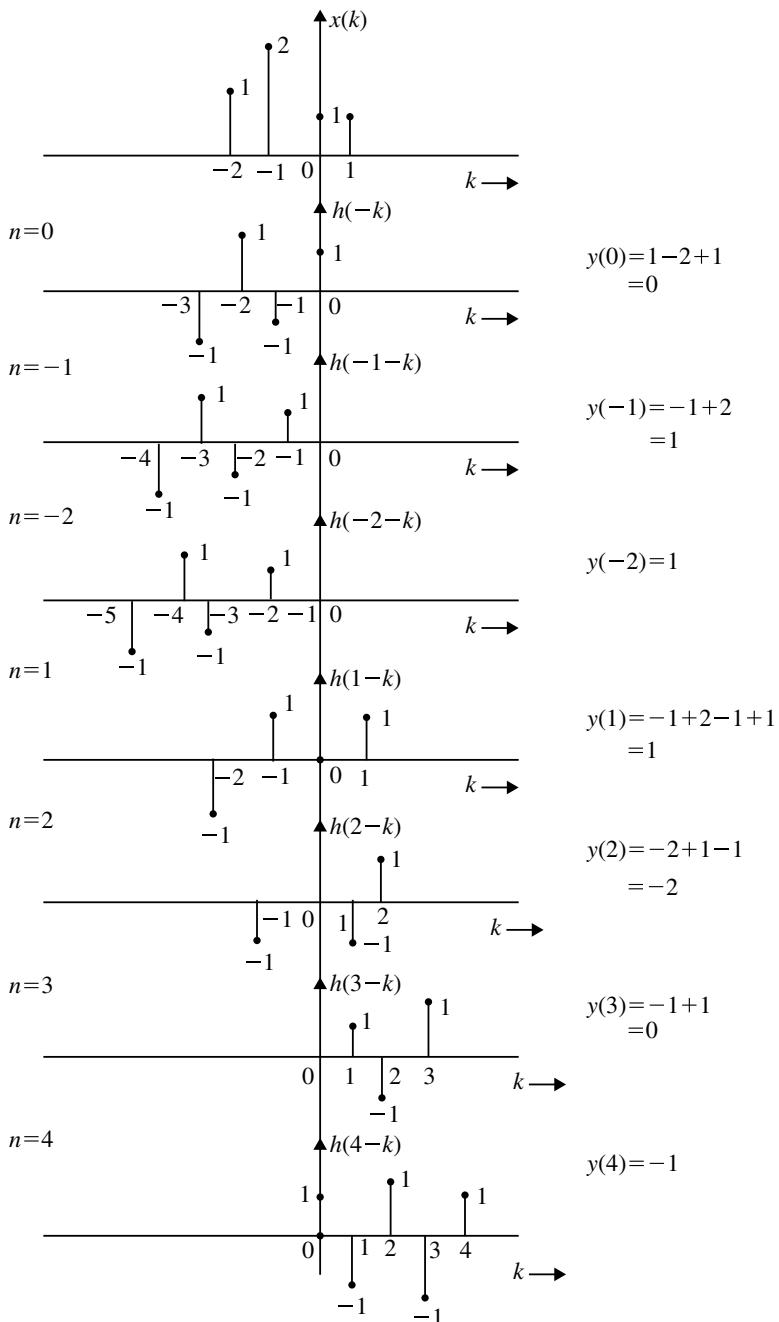


Fig. 2.13 Linear convolution of two sequences for Example 2.30

## 2.6 Fast Fourier Transform (FFT)

For spectral analysis of discrete signals, DFT approach is a very straight forward one. For larger values of  $N$ , DFT becomes tedious because of the huge number of mathematical operations required to perform. Consider the following DFT where  $N = 8$

$$X(k) = \sum_{n=0}^{7} x(n)e^{-jk\frac{2\pi n}{8}}, \quad k = 0, 1, \dots, 7 \quad (2.69)$$

Substituting  $(k2\pi/8) = K$  in the above equation we get,

$$\begin{aligned} X(k) = & x(0)e^{-jK0} + x(1)e^{-jK} + x(2)e^{-jK2} + x(3)e^{-jK3} + x(4)e^{-jK4} \\ & + x(5)e^{-jK5} + x(6)e^{-jK6} + x(7)e^{-jK7} \quad k = 0, 1, \dots, 7 \end{aligned} \quad (2.70)$$

Equation (2.70) has eight terms in the right hand side in which each term contains multiplication of a real term with complex exponential. Thus, for example  $x(1)e^{-jk} = x(1)[\cos K - j \sin K]$  requires two multiplications and one addition for each value of  $K$  where  $K = \frac{2\pi k}{8}$ ,  $k = 0, 1, 2, \dots, 7$ . Thus, in Eq. (2.70) each term in the right-hand side requires eight complex multiplications and seven additions. The eight-point DFT therefore requires  $8 \times 8 = 8^2 = 64$  complex multiplications  $8 \times 7 = 8(8 - 1) = 56$  additions.

In general, for an  $N$ -point DFT,  $N^2$  multiplications and  $N(N - 1)$  additions are required. For  $N = 1024$ , about  $10^8$  multiplications and equal number of additions are required which results in computational burden. Further such a huge number of mathematical operations limit the bandwidth of digital signal processors. Several algorithms have been developed to reduce the computation burden and ease the implementation of DFT. The algorithm developed by Cooley and Tukey in 1965 is the most efficient one and is called fast Fourier transform (FFT). The application FFT algorithms are discussed below with illustrated examples.

### 2.6.1 Radix-2 FFT Algorithm

For efficient computation of DFT several algorithms have been developed based on divide and conquer methods. However, the method is applicable for  $N$  not being a prime number. Consider the case when  $N = r_1r_2r_3\dots r_v$  where the  $\{r_j\}$  are prime. If  $r_1 = r_2 = r_3 = \dots = r$ , then  $N = r^v$ . In such a case the DFTs are of size  $r$ . The number  $r$  is called the radix of the FFT algorithm. The most widely used FFT algorithms are radix-2 and radix-4 algorithms and are discussed in the following sections.

For performing radix-2 FFT, the value of  $N$  should be such that,  $N = 2^m$ . Here the decimation can be performed  $m$  times, where  $m = \log_2^N$ .

In direct computation of  $N$ -point DFT, the total number of complex addition are  $N(N - 1)$  and total number of complex multiplications are  $N^2$ . In radix-2 FFT, the total number of complex additions is reduced to  $N \log_2^N$  and total number of complex multiplications is  $(\frac{N}{2}) \log_2^N$ . Comparison of number of computations by DFT and FFT is shown in Table 2.2.

### 2.6.1.1 Decimation in Time (DIT) Radix-2 FFT

Consider Eqs. (2.71) and (2.72) which are given below.

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{(-j2\pi kn)/N} \quad (2.71)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{(j2\pi kn)/N} \quad (2.72)$$

In direct evaluation of spectral components, the number of complex multiplications and additions required are  $N^2$  and  $N(N - 1)$  respectively as stated in Sect. 2.6. Such a huge number of mathematical operations limit the BW of digital signal processors. Classical DFT approach does not use the two important properties of twiddle factor, namely symmetry and periodicity properties which are given below:

$$\begin{aligned} W_N^{k+N/2} &= -W_N^k \\ W_N^{k+N} &= W_N^k \end{aligned}$$

Radix-2 FFT algorithm exploits these two properties thereby removing redundant mathematical operations. However the results obtained using FFT algorithms are exactly the same as that of DFT. Further, the efficiency of FFT algorithm increases

**Table 2.2** Comparison of number of computations by DFT and FFT

Number of points $N$	Direct computation		Radix-2 FFT	
	Addition $N(N - 1)$	Multiplication $N^2$	Addition $N \log_2^N$	Multiplication $(\frac{N}{2}) \log_2^N$
4	12	16	8	4
8	56	64	24	12
16	240	256	64	32
32	992	1024	160	80
64	4032	4096	384	192
128	16,256	16,384	896	448

as  $N$  is increased. For example, if  $N = 512$ , DFT requires nearly 110 times more multiplications than FFT algorithm. The basic principle of FFT algorithm is therefore to decompose DFT into successively smaller DFTs. The manner in which this decomposition is done leads to different FFT algorithms. The two basic classes of algorithms are:

1. Decimation in time (DIT)
2. Decimation in frequency (DIF)

In the algorithm developed by DIT, the sequence  $x(n)$  is decomposed into successively smaller subsequences. In DIF algorithm, the sequence of DFT coefficients  $x(k)$  is decomposed into smaller subsequences. DIT radix-2 FFT algorithm is discussed in this section.

In decimation in time (DIT) algorithm, the time domain sequence  $x(n)$  is decimated and smaller point DFTs are performed. The results of smaller point DFTs are combined to get the result of  $N$ -point DFT.

In DIT radix-2 FFT, the time domain sequence is decimated into two-point sequences. For each two-point sequence, the two-point DFT is computed. The results of two-point DFTs are used to compute four-point DFTs. Two numbers of four-point DFTs are combined to get an eight-point DFT. This process is continued until we get  $N$ -point DFT. In general we can say that, in decimation in time algorithm, the  $N$ -point DFT can be realized from two numbers of  $\frac{N}{2}$ -point DFTs, and the  $\frac{N}{2}$ -point DFT can be realized from two numbers of  $\frac{N}{4}$ -point DFTs, and so on. This is explained as given below.

Let us consider  $N$ -point DFT of  $x(n)$  which is written as,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}, \quad k = 0, 1, 2, \dots, (N-1)$$

Let

$$W_N = e^{\frac{-j2\pi k}{N}},$$

where  $W_N$  = twiddle factor (or) phase factor. Therefore

$$\begin{aligned} e^{\frac{-j2\pi kn}{N}} &= (e^{-j2\pi})^{\frac{n\pi}{N}} = W_N^{nk} \\ X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, 2, \dots, (N-1) \end{aligned} \quad (2.73)$$

Let  $x(n)$  be  $N$ -sample sequence. Decimate  $x(n)$  into two sequences of  $\frac{N}{2}$  samples. Let the two sequences be  $f_1(n)$  and  $f_2(n)$ . Let  $f_1(n)$  be of even numbered samples of  $x(n)$  and  $f_2(n)$  that of odd numbered samples of  $x(n)$ . Thus

$$\begin{aligned}\therefore f_1(n) &= x(2n), \quad n = 0, 1, \dots, \left(\frac{N}{2} - 1\right) \\ f_2(n) &= x(2n + 1) \quad n = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right)\end{aligned}$$

Eq. (2.73) can be written as

$$\begin{aligned}X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \\ X(k) &= \sum_{n=\text{even}} x(n) W_N^{kn} + \sum_{n=\text{odd}} x(n) W_N^{kn}, \quad k = 0, 1, 2, \dots, N-1 \quad (2.74) \\ X(k) &= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{k(2n+1)} \\ W_N^{k(2n)} &= (e^{-j2\pi})^{\frac{k2n}{N}} = (e^{-j2\pi})^{\frac{kn}{\frac{N}{2}}} = W_{\frac{N}{2}}^{kn} \\ W_N^{k(2n+1)} &= (e^{-j2\pi})^{\frac{k(2n+1)}{N}} = (e^{-j2\pi})^{\frac{k2n}{N}} \cdot (e^{-j2\pi})^{\frac{k}{N}} \\ &= (e^{-j2\pi})^{\frac{kn}{\frac{N}{2}}} \cdot (e^{-j2\pi})^{\frac{k}{N}} \\ &= W_{\frac{N}{2}}^{kn} \cdot W_N^k\end{aligned}$$

Therefore

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n) W_{\frac{N}{2}}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{\frac{N}{2}}^{kn} W_N^k \quad (2.75)$$

$$X(k) = F_1(k) + W_N^k F_2(k) \quad (2.76)$$

where  $F_1(k)$  and  $F_2(k)$  are  $\frac{N}{2}$ -point DFT of  $f_1(n)$  and  $f_2(n)$  respectively.

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n) W_{\frac{N}{2}}^{kn}, \quad F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{\frac{N}{2}}^{kn} \quad (2.77)$$

$f_1(n)$  would result in the two  $\frac{N}{4}$ -point sequences and  $f_2(n)$  would result in another  $\frac{N}{4}$ -point sequences.

Let the decimated  $\frac{N}{4}$ -point sequences of  $f_1(n)$  be  $V_{11}(n)$  and  $f_2(n)$  be  $V_{12}(n)$ . That is,

$$V_{11}(n) = f_1(2n), \quad n = 0, 1, 2, \dots, \left(\frac{N}{4} - 1\right)$$

$$\begin{aligned} V_{12}(n) &= f_1(2n + 1), \quad n = 0, 1, 2, \dots, \left(\frac{N}{4} - 1\right) \\ V_{21}(n) &= f_2(2n), \quad n = 0, 1, 2, \dots, \left(\frac{N}{4} - 1\right) \\ V_{22}(n) &= f_2(2n + 1), \quad n = 0, 1, 2, \dots, \left(\frac{N}{4} - 1\right) \end{aligned}$$

Let

$$\begin{aligned} V_{11}(k) &= \frac{N}{4} \text{ point DFT of } V_{11}(n) \\ V_{12}(k) &= \frac{N}{4} \text{ point DFT of } V_{12}(n) \\ V_{21}(k) &= \frac{N}{4} \text{ point DFT of } V_{21}(n) \\ V_{22}(k) &= \frac{N}{4} \text{ point DFT of } V_{22}(n) \end{aligned}$$

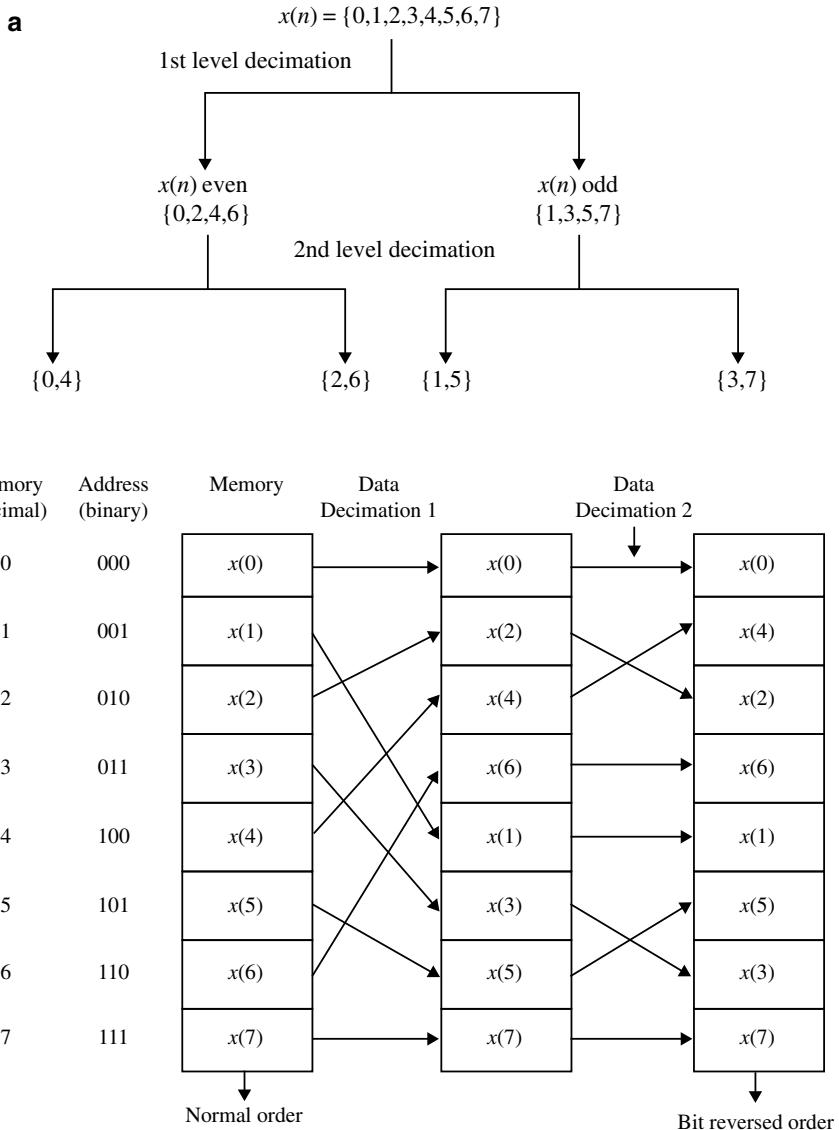
Therefore

$$\begin{aligned} F_1(k) &= V_{11}(k) + W_{\frac{N}{2}}^k V_{12}(k), \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \quad (2.78) \\ F_1\left(k + \frac{N}{4}\right) &= V_{11}(k) - W_{\frac{N}{2}}^k V_{12}(k), \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \\ F_2(k) &= V_{21}(k) + W_{\frac{N}{2}}^k V_{22}(k), \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \quad (2.79) \\ F_2\left(k + \frac{N}{4}\right) &= V_{21}(k) - W_{\frac{N}{2}}^k V_{22}(k), \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \end{aligned}$$

The decimation of the data sequence can be repeated again and again until the resulting sequences are reduced to two-point sequence.

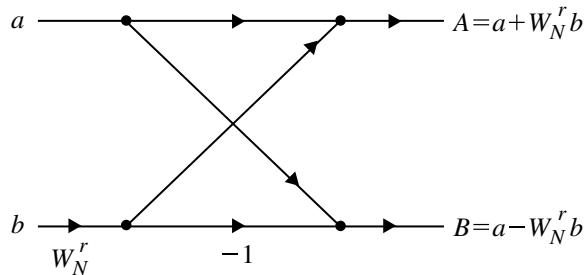
### **Eight-point DFT using radix-2 DIT-FFT**

Let us consider the computation of an  $N = 8$  point DFT. Here  $N = 8 = 2^3$ , and therefore the number of stages of computation is equal to 3. The given eight-point sequence is decimated to two-point sequences. For each two-point sequence, the two-point DFT is computed. From the result of two-point DFT, four-point DFT can be computed. From the result of four-point DFT, eight-point DFT can be computed. The block diagram of radix-2 DIT-FFT for  $N = 8$  is shown in Fig. 2.14a.



**Fig. 2.14** a Data sequence decimation by radix-2. b Shifting of data and bit reversal

**Fig. 2.15** Basic butterfly computation in the DIT-FFT algorithm



### Step 1: Before Decimation the Sequences are Arranged in Bit Reversed Order

Let us consider  $N = 8$  sequence. In the first-level decimation we have the sequence  $x(0), x(2), x(4), x(6), x(1), x(3), x(5), x(7)$  and in the second-level decimation, the sequence is  $x(0), x(4), x(2), x(6), x(1), x(5), x(3)$  and  $x(7)$ . This is represented in Fig. 2.14a.

The shifting of the input data sequences is to be arranged in a well-defined order. The index  $n$  of  $x(n)$  is expressed in binary form. The data point  $x(4)$  is expressed in binary form as  $x(100)$  and is placed in position  $m = 001$  or  $m = 1$  in the decimal array. Now the data  $x(n)$  after decimation is stored in bit reversed order as shown in Fig. 2.14b.

The basic computation in the above Fig. 2.14b involves butterfly operation which is illustrated in Fig. 2.15. Consider numbers  $a$  and  $b$ . The number  $b$  is multiplied by  $W_N^r$  and then added and the product subtracted from  $a$  to form a new complex numbers ( $A, B$ ). This basic computation is called butterfly because the flow graph resembles a butterfly. In each butterfly one complex multiplication and two complex additions are performed.

Natural order		Bit reversed order	
$x(0)$	$x(000)$	$x(000)$	$x(0)$
$x(1)$	$x(001)$	$x(100)$	$x(4)$
$x(2)$	$x(010)$	$x(010)$	$x(2)$
$x(3)$	$x(011)$	$x(110)$	$x(6)$
$x(4)$	$x(100)$	$x(001)$	$x(1)$
$x(5)$	$x(101)$	$x(101)$	$x(5)$
$x(6)$	$x(110)$	$x(011)$	$x(3)$
$x(7)$	$x(111)$	$x(111)$	$x(7)$

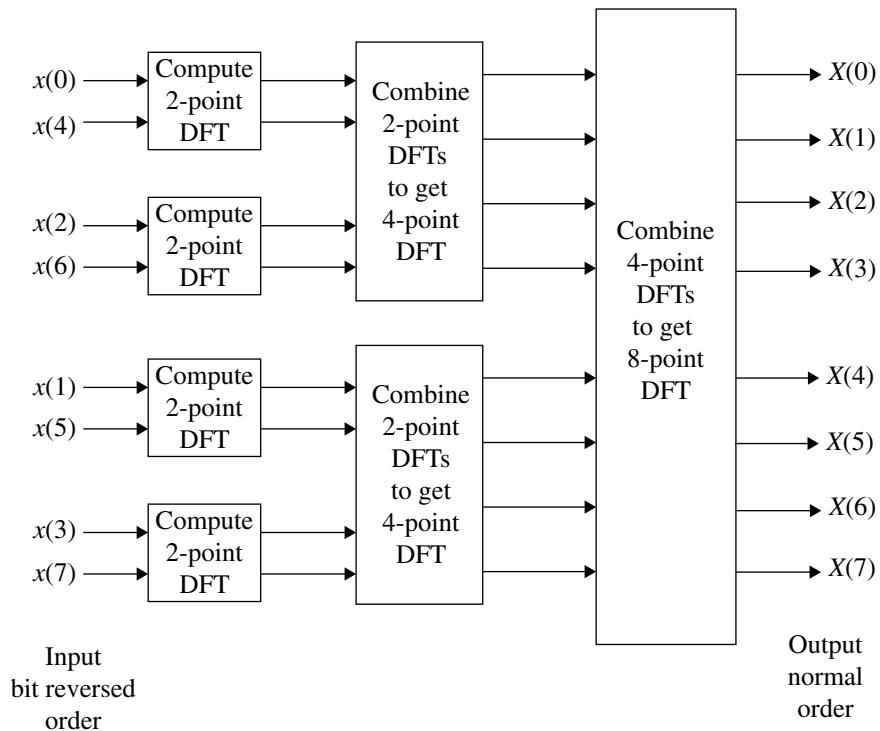
Let

$$x(n) = \text{eight-point sequence}$$

$$f_1(n), f_2(n) = \text{four-point sequences obtained from } x(n)$$

$$V_{11}(n), V_{12}(n) = \text{two-point sequences obtained from } f_1(n)$$

$$V_{21}(n), V_{22}(n) = \text{two-point sequences obtained from } f_2(n)$$



**Fig. 2.16** Block diagram of radix-2 DIT-FFT

Block diagram of radix-2 DIT-FFT is shown in Fig. 2.16.

### First Stage (two-point DFT Computation)

two-point DFT of  $V_{11}(n)$  is

$$V_{11}(k) = \sum_{n=0,1} V_{11}(n) W_{\frac{N}{4}}^{nk} \quad \text{for } k = 0, 1 \quad (2.80)$$

$$V_{11}(0) = \sum_{n=0,1} V_{11}(n) = V_{11}(0) + V_{11}(1) = x(0) + x(4)$$

$$\begin{aligned} V_{11}(1) &= \sum_{n=0,1} V_{11}(n) W_{\frac{N}{4}}^n = V_{11}(0) W_{\frac{N}{4}}^0 + V_{11}(1) W_{\frac{N}{4}}^1 \\ &= V_{11}(0) - V_{11}(1) W_{\frac{N}{4}}^0 = x(0) - x(4) \end{aligned}$$

Similarly,

$$V_{12}(k) = \sum_{n=0,1} V_{12}(n) W_{\frac{N}{4}}^{nk}, \quad k = 0, 1 \quad (2.81)$$

$$V_{12}(0) = \sum_{n=0,1} V_{12}(n) = V_{12}(0) + V_{12}(1) = x(2) + x(6)$$

$$\begin{aligned} V_{12}(1) &= \sum_{n=0,1} V_{12}(n) W_{\frac{N}{4}}^n = V_{12}(0) W_{\frac{N}{4}}^0 + V_{12}(1) W_{\frac{N}{4}}^1 \\ &= V_{12}(0) - V_{12}(1) \\ &= x(2) - x(6) \end{aligned}$$

$$V_{21}(k) = \sum_{n=0,1} V_{21}(n) W_{\frac{N}{4}}^{nk}, \quad k = 0, 1 \quad (2.82)$$

$$V_{21}(0) = \sum_{n=0,1} V_{21}(n) = V_{21}(0) + V_{21}(1) = x(1) + x(5)$$

$$\begin{aligned} V_{21}(1) &= \sum_{n=0,1} V_{21}(n) W_{\frac{N}{4}}^n = V_{21}(0) W_{\frac{N}{4}}^0 + V_{21}(1) W_{\frac{N}{4}}^1 \\ &= V_{21}(0) - V_{21}(1) \\ &= x(1) - x(5) \end{aligned}$$

$$V_{22}(k) = \sum_{n=0,1} V_{22}(n) W_{\frac{N}{4}}^{nk}, \quad k = 0, 1 \quad (2.83)$$

$$V_{22}(0) = \sum_{n=0,1} V_{22}(n) = V_{22}(0) + V_{22}(1) = x(3) + x(7)$$

$$\begin{aligned} V_{22}(1) &= \sum_{n=0,1} V_{22}(n) W_{\frac{N}{4}}^n = V_{22}(0) W_{\frac{N}{4}}^0 + V_{22}(1) W_{\frac{N}{4}}^1 \\ &= V_{22}(0) - V_{22}(1) \end{aligned}$$

### The Second Stage Computation (four-point DFT)

Let

$$F_1(k) = \text{DFT}[f_1(n)] \quad \text{and} \quad F_2(k) = \text{DFT}[f_2(n)]$$

From the Eq. (2.81) we get,

$$\begin{aligned}
 F_1(k) &= V_{11}(k) + W_N^k V_{12}(k) \quad \text{for } k = 0, 1, 2, 3 \\
 F_1(0) &= V_{11}(0) + W_N^0 V_{12}(0) \\
 F_1(1) &= V_{11}(1) + W_N^1 V_{12}(1) \\
 F_1(2) &= V_{11}(2) + W_N^2 V_{12}(2) \\
 &= V_{11}(0) - W_N^2 V_{12}(0) \\
 F_1(3) &= V_{11}(3) + W_N^3 V_{12}(3) \\
 &= V_{11}(1) - W_N^1 V_{12}(1)
 \end{aligned}$$

Here  $V_{11}(k)$  and  $V_{12}(k)$  are periodic with periodicity 2. Therefore,

$$V_{11}(k+2) = V_{11}(k) \quad \text{and} \quad V_{12}(k+2) = V_{12}(k)$$

Similarly, from the Eq. (2.82) we get,

$$\begin{aligned}
 F_2(k) &= V_{21}(k) + W_N^k V_{22}(k) \quad \text{for } k = 0, 1, 2, 3 \\
 F_2(0) &= V_{21}(0) + W_N^0 V_{22}(0) \\
 F_2(1) &= V_{21}(1) + W_N^1 V_{22}(1) \\
 F_2(2) &= V_{21}(2) + W_N^2 V_{22}(2) \\
 &= V_{21}(0) - W_N^2 V_{22}(0) \\
 F_2(3) &= V_{21}(3) + W_N^3 V_{22}(3) \\
 &= V_{21}(1) - W_N^1 V_{22}(1)
 \end{aligned}$$

### The Third Stage of Computation (eight-point DFT)

Let

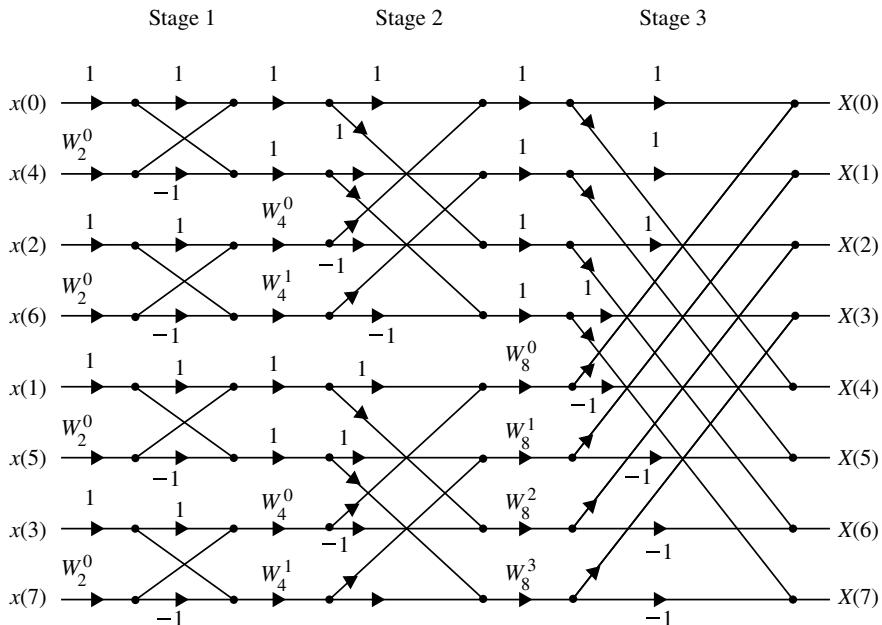
$$\begin{aligned}
 X(k) &= \text{DFT } x(n) \\
 \therefore X(k) &= F_1(k) + W_N^k F_2(k), \quad \text{for } k = 0, 1, 2, 3, 4, 5, 6, 7 \\
 X(0) &= F_1(0) + W_N^0 F_2(0) \\
 X(1) &= F_1(1) + W_N^1 F_2(1) \\
 X(2) &= F_1(2) + W_N^2 F_2(2) \\
 X(3) &= F_1(3) + W_N^3 F_2(3) \\
 X(4) &= F_1(4) + W_N^4 F_2(4) \\
 &= F_1(0) - W_N^0 F_2(0)
 \end{aligned}$$

$$\begin{aligned}
 X(5) &= F_1(5) + W_N^5 F_2(5) \\
 &= F_1(1) - W_N^1 F_2(1) \\
 X(6) &= F_1(6) + W_N^6 F_2(6) \\
 &= F_1(2) - W_N^2 F_2(2) \\
 X(7) &= F_1(7) + W_N^7 F_2(7) \\
 &= F_1(3) - W_N^3 F_2(3)
 \end{aligned}$$

$F_1(k)$  and  $F_2(k)$  are periodic with period is 4. Therefore,

$$F_1(k+4) = F_1(k) \quad \text{and} \quad F_2(k+4) = F_2(k)$$

The flow graph or butterfly diagram for eight-point DIT-FFT is shown in Fig. 2.17.



**Fig. 2.17** Eight-point radix-2 DIT-FFT butterfly diagram

The phase factor values are,

$$\begin{aligned}
 W_2^0 &= 1 \\
 W_4^0 &= e^{-j2\pi \times \frac{0}{4}} = 1 \\
 W_4^1 &= e^{-j2\pi \times \frac{1}{4}} = e^{-j\frac{\pi}{2}} = -j \\
 W_8^0 &= 1 \\
 W_8^1 &= e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \\
 W_8^2 &= e^{-j2\pi \times \frac{2}{8}} = e^{-j\frac{\pi}{2}} = -j \\
 W_8^3 &= e^{-j2\pi \times \frac{3}{8}} = e^{-j\frac{3\pi}{4}} = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}
 \end{aligned}$$

### Summary of steps of radix-2 DIT-FFT algorithm

1. The number of input samples  $N = 2^M$ , where  $M$  is an integer.
2. The input sequence is shuffled through bit reversal.
3. The number of stages in the flow graph is given by  $M = \log_2^N$ .
4. Each stage consists of  $N/2$  butterflies.
5. Input–output for each butterfly are separated by  $2^{m-1}$  samples, where  $m$  represents the stage index, i.e., for first stage  $m = 1$  and for second stage  $m = 2$  and so on.
6. The number of complex multiplications is given by  $\frac{N}{2} \log_2^N$ .
7. The number of complex additions is given by  $N \log_2^N$ .
8. The twiddle factor exponents are a function of the stage index  $m$  and is given by

$$k = \frac{Nt}{2^m}, \quad t = 0, 1, 2, \dots, 2^{m-1} - 1.$$

9. The number of sets or sections of butterflies in each stage is given by the formula  $2^{M-m}$ .
10. The exponent repeat factor (ERF), which is the number of times the exponent sequence associated with  $m$  is repeated is given by  $2^{M-m}$ .

#### 2.6.1.2 Decimation in Frequency (DIF) Radix-2 FFT

In decimation in frequency algorithm the frequency domain sequence  $X(k)$  is decimated. In this algorithm the  $N$ -point time domain sequence is converted to two numbers of  $\frac{N}{2}$ -point sequences. Then each  $\frac{N}{2}$ -point sequence is converted to two numbers of  $\frac{N}{4}$ -point sequences. This process is continued until to get  $\frac{N}{2}$  numbers of two-point sequences. Finally the two-point DFT of each two-point sequence is computed. The two-point DFTs of  $\frac{N}{2}$  numbers of two-point sequences will give  $N$ -samples, which is the  $N$ -point DFT of the time domain sequence.

Consider a  $N$ -point sequence  $x(n)$ . The  $N$ -point DFT of  $x(n)$  is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn} \quad (2.84)$$

$$\begin{aligned} &= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{k(n+\frac{N}{2})} \\ &= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{kn} \cdot W_N^{\frac{kN}{2}} \\ &= \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) W_N^{kn} + (-1)^k x\left(n + \frac{N}{2}\right) W_N^{kn} \right] \quad \left( \because W_N^{\frac{kN}{2}} = (-1)^k \right) \\ X(k) &= \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) + (-1)^k x\left(n + \frac{N}{2}\right) \right] W_N^{kn} \end{aligned} \quad (2.85)$$

Let us split  $X(k)$  into even and odd number sequences as follows.

$$X(2k) \text{ (even)} = \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) + (-1)^{2k} x\left(n + \frac{N}{2}\right) \right] W_N^{2kn} \quad (2.86)$$

$$\begin{aligned} &= \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) + x\left(n + \frac{N}{2}\right) \right] W_N^{kn}, \quad k = 0, 1, \dots, \left(\frac{N}{2} - 1\right) \\ X(2k+1) \text{ (odd)} &= \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) + (-1)x\left(n + \frac{N}{2}\right) \right] W_N^{(2k+1)n} \quad (2.87) \\ &= \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^{2kn} W_N^n \\ &= \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n W_N^{kn}, \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \end{aligned}$$

Let

$$g_1(n) = x(n) + x\left(n + \frac{N}{2}\right), \quad n = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right)$$

$$g_2(n) = \left[ x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n, \quad n = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right)$$

where  $g_1(n)$  and  $g_2(n)$  are  $\frac{N}{2}$ -point time domain sequences. Therefore

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n) W_{\frac{N}{2}}^{kn}, \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \quad (2.88)$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n) W_{\frac{N}{2}}^{kn}, \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \quad (2.89)$$

$X(2k)$  is  $\frac{N}{2}$ -point DFT of  $g_1(n)$  and  $X(2k+1)$  is  $\frac{N}{2}$ -point DFT of  $g_2(n)$

$$\therefore G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n) W_{\frac{N}{2}}^{kn}, \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \quad (2.90)$$

$$G_2(k) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n) W_{\frac{N}{2}}^{kn}, \quad k = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right) \quad (2.91)$$

The  $g_1(n)$  is  $\frac{N}{2}$ -point sequence, and it can be decimated into two numbers of  $\frac{N}{4}$ -point sequences.

$$\begin{aligned} \therefore G_1(k) &= \sum_{n=0}^{\frac{N}{2}-1} g_1(n) W_{\frac{N}{2}}^{kn} \\ &= \sum_{n=0}^{\frac{N}{4}-1} g_1(n) W_{\frac{N}{2}}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n + \frac{N}{4}\right) W_{\frac{N}{2}}^{k(n+\frac{N}{4})} \\ &= \sum_{n=0}^{\frac{N}{4}-1} \left[ g_1(n) + g_1\left(n + \frac{N}{4}\right) W_{\frac{N}{2}}^{k\frac{N}{4}} \right] W_{\frac{N}{2}}^{kn} \end{aligned}$$

Therefore,

$$G_1(2k) = \sum_{n=0}^{\frac{N}{4}-1} d_{11}(n) W_{\frac{N}{4}}^{kn} = D_{11}(k) \quad (2.92)$$

$$G_1(2k+1) = \sum_{n=0}^{\frac{N}{4}-1} d_{12}(n) W_{\frac{N}{4}}^{kn} = D_{12}(k) \quad (2.93)$$

$$G_2(2k) = \sum_{n=0}^{\frac{N}{4}-1} d_{21}(n) W_{\frac{N}{4}}^{kn} = D_{21}(k) \quad (2.94)$$

$$G_2(2k+1) = \sum_{n=0}^{\frac{N}{4}-1} d_{22}(n) W_N^{kn} = D_{22}(k) \quad (2.95)$$

where  $D_{11}(k)$ ,  $D_{12}(k)$ ,  $D_{21}(k)$  and  $D_{22}(k)$  are  $\frac{N}{4}$ -point DFTs of  $d_{11}(n)$ ,  $d_{12}(n)$ ,  $d_{21}(n)$  and  $d_{22}(n)$  respectively.

The decimation of the frequency domain sequence can be continued until the resulting sequence is reduced to two-point sequences.

### The eight-point DFT using radix-2 DIF-FFT

Let  $x(n)$  be an eight-point sequence where  $N = 8$

#### First Stage of Computation

In the first stage of computation, two numbers of four-point sequences  $g_1(n)$  and  $g_2(n)$  are obtained.

$$\begin{aligned} g_1(n) &= x(n) + x\left(n + \frac{N}{2}\right) \\ g_1(n) &= x(n) + x(n+4), \quad n = 0, 1, 2, 3 \end{aligned}$$

$$\begin{aligned} g_1(0) &= x(0) + x(4) \\ g_1(1) &= x(1) + x(5) \\ g_1(2) &= x(2) + x(6) \\ g_1(3) &= x(3) + x(7) \end{aligned}$$

and

$$\begin{aligned} g_2(n) &= \left[ x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n \\ g_2(n) &= [x(n) - x(n+4)] W_8^n, \quad n = 0, 1, 2, 3 \\ g_2(0) &= [x(0) - x(4)] W_8^0 \\ g_2(1) &= [x(1) - x(5)] W_8^1 \\ g_2(2) &= [x(2) - x(6)] W_8^2 \\ g_2(3) &= [x(3) - x(7)] W_8^3 \end{aligned}$$

#### Second Stage of Computation

In the second stage, 4 numbers of two-point sequence is obtained.

$$\begin{aligned} d_{11}(n) &= g_1(n) + g_1\left(n + \frac{N}{4}\right) \\ d_{11}(n) &= g_1(n) + g_1(n+2), \quad n = 0, 1 \\ d_{11}(0) &= g_1(0) + g_1(2) \end{aligned}$$

$$\begin{aligned}
d_{11}(1) &= g_1(1) + g_1(3) \\
d_{12}(n) &= \left[ g_1(n) - g_1\left(n + \frac{N}{4}\right) \right] W_4^n \\
&= [g_1(n) - g_1(n+2)] W_4^n, \quad n = 0, 1 \\
d_{12}(0) &= [g_1(0) - g_1(2)] W_4^0 \\
d_{12}(1) &= [g_1(1) - g_1(3)] W_4^1 \\
d_{21}(n) &= g_2(n) + g_2\left(n + \frac{N}{4}\right) \\
&= g_2(n) + g_2(2), \quad n = 0, 1 \\
d_{21}(n) &= g_2(0) + g_2(2) \\
d_{21}(1) &= g_2(1) + g_2(3) \\
d_{22}(n) &= \left[ g_2(n) - g_2\left(n + \frac{N}{4}\right) \right] W_4^n \\
&= [g_2(n) - g_2(n+2)] W_4^n, \quad n = 0, 1 \\
d_{22}(0) &= [g_2(0) - g_2(2)] W_4^0 \\
d_{22}(1) &= [g_2(1) - g_2(3)] W_4^1
\end{aligned}$$

### Third Stage Computation

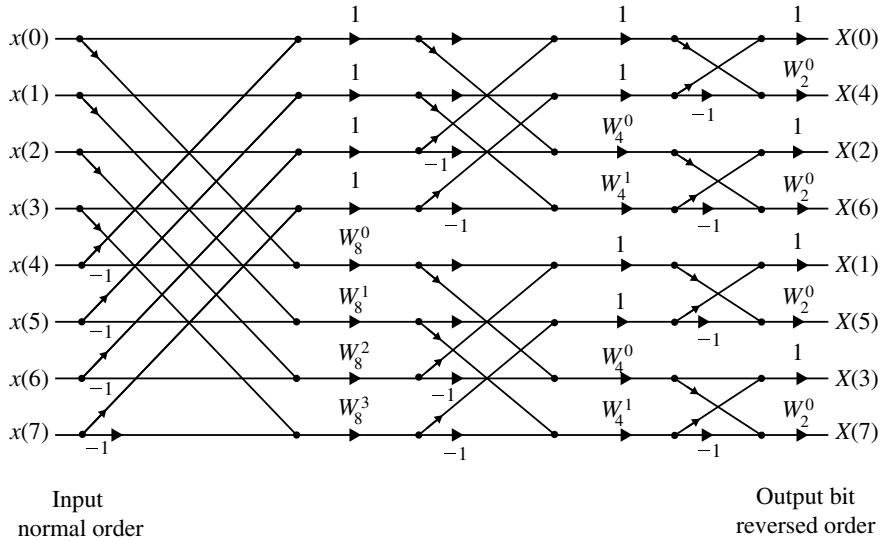
In the third stage, two-point DFTs are calculated.

$$\begin{aligned}
D_{11}(k) &= \sum_{n=0,1} d_{11}(n) W_2^{kn}, \quad k = 0, 1 \\
D_{11}(0) &= \sum_{n=0,1} d_{11}(n) = d_{11}(0) + d_{11}(1) \\
D_{11}(1) &= \sum_{n=0,1} d_{11}(n) W_2^n = [d_{11}(0) - d_{11}(1)] W_2^0
\end{aligned}$$

Similarly

$$\begin{aligned}
D_{12}(0) &= d_{12}(0) + d_{12}(1) \\
D_{12}(1) &= [d_{12}(0) - d_{12}(1)] W_2^0 \\
D_{21}(0) &= d_{21}(0) + d_{21}(1) \\
D_{21}(1) &= [d_{21}(0) - d_{21}(1)] W_2^0 \\
D_{22}(0) &= d_{22}(0) + d_{22}(1) \\
D_{22}(1) &= [d_{22}(0) - d_{22}(1)] W_2^0
\end{aligned}$$

The signal flow graph or butterfly diagram of all three stages is shown in Fig. 2.18.



**Fig. 2.18** Eight-point radix-2 DIF-FFT Butterfly diagram

### Summary of steps of radix-2 DIF-FFT algorithm

1. The number of input samples  $N = 2^M$ , where  $M$  is an integer.
2. The input sequence is in normal order.
3. The number of stages in the flow graph is given by  $M = \log_2^N$ .
4. Each stage consists of  $N/2$  butterflies.
5. Input–output for each butterfly are separated by  $2^{m-1}$  samples, where  $m$  represents the stage index, i.e., for first stage  $m = 1$  and for second stage  $m = 2$  and so on.
6. The number of complex multiplications is given by  $\frac{N}{2} \log_2^N$ .
7. The number of complex addition is given by  $N \log_2^N$ .
8. The twiddle factor exponents are a function of the stage index  $m$  and is given by

$$k = \frac{Nt}{2^{M-m+1}}, \quad t = 0, 1, 2, \dots, 2^{m-1} - 1.$$

9. The number of sets or sections of butterflies in each stage is given by the formula  $2^{m-1}$ .
10. The exponent repeat factor (ERF), which is the number of times the exponent sequence associated with  $m$  is repeated is given by  $2^{m-1}$ .

### 2.6.1.3 Comparison of DIF and DIT

#### Difference Between DIF and DIT

- (i) In DIT, the input is bit reversed while the output is in normal order. For DIF, the input is normal order, while output is in bit reversed order.
- (ii) Considering the butterfly diagram, in DIF, the complex multiplication takes place after the add–subtract operation.

#### Similarities

- (i) Both algorithms require same number of operations to compute DFT.
- (ii) Both algorithms require bit reversal at some place during computation.

## 2.6.2 Radix-4 FFT Algorithms

The decimation in time (DIT) radix-4 FFT recursively partitions a DFT into four quarter-length DFTs of groups of every fourth time sample. The outputs of these shorter FFTs are reused to compute many outputs, thus greatly reducing the total computation cost. In the decimation in frequency radix-4 FFT groups, every fourth output sample is decimated into shorter-length DFTs to save computations.

### 2.6.2.1 Radix-4 DIT-FFT Algorithms

Radix-4 DIT rearranges the DFT equation into four parts: sums over all groups of every fourth discrete time index  $n = [0, 4, 8, \dots, N - 4]$ ,  $n = [15, 9, \dots, N - 3]$ ,  $n = [2, 6, 10, \dots, N - 2]$  and  $n = [3, 7, 11, \dots, N - 1]$ , i.e.,

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} \\ &= \sum_{n=0}^{\frac{N}{4}-1} x(4n)e^{-j\frac{2\pi k(4n)}{N}} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+1)e^{-j\frac{2\pi k(4n+1)}{N}} \\ &\quad + \sum_{n=0}^{\frac{N}{4}-1} x(4n+2)e^{-j\frac{2\pi k(4n+2)}{N}} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+3)e^{-j\frac{2\pi k(4n+3)}{N}} \end{aligned} \tag{2.96}$$

$$\begin{aligned} &= \sum_{n=0}^{\frac{N}{4}-1} x(4n)e^{-j\frac{2\pi k4n}{N}} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+1)e^{-j\frac{2\pi k4N}{N}} \cdot e^{-j\frac{2\pi k}{N}} \\ &\quad + \sum_{n=0}^{\frac{N}{4}-1} x(4n+2)e^{-j\frac{2\pi 4n}{N}} \cdot e^{-j\frac{2\pi k \cdot 2}{N}} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+3)e^{-j\frac{2\pi 4n}{N}} \cdot e^{-j\frac{2\pi k \cdot 3}{N}} \end{aligned} \tag{2.97}$$

$$= X_1(k) + W_N^k X_2(k) + W_N^{2k} X_3(k) + W_N^{3k} X_4(k)$$

where

$$X_1(k) = \sum_{n=0}^{\frac{N}{4}-1} x(4n) W_N^{4nk} \quad (2.98)$$

$$X_2(k) = \sum_{n=0}^{\frac{N}{4}-1} x(4n+1) W_N^{4nk} \quad (2.99)$$

$$X_3(k) = \sum_{n=0}^{\frac{N}{4}-1} x(4n+2) W_N^{4nk} \quad (2.100)$$

$$X_4(k) = \sum_{n=0}^{\frac{N}{4}-1} x(4n+3) W_N^{4nk} \quad (2.101)$$

### 16-Point radix-4 DIT-FFT

Here  $N = 16 = 4^2$ , and the number of stages are two.

$$X(k) = X_1(k) + W_{16}^k X_2(k) + W_{16}^{2k} X_3(k) + W_{16}^{3k} X_4(k) \quad (2.102)$$

#### Stage I

$$\begin{aligned} X_1(k) &= \sum_{n=0}^3 x(4n) W_{16}^{4nk}, \quad k = 0, 1, 2, 3 \\ &= x(0) + x(4) W_{16}^{4k} + x(8) W_{16}^{8k} + x(12) W_{16}^{12k} \end{aligned} \quad (2.103)$$

$$\begin{aligned} X_2(k) &= \sum_{n=0}^3 x(4n+1) W_{16}^{4nk} \\ &= x(1) + x(5) W_{16}^{4k} + x(9) W_{16}^{8k} + x(13) W_{16}^{12k} \end{aligned} \quad (2.104)$$

$$\begin{aligned} X_3(k) &= \sum_{n=0}^3 x(4n+2) W_{16}^{4nk} \\ &= x(2) + x(6) W_{16}^{4k} + x(10) W_{16}^{8k} + x(14) W_{16}^{12k} \end{aligned} \quad (2.105)$$

$$\begin{aligned} X_4(k) &= \sum_{n=0}^3 x(4n+3) W_{16}^{4nk} \\ &= x(3) + x(7) W_{16}^{4k} + x(11) W_{16}^{8k} + x(15) W_{16}^{12k} \end{aligned} \quad (2.106)$$

**Stage II**

$$[X_i(k+4) = X_i(k)]$$

$$\begin{aligned}
X(k) &= X_1(k) + W_{16}^k X_2(k) + W_{16}^{2k} X_3(k) + W_{16}^{3k} X_4(k) \\
X(0) &= X_1(0) + X_2(0) + X_3(0) + X_4(0) \\
X(1) &= X_1(1) + W_{16}^1 X_2(1) + W_{16}^2 X_3(1) + W_{16}^3 X_4(1) \\
X(2) &= X_1(2) + W_{16}^2 X_2(2) + W_{16}^4 X_3(2) + W_{16}^6 X_4(2) \\
X(3) &= X_1(3) + W_{16}^3 X_2(3) + W_{16}^6 X_3(3) + W_{16}^9 X_4(3) \\
X(4) &= X_1(0) + W_{16}^4 X_2(0) + W_{16}^8 X_3(0) + W_{16}^{12} X_4(0) \\
X(5) &= X_1(1) + W_{16}^5 X_2(1) + W_{16}^{10} X_3(1) + W_{16}^{15} X_4(1) \\
X(6) &= X_1(2) + W_{16}^6 X_2(2) + W_{16}^{12} X_3(2) + W_{16}^{18} X_4(2) \\
X(7) &= X_1(3) + W_{16}^7 X_2(3) + W_{16}^{14} X_3(3) + W_{16}^{21} X_4(3) \\
X(8) &= X_1(0) + W_{16}^8 X_2(0) + W_{16}^{16} X_3(0) + W_{16}^{24} X_4(0) \\
X(9) &= X_1(1) + W_{16}^9 X_2(1) + W_{16}^{18} X_3(1) + W_{16}^{27} X_4(1) \\
X(10) &= X_1(2) + W_{16}^{10} X_2(2) + W_{16}^{20} X_3(2) + W_{16}^{30} X_4(2) \\
X(11) &= X_1(3) + W_{16}^{11} X_2(3) + W_{16}^{22} X_3(3) + W_{16}^{33} X_4(3) \\
X(12) &= X_1(0) + W_{16}^{12} X_2(0) + W_{16}^{24} X_3(0) + W_{16}^{36} X_4(0) \\
X(13) &= X_1(1) + W_{16}^{13} X_2(1) + W_{16}^{26} X_3(1) + W_{16}^{39} X_4(1) \\
X(14) &= X_1(2) + W_{16}^{14} X_2(2) + W_{16}^{28} X_3(2) + W_{16}^{42} X_4(2) \\
X(15) &= X_1(3) + W_{16}^{15} X_2(3) + W_{16}^{30} X_3(3) + W_{16}^{45} X_4(3)
\end{aligned}$$

16-point radix-4 DIT-FFT butterfly diagram is shown in Fig. 2.19.

**In radix-4 FFT**

$$\begin{aligned}
N &= 4^m \\
M &= \log_4^N = \frac{\log_2^N}{2}
\end{aligned}$$

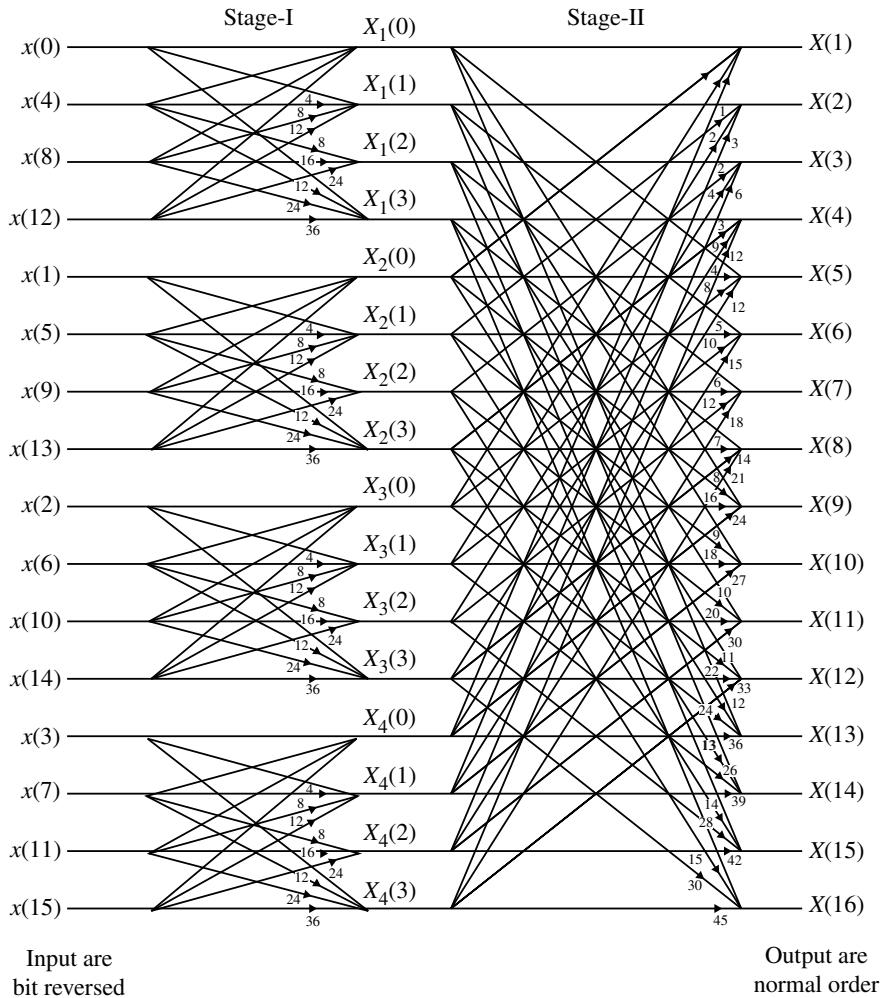
where  $N/4$  butterflies per stage.

Number of complex multiplications

$$3 \frac{N}{4} \frac{\log_2^N}{2} = \frac{3}{8} N \log_2^N \quad (\text{75\% of radix-2 FFT})$$

Number of complex additions

$$8 \frac{N}{4} \frac{\log_2^N}{2} = N \log_2^N \quad (\text{same as a radix-2 FFT})$$



**Fig. 2.19** 16-Point radix-4 DIT-FFT butterfly diagram

### 2.6.2.2 Radix-4 Decimation in Frequency FFT

$N$ -point DFT formula breaking into four smaller DFTs is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x(n) W_N^{kn} + \sum_{n=\left(\frac{N}{4}\right)}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\left(\frac{N}{2}\right)}^{3\left(\frac{N}{4}\right)-1} x(n) W_N^{kn}$$

$$+ \sum_{n=\left(\frac{3N}{4}\right)}^{N-1} x(n) W_N^{kn} \quad (2.107)$$

$$\begin{aligned} &= \sum_{n=0}^{\frac{N}{4}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{4}\right) W_N^{k(n+\frac{N}{4})} \\ &\quad + \sum_{n=0}^{\frac{N}{4}-1} x\left(n + \frac{N}{2}\right) W_N^{k(n+\frac{N}{2})} + \sum_{n=0}^{\frac{N}{4}-1} x\left(n + \frac{3N}{4}\right) W_N^{k(n+\frac{3N}{4})} \end{aligned} \quad (2.108)$$

$$\begin{aligned} &= \sum_{n=0}^{\frac{N}{4}-1} x(n) W_N^{kn} + W_N^{kn/4} \sum_{n=0}^{\frac{N}{4}-1} x\left(n + \frac{N}{4}\right) W_N^{kn} + W_N^{kn/2} \\ &\quad + \sum_{n=0}^{\frac{N}{4}-1} x\left(n + \frac{N}{2}\right) W_N^{nk} + W_N^{3Nk/4} \sum_{n=0}^{\frac{N}{4}-1} x\left(n + \frac{3N}{4}\right) W_N^{kn} \end{aligned} \quad (2.109)$$

$$W_N^{kN/4} = e^{-j\frac{2\pi k N}{4N}} = (e^{-j\frac{\pi}{2}})^k = (-j)^k$$

$$W_N^{kN/2} = (-1)^k$$

$$W_N^{3kN/4} = j^k$$

Substituting these twiddle factors in Eq. (2.92) we get,

$$X(k) = \sum_{n=0}^{\frac{N}{4}-1} \left[ x(n) + (-j)^k x\left(n + \frac{N}{4}\right) + (-1)^k x\left(n + \frac{N}{2}\right) + (j)^k x\left(n + \frac{3N}{4}\right) \right] W_N^{kn}$$

The four  $N/4$ -point DFTs are written as,

$$\begin{aligned} X(4k) &= \sum_{n=0}^{\frac{N}{4}-1} \left[ x(n) + x\left(n + \frac{N}{4}\right) + x\left(n + \frac{N}{2}\right) + x\left(n + \frac{3N}{4}\right) \right] W_N^{4kn} \\ X(4k) &= \sum_{n=0}^{\frac{N}{4}-1} \left[ x(n) + x\left(n + \frac{N}{4}\right) + x\left(n + \frac{N}{2}\right) + x\left(n + \frac{3N}{4}\right) \right] W_N^0 \cdot W_N^{kn} \end{aligned} \quad (2.110)$$

$$X(4k+1) = \sum_{n=0}^{\frac{N}{4}-1} \left[ x(n) - jx\left(n + \frac{N}{4}\right) - x\left(n + \frac{N}{2}\right) + jx\left(n + \frac{3N}{4}\right) \right] W_N^n \cdot W_N^{kn} \quad (2.111)$$

$$X(4k+2) = \sum_{n=0}^{\frac{N}{4}-1} \left[ x(n) - x\left(n + \frac{N}{4}\right) + x\left(n + \frac{N}{2}\right) - x\left(n + \frac{3N}{4}\right) \right] W_N^{2n} \cdot W_{\frac{N}{4}}^{kn} \quad (2.112)$$

$$X(4k+3) = \sum_{n=0}^{\frac{N}{4}-1} \left[ x(n) + jx\left(n + \frac{N}{4}\right) - x\left(n + \frac{N}{2}\right) - jx\left(n + \frac{3N}{4}\right) \right] W_N^{3n} \cdot W_{\frac{N}{4}}^{kn} \quad (2.113)$$

### 16-Point DFT using radix-4 DIF-FFT

Let  $x(n)$  be an 16-point sequence. The samples of  $x(n)$  are  $x(0), x(1), x(2), \dots, x(15)$ . Here  $N = 16$ .

#### Stage I

In the first stage four numbers of four-point sequences  $g_1(n)$ ,  $g_2(n)$ ,  $g_3(n)$  and  $g_4(n)$  are obtained from Eqs. (2.93)–(2.96) respectively. Thus,

$$g_1(n) = \left[ x(n) + x\left(n + \frac{N}{4}\right) + x\left(n + \frac{N}{2}\right) + x\left(n + \frac{3N}{4}\right) \right] W_N^0$$

$n = 0, 1, 2, 3$  and  $N = 16$ .

$$g_1(0) = [x(0) + x(4) + x(8) + x(12)] W_{16}^0$$

$$g_1(1) = [x(1) + x(5) + x(9) + x(13)] W_{16}^0$$

$$g_1(2) = [x(2) + x(6) + x(10) + x(14)] W_{16}^0$$

$$g_1(3) = [x(3) + x(7) + x(11) + x(15)] W_{16}^0$$

$$g_2(n) = \left[ x(n) - jx\left(n + \frac{N}{4}\right) - x\left(n + \frac{N}{2}\right) + jx\left(n + \frac{3N}{4}\right) \right] W_N^n$$

$$g_2(0) = [x(0) - jx(4) - x(8) + jx(12)] W_{16}^0$$

$$g_2(1) = [x(1) - jx(5) - x(9) + jx(13)] W_{16}^1$$

$$g_2(2) = [x(2) - jx(6) - x(10) + jx(14)] W_{16}^2$$

$$g_2(3) = [x(3) - jx(7) - x(11) + jx(15)] W_{16}^3$$

$$g_3(n) = \left[ x(n) - x\left(n + \frac{N}{4}\right) + x\left(n + \frac{N}{2}\right) - x\left(n + \frac{3N}{4}\right) \right] W_N^{2n}$$

$$g_3(0) = [x(0) - x(4) + x(8) - x(12)] W_{16}^0$$

$$\begin{aligned}g_3(1) &= [x(1) - x(5) + x(9) - x(13)] W_{16}^2 \\g_3(2) &= [x(2) - x(6) + x(10) - x(14)] W_{16}^4 \\g_3(3) &= [x(3) - x(7) + x(11) - x(15)] W_{16}^6\end{aligned}$$

$$g_4(n) = \left[ x(n) + jx\left(n + \frac{N}{4}\right) - x\left(n + \frac{N}{2}\right) - jx\left(n + \frac{3N}{4}\right) \right] W_N^{3n}$$

$$\begin{aligned}g_4(0) &= [x(0) - jx(4) - x(8) - jx(12)] W_{16}^0 \\g_4(1) &= [x(1) - jx(5) - x(9) - jx(13)] W_{16}^3 \\g_4(2) &= [x(2) - jx(6) - x(10) - jx(14)] W_{16}^6 \\g_4(3) &= [x(3) - jx(7) - x(11) - jx(15)] W_{16}^9\end{aligned}$$

## Stage II

The four four-point DFTs are

$$G_1(k) = \sum_{n=0}^{\frac{N}{4}-1} g_1(n) W_{N/4}^{kn} \quad k = 0, 1, \dots, 3$$

$$\begin{aligned}G_1(k) &= \sum_{n=0}^3 g_1(n) W_4^{kn} \\G_1(0) &= \sum_{n=0}^3 g_1(n) = g_1(0) + g_1(1) + g_1(2) + g_1(3) \\G_1(1) &= \sum_{n=0}^3 g_1(n) W_4^n \\&= g_1(0) W_4^0 + g_1(1) W_4^1 + g_1(2) W_4^2 + g_1(3) W_4^3\end{aligned}$$

$$\begin{aligned}G_1(2) &= \sum_{n=0}^3 g_1(n) W_4^{2n} \\&= g_1(0) W_4^0 + g_1(1) W_4^2 + g_1(2) W_4^0 + g_1(3) W_4^2 \\G_1(3) &= \sum_{n=0}^3 g_1(n) W_4^{3n} \\&= g_1(0) W_4^0 + g_1(1) W_4^3 + g_1(2) W_4^2 + g_1(3) W_4^1\end{aligned}$$

Similarly

$$\begin{aligned} G_2(k) &= \sum_{n=0}^{\frac{N}{4}-1} g_2(n) W_{N/4}^{kn} \quad k = 0, 1, \dots, 3 \\ G_3(k) &= \sum_{n=0}^{\frac{N}{4}-1} g_3(n) W_{N/4}^{kn} \quad k = 0, 1, \dots, 3 \\ G_4(k) &= \sum_{n=0}^{\frac{N}{4}-1} g_4(n) W_{N/4}^{kn} \quad k = 0, 1, \dots, 3 \end{aligned}$$

16-point radix-4 DIF-FFT butterfly diagram is shown in Fig. 2.20.

$$\text{Number of complex multiplications} = \frac{3}{8} N \log_2^N \quad (2.114)$$

$$\text{Number of complex additions} = N \log_2^N \quad (\text{same as DIT algorithm})$$

### 2.6.2.3 Four-Point DFT Using Radix-4 FFT Algorithm

$$\begin{aligned} X(k) &= \sum_{n=0}^3 x(n) W_N^{kn} \quad k = 0, \dots, 3 \\ X(0) &= \sum_{n=0}^3 x(n) W_N^0 = x(0) + x(1) + x(2) + x(3) \end{aligned} \quad (2.115)$$

$$X(1) = \sum_{n=0}^3 x(n) W_N^n = x(0) W_4^0 - jx(1) W_4^0 - x(2) W_4^0 + jx(3) W_4^0 \quad (2.116)$$

$$X(2) = \sum_{n=0}^3 x(n) W_N^{2n} = x(0) W_4^0 - x(1) W_4^0 + x(2) W_4^0 - x(3) W_4^0 \quad (2.117)$$

$$X(3) = \sum_{n=0}^3 x(n) W_N^{3n} = x(0) W_4^0 + jx(1) W_4^0 - x(2) W_4^0 - jx(3) W_4^0 \quad (2.118)$$

Four-point radix-4 FFT butterfly diagram is shown in Fig. 2.21.

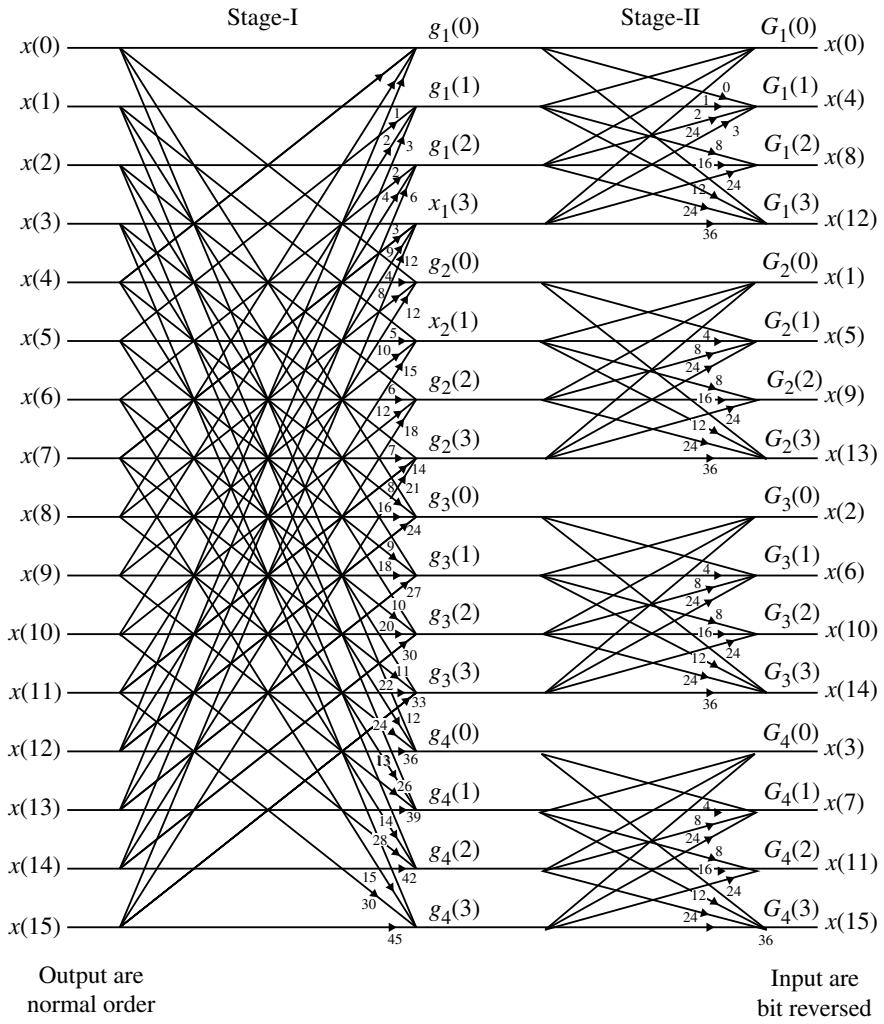


Fig. 2.20 16-Point radix-4 DIF-FFT butterfly diagram

In matrix form, the four-point DFT is written as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \quad (2.119)$$

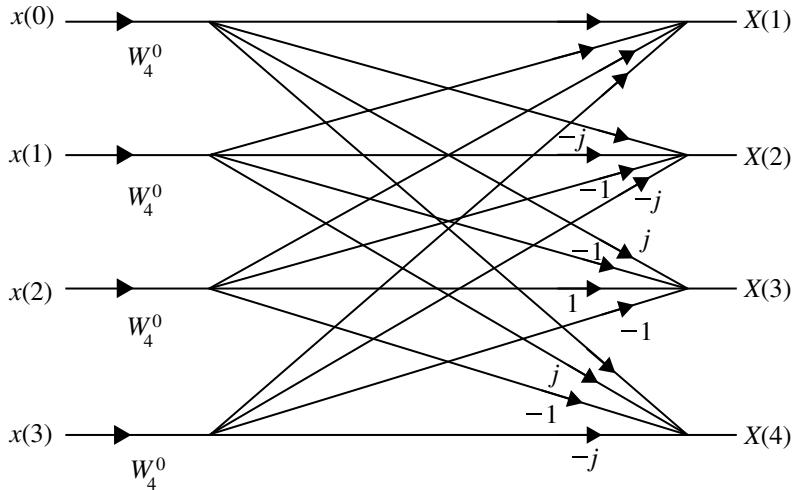


Fig. 2.21 Four-point radix-4 FFT butterfly diagram

or

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} W_4^0 x(0) \\ W_4^0 x(1) \\ W_4^0 x(2) \\ W_4^0 x(3) \end{bmatrix} \quad (2.120)$$

Therefore in general, the four  $N/4$ -point DFT of  $X(k)$  in matrix form is

$$\begin{bmatrix} X(0, q) \\ X(1, q) \\ X(2, q) \\ X(3, q) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} W_N^0 F(0, q) \\ W_N^q F(1, q) \\ W_N^{2q} F(2, q) \\ W_N^{3q} F(3, q) \end{bmatrix} \quad (2.121)$$

The basic butterfly computation in radix-4 FFT algorithm is shown in Fig. 2.22.

### 2.6.3 Computation of IDFT through FFT

The IDFT of an  $N$ -point sequence  $X(k)$  is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j \frac{2\pi k n}{N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} \quad \left[ \because W_N = e^{-j \frac{2\pi}{N}} \right]$$

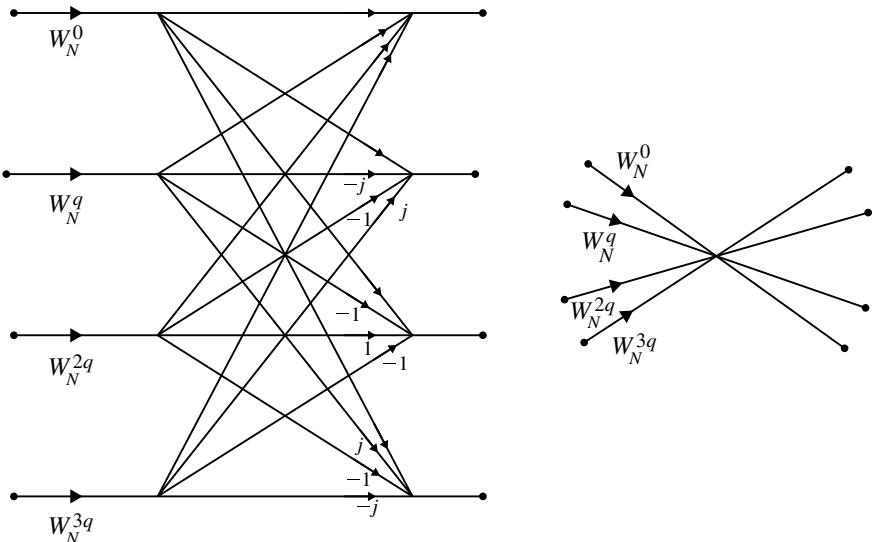
Taking the conjugate and multiplying by  $N$ , we get

$$\begin{aligned} Nx^*(n) &= \sum_{k=0}^{N-1} X^*(k) W_N^{nk} \\ x^*(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) W_N^{nk} \\ x(n) &= \frac{1}{N} \left[ \sum_{k=0}^{N-1} X^*(k) W_N^{nk} \right]^* \end{aligned} \quad (2.122)$$

where  $\sum_{k=0}^{N-1} X^*(k) W_N^{nk}$  is  $N$ -point DFT of  $X^*(k)$ .

Hence, in order to compute IDFT of  $X(k)$  the following procedure is followed:

1. Take conjugate of  $X(k)$  [i.e., determine  $X^*(k)$ ].
2. Compute the  $N$ -point DFT of  $X^*(k)$  using radix-2 FFT.
3. Take conjugate of the output sequence of FFT.
4. Divide the sequence obtained in step 3 by  $N$ . The resultant sequence is  $x(n)$ .



**Fig. 2.22** Basic butterfly in a radix-4 FFT algorithm

**Example 2.31**

An eight-point sequence is given by  $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$ . Compute eight-point DFT of  $x(n)$  by (i) radix-2 DIT-FFT and (ii) radix-2 DIF-FFT. Also sketch the magnitude and phase spectrum.

(Madras University, October, 2002)

(Anna University, December, 2002)

**Solution** Given

$$x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$$

$$\begin{matrix} x(0) & x(1) & x(2) & x(3) & x(4) & x(5) & x(6) & x(7) \end{matrix}$$

$$W_2^0 = 1$$

$$W_4^0 = 1$$

$$W_4^1 = e^{-j\frac{2\pi}{4}} = -j$$

$$W_8^0 = 1$$

$$W_8^1 = e^{-j\frac{2\pi}{8}} = \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j\frac{2\pi \times 2}{8}} = -j$$

$$W_8^3 = e^{-j\frac{2\pi \times 3}{8}} = -\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}}$$

**(i) DIT-FFT**

$$X(k) = \{12, -j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

Input	Stage-I	Stage-II	Stage-III	Output
2	$2 + 1 = 3$	$3 + 3 = 6$	$6 + 6 = 12$	$X(0)$
1	$2 - 1 = 1$	$1 + (-j) = 1 - j$	$1 - j + (\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}})(1 - j) = 1 - j2.414$	$X(1)$
2	$2 + 1 = 3$	$3 - 3 = 0$	$0 + 0 = 0$	$X(2)$
1	$2 - 1 = 1$	$1 + (-1)(-j) = 1 + j$	$1 + j + (\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}})(1 + j) = 1 - j0.414$	$X(3)$
2	$2 + 1 = 3$	$3 + 3 = 6$	$6 - 6 = 0$	$X(4)$
1	$2 - 1 = 1$	$1 + (-j) = 1 - j$	$1 - j - (\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}})(1 - j) = 1 - j0.414$	$X(5)$
2	$2 + 1 = 3$	$3 - 3 = 0$	$0 - 0 = 0$	$X(6)$
1	$2 - 1 = 1$	$1 + (-1)(-j) = 1 + j$	$1 + j - (\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}})(1 + j) = 1 + j2.414$	$X(7)$

**(ii) DIF-FFT**

Butterfly diagram of Example 2.31(i) is shown in Fig. 2.23.

Butterfly diagram of Example 2.31(ii) is shown in Fig. 2.24.

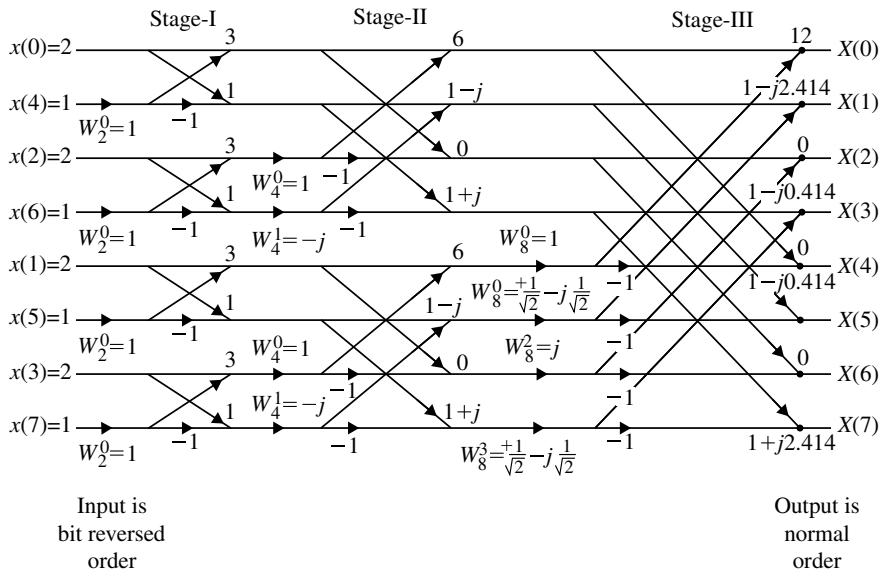


Fig. 2.23 Butterfly diagram of Example 2.31(i)

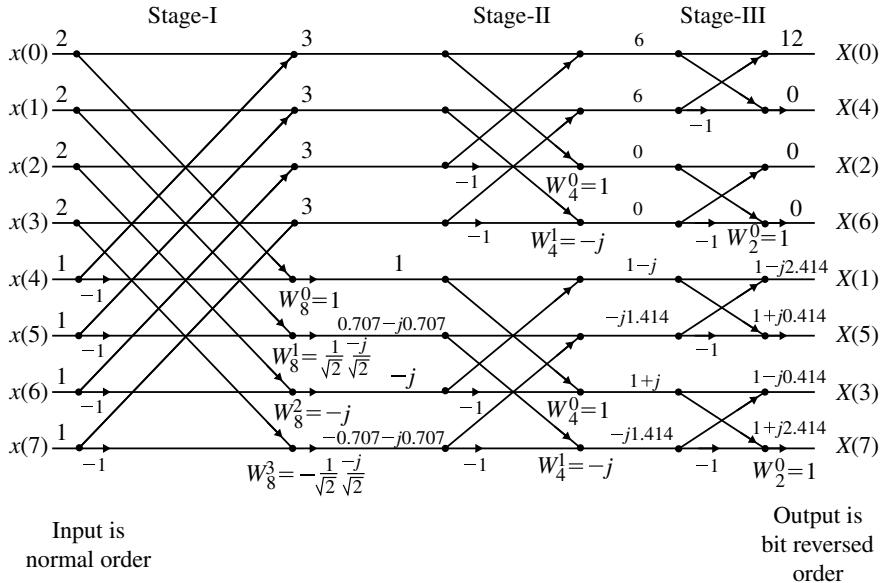


Fig. 2.24 Butterfly diagram of Example 2.31(ii)

$$\begin{aligned} X(k) &= \{12, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\} \\ |X(k)| &= \{12, 2.61, 0, 1.08, 0, 1.08, 0, 2.61\} \\ \angle X(k) &= \{0, -0.37\pi, 0, -0.12\pi, 0, 0.12\pi, 0, 0.37\pi\} \end{aligned}$$

Input	Stage-I	Stage-II	Stage-III	Output
2	$2 + 1 = 3$	$3 + 3 = 6$	$6 + 6 = 12$	$X(0)$
2	$2 + 1 = 3$	$3 + 3 = 6$	$6 - 6 = 0$	$X(4)$
2	$2 + 1 = 3$	$(3 - 3)(j) = 0$	$0 + 0 = 0$	$X(2)$
2	$2 + 1 = 3$	$(3 - 3)(-j) = 0$	$0 - 0 = 0$	$X(6)$
1	$(2 - 1)1 = 1$	$1 - j$	$1 - j - j1.414 =$	$X(1)$
			$1 - j2.414$	
1	$(2 - 1)\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) =$ $\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} =$ $-j1.414$	$1 - j + j1.414 =$ $1 + j0.414$	$X(5)$
1	$(2 - 1)(-j) = -j$	$1 - (-j) = 1 + j$	$1 + j - j1.414 =$ $1 - j0.414j$	$X(3)$
1	$(2 - 1)\left(\frac{-1}{\sqrt{2}} - \frac{-j}{\sqrt{2}}\right) =$ $\frac{-1}{\sqrt{2}} - \frac{-j}{\sqrt{2}}$	$\left(\frac{-1}{\sqrt{2}} - \frac{-j}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) - j =$ $= -j1.414$	$1 + j + j1.414 =$ $1 + j2.414$	$X(7)$

### Example 2.32

In an LTI system the input  $x(n) = \{1, 1, 1\}$  and the impulse response  $h(n) = \{-1, -1\}$ . Determine the response of LTI system by radix-2 DIT-FFT.

**Solution** The response of LTI system is given by linear convolution of input  $x(n)$  and impulse response  $h(n)$ . That is

$$y(n) = x(n) * h(n)$$

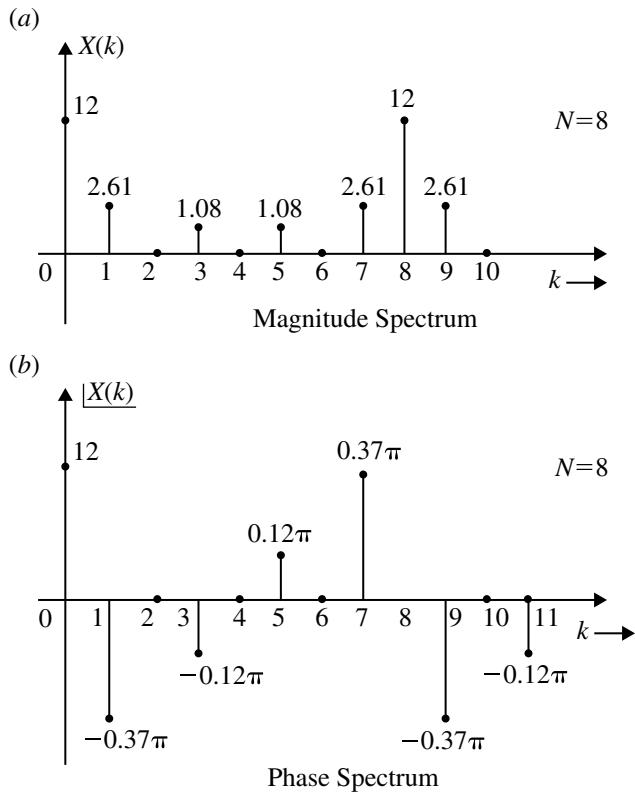
where  $N_1 = 3$  and  $N_2 = 2$ . The length of output sequence  $y(n)$  is  $N = N_1 + N_2 - 1 = 3 + 2 - 1 = 4$ . Therefore, the given sequence  $x(n)$  and  $h(n)$  are converted into four-point sequences by appending zeros.

$$\begin{aligned} x(n) &= \{1, 1, 1, 0\} \quad \text{and} \quad h(n) = \{-1, -1, 0, 0\} \\ y(n) &= x(n) * h(n) \\ Y(k) &= X(k) * H(k) \\ y(n) &= \text{IDFT}\{Y(k)\} \\ y(n) &= \text{IDFT}\{X(k)H(k)\} \end{aligned}$$

#### Step 1: To Determine $X(k)$

$$x(n) = \{1, 1, 1, 0\}$$

Therefore, by referring to butterfly diagram we get



**Fig. 2.25** Magnitude and phase spectrum of Example 2.31

Magnitude and phase spectrum of Example 2.31 is shown in Fig. 2.25.

$$X(k) = \{3, -j, 1, j\}$$

### Step 2: To Determine $H(k)$

$$h(n) = \{-1, -1, 0, 0\}$$

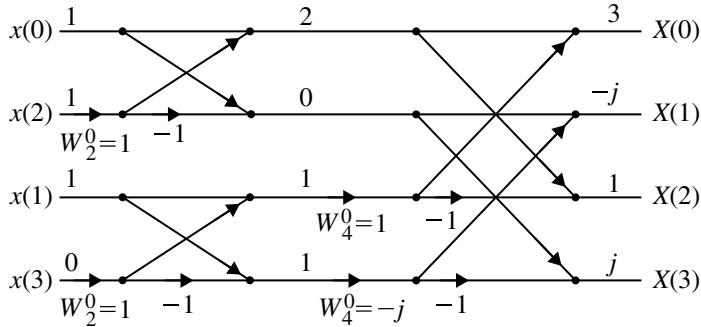
Therefore, by referring to Fig. 2.27 we get,

$$H(k) = \{2, -1 + j, 0, -1 - j\}$$

### Step 3: To Determine the Product $X(k)H(k)$

Let

$$Y(k) = X(k)H(k)$$

**Fig. 2.26** Butterfly diagram of  $X(k)$ 

where

$$\begin{aligned} k = 0, \quad Y(0) &= X(0)H(0) = -6 \\ k = 1, \quad Y(1) &= X(1)H(1) = 1 + j \\ k = 2, \quad Y(2) &= X(2)H(2) = 0 \\ k = 3, \quad Y(3) &= X(3)H(3) = 1 - j \end{aligned}$$

Therefore,

$$Y(k) = \{-6, 1 + j, 0, 1 - j\}$$

Butterfly diagram of  $X(k)$  is shown in Fig. 2.26.

#### Step 4: To Determine IDFT of $Y(k)$

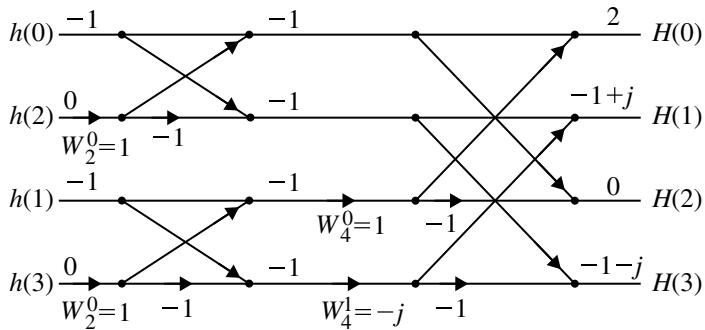
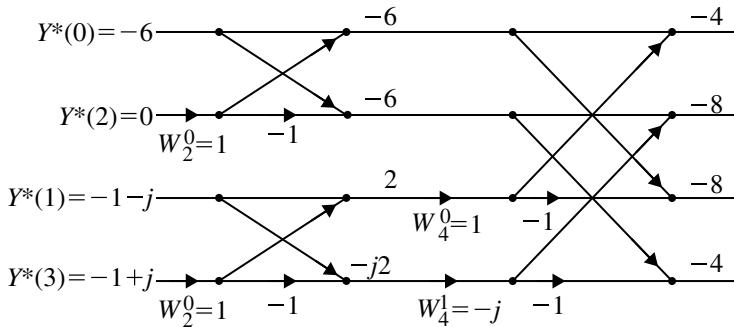
The IDFT of  $Y(k)$  is given by

$$\begin{aligned} y(n) &= \frac{1}{N} \left[ \sum_{k=0}^{N-1} Y^*(k) W_N^{nk} \right]^* \\ \therefore Y^*(k) &= \{-6, 1 - j, 0, 1 + j\} \end{aligned}$$

The four-point DFT of  $Y^*(k)$  is,

Therefore, by referring to Fig. 2.28 we get,

$$\begin{aligned} y(n) &= \frac{1}{4} \{4\text{-point DFT } Y^*(k)\}^* \\ &= \frac{1}{4} \{-4, -8, -8, -4\}^* \\ &= \frac{1}{4} \{-4, -8, -8, -4\} \end{aligned}$$

**Fig. 2.27** Butterfly diagram of  $H(k)$ **Fig. 2.28** Butterfly diagram of  $Y^*(k)$ 

$$y(n) = \{-1, -2, -2, -1\}$$

**Example 2.33**

Find the DFT of sequence

$$x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$$

using DIT and DIF algorithm.

**Solution** The twiddle factors are

$$\begin{aligned} W_2^0 &= 1, & W_4^0 &= 1, & W_4^1 &= -j \\ W_8^0 &= 0, & W_8^1 &= e^{-j\frac{2\pi}{8}} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \\ W_8^2 &= e^{-j\frac{2\pi \times 2}{8}} = -j \\ W_8^3 &= e^{-j\frac{2\pi \times 3}{8}} \end{aligned}$$

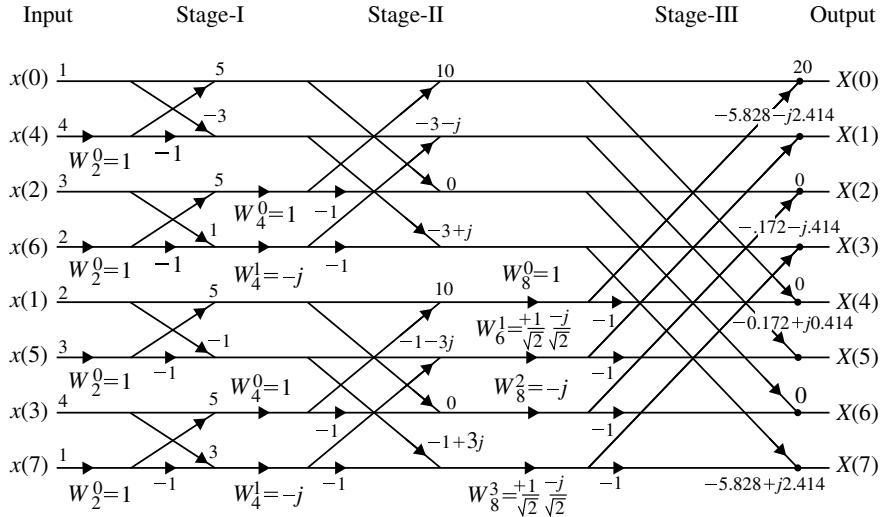


Fig. 2.29 Butterfly diagram of Example 2.33(i)

$$= -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$$

The butterfly diagram is shown in Fig. 2.29.

### (i) Using DIT Algorithm

Input	Stage-I	Stage-II	Output
1	$1 + 4 = 5$	$5 + 5 = 10$	$10 + 10 = 20$
4	$1 - 4 = -3$	$-3 - j$	$-3 - j + (-1 - 3j)(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}})$ $= -5.828 - j2.414$
3	$3 + 2 = 5$	$5 - 5 = 0$	$0 + 0 = 0$
2	$3 - 2 = 1$	$-3 + j$	$-3 + j + (-1 + 3j)(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}) =$ $-0.172 - j0.414$
2	$2 + 3 = 5$	$5 + 5 = 10$	$10 - 10 = 0$
3	$2 - 3 = -1$	$-1 + (3)(-j) = -1 - 3j$	$-3 - j - (-1 - 3j)(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}) =$ $-0.172 + j0.414$
4	$4 + 1 = 5$	$5 - 5 = 0$	$0 + 0 = 0$
1	$4 - 1 = 3$	$-1 + (3)(-j)(-1) = -1 + 3j$	$-3 + j - (-1 + 3j)(\frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}) =$ $-5.828 + j2.414$

$$X(k) =$$

$$\{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$$

Butterfly diagram of Example 2.33 using DIF algorithm is shown in Fig. 2.30.

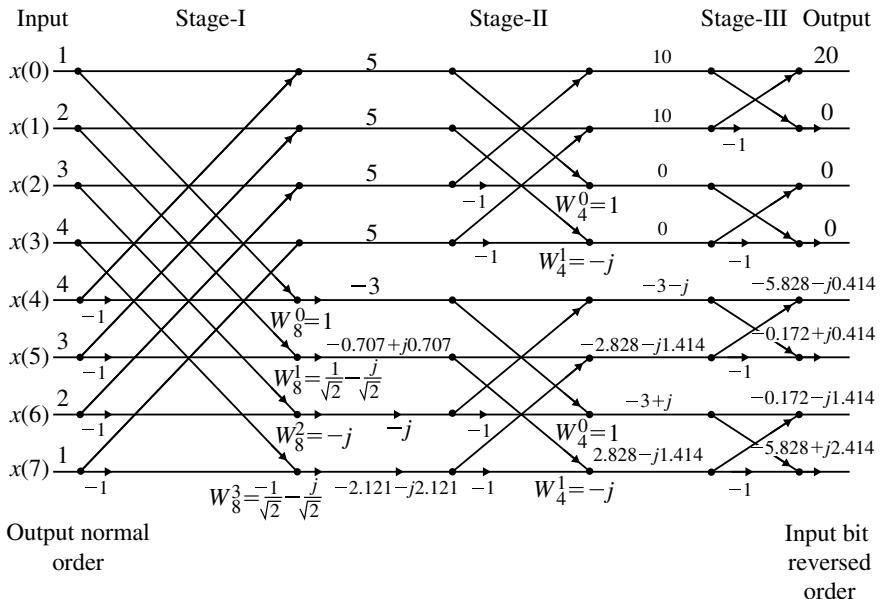


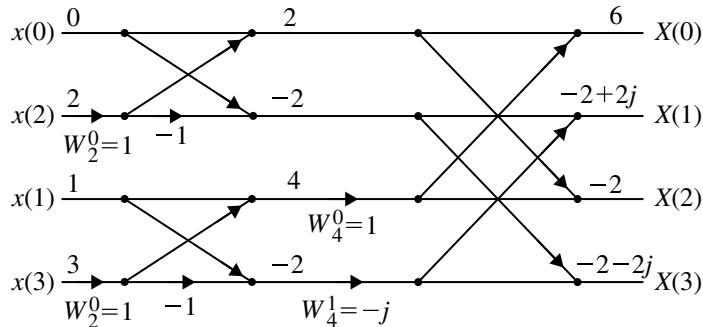
Fig. 2.30 Butterfly diagram of Example 2.33(ii)

### (ii) Using DIT Algorithm

$$X(k) =$$

$$\{20, -5.828-j2.414, 0, -0.172v j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414\}$$

Input	Stage-I	Stage-II	Output
1	$1 + 4 = 5$	$5 + 5 = 10$	$10 + 10 = 20$
2	$2 + 3 = 5$	$5 + 5 = 10$	$10 - 10 = 0$
3	$3 + 2 = 5$	$(5 - 5)(1) = 0$	$0 + 0 = 0$
4	$4 + 1 = 5$	$(5 - 5)(-j) = 0$	$0 - 0 = 0$
4	$(1 - 4)1 = -3$	$-3 - j$	$-3 - j - 2.828 - j1.414$
			$= -5.828 - j0.414$
3	$(2 - 3)(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})$ $= -0.707 + j0.707$	$-0.707 + j0.707 - 2.121 - j2.121$ $= -2.828 - j1.414$	$-3 - j + 2.828 + j1.414$ $= -0.172 + j0.414$
2	$(3 - 2)(-j) = -j$	$-3 + j$	$-3 + j + 2.828 - j1.414$ $= -0.172 - j0.414$
1	$(4 - 1)(\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})$ $= -2.121 - j2.121$	$-0.707 + j0.707 - (-2.121 - j2.121)$ $= (-j) = 2.828 - j1.414$	$-3 + j - 2.828 + j1.414$ $= -5.828 + j2.414$



**Fig. 2.31** Butterfly diagram of  $X(k)$  for Example 2.34

Butterfly diagram of Example 2.34 is shown in Fig. 2.31.

### Example 2.34

Compute four-point of sequence

$$x(n) = \{0, 1, 2, 3\}$$

using DIT algorithm.

**Solution** The twiddle factors are

$$W_2^0 = 1, \quad W_4^0 = 1, \quad W_4^1 = -j$$

$$X(k) = \{6, -2 + 2j, -2, -2 - 2j\}$$

### Example 2.35

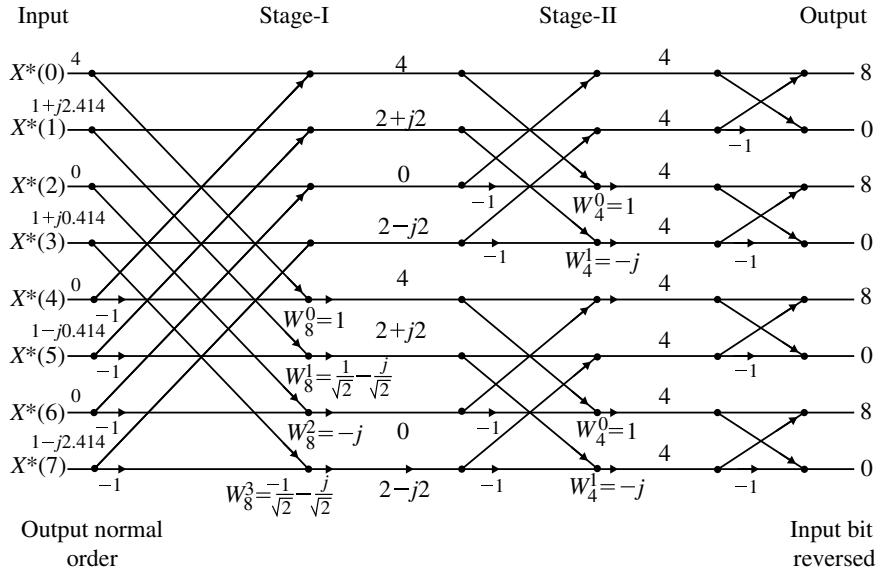
Find the IDFT of the sequence

$$X(k) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

using DIT algorithm.

**Solution**

$$X^*(k) = \{4, 1 + j2.414, 0, 1 + j0.414, 0, 1 - j0.414, 0, 1 - j2.414\}$$



**Fig. 2.32** Butterfly diagram of  $X^*(k)$

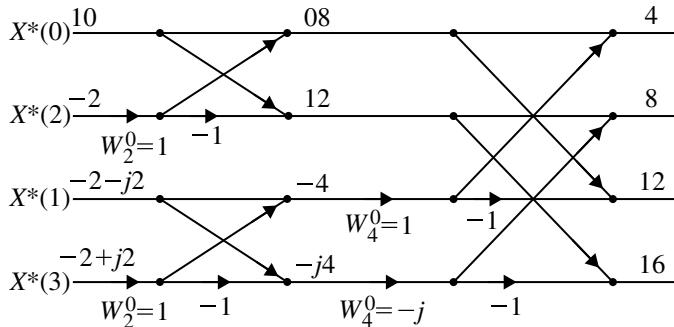
The eight-point butterfly diagram of  $X^*(k)$  is shown in Fig. 2.32.

Input	Stage-I	Stage-II	Output
4	$4 + 0 = 4$	$4 + 0 = 4$	$4 + 4 = 8$
$1 + j2.414$	$1 + j2.414 + 1 - j0.414 = 2 + j2$	$2 + j2 + 2 - j2 = 4$	$4 - 4 = 0$
0	$0 + 0 = 0$	$4 - 0 = 4$	$4 + 4 = 8$
$1 + j0.414$	$1 + j0.414 + 1 - j2.414 = 2 - j2$	$2 + j2 - (2 - j2)(-j) = 4$	$4 - 4 = 0$
0	$4 - 0 = 4$	$4 + 0 = 4$	$4 + 4 = 8$
$1 - j0.414$	$1 + j2.414 - (1 - j0.414) \left( \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \right) = 2 + j2$	$2 + j2 + 2 - j2 = 4$	$4 - 4 = 0$
0	$0 - 0 = 0$	$4 - 0 = 4$	$4 + 4 = 8$
$1 - j2.414$	$1 - j0.414 - (1 - j2.414) \left( \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \right) = 2 - j2$	$2 + j2 + 2 - (2 - j2)(-j) = 4$	$4 - 4 = 0$

Eight-point DFT of  $X^*(k)$  is

$$\begin{aligned} x(n) &= \frac{1}{8} \{ \text{8-point DFT of } X^*(k) \}^* \\ &= \frac{1}{8} \{ 8, 8, 8, 8, 0, 0, 0, 0 \}^* \end{aligned}$$

$x(n) = \{ 1, 1, 1, 1, 0, 0, 0, 0 \}$



**Fig. 2.33** Butterfly diagram of  $X^*(k)$

### Example 2.36

Find the IDFT of the sequence

$$X(k) = \{10, -2 + j2, -2, -2 - j2\}$$

using DIT algorithm.

### Solution Given

$$X(k) = \{10, -2 + j2, -2, -2 - j2\}$$

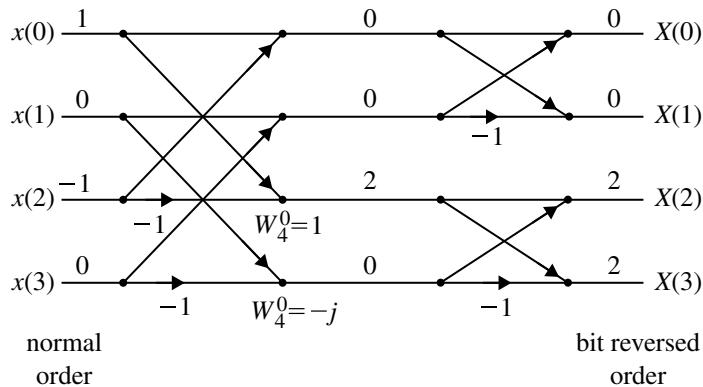
$$X^*(k) = \{10, -2 - j2, -2, -2 + j2\}$$

The butterfly diagram of  $X^*(k)$  is shown in Fig. 2.33.

Four-point DFT of  $X^*(k) = \{4, 12, 8, 16\}$

$$\begin{aligned} x(n) &= \frac{1}{4} \{4\text{-point DFT of } X^*(k)\}^* \\ &= \frac{1}{4} \{4, 8, 12, 16\}^* \end{aligned}$$

$x(n) = \{1, 2, 3, 4\}$



**Fig. 2.34** Butterfly diagram of  $X(k)$  for Example 2.37

Butterfly diagram of Example 2.37 is shown in Fig. 2.34.

### Example 2.37

Compute the DFT of the sequence

$$x(n) = \cos\left(\frac{n\pi}{2}\right)$$

where  $N = 4$ , using DIF-FFT algorithm.

**Solution** Given

$$\begin{aligned} x(n) &= \cos\left(\frac{n\pi}{2}\right), & N &= 4 \\ x(0) &= 1 \\ x(1) &= 0 \\ x(2) &= -1 \\ x(3) &= 0 \end{aligned}$$

Therefore,

$$x(n) = \{1, 0, -1, 0\}$$

Therefore,

$$X(k) = \{0, 2, 0, 2\}$$

**Example 2.38**

Using the decimation in frequency FFT flow graph compute DFT of

$$x[n] = \cos\left(\frac{n\pi}{4}\right) \quad \text{for } 0 \leq n \leq 7$$

(Anna University, December, 2007)

**Solution** Given

$$x[n] = \cos\left(\frac{n\pi}{4}\right) \quad \text{for } 0 \leq n \leq 7$$

$$x[n] = \{1, 0.707, 0, -0.707, -1, -0.07, 0, 0.707\}$$

The butterfly diagram is shown in Fig. 2.35. From Fig. 2.35,  $X(k)$  is obtained as,

$$X(k) = \{0, 4, 0, 0, 0, 0, 0, 0, 4\}$$

**Example 2.39**

Draw the butterfly diagram using eight-point DIT-FFT for the following sequence

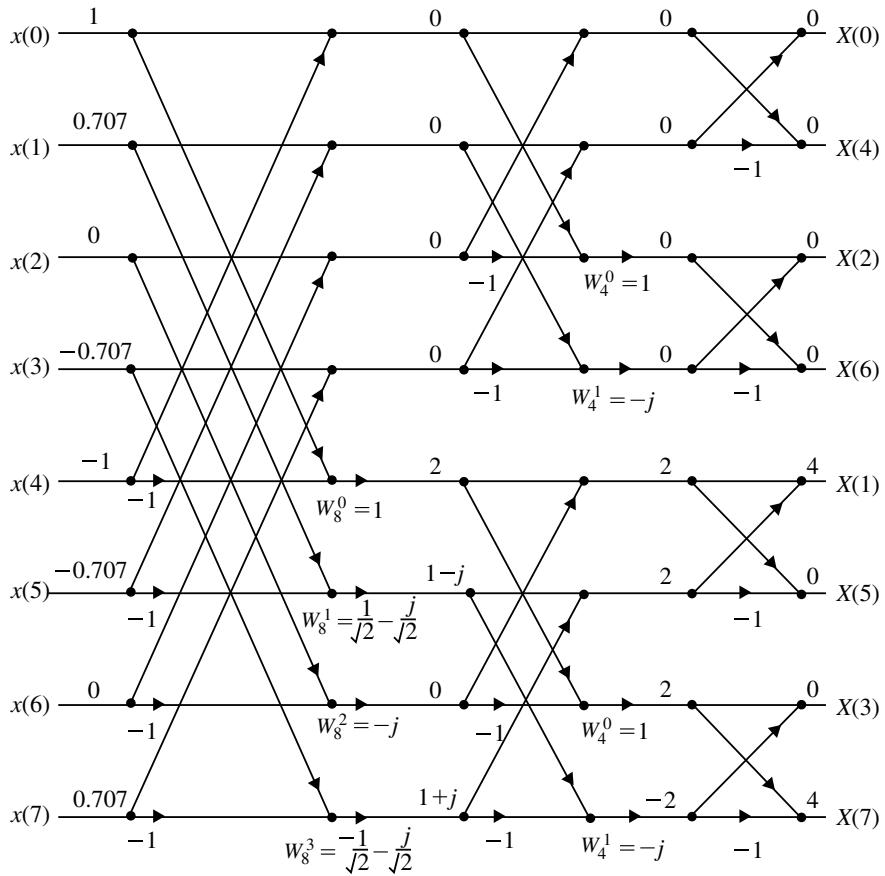
$$x[n] = \{1, 0, 0, 0, 0, 0, 0, 0\}$$

((Anna University, June, 2007)

**Solution** The butterfly diagram for Example 2.39 is shown in Fig. 2.36.

Input	Stage-I	Stage-II	Output
1	$1 + 0 = 1$	$1 + 0 = 1$	$1 + 0 = 1$
0	$0 + 1 = 1$	$1 + 0 = 1$	$1 + 0 = 1$
0	$0 + 0 = 0$	$0 + 1 = 1$	$1 + 0 = 1$
0	$0 + 0 = 0$	$0 + 1 = 1$	$1 + 0 = 1$
0	$0 + 0 = 0$	$0 + 0 = 0$	$0 + 1 = 1$
0	$0 + 0 = 0$	$0 + 0 = 0$	$0 + 1 = 1$
0	$0 + 0 = 0$	$0 + 0 = 0$	$0 + 1 = 1$
0	$0 + 0 = 0$	$0 + 0 = 0$	$0 + 1 = 1$

$$X(k) = \{1, 1, 1, 1, 1, 1, 1, 1\}$$



**Fig. 2.35** Butterfly diagram for Example 2.38

### Example 2.40

Compute the eight-point DFT of the sequence

$$x[n] = \{0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0\}$$

using the in place radix-2 DIF-FFT algorithm.

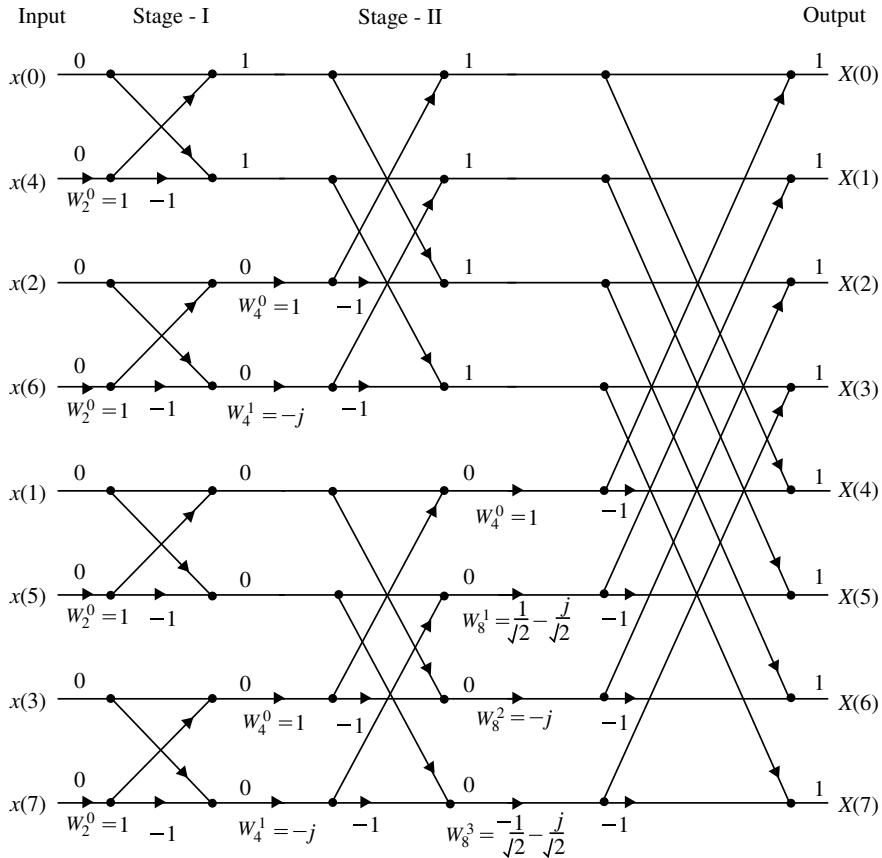


Fig. 2.36 Butterfly diagram for Example 2.39

### Solution

The twiddle factors are

$$\begin{aligned}
 W_8^0 &= 1, \\
 W_8^1 &= \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \\
 W_8^2 &= -j \\
 W_8^3 &= -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \\
 W_4^0 &= 1 \\
 W_4^1 &= -j \\
 W_2^0 &= 1
 \end{aligned}$$

The butterfly diagram is shown in Fig. 2.37.

Input	Stage-I	Stage-II	Output
0.5	$0.5 + 0 = 0.5$	$0.5 + 0.5 = 1$	$1 + 1 = 2$
0.5	$0.5 + 0 = 0.5$	$0.5 + 0.5 = 1$	$1 - 1 = 0$
0.5	$0.5 + 0 = 0.5$	$(0.5 - 0.5)1 = 0$	$0 + 0 = 0$
0.5	$0.5 + 0 = 0.5$	$(0.5 - 0.5)(-j) = 0$	$0 - 0 = 0$
0	$(0.5 + 0)1 = 0.5$	$0.5 - 0.5j$	$0.5 - 0.5j - 0.707j$
			$= 0.5 - 1.207j$
0	$(0.5 + 0) \left( \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \right)$ $= 0.3535 - j0.3525$	$(0.3525 - j0.3525)$ $+ (-0.3525 - j0.3525) = -0.707j$	$0.5 - 0.5j + 0.707j$ $= 0.5 + 0.207j$
0	$(0.5 + 0)(-j) = -0.5j$	$0.5 + 0.5j$	$0.5 + 0.5j - 0.707j$ $= 0.5 - 0.207j$
0	$(0.5 + 0) \left( \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \right)$ $= -0.3535 - j0.3525$	$[(0.3525 - j0.3525)$ $- (-0.3525 - j0.3525)](-j)$	$0.5 + 0.5j + 0.707j$ $= 0.5 + 1.207j$
			$= -0.707j$

$$X(k) = \{2, 0.5 - 1.207j, 0, 0.5 - 0.207j, 0, 0.5 + 0.207j, 0, 0.5 + 1.207j\}$$

### Example 2.41

Find the eight-point DFT of the given sequence

$$x[n] = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

using DIF radix-2 FFT algorithm.

(Anna University, May, 2005)

### Solution

Input	Stage-I	Stage-II	Output
0	$0 + 4 = 4$	$8 + 4 = 12$	$12 + 16 = 28$
1	$1 + 5 = 6$	$6 + 10 = 16$	$12 - 16 = -4$
2	$2 + 6 = 8$	$4 - 8 = -4$	$-4 + 4j$
3	$3 + 7 = 10$	$(6 - 10) - j = 4j$	$-4 - 4j$
4	$0 - 4 = -4$	$-4 + 4j$	$-4 + 4j + \frac{8j}{\sqrt{2}}$
			$= -4 + 9.656j$
5	$(1 - 5) \left( \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \right)$ $= \frac{-4}{\sqrt{2}} + \frac{4j}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}} + \frac{4j}{\sqrt{2}} + \frac{4}{\sqrt{2}} + \frac{4j}{\sqrt{2}} = \frac{8j}{\sqrt{2}}$	$-4 + 4j - \frac{8j}{\sqrt{2}}$
			$= -4 - 1.656j$
6	$(2 - 6)(-j) = 4j$	$-4 - 4j$	$-4 + 4j + \frac{8j}{\sqrt{2}}$
			$= -4 + 1.656j$
7	$(3 - 7) \left( -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} \right)$ $= \frac{4}{\sqrt{2}} + \frac{4j}{\sqrt{2}}$	$\left( -\frac{4}{\sqrt{2}} + \frac{4j}{\sqrt{2}} - \frac{4}{\sqrt{2}} - \frac{4j}{\sqrt{2}} \right) = -j = \frac{8j}{\sqrt{2}}$	$-4 - 4j - \frac{8j}{\sqrt{2}}$
			$= -4 - 9.656j$

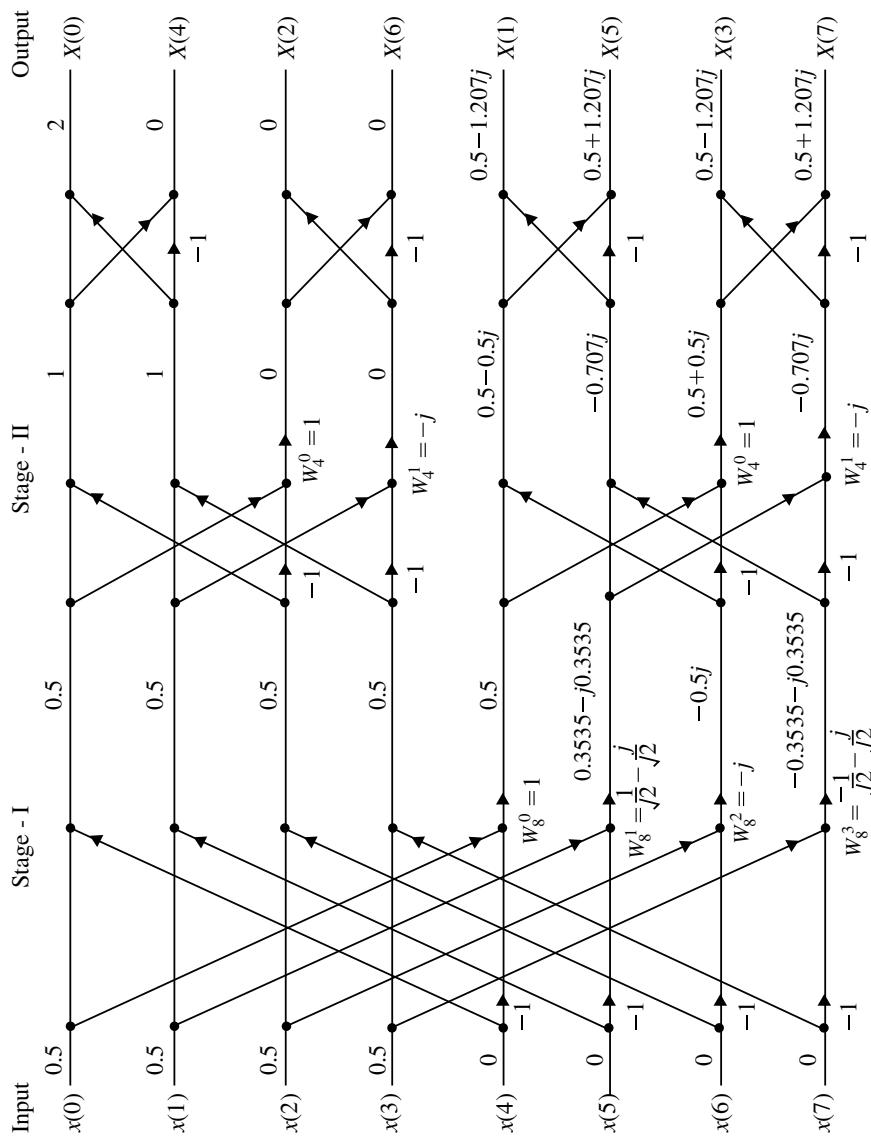


Fig. 2.37 Butterfly diagram for Example 2.40

$$X(k) = \left\{ 28, -4 + 9.656j, -4 + 4j, -4 + 1.656j, -4, -4 - 1.656j, \right. \\ \left. -4 - 4j, -4 - 9.656j \right\}$$

The butterfly diagram is shown in Fig. 2.38.

### Example 2.42

Find the response of an LTI system with impulse response  $h(n) = \{2, 1, 3\}$  for the input  $x(n) = \{1, 2\}$  using DIT radix-2 FFT algorithm.

*(Anna University, December, 2007)*

**Solution** The response of LTI system is given by linear convolution of input  $x(n)$  and impulse  $h(n)$  i.e.,

$$y(n) = x(n) * h(n)$$

$N_1 = 2$  and  $N_2 = 3$ . The length of output sequence  $y(n)$  is  $N = N_1 + N_2 - 1 = 2 + 3 - 1 = 4$ . Therefore, the given sequence  $x(n)$  and  $h(n)$  are converted into four-point sequence by appending zeros.

$$x(n) = \{1, 2, 0, 0\} \quad \text{and} \quad h(n) = \{2, 1, 3, 0\}$$

$$y(n) = \text{IDFT}\{X(k)H(k)\}$$

**Step 1:** To determine  $X(k)$  (Refer Fig. 2.39a)

$$x(n) = \{1, 2, 0, 0\}$$

$$X(k) = \{3, 1 - 2j, -1, 1 + 2j\}$$

**Step 2:** To determine  $H(k)$  (Refer Fig. 2.39b)

$$h(n) = \{2, 1, 3, 0\}$$

$$H(k) = \{6, -1 - j, 4, -1 + j\}$$

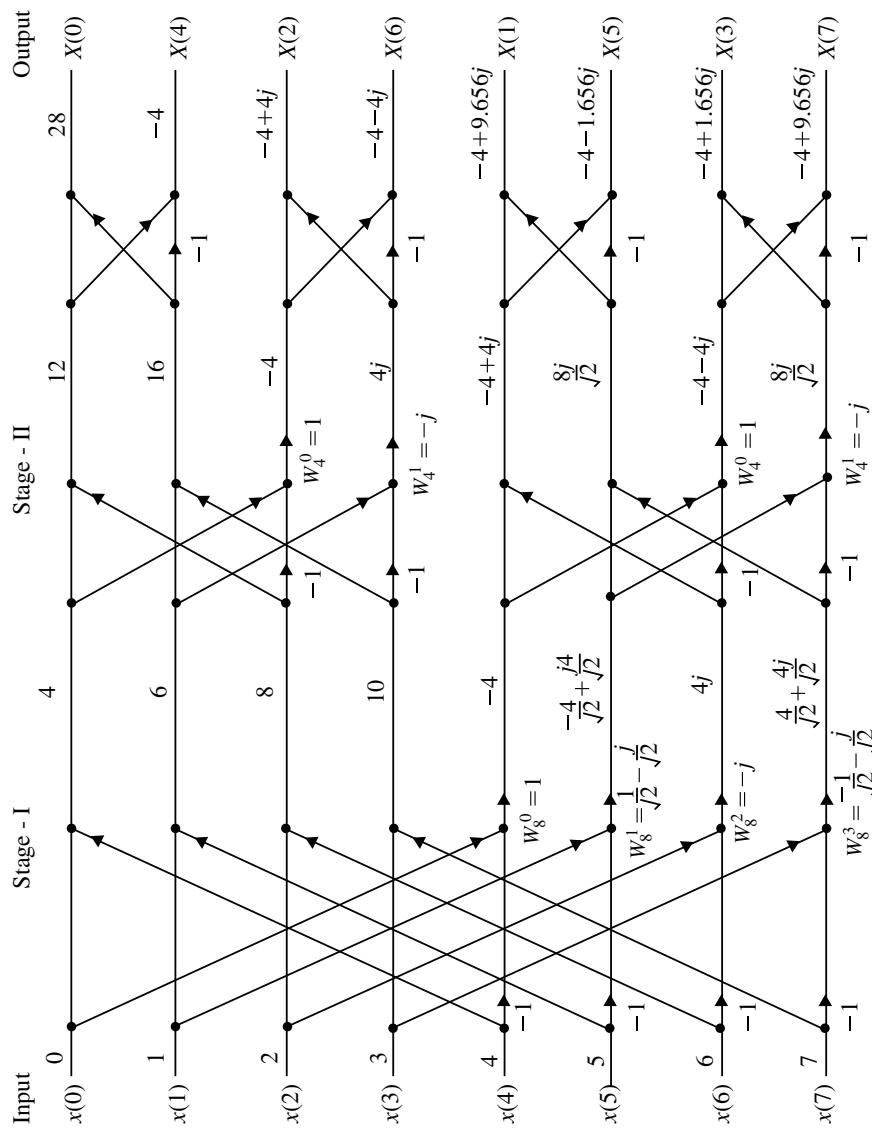
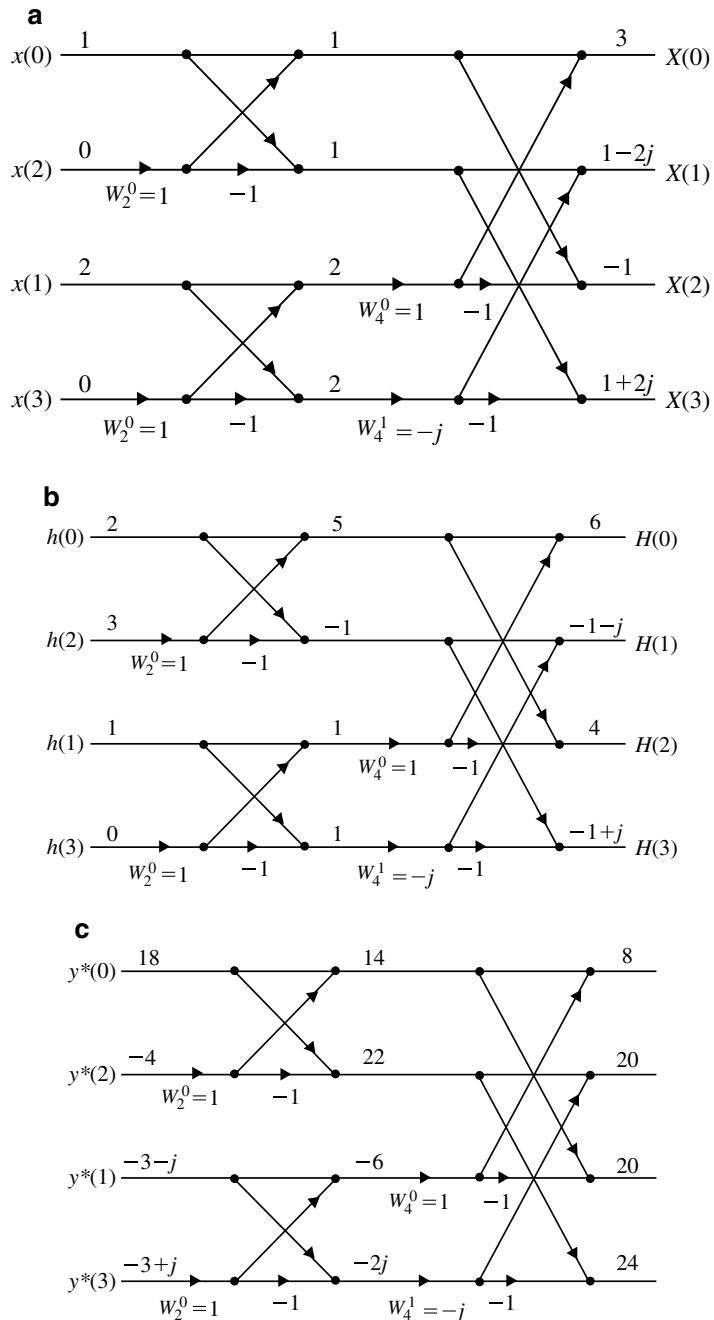


Fig. 2.38 Butterfly diagram for Example 2.41



**Fig. 2.39** **a** Butterfly diagram for Example 2.42. **b** Butterfly diagram for Example 2.42. **c** Butterfly diagram for Example 2.42

**Step 3:**

$$\begin{aligned}
 Y(k) &= X(k)H(k) \\
 &= \{3, (1-2j), -1, (1+2j)\}\{6, (-1-j), 4, (-1+j)\} \\
 &= \{3 \times 6, (1-2j)(-1-j), -1 \times 4, (1+2j)(-1+j)\} \\
 Y(k) &= \{18, -3+j, -4, -3-j\}
 \end{aligned}$$

**Step 4: To determine IDFT of  $Y(k)$**  (Refer Fig. 2.39(c))

The IDFT of  $Y(k)$  is given by

$$\begin{aligned}
 y(n) &= \frac{1}{N} \left[ \sum_{k=0}^{N-1} Y^*(k) W_N^{nk} \right]^* \\
 &= \frac{1}{N} [N \text{ point DFT of } Y^*(k)]^* \\
 Y^*(k) &= \{18, -3-j, -4, -3+j\}
 \end{aligned}$$

The four-point DFT of  $Y^*(k)$  is

Therefore,

$$\begin{aligned}
 y(n) &= \frac{1}{N} [4 \text{ point DFT of } Y^*(k)]^* \\
 &= \frac{1}{4} \{8, 20, 20, 24\}^* \\
 &= \frac{1}{4} \{8, 20, 20, 24\}
 \end{aligned}$$

$y(n) = \{2, 5, 5, 6\}$

**Example 2.43**

Develop a radix-3 DIF-FFT algorithm for evaluating the DFT for  $N = 9$ .

**Solution** For  $N = 9 = 3 \cdot 3$

$$\begin{aligned}
 X(k) &= \sum_{n=0}^2 x(3n) W_9^{3nk} + \sum_{n=0}^2 x(3n+1) W_9^{(3n+1)k} \\
 &\quad + \sum_{n=0}^2 x(3n+2) W_9^{(3n+2)k} \\
 X(k) &= X_1(k) + W_9^k X_2(k) + W_9^{2k} X_3(k)
 \end{aligned}$$

where

$$\begin{aligned} X_1(k) &= \sum_{n=0}^2 x(3n) W_9^{3nk} = x(0) + x(3) W_9^{3k} + x(6) W_9^{6k} \quad k = 0, \dots, 2 \\ X_2(k) &= \sum_{n=0}^2 x(3n+1) W_9^{3nk} = x(1) + x(4) W_9^{3k} + x(7) W_9^{6k} \quad k = 0, \dots, 2 \\ X_3(k) &= \sum_{n=0}^2 x(3n+2) W_9^{3nk} = x(2) + x(5) W_9^{3k} + x(8) W_9^{6k} \quad k = 0, \dots, 2 \end{aligned}$$

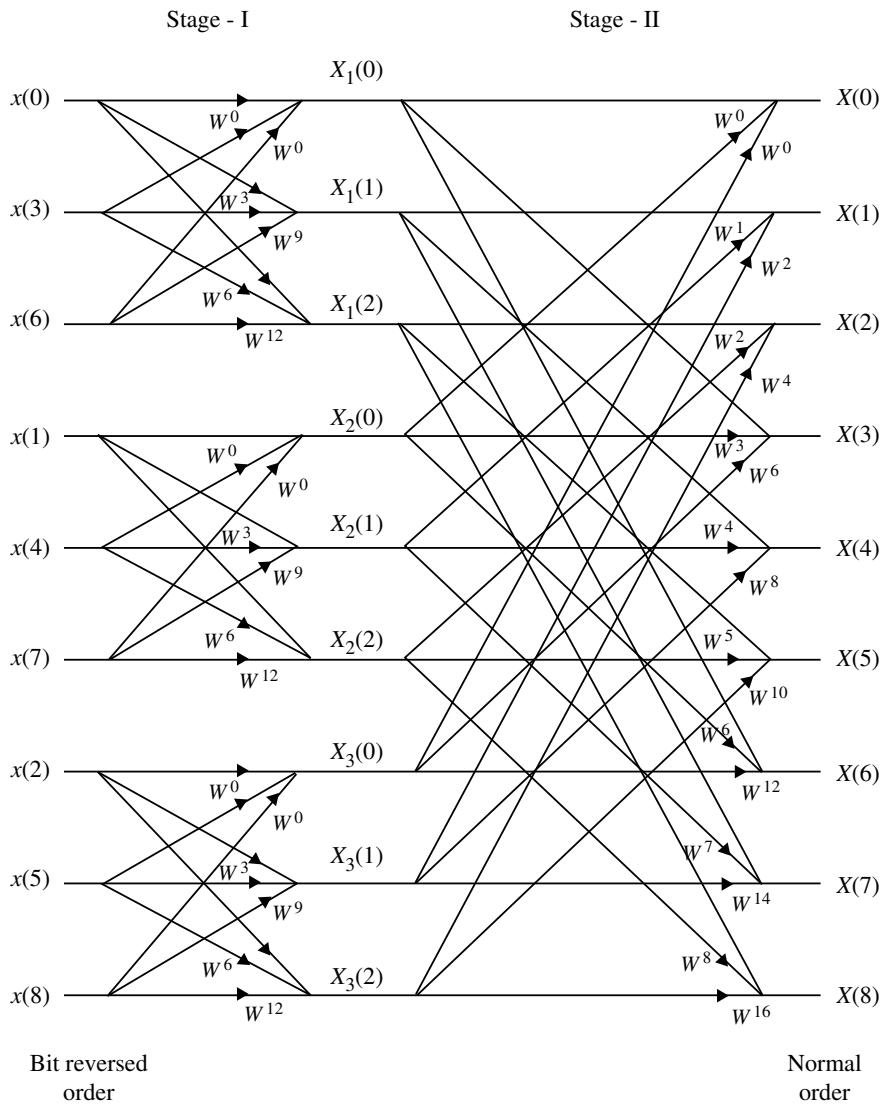
The butterfly diagram is shown in Fig. 2.40.

### Stage-I: Computation

$$\begin{aligned} X_1(k) &= \sum_{n=0}^2 x(3n) W_9^{3nk} \\ &= x(0) + x(3) W_9^{3k} + x(6) W_9^{6k} \quad k = 0, \dots, 2 \end{aligned}$$

$$\begin{aligned} X_1(0) &= x(0) + x(3) W_9^0 + x(6) W_9^0 \\ X_1(1) &= x(0) + x(3) W_9^3 + x(6) W_9^6 \\ X_1(2) &= x(1) + x(3) W_9^6 + x(6) W_9^{12} \\ X_2(k) &= x(1) + x(4) W_9^{3k} + x(7) W_9^{6k}, \quad k = 0, \dots, 2 \\ X_2(0) &= x(1) + x(3) W_9^0 + x(7) W_9^0 \end{aligned}$$

$$\begin{aligned} X_2(1) &= x(1) + x(3) W_9^3 + x(7) W_9^6 \\ X_2(2) &= x(1) + x(3) W_9^6 + x(7) W_9^{12} \\ X_3(k) &= x(2) + x(5) W_9^{3k} + x(8) W_9^{6k}, \quad k = 0, \dots, 2 \\ X_3(0) &= x(2) + x(5) W_9^0 + x(8) W_9^0 \\ X_3(1) &= x(2) + x(5) W_9^3 + x(8) W_9^6 \\ X_3(2) &= x(2) + x(5) W_9^6 + x(8) W_9^{12} \end{aligned}$$

**Fig. 2.40** Radix-3 DIT-FFT flow diagram for  $N = 9$

**Stage-II: Computation**

$$\begin{aligned}
 X(k) &= X_1(k) + W_9^k X_2(k) + W_9^{2k} X_3(k) \quad k = 0, \dots, 8 \\
 X(0) &= X_1(0) + W_9^0 X_2(0) + W_9^0 X_3(0) \\
 X(1) &= X_1(1) + W_9^1 X_2(1) + W_9^2 X_3(1) \\
 X(2) &= X_1(2) + W_9^2 X_2(2) + W_9^4 X_3(2) \\
 X(3) &= X_1(0) + W_9^3 X_2(0) + W_9^6 X_3(0) \\
 X(4) &= X_1(1) + W_9^4 X_2(1) + W_9^8 X_3(1) \\
 X(5) &= X_1(2) + W_9^5 X_2(2) + W_9^{10} X_3(2) \\
 X(6) &= X_1(0) + W_9^6 X_2(0) + W_9^{12} X_3(0) \\
 X(7) &= X_1(1) + W_9^7 X_2(1) + W_9^{14} X_3(1) \\
 X(8) &= X_1(2) + W_9^8 X_2(2) + W_9^{16} X_3(2)
 \end{aligned}$$


---

**Example 2.44**

Develop DIT FFT algorithm for decomposing the DFT for  $N = 6$  and draw the flow diagrams for

- (a)  $N = 2 \cdot 3$
  - (b)  $N = 3 \cdot 2$
  - (c) Also, by using the FFT algorithm developed in part (b); evaluate the DFT values for  $x(n) = \{1, 2, 3, 4, 5, 6\}$ .
- 

**Solution**

- (a) For  $N = 6 = 2 \cdot 3$ , where number of sequence ( $m_1 = 2$ ), number of elements in each subsequences ( $N_1 = 3$ ).

$$X(k) = \sum_{n=0}^2 x(2n) W_6^{2nk} + \sum_{n=0}^2 x(2n+1) W_6^{(2n+1)k} + \sum_{n=0}^2 x(2) W_6^{2nk} + W_6^k \sum_{n=0}^2 x(2n+1) W_6^{2nk}$$

Also,  $X_i(k+3) = X_i(k)$

$$X(k) = X_1(k) + W_6^k X_2(k), \quad k = 0, \dots, 5$$

where

$$\begin{aligned}
 X_1(k) &= \sum_{n=0}^2 x(2n) W_6^{2nk} \quad k = 0, \dots, 2 \\
 X_2(k) &= \sum_{n=0}^2 x(2n+1) W_6^{2nk} \quad k = 0, \dots, 2
 \end{aligned}$$

The butterfly diagram is shown in Fig. 2.41.

### Stage-I: Computation

$$\begin{aligned}
 X_1(k) &= \sum_{n=0}^2 x(2n) W_6^{2nk} \quad k = 0, \dots, 2 \\
 X_1(k) &= x(0) + x(2) W_6^{2k} + x(4) W_6^{4k} \quad k = 0, \dots, 2 \\
 X_1(0) &= x(0) + x(2) W_6^0 + x(4) W_6^0 \\
 X_1(1) &= x(0) + x(3) W_6^2 + x(4) W_6^4 \\
 X_1(2) &= x(1) + x(3) W_6^4 + x(4) W_6^8 \\
 X_1(2) &= x(0) + x(3) W_6^4 + x(4) W_6^2 \\
 X_2(k) &= \sum_{n=0}^2 x(2n+1) W_6^{2nk} \quad k = 0, \dots, 2 \\
 X_2(k) &= x(1) + x(3) W_6^{2k} + x(5) W_6^{4k} \quad k = 0, \dots, 2 \\
 X_2(0) &= x(1) + x(3) W_6^0 + x(5) W_6^0 \\
 X_2(1) &= x(1) + x(3) W_6^2 + x(5) W_6^4 \\
 X_2(2) &= x(1) + x(3) W_6^4 + x(5) W_6^8 \\
 X_2(2) &= x(1) + x(3) W_6^4 + x(5) W_6^2
 \end{aligned}$$

### Stage-II: Computation

$$\begin{aligned}
 X(k) &= X_1(k) + W_6^k X_2(k), \quad k = 0, \dots, 5 \\
 X(0) &= X_1(0) + W_6^0 X_2(0) \\
 X(1) &= X_1(1) + W_6^1 X_2(1) \\
 X(2) &= X_1(2) + W_6^2 X_2(2) \\
 X(3) &= X_1(3) + W_6^3 X_2(3) = X_1(0) + W_6^3 X_2(0) \\
 X(4) &= X_1(4) + W_6^4 X_2(4) = X_1(1) + W_6^4 X_2(1) \\
 X(5) &= X_1(5) + W_6^5 X_2(5) = X_1(2) + W_6^5 X_2(2)
 \end{aligned}$$

- (b) For  $N = 6 = 2 \cdot 3$ , where number of sequence ( $m_1 = 3$ ), number of elements in each subsequences ( $N_1 = 2$ ).

$$\begin{aligned}
 X(k) &= \sum_{n=0}^1 x(3n) W_6^{3nk} + \sum_{n=0}^1 x(3n+1) W_6^{(3n+1)k} + \sum_{n=0}^1 x(3n+2) W_6^{(3n+2)k} \\
 &= \sum_{n=0}^1 x(3n) W_6^{3nk} + W_6^k \sum_{n=0}^1 x(3n+1) W_6^{3nk} + W_6^{2k} \sum_{n=0}^1 x(3n+2) W_6^{3nk} \\
 X(k) &= X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k)
 \end{aligned}$$

where

$$\begin{aligned} X_1(k) &= \sum_{n=0}^1 x(3n) W_6^{3nk} = x(0) + x(3) W_6^{3k} \\ X_2(k) &= \sum_{n=0}^1 x(3n+1) W_6^{3nk} = x(1) + x(4) W_6^{3k} \\ X_3(k) &= \sum_{n=0}^1 x(3n+2) W_6^{3nk} = x(2) + x(5) W_6^{3k} \end{aligned}$$

Also,  $X_i(k+2) = X_i(k)$ .

### Stage-I: Computation

$$\begin{aligned} X_1(k) &= x(0) + x(3) W_6^{3k} & k = 0, 1 \\ X_1(0) &= x(0) + x(3) W_6^0 \\ X_1(1) &= x(0) + x(3) W_6^3 \\ X_2(k) &= x(1) + x(4) W_6^{3k} & k = 0, 1 \\ X_2(0) &= x(1) + x(4) W_6^0 \\ X_2(1) &= x(1) + x(4) W_6^3 \\ X_3(k) &= x(2) + x(5) W_6^{3k} & k = 0, 1 \\ X_3(0) &= x(2) + x(5) W_6^0 \\ X_3(1) &= x(2) + x(5) W_6^3 \end{aligned}$$

The butterfly diagram is shown in Fig. 2.42.

### Stage-II: Computation

$$\begin{aligned} X(k) &= X_1(k) + W_6^k X_2(k) + W_6^{2k} X_3(k), & k = 0, \dots, 5 \\ X(0) &= X_1(0) + W_6^0 X_2(0) + W_6^0 X_3(0) \\ X(1) &= X_1(1) + W_6^1 X_2(1) + W_6^2 X_3(1) \\ X(2) &= X_1(2) + W_6^2 X_2(2) + W_6^4 X_3(2) \\ &= X_1(0) + W_6^2 X_2(0) + W_6^4 X_3(0) \\ X(3) &= X_1(3) + W_6^3 X_2(3) + W_6^6 X_3(3) \\ &= X_1(0) + W_6^3 X_2(1) + W_6^0 X_3(0) \\ X(4) &= X_1(4) + W_6^4 X_2(4) + W_6^8 X_3(4) \\ &= X_1(0) + W_6^4 X_2(0) + W_6^2 X_3(0) \\ X(5) &= X_1(5) + W_6^5 X_2(5) + W_6^{10} X_3(5) \\ &= X_1(1) + W_6^5 X_2(1) + W_6^4 X_3(1) \end{aligned}$$

(c) To evaluate the DFT values for  $x(n)$  (Refer Fig. 2.43)

$$x(n) = \{1, 2, 3, 4, 5, 6\}$$

by using the FFT algorithm developed in part (b) twiddle factors are,

$$\begin{aligned}W_6^0 &= 1 \\W_6^1 &= e^{-\frac{j2\pi}{6}} = 0.5 - j0.866 \\W_6^2 &= e^{-\frac{j2\pi \cdot 2}{6}} = -0.5 - j0.866 \\W_6^3 &= e^{-\frac{j2\pi \cdot 3}{6}} = -1 \\W_6^4 &= e^{-\frac{j2\pi \cdot 4}{6}} = -0.5 + j0.866 \\W_6^5 &= e^{-\frac{j2\pi \cdot 5}{6}} = 0.5 + j0.866\end{aligned}$$

Input	Stage-I	Stage-II (or) Output
0	$1 + 4 = 5$	$5 + 7 + 9 = 21 = X(0)$
4	$1 - 4 = -3$	$-3 + (0.5 - j0.866)(-3) + (-0.5 - j0.866)(-3)$ $= -3 + j5.196 = X(1)$
2	$2 + 5 = 7$	$5 + (0.5 - j0.866)(7) + (-0.5 + j0.866)(9)$ $= -3 + j1.732 = X(2)$
5	$2 - 5 = -3$	$-3 + (-1)(-3) + (1)(-3) = -3 = X(3)$
3	$3 + 6 = 9$	$5 + (0.5 + j0.866)(7) + (-0.5 - j0.866)(9)$ $= -3 - j1.732 = X(4)$
6	$3 - 6 = -3$	$-3 + (0.5 + j0.866)(-3) + (-0.5 + j0.866)(-3)$ $= -3 - j5.196 = X(5)$

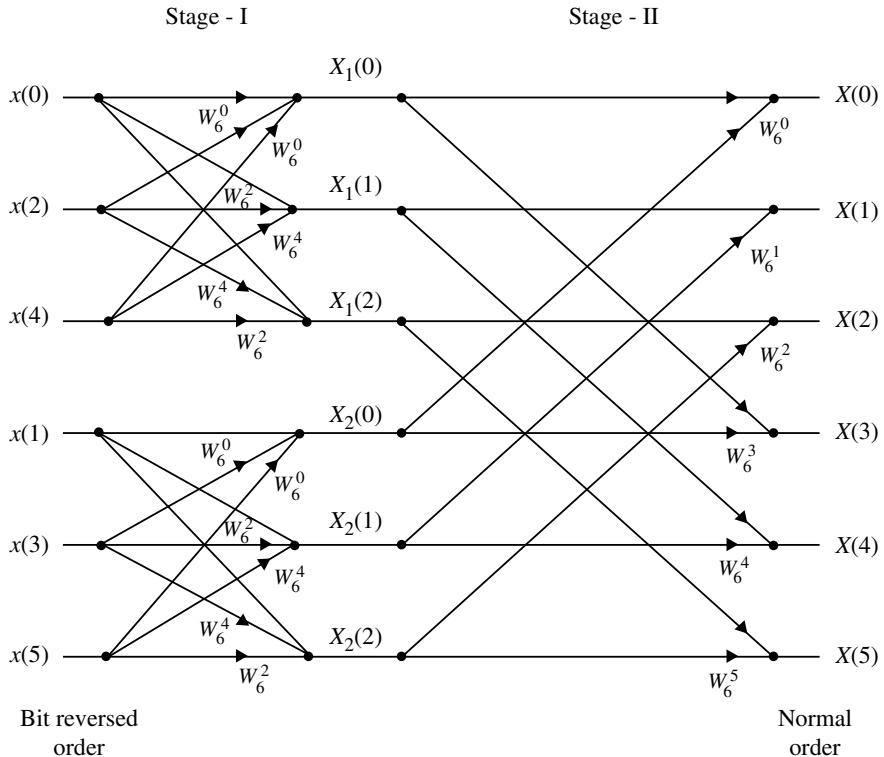
$$X(k) = \{21, -3 + j5.196, -3 + j1.732, -3, -3 - j1.732, -3 - j5.196\}$$

### Example 2.45

Develop the DIT-FFT algorithm for decomposing the DFT for  $N = 12$  and draw the flow diagrams.

**Solution** For  $N = 12 = 3 \cdot 4$  where  $m_1 = 3$  and  $N_1 = 4$

$$\begin{aligned}X(k) &= \sum_{n=0}^3 x(3n) W_{12}^{3nk} + \sum_{n=0}^3 x(3n+1) W_{12}^{(3n+1)k} + \sum_{n=0}^3 x(3n+2) W_{12}^{(3n+2)k} \\&= \sum_{n=0}^3 x(3n) W_{12}^{3nk} + W_{12}^k \sum_{n=0}^3 x(3n+1) W_{12}^{3nk} + W_{12}^{2k} \sum_{n=0}^3 x(3n+2) W_{12}^{3nk}\end{aligned}$$



**Fig. 2.41** DIT-FFT flow diagram for  $N = 6 = 2 \cdot 3$

$$X(k) = X_1(k) + W_{12}^k X_2(k) + W_{12}^{2k} X_3(k)$$

where

$$\begin{aligned}
 X_1(k) &= \sum_{n=0}^3 x(3n) W_{12}^{3nk} \quad k = 0, \dots, 3 \\
 &= x(0) + W_{12}^{3k} x(3) + W_{12}^{6k} x(6) + W_{12}^{9k} x(9) \\
 X_2(k) &= \sum_{n=0}^3 x(3n+1) W_{12}^{3nk} \\
 &= x(1) + x(4) W_{12}^{3k} + x(7) + W_{12}^{6k} + x(0) W_{12}^{7k} \\
 X_3(k) &= \sum_{n=0}^3 x(3n+2) W_{12}^{3nk} \\
 &= x(2) + W_{12}^{3k} x(5) + W_{12}^{6k} x(8) + W_{12}^{9k}
 \end{aligned}$$

The butterfly diagram is shown in Fig. 2.44.

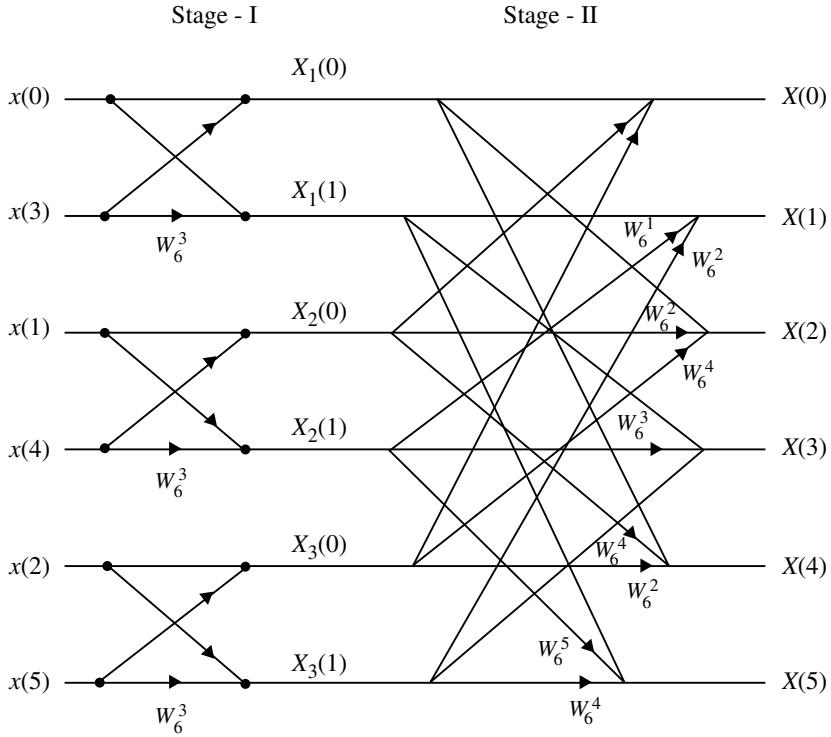
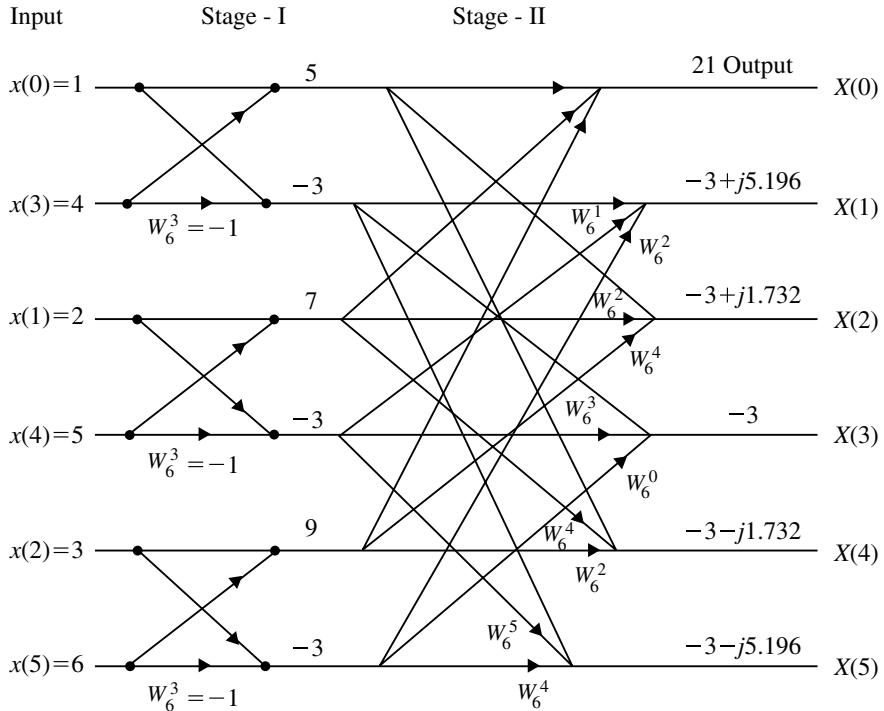


Fig. 2.42 DIT-FFT flow diagram for  $N = 6 = 3 \cdot 2$

### Stage-I: Computation

$$\begin{aligned}
 X_1(k) &= x(0) + W_{12}^{3k}x(3) + W_{12}^{6k}x(6) + W_{12}^{9k}x(9) \quad k = 0, \dots, 3 \\
 X_1(0) &= x(0) + x(3) + x(6) + x(9) \\
 X_1(1) &= x(0) + W_{12}^3x(3) + W_{12}^6x(6) + W_{12}^9x(9) \\
 X_1(2) &= x(0) + W_{12}^6x(3) + W_{12}^0x(6) + W_{12}^6x(9) \\
 X_1(3) &= x(0) + W_{12}^9x(3) + W_{12}^6x(6) + W_{12}^3x(9) \\
 X_2(k) &= x(1) + x(4)W_{12}^{3k} + x(7)W_{12}^{6k} + x(10)W_{12}^{9k} \quad k = 0, \dots, 5 \\
 X_2(0) &= x(1) + x(4) + x(7) + x(10) \\
 X_2(1) &= x(1) + x(4)W_{12}^3 + x(7)W_{12}^6 + x(10)W_{12}^9 \\
 X_2(2) &= x(1) + x(4)W_{12}^6 + x(7)W_{12}^0 + x(10)W_{12}^6 \\
 X_2(3) &= x(1) + x(4)W_{12}^9 + x(7) + W_{12}^6 + x(10)W_{12}^3 \\
 X_3(k) &= x(2) + x(5)W_{12}^{3k} + x(8)W_{12}^{6k} + x(11)W_{12}^{9k} \quad k = 0, \dots, 5 \\
 X_3(0) &= x(2) + x(5) + x(8) + x(11) \\
 X_3(1) &= x(2) + x(5)W_{12}^3 + x(8)W_{12}^6 + x(11)W_{12}^9
 \end{aligned}$$



**Fig. 2.43** Butterfly diagram for Example 2.44(c)

$$X_3(2) = x(2) + x(5)W_{12}^6 + x(8)W_{12}^0 + x(11)W_{12}^6$$

$$X_3(3) = x(2) + x(5)W_{12}^9 + x(8)W_{12}^6 + x(11)W_{12}^3$$

### Stage-II: Computation

$$X(k) = X_1(k) + W_{12}^k X_2(k) + W_{12}^{2k} X_3(k), \quad k = 0, \dots, 11$$

$$X(0) = X_1(0) + X_2(0) + X_3(0)$$

$$X(1) = X_1(1) + W_{12}^1 X_2(1) + W_{12}^2 X_3(1)$$

$$X(2) = X_1(2) + W_{12}^2 X_2(2) + W_{12}^4 X_3(2)$$

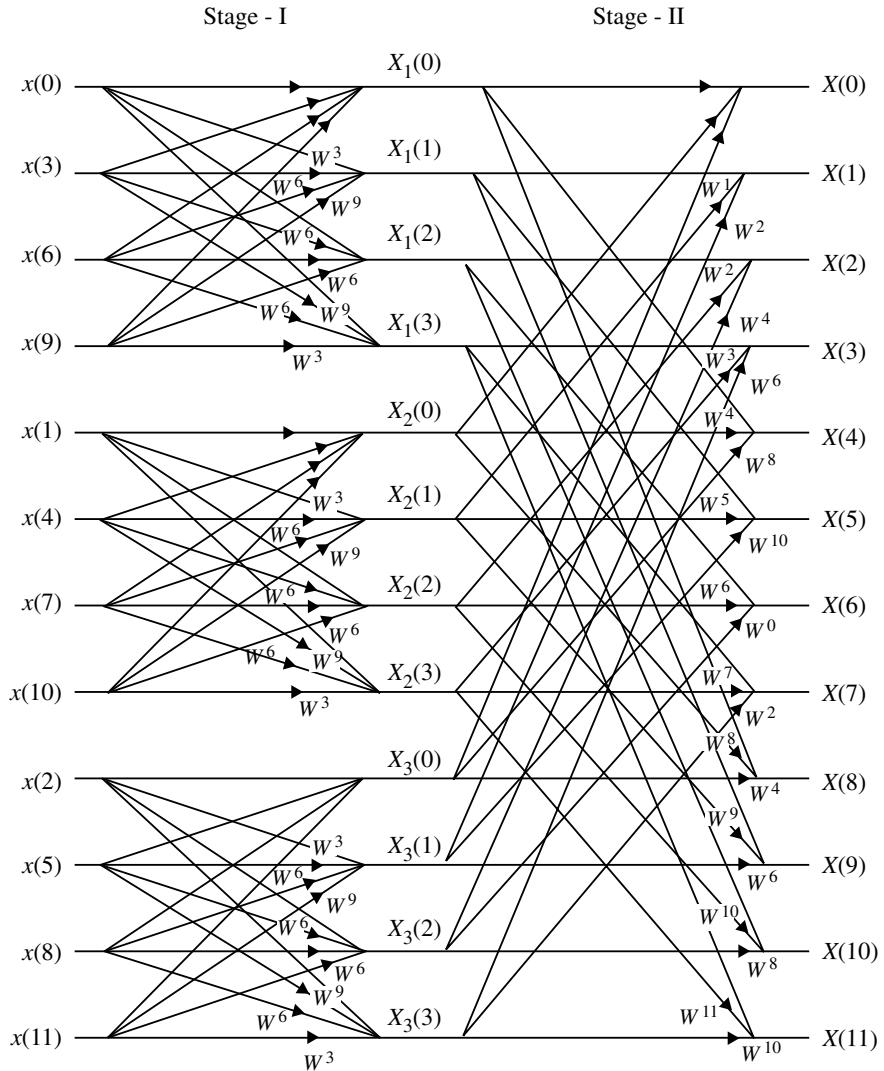
$$X(3) = X_1(3) + W_{12}^3 X_2(3) + W_{12}^6 X_3(3)$$

$$X(4) = X_1(0) + W_{12}^4 X_2(0) + W_{12}^8 X_3(0)$$

$$X(5) = X_1(1) + W_{12}^5 X_2(1) + W_{12}^{10} X_3(1)$$

$$X(6) = X_1(2) + W_{12}^6 X_2(2) + W_{12}^0 X_3(2)$$

$$X(7) = X_1(3) + W_{12}^7 X_2(3) + W_{12}^2 X_3(3)$$



**Fig. 2.44** Flow diagram of 12-point composite radix-FFT

$$X(8) = X_1(0) + W_{12}^8 X_2(0) + W_{12}^4 X_3(0)$$

$$X(9) = X_1(1) + W_{12}^9 X_2(1) + W_{12}^6 X_3(1)$$

$$X(10) = X_1(2) + W_{12}^{10} X_2(2) + W_{12}^8 X_3(2)$$

$$X(11) = X_1(3) + W_{12}^{11} X_2(3) + W_{12}^{10} X_3(3)$$

**Example 2.46**

Develop a radix-4 DIT-FFT algorithm for evaluating the DFT for  $N = 16$ , and hence, determine the 16-point DFT of the sequence

$$x(n) = \{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1\}$$

**Solution**

For  $N = 16 = 4 \cdot 4$  where  $m_1 = 4$  and  $N_1 = 4$

$$\begin{aligned} X(k) &= \sum_{n=0}^3 x(4n)W_{16}^{4nk} + \sum_{n=0}^3 x(4n+1)W_{16}^{(4n+1)k} \\ &\quad + \sum_{n=0}^3 x(4n+2)W_{16}^{(4n+2)k} + \sum_{n=0}^3 x(4n+3)W_{16}^{(4n+3)k} \\ X_1(k) &= \sum_{n=0}^3 x(4n)W_{16}^{4nk} + W_{16}^k \sum_{n=0}^3 x(4n+1)W_{16}^{4nk} \\ &\quad + W_{16}^{2k} \sum_{n=0}^3 x(4n+2)W_{16}^{4nk} + W_{16}^{3k} \sum_{n=0}^3 x(4n+3)W_{16}^{4nk} \\ &= X_1(k) + W_{16}^k X_2(k) + W_{16}^{2k} X_3(k) + W_{16}^{3k} X_4(k) \end{aligned}$$

The twiddle factors are,

$$W_{16}^0 = 1$$

$$W_{16}^1 = e^{-j\frac{2\pi}{16}} = (0.923 - j0.382)$$

$$W_{16}^2 = e^{-j\frac{2\pi \cdot 2}{16}} = 0.707 - j0.707$$

$$W_{16}^3 = e^{-j\frac{2\pi \cdot 3}{16}} = 0.3826 - j0.923$$

$$W_{16}^4 = e^{-j\frac{2\pi \cdot 4}{16}} = -j$$

$$W_{16}^5 = -0.3826 - j0.9238$$

$$W_{16}^6 = e^{-j\frac{2\pi \cdot 6}{16}} = -0.707 - j0.707$$

$$W_{16}^7 = e^{-j\frac{2\pi \cdot 7}{16}} = 0.9238 - j0.382$$

The radix-4 DIT-FFT algorithm is shown in Fig. 2.45.

Input	Stage-I	Stage-II (or) Output
0	$0 + 0 + 0 + 0 = 0$	$0 + 4 + 0 + 4 = 8 = X(1)$
0	$0 + 0 + 0 + 0 = 0$	$0 + (0.923 - j0.382)(-j2) + (0.707 - j0.707)(0) + (0.3826 - j0.923)(-j2)$ $= -2.61 - j2.61 = X(1)$
0	$0 + 0 + 0 + 0 = 0$	$0 + 0 + 0 + 0 = 0 = X(2)$
0	$0 + 0 + 0 + 0 = 0$	$0 + 0 + 0 + 0 = 0 = X(3)$
1	$1 + 1 + 1 + 1 = 4$	$0 + (-j)4 + (-1)(0) + (j)4 = 0 = X(4)$
1	$1 - j - 1 - j = -2j$	$0 + (-j2)(-0.3826 - j0.9238) + 0 + (-j2)(0.9038 + j0.382) = -j1.0824 - 1.0836$ $= X(5)$
1	$1 + 1(-1) + 1(1) + 1(-1) = 0$	$0 + 0 + 0 + 0 = 0 = X(6)$
1	$1 + 1(-j) + 1(-1) + 1(j) = 0$	$0 + 0 + 0 + 0 = 0 = X(7)$
0	0	$0 + 4(-1) + 0 + 4(1) = 0 = X(8)$
0	0	$0 + (-j2)(-0.923 + j0.9826) + 0 + (-j2)(-0.382 + j0.923) = 2.6112 + j2.6112$ $= X(9)$
0	0	$0 + 0 + 0 + 0 = 0 = X(10)$
0	0	$0 + 0 + 0 + 0 = 0 = X(11)$
1	$1 + 1 + 1 + 1 = 4$	$0 + 4(-j) + 0 + j4 = 0 = X(12)$
1	$1 + 1(-j) + 1(-1) + 1(-1) = -j2$	$0 + (-j2)(0.3826 + j0.923) + 0 + (-j2)(-0.9238 - j0.382) = 1.0818 + j1.0848$ $= X(13)$
1	$1 + 1(-1) + 1(1) + 1(-1) = 0$	$0 + 0 + 0 + 0 = 0 = X(14)$
1	$1 + 1(-j) + 1(-1) + 1(j) = 0$	$0 + 0 + 0 + 0 = 0 = X(15)$

$$W_{16}^8 = e^{-\frac{j2\pi \cdot 8}{16}} = -1$$

$$W_{16}^9 = e^{-\frac{j2\pi \cdot 9}{16}} = -0.923 + j0.3826$$

$$W_{16}^{10} = -0.707 + j0.707$$

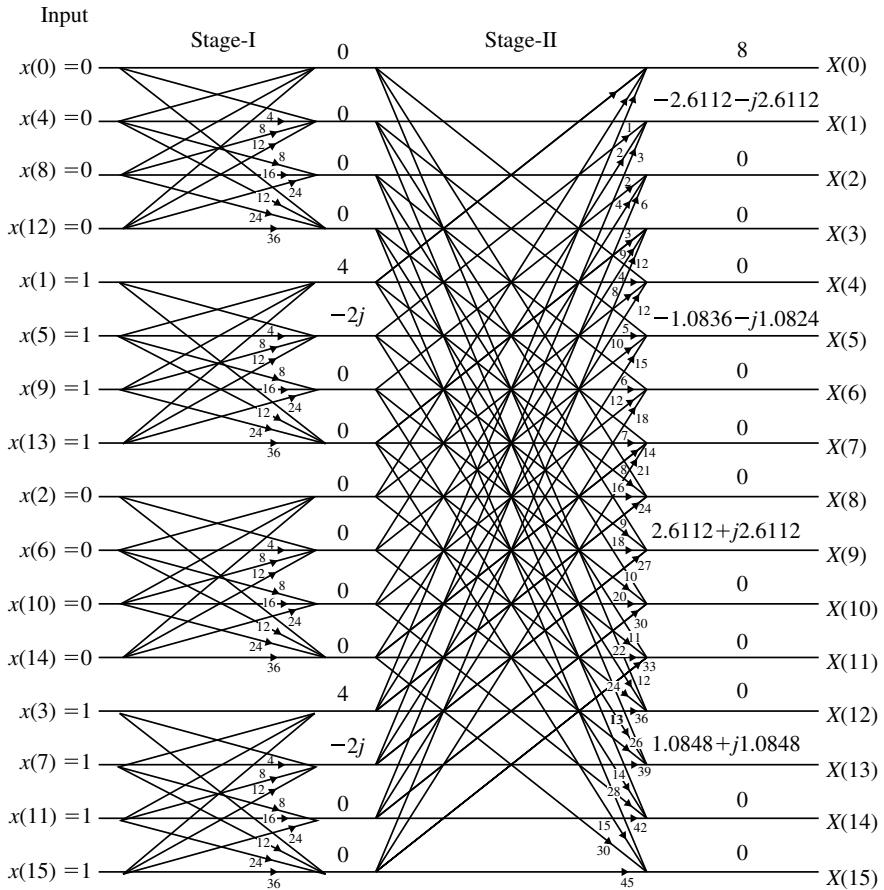
$$W_{16}^{11} = -0.3826 + j0.923$$

$$W_{16}^{12} = -j$$

$$W_{16}^{13} = 0.3826 + j0.923$$

$$W_{16}^{14} = 0.707 + j0.707$$

$$W_{16}^{15} = 0.9238 + j0.382$$



**Fig. 2.45** Radix-4 DIT-FFT algorithm for Example 2.46

### Example 2.47

Develop a DIF-FFT algorithm for decomposing the DFT for  $N = 6$  and draw the flow diagram for (a)  $N = 3 \cdot 2$  and (b)  $N = 2 \cdot 3$ .

### Solution

(a) To develop DIF-FFT algorithm for  $N = 3 \cdot 2$

$$X(k) = \sum_{n=0}^5 x(n) W_6^{nk} = \sum_{n=0}^2 x(n) W_6^{nk} + \sum_{n=3}^5 x(n) W_6^{nk}$$

$$\begin{aligned}
&= \sum_{n=0}^2 x(n) W_6^{nk} + \sum_{n=0}^2 x(n+3) W_6^{(n+3)k} \\
X(k) &= \sum_{n=0}^2 [x(n) + x(n+3) W_6^{3k}] W_6^{nk} \\
X(2k) &= \sum_{n=0}^2 [x(n) + x(n+3) W_6^{6k}] + W_6^{2nk} \\
&= \sum_{n=0}^2 [x(n) + x(n+3)] W_6^{2nk} \\
X(2k+1) &= \sum_{n=0}^2 [x(n) + x(n+3) W_6^{3(2k+1)}] W_6^{(2k+1)n} \\
&= \sum_{n=0}^2 [x(n) - x(n+3)] W_6^n W_6^{2nk}
\end{aligned}$$

Let  $g(n) = x(n) + x(n+3)$ ,  $h(n) = x(n) - x(n+3)$

$$\begin{aligned}
X(2k) &= \sum_{n=0}^2 g(n) W_6^{2nk} \\
X(2k+1) &= \sum_{n=0}^2 h(n) W_6^n W_6^{2nk}
\end{aligned}$$

The butterfly diagram is shown in Fig. 2.46.

### Stage I

$$\begin{aligned}
g(0) &= x(0) + x(3) & h(0) &= x(0) - x(3) \\
g(1) &= x(1) + x(4) & h(1) &= x(1) - x(4) \\
g(2) &= x(2) + x(5) & h(2) &= x(2) - x(5)
\end{aligned}$$

### Stage II

$$\begin{aligned}
X(0) &= g(0) + g(1) + g(2) \\
X(2) &= g(0) + g(1) W_6^2 + g(2) W_6^4 \\
X(4) &= g(0) + g(1) W_6^4 + g(2) W_6^8 \\
X(1) &= h(0) + h(1) W_6^1 + h(2) W_6^2 \\
X(3) &= h(0) + h(1) W_6^1 W_6^2 + h(2) W_6^2 W_6^4 \\
X(5) &= h(0) + h(1) W_6^1 W_6^4 + h(2) W_6^2 W_6^8
\end{aligned}$$

The DIF-FFT flow diagram for decomposing the DFT is shown in Fig. 2.47 for  $N = 6 = 3 \times 2$ .

- (b) To develop DIF-FFT algorithm for  $N = 2 \cdot 3$

$$\begin{aligned} X(k) &= \sum_{n=0}^5 x(n) W_6^{nk} = \sum_{n=0}^1 x(n) W_6^{nk} + \sum_{n=2}^3 x(n) W_6^{nk} + \sum_{n=4}^5 x(n) W_6^{nk} \\ &= \sum_{n=0}^1 [x(n) + x(n+2) W_6^{2k} + x(n+4) W_6^{4k}] W_6^{nk} \\ X(3k) &= \sum_{n=0}^1 [x(n) + x(n+2) + x(n+4)] W_6^{3nk} \\ \\ X(3k+1) &= \sum_{n=0}^1 [x(n) + x(n+2) W_6^2 + x(n+4) W_6^4] W_6^n W_6^{3nk} \\ X(3k+2) &= \sum_{n=0}^1 [x(n) + x(n+2) W_6^4 + x(n+4) W_6^2] W_6^{2n} W_6^{3nk} \end{aligned}$$

Let

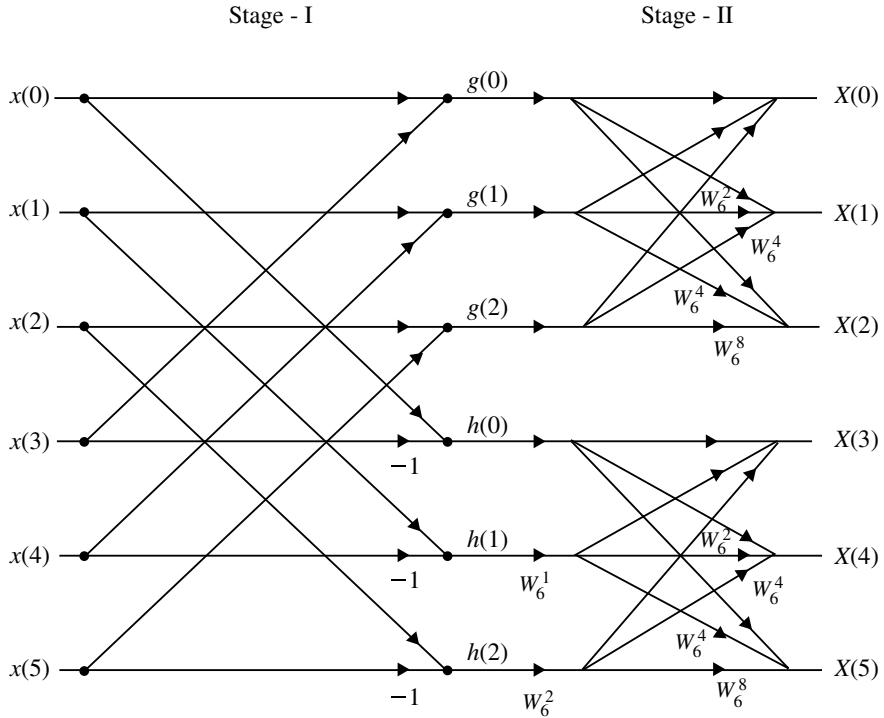
$$\begin{aligned} f(n) &= x(n) + x(n+2) + x(n+4), \quad n = 0, 1, \dots \\ g(n) &= x(n) + x(n+2) + W_6^2 + x(n+4) W_6^4, \quad n = 0, 1, \dots \\ h(n) &= x(n) + x(n+2) W_6^4 + x(n+4) W_6^2, \quad n = 0, 1, \dots \end{aligned}$$

### Stage I

$$\begin{aligned} f(0) &= x(0) + x(2) + x(4) \\ f(1) &= x(1) + x(3) + x(5) \\ g(0) &= x(0) + x(2) W_6^2 + x(4) W_6^4 \\ g(1) &= x(1) + x(3) W_6^2 + x(5) W_6^4 \end{aligned}$$

The DIF-FFT flow diagram for decomposing the DFT for  $N = 6 = 2 \times 3$  is shown in Fig. 2.47.

$$\begin{aligned} h(0) &= x(0) + x(2) W_6^4 + x(4) W_6^2 \\ h(1) &= x(1) + x(3) W_6^4 + x(5) W_6^2 \end{aligned}$$



**Fig. 2.46** DIF-FFT flow diagram for decomposing the DFT for  $N = 6 = 3 \cdot 2$

### Stage II

$$\begin{aligned}
 X(0) &= f(0) + f(1)W_6^0 \\
 X(1) &= g(0) + g(1)W_6^1 \\
 X(2) &= h(0) + h(1)W_6^2 \\
 X(3) &= f(0) + f(1)W_6^0W_6^3 \\
 X(4) &= g(0) + g(1)W_6^1W_6^3 \\
 X(5) &= h(0) + h(1)W_6^2W_6^3
 \end{aligned}$$

### Example 2.48

Develop a radix-3 DIF-FFT algorithm for evaluating the DFT for  $N = 9$ .

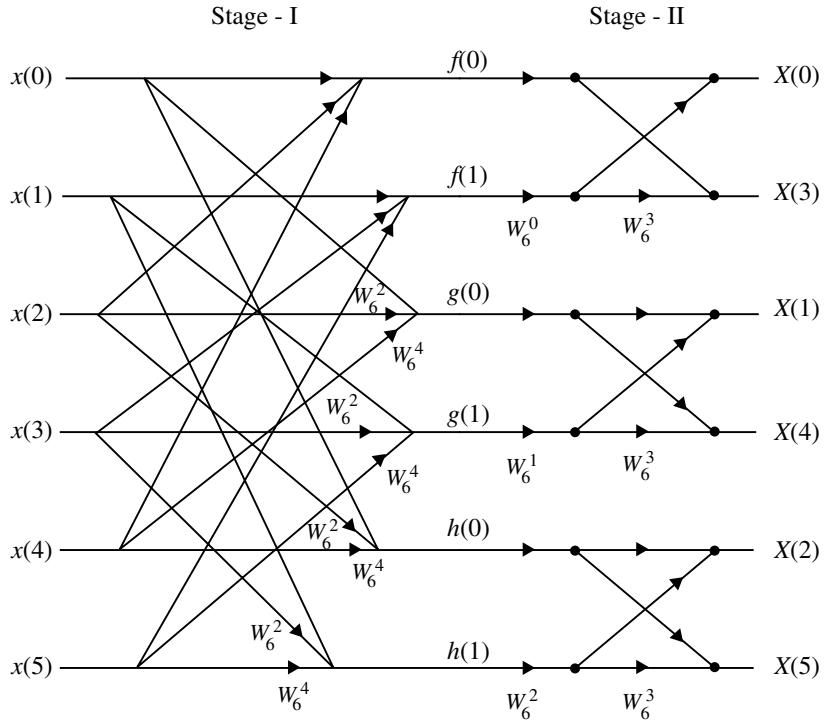
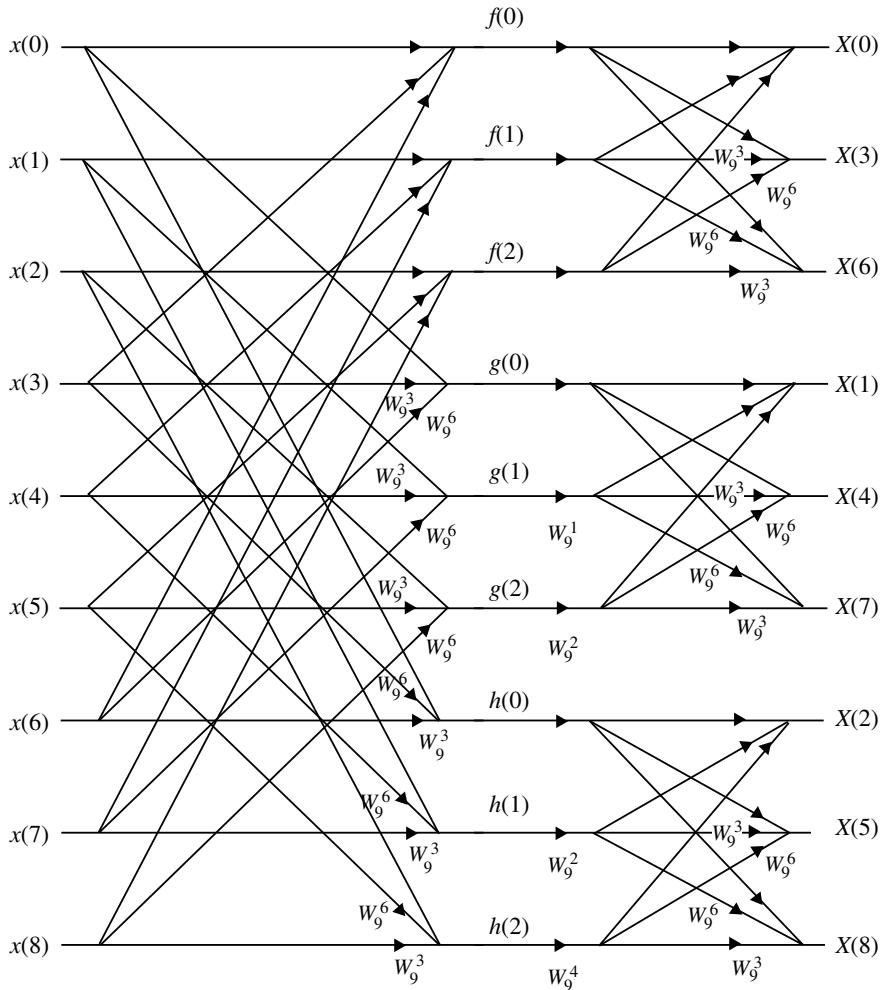


Fig. 2.47 DIF-FFT flow diagram for decomposing the DFT for \$N = 6 = 2 \cdot 3\$

**Solution** To develop a radix-3 DIF-FFT algorithm for \$N = 9 = 3 \cdot 3\$

$$\begin{aligned}
 X(k) &= \sum_{n=0}^2 x(n) W_9^{nk} + \sum_{n=0}^2 x(n+3) W_9^{(n+3)k} + \sum_{n=0}^2 x(n+6) W_9^{(n+6)k} \\
 X(k) &= \sum_{n=0}^2 [x(n) + x(n+3) W_9^{3k} + x(n+6) W_9^{6k}] W_9^{nk} \\
 X(3k) &= \sum_{n=0}^2 [x(n) + x(n+3) + x(n+6)] W_9^{3nk} \\
 &= \sum_{n=0}^2 f(n) W_9^{3nk} \\
 X(3k+1) &= \sum_{n=0}^2 [x(n) + x(n+3) W_9^3 + x(n+6) W_9^6] W_9^n W_9^{3nk} \\
 &= \sum_{n=0}^2 g(n) W_9^n W_9^{3nk}
 \end{aligned}$$



**Fig. 2.48** A radix-3 DIF-FFT flow diagram for decomposing the DFT for  $N = 9 = 3 \cdot 3$

$$\begin{aligned}
 X(3k+2) &= \sum_{n=0}^2 [x(n) + x(n+3)W_9^6 + x(n+6)W_9^3]W_9^{2n}W_9^{3nk} \\
 &= \sum_{n=0}^2 h(n)W_9^{2n}W_9^{3nk}
 \end{aligned}$$

The butterfly diagram is shown in Fig. 2.48.

**Stage I**

$$\begin{aligned}
 f(0) &= x(0) + x(3) + x(6) \\
 f(1) &= x(1) + x(4) + x(7) \\
 f(2) &= x(2) + x(5) + x(8) \\
 g(0) &= x(0) + x(3)W_9^3 + x(6)W_9^6 \\
 g(1) &= x(1) + x(4)W_9^3 + x(7)W_9^6 \\
 g(2) &= x(2) + x(5)W_9^3 + x(8)W_9^6 \\
 h(0) &= x(0) + x(3)W_9^6 + x(6)W_9^3 \\
 h(1) &= x(1) + x(4)W_9^6 + x(7)W_9^3 \\
 h(2) &= x(2) + x(5)W_9^6 + x(8)W_9^3
 \end{aligned}$$

**Stage II**

$$\begin{aligned}
 X(0) &= f(0) + f(1) + f(2) \\
 X(3) &= f(0) + f(1)W_9^3 + f(2)W_9^6 \\
 X(6) &= f(0) + f(1)W_9^0 + f(2)W_9^3 \\
 X(1) &= g(0) + g(1)W_9^1 + g(2)W_9^2 \\
 X(4) &= g(0) + g(1)W_9^1W_9^3 + g(2)W_9^2W_9^6 \\
 X(7) &= g(0) + g(1)W_9^1W_9^6 + g(2)W_9^2W_9^3 \\
 X(2) &= h(0) + h(1)W_9^2 + h(2)W_9^4 \\
 X(5) &= h(0) + h(1)W_9^2W_9^3 + h(2)W_9^4W_9^6 \\
 X(8) &= h(0) + h(1)W_9^2W_9^6 + h(2)W_9^4W_9^3
 \end{aligned}$$

#### **2.6.4 Use of the FFT Algorithm in Linear Filtering and Correlation**

An important application of the FFT algorithm is in FIR linear filtering of long data sequences. The response of an LTI system for any arbitrary input is given by linear convolution of the input and the impulse response of the system. If one of the sequences (either the input sequence or impulse response sequence) is very much larger than the other, then it is very difficult to compute the linear convolution using DFT for the following reasons:

1. The entire sequence should be available before convolution can be carried out.  
This makes long delay in getting the output.
2. Large amounts of memory is required to store the sequences.

The above problems can be overcome by linear filtering of longer sequence into the size of smaller sequences. Then the linear convolution of each section of

longer sequences and the smaller sequence is performed. The output sequences obtained from the convolutions of the sections are combined to get the overall output sequences. There are two methods to perform the linear filtering. They are,

1. Overlap add method.
2. Overlap save method.

#### **2.6.4.1 Overlap Add Method**

Let  $N_1$  be the length of longer sequence and  $N_2$ , the length of smaller sequence. Let the longer sequence be divided into sections of size  $N_3$  samples. (*Note:* Normally the longer sequence is divided into sections of size same as that of smaller sequence.).

The linear convolution of each section with smaller sequence will produce an output sequence of size  $N_3 + N_2 - 1$  samples. In this method last  $N_2 - 1$  samples of each output sequence overlaps with the first  $N_2 - 1$  samples of next section. While combining the output sequences of the various sectioned convolutions, the corresponding samples of overlapped regions are added and the samples of non-overlapped regions are retained as such.

#### **2.6.4.2 Overlap Save Method**

Let  $N_1$  be the length of longer sequence and  $N_2$ , the length of smaller sequence. Let the longer sequence be divided into sections of size  $N_3$  samples.

In overlap save method, the result of linear convolution is obtained by circular convolution. Hence, each section of longer sequence and the smaller sequence are converted to the size of the output sequence of size  $N_3 + N_2 - 1$  samples. The smaller sequence is converted to size of  $N_3 + N_2 - 1$  samples by appending with zeros. The conversion of each section of longer sequence to the size  $N_3 + N_2 - 1$  samples can be performed in two different methods.

##### **Method I**

In this method the first  $N_2 - 1$  samples of a section are appended as last  $N_2 - 1$  samples of the previous section. The circular convolution of each section will produce an output sequence of size  $N_3 + N_2 - 1$  samples. In this output the first  $N_2 - 1$  samples are discarded and the remaining samples of the output of sectioned convolution are saved as the overall output sequence.

##### **Method II**

In this method the last  $N_2 - 1$  samples of a section are appended as last  $N_2 - 1$  samples of the next section. (i.e., the overlapping samples are placed at the beginning of the section). The circular convolution of each section will produce an output sequence of size  $N_3 + N_2 - 1$  samples. In this output the last  $N_2 - 1$  samples are

discarded and the remaining samples of the output of sectioned convolution are saved as the overall output sequence.

---

### Example 2.49

Perform the linear convolution of the following sequence by (i) overlap add method and (ii) overlap save method.

$$x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\} \quad \text{and} \quad h(n) = \{-1, 1\}$$


---

### Solution

#### Overlap Add Method

Here  $x(n)$  is a longer sequence when compared to  $h(n)$ . Hence,  $x(n)$  is sectioned into sequences of size equal to  $h(n)$ .

Let  $x(n)$  be sectioned into three sequences, each consisting of two samples of  $x(n)$  as

$$\begin{array}{lll} x_1(n) = 1; & n = 0 & x_2(n) = -2; & n = 3 & x_3(n) = 4; & n = 6 \\ & -1; & n = 1 & +3; & n = 4 & -4; & n = 7 \\ & 2; & n = 2 & -3; & n = 5 & 0; & n = 8 \end{array}$$

Let  $y_1(n)$ ,  $y_2(n)$ ,  $y_3(n)$  and  $y_4(n)$  be the output of linear convolution of  $x_1(n)$ ,  $x_2(n)$ ,  $x_3(n)$  and  $x_4(n)$  with  $h(n)$  respectively.

#### Convolution of Section I

$$\begin{aligned} y_1(n) &= x_1(n) * h(n) \\ Y_1(k) &= X_1(k)H(k) \\ y_1(n) &= IDFT [X_1(k)H(k)] \end{aligned}$$

$$X_1(k) \Rightarrow x_1(n) = \{1, -1, 2\}$$

and

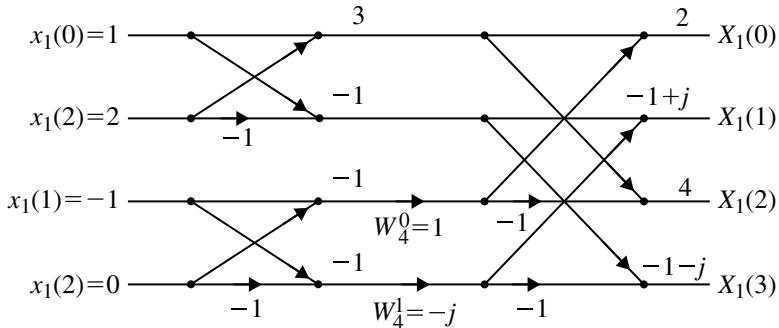
$$h(n) = \{-1, 1\}$$

Length of  $y_1(n) = 3 + 2 - 1 = 4$ ;

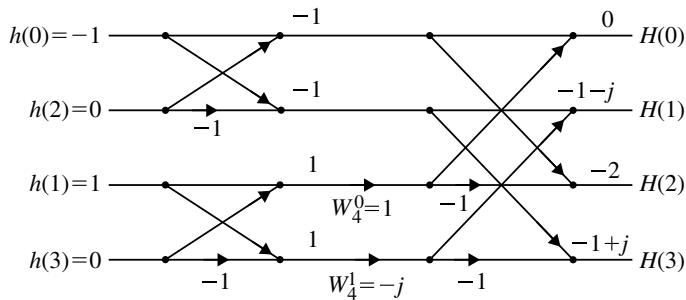
$$n_1 = 0 \quad \text{and} \quad n_2 = 0$$

so

$$n = n_1 + n_2 = 0.$$

**Fig. 2.49** Butterfly diagram of  $X(k)$ 

Butterfly diagram of  $X(k)$  of Example 2.49 is shown in Fig. 2.49.

**Fig. 2.50** Butterfly diagram of  $H(k)$ 

Butterfly diagram of  $H(k)$  of Example 2.49 is shown in Fig. 2.50.

Therefore,

$$x_1(n) = \{1, -1, 2, 0\}$$

and

$$h(n) = \{-1, 1, 0, 0\}$$

Therefore

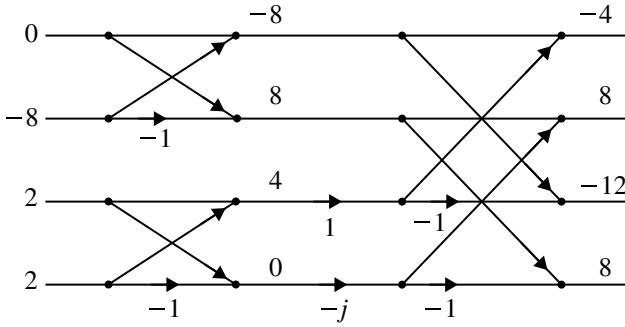
$$X_1(k) = \{2, -1 + j, 4, -1 - j\}$$

$$\therefore H(k) = \{0, -1 - j, -2, -1 + j\}$$

$$\begin{aligned} X_1(k)H(k) &= \{0, (-1 + j)(-1 - j), -8, (-1 - j)(-1 + j)\} \\ &= \{0, 2, -8, 2\} \end{aligned}$$

**IDFT**  $[X_1(k)H(k)] \Rightarrow$

$$\{X_1(k)H(k)\}^* = \{0, 2, -8, 2\}$$



**Fig. 2.51** Butterfly diagram of  $[X_1(k)H(k)]^*$

Butterfly diagram of  $[X_1(k)H(k)]^*$  of Example 2.49 is shown in Fig. 2.51.

DFT of  $[X_1(k)H(k)]^*$

$$\{-4, 8, -12, 8\}$$

$$\begin{aligned} \therefore y_1(n) &= \frac{1}{n} \{-4, 8, -12, 8\} \\ &= \frac{1}{4} \{-4, 8, -12, 8\} = \{-1, 2, -3, 2\} \end{aligned}$$

↑

### Convolution of Section II

$$x_2(n) = \{-2, 3, -3\} \quad \text{and} \quad h(n) = \{-1, 1\}$$

$$N = 3 + 2 - 1 = 4$$

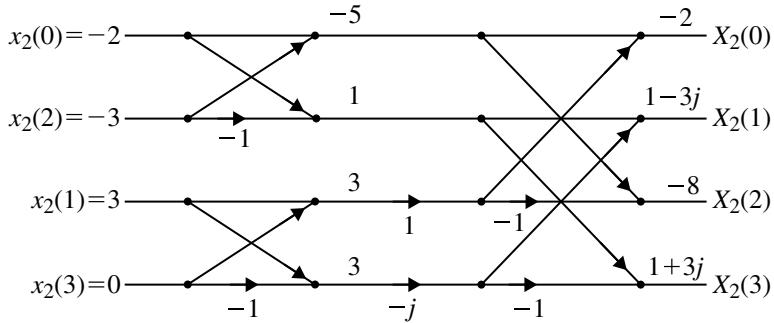
$$x_2(n) = \{-2, 3, -3, 0\} \quad \text{and} \quad h(n) = \{-1, 1, 0, 0\}$$

$$n_1 = 3, n_2 = 0 \text{ and } n = n_1 + n_2 = 3.$$

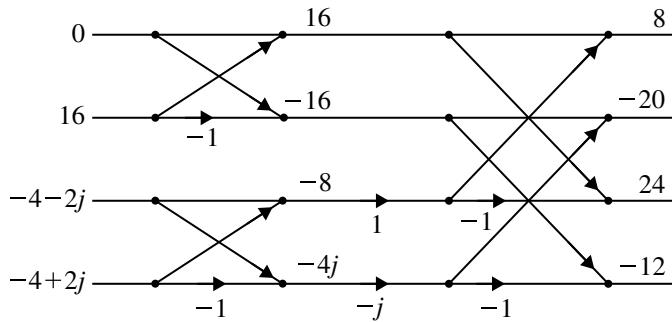
$$\begin{aligned} X_2(k) &= \{-2, 1 - 3j, -8, 1 + 3j\} \\ X_2(k)H(k) &= \{0, (-1 - j)(1 - 3j), (-2)(-8), (-1 + j)(1 + 3i)\} \\ &= \{0, -4 + 2j, 16, -4 - 2j\} \end{aligned}$$

IDFT  $[X_2(k)H(k)] \Rightarrow$

$$\{X_2(k)H(k)\}^* = \{0, -4 - 2j, 16, -4 + 2j\}$$

**Fig. 2.52** Butterfly diagram of  $X_2(k)$ 

Butterfly diagram of  $X_2(k)$  of Example 2.49 is shown in Fig. 2.52.

**Fig. 2.53** Butterfly diagram of  $[X_2(k)H(k)]^*$ 

Butterfly diagram of  $[X_2(k)H(k)]^*$  of Example 2.49 is shown in Fig. 2.53

DFT  $\{X_2(k)H(k)\}^*$

Therefore,

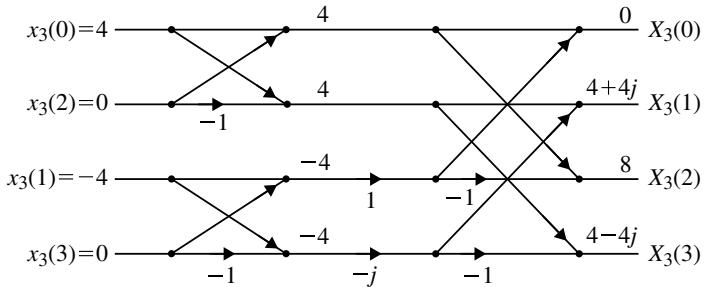
$$\begin{aligned} y_2(n) &= \frac{1}{N} \{DFT [X_2(k)H(k)]^*\}^* = \frac{1}{4} \{8, -20, 24, -12\} \\ &= \{2, -5, 6, -3\} \\ &\quad \uparrow \\ &\quad n = 3 \end{aligned}$$

### Convolution of Section III

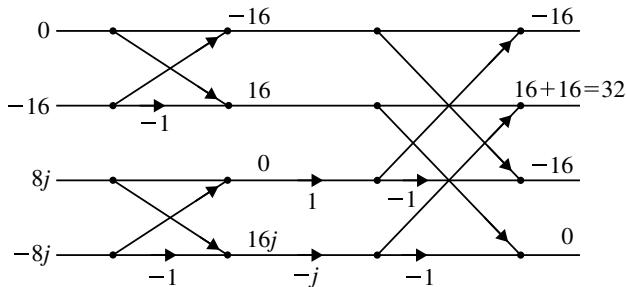
$$x_3(n) = \{4, -4, 0\} \quad \text{and} \quad h(n) = \{-1, 1\}$$

$N = 3 + 2 - 1 = 4$ ,  $n_1 = 6$ ,  $n_2 = 0$  so  $n = n_1 + n_2 = 6$ . Therefore

$$x_3(n) = \{4, -4, 0, 0\} \quad \text{and} \quad h(n) = \{-1, 1, 0, 0\}$$

**Fig. 2.54** Butterfly diagram of  $X_3(k)$ 

Butterfly diagram of  $X_3(k)$  of Example 2.49 is shown in Fig. 2.54.

**Fig. 2.55** Butterfly diagram of  $[x_3(k)H(k)]^*$ 

Butterfly diagram of  $[x_3(k)H(k)]^*$  of Example 2.49 is shown in Fig. 2.55

$$X_3(k) = \{0, 4 + 4j, 8, 4 - 4j\}$$

$$\begin{aligned} X_3(k)H(k) &= \{0, (-1 - j)(4 + 4j), -2(8), (-1 + j)(4 - 4j)\} \\ &= \{0, -8j, -16, 8j\} \end{aligned}$$

**IDFT**  $[X_2(k)H(k)] \Rightarrow$

$$[x_3(k)H(k)]^* = \{0, 8j, -16, -8j\}$$

DFT  $\left[\{x_3(k)H(k)\}^*\right]$

$$\begin{aligned} y_3(n) &= \frac{1}{N} [1DFT [x_3(k)H(k)]] \\ &= \frac{1}{N} [DFT [[x_3(k)H(k)]^*]]^* \\ &= \frac{1}{4} \{-16, 32, -16, 0\}^* \\ &= \{-4, 8, -4, 0\} \\ &\quad \uparrow \\ n &= 6 \end{aligned}$$

The combined output of each convolution is shown in Table 2.3.

**Table 2.3** Combined output of each convolution section

$n$	0	1	2	3	4	5	6	7	8
$y_1(n)$	-1	2	-3	2					
$y_2(n)$				2	-5	6	-3		
$y_3(n)$							-4	8	-4
$y(n)$	-1	2	-3	4	-5	6	-7	8	-4

To combine the output of the convolution of each section  $N_2 - 1$  samples are overlapped. The overlapped portion (or samples) are added while combining the output. Here  $N_2 = 2$  and  $N_2 - 1 = 1$ . Samples overlapped that are added.

$$\therefore y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, -4\}$$

The above result can be verified as follows:

	1	-1	2	-2	3	-3	4	-4
							-1	1
	1	-1	2	-2	3	-3	4	-4
-1	1	-2	2	-3	3	-4	4	
-1	2	-3	4	-5	6	-7	8	-4

$$y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, -4\}$$

↑

## II Overlap Save Method

### Method I

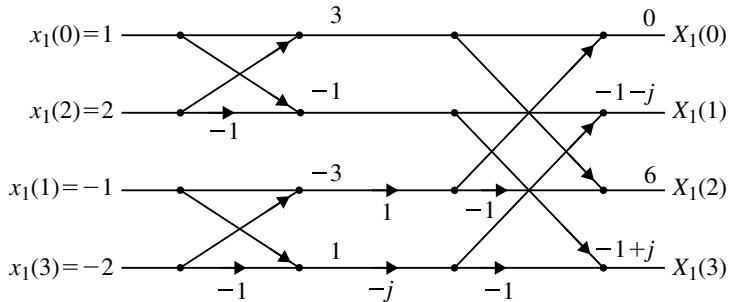
In this the  $N_2 - 1$  overlapping samples are placed at the beginning of the section. Number of overlapping samples are  $N_2 - 1 = 2 - 1 = 1$  sample.

$$\begin{array}{lll}
 x_1(n) = 1; & n = 0 & x_2(n) = -2; & n = 3 & x_3(n) = 4; & n = 6 \\
 -1; & n = 1 & & +3; & n = 4 & -4; & n = 7 \\
 2; & n = 2 & & -3; & n = 5 & 0; & n = 8 \\
 -2; & n = 2 & & 4; & n = 6 & 0; & n = 9
 \end{array}$$

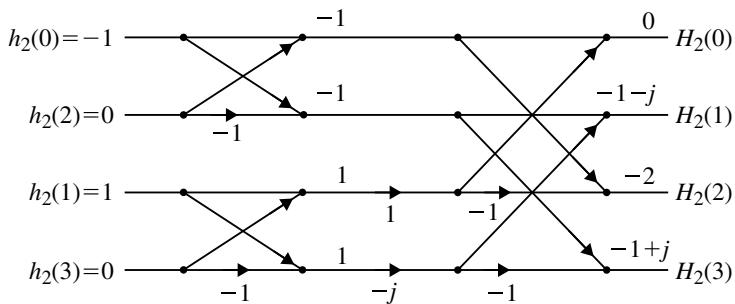
### Convolution of Section-I

$$x_1(n) = \{1, -1, 2, -2\} \quad \text{and} \quad h(n) = \{-1, 1, 0, 0\}$$



**Fig. 2.56** Butterfly diagram of  $X_1(k)$ 

Butterfly diagram of  $X_1(k)$  of Example 2.49 is shown in Fig. 2.56.

**Fig. 2.57** Butterfly diagram of  $H(k)$ 

Butterfly diagram of  $H(k)$  of Example 2.49 is shown in Fig. 2.57.  
at  $n = 0$ .

$$X_1(k) = \{0, -1 - j, 6, -1 + j\}$$

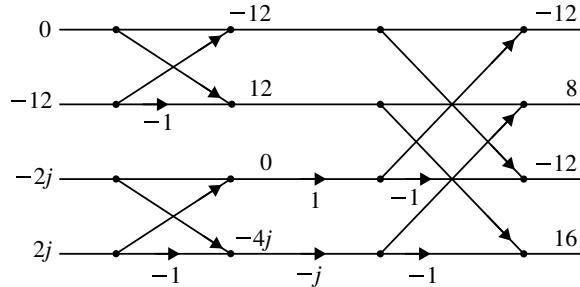
$$H(k) = \{0, -1 - j, -2, -1 + j\}$$

$$\begin{aligned} X_1(k)H(k) &= \{0, (-1 - j)(-1 - j), (-2)(6), (-1 + j)(-1 + j)\} \\ &= \{0, 2j, -12, -2j\} \end{aligned}$$

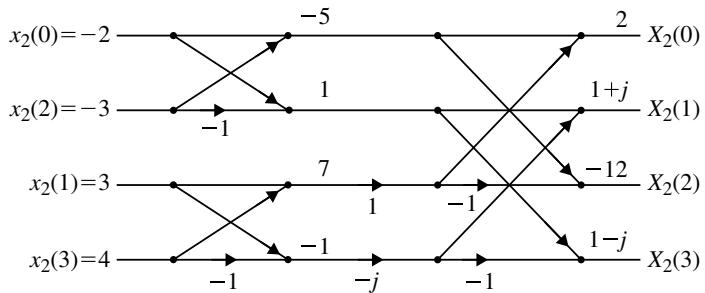
$$\{X_1(k)H(k)\}^* = \{0, -2j, -12, 2j\}$$

$$\text{DFT} [[X_1(k)H(k)]^*]$$

$$\{-12, 8, -12, 16\}$$

**Fig. 2.58** Butterfly diagram of  $[X_1(k)H(k)]^*$ 

Butterfly diagram of  $[X_1(k)H(k)]^*$  of Example 2.49 is shown in Fig. 2.58.

**Fig. 2.59** DFT Computation of  $X_2(k)$ 

DFT Computation of  $X_2(k)$  of Example 2.49 is shown in Fig. 2.59.

$$y_1(n) = \frac{1}{N} \{DFT [[x(k)H(k)]^*]\}^* = \frac{1}{4} \{-12, 8, -12, 16\}^*$$

$$y_1(n) = \{-3, 2, -3, 4\}$$

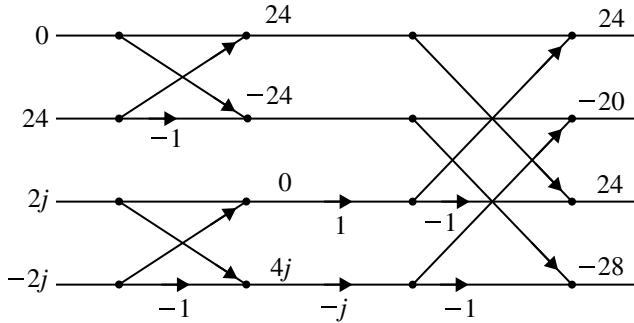
$\uparrow$   
 $n = 0$

### Convolution of Section-II

$$x_2(n) = \{-2, 3, -3, 4\} \quad \text{and} \quad h(n) = \{-1, 1, 0, 0\}$$

$\uparrow$   
 $n = 3$

$$X_2(k) = \{2, 1 + j, -12, 1 - j\}$$



**Fig. 2.60** DFT computation of  $[X_2(k)H(k)]^*$

DFT computation of  $[X_2(k)H(k)]^*$  of Example 2.49 is shown in Fig. 2.60.

$$\begin{aligned} X_2(k)H(k) &= \{0, (1+j)(-1-j), (-12)(-2), (-1+j)(1-j)\} \\ &= \{0, -2j, 24, 2j\} \\ [X_2(k)H(k)] &= \{0, 2j, 24, -2j\} \end{aligned}$$

DFT  $\left[ [X_2(k)H(k)]^* \right]$

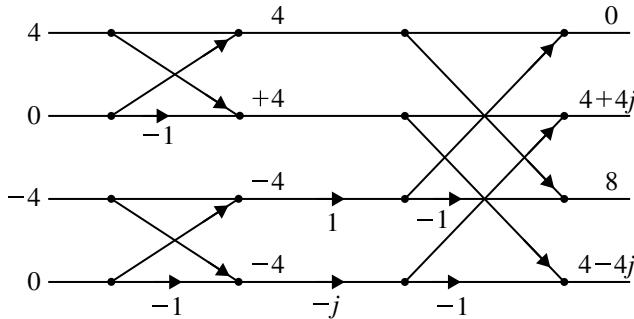
$$\begin{aligned} &\{24, -20, 24, -28\} \\ y_2(n) &= \frac{1}{4} \{24, -20, 24, -28\} \\ &= \{6, -5, 6, -7\} \\ &\quad \uparrow \\ &\quad n = 3 \end{aligned}$$

### Convolution of Section-III

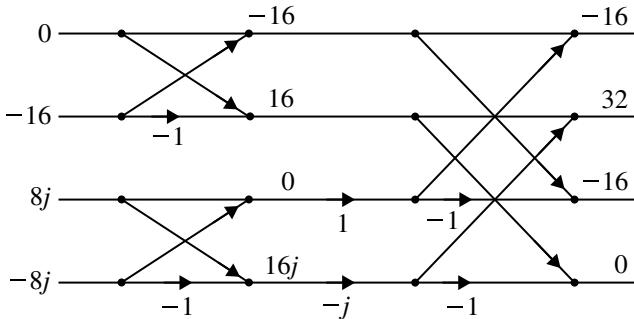
$$\begin{aligned} x_3(n) &= \{4, -4, 0, 0\} \quad \text{and} \quad h(n) = \{-1, 1, 0, 0\} \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad n = 6 \end{aligned}$$

$$X_3(k) = \{0, 4 + 4j, 8, 4 - 4j\}$$

$$\begin{aligned} X_3(k)H(k) &= \{0, (4+4j)(-1-j), 8(-2), (4-4j)(-1+j)\} \\ &= \{0, -8j, -16, 8j\} \\ \{X_3(k)H(k)\} &= \{0, 8j, -16, -8j\} \end{aligned}$$

**Fig. 2.61** DFT Computation of  $X_3(k)$ 

DFT Computation of  $X_3(k)$  of Example 2.49 is shown in Fig. 2.61.

**Fig. 2.62** DFT Computation of  $[X_3(k)H(k)]^*$ 

DFT Computation of  $[X_3(k)H(k)]^*$  of Example 2.49 is shown in Fig. 2.62.

DFT  $\{[X_3(k)H(k)]^*\}$

$$\begin{aligned}
 y_3(n) &= \frac{1}{4} \{-16, 32, -16, 0\}^* \\
 &= \{-4, 8, -4, 0\} \\
 &\quad \uparrow \\
 n &= 6
 \end{aligned}$$

To combine the output of the convolution of each section, the first  $N_2 - 1$  samples are overlapped that are discarded.

Here  $y(n)$  is linear convolution of  $x(n)$  and  $h(n)$ . It can be observed that the results of both the methods are same, except the first  $N_2 - 1$  samples.

The combined output of each convolution is shown in Table 2.4.

**Table 2.4** Combined output of each convolution section in Method I

$n$	0	1	2	3	4	5	6	7	8	9
$y_1(n)$	-X	2	-3	4						
$y_2(n)$				X	-5	6	-7			
$y_3(n)$	3						-X	8	-4	0
$y(n)$	3	2	-3	4	-5	6	-7	8	-4	0

**Table 2.5** Combined output of each convolution in Method II

$n$	0	1	2	3	4	5	6	7	8	9
$y_1(n)$	-1	2	-3	X						
$y_2(n)$				4	-5	6	-X			
$y_3(n)$							-7	8	-4	X
$y(n)$	-1	2	-3	4	-5	6	-7	8	-4	x

## Method II

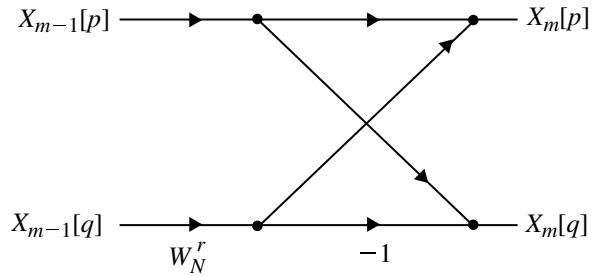
$$\begin{array}{lll}
 x_1(n) = 1; & n = 0 & x_2(n) = -2; & n = 3 & x_3(n) = 4; & n = 7 \\
 & -1; & n = 1 & +3; & n = 4 & -4; & n = 8 \\
 & 2; & n = 2 & -3; & n = 5 & 0; & n = 9 \\
 & 0; & n = 2 & 4; & n = 6 & 0; & n = 10
 \end{array}$$

$h(n) = \{-1, 1, 0, 0\}$ . Find  $y_1(n)$ ,  $y_2(n)$  and  $y_3(n)$  using FFT algorithm. Same above method. Result is shown in Table 2.5.

## 2.7 In-Plane Computation

The flow graph of Fig. 2.63 describes an algorithm for the computation of the DFT. In the flow graph the branches connecting the nodes and the transmittance of each of these branches. No matter how the nodes in the flow graph are rearranged, it will always represent the same computation provided that the connection between the nodes and the transmittance of the connection are maintained. The particular form for the flow graph in Fig. 2.63 arose out of deriving the algorithm by separating the original sequences into the even-numbered and odd-numbered points and then continuing to create smaller and smaller subsequences in the same way. An interesting by-product of this derivation is that this flow graph, in addition to describing an efficient procedure for computing the discrete Fourier transform, also suggests a useful way of storing the original data and storing the results of the computation in intermediate arrays.

When implementing the computation depicting in Fig. 2.63 we can imagine the use of two arrays of (complex) storage registers, one for the arrays being computed and one for the data being used in the computation. For example, in computing the

**Fig. 2.63** Flow graph

first array in Fig. 2.63, one set of storage registers would contain the input data and the second set would contain the computed results for the first stage. We denote the sequence of complex numbers resulting from the  $n$ th stage of computation as  $X_m(l)$  where  $l = 0, 1, \dots, N - 1$ , and  $m = 1, 2, \dots, v[v = \log_2^N]$ . Furthermore, for convenience we define the set of input samples of  $X_0[l]$ . We can think of  $X_{m-1}[l]$  as the input array and  $X_m(l)$  as the output array for the  $m$ th stage computation. Thus, for the case  $N = 8$  as in Fig. 2.63, we get

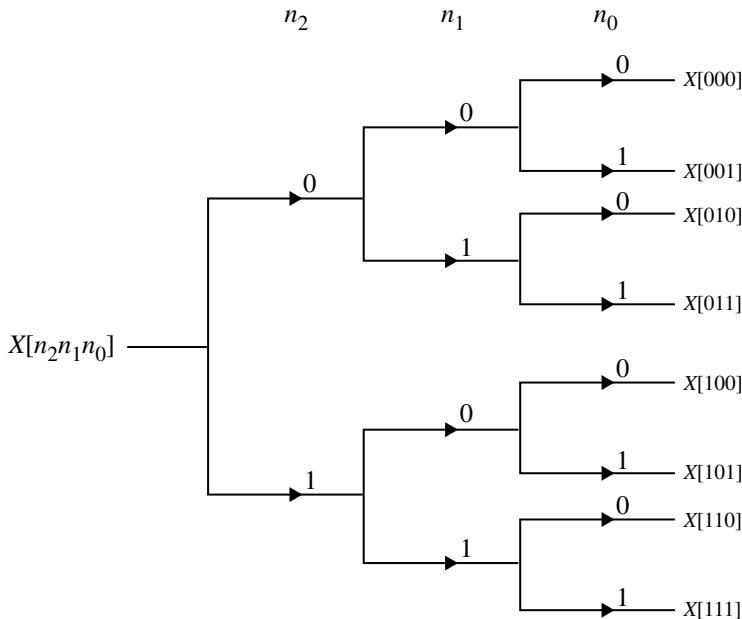
$$\begin{array}{ll} X_0[0] = x[0] & X_0[4] = x[1] \\ X_0[1] = x[4] & X_0[5] = x[5] \\ X_0[2] = x[2] & X_0[6] = x[3] \\ X_0[3] = x[6] & X_0[7] = x[7] \end{array}$$

Using this notation, the basic butterfly diagram is drawn as shown in Fig. 2.63, with the associated equation as,

$$\begin{aligned} X_m[p] &= X_{m-1}[p] + W_N^r X_{m-1}[q] \\ X_m[q] &= X_{m-1}[p] - W_N^r X_{m-1}[q] \end{aligned}$$

In above equations  $p$ ,  $q$  and  $r$  vary form stage to stage in a manner that is readily inferred from Fig. 2.15. It is clear from Fig. 2.15 and Fig. 2.63 that only the complex numbers in locations  $p$  and  $q$  of the  $(m - 1)$ st array are required to compute the elements  $p$  and  $q$  of the  $m$ th array. Thus only one complex array of  $N$  storage registers is physically necessary to implement the complete computation. That is  $X_m[p]$  and  $X_m[q]$  are stored in the same storage registers as  $X_{m-1}[p]$  and  $X_{m-1}[q]$  respectively. This kind of computation is commonly referred to as an “in-plane computation”.

If  $(n_2, n_1, n_0)$  is the binary representation of the index of the sequence  $x[n]$ , then the sequence value  $x[n_2, n_1, n_0]$  is stored in the array position  $X_0[n_0, n_1, n_2]$ . That is, in determining the position of  $x[n_2, n_1, n_0]$  in the input array we must reverse the order of the bits of the index  $n$ . Let us first consider the process depicted in Fig. 2.64 for sorting a data sequence in normal order by successive examination of the bits representing the data index. If the most significant bit of the data index is zero,  $x[n]$  belongs to the top half of the sorted array, otherwise it belongs to the bottom half. Next the top half and bottom half subsequences can be sorted by examining the second most significant bit, and so on.

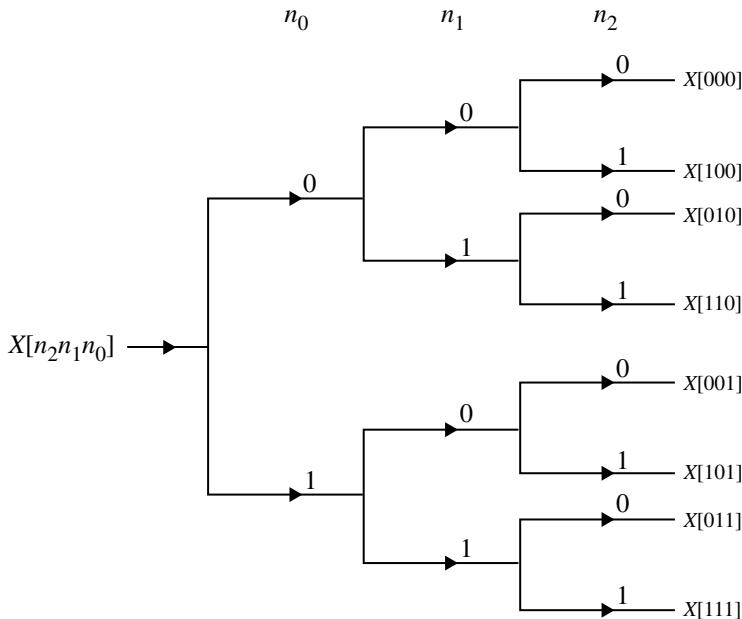


**Fig. 2.64** Tree diagram depicting normal-order sorting

Let us now consider the process depicted in Fig. 2.63 for sorting a data sequence in bit reversed order. If the least significant bit is 0, the sequence value corresponds to an even-numbered sample and therefore will appear in the top half and if the least significant bit is 1, the sequence value corresponds to an odd-numbered sample and consequently will appear in the bottom half of the array  $X_0[l]$ . Next the even and odd indexed subsequences are sorted into their even and odd indexed parts, and this can be done by examining the second least significant bit in the index. This process is repeated until  $N$  subsequences is depicted by the tree diagram Fig. 2.65.

## Summary

- For spectral analysis of discrete signals, DFT approach is a very straightforward one. However, for larger values of  $N$ , DFT becomes tedious because of the huge number of mathematical operations required to perform. Such huge number of mathematical operations limit the bandwidth of the digital signal processors.
- Several algorithms have been developed to reduce the computational burden and ease the implementation of DFT. The algorithm developed by Cooley and Tukey is the most efficient one and is called fast Fourier transform (FFT).
- The most widely used FFT algorithms are radix-2 and radix-4.



**Fig. 2.65** Tree diagram depicting bit-reversed-order sorting

- Radix-2 FFT algorithm exploits the two important properties of twiddle factors, namely symmetry and periodicity thereby removing redundant mathematical operations.
- The basic principle of FFT algorithm is to decompose DFT into successively smaller DFTs. The manner in which this decomposition is done leads to different FFT algorithms. Decimation in time (DIT) and decimation in frequency (DIF) are the two basic classes of algorithm.
- In the algorithm in DIT, the sequence  $x(n)$  is decomposed into successively smaller subsequences. Similarly in DIF algorithm, the sequence of DFT coefficients  $X(k)$  is decomposed in smaller subsequences.

### Short Questions and Answers

#### 1. Define DFT of a discrete time sequence (or) analysis equation of DFT.

(Nov./Dec. 2003)

**Ans:** The DFT is used to convert a finite discrete time sequence  $x(n)$  to an  $N$ -point frequency domain sequence denoted by  $X(k)$ . The  $N$ -point DFT of a finite duration sequence  $x(n)$  of length  $L$ , where  $L \leq N$  is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} \quad \text{for } k = 0, 1, 2, \dots, (N-1)$$

**2. Define IDFT (or) synthesis equation of DFT.** (Nov./Dec. 2003)

**Ans:** The IDFT is used to convert the  $N$ -point frequency domain sequence  $X(k)$  to an  $N$ -point time domain sequence. The IDFT of the sequence  $X(k)$  of length  $N$  is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}} \quad \text{for } k = 0, 1, 2, \dots, (N-1)$$

**3. What is the drawback in Fourier transform and how it is overcome?**

**Ans:** The drawback in Fourier transform is that it is a continuous function of  $\omega$  and so it cannot be processed by digital system. This drawback is overcome by using discrete Fourier transform. The DFT converts the continuous function of  $\omega$  to a discrete function of  $\omega$ .

**4. Give any two applications of DFT (or the importance of DFT).**

**Ans:**

- (i) The DFT is used for spectral analysis of signals using a digital computer.
- (ii) The DFT is used to perform filtering operation on signals using digital computer.

**5. What is the relationship between  $z$ -transform and DFT?**

**Ans:** Let  $N$ -point DFT of  $x(n)$  be  $X(k)$  and the  $z$ -transform of  $x(n)$  be  $X(z)$ . The  $N$ -point sequence  $X(k)$  can be obtained from  $X(z)$  by evaluating  $X(z)$  at  $N$  equally spaced points around the unit circle. That is

$$X(k) = X(z) \Big|_{Z=e^{j\frac{2\pi kn}{N}}} \quad \text{for } k = 0, 1, 2, \dots, (N-1)$$

**6. Why linear convolution is important in DSP?**

**Ans:** The response or output of LTI discrete time system for any input  $x(n)$  is given by linear convolution of the input  $x(n)$  and the impulse response  $h(n)$  of the system. (This means that if the impulse response of a system is known, then the response of the system for any input can be determined by convolution operation).

**7. Write the properties of linear convolution.**

**Ans:** The linear convolution has the following properties:

1. Commutative property:  $x(n) * h(n) = h(n) * x(n)$ .
2. Associative property:  $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$ .
3. Distributive property:  $x(n) * (h_1(n) + h_2(n)) = x(n) * h_1(n) + x(n) * h_2(n)$ .

### 8. What is circular convolution?

**Ans:** The convolution of two periodic sequences with periodicity  $N$  is called circular convolution. If  $x_1(n)$  and  $x_2(n)$  are two periodic sequences with  $N$  samples in a period, then the circular convolution of  $x_1(n)$  and  $x_2(n)$  is defined as,

$$x_1(n) * x_2(n) = \sum_{n=0}^{N-1} x_1(n)x_2(m-n)$$

### 9. What is zero padding? Why it is needed?

(Nov./Dec. 2003)

**Ans:** Appending zeros to a sequence in order to increase the size or length of the sequence is called zero padding. In circular convolution, when the two input sequences are of different size, then they are converted to equal size by zero padding.

### 10. List the difference between linear convolution and circular convolution.

**Ans:**

	Linear convolution	Circular convolution
1.	The length of the input sequences can be different	The length of the input sequences should be same
2.	Zero padding is not required	If the length of the input sequences are different, then zero padding is required
3.	The input sequences need not be periodic	At least one of the input sequences should be periodic or should periodically be extended
4.	The output sequence is non-periodic	The output sequence is periodic. The periodicity is same as that of input sequence
5.	The length of the output sequence will be greater than the length of input sequences	The length of the input and output sequences are same

### 11. Why circular convolution is important in DSP?

**Ans:** The Discrete Fourier Transform is used for the analysis and design of discrete time systems using digital computers. The DFT supports only circular convolution. Hence, when DFT techniques are employed, the results of linear convolution are obtained only via circular convolution.

### 12. How will you perform linear convolution via circular convolution?

(Anna University, May/June 2006)

**Ans:** The linear convolution of two sequences of length  $N_1$  and  $N_2$  produces an output sequence of length  $N_1 + N_2 - 1$ . To perform linear convolution via circular convolution, convert the input sequences to  $N_1 + N_2 - 1$ -point sequences by padding with zeros. Now the circular convolution of the  $N_1 + N_2 - 1$ -point sequences will give an output sequence, which is same as that of linear convolution of the original two sequences of length  $N_1$  and  $N_2$ .

### 13. What is sectioned convolution?

**Ans:** In linear convolution of two sequences, if one of the sequences is very much larger than the other, then it is very difficult to compute the linear convo-

lution using DFT. In such cases, the longer sequence is sectioned into size of smaller sequence. Then the linear convolution of each section of longer sequence and the smaller sequence is performed. The output sequence obtained from the convolution of all the sections are combined to get the overall output sequence. This technique of convolution is called sectioned convolution.

#### **14. Why sectioned convolution is performed?**

**Ans:** In linear convolution of two sequences, if one of the sequences is very much larger than the other, then it is very difficult to compute the linear convolution using DFT for the following reasons:

- (i) The entire sequence should be available before convolution can be carried out. This makes long delay in getting the output,
- (ii) Large amount of memory is required to store the sequences.

#### **15. Compare the overlap add and overlap save method of sectioned convolutions.**

	Overlap add method	Overlap save method
1.	Linear convolution of each section of longer sequence with smaller sequence is performed	Circular convolution of each section of longer sequence with smaller sequence is performed. (after converting them to the size of output sequence)
2.	Zero padding is not required	Zero padding is required to convert the input sequence to the size of output sequence
3.	Overlapping of samples of input sections are not required	The $N_2 - 1$ samples of an input section of longer sequence is overlapped with next input section
4.	The overlapped samples in the output of sectioned convolutions are added to get the overall output	Depending on the method of overlapping the input samples, either the last $N_2 - 1$ samples or the first $N_2 - 1$ samples of the output sequence of each sectioned convolution are discarded

#### **16. In what way zero padding is implemented in overlap save method?**

**Ans:** In overlap save method, the zero padding is employed to convert the small input sequence to the size of the output sequence of each sectioned convolution. The zero padding is also employed to convert either the last section or the first section of the longer input sequence to the size of the output sequence of each sectioned convolution.

#### **17. What is FFT?**

*(Anna University, Nov./Dec. 2006)*

**Ans:** The FFT is a method for computing the DFT with reduced number of calculations. The computational efficiency is achieved by employing divide and conquer approach. This is based on the decomposition of an  $N$ -point DFT into successively smaller DFTs.

In an  $N$ -point sequence, if  $N$  can be expressed as  $N = r^m$ , then the sequence can be decimated into  $r$ -point sequence, and  $r$ -point DFTs are computed. From the results of  $r$ -point DFTs, the  $r^2$ -point DFTs are computed. From the results of  $r^2$ -point DFT, the  $r^3$ -point DFTs are computed and so on, until we get  $r^m$ -point

DFT. Hence, the number of stages of computation is  $m$ . The number  $r$  is called the radix of the FFT algorithm.

**18. What is radix-2 FFT?**

**Ans:** The radix-2 FFT is an efficient algorithm for computing  $N$ -point DFT of an  $N$ -point sequence. In radix-2 FFT the  $N$ -point sequence is decimated into two-point sequences and two-point DFT for each decimated sequence is computed. From the results of two-point DFTs, the four-point DFTs are computed. From the results of four-point DFTs, the eight-point DFTs are computed and so on until we get  $N$ -point DFT.

**19. How many multiplications and additions are involved in radix-2 FFT?**

**Ans:** For performing radix-2 FFT, the value of  $N$  should be such that,  $N = 2^m$ . The total number of complex additions are  $N \log_2 N$  and the total number of complex multiplications are  $(N/2) \log_2 N$ .

**20. Calculate the percentage saving in calculations in a 512-point radix-2 FFT, when compared to direct DFT.**

**Ans:** Direct computation of DFT:

$$\begin{aligned}\text{Number of complex additions} &= N(N - 1) = 512 \times (512 - 1) \\ &= 2,61,632\end{aligned}$$

$$\text{Number of complex multiplications} = N^2 = 512 \times 512 = 2,61,144$$

Radix-2 FFT:

$$\text{Number of complex additions} = N \log_2 N = 4,608$$

$$\text{Number of complex multiplications} = (N/2) \log_2 N = 2,304$$

Percentage saving in additions

$$\begin{aligned}&= 100 - \frac{\text{Number of additions in radix-2 FFT}}{\text{Number of additions in direct DFT}} \times 100 \\ &= 100 - \frac{(4,608)}{(2,61,632)} \times 100 = 98\%\end{aligned}$$

Percentage saving in multiplications

$$\begin{aligned}&= 100 - \frac{\text{Number of multiplications in radix-2 FFT}}{\text{Number of multiplications in direct DFT}} \times 100 \\ &= 100 - \frac{(2,304)}{(2,61,144)} \times 100 = 99.1\%\end{aligned}$$

**21. Draw and explain the basic butterfly diagram or flow graph of DIT radix-2 FFT.**

**Ans:** The basic butterfly diagram of DIT radix-2 FFT is shown in Fig. 2.66. It performs the following operations. Here  $a$  and  $b$  are input complex numbers and  $A$  and  $B$  are output complex numbers.

- (i) Input complex number  $b$  is multiplied by the phase factor  $W_N^k$ .
- (ii) The product  $bW_N^k$  is added to the input complex number  $a$  to form a new complex number  $A$ .
- (iii) The product  $bW_N^k$  is subtracted from the input complex number  $a$  to form a new complex number  $B$ .

## 22. What is DIF radix-2 FFT?

**Ans:** The DIF radix-2 FFT is an efficient algorithm for computing DFT. In this algorithm the  $N$ -point time domain sequence is converted to two numbers of  $N/2$ -point sequences. Then each  $N/2$ -point sequence is converted to two numbers of  $N/4$ -point sequences. This process is continued until we get  $N/2$  numbers of two-point sequences. Now the two-point DFTs of  $N/2$  numbers of two-point sequences will give  $N$  samples, which is the  $N$ -point DFT of the time domain sequence. Here the equations for forming  $N/2$ -point sequences,  $N/4$ -point sequences, etc., are obtained by decimation of frequency domain sequences. Hence, this method is called DIF.

## 23. Compare the DIT and DIF radix-2 FFT.

**Ans:**

	DIT radix-2 FFT	DIF radix-2 FFT
1.	The time domain sequence is decimated	The frequency domain sequence is decimated
2.	When the input is in bit reversed order, the output will be in normal order and <i>vice versa</i>	When the input is in normal order, the output will be in bit reversed order and <i>vice versa</i>
3.	In each stage of computation, the phase factor are multiplied before add and subtract operation	In each stage of computation, the phase factor are multiplied after add and subtract operation
4.	The value of $N$ should be expressed such that $N = 2m$ and this algorithm consists of $m$ stages of computations	The value of $N$ should be expressed such that $N = 2m$ and this algorithm consists of $m$ stages of computations
5.	Total number of arithmetic operations are $N \log_2 N$ complex additions and $(N/2) \log_2 N$ complex multiplications	Total number of arithmetic operations are $N \log_2 N$ complex additions and $(N/2) \log_2 N$ complex multiplications

## 24. How will you compute IDFT using radix-2 FFT algorithm?

**Ans:** Let  $x(n)$  be an  $N$ -point sequence and  $X(k)$  be the  $N$ -point DFT of  $x(n)$ . The sequence  $x(n)$  can be computed from the sequence,

$$x(n) = \frac{1}{N} \left( \sum_{k=0}^{N-1} X^*(k) W_N^{nk} \right)^*$$

The following procedure can be used to determine  $x(n)$  using radix-2 FFT algorithm.

- (i) Take conjugate of  $X(k)$  (i.e., determine  $X^*(k)$ ).
- (ii) Compute  $N$ -point DFT of  $X^*(k)$  using radix-2 FFT.
- (iii) Take conjugate of the output sequence from FFT.
- (iv) Divide the sequence obtained in Step (iii) by  $N$ . The resultant sequence is  $x(n)$ .

## 25. What is direct or slow convolution and fast convolution?

**Ans:** The response of an LTI system is given by convolution of input and impulse response. The computation of the response of LTI system by convolution sum formula is called slow convolution because it involves very large number of calculations. The number of calculations in DFT computation of the response of LTI system by FFT algorithm is called fast convolution.

## 26. Why FFT is needed?

**Ans:** The FFT is needed to compute DFT with reduced number of calculations. The DFT is required for spectrum analysis and filtering operations on the signals using digital computers.

## 27. How FFT is faster?

**Ans:** FFT algorithm exploits the two important properties if twiddle factors namely symmetry and periodicity thereby removing redundant mathematical operation. It requires  $N/2 \log_2^N$  complex multiplication and  $N \log_2^N$  complex addition for computing  $N$  point DFT. FFT reduces the computation time required to compute a discrete Fourier transform and improves the performance by a factor 100 or more over direct evaluation of the DFT.

## 28. Compute DFT of the sequence $x(n) = e^{-n}$ , $0 \leq n \leq 4$

**Ans:**

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}} \\ &= \sum_{n=0}^4 e^{-n}e^{-\frac{j2\pi kn}{5}} = \sum_{n=0}^4 \left[ e^{-1-\frac{j2\pi k}{5}} \right]^n \\ X(k) &= \frac{1 - e^{-\left(1+\frac{j2\pi k}{5}\right)^5}}{1 - e^{-\left(1+\frac{j2\pi k}{5}\right)}} \end{aligned}$$

## 29. Calculate the number if complex multiplication for direct evaluation of eight-point DFT.

**Ans:** The number of complex multiplication required using direct computation is

$$N^2 = 8^2 = 64$$

**30. Distinguish between DFT and DTFT.**

**Ans:** DFT:

- Obtained by performing sampling operation in both the time and frequency domains.
- Discrete frequency spectrum (Discrete function of  $\omega$ ).

DTFT:

- Sampling is performed only in time domain.
- Continuous frequency spectrum (continuous function of  $\omega$ ).

**31. Give computation efficiency of 1024-point FFT over 1024-point DFT.**

**Ans:** The number of complex multiplication required using direct computation is

$$N^2 = (1024)^2 = 1048576$$

The number of complex multiplication required using FFT is

$$\begin{aligned} \frac{N}{2} \log_2^N &= \frac{1024}{2} \log_2^{1024} \\ &= 512 \log_2^{2^{10}} \\ &= 5120 \end{aligned}$$

Computation efficiency in multiplication

$$\begin{aligned} &= 100 - \frac{5120}{1048576} \times 100 \\ &= 99.5\% \end{aligned}$$

The number of complex multiplication required using direct computation is

$$N(N - 1) = 1024(1024 - 1) = 1047552$$

The number of complex multiplication required using FFT is

$$\begin{aligned} N \log_2^N &= 1024 \log_2^{1024} \\ &= 1024 \log_2^{2^{10}} \\ &= 10240 \end{aligned}$$

Computation efficiency in addition

$$\begin{aligned} &= 100 - \frac{10240}{1048576} \times 100 \\ &= 99\% \end{aligned}$$

**32. Write the analysis and synthesis equation of DFT.**

**Ans:** Analysis equation of DFT is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, \quad n = 0, 1, 2, \dots, N-1$$

Synthesis equation of DFT is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N} j}, \quad n = 0, 1, 2, \dots, N-1$$

**33. Determine the response of the system with  $y(n) = x(n - 1)$  for the input signal.**

$$x(n) = \begin{cases} |n|, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

**Ans:**

$$x(n) = \begin{cases} |n|, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$$x(n) = \{3, 2, 1, 0, 1, 2, 3\}$$



$$y(n) = x(n - 1)$$

**34. Calculate the multiplication reduction factor  $\alpha$  in computing 1024-point DFT in a radix-2 FFT algorithm.**

**Ans:** The number of complex multiplication required using direct computation is

$$N^2 = (1024)^2 = 1048576$$

The number of complex multiplication required using FFT is

$$\begin{aligned} \frac{N}{2} \log_2^N &= \frac{1024}{2} \log_2^{1024} \\ &= 512 \log_2^{2^{10}} \\ &= 5120 \end{aligned}$$

Multiplication reduction factor

$$\alpha = \frac{1048576}{5120} = 204.8$$

**35. Find the values of  $W_N^k$  when  $N = 8$  and  $k = 2$  and also for  $k = 3$ .**

$$W_N^k = e^{-\frac{j2\pi k}{N}}$$

**Ans:** For  $N = 8$  and  $k = 2$ ,

$$W_N^k = W_8^2 = e^{-\frac{j2\pi 2}{8}} = -j$$

For  $N = 8$  and  $k = 3$ ,

$$W_N^k = W_8^3 = e^{-\frac{j2\pi 3}{8}} = -0.707 - j0.707$$

**36. State and prove Parsevals' relation DFT.**

**Ans:** For complex-valued sequence  $x(n)$  and  $y(n)$ , if

$$x(n) \longleftrightarrow_N^{\text{DFT}} X(k) \quad \text{and} \quad y(n) \longleftrightarrow_N^{\text{DFT}} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k)$$

**Proof**

$$\begin{aligned} \text{IDFT}\{R_{xy}(k)\} &= \Gamma_{xy}(l) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k)e^{\frac{j2\pi kl}{N}} \end{aligned}$$

using cross correlation

$$\begin{aligned} \sum_{k=0}^{N-1} x(n)Y^*(n-l) &= \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k)e^{\frac{j2\pi kl}{N}} \\ \sum_{k=0}^{N-1} x(n)Y^*(n) &= \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k) \quad [\text{when } l=0] \\ \sum_{k=0}^{N-1} x(n)Y^*(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k) \end{aligned}$$

If  $x(n) = y(n)$

$$\sum_{k=0}^{N-1} x(n)x^*(n-l) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)X^*(k)$$

$$\sum_{k=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

**37. What do you mean by the term “bit reversal” as applied to FFT?**

**Ans:** In FFT, before decimation the sequences are arranged in bit reversed order. Bit reversal process for  $N = 8$  is

Normal order		Bit reversal order	
Input sample	Binary representation	Bit reversal binary	Bit reversed sample input
$x(0)$	000	000	$x(0)$
$x(1)$	001	100	$x(4)$
$x(2)$	010	010	$x(2)$
$x(3)$	011	110	$x(6)$
$x(4)$	100	001	$x(1)$
$x(5)$	101	101	$x(5)$
$x(6)$	110	011	$x(3)$
$x(7)$	111	111	$x(7)$

**38. The first five DFT coefficients of a sequence  $x(n)$  are  $X(0) = 20$ ,  $X(1) = 5 + j2$ ,  $X(2) = 20$ ,  $X(3) = 0.2 + j0.4$  and  $X(4) = 0$ . Determine the remaining DFT coefficients.**

**Ans:** Let  $N = 8$ . We have  $X(k) = X^*(N - k)$ . The first five DFT coefficients are given, and the remaining DFT coefficients are obtained as,

$$X(5) = X^*(8 - 5) = X^*(3) = 0.2 - j0.4$$

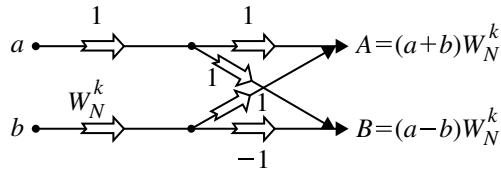
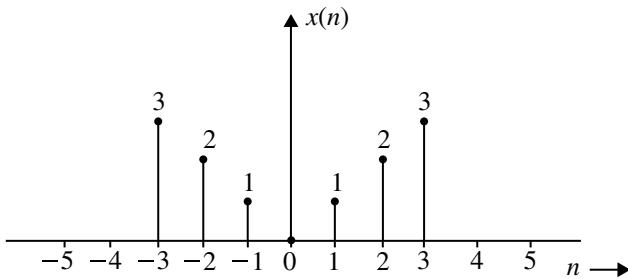
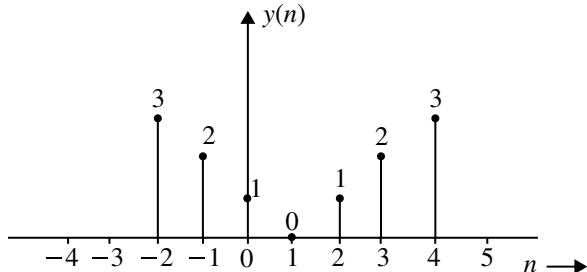
$$X(6) = X^*(8 - 6) = X^*(2) = 20$$

$$X(7) = X^*(8 - 7) = X^*(1) = 5 - j2$$

**39. What are the advances of FFT algorithm over direct computation of DFT?**

**Ans:**

1. FFT algorithm reduces the number of complex multiplication and complex addition operation.
2. It reduces the computation time required to compute DFT.

**Fig. 2.66** Butterfly diagram for question 21**Fig. 2.67** Input sequence for question 33**Fig. 2.68** Output sequence for question 33

The input sequence of question 33 is shown in Fig. 2.67. The output sequence of question 33 is shown in Fig. 2.68.

### Long Answer Type Questions

- Find eight-point DFT of the sequence  $x(n) = a^n$  where  $a = 3$ .
- Compute the DFTs of the following sequences where  $N = 4$  using DIT algorithm.
  - $x(n) = 2^n$
  - $x(n) = 2^{-n}$
  - $x(n) = \sin\left(\frac{n\pi}{2}\right)$
  - $x(n) = \cos\left(\frac{n\pi}{2}\right)$
- Compute the 16-point DFTs of the following sequences:

$$x(n) = \cos\left(\frac{\pi}{2}\right), \quad 0 \leq n \leq 15$$

using DIT algorithm.

4. Compute the FFT for the sequence  $x(n) = n^2 + 1$  where  $N = 8$  using DIT algorithm..
5. Compute the DFT coefficients of a finite duration sequence  $\{0,1,2,3,0,0,0,0\}$ .
6. Compute the IDFT of the following sequences using (a) DIT algorithm and (b) DIF algorithm.
  - (i)  $\{5, 0, 1 - j, 0, 1, 0, 1 + j, 0\}$
  - ii)  $\{1, 1 + j, 1 - j2, 1, 0, 1 + j2, 1 + j\}$
7. Derive the equation for radix-4 FFT for  $N = 4$  and draw the butterfly diagram.
8. Derive the DFT of the sample data sequence  $x(n) = \{1, 1, 1, 0\}$  and compute the corresponding amplitude and phase spectrum.
9. Compute the DFT for the sequence  $\{1, 2, 0, 0, 0, 2, 1, 1\}$  using radix-2 DIF-FFT algorithm.
10. Prove that multiplication of the DFT's of two sequences is equivalent to the DFT of the circular convolution of the two sequences in time domain.
11. Using DFT-IDFT method, perform circular convolution of the two sequences  $x(n) = \{1, 2, 0, 1\}$  and  $h(n) = \{2, 2, 1, 1\}$ .
12. By means of DFT and IDFT, determine the response of an FIR filter with impulse response  $h(n) = \{1, 2, 3\}$  to the input sequence  $x(n) = \{1, 2, 2, 1\}$ .
13. Develop a DIT-FFT algorithm for  $N = 12 = 3 \cdot 2 \cdot 2$  and draw the signal flow diagram.
14. The first five DFT coefficient are  $\{22, -7.5353 - j3.1213, 1 + j, -0.4645 - j1.1213, 0\}$ . Determine the remaining DFT coefficients for  $N = 16$ .
15. Determine the eight-point DFT of the sequence

$$x(n) = \begin{cases} 1, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

16. Convolve the following sequences using (i) overlap-add method and (ii) overlap save method.

$$x(n) = \{1, -1, 2, 1, 2, -1, 1, 3, 1\} \quad \text{and} \quad h(n) = \{1, 2, 1\}$$

17. An input sequence  $x(n) = \{2, 1, 0, 1, 2\}$  is applied to a DSP system having an impulse sequence  $h(n) = \{5, 3, 2, 1\}$ . Determine the output sequence produced by (i) linear convolution and (ii) verify the same through circular convolution.

18. Determine the DFT of the sequence

$$x(n) = \begin{cases} 1, & 2 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

Assume that  $N = 10$ .

19. Derive and draw the radix-4 DIT and DFT algorithm for FFT of 8 points.
20. Discuss in detail the use of FFT algorithm in linear filtering.

# Chapter 3

## Design of IIR Digital Filters



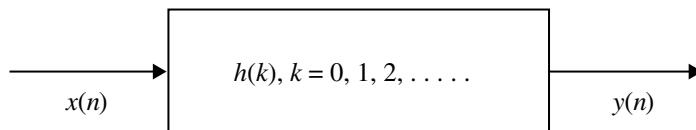
### Learning Objectives

After completing this chapter, you should be able to:

- ✖ study the characteristics and classify digital filters.
- ✖ establish the causality and stability conditions necessary when analog filters are converted into digital filters.
- ✖ design digital filters from analog filters using impulse invariance and bilinear transformation methods.
- ✖ design digital lowpass filters by Butterworth, Chebyshev approximations and approximation of derivatives.
- ✖ study analog and digital frequency transformations and design filters of different types.
- ✖ draw frequency response of digital filters from the system function.
- ✖ realize the structure of digital filters in different forms.

### 3.1 Introduction

A filter is a network or system that changes the frequency response (amplitude vs. frequency and phase vs. frequency) characteristics of a signal in a desired manner. The filter extracts information from the signal by removing noise. It is also used to separate two or more signals which can make use of the available communication channel very efficiently. Filters are broadly classified as analog filters and digital filters. Analog filters operate on analog signals. They are governed by differential equations. They consist of electric components like resistors, capacitors and inductors. On the other hand, digital filters operate on digital samples of the signal. They are governed by linear difference equation. They consist of adders, multipliers and delay implemented in logic.



**Fig. 3.1** A digital filter

Digital filters play very important role in Digital Signal Processing. They are extensively used in data transmission, data compression, echo cancelation, image processing, speech processing, telephone, digital audio, biomedical signal processing, etc. Digital filters have the following advantages over analog filters.

### 3.1.1 *Advantages*

1. Digital filters are used in low-frequency devices used in biomedical applications where analog filters cannot be used.
2. They have linear phase response characteristic.
3. The performance of digital filters does not vary with environmental changes.
4. Filtered and unfiltered data can be saved for future use.
5. They are cheap and consume low power. Precision is high compared to analog filters.
6. The frequency response can be automatically adjusted using programmable processor.

### 3.1.2 *Disadvantages*

The disadvantages of digital filters include the following:

1. The time taken for the design and hardware development is longer compared to analog filters.
2. The maximum bandwidth of signals that can be handled by digital filters is lower compared to analog filters.
3. Digital filters are subject to ADC noise due to quantization and round off noise produced during computation.

### 3.2 IIR and FIR Filters

Digital filters are broadly classified as infinite impulse response (IIR) and finite impulse response (FIR) filters. The impulse response of the filters can be represented by the sequence  $h(k)$ , ( $k = 0, 1, 2, \dots$ ) as in Fig. 3.1.

From Fig. 3.1, the following equations are written for the convolution sum.

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \quad (3.1)$$

$$y(n) = \sum_{k=0}^{N-1} h(k)x(n-k) \quad (3.2)$$

Equation (3.1) gives the output of an IIR filter while Eq. (3.2) gives the output of an FIR filter, where the response is of infinite and finite duration, respectively. The output of IIR filter using Eq. (3.1) cannot be computed in practice because the length of the impulse is too long. Equation (3.1) is therefore modified and expressed in recursive form as given below:

$$y(n) = \sum_{R=0}^N b_k x(n-k) - \sum_{k=1}^M a_k y(n-k) \quad (3.3)$$

In Eq. (3.3),  $a_k$  and  $b_k$  are the coefficients of IIR filter. Equations (3.2) and (3.3) are used to design FIR and IIR filters, respectively. It is to be noted that, the output of the IIR filter (recursive filter) as seen from Eq. (3.3), depends on past outputs as well as present and past input samples. Further, Eq. (3.3) shows the inherent feedback present in the IIR filter. On the other hand, the output of the FIR filter (non-recursive filter) as seen from Eq. (3.2) depends on the past and present values of the input samples. Equations (3.2) and (3.3) can also be represented in terms of the transfer function as given below:

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k} \quad (3.4)$$

$$H(z) = \frac{\sum_{k=0}^N b_k z^{-k}}{1 + \sum_{k=1}^M a_k z^{-k}} \quad (3.5)$$

Equations (3.4) and (3.5), respectively, are the transfer functions of FIR and IIR filters.

### 3.3 Basic Features of IIR Filters

The transfer function of IIR filter is given in Eq. (3.5). For the IIR filter design, it is necessary to find suitable values for the coefficients  $a_k$  and  $b_k$  so that the desired frequency response is obtained. The design stages for digital IIR filters are as follows:

1. Filter specification: The function of the filter (lowpass, highpass, bandpass, band rejected filter) and the desired performance are given.
2. Calculation of coefficients  $a_k$  and  $b_k$ .
3. Structure realization.
4. Finite word length effects analysis and solutions.
5. Hardware and software implementation.

### 3.4 Performance Specifications

For the design of IIR filters, the following specifications are normally followed.

1. Signal characteristics.
2. The frequency response characteristics of the filter.
3. Cost constraints.
4. Mode of implementation.

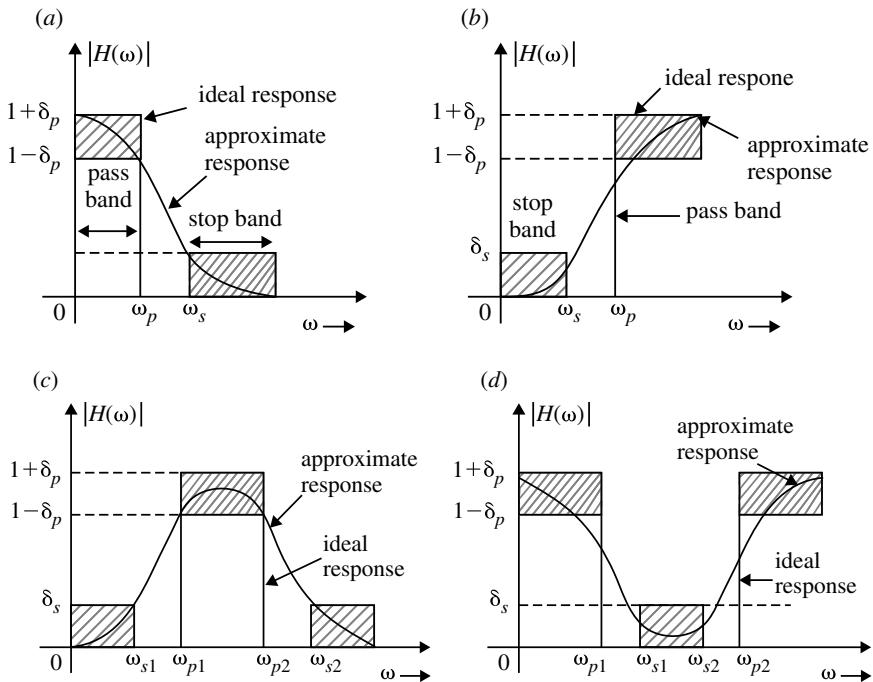
The frequency response specifications of frequency selective filters such as lowpass, highpass, bandpass and bandstop filters are expressed in the form of tolerance scheme as shown in Fig. 3.2.

Analog filter design is a well-established one. Several techniques have been developed to design digital filters which are all based on converting an analog filter into a digital filter. The digital filter is therefore first designed in the analog domain, and then the design is converted into the digital domain. For an analog filter, the system function can be represented by the following equation:

$$\begin{aligned} H_a(s) &= \frac{B(s)}{A(s)} \\ &= \frac{\sum_{k=0}^M \beta_k s^k}{\sum_{k=0}^N \alpha_k s^k} \end{aligned} \quad (3.6)$$

where  $[\alpha_k]$  and  $[\beta_k]$  are the coefficients of the filter. The system function  $H_a(s)$  can also be written by its impulse response as given below:

$$H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt \quad (3.7)$$



**Fig. 3.2** Magnitude response. **a** Lowpass, **b** highpass, **c** bandpass and **d** bandstop filter

The analog filter input and output equation can also be represented by the following equations:

$$\sum_{k=0}^N \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M \beta_k \frac{d^k x(t)}{dt^k} \quad (3.8)$$

where  $x(t)$  is input signal and  $y(t)$  is the output response. The above equations, which represent the characteristics of analog filter in three different forms, are converted to their equivalent forms in digital domain while designing a digital filter. It is to be noted that the conversion becomes valid and effective if the following conditions are satisfied. If  $H_a(s)$  is the system function of a linear time invariant stable system, then all its poles should fall in left half of the  $s$ -plane (LHP). On that basis, when analog filter is converted to digital filter, then it should satisfy the following stability conditions:

1. The  $j$  axis in the  $s$ -plane should map into unit circle in the  $z$ -plane. This results in a direct relationship between the two frequency variables in  $s$ -domain and  $z$ -domain.
2. The LHP of the  $s$ -plane should map into the interior of the unit circle in the  $z$ -plane. This guarantees the stability of the digital filter when converted from analog filter.

3. The T.F.s  $H_a(s)$  and  $H(z)$  should be rational, so that the filter is causal.

The following methods of digital filter design by converting analog filter are described:

1. Impulse invariance transform method.
2. Bilinear transformation method.

### 3.5 Impulse Invariance Transform Method

The impulse response of the digital filter is obtained by uniformly sampling the impulse response of the analog filter.

$$\begin{aligned} h(n) &= h_a(t) \Big|_{t=nT} = h_a(nT) \\ H_a(s) &= L[h_a(t)] \end{aligned} \quad (3.9)$$

When  $H_a(s)$  has distinct poles, it can be expressed by partial fraction expression as

$$\begin{aligned} H_a(s) &= \sum_{i=1}^N \frac{A_i}{s - p_i} = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \cdots + \frac{A_N}{s - p_N} \\ h_a(t) &= A_1 e^{p_1 t} u_a(t) + A_2 e^{p_2 t} u_a(t) + \cdots + A_N e^{p_N t} u_a(t) \end{aligned} \quad (3.10)$$

$$\begin{aligned} \therefore h(n) &= h_a(t) \Big|_{t=nT} \\ h(n) &= A_1 e^{p_1 nT} u_a(nT) + A_2 e^{p_2 nT} u_a(nT) + \cdots + A_N e^{p_N nT} u_a(nT) \end{aligned} \quad (3.11)$$

$$\begin{aligned} H(z) &= Z[h(n)] \\ &= A_1 \frac{1}{1 - e^{p_1 T} z^{-1}} + A_2 \frac{1}{1 - e^{p_2 T} z^{-1}} + \cdots + A_N \frac{1}{1 - e^{p_N T} z^{-1}} \\ &= \sum_{i=1}^N \frac{A_i}{1 - e^{p_i T} z^{-1}} \end{aligned} \quad (3.12)$$

Therefore, by impulse invariant transformation

$\frac{1}{s - p_i}$  is transformed to  $\frac{1}{1 - e^{p_i T} z^{-1}}$

(3.13)

### 3.5.1 Relation Between Analog and Digital Filter Poles

Let the analog poles be at  $s = p_i$  where  $i = 1, 2, \dots, N$ . The digital poles are given by roots of term  $(1 - e^{p_i T} z^{-1})$ , i.e.,

$$z = e^{p_i T}, \quad i = 1, 2, \dots, N \quad (3.14)$$

Any point in the  $s$ -plane can be expressed as

$$\begin{aligned} s &= \sigma_1 + j\Omega_1 \\ z &= e^{\sigma_1 T} \\ z &= e^{sT} \end{aligned} \quad (3.15)$$

i.e., analog pole at  $s = p_i$  is transformed to digital pole at  $z = e^{p_i T}$

$$\begin{aligned} z &= e^{(\sigma_1 + j\Omega_1)T} \\ z &= e^{\sigma_1 T} \cdot e^{j\Omega_1 T} \\ |z| \angle z &= e^{\sigma_1 T} \cdot e^{j\Omega_1 T} \\ |z| &= e^{\sigma_1 T} \quad \text{and} \quad \angle z = \Omega_1 T \end{aligned} \quad (3.16)$$

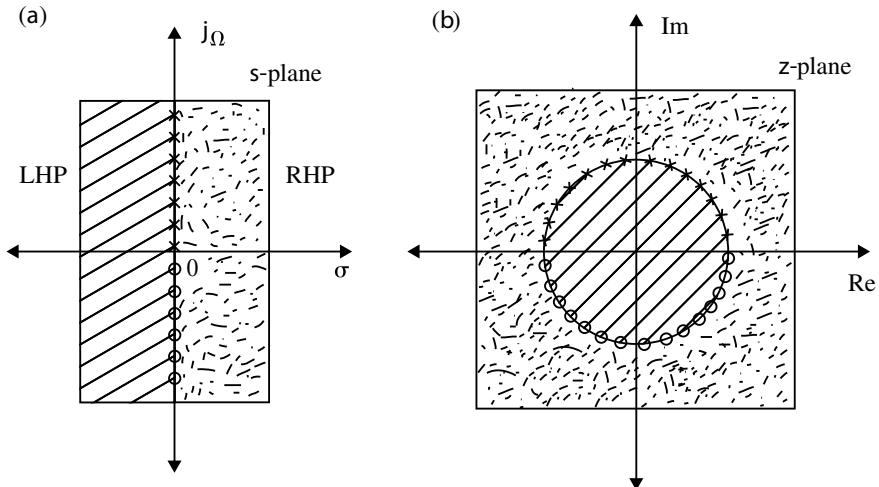
- (i) If  $\sigma_1 < 0$ , then the analog pole  $s$  lies on left half of  $s$ -plane. In this case,  $|z| < 1$ , and hence the corresponding digital pole  $z$  will lie inside the unit circle in the  $z$ -plane.
- (ii) If  $\sigma_1 = 0$ , then the analog pole  $s$  lies on imaginary axis of  $s$ -plane. Therefore,  $|z| = 1$ , and hence the corresponding digital pole  $z$  will lie on the unit circle in the  $z$ -plane.
- (iii) If  $\sigma_1 > 0$ , then the analog pole  $s$  lies on right half of  $s$ -plane. In this case,  $|z| = 1$ , and hence the corresponding digital pole  $z$  will lie outside the unit circle in  $z$ -plane as shown in Fig. 3.3.

The stability of a filter is related to the location of the poles. For a stable analog filter the poles should lie on the left half of the  $s$ -plane. Since the left half of  $s$ -plane maps inside the unit circle in  $z$ -plane we can say that, for a stable digital filter the poles should lie inside the unit circle in  $z$ -plane.

### 3.5.2 Relation Between Analog and Digital Frequency

Let  $\Omega$  be the analog frequency in rad/s, and  $\omega$  is digital frequency in rad/s.

$$\begin{aligned} z &= r e^{j\omega} \\ s &= \sigma + j\Omega \end{aligned} \quad (3.17)$$



**Fig. 3.3** *s*-plane to *z*-plane mapping

By impulse invariant transformation, the following equations are written

$$\begin{aligned} z &= e^{sT} \\ re^{j\omega} &= e^{(\sigma+j\Omega)T} \\ re^{j\omega} &= e^{\sigma T} \cdot e^{j\Omega T} \end{aligned}$$

$$\boxed{\omega = \Omega T \quad \text{or} \quad \Omega = \frac{\omega}{T}} \quad (3.18)$$

The mapping of analog-to-digital frequency is not one to one. Since  $\omega$  is unique over the range  $(-\pi, \pi)$ , i.e.,  $\omega = \Omega T$  implies that the interval  $-\pi/T \leq \Omega \leq (\pi/T)$  maps into the corresponding values of  $-\pi \leq \omega \leq \pi$ . This reflects the effects of aliasing in sampling.

The following impulse invariant transformations from *s*-plane to *z*-plane are used for different pole locations:

1. For distinct poles

$$\frac{1}{s - p_i} \longrightarrow \frac{1}{1 - e^{p_i T} z^{-1}} \quad (3.19)$$

2. For multiples poles

$$\frac{1}{(s + p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dp_i^{m-1}} \left( \frac{1}{1 - e^{-p_i T} z^{-1}} \right) \quad (3.20)$$

### 3. For complex poles

$$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad (3.21)$$

$$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad (3.22)$$

## 3.6 Bilinear Transformation Method

IIR filter design by means of impulse invariance method has a serve limitation in that it can be applied only for lowpass filters and a few limited bandpass filters. This serious limitation is overcome by bilinear transformation which is described below.

The bilinear transformation is a conformal mapping that transforms the imaginary axis of  $s$ -plane into the unit circle in the  $z$ -plane only once, thus avoiding aliasing of frequency components. In this mapping all points in the left half of  $s$ -plane are mapped inside the unit circle in the  $z$ -plane and all points in the right half of  $s$ -plane are mapped outside the unit circle in the  $z$ -plane.

Consider an analog filter with the following system function

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{(s+a)} \quad (3.23)$$

Equation (3.23) is written in differential equation form as

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \quad (3.24)$$

$y(t)$  can be expressed in terms of its derivative  $y'(t)$  with the initial condition  $y(t_0)$  as given below.

$$y(t) = \int_{t_0}^t y'(\tau)d\tau + y(t_0). \quad (3.25)$$

By using the trapezoidal formula at  $t = nT$  and  $t_0 = nT - T$ , the above integral is approximated as

$$y(nT) = \frac{T}{2}[y'(nT) + y'(nT - T)] + y(nT - T) \quad (3.26)$$

Now Eq. (3.24) is written as follows.

$$y'(nT) = -ay(nT) + bx(nT) \quad (3.27)$$

Substituting Eq. (3.26) in Eq. (3.27), the following difference equation for the discrete system is obtained.

$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}[x(n) + x(n-1)] \quad (3.28)$$

Taking  $z$ -transform on both sides of the above equation we get,

$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}[1 + z^{-1}]X(z) \quad (3.29)$$

Rearranging the terms, the discrete system function is written as follows.

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{\left(\frac{bT}{2}\right)(1+z^{-1})}{\left(1 + \frac{aT}{2}\right) - \left(1 - \frac{aT}{2}\right)z^{-1}} \\ &= \frac{b}{\left[\frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a\right]} \end{aligned} \quad (3.30)$$

Comparing Eq. (3.23) with Eq. (3.30), the following relationship becomes valid while mapping from  $s$ -plane to  $z$ -plane.

$$s = \frac{2}{T} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)$$

(3.31)

Equation (3.31) is called the bilinear transformation. This holds good for any  $N$ th order system. The following examples illustrate the method of transforming  $H_a(s)$  to digital system function  $H(z)$ .

### Example 3.1

All-pole analog filters have transfer function

$$H(s) = \frac{1}{(s^2 + 5s + 6)}$$

Find  $H(z)$  by impulse invariant technique. Assume  $T = 1$  s.

(Anna university, April, 2005)

**Solution**

$$(s^2 + 5s + 6) = (s + 2)(s + 3)$$

$$H(s) = \frac{1}{(s + 2)(s + 3)}$$

$$= \frac{A}{(s + 2)} + \frac{B}{(s + 3)}$$

$$H(s) = \frac{1}{(s + 2)} - \frac{1}{(s + 3)}$$

By applying impulse invariant transformation we get

$$H(z) = \frac{1}{(1 - e^{-2T}z^{-1})} - \frac{1}{(1 - e^{-3T}z^{-1})}$$

Substituting  $T = 1$ , we get

$$H(z) = \frac{1}{(1 - 0.135z^{-1})} - \frac{1}{(1 - 0.05z^{-1})}$$

$$H(z) = \frac{0.085z^{-1}}{(1 - 0.135z^{-1})(1 - 0.05z^{-1})}$$

**Example 3.2**

Design a digital filter using

$$H(s) = \frac{1}{s^2 + 9s + 18}$$

with  $T = 0.2\text{ s}$ .

(Anna University, May, 2007)

**Solution**

$$(s^2 + 9s + 18) = (s + 3)(s + 6)$$

$$H(s) = \frac{1}{(s + 3)(s + 6)}$$

$$= \frac{1}{3} \left[ \frac{1}{s + 3} - \frac{1}{s + 6} \right]$$

$$H(z) = \frac{1}{3} \left[ \frac{1}{(1 - e^{-3T}z^{-1})} - \frac{1}{(1 - e^{-6T}z^{-1})} \right]$$

Substituting  $T = 0.2$  s. we get

$$H(z) = \frac{1}{3} \left[ \frac{1}{(1 - 0.55z^{-1})} - \frac{1}{(1 - 0.3z^{-1})} \right]$$

$$H(z) = \frac{0.083z^{-1}}{(1 - 0.55z^{-1})(1 - 0.3z^{-1})}$$

### Example 3.3

Consider the following TF of an analog filter

$$H(s) = \frac{(s + 1)}{(s + 1)^2 + 1}$$

Convert the analog filter into a digital IIR filter by impulse invariance method for  $T = 1$ .

#### Solution Method 1

$$\begin{aligned} H(s) &= \frac{(s + 1)}{(s + 1)^2 + 1} \\ &= \frac{(s + 1)}{s^2 + 2s + 2} \\ &= \frac{(s + 1)}{(s + 1 + j)(s + 1 - j)} \\ &= \frac{A_1}{(s + 1 + j)} + \frac{A_1^*}{(s + 1 - j)} \\ A_1 &= \frac{A(s + 1)}{(s + 1 + j)} \Big|_{s=-1+j} = \frac{1}{2} \\ A_1^* &= \frac{1}{2} \end{aligned}$$

By impulse invariant transformation the TF of digital IIR filter is obtained as follows:

$$H(z) = \frac{1}{2} \left[ \frac{1}{1 - e^{-(1+j)T} z^{-1}} \right] + \frac{1}{2} \left[ \frac{1}{1 - e^{(-1+j)T} z^{-1}} \right]$$

Substituting  $T = 1$ , we get

$$\begin{aligned} H(z) &= \frac{1}{2} \left[ \frac{1}{1 - 0.3678e^{-j}z^{-1}} \right] + \frac{1}{2} \left[ \frac{1}{1 - 0.3678e^{j}z^{-1}} \right] \\ &= \frac{1 - 0.3678e^{j}z^{-1} + 1 - 0.3678e^{-j}z^{-1}}{2(1 - 0.3678e^{-j}z^{-1})(1 - 0.3678e^{j}z^{-1})} \\ &= \frac{1 - 0.3678 \cos 1z^{-1}}{1 - 0.3678z^{-1}(e^j + e^{-j}) + 0.1353z^{-2}} \end{aligned}$$

$$H(z) = \frac{(1 - 0.2z^{-1})}{(1 - 0.4z^{-1} + 0.1353z^{-2})}$$

**Method 2:** Using Eq. (3.21), we get

$$\frac{(s + a)}{(s + a)^2 + b^2} = \frac{1 - e^{-aT} \cos bTz^{-1}}{1 - 2e^{-aT} \cos bTz^{-1} + 2e^{-2aT}z^{-2}}.$$

Here,  $a = 1$ ;  $b = 1$ ;  $T = 1$ .

$$H(z) = \frac{(1 - 0.2z^{-1})}{(1 - 0.4z^{-1} + 0.1353z^{-2})}$$

### Example 3.4

Using impulse invariant method find  $H(z)$  at  $T = 1$  s.

$$H(z) = \frac{2}{(s^2 + 8s + 15)}$$

(Anna University, December, 2003)

### Solution

$$\begin{aligned} (s^2 + 8s + 15) &= (s + 3)(s + 5) \\ H(s) &= \frac{2}{(s + 3)(s + 5)} \\ &= \frac{1}{s + 3} - \frac{1}{s + 5} \end{aligned}$$

$$\begin{aligned}
 H(z) &= \frac{1}{(1 - e^{-3T}z^{-1})} - \frac{1}{(1 - e^{-5T}z^{-1})} \\
 &= \frac{1}{(1 - 0.05z^{-1})} - \frac{1}{(1 - 6.73 \times 10^{-3}z^{-1})}
 \end{aligned}$$

$$H(z) = \frac{0.04326z^{-1}}{(1 - 0.05z^{-1})(1 - 6.73 \times 10^{-3}z^{-1})}$$

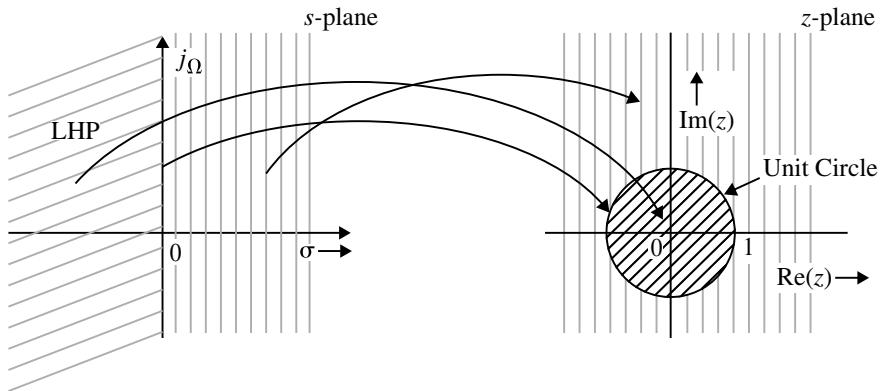
Hence, in the  $s$ -domain transfer function, if  $s$  is substituted by the term  $\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$ , then the resulting transfer function will be  $z$ -domain transfer function.

### 3.6.1 Relation Between Analog and Digital Filter Poles

The mapping of  $s$ -domain function to  $z$ -domain function by bilinear transformation is a one-to-one mapping; i.e., for every point in  $s$ -plane, there is exactly a corresponding point in  $z$ -plane and vice versa.

$$\begin{aligned}
 s &= \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} & (3.32) \\
 \frac{Ts}{2} &= \frac{1-\frac{1}{z}}{1+\frac{1}{z}} \\
 \frac{Ts}{2} &= \frac{z-1}{z+1} \\
 \frac{Ts}{2}(z+1) &= z-1 \\
 \frac{Ts}{2}z - z &= -1 - \frac{Ts}{2} \\
 -z \left(1 - \frac{Ts}{2}\right) &= -\left(1 + \frac{Ts}{2}\right)
 \end{aligned}$$

$$z = \frac{1 + \frac{Ts}{2}}{1 - \frac{Ts}{2}} \quad (3.33)$$



**Fig. 3.4** Mapping of  $s$ -plane into  $z$ -plane in bilinear transformation

Let  $s = \sigma_1 + j\Omega_1$

$$z = \frac{1 + \frac{T}{2}(\sigma_1 + j\Omega_1)}{1 - \frac{T}{2}(\sigma_1 + j\Omega_1)} = \frac{(1 + \frac{T}{2}\sigma_1) + j\frac{T}{2}\Omega_1}{(1 - \frac{T}{2}\sigma_1) + j\frac{T}{2}\Omega_1} \quad (3.34)$$

$$|z| = \sqrt{\frac{(1 + \frac{T}{2}\sigma_1)^2 + (\frac{T}{2}\Omega_1)^2}{(1 - \frac{T}{2}\sigma_1)^2 + (\frac{T}{2}\Omega_1)^2}} \quad (3.35)$$

1. If  $\sigma_1 < 0$  (i.e.,  $\sigma_1$  is negative), then the point  $s = \sigma_1 + j\Omega_1$  lies on the left half of  $s$ -plane. In this case,  $|z| < 1$ , and hence the corresponding point in  $z$ -plane will lie inside the unit circle in  $z$ -plane.
2. If  $\sigma_1 = 0$  (i.e., real part is zero), then the point  $s = \sigma_1 + j\Omega_1$  lies on the imaginary axis in the  $s$ -plane. In this case,  $|z| = 1$ , and hence the corresponding point in  $z$ -plane will lie on the unit circle in  $z$ -plane.
3. If  $\sigma_1 > 0$  (i.e.,  $\sigma_1$  is positive), then the point  $s = \sigma_1 + j\Omega_1$  lies on the right half of  $s$ -plane. In this case,  $|z| > 1$ , and hence the corresponding point in  $z$ -plane will lie outside the unit circle in  $z$ -plane.

The above three transformations are represented in Fig. 3.4.

### 3.6.2 Relation Between Analog and Digital Frequency

Let  $s = j\Omega$  be points on imaginary axis and the corresponding points on the  $z$ -plane on unit circle are given by  $z = e^{j\omega}$  where  $\Omega$  is the analog frequency and  $\omega$  is the digital frequency. Using bilinear transformation

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

and substituting  $s = j\Omega$  and  $z = e^{j\omega}$ , we get

$$\begin{aligned} j\Omega &= \frac{2}{T} \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{2}{T} \left[ \frac{e^{(j\omega)/2} \cdot e^{-(j\omega)/2} - e^{-j\omega}}{e^{(j\omega)/2} \cdot e^{-(j\omega)/2} + e^{-j\omega}} \right] \\ &= \frac{2 e^{-(j\omega)/2} (e^{(j\omega)/2} - e^{-(j\omega)/2})}{T e^{-(j\omega)/2} (e^{(j\omega)/2} + e^{-(j\omega)/2})} \\ &= \frac{2 (e^{(j\omega)/2} - e^{-(j\omega)/2})/2}{T (e^{(j\omega)/2} + e^{-(j\omega)/2})/2} \end{aligned}$$

$$\begin{aligned} \Omega &= \frac{2}{T} \frac{(e^{(j\omega)/2} - e^{-(j\omega)/2})/2j}{\cos(\omega/2)} \\ &= \frac{2}{T} \frac{\sin(\omega/2)}{\cos(\omega/2)} = \frac{2}{T} \tan \frac{\omega}{2} \end{aligned}$$

Analog frequency

$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$

(3.36)

Digital frequency

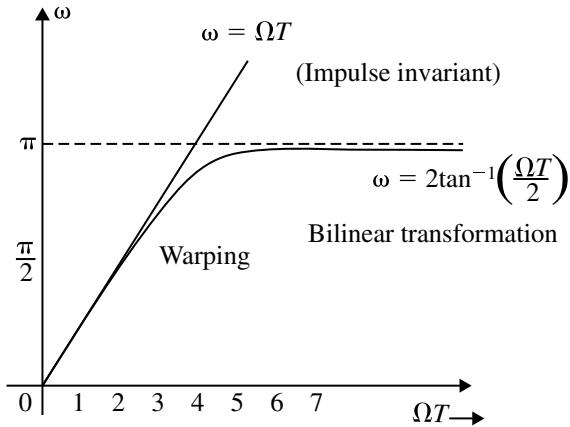
$\omega = 2 \tan^{-1} \frac{\Omega T}{2}$

(3.37)

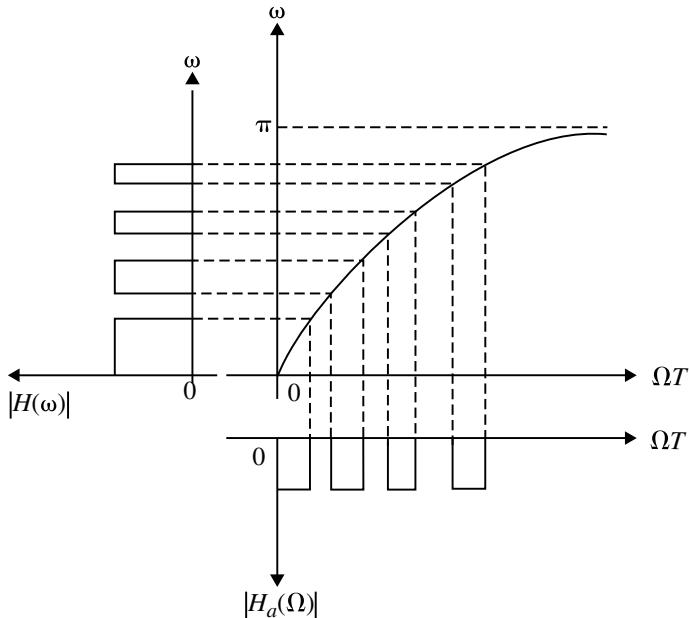
The analog frequency  $\Omega$  and digital frequency  $\omega$  have a nonlinear relationship and is shown in Fig. 3.5 because the entire negative imaginary axis in the  $s$ -plane ( $\Omega = -\infty$  to 0) is mapped into the lower half of unit circle in  $z$ -plane ( $\omega = -\pi$  to 0) and the entire positive imaginary axis in the  $s$ -plane ( $\Omega = 0$  to  $\infty$ ) is mapped into upper half of unit circle in  $z$ -plane ( $\omega = 0$  to  $\pi$ ). This nonlinear mapping introduces a distortion in the frequency axis which is called frequency warping.

### 3.6.3 Effect of Warping on the Magnitude Response

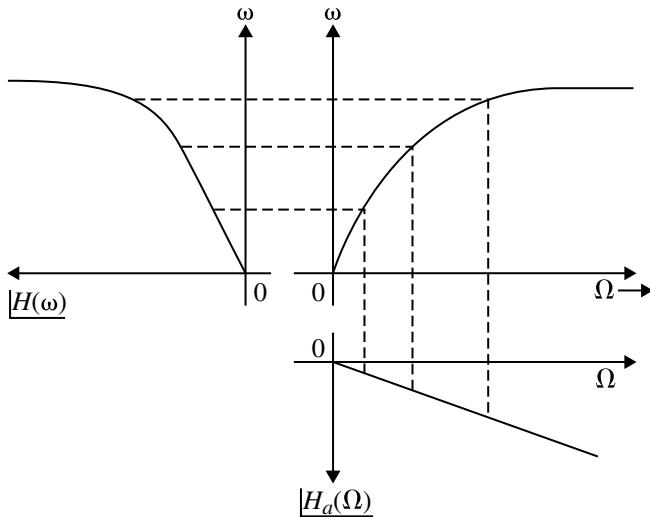
Consider an analog filter with a number of passbands. The corresponding digital filter will have same number of passbands, but with disproportionate bandwidth as shown in Fig. 3.6. The passband in the analog filter is of constant width with regular



**Fig. 3.5** Correspondence between analog and digital frequencies



**Fig. 3.6** Effect of warping on the magnitude response. (Equally spaced analog passbands are pushed together at high frequency in the digital domain)



**Fig. 3.7** Effect of warping on the phase response

intervals. However, the passbands for the digital equivalent are somewhat pushed together.

The effect of warping on amplitude responses can be eliminated by pre-warping the analog filter. In this method, the specified digital frequencies are converted to analog equivalent ( $\Omega = \frac{2}{T} \tan(\omega/2)$ ). These analog frequencies are called pre-warps frequencies. Using the pre-warps frequencies, the analog filter transfer function is designed and then it is transformed to digital filter transfer function.

### 3.6.4 Effect of Warping on the Phase Response

Consider an analog filter with linear phase. The phase response of corresponding digital filter will be nonlinear as shown in Fig. 3.7.

From the above, it can be stated that, the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude, but the phase response of the analog filter is not preserved. Therefore, the bilinear transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values. A linear phase analog filter cannot be transformed to a linear phase digital filter using bilinear transformation.

**Example 3.5**

Convert the analog filter with system transfer function

$$H_a(s) = \frac{2}{(s+1)(s+2)}$$

into a digital IIR filter by impulse invariance method, when (i)  $T = 1$  s. and (ii)  $T = 0.1$  s.

(Anna University, December, 2004)

**Solution** Given

$$\begin{aligned} H_a(s) &= \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \\ A &= \left. \frac{2}{s+2} \right|_{s=-1} = 2 \\ B &= \left. \frac{2}{s+1} \right|_{s=-2} = -2 \\ H_a(s) &= \frac{2}{(s+1)} - \frac{2}{(s+2)} = \frac{2}{s-(-1)} - \frac{2}{s-(-2)} \end{aligned}$$

By impulse invariant transformation, we know that

$$\frac{A_p}{s-p_i} \xrightarrow{\text{is transformed to}} \frac{A_i}{1-e^{p_i T} z^{-1}}$$

$$H(z) = \frac{2}{1-e^{p_1 T} z^{-1}} - \frac{2}{1-e^{p_2 T} z^{-1}} \quad (\text{where } p_1 = -1, p_2 = -2)$$

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} - \frac{2}{1-e^{-2T} z^{-1}}$$

(i) When  $T = 1$  s

$$\begin{aligned} H(z) &= \frac{2}{1-e^{-1} z^{-1}} - \frac{2}{1-e^{-2} z^{-1}} \\ &= \frac{2}{1-0.3678 z^{-1}} - \frac{2}{1-0.1353 z^{-1}} \end{aligned}$$

$$H(z) = \frac{0.465 z^{-1}}{(1-0.3678 z^{-1})(1-0.1353 z^{-1})}$$

(ii) When  $T = 0.1$  s

$$\begin{aligned} H(z) &= \frac{2}{1 - e^{-0.1}z^{-1}} - \frac{2}{1 - e^{-0.2}z^{-1}} \\ &= \frac{2}{1 - 0.9048z^{-1}} - \frac{2}{1 - 0.8187z^{-1}} \end{aligned}$$

$$H(z) = \frac{0.1722z^{-1}}{(1 - 0.9048z^{-1})(1 - 0.8187z^{-1})}$$

### Example 3.6

Convert the analog filter with system transfer function

$$H_a(s) = \frac{(s + 0.1)}{(s + 0.1)^2 + 9}$$

into a digital IIR filter by using the impulse invariant method, when  $T = 1$  s

*(Anna University, May, 2006)*

**Solution** Given

$$\begin{aligned} H(s) &= \frac{(s + 0.1)}{(s + 0.1)^2 + 9} \\ &= \frac{s + 0.1}{s^2 + 0.2s + 9.01} \\ &= \frac{s + 0.1}{[s - (-0.1 + j3)][s - (-0.1 - j3)]} \\ &= \frac{A_1}{s - (-0.1 + j3)} + \frac{A_1^*}{s - (-0.1 - j3)} \\ A_1 &= \left. \frac{s + 0.1}{s - (-0.1 - j3)} \right|_{s=-0.1+j3} = \frac{1}{2} \\ A_1^* &= \frac{1}{2} \\ H(s) &= \frac{1/2}{s - (-0.1 + j3)} + \frac{1/2}{s - (-0.1 - j3)} \end{aligned}$$

By impulse invariant transformation we get,

$$\begin{aligned}
H(z) &= \frac{1/2}{1 - e^{(-0.1+j3)T}z^{-1}} + \frac{1/2}{1 - e^{(-0.1-j3)T}z^{-1}} \\
&= \frac{0.5 - 0.5e^{-0.1T}e^{-j3T}z^{-1} + 0.5 - 0.5e^{-0.1T}e^{j3T}z^{-1}}{(1 - e^{(-0.1+j3)T}z^{-1})(1 - e^{(-0.1-j3)T}z^{-1})} \\
&= \frac{1 - 0.5e^{-0.1T}z^{-1}(e^{j3T} + e^{-j3T})}{1 - e^{-0.1T} - e^{-j3T}z^{-1}e^{-0.1T}e^{j3T}z^{-1} + e^{-0.1T}e^{j3T}e^{-0.1T}e^{-j3T}z^{-2}} \\
&= \frac{1 - 0.5e^{-0.1T}z^{-1}2\cos 3T}{1 - e^{-0.1T}z^{-1}(e^{j3T} + e^{-j3T}) + e^{-0.2T}z^{-2}} \\
&= \frac{1 - \cos(3T)e^{-0.1T}z^{-1}}{1 - 2\cos(3T)e^{-0.1T}z^{-1} + e^{-0.2T}z^{-2}}
\end{aligned}$$

For  $T = 1$  s

$$H(z) = \frac{1 - \cos(3)e^{-0.1}z^{-1}}{1 - 2\cos(3)e^{-0.1}z^{-1} + e^{-0.2}z^{-2}}$$

where 3 is in radian

$$H(z) = \frac{1 + 0.8959z^{-1}}{1 + 1.7915z^{-1} + 0.8187z^{-2}}$$

### Example 3.7

Apply the bilinear transformation to

$$H_a(s) = \frac{2}{(s+1)(s+2)}$$

with  $T = 1$  s and find  $H(z)$ .

**Solution** Given

$$H_a(s) = \frac{2}{(s+1)(s+2)}$$

put  $s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$  in  $H_a(s)$  to get  $H(z)$

$$\begin{aligned}
H(z) &= \frac{2}{\left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} + 1\right) \left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} + 2\right)} \\
&= \frac{2}{\left(\frac{2-2z^{-1}+1+z^{-1}}{1+z^{-1}}\right) \left(\frac{2-2z^{-1}+2+2z^{-1}}{1+z^{-1}}\right)}
\end{aligned}$$

$$\begin{aligned} H(z) &= \frac{2(1+z^{-1})^2}{(3-z^{-1})4} \\ &= \frac{0.5(1+z^{-1})^2}{3-z^{-1}} \end{aligned}$$

$$H(z) = \frac{-0.5(z^{-1} + 1)^2}{(z^{-1} - 3)}$$

**Example 3.8**

The normalized transfer function of an analog filter is given by

$$Ha(s_n) = \frac{1}{s_n^2 + 1.414s_n + 1}$$

convert the analog filter to a digital filter with a cutoff frequency of  $0.4\pi$ , using bilinear transformation.

*(Anna University, April, 2006)*

**Solution** To preserve the magnitude response the pre-warping of analog filter has to be performed. For this analog cutoff frequency is determined. The analog transfer function is unnormalized by replacing  $s_n = s/\Omega_c$  using this analog cutoff frequency. Then the analog transfer function is converted to digital filter transfer function using bilinear transformation.

Given  $\omega_c = 0.4\pi$  rad/s. Let  $T = 1$  s

$$\begin{aligned} \Omega_c &= \frac{2}{T} \tan\left(\frac{0.4\pi}{2}\right) \\ &= 2 \tan\left(\frac{0.4\pi}{2}\right) = 1.453 \text{ rad/s.} \end{aligned}$$

Normalized analog transfer function is,

$$Ha(s_n) = \frac{1}{s_n^2 + 1.414s_n + 1}$$

The analog transfer function is unnormalized by replacing  $s_n$  by  $s/\Omega_c$

$$Ha(s) = \frac{1}{\left(\frac{s}{\Omega_c}\right)^2 + 1.414\left(\frac{s}{\Omega_c}\right) + 1}$$

$$\begin{aligned}
 &= \frac{\Omega_c^2}{s^2 + 1.414\Omega_c s + \Omega_c^2} \\
 &= \frac{2.11}{s^2 + 2.05s + 2.11}
 \end{aligned}$$

$$\begin{aligned}
 H(z) &= H_a(s) \Big|_{s=\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} \\
 &= \frac{2.11}{\left[ \frac{2(1-z^{-1})}{1+z^{-1}} \right]^2 + 2.05 \left[ \frac{2(1-z^{-1})}{1+z^{-1}} \right] + 2.11} \\
 &= \frac{2.11(1+z^{-1})^2}{4(1-z^{-1})^2 + 4.1(1-z^{-1})(1+z^{-1}) + 2.11(1+z^{-1})^2} \\
 H(z) &= \frac{2.11(1+2z^{-1}+z^{-2})}{10.21 - 3.78z^{-1} + 2.01z^{-2}} \\
 &= \frac{2.11(1+2z^{-1}+z^{-2})}{10.21(1 - 0.37z^{-1} + 0.197z^{-2})}
 \end{aligned}$$

$$H(z) = \frac{0.207(1+2z^{-1}+z^{-2})}{1 - 0.37z^{-1} + 0.197z^{-2}}$$

### 3.7 Specifications of the Lowpass Filter

Let  $\omega_p$  be passband digital frequency in rad/s and  $\omega_s$  be the stopband digital frequency in rad/s Let

$$A_p = |H(\omega)|_{\omega=\omega_p} \quad \text{and} \quad A_s = |H(\omega)|_{\omega=\omega_s} \quad (3.38)$$

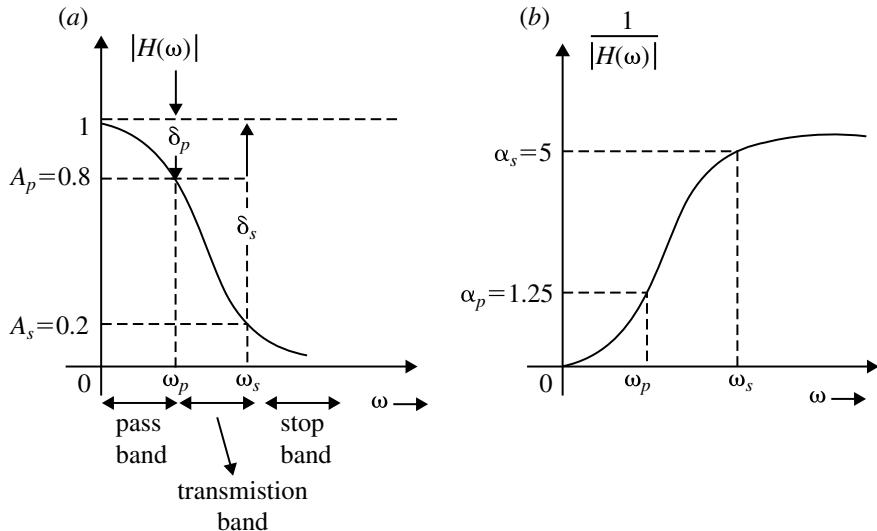
where  $A_p$  is the gain at the passband frequency and  $A_s$  is the gain at the stopband frequency. Let

$$\alpha_1 = \frac{1}{A_p} \quad \text{and} \quad \alpha_2 = \frac{1}{A_s} \quad (3.39)$$

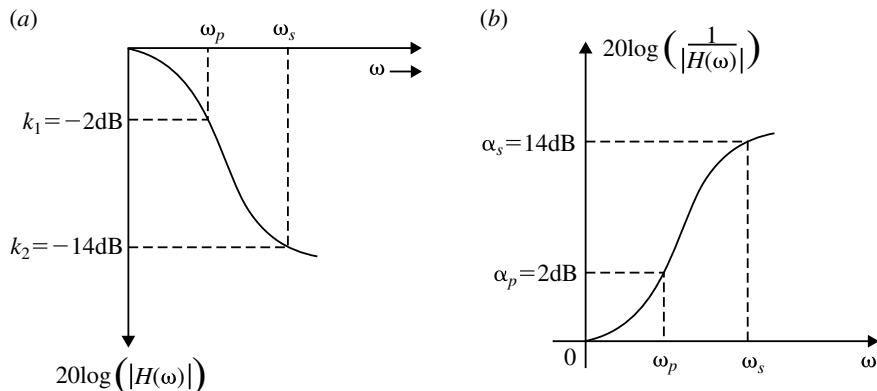
where  $\alpha_1$  is attenuation at the passband frequency and  $\alpha_2$  is attenuation at the stopband frequency.

The maximum values of normalized gain are unity and so  $A_p, A_s$  are less than 1 and  $\alpha_1, \alpha_2$  are greater than 1. The magnitude response of lowpass filter in terms of gain and attenuation is shown in Fig. 3.8.

The magnitude response in dB is shown in Fig. 3.9. Let  $k_1$  be gain in dB at passband frequency and  $k_2$  the gain in dB at stopband frequency.



**Fig. 3.8** Gain  $V_s\omega$  and attenuation  $V_p\omega$



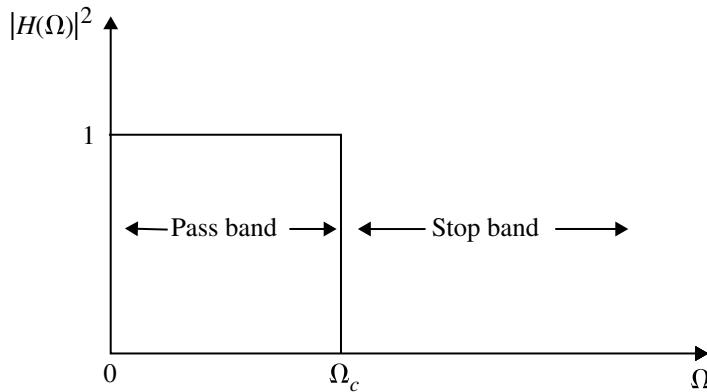
**Fig. 3.9** Gain and attenuation  $V_s\omega$  in dB

$$k_1 = 20 \log A_p \quad \text{and} \quad k_2 = 20 \log A_s \quad (3.40)$$

$$A_p = 10^{k_1/20} \quad \text{and} \quad A_s = 10^{k_2/20} \quad (3.41)$$

Let  $\alpha_1$  be attenuation in dB at  $\omega_1$ ,  $\alpha_2$  be attenuation in dB at  $\omega_2$

$$\begin{aligned} \alpha_p &= 20 \log \left( \frac{1}{A_p} \right) \quad \text{and} \quad \alpha_s = 20 \log \left( \frac{1}{A_s} \right) \\ \alpha_p &= -20 \log A_p \quad \text{and} \quad \alpha_s = -20 \log A_s \end{aligned} \quad (3.42)$$



**Fig. 3.10** Magnitude square frequency response of an ideal lowpass filter

Sometimes the specifications are given in terms of passband ripple  $\delta_p$  and stopband ripple  $\delta_s$ . In this case dB gain and attenuation can be estimated as

$$\begin{aligned} k_1 &= 20 \log(1 - \delta_p) \quad \text{and} \quad \alpha_p = -20 \log(1 - \delta_p) \\ k_2 &= 20 \log \delta_s \quad \text{and} \quad \alpha_s = -20 \log \delta_s \end{aligned} \quad (3.43)$$

### 3.8 Design of Lowpass Digital Butterworth Filter

On many practical applications, realization of  $H(z)$  from analog transfer function  $H(z)$  may not be available and it has to be determined from the specifications of the desired digital filters. Normally the cutoff frequency  $\Omega_c$ , of the filter, the passband attenuation and stopband attenuation are given. The system function  $H(s)$  is obtained to satisfy the above specifications. For realization the filter response characteristics have to be approximated. There are several approximation techniques available for the filter design. The most widely used approximation techniques are:

1. Butterworth filter.
2. Chebyshev filter.
3. Approximation of derivatives.

The analog filter transfer function  $H(s)$  obtained using any of these methods is converted to digital filter transfer function by using either impulse invariant transformation or bilinear transformation described earlier. The characteristics of Butterworth and Chebyshev filters are ideally suited to design lowpass filter. The design procedure for Butterworth filter is described below.

Let us consider the squared magnitude response of an ideal lowpass filter shown in Fig. 3.10.  $\Omega_c$  is the cutoff frequency of the filter. This ideal lowpass filter characteristics is however not physically realizable.

### 3.8.1 Analog Butterworth Filter

The analog Butterworth filter is designed by approximating the ideal frequency response using an error function. The error function is selected such that magnitude is maximally flat in the passband and monotonically decreasing in the stopband.

The magnitude response of lowpass filter obtained by the approximation is given by

$$|H_a(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad (3.44)$$

We know that  $s = j\Omega$ . Substituting  $\Omega$  by  $s/j$  in Eq. (3.44)

$$H_a(s)H_a(-s) = \frac{1}{1 + \left(\frac{s/j}{\Omega_c}\right)^{2N}} = \frac{1}{1 + \left(-\frac{s^2}{\Omega_c^2}\right)^N} \quad (3.45)$$

Let  $s/\Omega_c = s_n$  is the normalized function. Therefore, the normalized transfer function is

$$H_a(s_n)H_a(-s_n) = \frac{1}{1 + (-s_n^2)^N} \quad (3.46)$$

The normalized transfer function has  $2N$  poles. For a stable and causal filter, the poles should lie on the left half of  $s$ -plane. when  $N$  is even, all poles are complex and exit as conjugate pair. When  $N$  is odd, one of the poles is real and all other poles are complex and exit as conjugate pair.

From Eq. (3.46), it is possible to locate the poles of the analog filter in the  $s$ -plane. For odd number of  $N$ ,

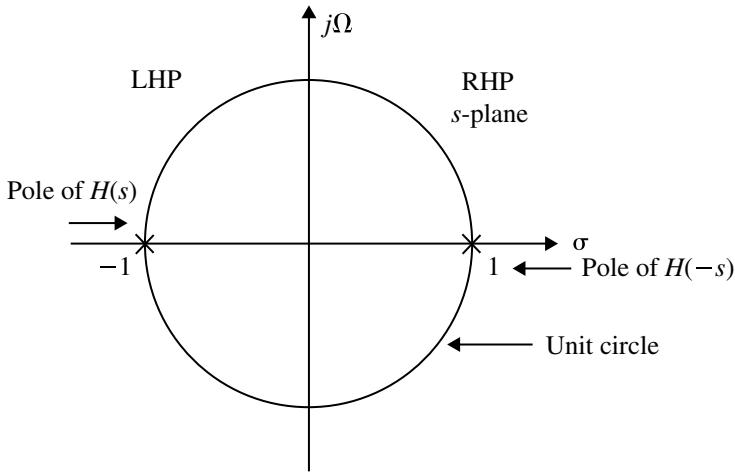
$$\begin{aligned} 1 + (-s^2)^N &= 0 \\ s^{2N} &= 1 \end{aligned}$$

For  $N = 1$ ,

$$\begin{aligned} s^2 &= 1 \\ s_1 &= 1 \\ s_2 &= -1 \end{aligned}$$

The poles are located as shown in Fig. 3.11.

For a stable filter, only LHP poles along are taken and the denominator of the TF is written as



**Fig. 3.11** Pole location of first-order filter

$$H(s) = \frac{1}{(s + 1)}$$

For  $N = 3$ ,

$$s^6 = 1$$

The magnitude of the six poles is 1, and their phase angle is given as

$$\phi_k = \frac{(2k)\pi}{2N}, \quad k = 0, 1, 2, 3, 4, 5$$

Thus,

$$s_1 = 1\angle 0^\circ; s_2 = 1\angle 60^\circ; s_3 = 1\angle 120^\circ; s_4 = 1\angle 180^\circ; s_5 = 1\angle 240^\circ; s_6 = 1\angle 300^\circ$$

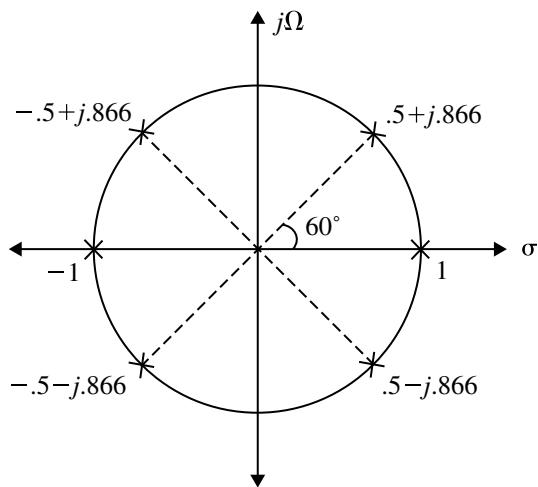
The pole locations are shown in Fig. 3.12.

Considering only LHP poles, the LF of the filter is obtained as

$$H(s) = \frac{1}{(s + 1)(s + 0.5 + j0.866)(s + 5 - j0.866)}$$

$$H(s) = \frac{1}{(s + 1)(s^2 + s + 1)}$$

**Fig. 3.12** Pole location of third-order filter



Now let us consider  $N$  being even. The following numerator polynomial for  $H(s)$  is written.

$$\begin{aligned} 1 + (-s^2)^N &= 0 \\ s^{2N} &= 1 \\ s_k &= 1\angle(2k+1)\pi/2N; \quad k = 0, 1, \dots, 2N \end{aligned}$$

For a second-order filter  $N = 2$ . The poles of  $H(s)$  are located as follows:

$$s_1 = 1\angle45^\circ; \quad s_2 = 1\angle135^\circ; \quad s_3 = 1\angle225^\circ; \quad s_4 = 1\angle315^\circ;$$

The pole locations are shown in Fig. 3.13.

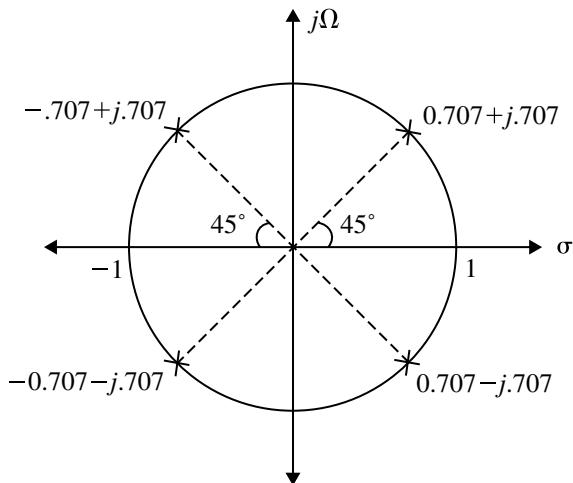
Considering only the LHP poles, the TF of the filter is obtained as given below

$$H(s) = \frac{1}{(s + 0.707 + j0.707)(s + 0.707 - j0.707)}$$

$$H(s) = \frac{1}{(s^2 + \sqrt{2}s + 1)}$$

Now consider  $N = 4$ ,

**Fig. 3.13** Pole location of a second-order filter



$$s_k = 1\angle(2k + 1)\pi/2N; \quad k = 0, 1, 2, 3, 4, 5, 6, 7$$

$$s_1 = 1\angle 22.5^\circ; \quad s_2 = 1\angle 67.5^\circ; \quad s_3 = 1\angle 112.5^\circ; \quad s_4 = 1\angle 157.5^\circ;$$

$$s_5 = 1\angle 202.5^\circ; \quad s_6 = 1\angle 247.5^\circ; \quad s_7 = 292.5^\circ; \quad s_8 = 1\angle 337.5^\circ$$

$$s_1 = 1\angle 22.5^\circ = \cos 22.5^\circ + j \sin 22.5^\circ = 0.9239 + j.3827$$

$$s_2 = \cos 67.5^\circ + j \sin 67.5^\circ = -0.3827 + j.9239$$

$$s_3 = \cos 112.5^\circ + j \sin 112.5^\circ = -0.3827 + j.9239$$

$$s_4 = \cos 157.5^\circ + j \sin 157.5^\circ = -0.9239 + j.3827$$

$$s_5 = \cos 202.5^\circ + j \sin 202.5^\circ = -0.9239 - j.7071$$

$$s_6 = \cos 247.5^\circ + j \sin 247.5^\circ = -0.3827 - j.9239$$

$$s_7 = \cos 292.5^\circ + j \sin 292.5^\circ = 0.3227 - j.9239$$

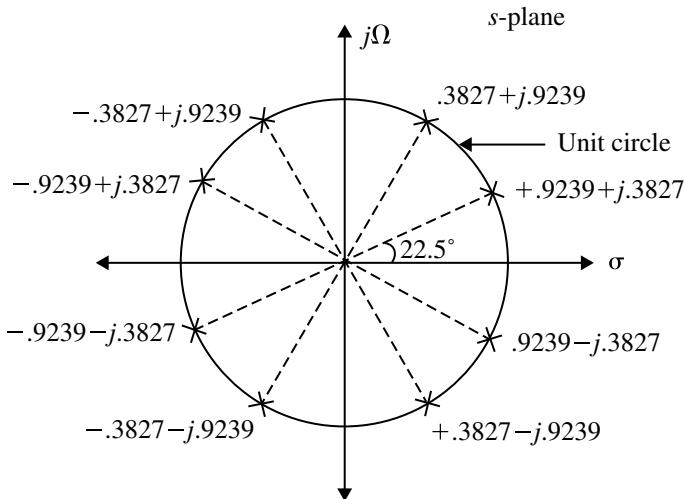
$$s_8 = \cos 337.5^\circ + j \sin 337.5^\circ = 0.9239 - j.3827$$

The pole locations are given in Fig. 3.14.

Considering the LHP poles only, the TF of the filter is obtained as

$$H(s) = \frac{1}{[(s + 0.3827 + j.9239)(s + 0.3827 - j.9239) \\ \times (s + 0.9239 + j.3827)(s + 0.9239 - j.3827)]}$$

$H(s) = \frac{1}{(s + 0.7654s + 1)(s + 1.8478s + 1)}$
---



**Fig. 3.14** Pole location of fourth-order filter

**Table 3.1** Butterworth polynomial for different  $N$

$N$	Denominator of $H(s)$
1	$(s + 1)$
2	$(s^2 + \sqrt{2}s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 1.9319s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.517s + s + 1)$
7	$(s + 1)(s^2 + 1.8019s + 1)(s^2 + 1.247s + 1)(s^2 + 0.44s + 1)$

The Butterworth polynomial for various values of  $N$  for  $\Omega_c = 1$  rad/s is given in Table 3.1.

The Butterworth polynomials given in Table 3.1 are called **normalized polynomials, and the corresponding poles are normalized poles**. Here  $\Omega_c = 1$  rad/s. If  $\Omega_c \neq 1$ , the polynomial is called unnormalized polynomial and  $s$  is replaced by  $s/\Omega_c$  in  $H(s)$ .

### 3.8.1.1 Determination of Order $N$ and Cutoff Frequency ‘ $\Omega_c$ ’ of Butterworth Filter

Determination of order  $N$  and cutoff frequency  $\Omega_c$  is very important in the design of Butterworth filter. Let the following specifications be considered.

$$\begin{aligned} A_p &\leq |H_a(\Omega)| \leq 1 \quad 0 \leq \Omega \leq \Omega_p \\ |H_a(\Omega)| &\leq A_s \quad \Omega_s \leq \Omega \end{aligned}$$

Consider the squared magnitude response of Butterworth filter which is given below:

$$|H_a(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Since  $1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N} \gg 1$ , the above equation is written as

$$|H_a(\Omega)|^2 = \frac{1}{\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

According to the given specifications, the following equations are written:

$$\begin{aligned} \frac{1}{\left(1 + \frac{\Omega_p}{\Omega_c}\right)^{2N}} &\geq A_p^2 \\ \frac{1}{\left(1 + \frac{\Omega_s}{\Omega_c}\right)^{2N}} &\leq A_s^2 \end{aligned}$$

The above equations are rearranged and written as follows:

$$\begin{aligned} \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} &\leq \frac{1}{A_p^2} - 1 \\ \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} &\geq \frac{1}{A_s^2} - 1 \end{aligned}$$

Dividing the equation one by the other after considering the equalities at  $\Omega_c$  we get

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{\left(\frac{1}{A_s^2} - 1\right)}{\left(\frac{1}{A_p^2} - 1\right)}$$

After taking logarithms on both sides, we solve for  $N$  as,

$$N = \frac{1}{2} \frac{\log \left[ \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right) \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)}$$

**Note:** If the alternations  $A_p$  and  $A_s$  are given in dB, then  $A_p$  and  $A_s$  are determined as follows:

$$\begin{aligned} 20 \log A_p &= -\text{dB of passband attenuation} \\ 20 \log A_s &= -\text{dB of stopband attenuation} \end{aligned}$$

Now to determine  $\Omega_c$ , any of the equation for passband or stopband attenuation is considered.

$$\begin{aligned} \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} &= \frac{1}{A_p^2} - 1 \\ \left(\frac{\Omega_p}{\Omega_c}\right) &= \left(\frac{1}{A_p^2} - 1\right)^{\frac{1}{2N}} \end{aligned}$$

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}} \quad \text{or} \quad \Omega_c = \frac{\Omega_s}{\left(\frac{1}{A_s^2} - 1\right)^{1/2N}}$$

The above equations may not give identical values for  $\Omega_c$ . Sometimes, the average of these two frequencies are taken to determine  $\Omega_c$  which is given below:

$$\Omega_c = \frac{1}{2} \left[ \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}} + \frac{\Omega_s}{\left(\frac{1}{A_s^2} - 1\right)^{1/2N}} \right]$$

Therefore, the transfer function of the Butterworth filters will be product of second-order factors. The analog filter transfer function of normalized and unnormalized Butterworth lowpass filter is also expressed as given below.

### Normalized Butterworth Lowpass Filter Transfer Function

Let  $N$  be the order of the filter. When  $N$  is even

$$H_a(s) = \prod_{k=1}^{N/2} \frac{1}{s_n^2 + b_k s_n + 1} \quad (3.47)$$

When  $N$  is odd

$$H_a(s) = \frac{1}{s_n + 1} \prod_{k=1}^{(N-1)/2} \frac{1}{s_n^2 + b_k s_n + 1} \quad (3.48)$$

where

$$b_k = 2 \sin \left( \frac{(2k-1)\pi}{2N} \right) \quad (3.49)$$

### Unnormalized Butterworth Lowpass Filter Transfer Function

The unnormalized transfer function is obtained by replacing  $s_n$  by  $s/\Omega_c$ , where  $\Omega_c = 3$  dB cutoff frequency. When  $N$  is even

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \quad (3.50)$$

When  $N$  is odd

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \quad (3.51)$$

where

$$b_k = 2 \sin \left( \frac{(2k-1)\pi}{2N} \right) \quad (3.52)$$

### 3.8.2 Frequency Response of Butterworth Filter

The frequency response of Butterworth filter depends on the order  $N$ . The approximated magnitude response approaches the ideal response as the value of  $N$  increases.

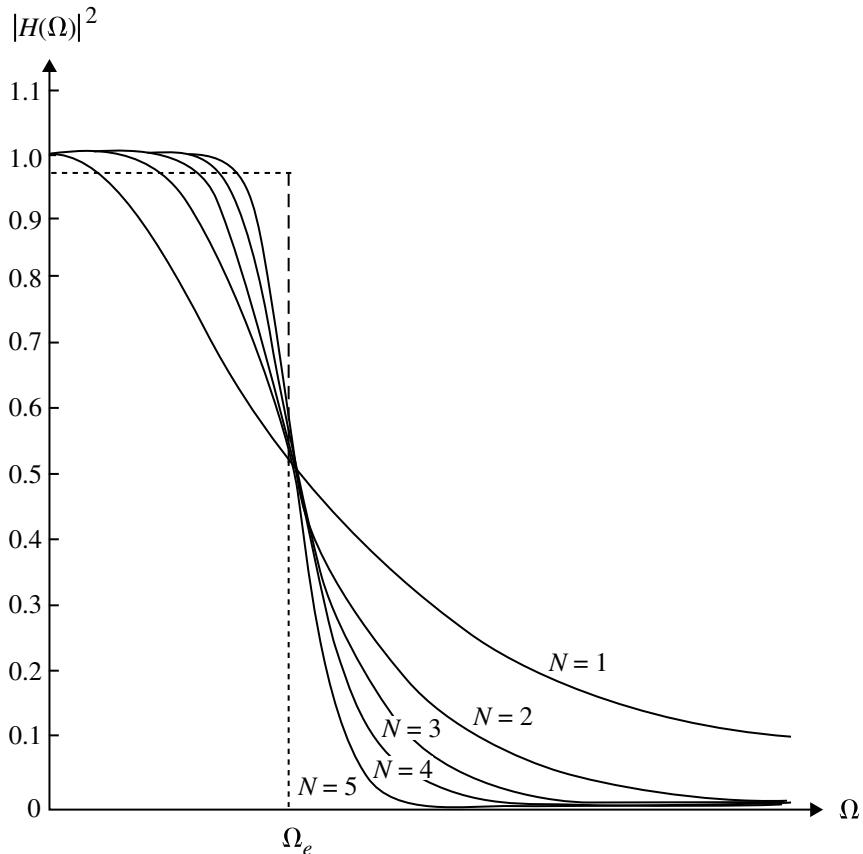
$A_p$  is gain at the passband frequency  $\omega_p$ , and  $A_s$  is the stopband frequency  $\omega_s$ . The order of the filter is given by

$$N \geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right) \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)} \quad (3.53)$$

The frequency response of Butterworth filter is shown in Fig. 3.15.

### 3.8.3 Properties of Butterworth Filters

1. The Butterworth filters are all-pole designs (i.e., the zeros of the filter exist at infinity).
2. At the cutoff frequency  $\Omega_c$ , the magnitude is  $1/\sqrt{2}$  or  $-3$  dB.
3. The filter order  $N$  completely specifies the filter.
4. The magnitude is flat at the origin.
5. The magnitude is monotonically decreasing function of  $\Omega$ .
6. The magnitude response approaches the ideal response as the value of  $N$  increases.



**Fig. 3.15** Frequency response of Butterworth filter

### 3.8.4 Design Procedure for Lowpass Digital Butterworth Filters

Let  $A_p$  be gain at passband frequency  $\omega_p$  and  $A_s$  be gain at stopband frequency  $\omega_s$ . Let  $\Omega_p$  is analog frequency corresponding to  $\omega_p$  and  $\Omega_s$  is analog frequency corresponding to  $\omega_s$ .

**Step 1.** Choose either bilinear or impulse invariant transformation.

**Step 2.** Calculate the ratio  $\Omega_s/\Omega_p$ . For bilinear transformation,

$$\frac{\Omega_s}{\Omega_p} = \frac{\tan(\omega_s/2)}{\tan(\omega_p/2)}$$

For impulse invariant,

$$\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p}$$

**Step 3.** Determine the order  $N$  of the filter using the following equation:

$$N \geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right) \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)}$$

**Step 4.** Calculate the analog cutoff frequency  $\Omega_c$ .

For bilinear transformation,

$$\Omega_c = \frac{2}{T} \frac{\tan(\omega_p/2)}{\left[ \frac{1}{A_p^2} - 1 \right]^{1/2N}}$$

For impulse invariant,

$$\Omega_c = \frac{\omega_p/T}{\left[ \frac{1}{A_p^2} - 1 \right]^{1/2N}}$$

**Step 5.** Determine the analog transfer function of the filter.

When  $N$  is even

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

When  $N$  is odd

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where

$$b_k = 2 \sin \left( \frac{(2k-1)\pi}{2N} \right)$$

*Note:* For normalized case take  $\Omega_c = 1$  rad/s.

**Step 6.** Using the chosen transformation determine  $H(z)$ .

**Step 7.** Realize the digital filter transfer function  $H(z)$  by a suitable structure.

**Example 3.9**

The specification of the desired lowpass filter is

$$\begin{aligned} 0.8 &\leq |H(\omega)| \leq 1.0; \quad 0 \leq \omega \leq 0.2\pi \\ |H(\omega)| &\leq 0.2; \quad 0.32\pi \leq \omega \leq \pi \end{aligned}$$

Design Butterworth digital filter using impulse invariant transformation.

(Anna University, April, 2006)

**Solution** Given  $A_p = 0.8$  at  $\omega_p = 0.2\pi$  rad/s and  $A_s = 0.2$  at  $\omega_s = 0.32\pi$

$$\begin{aligned} \frac{\Omega_s}{\Omega_p} &= \frac{\omega_s}{\omega_p} = \frac{0.32\pi}{0.2\pi} = 1.6 \\ N &\geq \frac{1}{2} \frac{\log \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right)}{\log \left( \frac{\Omega_s}{\Omega_p} \right)} \\ &\geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{(0.2)^2} - 1 \right) / \left( \frac{1}{(0.8)^2} - 1 \right) \right]}{\log 1.6} \\ N &\geq 3.9931 \end{aligned}$$

Choose the order of the filter  $N = 4$ . The analog cutoff frequency is,

$$\Omega_c = \frac{\omega_p/T}{\left( \frac{1}{A_p^2} - 1 \right)^{1/2N}}$$

Let  $T = 1$  s

$$\Omega_c = \frac{\omega_p}{\left( \frac{1}{A_p^2} - 1 \right)^{1/2N}} = \frac{0.2\pi}{\left( \frac{1}{(0.8)^2} - 1 \right)^{1/8}} = 0.675 \text{ rad/s.}$$

For  $N$  is even, the analog transfer function is

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where

$$b_k = 2 \sin \left( \frac{(2k-1)\pi}{2N} \right)$$

$$b_1 = 2 \sin\left(\frac{\pi}{8}\right) = 0.765 \text{ and } b_2 = 2 \sin\left(\frac{3\pi}{8}\right) = 1.848$$

$$\begin{aligned} H_a(s) &= \prod_{k=1}^2 \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \\ &= \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2} \times \frac{\Omega_c^2}{s^2 + b_2 \Omega_c s + \Omega_c^2} \\ &= \frac{(0.675)^2}{s^2 + (0.765 \times 0.675)s + (0.675)^2} \times \frac{(0.675)^2}{s^2 + (1.848)(0.675)s + (0.675)^2} \end{aligned}$$

$$\begin{aligned} H_a(s) &= \frac{0.2076}{s^2 + 0.516s + 0.456)(s^2 + 1.247s + 0.456)} \\ &= 0.2076 \Big/ [s - (-0.26 + j0.62)][s - (-0.26 - j0.62)] \\ &\quad \times [s - (-0.62 + j0.26)][s - (-0.6 - j0.26)] \end{aligned}$$

Putting into partial fraction we get,

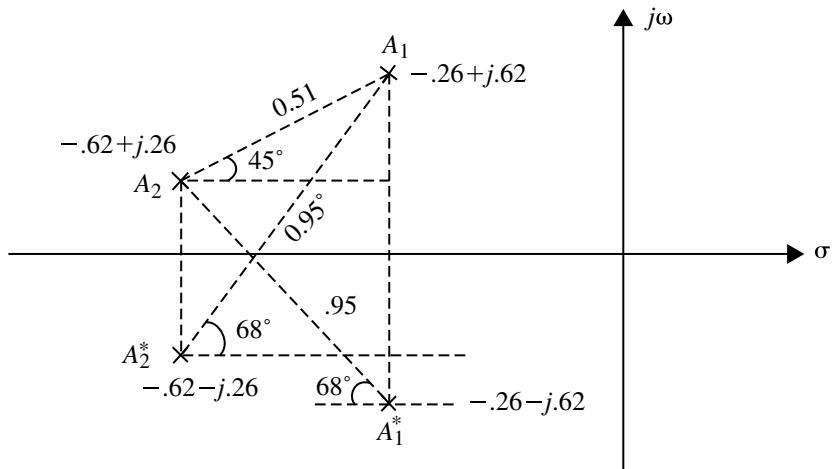
$$\begin{aligned} H_a(s) &= \frac{A_1}{s - (-0.26 + j0.62)} + \frac{A_1^*}{s - (-0.26 - j0.62)} \\ &\quad + \frac{A_2}{s - (-0.62 + j0.26)} + \frac{A_2^*}{s - (-0.62 - j0.26)} \\ A_1 &= \frac{0.2076}{[s - (-0.26 - j0.62)][s - (-0.62 + j0.26)][s - (-0.62 - j0.26)]} \Big|_{s=-0.26+j0.62} \\ A_1 &= -0.32 + j0.13 \\ A_1^* &= -0.32 - j0.13 \end{aligned}$$

Similarly

$$A_2 = 0.32 + j0.76 \text{ and } A_2^* = 0.32 - j0.76$$

The residues  $A_1$ ,  $A_1^*$ ,  $A_2$  and  $A_2^*$  may be calculated by graphical method. The poles are located in the  $s$ -plane as shown in Fig. 3.16.

$$\begin{aligned} A_1 &= \frac{0.2076}{1.24\angle 90^\circ \times 0.51\angle 45^\circ \times 0.95\angle 68^\circ} = 0.3455\angle -203^\circ \\ &= 0.3455(\cos 203^\circ - j \sin 203^\circ) = -0.32 + j0.13 \\ A_1^* &= -0.32 - j0.13 \end{aligned}$$



**Fig. 3.16** Determination of residues by graphical method

$$A_2 = \frac{0.2076}{0.52\angle 90^\circ 0.51\angle 225^\circ 0.95\angle 112^\circ} = 0.824\angle -67^\circ \\ = 0.824(\cos 67^\circ - j \sin 67^\circ)$$

$$A_2 = 0.32 + j0.76$$

$$A_2^* = 0.32 - j0.76$$

$$H_a(s) = \frac{-0.32 + j0.13}{s - (-0.26 + j0.62)} + \frac{-0.32 - j0.13}{s - (-0.26 - j0.62)} \\ + \frac{0.32 + j0.76}{s - (-0.62 + j0.26)} + \frac{0.32 - j0.76}{s - (-0.62 - j0.26)}$$

$$H(z) = H_a(s) \Bigg|_{\frac{A_i}{s-p_i}} \longrightarrow \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

Let  $T = 1$  s

$$H(z) = \frac{-0.32 + j0.13}{1 - e^{-0.26+j0.62} z^{-1}} + \frac{-0.32 - j0.13}{1 - e^{-0.26-j0.62} z^{-1}} \\ + \frac{0.32 + j0.76}{1 - e^{-0.62+j0.26} z^{-1}} + \frac{0.32 - j0.76}{1 - e^{-0.62-j0.26} z^{-1}}$$

The above equation is simplified as

$$H(z) = \frac{-0.64 + 0.28z^{-1}}{1 - 1.26z^{-1} + 0.59z^{-2}} + \frac{0.64 - 0.12z^{-1}}{1 - 1.04z^{-1} + 0.29z^{-2}}$$

The above system function is realized in direct form-II and is shown in Fig. 3.17.

**Note:** For structure realization refer Sect. 3.3.

$$H(z) = \frac{-0.64 + 0.28z^{-1}}{1 - 1.26z^{-1} + 0.59z^{-2}} + \frac{0.64 - 0.12z^{-1}}{1 - 1.04z^{-1} + 0.29z^{-2}}$$

### Example 3.10

Design a digital Butterworth filter satisfying the following constraints

$$\begin{aligned} 0.707 &\leq |H(e^{j\omega})| \leq 1 && \text{for } 0 \leq \omega \leq \frac{\pi}{2} \\ |H(e^{j\omega})| &\leq 0.2 && \text{for } \frac{3\pi}{4} \leq \omega \leq \pi \end{aligned}$$

with  $T = 1$  s using (a) the bilinear transformation; (b) impulse invariant method. Realize the filter in each case using the most convenient realization form.

(Anna University, May, 2007)

### Solution

#### (a) Bilinear Transformation

Given  $A_p = 0.707$  at  $\omega_p = (\pi/2)$  and  $A_s = 0.2$  at  $\omega_s = (3\pi/4)$

$$\frac{\Omega_p}{\Omega_s} = \frac{\tan \frac{\omega_p}{2}}{\tan \frac{\omega_s}{2}} = \frac{\tan \frac{3\pi}{8}}{\tan \frac{\pi}{4}} = 2.414$$

The order of the filter

$$\begin{aligned} N &\geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right) \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)} \\ &= \frac{1}{2} \frac{\log \left[ \left( \frac{1}{0.2^2} - 1 \right) / \left( \frac{1}{0.707^2} - 1 \right) \right]}{\log 2.414} \\ N &\geq 1.803 \end{aligned}$$

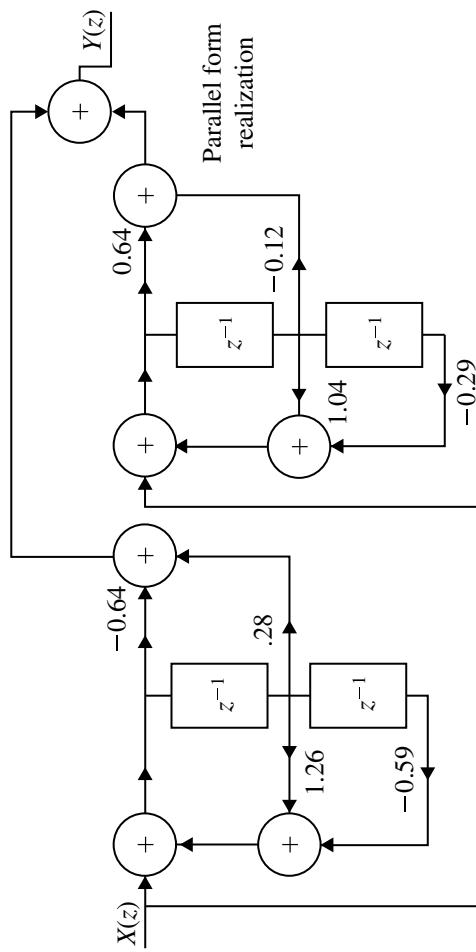


Fig. 3.17 Parallel form realization of  $H(z)$  for Example 3.9

Let  $N = 2$ . The analog cutoff frequency

$$\begin{aligned}\Omega_c &= \frac{2}{T} \frac{\tan(\omega_p/2)}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}} \\ &= 2 \frac{\tan(\pi/4)}{\left(\frac{1}{(0.707)^2} - 1\right)^{1/4}} = 2 \text{ rad/s.}\end{aligned}$$

The analog transfer function when  $N = \text{even}$  is

$$\begin{aligned}H_a(s) &= \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \quad b_k = 2 \sin \left[ \frac{(2k-1)\pi}{2N} \right] = 2 \sin \left( \frac{\pi}{4} \right) = 1.414 \\ &= \frac{4}{s^2 + 2.828s + 4}\end{aligned}$$

By using bilinear transformation  $H(z)$  is obtained as

$$\begin{aligned}H(z) &= H_a(s) \Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{4}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 2.828 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 4} \\ &= \frac{4(1+z^{-1})^2}{4(1-z^{-1})^2 + 5.656(1-z^{-2}) + 4(1+z^{-1})^2} \\ &= \frac{0.2929(1+z^{-1})^2}{1+0.1716z^{-2}}\end{aligned}$$

$$H(z) = \frac{0.2929(1+2z^{-1}+z^{-2})}{1+0.1716z^{-2}}$$

The above system is realized in direct form-II and is shown in Fig. 3.18a.

### (b) Impulse Invariant Method

Given  $A_p = 0.707$  at  $\omega_p = (\pi/2)$  and  $A_s = 0.2$  at  $\omega_s = (3\pi/4)$

$$\frac{\omega_s}{\omega_p} = \frac{3\pi/4}{\pi/2} = \frac{3}{2}$$

$$N \geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{(0.2)^2} - 1 \right) / \left( \frac{1}{(0.707)^2} - 1 \right) \right]}{\log \left( \frac{3}{2} \right)}$$

$$N \geq 3.924$$

$$N = 4$$

$$\begin{aligned}\Omega_c &= \left( \frac{1}{A_p^2 - 1} \right)^{\frac{1}{2N}} \cdot \Omega_p \\ &= \left( \frac{1}{0.707^2 - 1} \right)^{\frac{1}{8}} \cdot \frac{\pi}{2} \\ &= 1.57\end{aligned}$$

The transfer function of analog filter for  $N$  being even is,

$$H_a(s) = \prod_{k=1}^2 \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

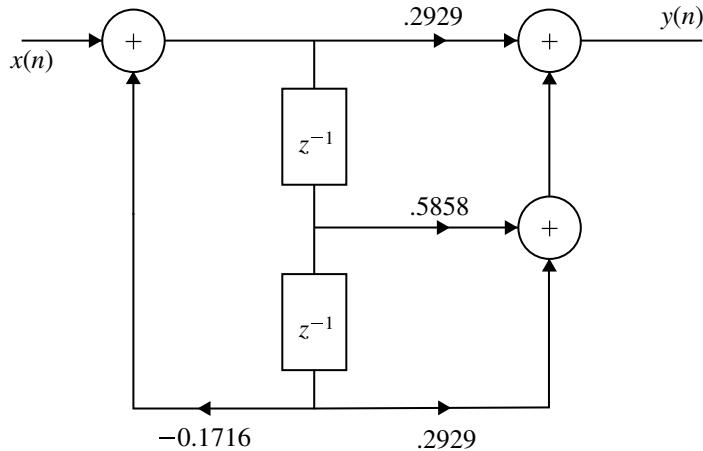
where  $b_1 = 0.7654$  and  $b_2 = 1.8478$

$$\begin{aligned}H_a(s) &= \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)} \\ &= \frac{A}{s+1.45+j0.6} + \frac{A^*}{s+1.45-j0.6} + \frac{B}{s+0.6+j1.45} + \frac{B^*}{s+0.6-j1.45} \\ A &= \frac{(1.57)^4}{(s + 1.45 - j0.6)(s + 0.6 + j1.45)(s + 0.6 - j1.45)} \Big|_{s=-1.45-j0.6} \\ &= 0.7253 + j1.754 \\ B &= \frac{(1.57)^4}{(s + 1.45 + j0.6)(s + 1.45 - j0.6)(s + 0.6 - j1.45)} \Big|_{s=-0.6-j1.45} \\ &= -0.7253 - j0.3\end{aligned}$$

$$\begin{aligned}H_a(s) &= \frac{0.7253 + j1.754}{s + 1.45 + j0.6} + \frac{0.7253 - j1.754}{s + 1.45 - j0.6} \\ &\quad + \frac{-0.7253 - 0.3j}{s + 0.6 + j1.45} + \frac{-0.7253 + 0.3j}{s + 0.6 - j1.45}\end{aligned}$$

Impulse invariant transformation is

$$\frac{1}{s - p_i} \implies \frac{1}{1 - e^{p_i T} z^{-1}}$$



**Fig. 3.18** Direct form-II realization of  $H(z)$  for Example 3.10a

let  $T = 1 \text{ s}$

$$H(z) = \frac{0.7253 + j1.754}{1 - e^{-1.45} \cdot e^{-j0.6} z^{-1}} + \frac{0.7253 - j1.754}{1 - e^{-1.45} \cdot e^{+j0.6} z^{-1}} \\ + \frac{-0.7253 - 0.3j}{1 - e^{-0.6} \cdot e^{-j1.45} z^{-1}} + \frac{-0.7253 + 0.3j}{1 - e^{-0.6} \cdot e^{j1.45} z^{-1}}$$

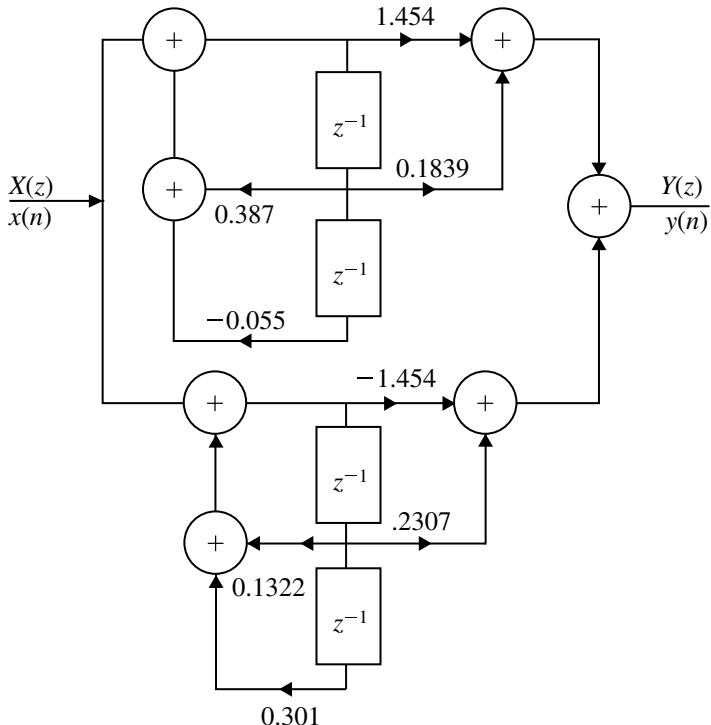
$$H(z) = \frac{1.454 + 0.1839z^{-1}}{1 - 0.387z^{-1} + 0.055z^{-2}} + \frac{-1.454 + 0.2307z^{-1}}{1 - 0.1322z^{-1} + 0.301z^{-2}}$$

This can be realized using parallel form as shown in Fig. 3.19.

### Example 3.11

For the given specification design an analog Butterworth filter

$$0.9 \leq |H(j\Omega)| \leq 1 \quad \text{for } 0 \leq \Omega \leq 0.2\pi \\ |H(j\Omega)| \leq 0.2 \quad \text{for } 0.4\pi \leq \Omega < \pi$$



**Fig. 3.19** Parallel form realization of  $H(z)$  for Example 3.10

**Solution** Given  $\Omega_p = 0.2\pi$ ,  $\Omega_s = 0.4\pi$ ,  $A_p = 0.9$ ,  $A_s = 0.2$ .

$$N \geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right) \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)}$$

$$N \geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{(0.2)^2} - 1 \right) / \left( \frac{1}{(0.9)^2} - 1 \right) \right]}{\log \left( \frac{0.4\pi}{0.2\pi} \right)}$$

$$N \geq 3.34$$

$$N = 4$$

$$\begin{aligned} \Omega_c &= \frac{\Omega_p}{(10^{0.1} - 1)^{1/2N}} \quad \text{or} \quad \Omega_c = \frac{\Omega_p}{\left( \frac{1}{A_p^2} - 1 \right)^{1/2N}} \\ &= \frac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi \end{aligned}$$

The analog transfer function

$$\begin{aligned} H_a(s) &= \prod_{k=1}^2 \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \\ &= \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2} \cdot \frac{\Omega_c^2}{s^2 + b_2 \Omega_c s + \Omega_c^2} \\ b_k &= 2 \sin \left( \frac{(2k-1)\pi}{2N} \right), \quad k = 1, 2, \end{aligned}$$

where  $b_1 = 0.7653$  and  $b_2 = 1.8477$

$$H_a(s) = \frac{0.323}{(s^2 + 0.577s + 0.0576\pi^2)(s^2 + 1.393s + 0.0576\pi^2)}$$

### Example 3.12

Determine the order and the poles of lowpass Butterworth analog filter that has a 3 dB attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.

**Solution** Given  $-20 \log A_p = 3$  dB,  $\Omega_p = 2\pi \times 500 = 1000\pi$  rad/s and  $-20 \log A_s = 40$  dB,  $\Omega_s = 2\pi \times 1000 = 2000\pi$  rad/s.

$$\begin{aligned} 20 \log A_p &= -3 \\ A_p &= 0.707 \\ 20 \log A_s &= -40 \\ A_s &= 0.01 \end{aligned}$$

$$\begin{aligned} N &\geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right) \right]}{\log \left( \frac{\Omega_s}{\Omega_p} \right)} \\ N &\geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{(0.01)^2} - 1 \right) / \left( \frac{1}{(0.707)^2} - 1 \right) \right]}{\log \left( \frac{2000\pi}{1000\pi} \right)} \\ N &\geq 6.64319 \\ N &= 7 \end{aligned}$$

The poles of Butterworth filter are given by  $s_k = \Omega_c e^{j\Phi_k}$

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_1^2} - 1\right)^{1/2N}} = \frac{1000\pi}{\left(\frac{1}{(0.707)^2} - 1\right)^{1/14}} = 1000\pi$$

$$s_k = 1000\pi e^{j\Phi_k}, \quad k = 1, \dots, 7$$

where

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}, \quad k = 1, \dots, 7$$

The normalized poles of analog Butterworth filter are obtained as follows. (This can be taken from Table 3.1 also.)

$$\text{For } k = 1, \quad \phi_1 = \frac{\pi}{2} + \frac{\pi}{14} = 102.865^\circ$$

$$\text{For } k = 2, \quad \phi_2 = 128.57^\circ$$

$$\text{For } k = 3, \quad \phi_3 = 154^\circ$$

$$\text{For } k = 4, \quad \phi_4 = 180^\circ$$

The normalized LHP poles are,

$$s_1 = 1e^{j102.865^\circ} = -0.2265 + j0.9749$$

$$s_2 = e^{j128.57^\circ} = -0.6235 + j0.7818$$

$$s_3 = e^{j154^\circ} = -0.8987 + j0.4384$$

$$s_4 = e^{j180^\circ} = -1$$

$$s_5 = s_3^* = -0.8987 - j0.4384$$

$$s_6 = s_2^* = -0.6235 - j0.7818$$

$$s_7 = s_1^* = -0.2265 - j0.9749$$

These poles in LHP of the  $s$ -plane are shown in Fig. 3.20.

The transfer function of analog Butterworth normalized filter is given as

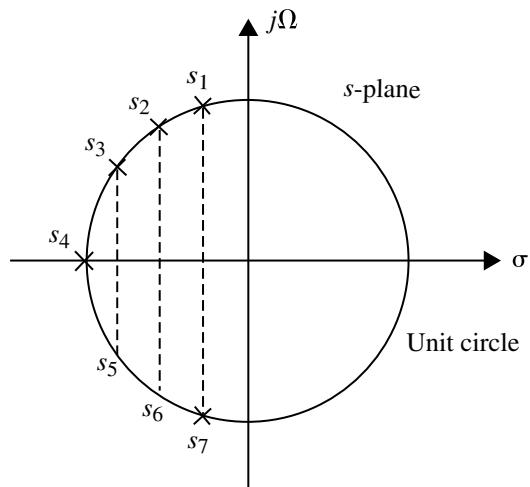
$$H_a(s) = \frac{1}{(s + s_4)(s + s_1)(s + s_7)(s + s_2)(s + s_6)(s + s_3)(s + s_5)}$$

$$= \frac{1}{[(s + 1)(s^2 + 1.8019s + 1)(s^2 + 1.247s + 1)(s^2 + 0.44s + 1)]}$$

The transfer function of the unnormalized filter is obtained by replaced  $s$  by

$$\frac{s}{\Omega_c} = \frac{s}{1000\pi}.$$

**Fig. 3.20** Location of normalized poles



Therefore

$$H_a(s) = \frac{(\Omega_c)^7}{[(s + \Omega_c)(s^2 + 1.8019\Omega_c s + \Omega_c^2)(s^2 + 1.247\Omega_c s + \Omega_c^2)(s^2 + 0.44\Omega_c s + \Omega_c^2)]}$$

$$H_a(s) = \frac{3 \times 10^{24}}{(s + 3140)(s^2 + 5658s + 9.87 \times 10^6)(s^2 + 3893.6s + 9.87 \times 10^6)} \left[ (s^2 + 1381.6s + 9.87 \times 10^6) \right]$$

### Example 3.13

If  $H(s) = \frac{1}{(s+1)(s+2)}$ , find  $H(z)$  using impulse invariance method for sampling frequency of 5 samples/s.

(Anna University, May, 2007)

#### Solution

$$H(s) = \frac{1}{(s+1)(s+2)}$$

The above function can be put into partial fraction as,

$$H(s) = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

The sampling frequency is given as  $F_s = 5$

$$T = \frac{1}{F_s} = \frac{1}{5} = 0.2 \text{ s}$$

Using impulse invariance we get,

$$\begin{aligned} H(z) &= \frac{1}{1 - e^{-T}z^{-1}} - \frac{1}{(1 - e^{2T}z^{-1})} \\ &= \frac{1}{1 - e^{-0.2}z^{-1}} - \frac{1}{(1 - e^{-0.4}z^{-1})} \\ &= \frac{1}{1 - 0.8187z^{-1}} - \frac{1}{(1 - 0.6703z^{-1})} \end{aligned}$$

$$H(z) = \frac{0.1484z^{-1}}{(1 - 1.489z^{-1} + 0.5488z^{-2})}$$

### Example 3.14

Convert the analog filter with system function  $H_a(s)$  into digital filter using bilinear transformation.

$$H_a(s) = \frac{(s + 0.3)}{(s + 0.3)^2 + 16}$$

(Anna University, November, 2006)

#### Solution

$$H_a(s) = \frac{(s + 0.3)}{(s + 0.3)^2 + 16} = \frac{(s + 0.3)}{(s + 0.3)^2 + 16}$$

The bilinear transformation is given by

$$s = \frac{2}{T} \frac{(1 - z^{-1})}{(1 + z^{-1})} = \frac{2}{T} \frac{(z - 1)}{(z + 1)}$$

Assuming  $T = 1$  and substituting for  $s$  in  $H_a(s)$  we get

$$\begin{aligned}
 H(z) &= \frac{2\frac{(z-1)}{(z+1)} + 0.3}{\left(2\frac{(z-1)}{(z+1)}\right)^2 + 0.6 + \frac{(z-1)}{(z+1)} + 16.09} \\
 &= \frac{[2(z-1) + 0.3(z+1)](z+1)}{4(z-1)^2 + 0.6(z-1)(z+1) + 16.09(z+1)} \\
 &= \frac{(2.3z^2 + 0.6z - 1.7)}{(4.6z^2 + 8.09z + 19.49)}
 \end{aligned}$$

$$H(z) = \frac{(-1.7z^{-2} + 0.6z^{-1} + 2.3)}{(19.4z^{-2} + 8.09z^{-1} + 4.6)}$$

**Example 3.15**

Find the analog T.F. for a digital Butterworth filter satisfying the following specifications.

$$\begin{aligned}
 0.7 &\leq |H(e^{j\omega})| \leq 1, \quad 0 \leq \omega \leq 0.2\pi \\
 |H(e^{j\omega})| &\leq 0.004, \quad 0.6\pi \leq \omega \leq \pi
 \end{aligned}$$

Assume  $T = 1$  s. Apply impulse invariant transformation.

(Anna University, December, 2007)

**Solution** Given  $A_p = 0.7$ ;  $\omega_p = 0.2\pi$  and  $A_s = 0.004$ ;  $\omega_s = 0.6\pi$

$$\begin{aligned}
 \frac{\Omega_s}{\Omega_p} &= \frac{0.6\pi}{0.2\pi} = 3 \\
 N &\geq \frac{1}{2} \frac{\log \left[ \left( \frac{1}{A_s^2} - 1 \right) / \left( \frac{1}{A_p^2} - 1 \right) \right]}{\log \frac{\Omega_s}{\Omega_p}} \\
 &\leq \frac{\frac{1}{2} \log \left[ \left( \frac{1}{(0.004)^2} - 1 \right) / \left( \frac{1}{(0.7)^2} - 1 \right) \right]}{\log 3} = 5
 \end{aligned}$$

$$\begin{aligned}
 \Omega_c &= \frac{1}{T} \frac{\omega_p}{\left( \frac{1}{A_p^2} - 1 \right)^{1/2N}} \\
 &= \frac{0.2\pi}{\left( \frac{1}{0.49} - 1 \right)^{0.1}} = 0.2\pi \text{ rad/s.} \\
 \Omega_c &= 0.6283
 \end{aligned}$$

For  $N = 5$  which is odd, the analog Butterworth filter is expressed by the following system function

$$H_a(s) = \frac{(\Omega_c)^5}{(s + \Omega_c)(s^2 + b_1\Omega_c s + \Omega_c^2)(s^2 + b_2\Omega_c s + \Omega_c^2)}$$

Since  $b_1 = 0.618$  and  $b_2 = 1.618$

$$H_a(s) = \frac{(0.6283)^5}{(s + 0.6283)(s^2 + 0.3883s + 0.395)(s^2 + s + 0.395)}$$

### Example 3.16

Design a third-order Butterworth digital filter using impulse invariant technique. Assume the sampling period  $T = 1$ s

**Solution** Given  $N = 3$ . If  $N$  is odd, the analog transfer function is

$$\begin{aligned} H_a(s) &= \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \\ &= \frac{\Omega_c}{s + \Omega_c} \cdot \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \quad (\text{Let } \Omega_c = 1 \text{ for normalized filter}). \\ &= \frac{1}{(s+1)(s^2 + b_1 s + 1)} \quad \left( b_k = 2 \sin \left[ \frac{(2k-1)\pi}{2N} \right] \right) \\ H_a(s) &= \frac{1}{(s+1)(s^2 + s + 1)} \quad (b_1 = 1) \\ &= \frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)} \\ &= \frac{A}{s+1} + \frac{B}{s+0.5+j0.866} + \frac{C}{s+0.5-j0.866} \end{aligned}$$

$$A = \frac{1}{s^2 + s + 1} \Big|_{s=-1} = 1$$

$$B = \frac{1}{(s+1)(s+0.5-j0.866)} \Big|_{s=0.5-j0.866} = -0.5 + j0.288$$

$$C = B^* = -0.5 - j0.288$$

$$H(z) = \frac{1}{s+1} + \frac{-0.5 + 0.288j}{s+0.5+j0.866} + \frac{-0.5 - 0.288j}{s+0.5-j0.866}$$

By impulse invariant transformation, the transfer function of the digital filter is,

$$H(z) = \frac{1}{1 - e^{-1}z^{-1}} + \frac{-0.5 + 0.288j}{1 - e^{-0.5} \cdot e^{-j0.866}z^{-1}} + \frac{-0.5 - 0.288j}{1 - e^{-0.5}e^{j0.866}z^{-1}}$$

$$H(z) = \frac{1}{1 - 0.368z^{-1}} + \frac{-1 + 0.66z^{-1}}{1 - 0.786z^{-1} + 0.368z^{-2}}$$

### Example 3.17

Apply impulse invariant method and find  $H(z)$  for

$$H(s) = \frac{s + a}{(s + a)^2 + b^2}$$

**Solution** The inverse Laplace transform of given function is

$$h(t) = \begin{cases} e^{-at} \cos bt, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$h(n) = h(t) \Big|_{t=nT}$$

$$h(n) = \begin{cases} e^{-anT} \cos(bnT), & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The digital transfer function is

$$H(z) = \sum_{n=0}^{\infty} e^{-anT} \cos bnT z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-anT} \left( \frac{e^{jbnT} + e^{-jbnT}}{2} \right) z^{-n}$$

$$H(z) = \frac{1}{2} \sum_{n=0}^{\infty} [ (e^{-(a-jb)T} z^{-1})^n + (e^{-(a+jb)T} z^{-1})^n ]$$

$$= \frac{1}{2} \left[ \frac{1}{1 - e^{-(a-jb)T} z^{-1}} + \frac{1}{1 - e^{-(a+jb)T} z^{-1}} \right]$$

$$= \frac{1}{2} \left[ \frac{1 - e^{-(a+jb)T} z^{-1} + 1 - e^{-(a-jb)T} z^{-1}}{(1 - e^{-(a-jb)T} z^{-1})(1 - e^{-(a+jb)T} z^{-1})} \right]$$

$$H(z) = \frac{1 - e^{-aT} \cos(bT)z^{-1}}{1 - 2e^{-aT} \cos(bT)z^{-1} + e^{-2aT}z^{-2}}$$

**Example 3.18**

An analog filter has a transfer function

$$H(s) = \frac{10}{s^2 + 7s + 10}$$

Design a digital filter equivalent to this using impulse invariant method for  $T = 0.2$  s

**Solution** Given

$$H(s) = \frac{10}{s^2 + 7s + 10} \quad \text{and} \quad T = 0.2 \text{ s}$$

$$H(s) = \frac{10}{(s+5)(s+2)} = \frac{A}{(s+5)} + \frac{B}{(s+2)}$$

$$A = \left. \frac{10}{(s+2)} \right|_{s=-5} = -3.33$$

$$B = \left. \frac{10}{(s+5)} \right|_{s=-2} = 3.33$$

$$H(s) = \frac{-3.33}{(s+5)} + \frac{3.33}{(s+2)}$$

$$H(z) = \frac{-3.33}{1 - e^{-5T}z^{-1}} + \frac{3.33}{1 - e^{-2T}z^{-1}}$$

$$= \frac{-3.33}{1 - e^{-1}z^{-1}} + \frac{3.33}{1 - e^{-0.4}z^{-1}}$$

$$= \frac{-3.33}{1 - 0.3678z^{-1}} + \frac{3.33}{1 - 0.67z^{-1}}$$

$$H(z) = \frac{1.006z^{-1}}{1 - 1.0378z^{-1} + 0.247z^{-2}}$$

**Example 3.19**

Use bilinear transform to design a first-order Butterworth LPF with 3 dB cutoff frequency of  $0.2\pi$ .

(Anna University, April, 2004)

**Solution** Given 3 dB cutoff frequency  $\omega_c = 0.2\pi = \omega_p$ .

The analog transfer function for first-order Butterworth filter is

$$\begin{aligned} H_a(s) &= \frac{\Omega_c}{s + \Omega_c} \\ \Omega_c &= \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right) = 2 \tan\left(\frac{0.2\pi}{2}\right) = 0.6498 \\ H_a(s) &= \frac{0.6498}{s + 0.6498} \end{aligned}$$

Using bilinear transformation,

$$H(z) = H_a(s) \Big|_{s=\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$

Assume  $T = 1$  s

$$\begin{aligned} H(z) &= \frac{0.6498}{s + 0.6498} \Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} [\because T = 1 \text{ s}] \\ &= \frac{0.6498}{\frac{2(1-z^{-1})}{(1+z^{-1})} + 0.6498} \\ &= \frac{0.6498(1+z^{-1})}{2(1-z^{-1}) + 0.6498(1+z^{-1})} \\ &= \frac{0.6498(1+z^{-1})}{2 - 2z^{-1} + 0.6498 + 0.6498z^{-1}} \\ &= \frac{0.6498(1+z^{-1})}{2.6498 - 1.3502z^{-1}} \\ &= \frac{0.6498(1+z^{-1})}{2.6498(1 - 0.5095z^{-1})} \end{aligned}$$

$$H(z) = \frac{0.2452(1+z^{-1})}{1 - 0.5095z^{-1}}$$

**Example 3.20**

Using bilinear transformation design a digital bandpass Butterworth filter with the following specifications:

Sampling frequency = 8 kHz

$\alpha_p = 2$  dB in the passband  $800 \text{ Hz} \leq f \leq 1000 \text{ Hz}$

$\alpha_s = 20$  dB in the stopband  $0 \leq f \leq 400 \text{ Hz}$  and  $2000 \leq f \leq \infty$

(Anna University, December, 2006)

**Solution**

$$\frac{\omega_1 T}{2} = \frac{2 \times \pi \times 400}{2 \times 8000} = \frac{\pi}{20} \quad \left[ T = \frac{1}{f_s} = \frac{1}{8000} \right]$$

$$\frac{\omega_l T}{2} = \frac{2 \times \pi \times 800}{2 \times 8000} = \frac{\pi}{10}$$

$$\frac{\omega_u T}{2} = \frac{2\pi \times 1000}{2 \times 8000} = \frac{\pi}{8}$$

$$\frac{\omega_2 T}{2} = \frac{2\pi \times 2000}{2 \times 8000} = \frac{\pi}{4}$$

Pre-warped analog frequencies are given by

$$\frac{\Omega_1 T}{2} = \tan \frac{\omega_1 T}{2} = \tan \frac{\pi}{20} = 0.1584$$

$$\frac{\Omega_l T}{2} = \tan \frac{\omega_l T}{2} = \tan \frac{\pi}{10} = 0.325$$

$$\frac{\Omega_u T}{2} = \tan \frac{\omega_u T}{2} = \tan \frac{\pi}{8} = 0.4142$$

$$\frac{\Omega_2 T}{2} = \tan \frac{\omega_2 T}{2} = \tan \frac{\pi}{4} = 1$$

First we design a prototype normalized compass filter and then use suitable transformation to obtain the transfer function of bandpass filter. To reduce computational complexity we use above values to find  $\Omega_r$  and substitute  $s = \frac{1-z^{-1}}{1+z^{-1}}$  for bilinear transformation ( $\because$  all the above frequencies contain the term  $T/2$ )

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1 (\Omega_u - \Omega_l)} \\ = \frac{-(0.1584)^2 + (0.325)(0.4142)}{0.1584(0.4142 - 0.325)}$$

$$A = 11.303$$

$$\begin{aligned} B &= \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)} = \frac{1 - (0.4142)(0.325)}{1(0.4142 - 0.325)} \\ &= \frac{1 - 0.1346}{0.0892} = \frac{0.865385}{0.0892} \\ &= 9.7016 \end{aligned}$$

$$\Omega_r = \min\{|A|, |B|\} = 9.7016$$

$$\begin{aligned} N &= \frac{\log_{10} \sqrt{\frac{10^2 - 1}{10^{0.2} - 1}}}{\log_{10}(9.7016)} = \frac{\log_{10}(13.01)}{\log_{10}(9.7016)} = \frac{1.1142}{0.9868} \\ &= 1.1290 \end{aligned}$$

$$N = 2$$

The second-order normalized Butterworth lowpass filter transfer function given by

$$H(s) = \frac{1}{s^2 + 1.4142s + 1}$$

The transformation for the bandpass filter is

$$\begin{aligned} s &\rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 0.1346}{s(0.0892)} \\ H(s) &= \frac{1}{s^2 + 1.4142s + 1} \Big|_{s=\frac{s^2+0.1346}{(0.0892)s}} \\ &= \frac{1}{\left(\frac{s^2+0.1346}{(0.0892)s}\right)^2 + 1.4142 \left(\frac{s^2+0.1346}{(0.0892)s}\right) + 1} \\ &= \frac{0.0079s^2}{s^4 + 0.2692s^2 + 0.0181 + 1.4142(0.0892)s(s^2 + 0.1346) + 0.0079s^2} \\ &= \frac{0.0079s^2}{s^4 + 0.2692s^2 + 0.0181 + 0.126s(s^2 + 0.1346) + 0.0079s^2} \\ &= \frac{0.0079s^2}{s^4 + 0.2692s^2 + 0.0181 + 0.126s^3 + 0.0169s + 0.0079s^2} \\ H(s) &= \frac{0.0079s^4}{s^4 + 0.126s^3 + 0.2771s^2 + 0.0169s + 0.0181} \\ H(z) &= H(s) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \\ &= \frac{0.0079 \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^4 + 0.126 \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 0.277 \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.169 \left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.0181} \end{aligned}$$

$$\begin{aligned}
&= 0.0079(1 - z^{-1})^2 \left/ \left[ (1 - z^{-1})^4 + 0.126(1 - z^{-1})^3(1 + z^{-1}) \right. \right. \\
&\quad + 0.2692(1 - z^{-1})^2(1 + z^{-1})^2 + 0.169(1 - z^{-1})(1 + z^{-1})^3 \\
&\quad \left. \left. + 0.0181(1 + z^{-1})^4 \right] \right. \\
&= 0.0079(1 - z^{-2})^2 \left/ \left[ 1 - 4z^{-1} + 6z^{-2} - 4z^{-3} + z^{-4} \right. \right. \\
&\quad + 0.126(1 - 2z^{-1} + 2z^{-3} - z^{-4}) + 0.2677(1 - 2z^{-2} \\
&\quad + z^{-4}) + 0.169(1 + 2z^{-1} - 2z^{-3} - z^{-4}) + 0.0181(1 + 4z^{-1} \\
&\quad \left. \left. + 6z^{-2} + 4z^{-3} + z^{-4} \right) \right] \\
&= \frac{0.0079(1 - z^{-2})^2}{7.1871 - 3.8416z^{-1} + 0.644z^{-2} - 4.086z^{-3} + 0.9908z^{-4}}
\end{aligned}$$

## 3.9 Design of Lowpass Digital Chebyshev Filter

### 3.9.1 Analog Chebyshev Filter

It is designed by approximating the ideal frequency response using error function.

The Chebyshev characteristic provides an alternative method of getting a suitable analog transfer function  $H_a(s)$  from which lowpass digital filters are designed using either bilinear transformation or impulse invariance. There are two types of Chebyshev filters available as type I and type II filters. The type I filter characteristic for odd value of  $N$  is shown in Fig. 3.21a, and type II characteristic for even value of  $N$  is shown in Fig. 3.21b. The characteristic of type I filters has equal ripple in the passband and monotonic in the stopband. They are all-pole filters. Type II Chebyshev filter characteristics are shown in Figs. 3.22a, b for odd and even values of  $N$ . These filters have equal ripples in the stopband and monotonic in the passband. Further, they have both poles and zeros in the transfer function.

### 3.9.2 Determination of the Order of the Chebyshev Filter

The type I of  $N$ th order Chebyshev filter can be expressed as,

$$|H(j\Omega)|^2 = \frac{1}{\left[ 1 + \epsilon^2 C_N^2 \left( \frac{\Omega}{\Omega_p} \right) \right]}; \quad N = 1, 2, \dots \quad (3.53a)$$

where  $\epsilon$  is a parameter which is the function of the ripple in the passband.  $C_N(x)$  is the Chebyshev polynomial of order  $N$ . In the passband and stopband it is defined as,

$$\begin{aligned} C_N(x) &= \cos(N \cos^{-1} x), \quad |x| \leq 1 \\ C_N(x) &= \cosh(N \cosh^{-1} x), \quad |x| > 1 \end{aligned}$$

The Chebyshev polynomial is also expressed by the following recursive formula.

$$C_N(x) = 2x C_{N-1}(x) - C_{N-2}(x), \quad N > 1$$

where  $C_0(x) = 1$  and  $C_1(x) = x$ .

Taking logarithm for Eq. (3.53a) we get

$$20 \log |H(\Omega)| = -10 \log \left[ 1 + \epsilon^2 C_N^2 \left( \frac{\Omega}{\Omega_p} \right) \right]$$

At passband  $\Omega = \Omega_p$

$$\begin{aligned} 20 \log |H(\Omega)| &= -\alpha_p \\ \alpha_p &= 10 \log \left[ 1 + \epsilon^2 C_N^2 \left( \frac{\Omega_p}{\Omega_p} \right) \right] \end{aligned}$$

Using the property of  $C_N(1) = 1$ , we get

$$\begin{aligned} \alpha_p &= 10 \log(1 + \epsilon^2) \\ \epsilon &= \sqrt{10^{0.1\alpha_p} - 1} \end{aligned}$$

At stopband  $\Omega = \Omega_s$ .

$$20 \log |H(j\Omega)| = -\alpha_s \log \left[ 1 + \epsilon^2 C_N^2 \left( \frac{\Omega_s}{\Omega_p} \right) \right]$$

$$\text{But } C_N = \left( \frac{\Omega_s}{\Omega_p} \right) = \cosh \left[ N \cosh^{-1} \frac{\Omega_s}{\Omega_p} \right]$$

$$\alpha_s = 10 \log \left[ 1 + \epsilon^2 \left\{ \cosh \left( N \cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right) \right) \right\}^2 \right]$$

But

$$\cosh \left( N \cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right) \right)^2 = \frac{10^{0.1\alpha_s} - 1}{\epsilon^2} = \frac{10^{0.1\alpha_s} - 1}{(10^{0.1\alpha_p} - 1)}$$

$$N \cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right) = \cosh^{-1} \left[ \frac{(10^{0.1\alpha_s}) - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2}$$

$$N \geq \frac{\cosh^{-1} \left[ \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2}}{\cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right)}$$

or

$$N \geq \frac{\cosh^{-1} \left[ \frac{1}{\epsilon} \left( \frac{1}{A_s^2} - 1 \right)^{1/2} \right]}{\cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right)}$$

**Note:**  $\cosh^{-1} x$  can be evaluated using the following identity:

$$\cosh^{-1} x = \ln[x + \sqrt{(x^2 - 1)}]$$

They are two types:

- (I) **Type-I:** The magnitude response is equiripple in the passband and monotonic in the stopband as shown in Fig. 3.21.
- (II) **Type-II:** The magnitude response is monotonic in passband and equiripple in stopband as shown in Fig. 3.22.

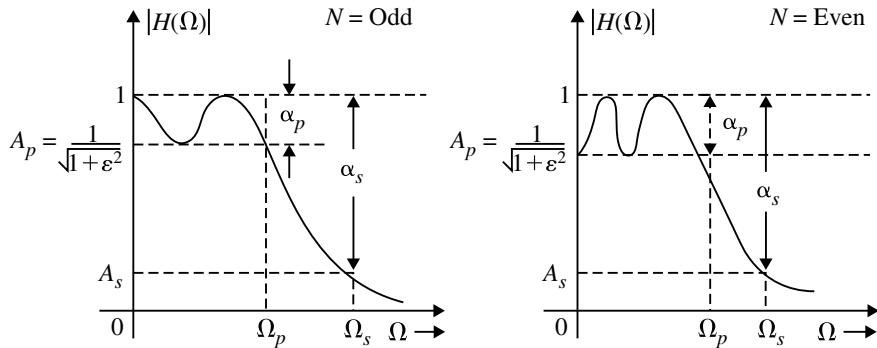
The magnitude response of type-I lowpass filter is given by

$$|H_a(\Omega)|^2 = \frac{1}{1 + \epsilon^2 C_N^2 \left( \frac{\Omega}{\Omega_c} \right)} \quad (3.54)$$

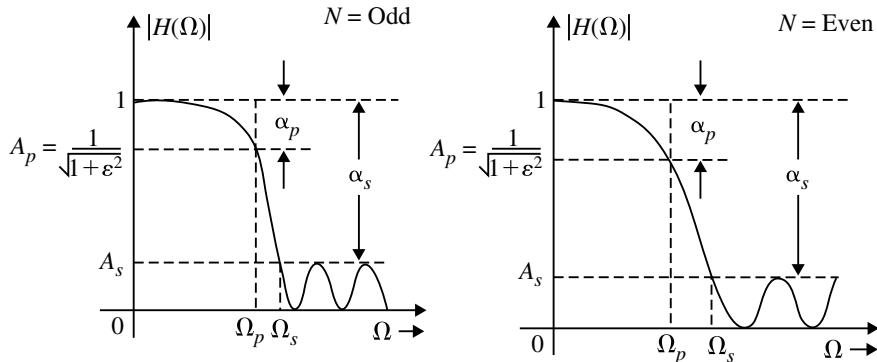
where  $\epsilon$  is attenuation constant and  $C_N$  is Chebyshev polynomial

$$\epsilon = \left[ \frac{1}{A_p^2} - 1 \right]^{1/2} \quad (3.55)$$

where  $s = j\Omega$  gives  $\Omega = s/j$ . Substitute  $\Omega = s/j$  in above equation



**Fig. 3.21** Frequency response of type-I Chebyshev filter



**Fig. 3.22** Frequency response of type-II Chebyshev filter. **a**  $N$  is odd and **b**  $N$  is even

$$\begin{aligned} H_a(s)H_a(-s) &= \frac{1}{1 + \epsilon^2 C_N^2 \left( \frac{s/j}{\Omega_c} \right)} \\ H_a(s)H_a(-s) &= \frac{1}{1 + \epsilon^2 C_N^2 (-js_n)} \end{aligned} \quad (3.56)$$

If  $N$  is even all poles are complex. When  $N$  is odd one of the poles is real and all other poles are complex.

### 3.9.3 Unnormalized Chebyshev Lowpass Filter Transfer Function

When  $N$  is even

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c^2 s + C_k \Omega_c^2} \quad (3.57)$$

When  $N$  is odd

$$H_a(s) = \frac{B_0 \Omega_c}{s + C_0 \Omega_c} \prod_{k=1}^{N-1/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c^2 s + C_k \Omega_c^2} \quad (3.58)$$

where

$$b_k = 2Y_N \sin\left(\frac{(2k-1)\pi}{2N}\right) \quad (3.59)$$

$$c_k = Y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right) \quad (3.60)$$

$$c_0 = Y_N \quad (3.61)$$

$$Y_N = \frac{1}{2} \left\{ \left[ \left( \frac{1}{\epsilon^2} - 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{1/N} - \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{-(1/N)} \right\} \quad (3.62)$$

For  $N$  is even, parameter  $B_k$  is evaluated by

$$H_a(s) \Big|_{s=0} = \frac{1}{(1+\epsilon^2)^{1/2}} \quad (3.63)$$

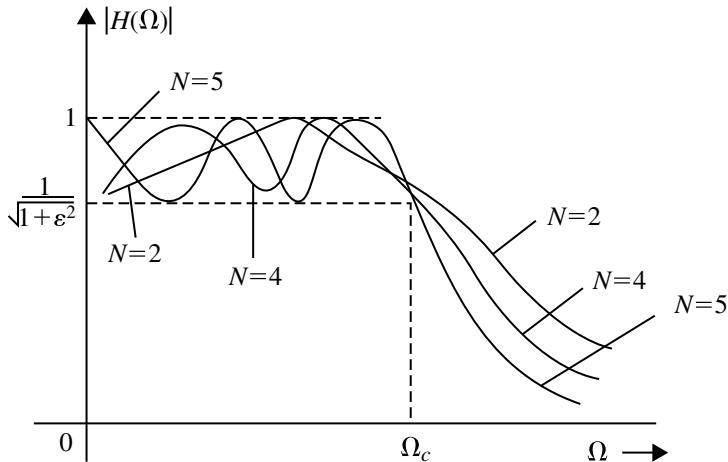
for  $N$  is odd,  $B_k$  is calculated by

$$H_a(s) \Big|_{s=0} = 1 \quad \text{and} \quad B_0 = B_1 = B_2 = \dots = B_k \quad (3.64)$$

### 3.9.4 Frequency Response of Chebyshev Filter

The frequency response of Chebyshev filter is shown in Fig. 3.23.

$$N \geq \frac{\cos h^{-1} \left\{ \frac{1}{\epsilon} \left( \frac{1}{A_s^2} - 1 \right)^{1/2} \right\}}{\cos h^{-1} \left( \frac{\Omega_s}{\Omega_p} \right)} \quad (3.65)$$



**Fig. 3.23** Frequency response of Chebyshev filter type-I filter

### 3.9.5 Properties of Chebyshev Filter (Type I)

1. The magnitude oscillates between  $1$  and  $\frac{1}{\sqrt{1+\epsilon^2}}$  with in passband (equiripple in the passband).
2. At cutoff frequency  $\Omega_c$ , the magnitude is  $\frac{1}{\sqrt{1+\epsilon^2}}$ .
3. Magnitude is monotonic outside the passband.
4. Chebyshev filter are all-pole design.
5. As  $N$  increases, it is approaches to ideal response.

### 3.9.6 Design Procedures for Lowpass Digital Chebyshev IIR Filter

1. Choose either bilinear or impulse invariant transformation.
2. Calculate the attenuation constant  $\epsilon$

$$\epsilon = \left[ \frac{1}{A_p^2} - 1 \right]^{1/2}$$

3. Calculate the ratio  $\Omega_s / \Omega_p$  using the following equation which is appropriate.

For bilinear transformation,  $\frac{\Omega_s}{\Omega_p} = \frac{\tan(\omega_s/2)}{\tan(\omega_p/2)}$

For impulse invariant,  $\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p}$

4. Decide the order  $N$  of the filter from

$$N \leq \frac{\cos h^{-1} \left\{ \frac{1}{\epsilon} \left[ \frac{1}{A_2^s} - 1 \right]^{1/2} \right\}}{\cos h^{-1} \left( \frac{\Omega_s}{\Omega_p} \right)}$$

5. Calculate the analog cutoff frequency  $\Omega_c$  from the equations given below:

For bilinear transformation,  $\Omega_c = \frac{2}{T} \frac{\tan(\omega_p/2)}{\left[ \frac{1}{A_p^2} - 1 \right]^{1/2N}}$

For impulse invariant,  $\Omega_c = \frac{(\omega_p/T)}{\left[ \frac{1}{A_p^2} - 1 \right]^{1/2N}}$

6. Determine analog transfer function, when  $N$  is even

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

When  $N$  is odd

$$H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{N-1/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

where

$$b_k = 2Y_N \sin \left( \frac{(2k-1)\pi}{2N} \right)$$

$$c_k = Y_N^2 + \cos^2 \left( \frac{(2k-1)\pi}{2N} \right)$$

$$c_0 = Y_N$$

$$Y_N = \frac{1}{2} \left\{ \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{1/N} - \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{-(1/N)} \right\}$$

$B_k$  can be calculated from

$$\begin{aligned} N = \text{even}; \quad H_a(s) \Big|_{s=0} &= \frac{1}{(1 + \epsilon^2)^{1/2}} \\ N = \text{odd}; \quad H_a(s) \Big|_{s=0} &= 1 \end{aligned}$$

(the normal practice is to take  $B_0 = B_1 = B_2, \dots, B_k$ ).

7. Convert  $H_a(s)$  into  $H(z)$  using the chosen transform (bilinear or impulse invariance).
8. Realize the digital filter transfer function by a suitable structure.

## 3.10 Frequency Transformation

### 3.10.1 Analog Frequency Transformation

The IIR lowpass analog filters are designed for the given specifications. By applying appropriate approximations and transformations lowpass digital filters are designed. By using frequency transformations it is possible to design lowpass filters with different passband frequencies, highpass filters, bandpass filters and bandstop filters. In the frequency transformation, the variable  $s$  of normalized ( $\Omega_c = 1$ ) lowpass filter is replaced by appropriate variable. From the normalized lowpass filter, the following filters are designed:

1. Lowpass to lowpass.
2. Lowpass to highpass filter.
3. Lowpass to bandpass filter.
4. Lowpass to band elimination filter (bandstop filter).

Let  $\Omega_p$  be the passband frequency of lowpass analog filter. Let  $\Omega'_p$  be the edge frequency of the lowpass filter which we wish to convert. The frequency transformation to achieve this is given by

$$s \rightarrow \frac{\Omega_p}{\Omega'_p}$$

(3.66)

### 3.10.1.1 Lowpass to Highpass Filter

The transformation that is made to convert the lowpass to highpass filter is given by

$$\boxed{s \rightarrow \frac{\Omega_p \Omega'_p}{s}} \quad (3.67)$$

where  $\Omega_p$  = Edge frequency of the prototype lowpass filter and  $\Omega'_p$  = Edge frequency of the highpass filter.

### 3.10.1.2 Lowpass to Bandpass Filter

Let  $\Omega_l$  be the lower band edge frequency of bandpass filter and  $\Omega_u$  be the upper band edge frequency of the same filter. The transfer function from lowpass filter to bandpass filter is achieved by the following transformation:

$$\boxed{s \rightarrow \Omega_p \frac{(s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)}} \quad (3.68)$$

### 3.10.1.3 Lowpass to Bandstop Filter

Let

$\Omega_p$  = Passband edge frequency of the given lowpass filter.

$\Omega_u$  = Upper band edge frequency of the bandstop filter.

$\Omega_l$  = Lower band edge frequency of the bandstop filter.

The transfer function from lowpass prototype to bandstop filter transfer function is achieved by the following transformation.

$$s \rightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{(s^2 + \Omega_u \Omega_l)} \quad (3.69)$$

## 3.10.2 Digital Frequency Transformation

If we design a lowpass digital filter and wish to convert to desired lowpass filter, high-pass filter, bandpass filter and bandstop filter, the following frequency transformation may be used.

### 3.10.2.1 Lowpass to Lowpass Filter

$$\boxed{z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}} \quad (3.70)$$

where

$$\alpha = \frac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}.$$

$\omega_p$  = Band edge frequency of the lowpass original filter.

$\omega'_p$  = Band edge frequency of the lowpass desired filter.

### 3.10.2.2 Lowpass to Highpass Filter

$$\boxed{z^{-1} \rightarrow \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}} \quad (3.71)$$

$$\alpha = \frac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$$

### 3.10.2.3 Lowpass to Bandpass Filter

$$\boxed{z^{-1} \rightarrow \frac{(z^{-2} - \alpha_1 z^{-1} + \alpha_2)}{(\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1)}} \quad (3.72)$$

$$\alpha_1 = \frac{2\alpha k}{k+1}$$

$$\alpha_2 = \frac{(k-1)}{(k+1)}$$

$$\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$$

$$k = \cot \frac{(\omega_u - \omega_l)}{2} \tan \frac{\omega_p}{2}$$

### 3.10.2.4 Lowpass to Bandstop Filter

$$\boxed{z^{-1} \rightarrow \frac{(z^{-2} - \alpha_1 z^{-1} + \alpha_2)}{(\alpha_2 z^{-1} - \alpha_1 z^{-1} + 1)}} \quad (3.73)$$

$$\begin{aligned}\alpha_1 &= \frac{2\alpha}{(k+1)} \\ \alpha_2 &= \frac{(1-k)}{(1+k)} \\ \alpha &= \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]} \\ k &= \tan \frac{(\omega_u - \omega_l)}{2} \tan \frac{\omega_p}{2}\end{aligned}$$

	Filter type	Transformation
1.	Lowpass	$s \rightarrow \frac{s}{\Omega_c}$
2.	Highpass	$s \rightarrow \frac{s}{\Omega_c}$
3.	Bandpass	$s \rightarrow \frac{Q(s^2 + \Omega_0^2)}{\Omega_0 s}$
4.	Bandstop	$s \rightarrow \frac{\Omega_0 s}{Q(s^2 + \Omega_0^2)}$

where  $\Omega_0$  is center frequency,  $\Omega_0 = \sqrt{\Omega_p \Omega_s}$  where  $Q$  is quality factor,  $Q = \frac{\Omega_0}{\Omega_s - \Omega_p}$ .

### Example 3.21

The specification of the desired lowpass filter is

$$\begin{aligned}0.8 \leq |H(\omega)| &\leq 1.0; & 0 \leq \omega < 0.2\pi \\ |H(\omega)| &\leq 0.2; & 0.32\pi \leq \omega \leq \pi\end{aligned}$$

Design Chebyshev digital filter using bilinear transformation.

**Solution** Given  $A_p = 0.8$  at  $\omega_p = 0.2\pi$  rad/s and  $A_s = 0.2$  at  $\omega_s = 0.32\pi$  rad/s  
For bilinear transformation we get,

$$\frac{\Omega_s}{\Omega_p} = \frac{\tan(\omega_s/2)}{\tan(\omega_p/2)} = \frac{\tan(0.32\pi/2)}{\tan(0.2\pi/2)} = 1.692$$

The attenuation constant  $\epsilon = \left(\frac{1}{A_p^2} - 1\right)^{1/2} = 0.75$

$$N \geq \frac{\cos h^{-1} \left\{ \frac{1}{\epsilon} \left[ \frac{1}{A_s^2} - 1 \right]^{1/2} \right\}}{\cos h^{-1} \frac{\Omega_s}{\Omega_p}} = \frac{\cos h^{-1} \left\{ \frac{1}{0.75} \left[ \frac{1}{0.2^2} - 1 \right]^{1/2} \right\}}{\cos h^{-1} 1.692}$$

$$= \frac{\cos h^{-1} 6.53}{\cos h^{-1} 1.692}$$

$$\begin{aligned}\cos h^{-1} 6.53 &= \ln(6.53 + \sqrt{6.53^2 - 1}) \\ &= 2.56\end{aligned}$$

$$\begin{aligned}\cos h^{-1} 1.692 &= \ln(1.692 + \sqrt{1.692^2 - 1}) \\ &= 1.1173\end{aligned}$$

$$N = \frac{2.56}{1.1173}$$

$$N \geq 2.295$$

$$N = 3$$

Let  $T = 1\text{s}$ . The analog cutoff frequency

$$\begin{aligned}\Omega_c &= \frac{2}{T} \frac{\tan(\omega_p/2)}{\left[ \frac{1}{A_p^2} - 1 \right]^{1/2N}} \\ \Omega_c &= 2 \frac{\tan(0.2\pi/2)}{\left[ \frac{1}{(0.8)^2} - 1 \right]^{1/6}} \\ \Omega_c &= 0.715 \text{ rad/s.}\end{aligned}$$

For  $N$  is odd, the analog transfer function

$$\begin{aligned}H_a(s) &= \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \\ &= \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2} \\ Y_N &= \frac{1}{2} \left\{ \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{1/N} - \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{-(1/N)} \right\}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left\{ \left[ \left( \frac{1}{0.75^2} + 1 \right)^{1/2} + \frac{1}{0.75} \right]^{1/3} - \left[ \left( \frac{1}{0.75^2} + 1 \right)^{1/2} + \frac{1}{0.75} \right]^{-1/(3)} \right\} \\
&= 0.7211 - 0.3467 = 0.3744
\end{aligned}$$

$$c_0 = Y_N = 0.3744$$

$$c_k = Y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}$$

$$c_1 = Y_N^2 + \cos^2 \left( \frac{\pi}{6} \right) = 0.3744^2 + \cos^2 \left( \frac{\pi}{6} \right) = 0.8902$$

$$b_k = 2Y_N \sin \left[ \frac{(2k-1)\pi}{2N} \right]$$

$$b_1 = 2 \times 0.3744 \sin \left[ \frac{\pi}{6} \right] = 0.3744$$

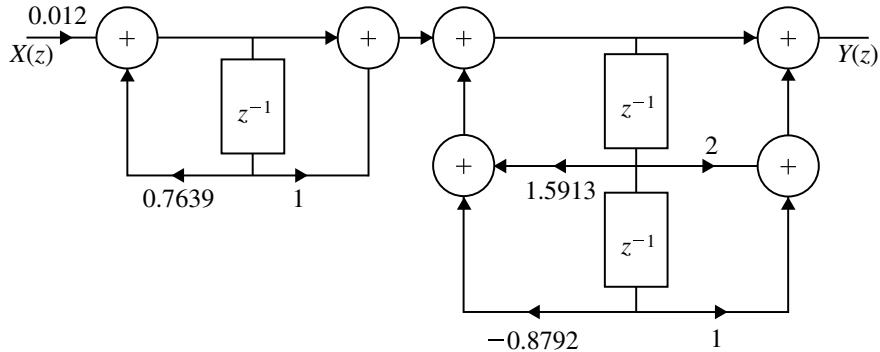
$$\begin{aligned}
H_a(s) &= \frac{0.715B_0}{(s + 0.3744 \times 0.715)} \times \frac{(0.715)^2 B_1}{s^2 + 0.3744 \times 0.715s + 0.8902 \times 0.715^2} \\
&= \frac{0.3655 B_0 B_1}{(s + 0.2677)(s^2 + 0.2677s + 0.4551)}
\end{aligned}$$

$$\begin{aligned}
H_a(s) \Big|_{s=0} &= 1 \\
\frac{0.3655 B_0 B_1}{0.2677 \times 0.4551} &= 1 \\
3B_0 B_1 &= 1 \\
B_0 B_1 &= \frac{1}{3} \\
B_0^2 &= \frac{1}{3} \\
B_0 &= 0.577 \\
\therefore B_0 &= B_1 = 0.577
\end{aligned}$$

$$\begin{aligned}
H_a(s) &= \frac{0.3655 \times 0.577^2}{(s + 0.2677)(s^2 + 0.2677s + 0.4551)} \\
&= \frac{0.1217}{(s + 0.2677)(s^2 + 0.2677s + 0.4551)}
\end{aligned}$$

$$H(z) = H_a(s) \Big|_{s=\frac{2(1-z^{-1})}{1+z^{-1}}} \quad (\text{Let } T = 1 \text{ s})$$

$$H(z) = \frac{0.1217}{\left( \frac{2(1-z^{-1})}{1+z^{-1}} + 0.2677 \right) \left( \left( \frac{2(1-z^{-1})}{1+z^{-1}} \right)^2 + 0.2677 \times \frac{2(1-z^{-1})}{1+z^{-1}} + 0.4551 \right)}$$



**Fig. 3.24** Cascade form realization of  $H(z)$  for Example 3.22

$$\begin{aligned}
 &= 0.1217(1 + z^{-1})^3 / ((2.2677 - 1.7323z^{-1})(4 + 4z^{-2} - 8z^{-1} + 0.5354 \\
 &\quad - 0.5354z^{-2} + 0.4551 + 0.4551z^{-2} + 0.9102z^{-1}) \\
 &= \frac{0.1217(1 + z^{-1})^3}{(2.2677 - 1.7323z^{-1})(4.4551 - 7.089z^{-1} + 3.9197z^{-2})} \\
 &= \frac{0.012(1 + z^{-1})^3}{(1 - 0.7639z^{-1})(1 - 1.5913z^{-1} + 0.8798z^{-2})}
 \end{aligned}$$

$$H(z) = \frac{0.012(1 + z^{-1})(1 + 2z^{-1} + z^{-2})}{(1 - 0.7639z^{-1})(1 - 1.5913z^{-1} + 0.8798z^{-2})}$$

The cascade form realization is shown in Fig. 3.24.

### Example 3.22

The specifications of the desired lowpass digital filter are:

$$\begin{aligned}
 0.9 &\leq |H(\omega)| \leq 1; & 0 \leq \omega < 0.25\pi \\
 |H(\omega)| &\leq 0.24; & 0.5\pi \leq \omega \leq \pi
 \end{aligned}$$

Design Chebyshev digital filter using impulse invariant transformation.

(Anna University, December, 2007)

**Solution** Given  $A_p = 0.9$  at  $\omega_p = 0.25\pi$  rad/s and  $A_s = 0.24$  at  $\omega_s = 0.5\pi$  rad/s.

$$\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p} = \frac{0.5\pi}{0.25\pi} = 2$$

$$\epsilon = \left( \frac{1}{A_p^2} - 1 \right)^{1/2} = \left( \frac{1}{(0.9)^2} - 1 \right)^{1/2} = 0.484$$

$$\begin{aligned} N &\geq \frac{\cosh^{-1} \left\{ \frac{1}{\epsilon} \left[ \frac{1}{A_s^2} - 1 \right]^{1/2} \right\}}{\cos h^{-1} \frac{\Omega_s}{\Omega_p}} \\ &\geq \frac{\cosh^{-1} \left\{ \frac{1}{0.484} \left[ \frac{1}{0.24^2} - 1 \right]^{1/2} \right\}}{\cos h^{-1} 2} \\ &\geq \frac{\cosh^{-1} 2.8126}{\cosh^{-1} 2} \\ N &\geq 2.136 \\ N &= 3 \end{aligned}$$

Let  $T = 1$  s

$$\begin{aligned} \Omega_c &= \frac{\omega_p/T}{\left[ \frac{1}{A_p^2} - 1 \right]^{1/2N}} \\ \Omega_c &= \frac{0.25\pi}{\left[ \frac{1}{(0.9)^2} - 1 \right]^{1/6}} \\ \Omega_c &= 1 \text{ rad/s.} \end{aligned}$$

The analog transfer function for  $N$  is odd is given by

$$\begin{aligned} H_a(s) &= \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{(N-1)/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \\ &= \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \cdot \frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2} \end{aligned}$$

$$\begin{aligned} Y_N &= \frac{1}{2} \left\{ \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{1/N} - \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{1/2} + \frac{1}{\epsilon} \right]^{-(1/N)} \right\} \\ &= \frac{1}{2} \left\{ \left[ \left( \frac{1}{0.484^2} + 1 \right)^{1/2} + \frac{1}{0.484} \right]^{1/3} \right\} \end{aligned}$$

$$-\left[ \left( \frac{1}{0.484^2} + 1 \right)^{1/2} + \frac{1}{0.484} \right]^{-(1/3)} \Bigg\}$$

$$= \frac{1}{2} \left\{ 1.6338 - \frac{1}{1.6338} \right\}$$

$$Y_N = 0.511$$

$$c_0 = Y_N = 0.511$$

$$c_k = Y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}$$

$$c_1 = 0.511^2 + \cos^2 \left( \frac{\pi}{6} \right) = 1.011$$

$$b_k = 2Y_N \sin \left[ \frac{(2k-1)\pi}{2N} \right]$$

$$b_1 = 2 \times 0.511 \sin \left[ \frac{\pi}{6} \right] = 0.511$$

$$\therefore H_a(s) = \frac{B_0 B_1}{(s + 0.511)(s^2 + 0.511s + 1.011)}$$

$$H_a(s) \Big|_{s=0} = 1$$

$$\frac{B_0 B_1}{0.511 \times 1.011} = 1 \quad (\text{Let } B_0 = B_1)$$

$$B_0^2 = \frac{1}{1.939}$$

$$B_0 = 0.719$$

$$\therefore B_0 = B_1 = 0.719$$

$$\begin{aligned} H_a(s) &= \frac{(0.719)^2}{(s + 0.511)(s^2 + 0.511s + 1.011)} \\ &= \frac{0.517}{(s + 0.511)(s^2 + 0.511s + 1.011)} \\ &= \frac{A}{(s + 0.511)} + \frac{Bs + C}{(s^2 + 0.511s + 1.011)} \end{aligned}$$

$$A(s^2 + 0.511s + 1.011) + (Bs + C)(s + 0.511) = 0.517$$

Put  $s = -0.511$

$$A(1.011) = 0.5166; \quad A = 0.511$$

Put  $s = 0$

$$1.01A + 0.51C = 0.517; \quad C = 0$$

Equating the coefficients of  $s^2$  we get

$$\begin{aligned} A + B &= 0 \\ B &= -0.511 \end{aligned}$$

$A = 0.511$ ,  $B = -0.511$  and  $C = 0$

$$\begin{aligned} H_a(s) &= \frac{0.511}{(s + 0.511)} - \frac{0.511s}{(s^2 + 0.511s + 1.011)} \\ &= \frac{0.511}{(s + 0.511)} - 0.511 \frac{s + 0.256 - 0.256}{(s + 0.256)^2 + 0.972^2} \\ &= \frac{0.511}{(s + 0.511)} - 0.511 \frac{s + 0.256}{(s + 0.256)^2 + 0.972^2} \\ &\quad + 0.0511 \cdot \frac{0.256}{(s + 0.256)^2 + 0.972^2} \\ &= \frac{0.511}{(s + 0.511)} - 0.511 \frac{s + 0.256}{(s + 0.256)^2 + 0.972^2} + \frac{0.0511}{0.972} \\ &\quad \times 0.256 \cdot \frac{0.972}{(s + 0.256)^2 + 0.972^2} \\ H(s) &= \frac{0.511}{(s + 0.511)} - 0.511 \frac{s + 0.256}{(s + 0.256)^2 + 0.972^2} + 0.135 \\ &\quad \times \frac{0.972}{(s + 0.256)^2 + 0.972} \\ H(z) &= \frac{0.511}{1 - e^{-0.511}z^{-1}} - 0.511 \frac{1 - e^{-0.256} \cos(0.972)z^{-1}}{1 - 2e^{-0.256}(\cos 0.972)z^{-1} + e^{-2 \times 0.256}z^{-2}} \\ &\quad + 0.135 \frac{e^{-0.256} \sin(0.972)z^{-1}}{1 - 2e^{-0.256}(\cos 0.972)z^{-1} + e^{-2 \times 0.256}z^{-2}} \\ &= \frac{0.511}{1 - 0.6z^{-1}} + \frac{-0.511 + 0.223z^{-1}}{1 - 0.873z^{-1} + 0.6z^{-2}} + \frac{0.086z^{-1}}{1 - 0.873z^{-1} + 0.6z^{-2}} \\ &= \frac{0.511}{1 - 0.6z^{-1}} + \frac{-0.511 + 0.223z^{-1} + 0.086z^{-1}}{1 - 0.873z^{-1} + 0.6z^{-2}} \end{aligned}$$

$H(z) = \frac{0.511}{1 - 0.6z^{-1}} + \frac{-0.511 + 0.309z^{-1}}{1 - 0.873z^{-1} + 0.6z^{-2}}$
---

**Example 3.23**

Determine the system lowest-order Chebyshev digital filter that meets the following specification:

- (a) 1 dB ripple in the passband  $0 \leq |\omega| \leq 0.3\pi$ .
- (b) Atleast 60 dB attenuation in the stopband  $0.35\pi \leq |\omega| \leq \pi$ . Use the bilinear transformation.

**Solution**

$$-20 \log A_p = 1 \text{ dB}, \quad \omega_p = 0.3\pi$$

$$-20 \log A_s = 60 \text{ dB}, \quad \omega_s = 0.35\pi$$

$$20 \log A_p = -1,$$

$$A_p = 0.89125,$$

$$20 \log A_s = -60,$$

$$A_s = 0.001$$

$$\begin{aligned} \text{The attenuation constant } \epsilon &= \left( \frac{1}{A_p^2} - 1 \right)^{\frac{1}{2}} \\ &= \left( \frac{1}{(0.89)^2} - 1 \right)^{\frac{1}{2}} = 0.5123 \\ \frac{\Omega_s}{\Omega_p} &= \frac{\tan\left(\frac{\omega_s}{2}\right)}{\tan\left(\frac{\omega_p}{2}\right)} = \frac{\tan\left(0.35\frac{\pi}{2}\right)}{\tan\left(0.3\frac{\pi}{2}\right)} = \frac{0.6128}{0.5045} \\ &= 1.203 \\ N &\geq \frac{\cosh^{-1} \left\{ \frac{1}{\epsilon} \left[ \frac{1}{A_s^2} - 1 \right]^{\frac{1}{2}} \right\}}{\cosh^{-1} \frac{\Omega_s}{\Omega_p}} \\ &\geq \frac{\cosh^{-1} \left\{ \frac{1}{0.5123} \left[ \frac{1}{(0.001)^2} - 1 \right]^{\frac{1}{2}} \right\}}{\cosh^{-1}(1.203)} \\ &\geq \frac{\cosh^{-1}(1951.98)}{\cosh^{-1}(1.203)} \\ &\geq \frac{8.2697}{0.6265} \\ N &\geq 13.1935 \end{aligned}$$

$N = 14$

**Example 3.24**

Design a Chebyshev filter with a maximum passband attenuation of 2.5 dB at  $\Omega_p = 20$  rad/s and the stopband attenuation of  $\Omega_s = 50$  rad/s.

(Anna University, May, 2007)

**Solution** Given

$$-20 \log A_p = 2.5, \quad \Omega_p = 20 \text{ rad/s}$$

$$A_p = 0.749,$$

$$A_p \approx 0.75$$

$$-20 \log A_s = 30, \quad \Omega_s = 50 \text{ rad/s}$$

$$A_s = 0.0316$$

$$\begin{aligned} \text{The attenuation constant } \epsilon &= \left( \frac{1}{A_p^2} - 1 \right)^{\frac{1}{2}} \\ &= \left( \frac{1}{(0.75)^2} - 1 \right)^{\frac{1}{2}} = 0.8819 \\ \frac{\Omega_s}{\Omega_p} &= \frac{50}{20} = 2.5 \\ N &\geq \frac{\cosh^{-1} \left\{ \frac{1}{\epsilon} \left[ \frac{1}{A_s^2} - 1 \right]^{\frac{1}{2}} \right\}}{\cosh^{-1} \left( \frac{\Omega_s}{\Omega_p} \right)} \\ &\geq \frac{\cosh^{-1} \left\{ \frac{1}{0.8819} \left[ \frac{1}{(0.0316)^2} - 1 \right]^{\frac{1}{2}} \right\}}{\cosh^{-1}(2.5)} \\ &\geq \frac{\cosh^{-1}(35.8654)}{\cosh^{-1}(2.5)} \\ &\geq \frac{4.2727}{1.5667} \\ &\geq 2.72 \\ N &= 3 \end{aligned}$$

The analog cutoff frequency

$$\Omega_c \approx \Omega_p = 20 \text{ rad/s}$$

For  $N$  is odd

$$\begin{aligned}
 H_a(s) &= \frac{B_0\Omega_c}{s + c_0\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2} \\
 &= \frac{B_0 \times 20}{s + (c_0 \times 20)} \prod_{k=1}^{\frac{3-1}{2}} \frac{B_k(20)^2}{s^2 + b_k 20s + c_k(20)^2} \\
 &= \frac{20B_0}{s + 20c_0} \times \frac{20B_1}{s^2 + b_1 20s + (20)^2 c_1} \\
 b_k &= 2Y_N \sin\left(\frac{(2k-1)\pi}{2N}\right) \\
 Y_N &= \frac{1}{2} \left\{ \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{\epsilon} \right]^{\frac{1}{N}} - \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{\epsilon} \right]^{-\left(\frac{1}{N}\right)} \right\} \\
 &= \frac{1}{2} \left\{ \left[ \left( \frac{1}{(0.8819)^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{0.8819} \right]^{\frac{1}{3}} - \left[ \left( \frac{1}{(0.8819)^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{0.8819} \right]^{-\frac{1}{3}} \right\} \\
 &= \frac{1}{2} \left\{ [1.51187 + 1.134]^{\frac{1}{3}} - [1.51187 + 1.134]^{-\frac{1}{3}} \right\} \\
 &= \frac{1}{2} \{1.3831 - 0.723\}
 \end{aligned}$$

$$Y_N = 0.33$$

$$\begin{aligned}
 b_1 &= 2Y_N \sin\left(\frac{(2-1)\pi}{2 \times 3}\right) \\
 &= 2 \times 0.33 \sin\left[\frac{\pi}{6}\right]
 \end{aligned}$$

$$b_1 = 0.33$$

$$c_0 = Y_N = 0.33$$

$$\begin{aligned}
 c_k &= Y_n^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right) \\
 c_1 &= (0.33)^2 + \cos^2\left[\frac{(2-1)\pi}{2 \times 3}\right] \\
 &= (0.33)^2 + \cos^2\left[\frac{\pi}{6}\right] \\
 &= (0.33)^2 + 0.75
 \end{aligned}$$

$$c_1 = 0.8589$$

$$H_a(s) = \frac{20B_0}{s + (20 \times 0.33)} \times \frac{20^2 B_1}{s^2 + (0.33)(20)s + ((20)^2 \times 0.8589)}$$

$$H_a(s) = \frac{20B_0}{s + 6.6} \times \frac{20^2 B_1}{s^2 + 6.6s + 343.56}$$

For  $N = \text{odd}$ ,  $B_k$  can be calculated from

$$\begin{aligned} H_a(s) \Big|_{s=0} &= 1, \quad [B_0 = B_1 = B_2 \cdots B_N] \\ \frac{20B_0}{6.6} \times \frac{20^2 B_1}{343.56} &= 1 \\ \frac{8000B_0^2}{2267.496} &= 1 \\ B_0^2 &= 0.2834 \\ B_0 &= 0.532 \\ B_0 &= B_1 = 0.532 \\ H_a(s) &= \frac{20 \times 0.532}{s + 6.6} \times \frac{20^2 \times 0.532}{s^2 + 6.6s + 343.56} \\ &= \frac{10.64}{s + 6.6} \cdot \frac{212.8}{s^2 + 6.6s + 343.56} \\ H_a(s) &= \frac{2265.76}{(s + 6.6)(s^2 + 6.6s + 343.56)} \end{aligned}$$

$$\begin{aligned} H(z) &= H_a(s) \Big|_{s=\frac{2}{T} \frac{(z-1)}{(z+1)}}, \quad \text{Let } T = 1 \text{ s} \\ &= \frac{2265.76}{\left(\frac{2(z-1)}{z+1} + 6.6\right) \left[\left(\frac{2(z-1)}{z+1}\right)^2 + 6.6 \times \frac{2(z-1)}{(z+1)} + 343.56\right]} \\ &= \frac{2265.76}{\left(\frac{2(z-1)+6.6(z+1)}{(z+1)}\right) \left(\frac{4(z-1)^2+13.2(z-1)(z+1)+343.56(z+1)^2}{(z+1)^2}\right)} \\ &= \frac{2265.76(z+1)^3}{[2(z-1) + 6.6(z+1)][4(z^2 - 2z + 1) + 13.2(z^2 - 1) + 343.56(z^2 + 2z + 1)]} \\ &= \frac{2265.76(z+1)^2}{[8.6z + 4.6][4z^2 - 8z + 2 + 13.2z^2 - 13.2 + 343.56z^2 + 687.12z + 343.56]} \end{aligned}$$

$$H(z) = \frac{2265.76(z+1)^2}{(8.6z + 4.6)[360.76z^2 + 687.72z + 332.36]}$$

**Example 3.25**

Design a digital Chebyshev filter to meet the constraints

$$\begin{aligned}\frac{1}{\sqrt{2}} \leq |H(\omega)| \leq 1; \quad 0 \leq \omega \leq 0.2\pi \\ 0 \leq |H(\omega)| \leq 0.1; \quad 0.5\pi \leq \omega \leq \pi\end{aligned}$$

by using bilinear transformation. Assume  $T = 1$  s

(Anna University)

**Solution** Given

$$A_p = 0.707 \text{ at } \omega_p = 0.2\pi \text{ rad/s}$$

$$A_s = 0.2 \text{ at } \omega_s = 0.5\pi \text{ rad/s}$$

$$\text{The attenuation constant } \epsilon = \left( \frac{1}{A_p^2} - 1 \right)^{\frac{1}{2}} = (2 - 1)^{\frac{1}{2}} \\ \epsilon = 1$$

$$\frac{\Omega_s}{\Omega_p} = \frac{\tan\left(\frac{\omega_s}{2}\right)}{\tan\left(\frac{\omega_p}{2}\right)} = \frac{\tan\left(\frac{0.5\pi}{2}\right)}{\tan\left(\frac{0.2\pi}{2}\right)} = \frac{1}{0.3249}$$

$$\frac{\Omega_s}{\Omega_p} = 3.0378$$

$$n \geq \frac{\cosh^{-1} \left\{ \frac{1}{\epsilon} \left[ \frac{1}{A_s^2} - 1 \right]^{\frac{1}{2}} \right\}}{\cosh^{-1} \frac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\cosh^{-1} \left\{ \frac{1}{1} \left[ \frac{1}{0.2^2} - 1 \right]^{\frac{1}{2}} \right\}}{\cosh^{-1}(3.078)}$$

$$\geq \frac{\cosh^{-1}(4.899)}{\cosh^{-1}(3.078)}$$

$$\geq \frac{2.2716}{1.7899} \geq 1.269$$

$$N = 2$$

Given  $T = 1$  s. The analog cutoff frequency

$$\Omega_c = \frac{2}{T} \frac{\tan\left(\frac{\omega_p}{2}\right)}{\left[\frac{1}{A_p^2} - 1\right]^{\frac{1}{2}N}}$$

$$\Omega_c = 2 \frac{\tan\left(\frac{0.2\pi}{2}\right)}{\left[\frac{1}{(0.707)^2} - 1\right]^{\frac{1}{4}}} = \frac{0.6498}{1} = 0.6498 \text{ rad/s}$$

For  $N$  is even

$$H_a(s) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

$$= \frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2}$$

$$Y_N = \frac{1}{2} \left\{ \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{\epsilon} \right]^{\frac{1}{N}} - \left[ \left( \frac{1}{\epsilon^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{\epsilon} \right]^{-\left(\frac{1}{N}\right)} \right\}$$

$$= \frac{1}{2} \left\{ \left[ \left( \frac{1}{1^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{1} \right]^{\frac{1}{2}} - \left[ \left( \frac{1}{1^2} + 1 \right)^{\frac{1}{2}} + \frac{1}{1} \right]^{-\frac{1}{2}} \right\}$$

$$= \frac{1}{2} \left\{ [4.4142 + 1]^{\frac{1}{2}} - [4.4142 + 1]^{-\frac{1}{2}} \right\}$$

$$= \frac{1}{2} \{1.554 - 0.6435\}$$

$$Y_N = 0.45525$$

$$b_k = 2Y_n \sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$b_1 = 2Y_N \sin\left(\frac{(2-1)\pi}{4}\right)$$

$$= 2Y_N \sin\left(\frac{\pi}{4}\right)$$

$$= 2 \times 0.45525 \sin\left(\frac{\pi}{4}\right)$$

$$b_1 = 0.6438$$

$$c_k = Y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right)$$

$$\begin{aligned}
c_1 &= (0.45525)^2 + \cos^2\left(\frac{\pi}{4}\right) \\
&= (0.45525)^2 + 0.5 \\
c_1 &= 0.707 \\
H_a(s) &= \frac{B_1 \Omega_c^2}{s^2 + 0.6438 \Omega_c s + 0.707 \Omega_c^2} \\
&= \frac{B_1 (0.6498)^2}{s^2 + 0.6438 \times 0.6498 s + 0.707 (0.6498)^2} \\
H_a(s) &\frac{B_1 0.4222}{s^2 + 0.4183 s + 0.2985}
\end{aligned}$$

For  $N$  is even.

$$\begin{aligned}
H_a(s) \Big|_{s=0} &= \frac{1}{(1+\epsilon^2)^{\frac{1}{2}}} \\
\frac{B_1 \times 0.4222}{0.2985} &= \frac{1}{(1+1^2)^{\frac{1}{2}}} \\
\frac{0.4222 B_1}{0.2985} &= 1 \\
B_1 &= \frac{0.2985}{0.4222} \\
B_1 &= 0.707 \\
H_a(s) &= \frac{0.707 \times 0.4222}{s^2 + 0.4183 s + 0.2985} \\
H_a(s) &= \frac{0.2985}{s^2 + 0.4183 s + 0.2985}
\end{aligned}$$

Using bilinear transformation we get

$$\begin{aligned}
H(z) &= H_a(s) \Big|_{s=\frac{2}{T} \frac{z-1}{z+1}}, \quad T = 1 \text{ s} \\
&= \frac{0.2985}{\left(2 \frac{z-1}{z+1}\right)^2 + 0.4183 \left(s \frac{(z-1)}{(z+1)}\right) 2 + 0.2985} \\
&= \frac{0.2985(z+1)^2}{4(z-1)^2 + 0.8366(z-1)(z+1) + 0.2985(z+1)^2} \\
&= \frac{0.2985(z^2 + 2z + 1)}{4(z^2 - 2z + 1) + 0.8366(z^2 - 1) + 0.2985(z^2 + 2z + 1)} \\
&= \frac{0.2985(z^2 + 2z + 1)}{4z^2 - 8z + 4 + 0.8366z^2 - 0.8366 + 0.2985z^2 + 0.597z + 0.2985}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{0.2985(z^2 + 2z + 1)}{5.1351z^2 - 7.403z + 3.4619} \\
 H(z) &= \frac{0.2985(z^{-2} + 2z^{-1} + 1)}{3.4619z^{-2} - 7.403z^{-1} + 5.1351} \\
 &= \frac{0.2985(z^{-2} + 2z^{-1} + 1)}{5.1351(0.674z^{-2} - 1.4416z^{-1} + 1)}
 \end{aligned}$$

$$H(z) = \frac{0.0581(z^{-2} + 2z^{-1} + 1)}{0.674z^{-2} - 1.4416z^{-1} + 1}$$

**Example 3.26**

The transfer function of an analog LPF is

$$H(z) = \frac{1}{s+1}$$

with a bandwidth of 1 rad/s. Use bilinear transform to design a digital filter with a bandwidth 20 Hz at a sampling frequency 600 Hz.

*(Anna University, December, 2005)*

**Solution** Given

$$\begin{aligned}
 H(z) &= \frac{1}{s+1}, \quad \Omega_p = 1 \text{ rad/s} \\
 f'_p &= 20 \text{ Hz} \\
 \Omega'_p &= 2\pi f'_p \\
 &= 2\pi \times 20 \\
 &= 125.66 \text{ rad/s.}
 \end{aligned}$$

Analog lowpass filter to lowpass filter transformation from Eq. (3.66) obtains by replacing  $s \rightarrow \frac{\Omega_p}{\Omega'_p} \cdot s$

$$\begin{aligned}
 H(s) &= \frac{1}{\frac{s}{125.66} + 1} \\
 &= \frac{125.66}{s + 125.66}
 \end{aligned}$$

The sampling frequency is given as  $f_s = 60 \text{ Hz}$

$$\begin{aligned} T &= \frac{1}{f_s} \\ &= \frac{1}{60} \\ &= 0.0167 \text{ s.} \end{aligned}$$

Using bilinear transformation we get,

$$\begin{aligned} H(z) &= H(s) \Big|_{s=\frac{2}{T} \frac{(1-z^{-1})}{1+z^{-1}}} \\ H(z) &= \frac{125.66}{s + 125.66} \Big|_s = \frac{2}{0.0167} \frac{1 - z^{-1}}{1 + z^{-1}} \\ &= \frac{125.66}{120 \frac{(1-z^{-1})}{(1+z^{-1})} + 125.66} \\ &= \frac{125.66(1 + z^{-1})}{120(1 - z^{-1}) + 125.66(1 + z^{-1})} \\ &= \frac{125.66(1 + z^{-1})}{120 - 120z^{-1} + 125.66 + 125.66z^{-1}} \\ &= \frac{125.66(1 + z^{-1})}{245.66 + 5.66z^{-1}} \\ &= \frac{125.66(1 + z^{-1})}{245.66(1 + 0.023z^{-1})} \end{aligned}$$

$$H(z) = \frac{0.512(1 + z^{-1})}{1 + 0.023z^{-1}}$$

### 3.11 IIR Filter Design by Approximation of Derivatives

One way of converting an analog filter into a digital filter is to approximate the differential equation governing the analog filter by an equivalent difference equation. Thus, the first-order differential equation can be approximated and written as follows.

$$\begin{aligned} \frac{dy(t)}{dt} \Big|_{t=nT} &= \frac{y(nT) - y(nT - T)}{T} \\ &= \frac{y(n) - y(n - 1)}{T} \end{aligned}$$

where  $T$  is the sampling interval. The above equation can be written with analog system function as  $H(s) = s$  and digital system function  $H(z) = \frac{(1-z^{-1})}{T}$ . The frequency domain equivalence is therefore written as

$$\boxed{s = \frac{(1-z^{-1})}{T}} \quad (3.74)$$

Thus, the system function for the digital IIR filter is written as

$$\boxed{H(z) = H_a(s) \Big|_{s=\left(\frac{1-z^{-1}}{T}\right)}} \quad (3.75)$$

The stable analog filter when converted into digital filter should also be stable. The mapping of the LHP in the  $s$ -plane to  $z$ -plane has to be investigated. Equation (3.74) can be written as

$$z = \frac{1}{(1-sT)}$$

Substituting  $s = j\Omega$  in the above equation we have

$$\begin{aligned} z &= \frac{1}{1+j\Omega T} \\ &= \frac{1+j\Omega T}{(1+j\Omega T)(1+j\Omega T)} \\ &= \frac{1}{(1+\Omega^2 T^2)} + \frac{j\Omega T}{(1+\Omega^2 T^2)} \end{aligned}$$

Let

$$\begin{aligned} z &= x + jy \\ x &= \frac{1}{(1+\Omega^2 T^2)} \quad \text{and} \quad y = \frac{\Omega T}{(1+\Omega^2 T^2)} \end{aligned}$$

Consider the following equation

$$\begin{aligned} \left(x - \frac{1}{2}\right)^2 + y^2 &= \left[\frac{1}{(1+\Omega^2 T^2)} - \frac{1}{2}\right]^2 + \frac{\Omega^2 T^2}{(1+\Omega^2 T^2)} \\ &= \left(\frac{1}{2}\right)^2. \end{aligned}$$

This is the equation of a circle in the  $z$ -plane with center at  $z = \frac{1}{2}$  and radius  $\frac{1}{2}$ . Thus the mapping takes the points in LHP of the  $s$ -plane (for stable analog filter) into corresponding points inside this circle in the  $z$ -plane as shown in Fig. 3.25.

From Fig. 3.25 it is evident that the mapping  $s = \frac{(1-z^{-1})}{T}$  takes LHP in the  $s$ -plane into points inside the circle of radius  $\frac{1}{2}$  and at center  $z = \frac{1}{2}$  in the  $z$ -plane. Thus, the location of the poles of the digital filter so designed are confined to a small frequency range which means it is suitable for the design of lowpass and bandpass filters of small resonant frequencies and not suitable for highpass digital filter design. The method is illustrated by the following numerical example.

### Example 3.27

Convert the analog bandpass filter with the system function

$$H_a(s) = \frac{1}{(s + 0.1)^2 + 5}; \quad T = 0.1 \text{ s}$$

into a digital IIR filter using the backward difference for the derivative.

#### Solution

$$H_a(s) = \frac{1}{(s + 0.1)^2 + 5}$$

Substituting

$$s = \frac{(1 + z^{-1})}{T} = (10 + 10z^{-1})$$

$$H(z) = \frac{1}{(10 + 10z^{-1})^2 + 5}$$

$$H(z) = \frac{9.34 \times 10^{-3}}{1 - 1.8877z^{-1} + 0.934z^{-2}}$$

The poles are at

$$z = 0.9438 \pm j0.209$$

$$= 0.9667 \angle \pm 12.5^\circ$$

The poles are within the circle at center  $z = \frac{1}{2}$  and radius  $\frac{1}{2}$ . Hence, the digital filter so designed is stable.

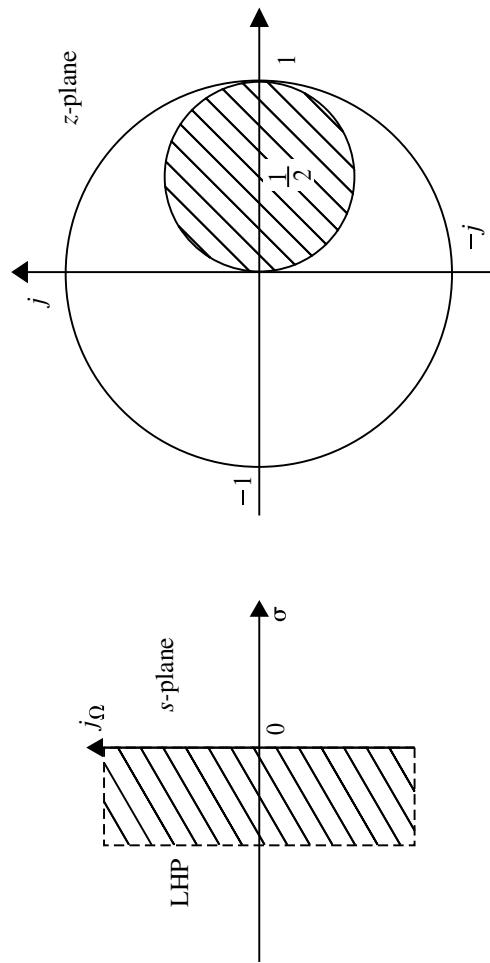


Fig. 3.25 Mapping from *s*-plane to *z*-plane by approximation of derivatives

### 3.12 Frequency Response from Transfer Function $H(z)$

The frequency response of a discrete time system can be determined from the transfer function  $H(z)$ . Several methods are available to achieve this. The geometric method of evaluation of frequency response is described here.

Consider the transfer function  $H(z)$  expressed in terms of poles and zeros are given below:

$$H(z) = \frac{K(z - z_1)(z - z_2) \cdots (z - z_n)}{(z - p_1)(z - p_2) \cdots (z - p_n)} = \prod_{i=1}^N \frac{k(z - z_i)}{(z - p_i)}$$

The frequency response is obtained by substituting  $z = e^{j\omega T}$  where  $0 \leq \omega \leq \omega_s/2$ ;  $\omega_s$  is sampling frequency; and  $T$  is sampling period.

The frequency response of  $H(z)$  written as  $H(e^{j\omega T})$  is given below:

$$H(e^{j\omega T}) = \frac{\prod_{i=1}^N k(e^{j\omega T} - z_i)}{\prod_{i=1}^N (e^{j\omega T} - p_i)}$$

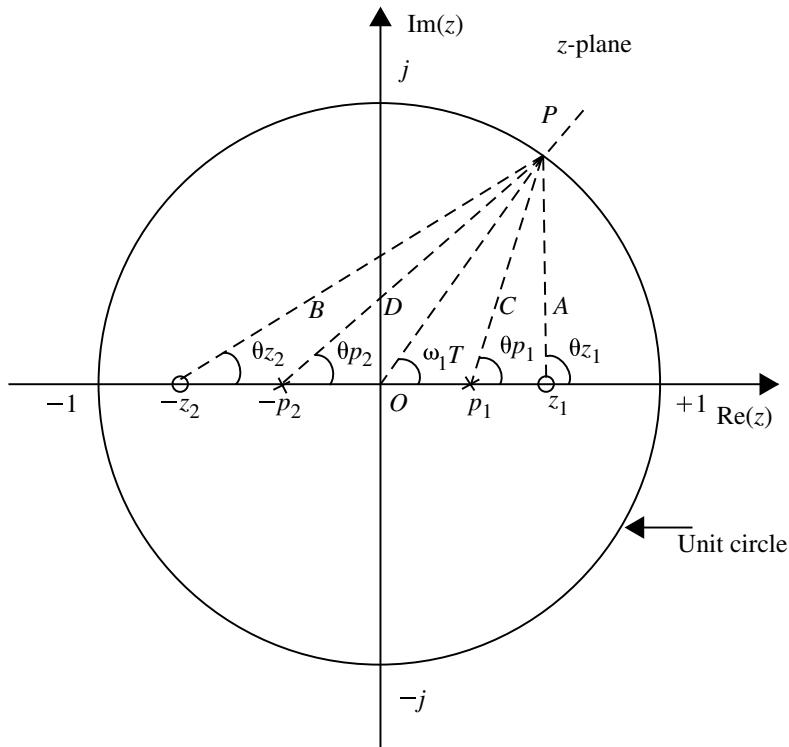
For simplicity consider a system function  $H(z)$  having two poles and two zeros. The pole-zero diagram is shown in Fig. 3.26. The zeros are located at  $z_1$  and  $-z_2$ , and the poles are located at  $p_1$  and  $p_2$ . Consider any point  $P$  on the unit circle. Draw a radial line from the origin to the point  $P$ . The line  $OP$  makes angle  $\theta$  from the reference axis.  $\theta$  and  $\omega T$  are related as follows.

$$\theta = \omega_1 T$$

Knowing  $\theta$  and the sampling period  $T$ , the frequency  $\omega_1$  at point  $P$  can be calculated. Draw radial lines from poles and zeros of  $H(z)$  to the point  $P$ . Then

$$H(e^{j\omega T}) = K \frac{A \angle \theta_{z_1} B \angle \theta_{z_2}}{C \angle \theta_{p_1} D \angle \theta_{p_2}}$$

Knowing  $K$ ,  $A$ ,  $B$ ,  $C$  and  $D$ ,  $|H(e^{j\omega T})|$  can be estimated. Similarly, knowing  $\theta_{z_1}$ ,  $\theta_{z_2}$ ,  $\theta_{p_1}$  and  $\theta_{p_2}$ ,  $\angle H(e^{j\omega T})$  can be estimated at  $\omega = \omega_1$ . By moving the point  $P$  on the unit circle, at  $\omega = \omega_2$ , the magnitude and phase of  $H(e^{j\omega T})$  can be estimated. Since the frequency response gets repeated for every  $\omega = \frac{\omega_s}{2}$ , it is necessary to estimate the magnitude and phase of  $H(e^{j\omega T})$  for the frequency interval  $0 \leq \omega \leq \frac{\omega_s}{2}$ . The following example illustrates the above procedure.



**Fig. 3.26** Pole–zero diagram of  $H(z)$

### Example 3.28

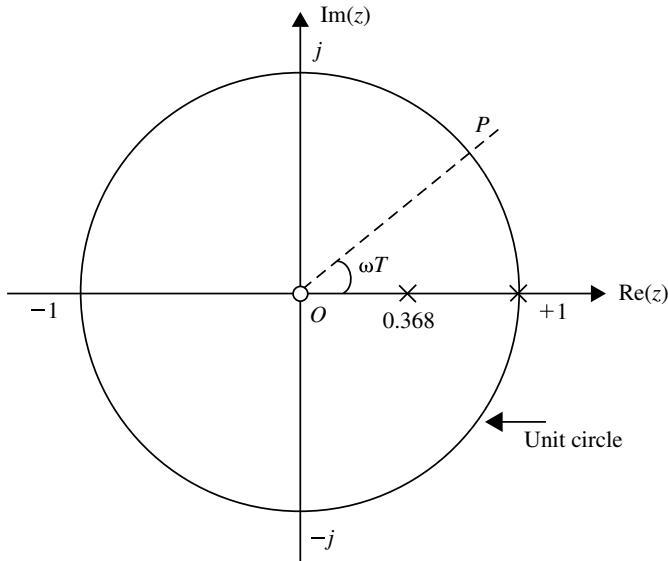
Consider the following system function of a certain discrete time system:

$$H(z) = \frac{0.632z}{(z - 1)(z - 0.368)}$$

Draw the frequency response plot. Assume  $T = 1$  s.

### Solution

The poles and zeros of  $H(z)$  are located in the  $z$ -plane as shown in Fig. 3.27, and a unit circle is drawn. Any point  $P$  is chosen on the unit circle, and radial line from the origin is drawn. From Fig. 3.27  $\omega T$  is found and knowing  $T = 1$ ,  $\omega$  is estimated  $|H(e^{j\omega T})|$  and  $\angle H(e^{j\omega T})$  are also estimated at point  $P$ . This is repeated upto  $\omega = \frac{\pi}{T} = \pi$ . The values are tabulated as shown below:



**Fig. 3.27** Pole–zero diagram of Example 3.28

$\omega$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$ H(e^{j\omega}) $	$\infty$	0.42	0.231	0.42	$\pi$
$ H(e^{j\omega}) $	$-90^\circ$	$-155^\circ$	$-180^\circ$	$-205^\circ$	$-270^\circ$

The frequency response polar plot of the discrete time system is shown in Fig. 3.28a. The magnitude and phase responses are shown in Figs. 3.28b and 3.28c, respectively.

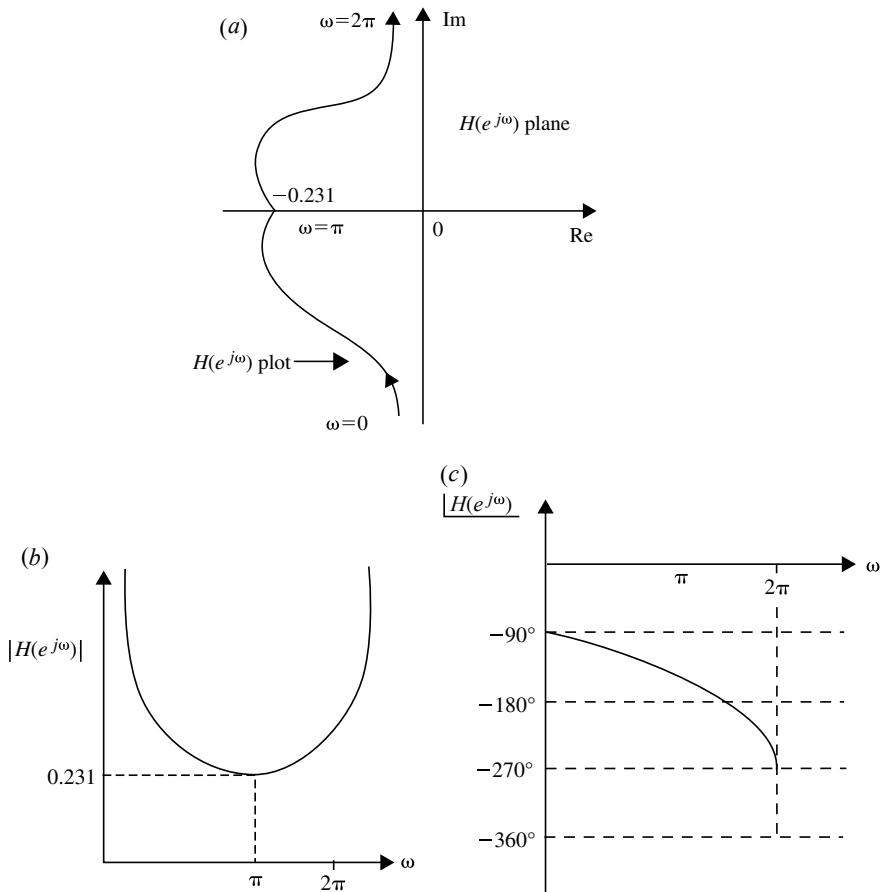
### 3.13 Structure Realization of IIR System

The IIR system can be described by the following difference equation.

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k) \quad (3.76)$$

Taking z-transform on both sides we get,

$$Y(z) = - \sum_{k=1}^N a_k z^{-k} Y(z) + \sum_{k=0}^M b_k z^{-k} X(z) \quad (3.77)$$



**Fig. 3.28** a Frequency response polar plot of discrete system. b Magnitude response. c Phase response

$$\begin{aligned}
 Y(z) + \sum_{k=1}^N a_k z^{-k} Y(z) &= \sum_{k=0}^M b_k z^{-k} X(z) \\
 Y(z) \left[ 1 + \sum_{k=1}^N a_k z^{-k} \right] &= X(z) \sum_{k=0}^M b_k z^{-k} \\
 \frac{Y(z)}{X(z)} &= H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}
 \end{aligned} \tag{3.78}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \tag{3.79}$$

where  $H(z) \rightarrow$  transfer function of IIR system.

The different types of structures for realizing IIR systems are:

- (i) Direct form-I structure.
- (ii) Direct form-II structure.
- (iii) Cascade form structure.
- (iv) Parallel form structure.
- (v) Transposed direct form realization.
- (vi) Lattice structure form.

### 3.13.1 Direct Form-I Structure

Consider the following difference equation of an IIR system.

$$\begin{aligned} y(n) &= -\sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k) \\ y(n) &= -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) \\ &\quad + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M) \end{aligned} \quad (3.80)$$

Taking  $z$ -transform, we get

$$\begin{aligned} Y(z) &= -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) \\ &\quad + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_M z^{-M} X(z) \end{aligned} \quad (3.81)$$

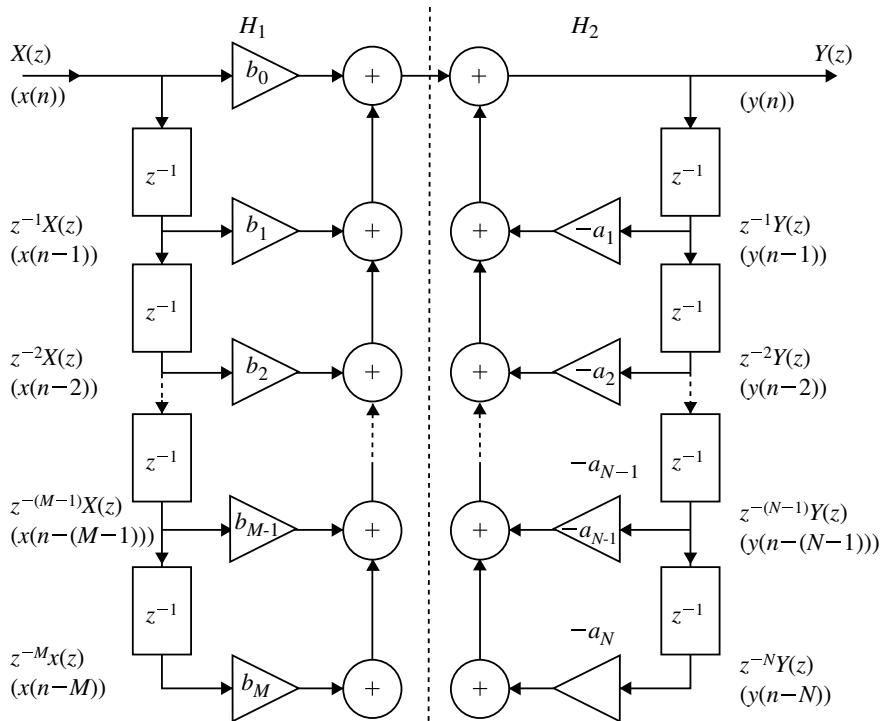
Equation (3.81) is represented in Fig. 3.29.

This structure uses different delays ( $z^{-1}$ ) for input and output. So more memory is required for realizing.

### 3.13.2 Direct Form-II Structure

Direct form-II structure uses less number of delays compared to Direct form-I. Consider the IIR difference Eq. (3.80)

$$\begin{aligned} y(n) &= -\sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k) \\ y(n) &= -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) \\ &\quad + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M) \end{aligned}$$

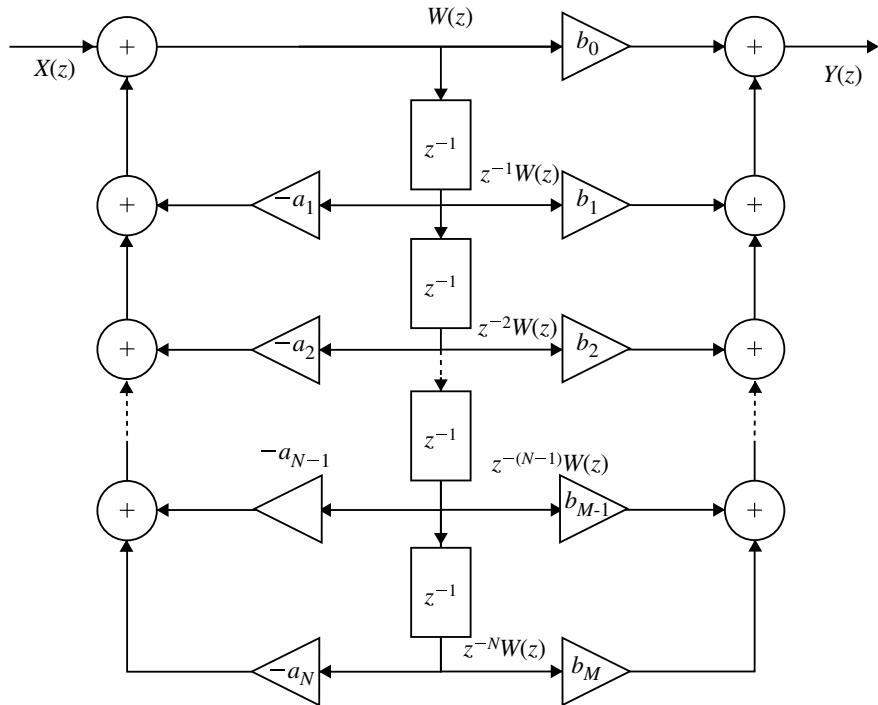


**Fig. 3.29** Direct form-I realization

Taking  $z$ -transform on both sides we get,

$$\begin{aligned} Y(z) &= -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) \\ &\quad + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_M z^{-M} X(z) \end{aligned}$$

$$\begin{aligned} Y(z) &+ a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_N z^{-N} Y(z) \\ &= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_M z^{-M} X(z) \\ Y(z)[1 &+ a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}] \\ &= X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}] \\ \frac{Y(z)}{X(z)} &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \end{aligned}$$



**Fig. 3.30** Direct form-II structure of IIR system for  $N = M$

Let

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \frac{Y(z)}{W(z)} \text{ where } \frac{W(z)}{X(z)} = \frac{1}{1 + a_1z^{-1} + \dots + a_Nz^{-N}} \quad (3.82)$$

$$\text{and } \frac{Y(z)}{W(z)} = b_0 + b_1z^{-1} + b_2z^{-2} + \dots + b_Mz^{-M} \quad (3.83)$$

Cross-multiplying Eq. (3.82) we get,

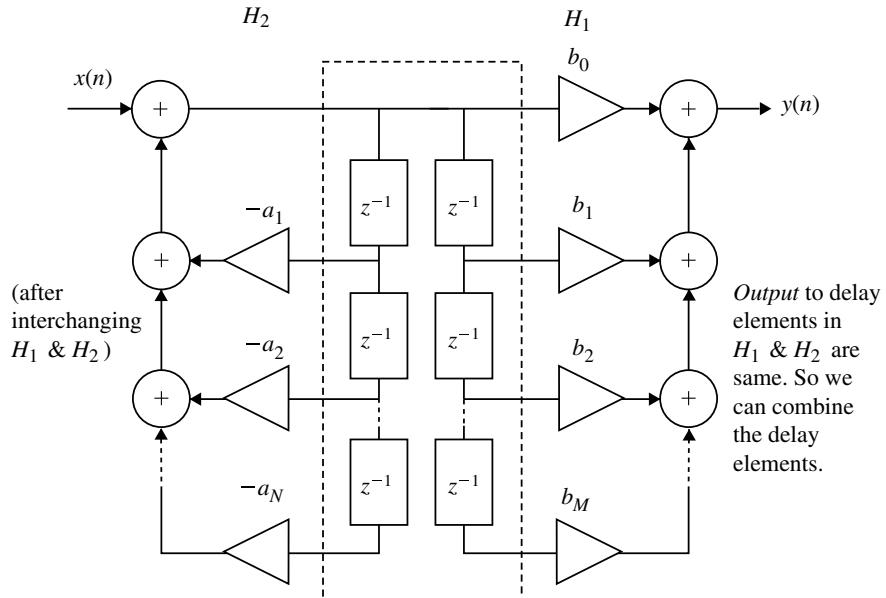
$$W(z) + a_1z^{-1}W(z) + a_2z^{-2}W(z) + \dots + a_Nz^{-N}W(z) = X(z)$$

$$\therefore W(z) = X(z) - a_1z^{-1}W(z) - a_2z^{-2}W(z) - \dots - a_Nz^{-N}W(z) \quad (3.84)$$

Cross-multiplying Eq. (3.84)

$$Y(z) = b_0W(z) + b_1z^{-1}W(z) + b_2z^{-2}W(z) + \dots + b_Mz^{-M}W(z) \quad (3.85)$$

Equations (3.84) and (3.85) represent IIR system in  $z$ -domain. It can be realized by direct form-II structure as shown in Fig. 3.30.



**Fig. 3.31** Conversion of Direct form-I to Direct form-II structure.

### 3.13.2.1 Conversion of DF-I Structure to DF-II Structure

DF-I can be converted to DF-II by interchanging the order of cascading of two systems  $H_1$  and  $H_2$ . It can be interchanged by linearity property.

Now we combine the delay elements to get a single system, and the resultant structure will be DF-II structure as shown in Fig. 3.31.

### 3.13.3 Cascade Form Realization

The transfer function  $H(z)$  can be expressed as a product of a number of second-order or first-order sections.

$$H(z) = \frac{Y(z)}{X(z)} = \prod_{i=1}^k H_i(z) \quad (3.86)$$

where

$$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1} + C_{2i}z^{-2}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}} \quad (\text{second-order section}) \quad (3.87)$$

**Fig. 3.32** Cascade realization of IIR system

or

$$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1}}{d_{0i} + d_{1i}z^{-1}} \quad (\text{first-order section}) \quad (3.88)$$

The individual second-order or first-order sections can be realized either in direct form-I or DF-II structures. The overall system is obtained by cascading the individual sections.

The cascade form is shown in Fig. 3.32. The difficulty in realizing the system in cascade form is:

- (i) Decision of pairing poles and zeros.
- (ii) Deciding the order of cascading the first- and second-order sections.
- (iii) Scaling multipliers should be provided between individual sections to prevent the filter variable from becoming too large or too small.

### 3.13.4 Parallel Form Realization

By applying partial fraction to  $H(z)$ , the system can be expressed as sum of first- and second-order sections as given below.

$$H(z) = \frac{Y(z)}{X(z)} = C + \sum_{i=1}^k H_i(z) \quad (3.89)$$

where

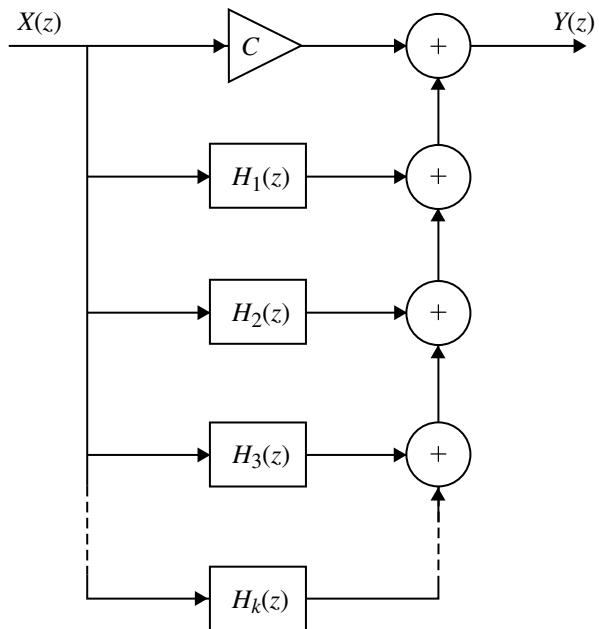
$$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1} + C_{2i}z^{-2}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}} \quad (\text{second-order section}) \quad (3.90)$$

or

$$H_i(z) = \frac{C_{0i}}{d_{0i} + d_{1i}z^{-1}} \quad (\text{first-order section}) \quad (3.91)$$

The individual first- and second-order sections can be realized either in DF-I or in DF-II structure. Then the overall system is obtained by connecting the individual sections in parallel as shown in Fig. 3.33.

**Fig. 3.33** Parallel form realization of IIR filter



### 3.13.5 Transposed Direct Form Realization

#### 3.13.5.1 Signal Flow Graph

It is a graphical representation of the relationship between the variables of a set of linear difference equations. The basic elements of signal flow graph are branches and nodes.

**Nodes:** Represent system variables, which is equal to the sum of incoming signals from all branches connecting to the node. Nodes having two types:

Source Nodes → have only outgoing branches.

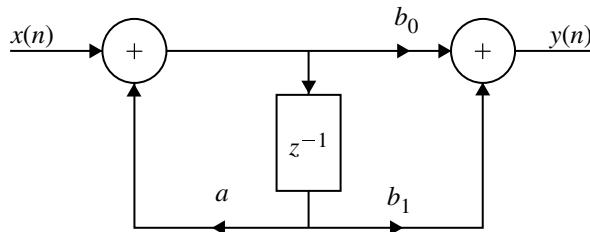
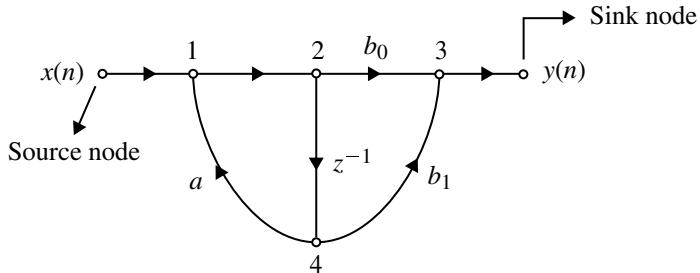
Sink Nodes → have only incoming branches.

Arrow head shows the direction of branch. The branch gain is indicated next to arrow head. The delay is indicated by the branch transmittance  $z^{-1}$ .

When branch transmittance is unity it is unlabelled. Consider the block diagram of first-order digital filter as shown in Fig. 3.34.

The signal flow graph is shown in Fig. 3.35.

Here we have 4 nodes out of this 2 nodes are running nodes, while other 2 are branching points. The delay is indicated by branch transmittance  $z^{-1}$ . Branch transmittance is indicated next to arrow head. The signal flow graph of first-order filter is shown in Fig. 3.35.

**Fig. 3.34** First-order system**Fig. 3.35** Signal flow graph of first-order system

### 3.13.6 Transposition Theorem and Transposed Structure

The transpose of a structure is defined by the following operations:

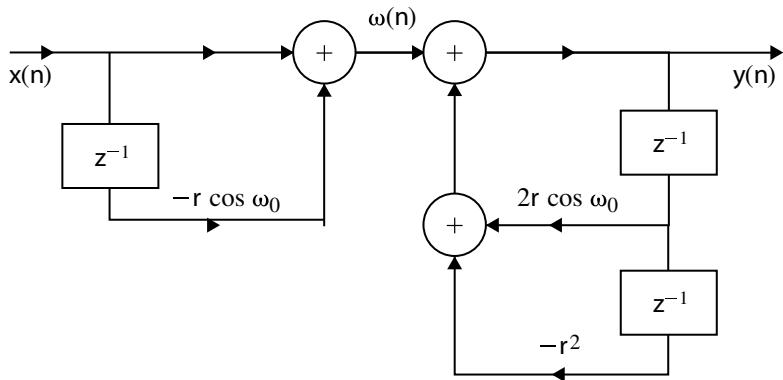
- (i) Reverse the direction of all branches in the signal flow graph.
- (ii) Interchange the inputs and outputs.
- (iii) Reverse the roles of all nodes in the flow graph.
- (vi) Summing points become branching points.
- (v) Branching points become running points.

According to transportation theorem, the system transfer function remains unchanged by transposition.

#### **Example 3.29**

Realize the second-order digital filter

$$y(n) = 2n \cos \omega_0 y(n-1) - r^2 y(n-2) + x(n) - r \cos \omega_0 x(n-1).$$



**Fig. 3.36** Direct form-I realization for Example 3.29

**Solution** Let

$$x(n) - r \cos(\omega_0)x(n-1) = \omega(n)$$

$$y(n) = 2r \cos \omega_0 y(n-1) - r^2 y(n-2) + \omega(n)$$

The realization is shown in Fig. 3.36.

### Example 3.30

Obtain the DF-I realization for the system described by difference equation

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$$

**Solution** Let

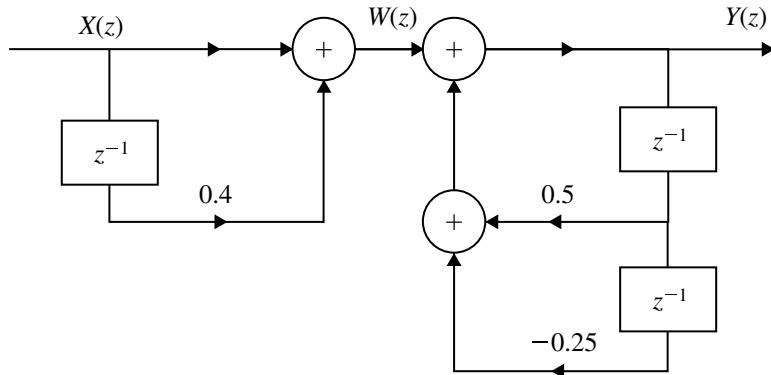
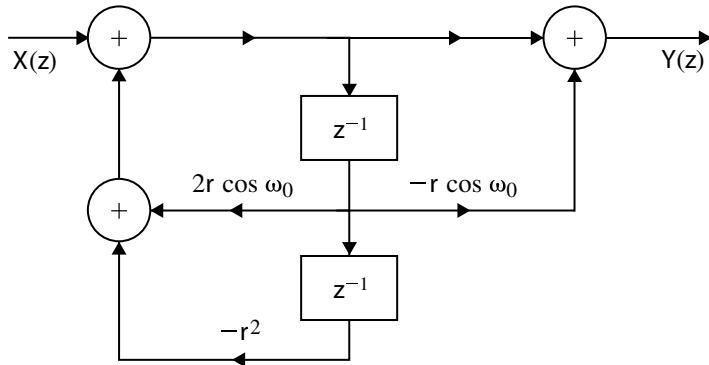
$$x(n) + 0.4x(n-1) = \omega(n)$$

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + \omega(n)$$

The direct form-I is shown in Fig. 3.37.

### Example 3.31

Realize the second-order system  $y(n) = 2r \cos \omega_0 y(n-1) - r^2 y(n-2) + x(n) - r \cos(\omega_0)x(n-1)$  in DF-II.

**Fig. 3.37** Direct form-I realization for Example 3.30**Fig. 3.38** Direct form-II realization for Example 3.31**Solution**

$$y(n) = 2r \cos \omega_0 y(n-1) - r^2 y(n-2) + x(n) - r \cos(\omega_0) x(n-1)$$

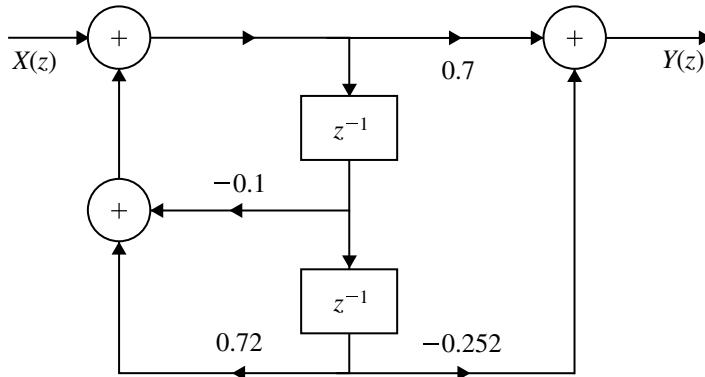
$$Y(z)(1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}) = X(z)(1 - r \cos \omega_0 z^{-1})$$

$$\frac{Y(z)}{X(z)} = \frac{1 - r \cos \omega_0 z^{-1}}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}}$$

The direct form-II realization is shown in Fig. 3.38.

**Example 3.32**

Determine the DF-II realization for the following system  $y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$ .



**Fig. 3.39** Direct form-II realization for Example 3.32

### Solution

$$Y(z)(1 + 0.1z^{-1} - 0.72z^{-2}) = X(z)(0.7 - 0.252z^{-2})$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

Direct form-II realization is shown in Fig. 3.39.

### Example 3.33

Realize the system with difference equation

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + \frac{1}{3}x(n-1) + x(n)$$

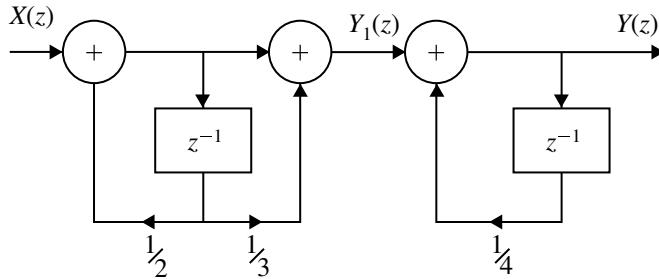
in cascade form.

### Solution

$$Y(z) \left( 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} \right) = X(z) \left( 1 + \frac{1}{3}z^{-1} \right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{1 + \frac{1}{3}z^{-1}}{\left( 1 - \frac{1}{2}z^{-1} \right) \left( 1 - \frac{1}{4}z^{-1} \right)} = H_1(z) \cdot H_2(z)$$



**Fig. 3.40** Cascade form realization for Example 3.33

$$H_1(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-2}}, \quad H_2(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

The cascade form realization is shown in Fig. 3.40.

### Example 3.34

Obtain the DF-I, DF-II, cascade and parallel form realization for the system.

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$$

### Solution

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$$

$$Y(z)[1 + 0.1z^{-1} - 0.2z^{-2}] = X(z)(3 + 3.6z^{-1} + 0.6z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} = \left( \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} \right)$$

### Cascade Form

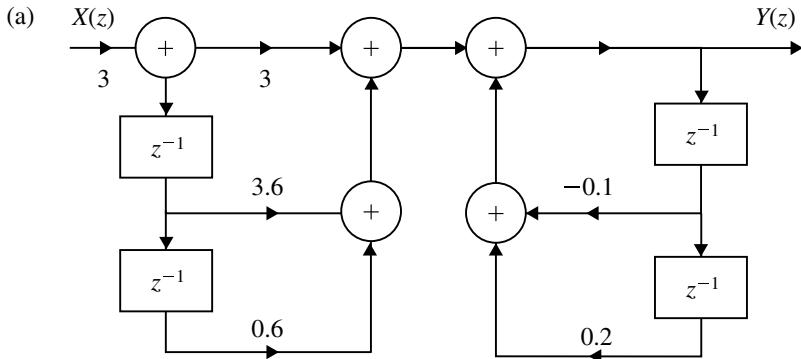
$$\begin{aligned} \frac{Y(z)}{X(z)} &= \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} \\ &= \frac{(3 + 0.6z^{-1})(1 + z^{-1})}{(1 + 0.5z^{-1})(1 - 0.4z^{-1})} \end{aligned}$$

Let

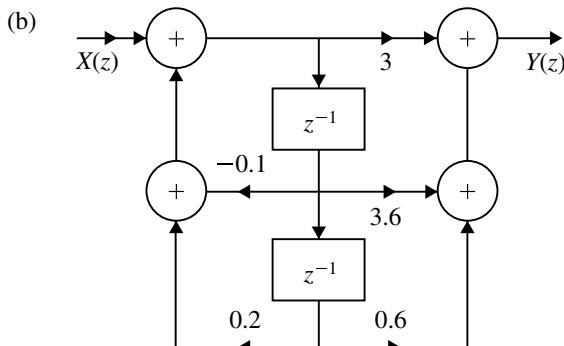
$$H_1(z) = \frac{3 + 0.6z^{-1}}{1 + 0.5z^{-1}}, \quad H_2(z) = \frac{1 + z^{-1}}{1 - 0.4z^{-1}}$$

Direct form-I and direct form-II are shown in Fig. 3.41 a, b respectively.

### DF-I



### DF-II



**Fig. 3.41 a, b** Direct form-I and direct form-II realization for Example 3.34

### Parallel Form

$$\begin{aligned} H(z) &= \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}} \\ &= -3 + \frac{3.9z^{-1} + 6}{1 + 0.1z^{-1} - 0.2z^{-2}} \end{aligned}$$

Consider

$$\begin{aligned} \frac{3.9z^{-1} + 6}{1 + 0.1z^{-1} - 0.2z^{-2}} &= \frac{3.9z^{-1} + 6}{(1 - 0.4z^{-1})(1 + 0.5z^{-1})} \\ &= \frac{A}{1 - 0.4z^{-1}} + \frac{B}{1 + 0.5z^{-1}} \end{aligned}$$

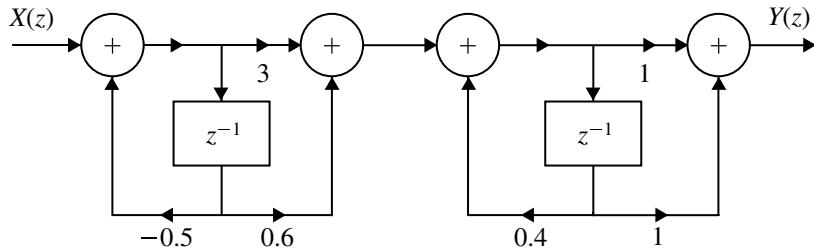
$$\Rightarrow A = (1 - 0.4z^{-1}) \cdot \frac{3.9z^{-1} + 6}{(1 - 0.4z^{-1})(1 + 0.5z^{-1})} \Big|_{z^{-1}=\frac{1}{0.4}} = 7$$

$$\Rightarrow B = (1 - 0.5z^{-1}) \cdot \frac{3.9z^{-1} + 6}{(1 - 0.4z^{-1})(1 + 0.5z^{-1})} \Big|_{z^{-1}=\frac{-1}{0.5}} = -1$$

$$\therefore H(z) = -3 + \frac{7}{1 - 0.4z^{-1}} - \frac{1}{1 + 0.5z^{-1}}$$

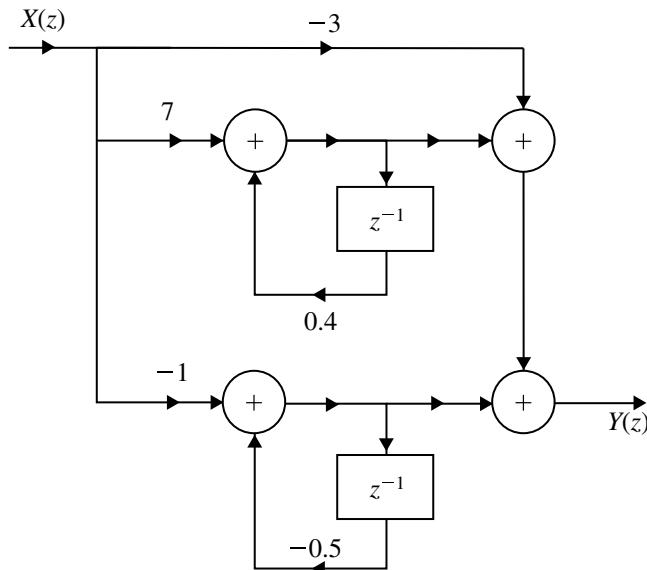
$$= C + H_1(z) + H_2(z)$$

Cascade form realization of Example 3.34 is shown in Fig. 3.42.



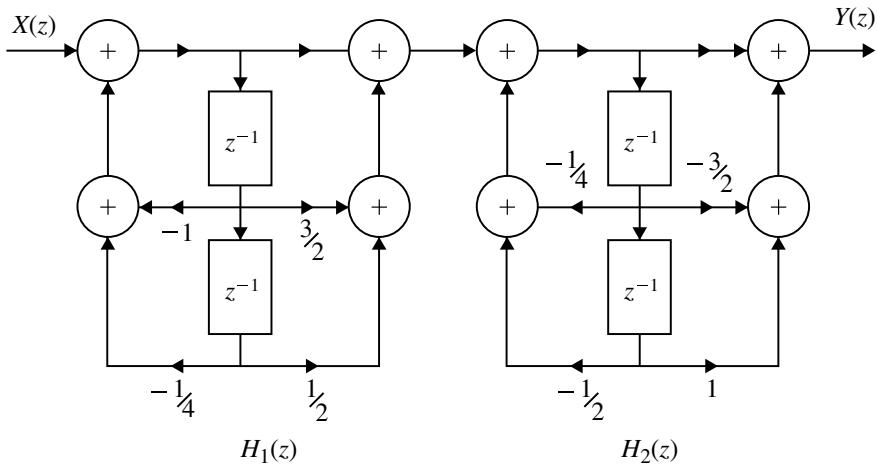
**Fig. 3.42** Cascade form realization for Example 3.34

Parallel form realization of Example 3.34 is shown in Fig. 3.43.



**Fig. 3.43** Parallel form realization for Example 3.34

Cascade form realization for Example 3.35 is shown in Fig. 3.44.



**Fig. 3.44** Cascade form realization for Example 3.35

### Example 3.35

Obtain the cascade realization for the following system.

$$H(z) = \frac{(1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2})(1 - \frac{3}{2}z^{-1} + z^{-2})}{(1 + z^{-1} + \frac{1}{4}z^{-2})(1 + \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2})}$$

**Solution** Let  $H(z) = H_1(z)H_2(z)$

$$H_1(z) = \frac{1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}{1 + z^{-1} + \frac{1}{4}z^{-2}}, \quad H_2(z) = \frac{1 - \frac{3}{2}z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}},$$

### Example 3.36

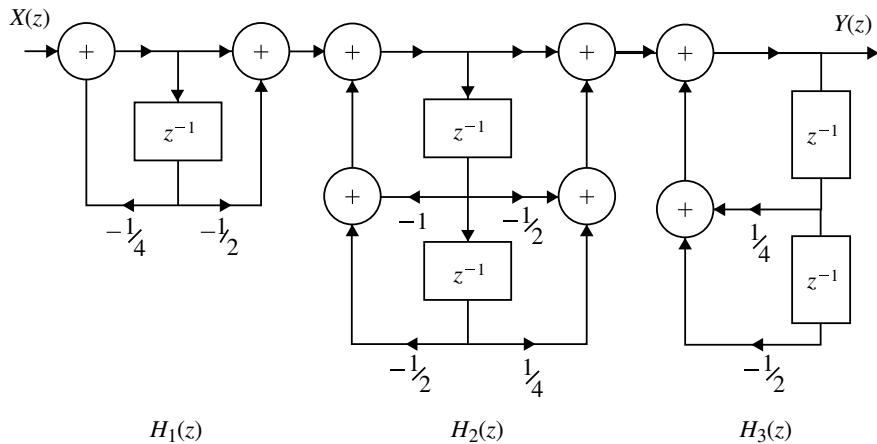
Obtain the cascade realization of the system

$$H(z) = \frac{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2})}{(1 + \frac{1}{4}z^{-1})(1 + z^{-1} + \frac{1}{2}z^{-2})(1 - \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2})}$$

**Solution** Let

$$H(z) = H_1(z)H_2(z)H_3(z)$$

Cascade form realization for Example 3.36 is shown in Fig. 3.45.



**Fig. 3.45** Cascade form realization for Example 3.36

where

$$H_1(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}; \quad H_2(z) = \frac{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}; \quad H_3(z) = \frac{1}{1 - \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}}$$

### Example 3.37

Obtain the direct form-I, direct form-II, cascade and parallel form realization of LTI system governed by the equation

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$$

(Anna University, April, 2004)

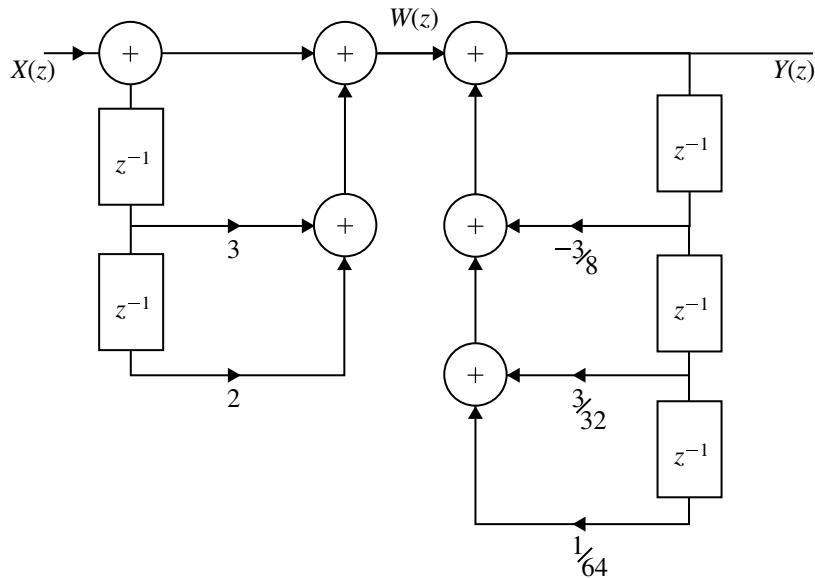
#### Solution Direct form-I

$$\begin{aligned} y(n) &= -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2) \\ Y(z) &= -\frac{3}{8}z^{-1}Y(z) + \frac{3}{32}z^{-2}Y(z) + \frac{1}{64}z^{-3}Y(z) + X(z) + 3z^{-1}X(z) + 2z^{-2}X(z) \end{aligned}$$

#### Direct form-II

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \times \frac{W(z)}{X(z)} = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}}$$

Direct form-I realization for Example 3.37 is shown in Fig. 3.46.



**Fig. 3.46** Direct form-I realization of  $H(z)$  for Example 3.37

where

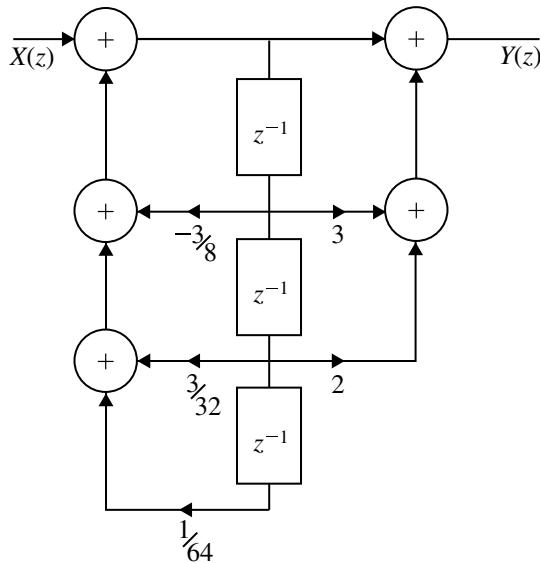
$$\begin{aligned} \frac{W(z)}{X(z)} &= \frac{1}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \\ \frac{Y(z)}{W(z)} &= 1 + 3z^{-1} + 2z^{-2} \\ X(z) &= W(z) \left( 1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3} \right) \end{aligned}$$

$$\begin{aligned} W(z) &= X(z) - \frac{3}{8}z^{-1}W(z) + \frac{3}{32}z^{-2}W(z) + \frac{1}{64}z^{-3}W(z) \\ Y(z) &= W(z) + 3z^{-1}W(z) + 2z^{-2}W(z) \end{aligned}$$

### Cascade Form Realization

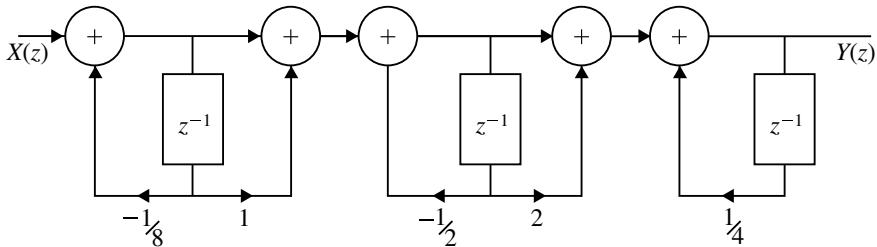
$$\begin{aligned} H(z) &= \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \\ &= \frac{(1 + z^{-1})(1 + 2z^{-1})}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \\ H(z) &= H_1(z)H_2(z)H_3(z) \end{aligned}$$

Direct form-II realization for Example 3.37 is shown in Fig. 3.47.



**Fig. 3.47** Direct form-II realization of  $H(z)$  for Example 3.37

Cascade form realization for Example 3.37 is shown in Fig. 3.48.



**Fig. 3.48** Cascade form realization of  $H(z)$  for Example 3.37

where

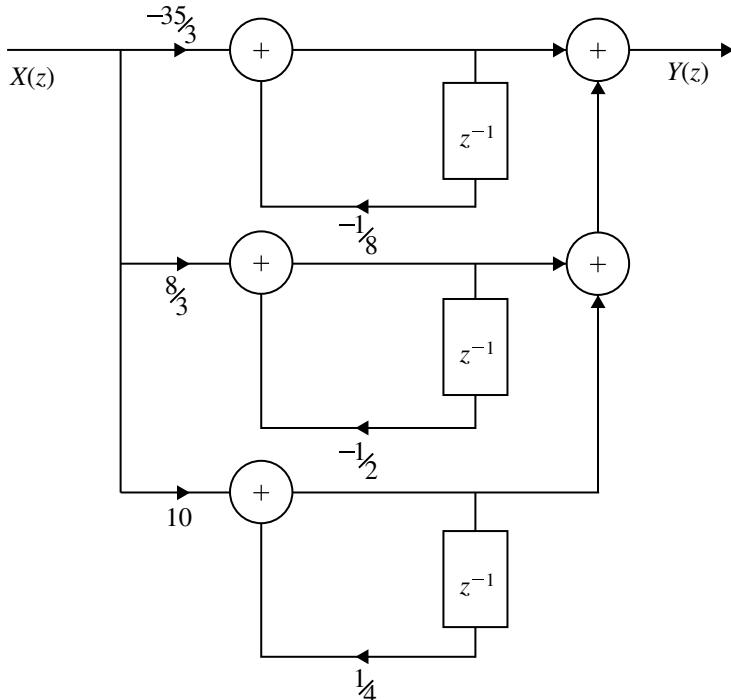
$$H_1(z) = \frac{1 + z^{-1}}{1 + \frac{1}{8}z^{-1}}; \quad H_2(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{2}z^{-1}}; \quad H_3(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

### Parallel Form

$$\begin{aligned} H(z) &= \frac{(1 + z^{-1})(1 + 2z^{-1})}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \\ &= \frac{A}{1 + \frac{1}{8}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}} + \frac{C}{1 - \frac{1}{4}z^{-1}} \end{aligned}$$

$$\begin{aligned}
 A &= \frac{(1+z^{-1})(1+2z^{-1})}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \Big|_{z^{-1}=-8} = -\frac{35}{3} \\
 B &= \frac{(1+z^{-1})(1+2z^{-1})}{(1+\frac{1}{8}z^{-1})(1-\frac{1}{4}z^{-1})} \Big|_{z^{-1}=-2} = \frac{8}{3} \\
 C &= \frac{(1+z^{-1})(1+2z^{-1})}{(1+\frac{1}{8}z^{-1})(1+\frac{1}{2}z^{-1})} \Big|_{z^{-1}=4} = 10 \\
 H(z) &= -\frac{35}{3} \left( \frac{1}{1+\frac{1}{8}z^{-1}} \right) + \frac{8}{3} \left( \frac{1}{1+\frac{1}{2}z^{-1}} \right) + 10 \left( \frac{1}{1-\frac{1}{4}z^{-1}} \right) \\
 H(z) &= H_1(z) + H_2(z) + H_3(z)
 \end{aligned}$$

Parallel form realization for Example 3.37 is shown in Fig. 3.49.



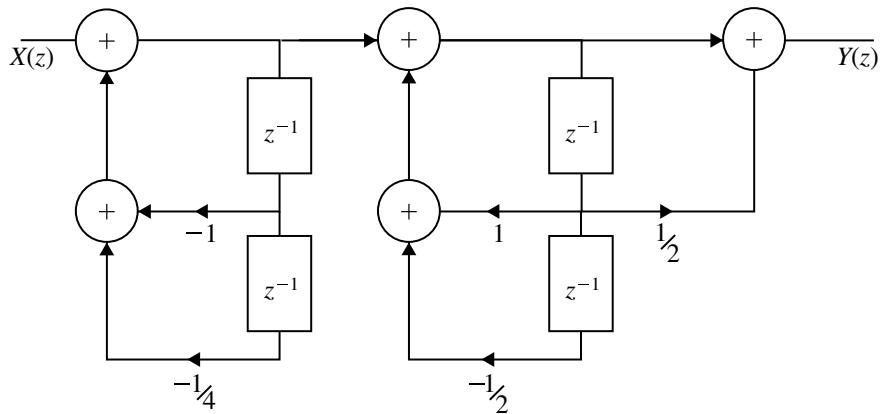
**Fig. 3.49** Parallel form realization of  $H(z)$  for Example 3.37

### Example 3.38

Realize the given system in cascade and parallel form

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{(1 - z^{-1} + \frac{1}{4}z^{-2})(1 - z^{-1} + \frac{1}{2}z^{-2})}$$

Cascade form realization for Example 3.38 is shown in Fig. 3.50.



**Fig. 3.50** Cascade form realization of  $H(z)$  for Example 3.38

### Solution Cascade Form Realization

$$\begin{aligned} H(z) &= \frac{1 + \frac{1}{2}z^{-1}}{(1 - z^{-1} + \frac{1}{4}z^{-2})(1 - z^{-1} + \frac{1}{2}z^{-2})} \\ &= \frac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}} \cdot \frac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \\ H(z) &= H_1(z)H_2(z) \end{aligned}$$

where

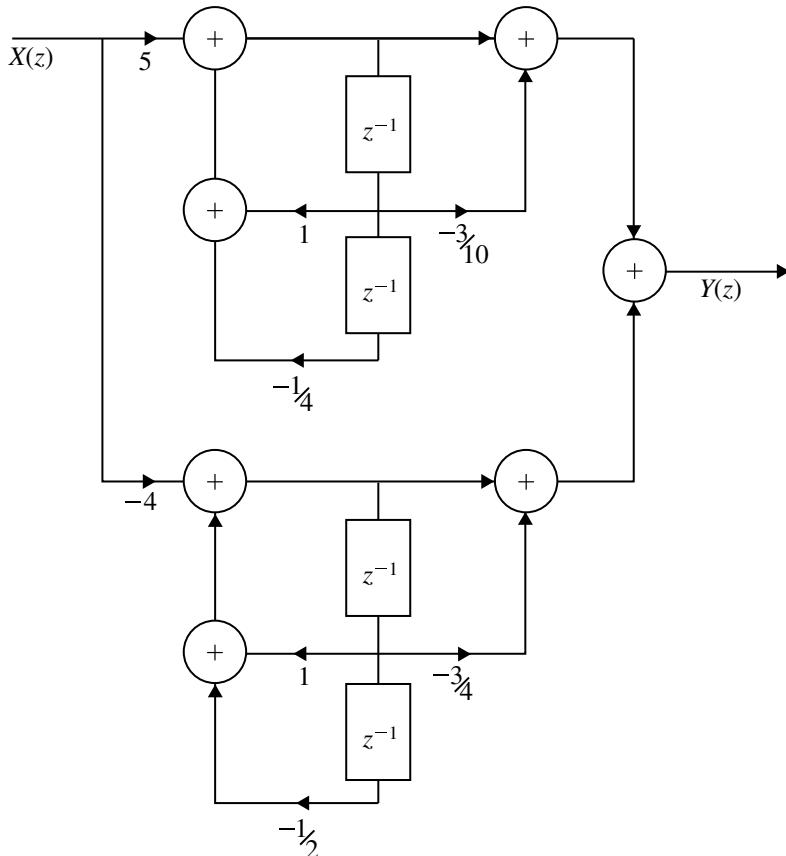
$$H_1(z) = \frac{1}{1 + z^{-1} + \frac{1}{4}z^{-2}}, \quad H_2(z) = \frac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

### Parallel Form Realization

$$\begin{aligned} H(z) &= \frac{1 + \frac{1}{2}z^{-1}}{(1 - z^{-1} + \frac{1}{4}z^{-2})(1 - z^{-1} + \frac{1}{2}z^{-2})} \\ &= \frac{z(z^3 + \frac{1}{2}z^2)}{(z^2 - z + \frac{1}{4})(z^2 - z + \frac{1}{2})} \\ \frac{H(z)}{z} &= \frac{Az + B}{z^2 - z + \frac{1}{4}} + \frac{Cz + D}{z^2 - z + \frac{1}{2}} \\ &= \frac{5z - \frac{3}{2}}{z^2 - z + \frac{1}{4}} + \frac{-4z + 3}{z^2 - z + \frac{1}{2}} \end{aligned}$$

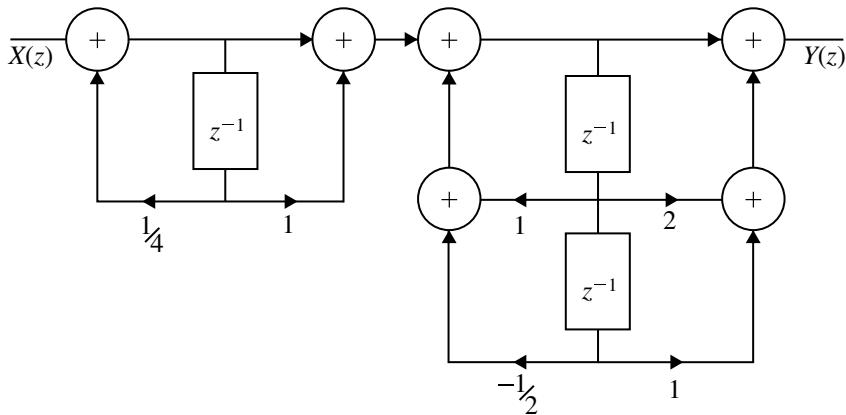
$$\begin{aligned}
 H(z) &= \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} \\
 &= H_1(z)H_2(z) \\
 H(z) &= \frac{5 - \frac{3}{2}z^{-1}}{1 - z^2 + \frac{1}{4}z^{-2}} + \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \\
 &= \frac{5(1 - \frac{3}{10}z^{-1})}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{-4(1 - \frac{3}{4}z^{-1})}{1 - z^{-1} + \frac{1}{2}z^{-2}}
 \end{aligned}$$

Parallel form realization for Example 3.38 is shown in Fig. 3.51.



**Fig. 3.51** Parallel form realization of  $H(z)$  for Example 3.38

Cascade form realization for Example 3.39 is shown in Fig. 3.52.



**Fig. 3.52** Cascade form realization of  $H(z)$  for Example 3.39

### Example 3.39

The transfer function of a system is given by

$$H(z) = \frac{(1+z^{-1})^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$$

Realize the system in cascade and parallel structures.

#### Solution Cascade Form Realization

$$\begin{aligned} H(z) &= \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \cdot \frac{(1+z^{-1})^2}{1-z^{-1}+\frac{1}{2}z^{-2}} \\ &= \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \cdot \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}} \\ H(z) &= H_1(z)H_2(z) \end{aligned}$$

where

$$H_1(z) = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

#### Parallel Form

$$H(z) = \frac{(1+z^{-1})^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$$

$$\begin{aligned}
H(z) &= \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}} \\
&= \frac{z^3 + 3z^2 + 3z + 1}{z^3 - \frac{5}{4}z^2 + \frac{3}{4}z - \frac{1}{8}} \\
&= 1 + \frac{\frac{17}{4}z^2 + \frac{9}{4}z + \frac{9}{8}}{z^3 - \frac{5}{4}z^2 + \frac{3}{4}z - \frac{1}{8}} \\
&= 1 + \frac{\frac{17}{4}z^2 + \frac{9}{4}z + \frac{9}{8}}{(z - \frac{1}{4})(z^2 - z + \frac{1}{2})} \\
\frac{\frac{17}{4}z^2 + \frac{9}{4}z + \frac{9}{8}}{(z - \frac{1}{4})(z^2 - z + \frac{1}{2})} &= \frac{A}{z - \frac{1}{4}} + \frac{Bz + C}{z^2 - z + \frac{1}{2}} \\
&= \frac{\frac{25}{4}}{z - \frac{1}{4}} + \frac{-2z + 8}{z^2 - z + \frac{1}{2}} \\
H(z) &= 1 + \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} + \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}
\end{aligned}$$

### Example 3.40

Determine the direct form-II and transposed direct form-II for the given system

$$y(n) = \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) + x(n) + x(n-1)$$

**Solution** Given

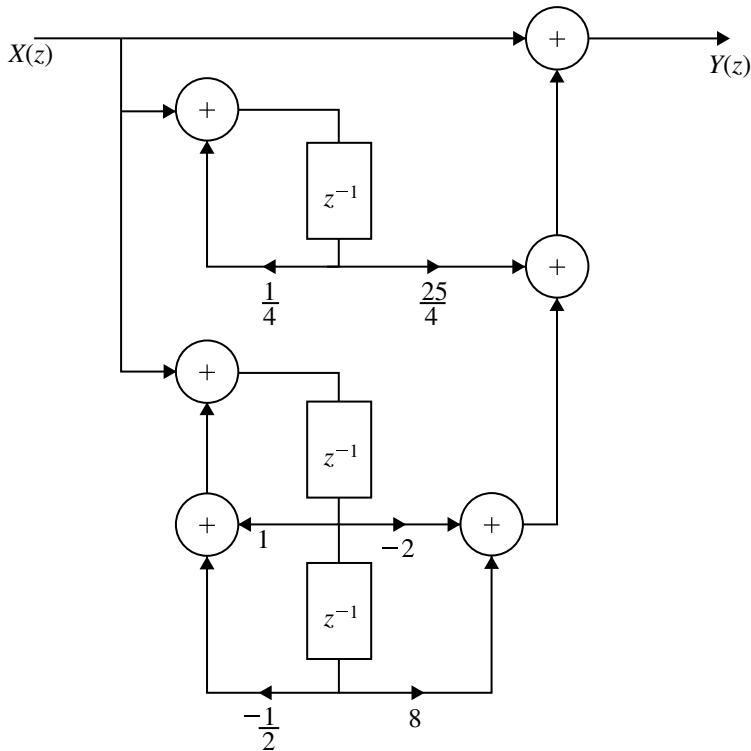
$$\begin{aligned}
Y(z) &= \frac{1}{2}z^{-1}Y(z) - \frac{1}{4}z^{-2}Y(z) + X(z) + z^{-1}X(z) \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}
\end{aligned}$$

### Direct Form-II Realization

To get transposed direct form-II do the following operation:

1. Change the direction of all branches.
2. Interchange the input and output.
3. Change the summing point to branching point and *vice versa*.

Parallel form realization for Example 3.39 is shown in Fig. 3.53.



**Fig. 3.53** Parallel form realization of  $H(z)$  for Example 3.39

### Example 3.41

Realize the system given by difference equation

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$$

in parallel form.

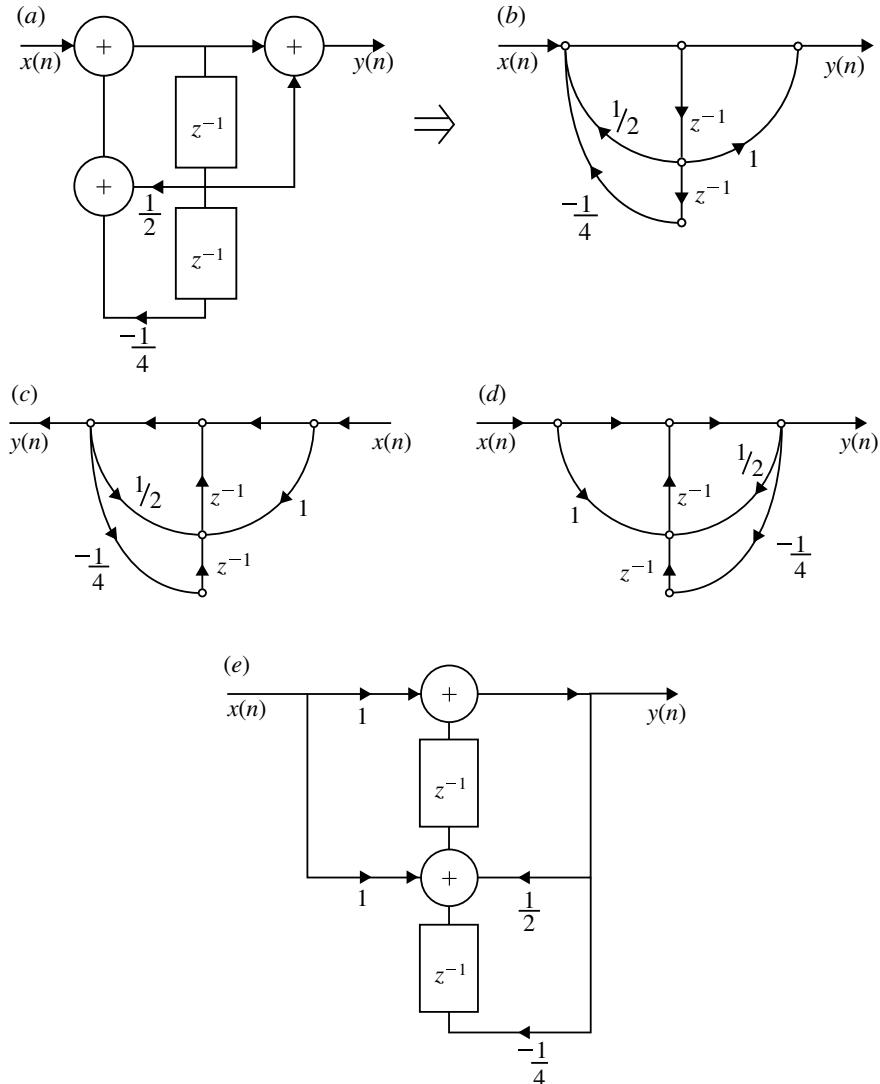
(Anna University, May, 2007)

### Solution

$$\begin{aligned} H(z) &= \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}} \\ &= 0.35 + \frac{0.35 - 0.035z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}} \\ &= 0.35 + \frac{0.35 - 0.035z^{-1}}{(1 + 0.9z^{-1})(1 - 0.8z^{-1})} \end{aligned}$$

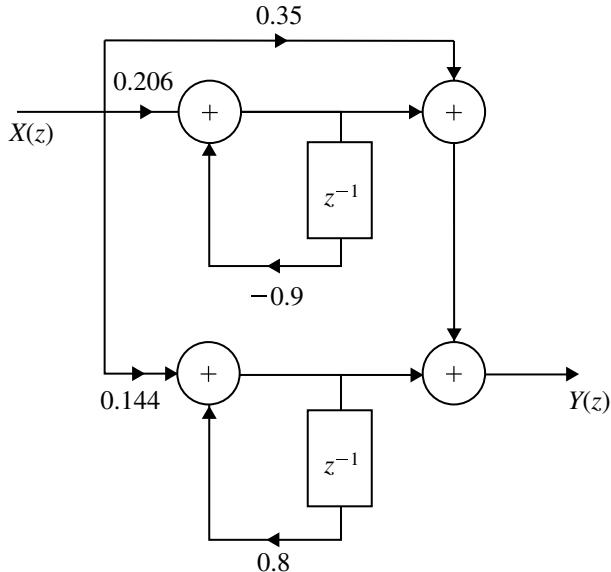
$$\begin{aligned}
 &= 0.35 + \frac{A}{(1 + 0.9z^{-1})} + \frac{B}{(1 - 0.8z^{-1})} \\
 &= 0.35 + \frac{0.206}{(1 + 0.9z^{-1})} + \frac{0.1444}{(1 - 0.8z^{-1})} \\
 &= C + H_1(z) + H_2(z)
 \end{aligned}$$

Transposed direct form-II for Example 3.40 is shown in Fig. 3.54.



**Fig. 3.54** Transposed direct form-II realization for Example 3.40

Parallel realization for Example 3.41 is shown in Fig. 3.55.



**Fig. 3.55** Parallel form realization of  $H(z)$  for Example 3.41

### 3.13.7 Lattice Structure of IIR System

Let us consider an all-pole system with system function

$$H(z) = \frac{1}{1 + \sum_{k=1}^N a_N z^{-k}} = \frac{1}{A_N(z)} \quad (3.92)$$

The difference equation for this IIR system is derived as follows.

$$\frac{Y(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^N a_N z^{-k}}$$

$$\begin{aligned}
 Y(z) + \sum_{k=1}^N a_N z^{-k} Y(z) &= X(z) \\
 Y(z) &= - \sum_{k=1}^N a_N z^{-k} Y(z) + X(z) \\
 y(n) &= - \sum_{k=1}^N a_N(k) y(n-k) + x(n)
 \end{aligned} \quad (3.93)$$

or

$$x(n) = y(n) + \sum_{k=1}^N a_N(k)y(n-k) \quad (3.94)$$

For  $N = 1$

$$x(n) = y(n) + a_1(1)y(n-1) \quad (3.95)$$

Equation (3.95) can be realized in lattice structure as shown in Fig. 3.56.

From Fig. 3.56, we can obtain

$$x(n) = f_1(n) \quad (3.96)$$

$$\begin{aligned} y(n) &= f_0(n) = f_1(n) - k_1 g_0(n-1) \\ &= x(n) - k_1 y(n-1) \end{aligned} \quad (3.97)$$

$$\begin{aligned} g_1(n) &= k_1 f_0(n) + g_0(n-1) \\ &= k_1 y(n) + y(n-1) \end{aligned} \quad (3.98)$$

$$k_1 = a_1(1) \quad (3.99)$$

Now, let us consider for the case  $N = 2$ , then

$$x(n) = f_2(n) \quad (3.100)$$

$$y(n) = x(n) - a_1(1)y(n-1) - a_2(2)y(n-2) \quad (3.101)$$

This output can also be obtained from a two-stage lattice structure as shown in Fig. 3.57 from which we have

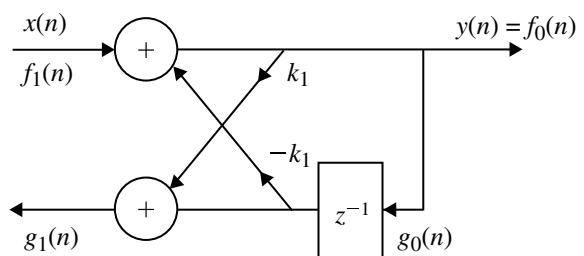
$$f_2(n) = x(n) \quad (3.102)$$

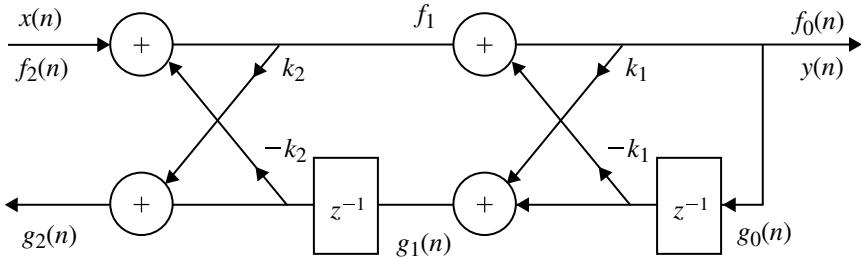
$$f_1(n) = f_2(n) - k_2 g_1(n-1) \quad (3.103)$$

$$g_2(n) = k_2 f_1(n) + g_1(n-1) \quad (3.104)$$

$$f_0(n) = f_1(n) - k_1 g_0(n-1) \quad (3.105)$$

**Fig. 3.56** Single stage all-pole lattice structure



**Fig. 3.57** Two stage all-pole lattice filter

$$g_1(n) = k_1 f_0(n) + g_0(n-1) \quad (3.106)$$

$$y(n) = f_0(n) = g_0(n) \quad (3.107)$$

$$\begin{aligned} &= f_1(n) - k_1 g_0(n-1) \\ &= f_2(n) - k_2 g_1(n-1) - k_1 g_0(n-1) \\ &= f_2(n) - k_2 [k_1 f_0(n-1) + g_0(n-2)] - k_1 g_0(n-1) \\ &= f_2(n) - k_1(1+k_2)y(n-1) - k_2 y(n-2) \end{aligned} \quad (3.108)$$

Similarly

$$g_2(n) = k_2 y(n) + k_1(1+k_2)y(n-1) + y(n-2) \quad (3.109)$$

Comparing Eqs. (3.101) and (3.108) we get

$$a_2(0) = 1, \quad a_2(1) = k_1(1+k_2), \quad a_2(2) = k_2 \quad (3.110)$$

For a  $N$ -stage IIR filter realized in lattice structure as shown in Fig. 3.58, we get

$$f_N(n) = x(n) \quad (3.111)$$

$$f_{m-1}(n) = f_m(n) - k_m g_{m-1}(n-1), \quad m = N, N-1, \dots, 1 \quad (3.112)$$

$$g_m(n) = k_m f_{m-1}(n) + g_{m-1}(n-1), \quad m = N, N-1, \dots, 1 \quad (3.113)$$

$$y(n) = f_0(n) = g_0(n) \quad (3.114)$$

### 3.13.8 Conversion from Direct Form to Lattice Structure

For a 3-stage IIR system

$$x(n) = f_3(n) = f_2(n) + k_3 g_2(n-1) \quad (3.115)$$

$$g_3(n) = k_3 f_2(n) + g_2(n-1) \quad (3.116)$$

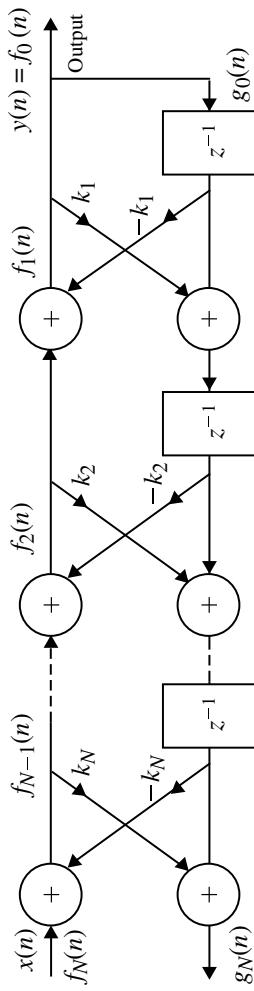


Fig. 3.58 All-pole lattice filter

from which

$$g_2(n - 1) = g_3(n) - k_3 f_2(n) \quad (3.117)$$

Substituting Eq. (3.117) in Eq. (3.115) we get,

$$f_3(n) = f_2(n) + k_3[g_3(n) - k_3 f_2(n)]$$

and

$$\begin{aligned} f_2(n) &= \frac{f_3(n) - k_3 g_3(n)}{1 - k_3^2} \\ &= \frac{\left[ \begin{array}{l} y(n) + a_3(1)y(n-1) + a_3(2)y(n-2) + a_3(3)y(n-3) \\ + k_3 a_3(3)y(n) - k_3 a_3(2)y(n-2) \\ - k_3 a_3(n)y(n-1) - a_3 y(n-3) \end{array} \right]}{1 - k_3^2} \\ &= y(n) + \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)}y(n-1) + \frac{a_3(2) - a_3(2)a_3(3)}{1 - a_3^2(3)}y(n-2) + \dots \end{aligned} \quad (3.118)$$

Comparing Eqs. (3.118) and (3.101) we have

$$a_2(0) = 1, \quad a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{a - a_3^2(3)}, \quad a_2(2) = \frac{a_3(2) - a_3(2)a_3(3)}{1 - a_3^2(3)} \quad (3.119)$$

In general

$$\begin{aligned} a_{m-1}(0) &= 1 \\ k_m &= a_m(m) \end{aligned}$$

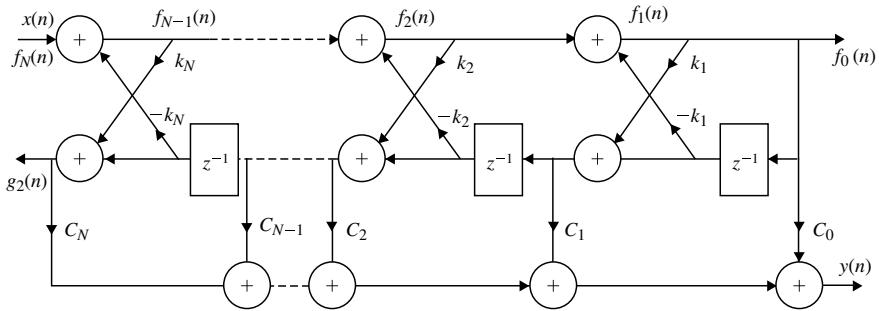
$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

(3.120)

Equation (3.120) can be used to convert direct form to lattice structure.

### 3.13.9 Lattice–Ladder Structure

A general IIR filter containing both poles and zeros can be realized using an all-pole lattice as the basic building block.



**Fig. 3.59** Lattice–ladder structure for realizing a pole–zero IIR filter

Consider an IIR filter with system function

$$H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum_{k=0}^N b_M(k)z^{-k}}{1 + \sum_{k=1}^N a_N(k)z^{-k}} \quad (3.121)$$

where  $N \geq M$ . A lattice structure for Eq. (3.121) can be constructed by filter realizing an all-pole lattice coefficients,  $k_m$ ,  $1 \leq M \leq N$  for the denominator  $A_N(z)$ , and then adding a ladder part as shown in Fig. 3.59 for  $M = N$ . The output of the ladder part can be expressed as a weighted linear combination of  $\{g_m(n)\}$ .

Now the output is given by

$$y(n) = \sum_{m=0}^M C_m g_m(n) \quad (3.122)$$

where  $\{c_m\}$  is called the ladder co-efficients and can be obtained using the recursive relation.

$$C_m = b_m - \sum_{i=m+1}^M C_i a_i(i-m); \quad m = M, M-1, \dots, 0 \quad (3.123)$$

### Example 3.42

Realize the following system function in lattice–ladder structure

$$H(z) = \frac{1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}{1 + z^{-1} + \frac{1}{4}z^{-2}}$$

**Solution**

$$\begin{aligned} \text{Given } B_M(z) &= 1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \\ A_N(z) &= 1 + z^{-1} + \frac{1}{4}z^{-2} \\ a_2(0) &= 1, \quad a_2(1) = 1, \quad a_2(2) = \frac{1}{4} \\ k_2 &= a_2(2) = \frac{1}{4} \end{aligned}$$

From Eq. (3.121) we have

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

For  $m = 2$  and  $k = 1$

$$\begin{aligned} k_1 &= a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)} \\ &= \frac{1 - \frac{1}{4} \cdot 1}{1 - \left(\frac{1}{4}\right)^2} = \frac{0.75}{0.9375} = 0.8 \end{aligned}$$

For  $m = 2$  and  $k = 0$

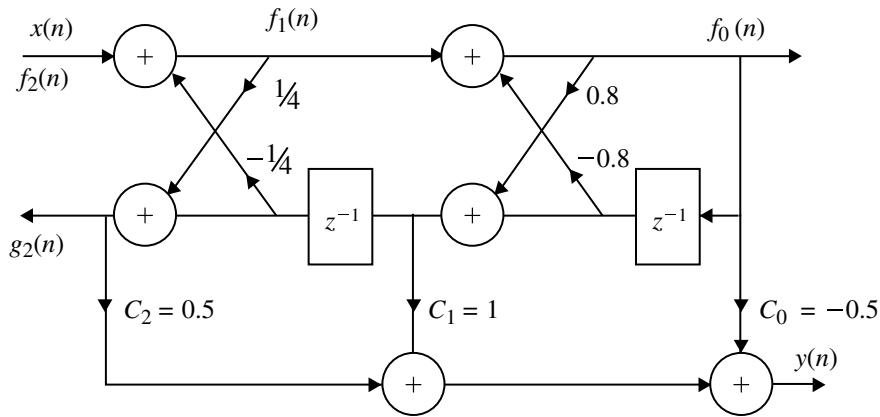
$$a_1(0) = \frac{a_2(0) - a_2(2)a_2(2)}{1 - a_2^2(2)} = \frac{1 - \left(\frac{1}{4}\right)^2}{1 - \left(\frac{1}{4}\right)^2} = 1$$

Therefore, for lattice structure

$$k_1 = 0.8, \quad k_2 = 0.25$$

For ladder structure

$$\begin{aligned} C_m &= b_m - \sum_{i=m+1}^M C_i a_i(i-m), \quad m = M, M-1, \dots, 0 \\ C_2 &= b_2 = \frac{1}{2} \\ C_1 &= b_1 - c_2 a_2(0) \\ &= \frac{3}{2} - \frac{1}{2}(1) = 1 \end{aligned}$$



**Fig. 3.60** Lattice-ladder form for the Example 3.42

$$\begin{aligned}
 C_0 &= b_0 - [C_1 a_1(0) + C_2 a_2(1)] \\
 &= 1 - \left[ 1 \cdot 1 + \frac{1}{2} \cdot 1 \right] \\
 &= 1 - \left[ 1 + \frac{1}{2} \right] = -0.5
 \end{aligned}$$

The lattice-ladder structure for the given pole-zero is shown in Fig. 3.60.

### Example 3.43

Convert the following all-pole IIR filter into a lattice structure.

$$H(z) = \frac{1}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$$

**Solution** Given

$$\begin{aligned}
 A_N(z) &= 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3} \\
 a_3(0) &= 1, \quad a_3(1) = \frac{13}{24}, \quad a_3(2) = \frac{5}{8}; \quad a_3(3) = \frac{1}{3} \\
 K_3 &= a_3(3) = \frac{1}{3}
 \end{aligned}$$

From Eq. (3.121) we have

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

For  $m = 2$ , and  $k = 1$

$$\begin{aligned} a_2(1) &= \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)} \\ &= \frac{\frac{13}{24} - \frac{1}{3}\left(\frac{5}{8}\right)}{1 - \left(\frac{2}{3}\right)^2} = \frac{3}{8} \end{aligned}$$

For  $m = 3$  and  $k = 2$

$$\begin{aligned} k_2 = a_2(2) &= \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)} \\ &= \frac{\frac{5}{8} - \frac{1}{3}\left(\frac{13}{24}\right)}{1 - \left(\frac{1}{3}\right)^2} = \frac{1}{2} \end{aligned}$$

For  $m = 2$  and  $k = 1$

$$\begin{aligned} k_1 = a_1(1) &= \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)} \\ &= \frac{\frac{3}{8} - \frac{1}{2}\left(\frac{3}{8}\right)}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{4} \end{aligned}$$

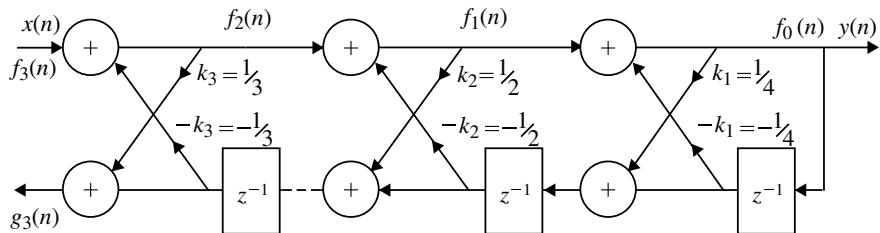
Therefore, for lattice structure

$$k_1 = \frac{1}{4}, \quad k_2 = \frac{1}{2}, \quad k_3 = \frac{1}{3}$$

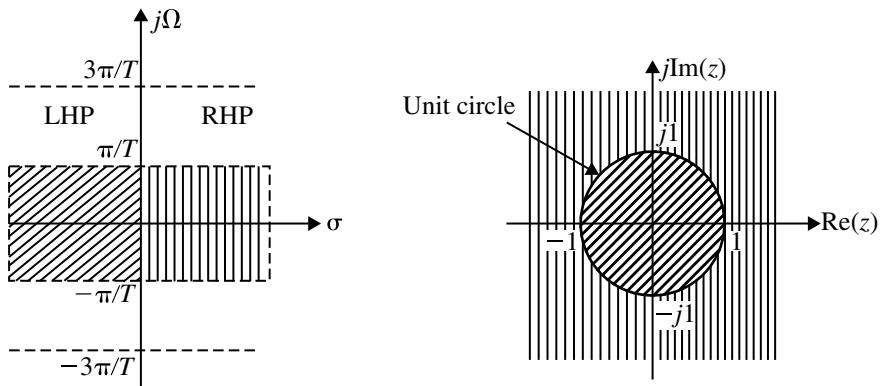
The lattice structure for the given all-pole filter is shown in Fig. 3.61 (Figs. 3.62, 3.63, 3.64, 3.65).

### Summary

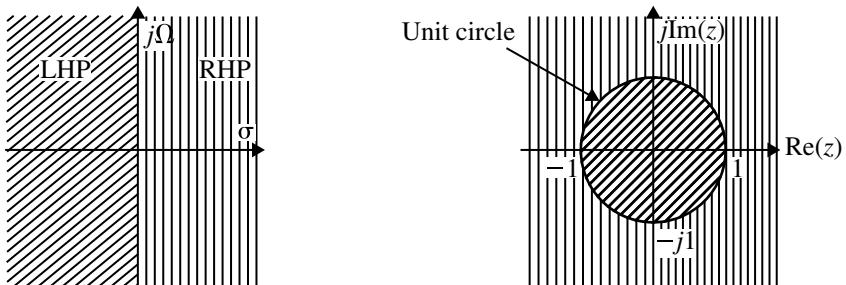
- Digital filters are broadly classified as infinite impulse response (IIR) and finite impulse response (FIR) filters. The design of IIR filter is presented in this chapter.
- The design methodology for analog filters is a well-developed one. By proper transformations, these analog filters can be converted to digital filters. Impulse invariant transformation and bilinear transformation are used to convert analog filters to digital filters.
- When stable analog filters are converted to digital filters, it is necessary that the poles of these digital filters lie within the unit circle in the  $s$ -plane, so that the filter is causal and stable.



**Fig. 3.61** Lattice structure for Example 3.43

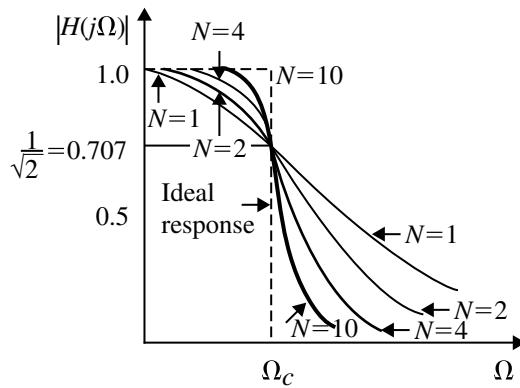


**Fig. 3.62** Mapping of  $s$ -plane into  $z$ -plane in impulse invariant transformation for question 16

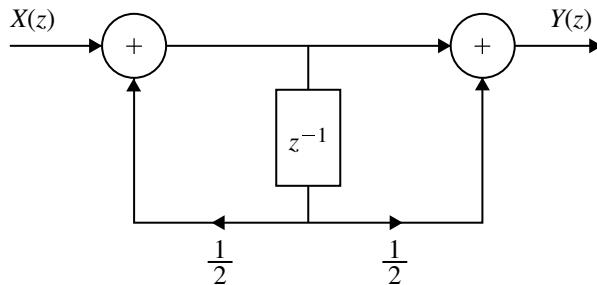


**Fig. 3.63** Mapping of  $s$ -plane to  $z$ -plane for question 26

- In the design of lowpass filters, when the specifications are given in the frequency domain it may not be always possible to get physically realizable stable filters. The magnitude response needs to be approximated.
- Several approximations to design lowpass filters are available in the literature. However, in this chapter, Butterworth and Chebyshev approximations and approximation of derivatives are used to design lowpass filters.



**Fig. 3.64** Magnitude response of Butterworth lowpass filter for various values of  $N$  for question 27



**Fig. 3.65** Realization of  $H(z) = \frac{(1+\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})}$  for question 32

- By analog-to-digital frequency transformation, it is possible to convert lowpass prototype analog filter into either a bandpass, highpass or stop band filter.
- Several methods such as direct form-I, direct form-II parallel form, cascade form are available to realize the structure of the digital filter. They are described in this chapter. Finally the method of getting frequency response plot for the system function  $H(z)$  is described.

### Short Questions and Answers

#### 1. What are the requirements for an analog filter to be stable and causal?

- The analog transfer function  $H(s)$  should be a rational function of  $s$ , and the co-efficients of  $s$  should be real, so that the system is causal.
- The poles should lie in the left half of  $s$ -plane, so that the system is stable.
- The number of zeros should be less than or equal to number of poles.

## 2. What are the requirements for a digital filter to be stable and causal?

- The digital transfer function  $H(z)$  should be a rational function of  $z$  and the co-efficients of  $z$  should be real, so that the system is causal.
- The poles should lie inside of the unit circle in  $z$ -plane, so that the system is stable.

## 3. Define ripples in a filter.

The limits of tolerance in the magnitude of passband and stopband are called ripples. The tolerance in passband is denoted by  $\delta_p$  and that in stopband is denoted by  $\delta_s$ .

## 4. Mention any two techniques for digitizing the transfer function of an analog filter.

The bilinear transformation and the impulse invariant transformations are the two techniques available for digitizing the analog filter transfer function.

## 5. Compare the digital and analog filters.

	Digital filter	Analog filter
1	Operates on digital samples of the signal	Operates on analog samples of the signal
2	It is governed by linear difference equation	It is governed by linear differential equation
3	It consists of adders, multipliers and delays implemented digital logic	It consists of electrical components like resistors, capacitors and inductors
4	In digital filters the filter co-efficients are designed to satisfy the desired frequency response	In analog filters the approximation problem is solved to satisfy the desired frequency response

## 6. What are the advantages and disadvantages of digital filters?

### Advantages:

- High thermal stability due to the absence of electrical components.
- The performance characteristics like accuracy, dynamic range, stability and tolerance can be enhanced by increasing the length of the registers.
- The digital filters are programmable.
- Multiplexing and adaptive filtering are possible.

### Disadvantages:

- The bandwidth of the discrete signal is limited by the sampling frequency.
- The performance of digital filter depends on the hardware used to implement the filter.

## 7. Mention the important features of IIR filters.

- The physically realizable IIR filter does not have linear phase.
- The IIR specifications include the desired characteristics for the magnitude response only.

**8. What is impulse invariant transformation?**

The transformation of analog filter to digital filter without modifying the impulse response of the filter is called impulse invariant transformation, i.e., in this transformation the impulse response of the digital filter will be sampled version of the impulse response of analog filters.

**9. What is the main objective of impulse invariant transformation?**

The main objective is to develop an IIR filter transfer function whose impulse response is the sampled version of impulse response of the analog filter. Therefore, the frequency response characteristics of analog filter are preserved.

**10. How analog poles are mapped to digital poles in impulse invariant/bilinear transformation?**

In impulse invariant/linear transformation the mapping of analog-to-digital poles are as follows:

- The analog poles on the left half of  $s$ -plane are mapped into the interior of unit circle in  $z$ -plane.
- The analog poles on the imaginary axis of  $s$ -plane are mapped into the unit circle in  $z$ -plane.
- The analog poles on the right half of  $s$ -plane are mapped into the exterior of unit circle in  $z$ -plane.

**11. What is the importance of poles in filter design?**

The stability of a filter is related to the location of the poles. For a stable analog filter the poles should lie on the left half of the  $s$ -plane. For a stable digital filter the poles should lie inside of the unit circle of  $z$ -plane.

**12. Write the impulse invariant transformation used to transform real poles with and without multiplicity.**

The impulse invariant transformation used to transform real pole (at  $s = -p_i$ ) without multiplicity is

$$\frac{1}{(s + p_i)} \xrightarrow{\text{is transformed to}} \frac{1}{1 - e^{-p_i T} z^{-1}}$$

**13. Write the impulse invariant transform used to transform complex conjugate poles.**

$$\begin{aligned} \frac{(s + a)}{(s + a)^2 + b^2} &\xrightarrow{\text{is transformed to}} \frac{1 - e^{-aT}(\cos bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}} \\ \frac{b}{(s + a)^2 + b^2} &\xrightarrow{\text{is transformed to}} \frac{1 - e^{-aT}(\sin bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}} \end{aligned}$$

**14. What is the relation between analog and digital frequency in impulse invariant transformation?**

The relation between analog and digital frequency in impulse invariant transformation is given by,

Digital frequency,  $\omega = \Omega T$

where  $\Omega$  = analog frequency and  $T$  = sampling time period.

**15. What is aliasing?**

The phenomena of high-frequency sinusoidal components acquiring the identity of low-frequency sinusoidal components after sampling is called aliasing (i.e., aliasing is higher frequencies impersonating lower frequencies). The aliasing problem will arise if the sampling rate does not satisfy the Nyquist sampling criteria.

**16. Why an impulse invariant transformation is not considered to be one-to-one?**

In impulse invariant transformation any strip of width  $(2\pi)/T$  in the  $s$ -plane for values of  $s$  in the range  $(2k - 1)\pi/T \leq \Omega \leq (2k + 1)\pi/T$  (where  $k$  is an integer) is mapped into the entire  $z$ -plane. The left half portion of each strip in  $s$ -plane maps into the interior of the unit circle in  $z$ -plane, right half portion of each strip in  $s$ -plane maps into the exterior of the unit circle in  $z$ -plane and the imaginary axis of each strip in  $s$ -plane maps into the unit circle in  $z$ -plane. Hence, the impulse invariant transformation is many-to-one.

**17. What is aliasing problem in impulse invariant method of designing digital filter? Why it is absent in bilinear transformation?**

In impulse invariant mapping, the analog frequencies in the interval  $(2k - 1)\pi/T \leq \Omega \leq (2k + 1)\pi/T$  (where  $k$  is an integer) map into corresponding values of digital frequencies in the interval  $-\pi \leq \omega \leq \pi$ . Hence, the mapping of  $\Omega$  to  $\omega$  is many-to-one.

This will result in high-frequency components acquiring the identity of the low-frequency components if the analog filter is not bandlimited. This effect is called aliasing. The aliasing can be avoided in bandlimited filters by choosing very small values of sampling time. The bilinear mapping is one-to-one mapping, and so there is no effect of aliasing.

**18. What is bilinear transformation?**

The bilinear transformation is a conformal mapping that transforms the  $s$ -plane to  $z$ -plane. In this mapping the imaginary axis of  $s$ -plane is mapped into the unit circle in  $z$ -plane, the left half of  $s$ -plane is mapped into interior of unit circle in  $z$ -plane and right half of  $s$ -plane is mapped into exterior of unit circle in  $z$ -plane. The bilinear mapping is a one-to-one mapping, and it is accomplished when

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

**19. What is the relation between analog and digital frequency in bilinear transformation?**

The relation between analog and digital frequency in bilinear transformation is given by,

$$\text{Digital frequency, } \omega = 2 \tan^{-1} \left( \frac{\Omega T}{2} \right)$$

where  $\Omega$  = analog frequency and  $T$  = sampling time period.

**20. What is frequency warping?**

In bilinear transformation, the relation between analog and digital frequencies is nonlinear. When the  $s$ -plane is mapped into  $z$ -plane using bilinear transformation, this nonlinear relationship introduces distortion in frequency axis, which is called frequency warping.

**21. What is pre-warping? Why it is employed?**

In IIR filter design using bilinear transformation, the conversion of the specified digital frequencies to analog frequencies is called pre-warping. The pre-warping is necessary to eliminate the effect of warping on amplitude response.

**22. Explain the technique of pre-warping.**

In IIR filter design using bilinear transformation, the specified digital frequencies are converted to analog frequencies which are called pre-warped frequencies. Using the pre-warped frequencies, the analog filter transfer function is designed and then it is transferred to digital filter transfer function.

**23. Compare the impulse invariant and bilinear transformation.**

	Impulse invariant	Bilinear transformation
1	It is many-to-one mapping	It is one-to-one mapping
2	The relation between analog and digital frequency is linear	The relation between analog and digital frequency is nonlinear
3.	To prevent the problem of aliasing the analog filters should be bandlimits	There is no problem of aliasing and the filters need not be bandlimits

**24. Obtain the impulse response of digital filter to correspond to an analog filter with impulse response  $h_a(t) = 0.3e^{-2t}$  and with a sampling rate of 1.0 kHz using impulse invariant method.**

Given that,  $h_a(t) = 0.3e^{-2t}$  and sampling frequency,  $F = 1\text{kHz} = 1 \times 10^3 \text{ Hz}$ .

$$\therefore \text{Sampling time, } T = \frac{1}{F} = \frac{1}{1 \times 10^3} = 10^{-3} \text{ s.}$$

$$\begin{aligned} \text{Impulse response of digital filter } h(n) &= h_a(t)|_{t=nT} = 0.3e^{-2t}|_{t=nT} = 0.3e^{-2nT} \\ &= 0.3(e^{-2T})^n = 0.3(e^{-2 \times 10^{-3}})^n \\ &= 0.3(0.998)^n; \quad \text{for } n \geq 0. \end{aligned}$$

25. Given that,  $H_a(s) = 1/(s + 1)$ . By impulse invariant method, obtain the digital filter transfer function and the difference equation of digital filter.
- Given that,  $H_a(s) = 1/(s + 1)$ . In impulse invariant transformation,

$$\frac{1}{s + p_i} \xrightarrow{\text{is transformed to}} \frac{1}{1 - e^{-p_i T} z^{-1}}$$

Let  $T = 1$  s. Transfer function of digital filter

$$H(z) = \frac{1}{1 - e^{-T} z^{-1}} = \frac{1}{1 - e^{-1} z^{-1}} = \frac{1}{1 - 0.368 z^{-1}}$$

We know that,

$$H(z) = \frac{Y(z)}{X(z)}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 0.368 z^{-1}}$$

On cross-multiplying we get

$$Y(z) - 0.368z^{-1}Y(z) = X(z)$$

$$\therefore Y(z) = X(z) + 0.368z^{-1}Y(z)$$

On taking inverse  $z$ -transform we get,

$$y(n) = x(n) + 0.368y(n - 1)$$

26. Sketch the mapping of  $s$ -plane to  $z$ -plane in bilinear transformation.
27. How does the order of the filter affect the frequency response of Butterworth filter?

The magnitude response of Butterworth filter is shown in Fig. 3.64, from which it can be observed that the magnitude response approaches the ideal response as the order of the filter is increased.

28. Write the transfer function of unnormalized Butterworth lowpass filter.
- When  $N$  is even, transfer function of analog lowpass Butterworth filter is,

$$H_a(s) = \sum_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

When  $N$  is odd, transfer function of analog lowpass Butterworth filter is,

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \sum_{k=1}^{(N-1)/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where

$$b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

$N$  = Order of the filter and  $\Omega_c$  = Analog cutoff frequency.

### 29. How will you choose the order $N$ for a Butterworth filter?

Calculate a parameter  $N_1$  using the following equation and correct it to nearest integer.

$$N_1 = \frac{1}{2} \frac{\log \left\{ \left[ \frac{1}{A_p^2} - 1 \right] / \left[ \frac{1}{A_s^2 - 1} \right] \right\}}{\log \left( \frac{\Omega_s}{\Omega_p} \right)}$$

Choose the order  $N$  of the filter such that  $N \geq N_1$ .

### 30. Write the properties of Butterworth filter.

- (i) The Butterworth filters have all-pole designs.
- (ii) At the cutoff frequency  $\Omega_c$ , the magnitude of normalized Butterworth filter is  $1/\sqrt{2}$ .
- (iii) The filter order  $N$  completely specifies the filter, and as the value of  $N$  increases the magnitude response approaches the ideal response.

### 31. Find the digital transfer function $H(z)$ by using impulse invariant method for the analog T.F. $H(s) = \frac{1}{(s+2)}$ . Assume $T = 0.1$ sec and $0.5$ s.

(Anna University, December, 2007)

$$H(z) = \frac{1}{(1 - 0.8187z^{-1})} \quad \text{for } T = 0.1 \text{ s.}$$

$$H(z) = \frac{1}{(1 - 0.3679z^{-1})} \quad \text{for } T = 0.5 \text{ s.}$$

### 32. Realize the following system using Direct Form-II.

(Anna University, December, 2007)

$$y(n) - \frac{1}{2}y(n-1) = x(n) + \frac{1}{2}x(n-1)$$

$$H(z) = \frac{(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})}$$

33. **Transform the single pole lowpass Butterworth filter with system function  $H(s) = \frac{\Omega_p}{(\frac{\Omega_p}{s} + \Omega_p)}$  into a bandpass filter with upper and lower band edge frequencies  $\Omega_u$  and  $\Omega_l$ , respectively.**

(Anna University, June, 2007)

$$H(s) = \frac{\Omega_p(s^2 + \Omega_l\Omega_u)}{\Omega_p(s^2 + \Omega_l + \Omega_u) + s(\Omega_u - \Omega_l)}$$

34. **Convert the analog bandpass filter.**

$$H_a(s) = \frac{1}{(s + 0.1)^2 + 9}$$

**into a digital IIR filter by use of the mapping  $s = \frac{1}{T} \frac{(z-1)}{(z+1)}$ .**

(Anna University, June, 2007)

$$H(z) = \frac{T^2(1 + 2z^{-1} + z^{-2})}{[9.01T^2 + (18.02T^2 - 2)z^{-1} + (9.01T^2 - 2T + 1)z^{-2}]}$$

35. **Write the equation for frequency transformation from lowpass to bandpass filter.**

(Anna University, June, 2007)

$$s \rightarrow \Omega_p \frac{(s^2 + \Omega_l\Omega_u)}{s(\Omega_u - \Omega_l)}$$

36. **Find the digital filter equivalent for  $H(s) = \frac{1}{(s+8)}$**

(Anna University, June, 2007)

$$H(z) = \frac{1}{(1 - e^{-8T}z^{-1})}$$

37. **What are the parameters (specifications) of a Chebyshev filter?**

(Anna University, June, 2007)

The parameters of a Chebyshev filter are (a) passband ripple, (b) passband cutoff frequency, (c) stopband frequency, (d) attenuation beyond stopband frequency.

**38. Give the location of poles of normalized Butterworth filter.**

(Anna University, June, 2007)

The poles of the normalized Butterworth filter occur on a unit circle at equally spaced points in the complex  $s$ -plane. The pole locations are identified as

$$s_k = e^{j\pi/2} e^{j(2k+1)\frac{\pi}{N}}, \quad k = 0, 1, 2 \dots (N-1)$$

$N$  is order of the filter. For stable filter only LHP poles are taken.

**39. State two advantages of bilinear transformation**

(Anna University, December, 2006)

The two advantages of bilinear transformation are:

- (a) It avoids aliasing in frequency components.
- (b) The transformation of stable analog filter results in a stable digital filter.

**40. What is Chebyshev approximation?**

In Chebyshev approximation, the error is defined as the difference between the ideal brickwall characteristic and the actual response and this is minimized over a prescribed band of frequencies. There are two types of Chebyshev transfer functions. In the type 1, approximation, the magnitude characteristic is equiripple in the passband and monotonic in the stopband. In type 2 approximation, the magnitude response is monotonic in the passband and equiripple in the stopband.

**41. What is Butterworth approximation?**

The frequency response characteristic of the lowpass Butterworth filter is monotonic in both the passband and stopband. The response approximates to the ideal response as the order  $N$  of the filter increases. (Flat Characteristics).

**42. What are the parameters that can be obtained from the Chebyshev filter specifications?**

For a given set of specifications, the order  $N$  of the filter can be determined. Knowing  $N$ , the poles and zeros can be located.

**43. Write down the transfer functions of a first-order Butterworth normalized lowpass filter and highpass filter.**

The normalized first-order Butterworth normalized lowpass filter is

$$h_a(s) = \frac{1}{(s + 1)}$$

The first-order highpass filter normalized transfer function is

$$h_a(s) = \frac{1}{\frac{1}{s} + 1} = \frac{s}{s + 1}$$

**44. Compare the lowpass Butterworth filter with lowpass Chebyshev filter.**

- (a) In the Butterworth filter, the frequency response decreases monotonically while the magnitude response of the Chebyshev filter contains ripples in the passband and monotonically decreases in the stopband.
- (b) The poles of the Butterworth filter lie on unit circle, whereas the poles of Chebyshev filter lie on ellipse.
- (c) The transition band is more in Butterworth filter compared to Chebyshev filter.
- (d) For the same specifications, the order of the Butterworth filter is higher than that of Chebyshev filter.

**45. Why IIR filters do not have linear phase?**

A linear phase filter must have a system function

$$H(z) = \pm z^{-N} H(z^{-1})$$

where  $z^{-N}$  represents a delay of  $N$  units. For this it is necessary to have a mirror image pole outside the unit circle for every pole inside the unit circle in the  $z$ -plane. This means the filter will be unstable. Hence, a causal and stable IIR filter cannot have linear phase if it is to be physically realizable.

**46. What are the disadvantages of Impulse invariance method of designing filters?**

If the sampling interval  $T$  is large, the IIR filter designed using impulse invariance method results in aliasing due to sampling. This method is not suitable to design highpass filter due to the spectrum aliasing which results from the sampling process.

**Long Answer Type Questions**

1. Explain the impulse invariance method of IIR filter design.
2. Explain bilinear transformation method of IIR filter design.
3. Draw the direct form-I and direct form-II structure of IIR system.
4. Convert the analog transfer function of the second-order Butterworth filter into digital transfer function using bilinear transformation.

5. What is meant by aliasing? Explain with example.
6. Describe the procedure for the design of digital filters from analog filters and advantages and disadvantages of digital filters.
7. State and prove the conditions satisfied by a stable and causal discrete time filter in the  $z$ -transform domain.
8. Design a Butterworth filter using impulse invariant method for the following specifications:

$$0.8 \leq |H(e^{j\omega})| \leq 1; \quad 0 \leq \omega \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.2; \quad 0.6 \leq \omega \leq \pi$$

9. Mention the advantages and disadvantages of FIR and IIR filters.
10. Design a Butterworth digital filter to meet the following constraints:

$$0.9 \leq |H(\omega)| \leq 1; \quad 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(\omega)| \leq 0.2; \quad \frac{3\pi}{4} \leq \omega \leq \pi$$

Use bilinear transformation mapping technique. Assume  $T = 1$  s.

11. Develop impulse invariant mapping technique for designing IIR filter.
12. Realize the given transfer function using direct form-I and parallel methods

$$H(z) = \frac{(4z^2 + 11z - 2)}{(z + 1)(z - 3)}.$$

13. Derive the equation for calculating the order of the Butterworth filter.
14. Design and realize a digital filter using bilinear transformation for the following specifications:
  - (a) Monotonic passband and stopband.
  - (b)  $-3$  dB cutoff at  $0.5\pi$  rad.
  - (c) magnitude down at least  $15$  dB at  $\omega = 0.75\pi$  rad.

15. Using impulse invariant method find  $H(z)$  at  $T = 1$  s.

$$H(s) = \frac{2}{s^2 + 8s + 15}$$

16. Enumerate the various steps involved in the design of LF digital Butterworth IIR filter.
17. Obtain direct form-II, cascade and parallel realizations of a discrete time system described by the difference equation

$$y(n) - \frac{5}{8}y(n-1) - \frac{1}{16}y(n-2) = x(n) - 3x(n-1) + 3x(n-2) - x(n-3)$$

(Anna University)

**Ans:** Direct Form-II

$$\frac{Y(z)}{X(z)} = \frac{(1 - 3z^{-1} + 3z^{-2} - z^{-3})}{(1 - \frac{5}{8}z^{-1} - \frac{1}{16}z^{-2})}$$

Cascade Form

$$H_1(z) = \frac{(1 - z^{-1})}{(1 - 0.7125z^{-1})}$$

$$H_2(z) = \frac{(1 - 2z^{-1} + z^{-2})}{(1 + 0.086z^{-1})}$$

Parallel Form

$$H(z) = -208 + 16z^{-1} + \frac{209 - 149z^{-1}}{(1 - \frac{5}{8}z^{-1} - \frac{1}{16}z^{-2})}$$

18. Determine the system function  $H(z)$  of the lowest-order Chebyshev digital filter that meets the following specifications.

- (a) 1 dB ripple in the passband  $0 \leq \omega \leq 0.3\pi$ .
- (b) At least 60 dB attenuation in the stopband  $0.35 \leq |\omega| \leq \pi$ . Use the bilinear transformation.

(Anna University)

19. Design a Chebyshev filter with a maximum passband attenuation of 2.5 dB at  $\Omega_p = 20$  rad/s and the stopband attenuation of 30 dB at  $\Omega_s = 50$  rad/s.

(Anna University)

20. Find the direct form-I and direct form-II realization of the filter

$$H(z) = \frac{1 - 2z^{-2} + 3z^{-3}}{1 - 0.2z^{-3}}$$

(Anna University, May, 2006)

21. Obtain the direct form-I, Canonic form and parallel form realization structures for the system given by the difference equation

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$$

(Anna University, June, 2006)

**Ans:** Direct Form-I and Canonic Form

$$H(z) = \frac{(0.7 - 0.252z^{-2})}{(1 + 0.1z^{-1} - 0.72z^{-2})}$$

Parallel Form

$$H(z) = 0.35 + \frac{0.1441}{(1 - 0.8z^{-1})} + \frac{0.2058}{(1 + 0.9z^{-1})}$$

# Chapter 4

## Finite Impulse Response (FIR) Filter Design



### Learning Objectives

After completing this chapter, you should be able to:

- ✖ Study the characteristic of practical frequency selective digital filters.
- ✖ Realize the finite impulse response (FIR) filter structure.
- ✖ Study the characteristics of FIR filters with linear phase.
- ✖ Study different type of symmetry of the impulse response of linear phase FIR filters.
- ✖ Design FIR filter by frequency sampling method.
- ✖ Design linear phase FIR filters using windows.
- ✖ Design FIR differentiators.

### 4.1 Introduction

In Chap. 3, the design of IIR filter was considered. These filters are designed by transformation from analog domain to digital domain. They are called recursive filters since feedback is used. Hence, IIR filters should be carefully designed to ensure that the system function  $H(z)$  is stable. In the case of FIR filter design which is non-recursive in nature, the stability problem does not arise. Since the system function  $H(z)$  is a polynomial of  $z^{-1}$  it always guarantees the stability of the filter. Further, unlike IIR filter design which has nonlinear phase characteristic, the FIR digital filter has linear phase. A filter with a nonlinear phase characteristic causes phase distortion in the signal which is undesirable in much practical applications such as data transmission, music and video. However, to meet the same frequency response specifications, the order of FIR filter transfer function is much higher than that of an IIR digital filter. Hence, FIR filters require more parameters more memory and has more computational complexity.

The basic approach to the design of FIR digital filter is based on direct approximation of the specified magnitude response in addition to the requirement of linear phase response. Unlike IIR filter, it does not have any connection with the analog filter. Linear phase FIR filters are designed based on whether the impulse response function  $h(n)$  is positive or negative and the order of the filter is odd or even. Thus there are four types of linear phase FIR filters. FIR filters are also designed based on Fourier coefficients which have infinite length. By truncating these coefficients of infinite Fourier series, FIR filters are designed. This is called windowing. Several windowing techniques are available. FIR filter design using the above techniques is discussed in this chapter.

### 4.1.1 LTI System as Frequency Selective Filters

An LTI system performs a type of filtering among the various frequency components at its input. The nature of this filtering is determined by the frequency response characteristics  $H(\omega)$ , which in turn depends on the choice of the system parameters [ $a_k$  and  $b_k$  coefficient]. Thus, by proper selection of the coefficients, we can design frequency selective filters that pass signals with frequency components in some bands while they attenuate signals that contain frequency components in other frequency bands.

In general an LTI system modifies the input spectrum  $X(\omega)$  according to its frequency response  $H(\omega)$  to yield an output signal with spectrum  $Y(\omega) = H(\omega)X(\omega)$ .  $H(\omega)$  acts as a weighting function or a spectral shaping function to the different frequency components in the input signal.

Let us consider a signal  $x(n)$  with frequency content in a band of frequencies  $\omega_1 < \omega < \omega_2$  (i.e., band limited signal)

$$X(\omega) = 0 \quad \text{for } \omega \geq \omega_2 \quad \text{and} \quad \omega \leq \omega_1 \quad (4.1)$$

If it is passed through a filter with frequency response,

$$H(\omega) = \begin{cases} Ce^{-j\omega\alpha}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise, where } C \text{ and } \alpha = +\text{ve constants} \end{cases} \quad (4.2)$$

The signal at the output of the filter has the spectrum which is expressed as,

$$Y(\omega) = X(\omega)H(\omega) \quad (4.3)$$

$$= X(\omega)Ce^{-j\omega\alpha} \quad (4.3)$$

$$= CX(\omega)e^{-j\omega\alpha} \quad (4.4)$$

Using the following identity we get,

$$\begin{aligned} F[x(n - \alpha)] &= X(\omega)e^{-j\omega\alpha} \\ \therefore Y(\omega) &= C F[x(n - \alpha)] \end{aligned}$$

Taking inverse Fourier transform on both sides we get,

$$y(n) = Cx(n - \alpha)$$

i.e., the filter output is simply a delayed and amplitude scaled version of the input signal. A pure delay is usually tolerable and is not considered a distortion of the signal. Likewise the amplitude scaling. The filter defined by

$$H(\omega) = \begin{cases} Ce^{-j\omega\alpha}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

is called an ideal filter.

$$\begin{aligned} H(\omega) &= |H(\omega)|\angle H(\omega) \\ |H(\omega)| &= C, \quad \angle H(\omega) = \theta(\omega) = -\alpha\omega \quad (4.5) \\ |H(\omega)| \text{ (amplitude)} &= \text{constant} \\ |H(\omega)| \text{ (phase)} &= \text{linear function of frequency } \omega \end{aligned}$$

In general any deviation of the frequency response characteristics of a linear filter from the ideal response results in signal distortion.

- (i) *Amplitude distortion*: If the filter has a variable magnitude frequency response characteristic in the passband, then the filter has amplitude distortion.
- (ii) *Phase distortion*: If the phase characteristic is not linear within the desired frequency band, the signal undergoes phase distortion.
- (iii) *Signal delay*: The derivative of the phase with respect to frequency has the units of delay

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = -\frac{d}{d\omega}(-\alpha\omega) = \alpha \quad (4.6)$$

Filters are classified according to their frequency response characteristics. The ideal frequency response  $H(\omega)$  of four major types of filters is:

(i) Ideal frequency response of LPF

$$H_d(\omega) = \begin{cases} C e^{-j\alpha\omega}, & -\omega_c \leq \omega \leq \omega_c \\ 0, & -\pi < \omega < -\omega_c \text{ and } \omega_c < \omega < \pi \end{cases} \quad (4.7)$$

(ii) Ideal frequency response of HPF

$$H_d(\omega) = \begin{cases} C e^{-j\alpha\omega}, & -\pi \leq \omega \leq -\omega_c, \omega_c \leq \omega \leq \pi, \\ 0, & -\omega_c < \omega < \omega_c \end{cases} \quad (4.8)$$

(iii) Ideal frequency response of BPF

$$H_d(\omega) = \begin{cases} C e^{-j\alpha\omega}, & -\omega_{c_2} \leq \omega \leq -\omega_{c_1} \text{ and } \omega_{c_1} \leq \omega \leq \omega_{c_2} \\ 0, & -\pi \leq \omega \leq -\omega_{c_2} \& -\omega_{c_1} \leq \omega \leq \omega_{c_1} \& \omega_{c_2} \leq \omega \leq \pi \end{cases} \quad (4.9)$$

(iv) Ideal frequency response of BSF

$$H_d(\omega) = \begin{cases} C e^{-j\alpha\omega}, & -\pi \leq \omega \leq -\omega_{c_2} \& -\omega_{c_1} \leq \omega \leq \omega_{c_1} \& \omega_{c_2} \leq \omega \leq \pi \\ 0, & -\omega_{c_2} < \omega < -\omega_{c_1} \text{ and } \omega_{c_1} < \omega < \omega_{c_2} \end{cases} \quad (4.10)$$

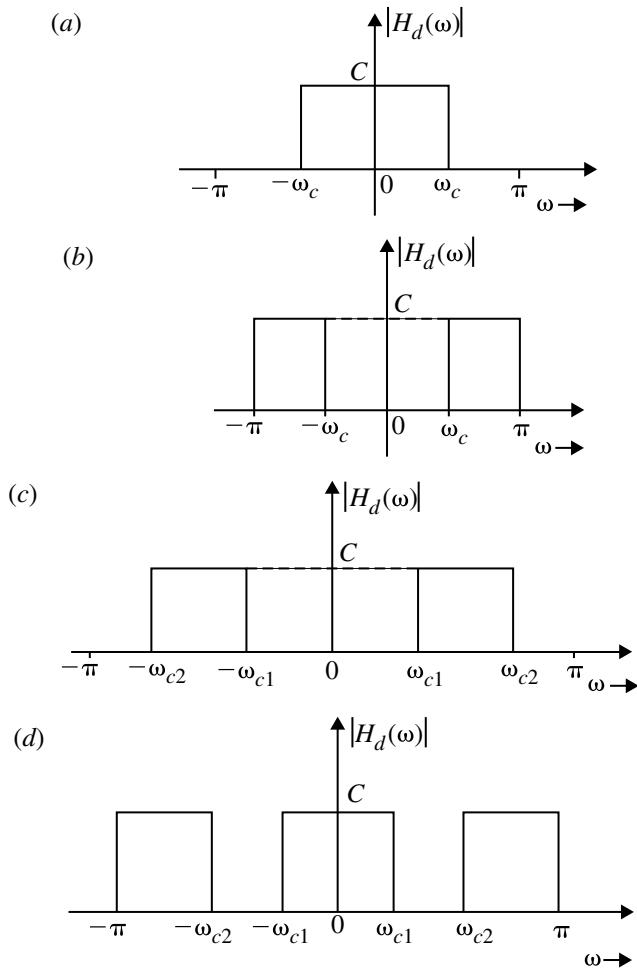
The magnitude response of ideal lowpass, highpass, bandpass and bandstop filters is shown in Fig. 4.1a–d respectively.

## 4.2 Characteristic of Practical Frequency Selective Filters

Causality has very important implications in the design of frequency selective filters. Causality implies that the frequency response characteristic  $H(\omega)$  of the filter cannot be zero, except at a finite set of points in the frequency range. In addition,  $H(\omega)$  cannot have an infinitely sharp cutoff from passband to stopband. That is  $H(\omega)$  cannot drop from unity to zero abruptly. Therefore, the ideal filters are non-causal and hence physically unrealizable for the real-time signal processing applications.

The input  $x(n)$  and output  $y(n)$  of a LTI system are governed by the following  $N^{\text{th}}$  order difference equation.

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)$$



**Fig. 4.1** Magnitude response of ideal lowpass, highpass, bandpass and bandstop filter

On taking Fourier transform on both sides we get

$$\begin{aligned}
 Y(\omega) &= -\sum_{k=1}^N a_k e^{-j\omega k} Y(\omega) + \sum_{k=0}^M b_k e^{-j\omega k} X(\omega) \\
 Y(\omega) \left[ 1 + \sum_{k=1}^N a_k e^{-j\omega k} \right] &= \sum_{k=0}^M b_k e^{-j\omega k} X(\omega) \\
 H(\omega) &= \frac{Y(\omega)}{X(\omega)} \\
 &= \frac{\sum_{k=0}^M b_k e^{-j\omega k}}{1 + \sum_{k=1}^N a_k e^{-j\omega k}}; \quad N \geq M \tag{4.11}
 \end{aligned}$$

which are causal and physically realizable. The basic digital filter design problem is to approximate any of the ideal frequency response characteristics with a system that has frequency response  $H(\omega)$  by properly selecting the coefficient  $\{a_k\}$  and  $\{b_k\}$ .

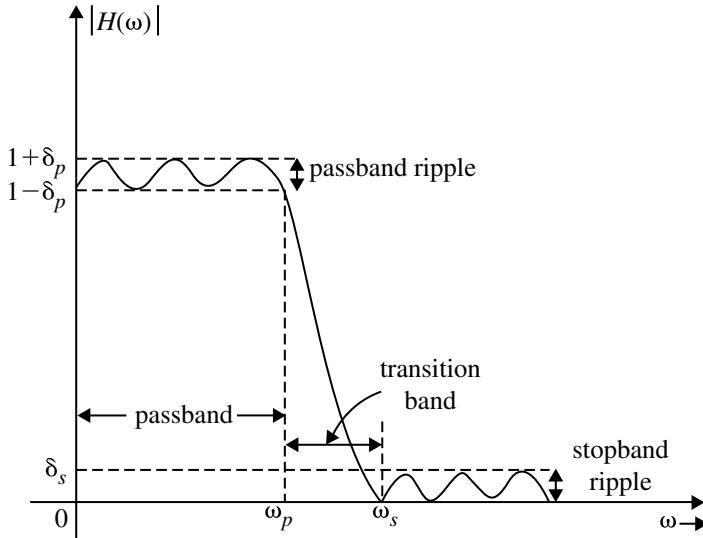
In practice it is not necessary to insist that the magnitude  $|H(\omega)|$  is constant in the entire passband of the filters. A small amount of ripple in the passband is usually tolerable. Similarly it is not necessary for the filter response  $|H(\omega)|$  to be zero in the stopband. A small nonzero value or a small amount of ripple in the stopband is also tolerable. The magnitude response of a practical lowpass filter is shown in Fig. 4.2.

The transition of the frequency response from passband to stopband defines the transition band or transition region of the filter. The band edge frequency  $\omega_p$  defines the edge of the passband, while the frequency  $\omega_s$  denotes the beginning of the stopband. Thus the width of the transition band is  $\omega_s - \omega_p$ . The width of the passband is usually called the bandwidth of the filter.

In any filter design problem, one may specify

- (i) The maximum passband ripple  $\delta_p$ .
- (ii) The maximum stopband ripple  $\delta_s$ .
- (iii) Passband edge frequency  $\omega_p$ .
- (iv) Stopband edge frequency  $\omega_s$ .

Based on these specifications, we may select the parameters  $a_k$  and  $b_k$  in the frequency response characteristic which best approximates the desired specifications.



**Fig. 4.2** Magnitude response of a practical lowpass filter

### 4.3 Structures for Realization of the FIR Filter

In general an FIR system is described by the following difference equation:

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k) \quad (4.12)$$

$$Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

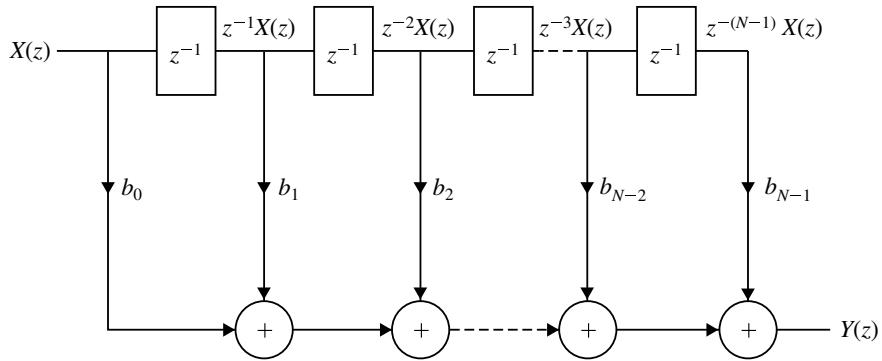
$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{N-1} z^{-(N-1)} \quad (4.13)$$

where  $H(z)$  is transfer function.

$$\begin{aligned} H(z) &= Z[h(n)] \\ &= \sum_{k=0}^{N-1} h(k) z^{-k} = h(0) + h(1) z^{-1} + h(2) z^{-2} + \cdots + h(N-1) z^{-(N-1)} \end{aligned} \quad (4.14)$$

On comparing Eqs. (4.13) and (4.14) we get

$$b_k = h(k) \quad \text{for } k = 0, 1, 2, 3, \dots, N-1 \quad (4.15)$$



**Fig. 4.3** Direct form realization

The different types of structures for realizing FIR systems are:

1. Direct form realization.
2. Cascade form realization.
3. Linear phase realization.

### 4.3.1 Direct Form Realization

The direct form structure that can be obtained from the  $z$ -domains equation governing the FIR system is shown in Fig. 4.3.

$$\begin{aligned}
 Y(z) &= \sum_{k=0}^{N-1} b_k z^{-k} X(z) \\
 &= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_{N-1} z^{-(N-1)} X(z)
 \end{aligned} \quad (4.16)$$

### 4.3.2 Cascade Form Realization

The transfer function of FIR system is  $(N - 1)^{\text{th}}$  order polynomial in  $z$ . This polynomial can be factorized into second-order factors (when  $N$  is odd) and the transfer function can be expressed as a product of second-order factors.

$$\begin{aligned}
 \text{When } N \text{ is odd, } H(z) &= \prod_{i=1}^{(N-1)/2} (c_{0i} + c_{1i} z^{-1} + c_{2i} z^{-2}) \\
 &= H_1 H_2 \dots H_{(N-1)/2}
 \end{aligned} \quad (4.17)$$

When  $N$  is even,  $H(z)$  will have one first-order and  $(N - 2)/2$  second-order section.

$$\begin{aligned} \text{When } N \text{ is even, } H(z) &= (c_{01} + c_{11}z^{-1}) \prod_{i=2}^{N/2} (c_{0i} + c_{1i}z^{-1} + c_{2i}z^{-2}) \quad (4.18) \\ &= H_1 H_2 \dots H_{N/2} \end{aligned}$$

### 4.3.3 Linear Phase Realization

In FIR systems for linear phase response, the impulse response should be symmetrical. For this,

$$h(k) = h(N - 1 - k) \quad (4.19)$$

#### When $N$ is Even

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k} = \sum_{k=0}^{(N/2)-1} h(k)z^{-k} + \sum_{k=(N/2)}^{N-1} h(k)z^{-k} \quad (4.20)$$

$$\text{Let } m = N - 1 - k \implies k = N - 1 - m$$

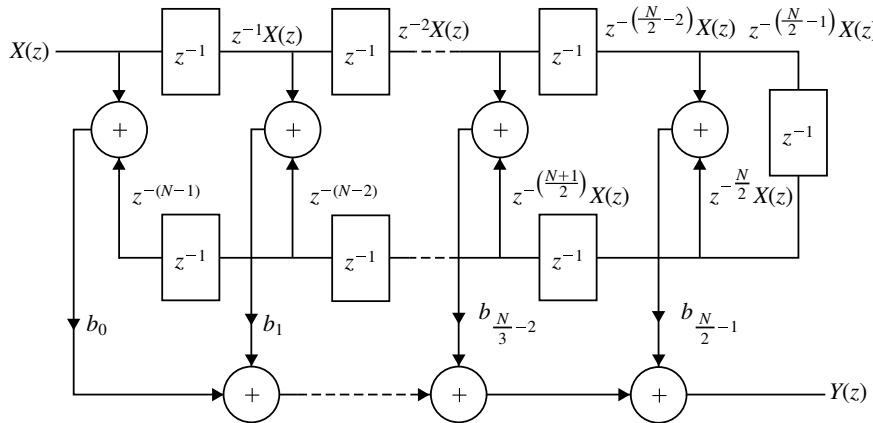
$$\text{When } k = \frac{N}{2}; \implies m = N - 1 - \frac{N}{2} = \frac{N}{2} - 1$$

$$\text{When } k = N - 1; \implies m = N - 1 - (N - 1) = 0$$

$$\therefore H(z) = \sum_{k=0}^{(N/2)-1} h(k)z^{-k} + \sum_{m=0}^{(N/2)-1} h(N - 1 - m)z^{-(N-1-m)}$$

On replacing  $m$  by  $k$  we get,

$$\begin{aligned} H(z) &= \sum_{k=0}^{(N/2)-1} h(k)z^{-k} + \sum_{k=0}^{(N/2)-1} h(N - 1 - k)z^{-(N-1-k)} \\ &= \sum_{k=0}^{(N/2)-1} h(k)z^{-k} + \sum_{k=0}^{(N/2)-1} h(k)z^{-(N-1-k)} \\ H(z) &= \sum_{k=0}^{(N/2)-1} h(k)[z^{-k} + z^{-(N-1-k)}] \end{aligned}$$



**Fig. 4.4** Linear phase realization when  $N$  is even

Therefore,

$$\begin{aligned}
 Y(z) &= \sum_{k=0}^{(N/2)-1} h(k)[z^{-k}X(z) + z^{-(N-1-k)}X(z)] \quad \text{where } h(k) = b_k \\
 &= b_0[X(z) + z^{-(N-1)}X(z)] + b_1[z^{-1}X(z) + z^{-(N-2)}X(z)] \\
 &\quad + \cdots + b_{(N/2)-1}[z^{-(N/2)-1}X(z) + z^{-(N/2)}X(z)]
 \end{aligned} \tag{4.21}$$

Equation (4.21) is realized in Fig. 4.4.

### When $N$ is Odd

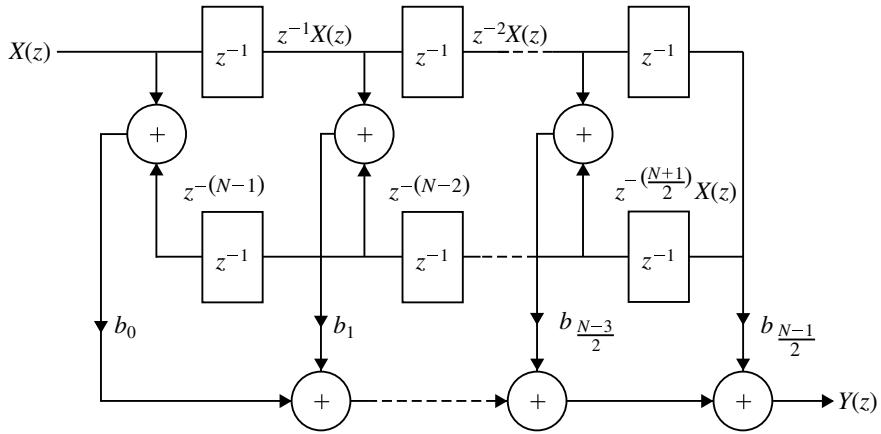
$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k} = \sum_{k=0}^{\frac{N-3}{2}} h(k)z^{-k} + h\left(\frac{N-1}{2}\right)z^{-\left(\frac{N-1}{2}\right)} + \sum_{k=\frac{N+1}{2}}^{N-1} z^{-1} \tag{4.22}$$

$$\text{Let } m = N - 1 - k \implies k = N - 1 - m$$

$$\text{When } k = \frac{N+1}{2}; \implies m = \frac{N-3}{2}$$

$$\text{When } k = N - 1; \implies m = 0$$

$$\therefore H(z) = \sum_{k=0}^{\frac{N-3}{2}} h(k)z^{-k} + h\left(\frac{N-1}{2}\right)z^{-\left(\frac{N-1}{2}\right)} \sum_{m=0}^{\frac{N-3}{2}} h(N - 1 - m)z^{-(N-1-m)}$$



**Fig. 4.5** Linear phase realization when  $N$  is odd

On replacing  $m$  by  $k$  we get

$$\begin{aligned}
 H(z) &= \sum_{k=0}^{\frac{N-3}{2}} h(k)z^{-k} + h\left(\frac{N-1}{2}\right)z^{-\left(\frac{N-1}{2}\right)} \sum_{k=0}^{\frac{N-3}{2}} h(N-1-k)z^{-(N-1-k)} \\
 &= \sum_{k=0}^{\frac{N-3}{2}} h(k)[z^{-k} + z^{-(N-1-k)}] + h\left(\frac{N-1}{2}\right)z^{-\left(\frac{N-1}{2}\right)} \quad \because (b_k = h(k)) \\
 Y(z) &= b_{\frac{N-1}{2}}z^{-\left(\frac{N-1}{2}\right)}X(z) + \sum_{k=0}^{\frac{N-3}{2}} b_k(z^{-k}X(z)) + z^{-(N-1-k)}X(z)
 \end{aligned}$$

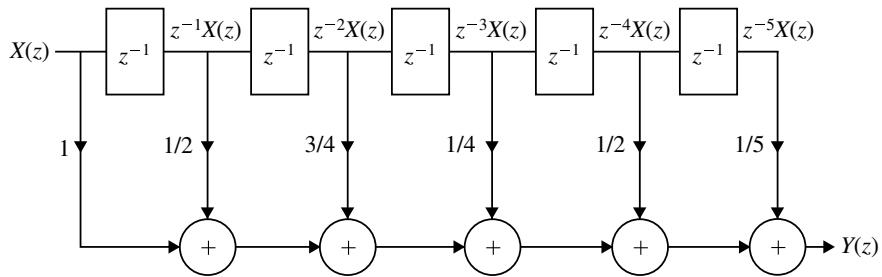
$$\begin{aligned}
 Y(z) &= b_{\frac{N-1}{2}}z^{-\left(\frac{N-1}{2}\right)}X(z) + b_0[X(z) + z^{-(N-1)}X(z)] + \dots \\
 &\quad + b_{\frac{N-3}{2}} \left[ z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-(N+2)}X(z) \right]
 \end{aligned}
 \tag{4.23}$$

Equation (4.23) is realized in Fig. 4.5.

### Example 4.1

Draw the direct form structure of the FIR system described by the following transfer function:

$$H(z) = 1 + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{1}{5}z^{-5}$$



**Fig. 4.6** Direct form (or) transversal realization of  $H(z)$  for Example 4.1

### Solution

$$Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{4}z^{-3}X(z) + \frac{1}{2}z^{-4}X(z) + \frac{1}{5}z^{-5}X(z)$$

The direct form or transversal realization of Example 4.1 is shown in Fig. 4.6.

### Example 4.2

Realize the FIR system in (1) Direct form (2) Cascade form.

$$H(z) = \left(1 + \frac{1}{2}z^{-1}\right) \left(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)$$

(Anna University, December, 2005)

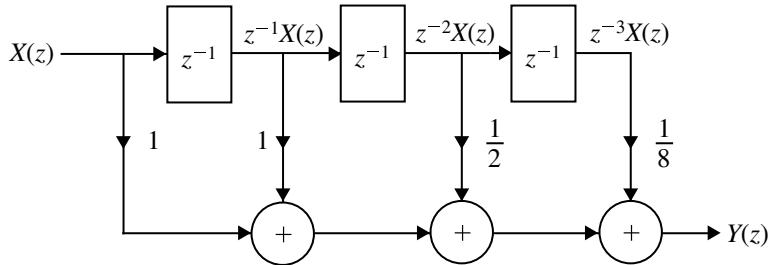
**Solution** Given

$$H(z) = \left(1 + \frac{1}{2}z^{-1}\right) \left(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)$$

(1) Direct form realization

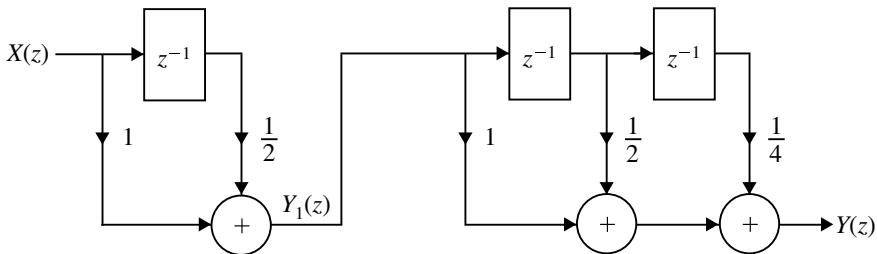
$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} \\ &= 1 + z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{8}z^{-3} \\ Y(z) &= X(z) + z^{-1}X(z) + \frac{1}{2}z^{-2}X(z) + \frac{1}{8}z^{-3}X(z) \end{aligned}$$

The direct form FIR filter for Example 4.2 is shown in Fig. 4.7.



**Fig. 4.7** Direct form realization of FIR filter for Example 4.2

Direct form realization for Example 4.2 is shown in Fig. 4.7.



**Fig. 4.8** Cascade form realization of FIR filter for Example 4.2

## (2) Cascade form realization

$$\begin{aligned} H(z) &= \left(1 + \frac{1}{2}z^{-1}\right) \left(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right) \\ &= H_1(z) \cdot H_2(z) \end{aligned}$$

where

$$\begin{aligned} H_1(z) &= 1 + \frac{1}{2}z^{-1} \\ H_2(z) &= 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} \\ h_1(n) &= \left\{1, \frac{1}{2}\right\} \\ h_2(n) &= \left\{1, \frac{1}{2}, \frac{1}{4}\right\} \end{aligned}$$

Here  $h_1(n)$  and  $h_2(n)$  do not satisfy the condition  $h(n) \neq h(\infty - 1 - \infty)$ . Therefore, both systems are realized in direct form. Cascade form realization of FIR filter for Example 4.2 is shown in Fig. 4.8.

**Example 4.3**

Obtain cascade realization of linear phase FIR filter having system function

$$H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right) \left(2 + \frac{1}{4}z^{-1} + 2z^{-2}\right)$$

using minimum number of multipliers.

(Anna University, April, 2005)

**Solution** Given

$$\begin{aligned} H(z) &= \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right) \left(2 + \frac{1}{4}z^{-1} + 2z^{-2}\right) \\ &= H_1(z)H_2(z) \end{aligned}$$

where

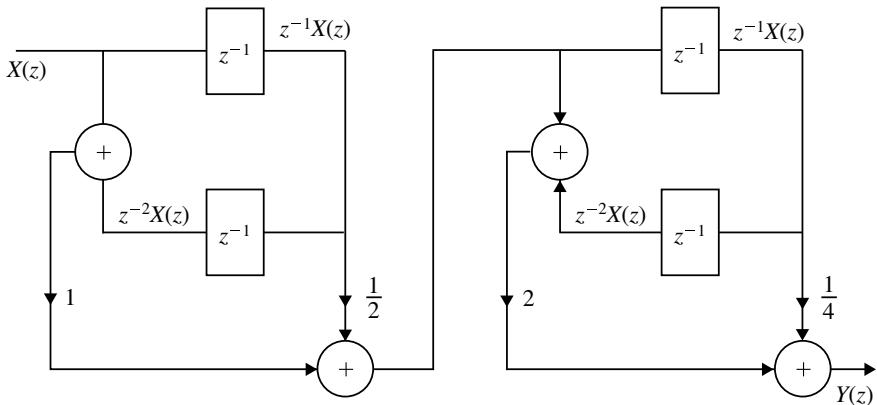
$$\begin{aligned} H_1(z) &= 1 + \frac{1}{2}z^{-1} + z^{-2}, \\ H_2(z) &= 2 + \frac{1}{4}z^{-1} - 2z^{-2} \\ h_1(n) &= \left\{1 + \frac{1}{2}, 1\right\} \text{ and } N = 3, \\ h_2(n) &= \left\{2 + \frac{1}{4}, 2\right\} \text{ and } N = 3 \end{aligned}$$

Here  $h_1(n)$  and  $h_2(n)$  satisfy the condition  $h(n) = h(N - 1 - n)$ .

Therefore,

$$H_1(z) = (1 + z^{-2}) + \frac{1}{2}z^{-1}$$

$$H_2(z) = 2(1 + z^{-2}) + \frac{1}{4}z^{-1}$$



**Fig. 4.9** Cascade connection of linear phase FIR filter of Example 4.3

Cascade form realization of FIR filter for Example 4.3 is shown in Fig. 4.9.

### Example 4.4

Realize the following system with minimum number of multipliers

$$(i) \quad H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$$

$$(ii) \quad H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$$

$$(iii) \quad H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right) \left(1 + \frac{1}{4}z^{-1} + z^{-2}\right)$$

### Solution

(i) Given

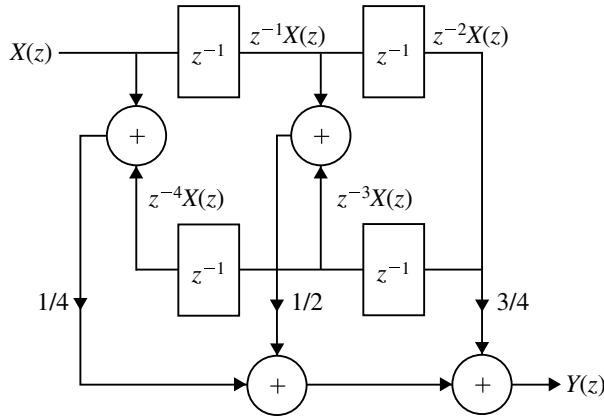
$$H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$$

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \dots$$

Comparing these two equations, we get the following impulse response:

$$h(n) = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4} \right\} \quad \text{and} \quad N = 5$$

Here  $h(n)$  satisfies the condition  $h(n) = h(N - 1 - n)$



**Fig. 4.10** Linear phase realization of  $H(z)$  for Example 4.4(i)

$$\begin{aligned}\therefore H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4} \\ &= \frac{Y(z)}{X(z)} = \frac{1}{4}(1 + z^{-4}) + \frac{1}{2}(z^{-1} + z^{-3}) + \frac{3}{4}z^{-2} \\ Y(z) &= \frac{1}{4}(X(z) + z^{-4}X(z)) + \frac{1}{2}(z^{-1}X(z) + z^{-3}X(z)) + \frac{3}{4}z^{-2}X(z)\end{aligned}$$

$Y(z)$  is realized as shown in Fig. 4.10.

(ii) Given

$$H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$$

Impulse response

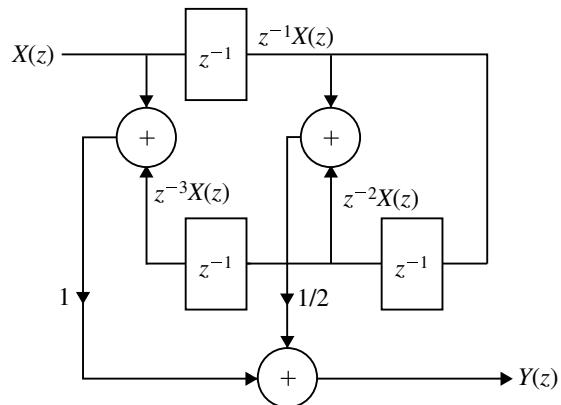
$$h(n) = \left\{ 1, \frac{1}{2}, \frac{1}{2}, 1 \right\} \quad \text{and} \quad N = 4$$

Here  $h(n)$  satisfies the condition

$$\begin{aligned}h(n) &= h(N - 1 - n) \\ \therefore \frac{Y(z)}{X(z)} &= [1 + z^{-3}] + \frac{1}{2}[z^{-1} + z^{-2}] \\ Y(z) &= [X(z) + z^{-3}X(z)] + \frac{1}{2}[z^{-1}X(z) + z^{-2}X(z)]\end{aligned}$$

$Y(z)$  is realized as shown in Fig. 4.11.

**Fig. 4.11** Linear phase realization of  $H(z)$  for Example 4.4(ii)



(iii) Given

$$\begin{aligned} H(z) &= \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right) \left(1 + \frac{1}{4}z^{-1} + z^{-2}\right) \\ &= H_1(z)H_2(z) \end{aligned}$$

where

$$\begin{aligned} H_1(z) &= 1 + \frac{1}{2}z^{-1} + z^{-2} = (1 + z^{-2}) + \frac{1}{2}z^{-1} \\ H_2(z) &= 1 + \frac{1}{4}z^{-1} + z^{-2} = (1 + z^{-2}) + \frac{1}{4}z^{-1} \end{aligned}$$

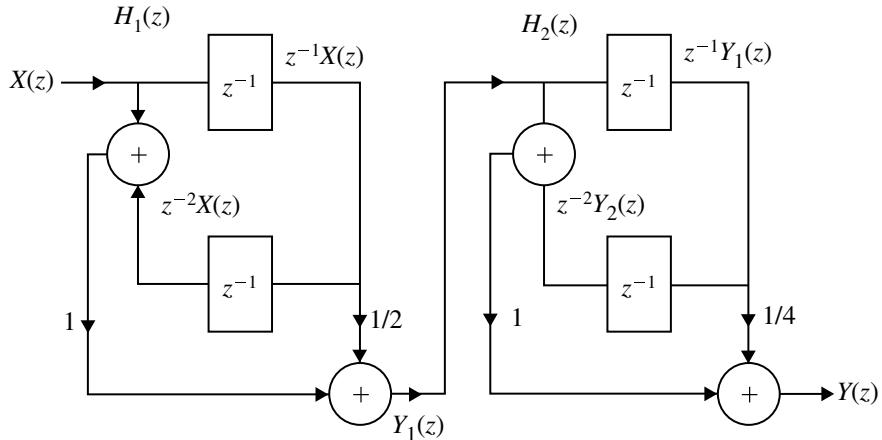
Consider

$$\begin{aligned} H_1(z) &= \frac{Y_1(z)}{X(z)} = (1 + z^{-2}) + \left(\frac{1}{2}z^{-1}\right) \\ Y_1(z) &= [X(z) + z^{-2}X(z)] + \frac{1}{2}z^{-1}X(z) \end{aligned}$$

Consider

$$\begin{aligned} H_2(z) &= \frac{Y(z)}{Y_1(z)} = (1 + z^{-2}) + \left(\frac{1}{4}z^{-1}\right) \\ Y(z) &= [Y_1(z) + z^{-2}Y_1(z)] + \frac{1}{4}z^{-1}Y_1(z) \end{aligned}$$

$Y(z)$  is realized as shown in Fig. 4.12.



**Fig. 4.12** Linear phase realization of  $H(z)$  for Example 4.4(iii)

#### 4.3.4 Lattice Structure of an FIR Filter

Let us consider an FIR filter with system function

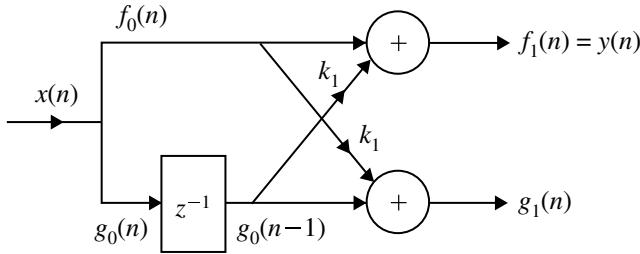
$$\begin{aligned}
 H(z) &= A_m(z) = 1 + \sum_{k=1}^m \alpha_m(k)z^{-k}, \quad m \geq 1 \\
 \frac{Y(z)}{X(z)} &= 1 + \sum_{k=1}^m \alpha_m(k)z^{-k} \\
 Y(z) &= X(z) + \sum_{k=1}^m \alpha_m(k)z^{-k}X(z)
 \end{aligned}$$

Taking inverse  $z$ -transform on both sides we get

$$y(n) = x(n) + \sum_{k=1}^m \alpha_m(k)x(n-k) \quad (4.24)$$

Interchanging the role of input and output in Eq. (4.24) we get

$$x(n) = y(n) + \sum_{k=1}^m \alpha_m(k)y(n-k) \quad (4.25)$$



**Fig. 4.13** Single stage all-zero lattice structure

We find that Eq. (4.25) describes an IIR system having the system function  $H(z) = \frac{1}{A_m(z)}$ , while the system described by the difference equation in Eq. (4.24) represents an FIR system with system function  $H(z) = A_m(z)$ .

Based on this, we use lattice structure system described in Chap. 3 to obtain a lattice structure for an all-zero FIR system by interchanging the role of input and output.

For an all-pole filter the input  $x(n) = f_N(n)$  and the output  $y(n) = f_0(n)$ . For an all-zero FIR filter of order  $M - 1$  the input  $x(n) = f_0(n)$  and the output  $y(n) = f_{M-1}(n)$ .

For  $m = 1$  the Eq. (4.24) reduces to

$$y(n) = x(n) + \alpha_1 x(n - 1) \quad (4.26)$$

This output can also be obtained from a single-stage lattice filter shown in Fig. 4.13 from which we have

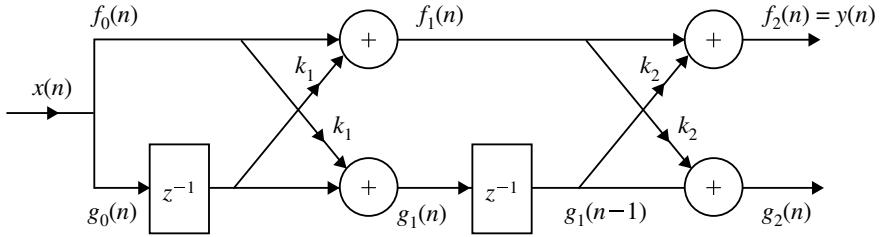
$$\begin{aligned} x(n) &= f_0(n) = g_0(n) \\ y(n) &= f_1(n) = f_0(n) + k_1 g_0(n - 1) \\ &= x(n) + k_1 x(n - 1) \end{aligned} \quad (4.27)$$

and

$$\begin{aligned} g_1(n) &= k_1 f_0(n) + g_0(n - 1) \\ &= k_1 x(n) + x(n - 1) \end{aligned} \quad (4.28)$$

Comparing Eq. (4.26) with Eq. (4.27) we get

$$\begin{aligned} \alpha_1(0) &= 1 \\ \alpha_1(1) &= k_1 \end{aligned}$$



**Fig. 4.14** Two stage all-zero lattice structure

Now let us consider an FIR filter for which  $m = 2$ .

$$y(n) = x(n) + \alpha_2(1)x(n-1) + \alpha_2(2)x(n-2) \quad (4.29)$$

By cascading two lattice stage as shown in Fig. 4.14 it is possible to obtain the output  $y(n)$ .

From Fig. 4.14 the output from second stage is obtained as,

$$\begin{aligned} y(n) &= f_2(n) = f_1(n) + k_2 g_1(n-1) \\ g_2(n) &= k_2 f_1(n) + g_1(n-1) \end{aligned} \quad (4.30)$$

Substituting for  $f_1(n)$  and  $g_1(n-1)$  from Eq. (4.28) in Eq. (4.30), we get

$$\begin{aligned} y(n) &= f_2(n) = x(n) + k_1 x(n-1) + k_2 [k_1 x(n-1) + x(n-2)] \\ &= x(n) + k_1(1+k_2)x(n-1) + k_2 x(n-2) \end{aligned} \quad (4.31)$$

Equation (4.31) is identical to Eq. (4.29) from which we have

$$\begin{aligned} \alpha_2(0) &= 1, \\ \alpha_2(2) &= k_2, \\ \alpha_2(1) &= k_1(1+k_2) \\ &= \alpha_1(1)[1+\alpha_2(2)] \end{aligned} \quad (4.32)$$

Similarly

$$g_2(n) = \alpha_2 x(n) + k_1(1+k_2)x(n-1) + x(n-2) \quad (4.33)$$

From Eqs. (4.31) and (4.33), we observe that the filter coefficients for the lattice filter that produces  $f_2(n)$  are  $\{1, \alpha_2(1), \alpha_2(2)\}$  while the coefficients for filter with output  $g_2(n)$  are  $\{\alpha_2(2), \alpha_2(1), 1\}$ . We also note that these two sets of coefficients are in reverse order.

For a  $m - 1$  stage filter

$$\begin{aligned} f_0(n) &= g_0(n) \\ f_m(n) &= f_{m-1}(n) + k_m g_{m-1}(n-1) \\ g_m(n) &= k_m f_{m-1}(n) + g_{m-1}(n-1), \quad m = 1, 2, \dots, m-1 \end{aligned} \quad (4.34)$$

The output of  $m - 1$  stage filter

$$y(n) = f_{m-1}(n)$$

#### 4.3.4.1 Conversion of Lattice Coefficients to Direct Form Filter Coefficients

For  $m = 3$  the Eq. (4.24) can be written as

$$y(n) = x(n) + \alpha_3(1)x(n-1) + \alpha_3(2)x(n-2) + \alpha_3(3)x(n-3) \quad (4.35)$$

We have

$$\begin{aligned} y(n) &= f_3(n) = f_2(n) + k_3 g_2(n-1) \\ &= x(n) + \alpha_2(1)x(n-1) + \alpha_2(2)x(n-2) + k_3 \alpha_2(2)x(n-1) \\ &\quad + k_3 \alpha_2(1)x(n-2) + k_3 x(n-3) \\ &= x(n) + [\alpha_2(1) + k_3 \alpha_2(1)]x(n-1) + [\alpha_2(2) + k_3 \alpha_2(1)]x(n-2) \\ &\quad + k_3 x(n-3) \end{aligned} \quad (4.36)$$

Comparing Eqs. (4.35) and (4.36) we get

$$\begin{aligned} \alpha_3(0) &= 1 \\ \alpha_3(1) &= \alpha_2(1) + k_3 \alpha_2(1) \\ &= \alpha_2(1) + \alpha_3(3)\alpha_2(1) \\ \alpha_3(2) &= \alpha_2(2) + k_3 \alpha_2(1) \\ &= \alpha_2(2) + \alpha_3(3)\alpha_2(1) \\ \alpha_3(3) &= k_3 \end{aligned} \quad (4.37)$$

From Eqs. (4.32) and (4.37) for a general case we find that

$$\begin{aligned} \alpha_m(0) &= 1 \\ \alpha_m(m) &= k_m \\ \alpha_m(k) &= \alpha_{m-1}(k) + k_m(m)\alpha_{m-1}(m-k) \end{aligned} \quad (4.38)$$

Equation (4.38) can be used to convert the lattice filter coefficients to direct form FIR filter coefficients.

---

### Example 4.5

Consider an FIR lattice filter coefficients

$$k_1 = \frac{1}{3}, \quad k_2 = \frac{1}{4}, \quad k_3 = \frac{1}{2}$$

Determine the FIR filter coefficients for the direct form structure.

---

**Solution** From the given data and from Eq. (4.38) we find that

$$\begin{aligned}\alpha_3(0) &= 1 \\ \alpha_3(3) &= k_3 = \frac{1}{2} \\ \alpha_1(1) &= k_1 = \frac{1}{3} \\ \alpha_2(2) &= k_2 = \frac{1}{4}\end{aligned}$$

From Eq. (4.38) we write

$$\alpha_m(k) = \alpha_{m-1}(k) + k_m \alpha_{m-1}(m-k)$$

For  $m = 2$  and  $k = 1$

$$\begin{aligned}\alpha_2(1) &= \alpha_1(1) + k_2 \alpha_1(1) \\ &= \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{12} = \frac{4+1}{12} = \frac{5}{12}\end{aligned}$$

For  $m = 3$  and  $k = 1$

$$\begin{aligned}\alpha_3(1) &= \alpha_2(1) + \alpha_3(3)\alpha_2(2) \\ &= \frac{5}{12} + \frac{1}{2} \cdot \frac{1}{4} \\ &= \frac{5}{12} + \frac{1}{8} \\ &= \frac{40+12}{96} \\ &= \frac{52}{96} = \frac{13}{24}\end{aligned}$$

For  $m = 2$  and  $k = 2$

$$\begin{aligned}\alpha_3(2) &= \alpha_3(2) + \alpha_3(3)\alpha_2(1) \\ &= \frac{1}{4} + \frac{1}{2} \cdot \frac{5}{12} \\ &= \frac{1}{4} + \frac{5}{24} \\ &= \frac{6+5}{24} \\ &= \frac{11}{24}\end{aligned}$$

$\alpha_3(0) = 1, \quad \alpha_3(1) = \frac{13}{24}, \quad \alpha_3(2) = \frac{11}{2}, \quad \alpha_3(3) = \frac{1}{2}$
---

#### 4.3.4.2 Conversion of Direct Form FIR Filter Coefficients to Lattice Coefficients

For a three-stage direct form FIR filter

$$y(n) = x(n) + \alpha_3(1)x(n-1) + \alpha_3(2)x(n-2) + \alpha_3(3)x(n-3) \quad (4.39)$$

For a three-stage lattice structure

$$\begin{aligned}y(n) &= f_3(n) = f_2(n) + k_3 g_2(n-1) \\ g_3(n) &= k_3 f_2(n) + g_2(n-1) \\ y(n) &= f_3(n) = f_2(n) + k_3 [g_3(n) - k_3 f_2(n)] \\ y(n) &= f_3(n) = f_2(n)[1 - k_3^2] + k_3 g_3(n) \\ f_2(n)[1 - k_3^2] &= f_3(n) - k_3 g_3(n)\end{aligned}$$

$$\begin{aligned}f_2(n) &= \frac{f_3(n) - k_3 g_3(n)}{1 - k_3^2} \\ &= \left[ x(n) + \alpha_3(1)x(n-1) + \alpha_3(2)x(n-2) + \alpha_3(3)x(n-3) \right. \\ &\quad \left. - k_3^2 x(n) - \alpha_3(3)\alpha_3(2)x(n-1) - \alpha_3(3)\alpha_3(1)x(n-2) \right. \\ &\quad \left. - \alpha_3(3)x(n-3) \right] / 1 - \alpha_3^2(3) \\ &= x(n) + \frac{\alpha_3(1) - \alpha_3(2)\alpha_3(2)}{1 - \alpha_3^2(2)} x(n-1) + \frac{\alpha_3(2) - \alpha_3(3)\alpha_3(1)}{1 - \alpha_3^2(3)} x(n-2) \quad (4.40)\end{aligned}$$

Comparing Eqs. (4.39) and (4.40) we get

$$\begin{aligned}\alpha_2(0) &= 1 \\ \alpha_2(1) &= \frac{\alpha_3(1) - \alpha_3(3)\alpha_3(2)}{1 - \alpha_3^2(3)} \\ \alpha_2(2) &= \frac{\alpha_3(2) - \alpha_3(3)\alpha_3(1)}{1 - \alpha_3^2(3)}\end{aligned}$$

In general for a  $m$ -stage filter

$$\begin{aligned}\alpha_{m-1}(0) &= 1 \\ k_m &= \alpha_m(m) \\ \alpha_{m-1}(k) &= \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)}, \quad 1 \leq k \leq m-1\end{aligned}\tag{4.41}$$

### Example 4.6

An FIR filter is given by the difference equation

$$y(n) = x(n) + \frac{4}{3}x(n-1) + \frac{1}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Determine its lattice form.

### Solution

$$y(n) = x(n) + \frac{4}{3}x(n-1) + \frac{1}{2}x(n-2) + \frac{2}{3}x(n-3)$$

$$\alpha_3(0) = 1, \quad \alpha_3(1) = \frac{4}{3}, \quad \alpha_3(2) = \frac{1}{2}, \quad \alpha_3(3) = \frac{2}{3}$$

From Eq. (4.41) we get

$$\begin{aligned}\alpha_2(0) &= 1 \\ k_3 &= \alpha_3(3) = \frac{2}{3} \\ \alpha_{m-1}(k) &= \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)}, \quad 1 \leq k \leq 2\end{aligned}$$

For  $m = 3$  and  $k = 1$

$$\begin{aligned}\alpha_2(1) &= \frac{\alpha_3(1) - \alpha_3(3)\alpha_3(2)}{1 - \alpha_3^2(3)} \\ &= \frac{\frac{4}{3} - \frac{2}{3} \cdot \frac{1}{2}}{1 - (\frac{2}{3})^2} \\ &= \frac{\frac{4}{3} - \frac{2}{6}}{1 - \frac{4}{9}} \\ &= 1.8\end{aligned}$$

For  $m = 3$  and  $k = 2$

$$\begin{aligned}k_2 = \alpha_2(2) &= \frac{\alpha_3(2) - \alpha_3(3)\alpha_3(1)}{1 - \alpha_3^2(3)} \\ &= \frac{\frac{1}{2} - \frac{2}{3} \cdot \frac{4}{3}}{1 - (\frac{2}{3})^2} \\ &= \frac{\frac{1}{2} - \frac{8}{9}}{1 - \frac{4}{9}} \\ &= -0.7\end{aligned}$$

For  $m = 2$  and  $k = 1$

$$\begin{aligned}k_1 = \alpha_1(1) &= \frac{\alpha_2(1) - \alpha_2(2)\alpha_2(1)}{1 - \alpha_2^2(2)} \\ &= \frac{1.8 - (-0.7)(1.8)}{1 - (-0.7)^2} \\ &= \frac{3.06}{0.51} \\ &= 6\end{aligned}$$

$$\boxed{k_1 = 6, \quad k_2 = -0.7, \quad k_3 = 0.6667}$$

The lattice structure is shown in Fig. 4.15.

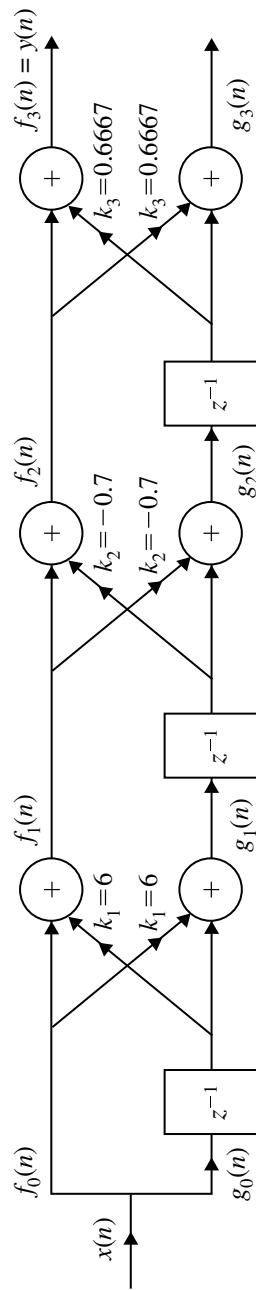


Fig. 4.15 Lattice structure for the Example 4.6

**Example 4.7a**

An FIR filter is given by the difference equation

$$y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Determine the lattice form.

(Anna University, May 2007)

**Solution** Given

$$\begin{aligned} y(n) &= 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3) \\ &= 2 \left[ x(n) + \frac{2}{5}x(n-1) + \frac{3}{4}x(n-2) + \frac{1}{3}x(n-3) \right] \\ &= k_0 \left[ 1 + \sum_{k=1}^3 \alpha_m(k)x(n-k) \right] \end{aligned}$$

where  $k_0 = 2$

$$\alpha_3(0) = 1, \quad \alpha_3(1) = \frac{2}{5}, \quad \alpha_3(2) = \frac{3}{4}, \quad \alpha_3(3) = \frac{1}{3}$$

From Eq. (4.24), we get

$$\begin{aligned} \alpha_2(0) &= 1 \\ k_3 &= \alpha_3(3) = \frac{1}{3} \\ \alpha_{m-1}(k) &= \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)}, \quad 1 \leq k \leq 2 \end{aligned}$$

For  $m = 3$  and  $k = 1$

$$\begin{aligned} \alpha_2(1) &= \frac{\alpha_3(1) - \alpha_3(3)\alpha_3(2)}{1 - \alpha_3^2(3)} \\ &= \frac{\frac{2}{5} - \frac{1}{3} \cdot \frac{3}{4}}{1 - (\frac{1}{3})^2} \\ \alpha_2(1) &= 0.16875 \end{aligned}$$

Lattice form realization of FIR filter for Example 4.7(a) is shown in Fig. 4.16.

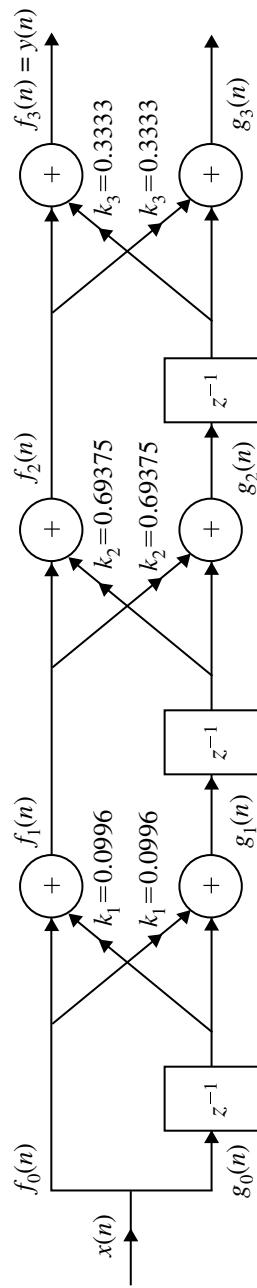


Fig. 4.16 Lattice Form for the Example 4.7(a)

For  $m = 3$  and  $k = 2$

$$\begin{aligned} k_2 &= \alpha_2(2) = \frac{\alpha_3(2) - \alpha_3(3)\alpha_3(1)}{1 - \alpha_3^2(3)} \\ &= \frac{\frac{3}{4} - \frac{1}{3} \cdot \frac{2}{5}}{1 - (\frac{1}{3})^2} \\ &= \frac{111}{160} \end{aligned}$$

$$k_2 = 0.69375$$

For  $m = 2$  and  $k = 1$

$$\begin{aligned} k_1 &= \alpha_1(1) = \frac{\alpha_2(1) - \alpha_2(2)\alpha_2(1)}{1 - \alpha_2^2(2)} \\ &= \frac{0.16875 - (0.69375)(0.16875)}{1 - (0.69375)^2} \end{aligned}$$

$$k_1 = 0.0996$$

Lattice form coefficients are,

$$k_1 = 0.0996, \quad k_2 = 0.69375, \quad k_3 = 0.3333.$$

### Example 4.7(b)

Given a three stage lattice filter with coefficients  $k_1 = \frac{1}{4}$  and  $k_2 = \frac{1}{4}, k_3 = \frac{1}{3}$ . Determine the FIR filter coefficients for the direct form structure?

(Anna University, May, 2007)

**Solution** From the given data and from Eq. (4.38), we can find that

$$\alpha_3(0) = 1, \quad \alpha_3(3) = k_3 = \frac{1}{3}$$

$$\alpha_1(1) = k_1 = \frac{1}{4}, \quad \alpha_2(2) = k_2 = \frac{1}{4}$$

We know that

$$\alpha_m(k) = \alpha_{m-1}(k) + k_m \alpha_{m-1}(m-k)$$

For  $m = 2$  and  $k = 1$

$$\begin{aligned}\alpha_2(1) &= \alpha_1(1) + k_2\alpha_1(1) \\ \alpha_2(1) &= \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{16} \\ &= \frac{4+1}{16} = \frac{5}{16}\end{aligned}$$

For  $m = 3$  and  $k = 1$

$$\begin{aligned}\alpha_3(1) &= \alpha_2(1) + \alpha_3(3)\alpha_2(2) \\ &= \frac{5}{16} + \frac{1}{3} \cdot \frac{1}{4} = \frac{5}{16} + \frac{1}{12} \\ &= \frac{60+16}{192} = \frac{76}{192} \\ &= 0.3958 \\ &= \frac{19}{48}\end{aligned}$$

For  $m = 3$  and  $k = 2$

$$\begin{aligned}\alpha_3(2) &= \alpha_2(2) + k_3\alpha_2(1) \\ &= \frac{1}{4} + \frac{1}{3} \cdot \frac{5}{16} \\ \alpha_3(2) &= \frac{1}{4} + \frac{5}{48} = \frac{12+5}{48} = \frac{17}{48}\end{aligned}$$

Direct form coefficients are,

$\alpha_3(0) = 1, \quad \alpha_3(1) = \frac{19}{48}, \quad \alpha_3(2) = \frac{17}{48}, \quad \alpha_3(3) = \frac{1}{3}$

## 4.4 FIR Filters

The filters designed by using finite number of samples of impulse response are called FIR filters. These finite number of samples are obtained from the infinite duration desired impulse response  $h_d(n)$ . Here  $h_d(n)$  is the inverse Fourier transform of  $H_d(\omega)$ , where  $H_d(\omega)$  is the ideal (desired) frequency response. The various methods of designing FIR filters differ only in the method of determining the samples of  $h(n)$  from the samples of  $h_d(n)$ . They are discussed ahead.

#### 4.4.1 Characteristics of FIR Filters with Linear Phase

Let  $h(n)$  be a causal finite duration sequence defined over the interval  $0 \leq n \leq N - 1$  and the samples of  $h(n)$  be real. The Fourier transform of  $h(n)$  is given by

$$H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \quad (4.42)$$

which is periodic with period  $2\pi$ . Therefore

$$H(\omega) = H(\omega + 2\pi m), \quad m = 0, \pm 1, \pm 2, \dots \quad (4.43)$$

with the constraint that  $h(n)$  is real and  $H(\omega)$  is complex.

$$H(\omega) = \pm|H(\omega)|e^{j\theta(\omega)}$$

The operators “ $\pm$ ” represents real part of  $H(\omega)$  taking on both +ve and -ve values. When  $h(n)$  is real, then  $|H(\omega)|$  is symmetric function and  $\angle H(\omega)$  or  $\angle\theta(\omega)$  is anti-symmetric function, i.e.,

$$\begin{aligned} |H(\omega)| &= |H(-\omega)| \\ \angle\theta(\omega) &= -\angle\theta(-\omega) \end{aligned} \quad (4.44)$$

For many practical FIR filters, exact linearity of phase is a desired goal.

Let us assume that the phase of  $H(\omega)$  is a linear function of  $\omega$ , i.e.,

$$\begin{aligned} \theta(\omega) &\propto \omega \\ \theta(\omega) &\propto -\alpha\omega, \quad -\pi \leq \omega \leq \pi \end{aligned} \quad (4.45)$$

where  $\alpha$  is a constant phase delay in samples. From Eqs. (4.43) and (4.45) we get

$$H(\omega) = \pm|H(\omega)|e^{-j\alpha\omega}$$

From Eq. (4.24) we get

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \\ \sum_{n=0}^{N-1} h(n)e^{-j\omega n} &= \pm|H(\omega)|e^{-j\alpha\omega} \\ \sum_{n=0}^{N-1} h(n)[\cos \omega n - j \sin \omega n] &= \pm|H(\omega)|[\cos \alpha\omega - j \sin \alpha\omega] \end{aligned} \quad (4.46)$$

On equating the real and imaginary parts we get,

$$\pm |H(\omega)| \cos \alpha\omega = \sum_{n=0}^{N-1} h(n) \cos \omega n \quad (4.47)$$

$$\pm |H(\omega)| \sin \alpha\omega = \sum_{n=0}^{N-1} h(n) \sin \omega n \quad (4.48)$$

Dividing the Eq. (4.20) by Eq. (4.48), we get

$$\frac{\sin \alpha\omega}{\cos \alpha\omega} = \frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} \quad (4.49)$$

$$\begin{aligned} \sin \alpha\omega \sum_{n=0}^{N-1} h(n) \cos \omega n &= \cos \alpha\omega \sum_{n=0}^{N-1} h(n) \sin \omega n \\ \sum_{n=0}^{N-1} h(n) [\sin \alpha\omega \cos \omega n - \cos \alpha\omega \sin \omega n] &= 0 \end{aligned} \quad (4.50)$$

$$\sum_{n=0}^{N-1} h(n) \sin(\alpha - n)\omega = 0 \quad (4.51)$$

The solution of Eq. (4.51) exists when

$$\alpha = \frac{N-1}{2} \quad \text{and} \quad h(n) = h(N-1-n), \quad 0 \leq n \leq N-1 \quad (4.52)$$

From the condition,  $\alpha = \frac{N-1}{2}$  we can say that for every value of  $N$  there is only one value of phase delay  $\alpha$  for which linear phase can be obtained easily.

From the condition,  $h(n) = h(N-1-n)$  we can say that for this value of  $\alpha$ , the  $h(n)$  has a special kind of symmetry. The symmetry impulse response is shown in Fig. 4.17.

The definition of linear phase filter  $\theta(\omega) = -\omega\alpha$  requires to have both constant group delay and constant phase delay.

If only constant group delay is required an another type of linear phase filter exists, in which, the phase of  $H(\omega)$  is a piece-wise linear function of  $\omega$ . For this case  $H(\omega)$  can be expressed as

$$H(\omega) = \pm |H(\omega)| e^{\pm j(\beta - \omega\alpha)} \quad (4.53)$$

Solution of Eq. (4.53) exists only if  $\alpha = \frac{N-1}{\alpha}$ ,  $\beta = \pm \frac{\pi}{2}$  and

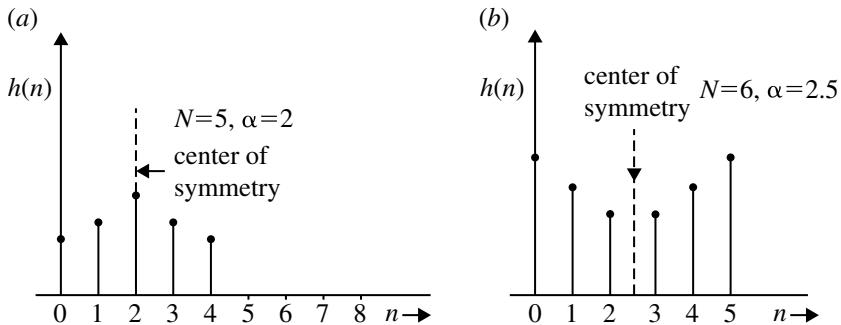


Fig. 4.17 Symmetry impulse response for  $N = \text{odd}$  and  $N = \text{even}$

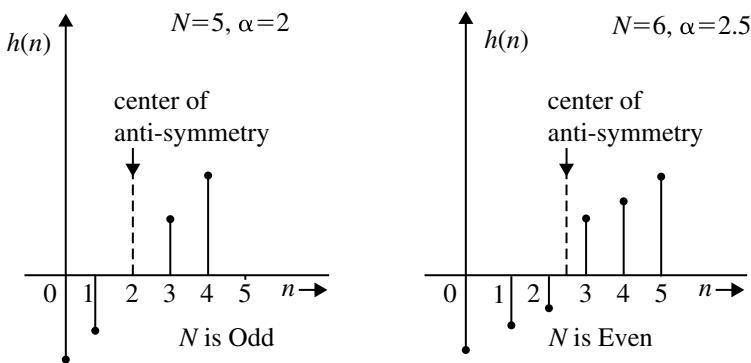


Fig. 4.18 Anti-symmetry impulse response for  $N = \text{odd}$  and  $N = \text{even}$

$$h(n) = -h(N-1-n) \quad \text{for } 0 \leq n \leq -1 \quad (4.54)$$

The anti-symmetric impulse response is shown in Fig. 4.18.

#### 4.4.1.1 Phase Delay and Group Delay

The phase delay and group delay are the two important parameters that characterize the frequency response characteristics of a digital filter. Suppose the system is excited by the following input which is sinusoidal of frequency  $\omega_0$  and amplitude  $A$ .

$$x(n) = A \cos(\omega_0 n + \phi)$$

For the LTI system, the output may be expressed by the following equation.

$$\begin{aligned} y(n) &= A|H(e^{j\omega_0})| \cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \Phi\right) \\ &= A|H(e^{j\omega_0})| \cos(\omega_0(n - \tau_p(\omega_0)) + \Phi) \end{aligned}$$

where

$$\boxed{\tau_p(\omega_0) = \frac{\theta(\omega_0)}{\omega_0}}$$

$\tau_p$  is called the phase delay. Now the output is a time delayed version of the input  $x(n)$ .

When the input signal contains many sinusoidal components with different frequencies which are different from harmonics, each component will go through different phase delays when they are passed through a LTI discrete system and the signal delay now is determined by what is named as group delay denoted by the letter  $\tau_g(\omega)$ . The group delay is defined as,

$$\boxed{\tau_g(\omega) = \frac{d\theta(\omega)}{d\omega}}$$

### Example 4.8

Determine the frequency response of FIR filter defined by  $y(n) = 0.25x(n) + x(n - 1) + 0.25x(n - 2)$ . Calculate the phase delay and group delay.

*(Anna University, December, 2005)*

**Solution** Given

$$y(n) = 0.25x(n) + x(n - 1) + 0.25x(n - 2)$$

Taking Fourier transform on both sides

$$Y(e^{j\omega}) = 0.25X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) + 0.25e^{-j2\omega}X(e^{j\omega})$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 0.25 + e^{-j\omega} + 0.25e^{-2j\omega} \\ &= e^{-j\omega}(0.25e^{j\omega} + 1 + 0.25e^{-j\omega}) \\ &= e^{-j\omega}(1 + 0.5 \cos \omega) \\ H(e^{j\omega}) &= e^{-j\omega}\bar{H}(e^{j\omega}) \end{aligned}$$

We know that

$$H(e^{j\omega}) = e^{j\theta(\omega)} \tilde{H}(e^{j\omega})$$

Comparing these two equations, we get

$$\theta(\omega) = -\omega$$

The phase delay

$$\boxed{\tau_p = \frac{-\theta(\omega)}{\omega} = \frac{\omega}{\omega} = 1}$$

The group delay

$$\boxed{-\frac{d\theta(\omega)}{d\omega} = -\frac{d}{d\omega}(-\omega) = 1}$$

#### 4.4.2 Frequency Response of Linear Phase FIR Filter

Depending on the value of  $N$  and the type of symmetry of the filter impulse response there are four possible types of linear phase. The following are the four cases of impulse response for the linear phase FIR filters.

- Case I. Symmetric impulse response when  $N$  is odd.
- Case II. Symmetric impulse response when  $N$  is even.
- Case III. Anti-symmetric impulse response when  $N$  is odd.
- Case IV. Anti-symmetric impulse response when  $N$  is even.

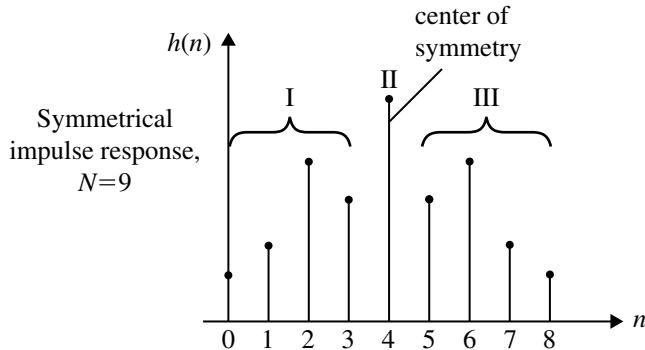
##### 4.4.2.1 Symmetric Impulse Response of the Linear Phase FIR Filters When $N$ Is Odd

The Fourier transform of  $h(n)$  is

$$H(\omega) = H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

Since the impulse response of FIR filter has only  $N$  samples, the limits of summation can be changed to  $n = 0$  to  $N - 1$

$$H(\omega) = \sum_{n=-\infty}^{N-1} h(n)e^{-j\omega n}$$



**Fig. 4.19** Symmetry impulse response when  $N = \text{odd}$

Let impulse response  $h(n)$  be symmetric and it has odd number of samples. Let  $N = 9$ . The symmetrical impulse response is shown in Fig. 4.19.

When  $N$  is an odd number, the symmetrical impulse response will have the center of symmetry at  $n = (N - 1)/2$ . Hence,  $H(e^{j\omega})$  can be expressed as

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left[\frac{N-1}{2}\right]e^{-j\omega[\frac{N-1}{2}]} + \sum_{n=(N+1)/2}^{N-1} h(n)e^{-j\omega n} \quad (4.55)$$

Let  $m = N - 1 - n$ . Therefore,  $n = N - 1 - m$ .

$$\text{When } n = \frac{N+1}{2}; \quad m = N - 1 - \frac{N+1}{2} = \frac{N-3}{2}$$

$$\text{When } n = N - 1; \quad m = N - 1 - (N - 1) = 0$$

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left[\frac{N-1}{2}\right]e^{-j\omega[\frac{N-1}{2}]} + \sum_{m=0}^{\frac{N-3}{2}} h(N - 1 - m)e^{-j\omega(N-1-m)}$$

Replacing  $m$  by  $n$  we get

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left[\frac{N-1}{2}\right]e^{-j\omega[\frac{N-1}{2}]} + \sum_{n=0}^{\frac{N-3}{2}} h(N - 1 - n)e^{-j\omega(N-1-n)}$$

For symmetrical impulse response  $h(N - 1 - n) = h(n)$ . Therefore

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left[\frac{N-1}{2}\right]e^{-j\omega[\frac{N-1}{2}]} + \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{(-j\omega(N-1)+j\omega n)} \\
H(\omega) &= e^{-j\omega[\frac{N-1}{2}]} \left[ h\left[\frac{N-1}{2}\right] + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[ e^{j\omega[\frac{N-1}{2}]-j\omega n} + e^{-j\omega[\frac{N-1}{2}]+j\omega n} \right] \right] \\
&= e^{-j\omega[\frac{N-1}{2}]} \left[ h\left[\frac{N-1}{2}\right] + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[ e^{j\omega[\frac{N-1}{2}-n]} + e^{-j\omega[\frac{N-1}{2}-n]} \right] \right] \\
&= e^{-j\omega[\frac{N-1}{2}]} \left[ h\left[\frac{N-1}{2}\right] + \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos \omega \left( \frac{N-1}{2} - n \right) \right] \\
\text{Let } k &= \frac{N-1}{2} - n; \quad n = \frac{N-1}{2} - k \\
\text{When } n &= 0, k = \frac{N-1}{2} \text{ and} \\
\text{When } n &= \frac{N-3}{2}, k = 1 \\
&= e^{-j\omega[\frac{N-1}{2}]} \left[ h\left[\frac{N-1}{2}\right] + \sum_{k=1}^{\frac{N-1}{2}} 2 \left( \frac{N-1}{2} - k \right) \cos \omega k \right]
\end{aligned}$$

Replacing  $k$  by  $n$  we get

$$H(\omega) = e^{-j\omega[\frac{N-1}{2}]} \left[ h\left[\frac{N-1}{2}\right] + 2 \sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \cos \omega n \right] \quad (4.56)$$

Equation (4.56) is the frequency response of linear phase FIR filter when impulse response is symmetric and  $N$  is odd.

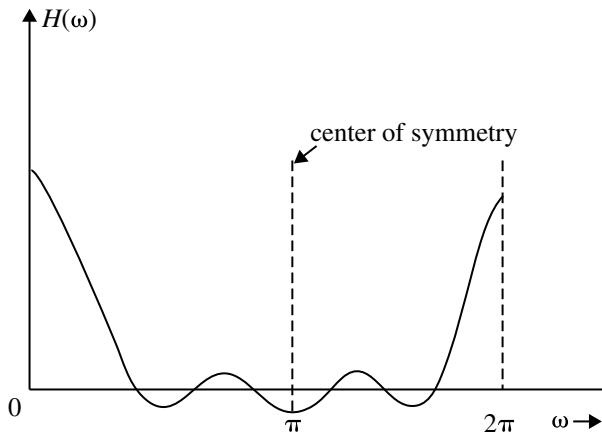
Magnitude function of  $H(\omega)$  is given by

$$|H(\omega)| = h\left[\frac{N-1}{2}\right] + 2 \sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \cos \omega n \quad (4.57)$$

Phase function of  $H(\omega)$  is given by

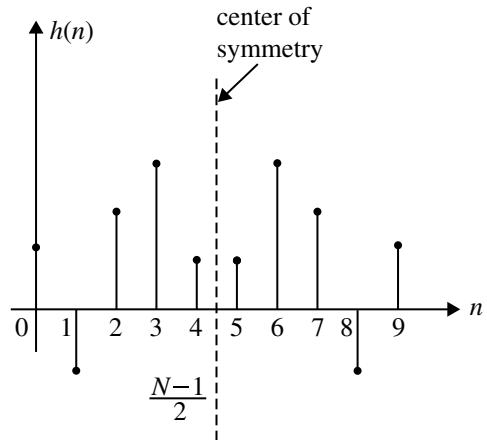
$$\angle H(\omega) = -\omega \left[ \frac{N-1}{2} \right] = -\omega \alpha \quad (4.58)$$

where  $\alpha = (N-1)/2$ . The magnitude response of  $H(\omega)$  is shown in Fig. 4.20.



**Fig. 4.20** Magnitude response of  $H(\omega)$

**Fig. 4.21** Symmetry impulse response when  $N = \text{even}$



The magnitude response of  $H(\omega)$  is symmetric with  $\omega = \pi$  when the impulse response is symmetric and  $N$  is odd.

#### 4.4.2.2 Symmetric Impulse Response of the Linear Phase FIR Filters When $N$ Is Even

Let  $h(n)$  be symmetric and impulse response, for  $N = 10$  is shown in Fig. 4.21.

The Fourier transform of  $h(n)$  is

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n} \end{aligned} \quad (4.59)$$

Let  $m = N - 1 - n \implies n = N - 1 - m$

$$\text{When } n = \frac{N}{2}; \implies m = N - 1 - \frac{N}{2} = \frac{N}{2} - 1$$

$$\text{When } n = N - 1; \implies m = N - 1 - N + 1 = 0$$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m)e^{-j\omega(N-1-m)}$$

replace  $m$  by  $n$  and  $h(n) = h(N-1-m)$

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega(N-1)+j\omega n} \\ &= e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=0}^{\frac{N}{2}-1} h(n) \left( e^{j\omega(\frac{N-1}{2})-j\omega n} + e^{-j\omega(\frac{N-1}{2})+j\omega n} \right) \right] \\ &= e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=0}^{\frac{N}{2}-1} h(n) \left( e^{j\omega(\frac{N-1}{2}-n)} + e^{-j\omega(\frac{N-1}{2}-n)} \right) \right] \\ &= e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cos \left( \frac{N-1}{2} - n \right) \omega \right] \quad \left( \because \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \right) \\ &= e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cos \left( \frac{N}{2} - n - \frac{1}{2} \right) \omega \right] \end{aligned}$$

$$\text{Let } k = \frac{N}{2} - n \implies n = \frac{N}{2} - k$$

$$\text{When } n = 0; \implies k = \frac{N}{2}$$

$$\text{When } n = \frac{N}{2} - 1; \implies k = 1$$

$$H(\omega) = e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - k\right) \cos \omega \left(k - \frac{1}{2}\right) \right]$$

Replacing  $k$  by  $n$  we get

$$H(\omega) = e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \cos \omega \left(n - \frac{1}{2}\right) \right] \quad (4.60)$$

Equation (4.60) is the frequency response of linear phase FIR filter when impulse response is symmetric and  $N$  is even.

Magnitude function of  $H(\omega)$  is given by

$$|H(\omega)| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right) \cos \left(n - \frac{1}{2}\right) \omega \quad (4.61)$$

Phase function of  $H(\omega)$  is given by

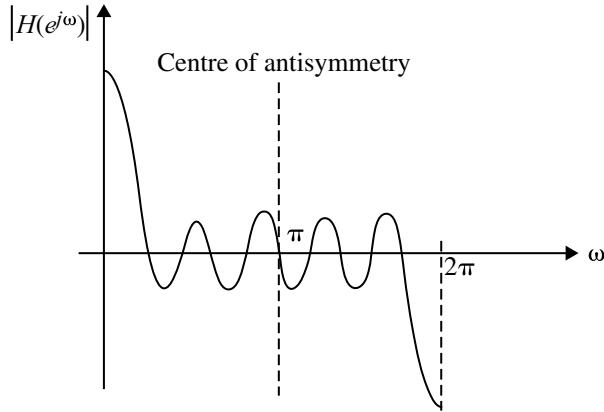
$$\angle|H(e^{j\omega})| = -\omega \left[ \frac{N-1}{2} \right] = -\omega\alpha \quad \text{where } \alpha = \frac{N-1}{2}$$

The magnitude function of  $H(\omega)$  is anti-symmetric with  $\omega = \pi$ , when impulse response is symmetric and  $N$  is even is shown in Fig. 4.22.

#### 4.4.2.3 Frequency Response of Linear Phase FIR Filters When Impulse Response Is Anti-symmetric When $N$ Is Odd

The Fourier transform of  $h(n)$  is,

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$



**Fig. 4.22** Magnitude function of  $H(\omega)$

Since the impulse response has only  $N$  samples, the limit of summation can be changed from  $N = 0$  to  $N - 1$ .

$$\therefore H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

The impulse response is anti-symmetric at  $n = (\frac{N-1}{2})$ , and  $h(\frac{N-1}{2}) = 0$ . Hence  $H(\omega)$  can be expressed as,

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega(\frac{N-1}{2})} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n} \\ &= \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n} \end{aligned} \quad (4.62)$$

$$\text{Let } m = N - 1 - n \implies n = N - 1 - m$$

$$\text{When } n = \frac{N+1}{2}; \implies m = N - 1 - \left(\frac{N+1}{2}\right) = \frac{N-3}{2}$$

$$\text{When } n = N - 1; \implies m = N - 1 - (N - 1) = 0$$

Using the above relations, Eq. (4.62) can be written as

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m)e^{-j\omega(N-1-m)}$$

On replacing  $m$  by  $n$  we get

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

For anti-symmetric impulse response,  $h(N-1-n) = -h(n)$ . Therefore,

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} (-h(n))e^{-j\omega(-n)-j\omega(N-1)} \\ H(\omega) &= \left[ \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[ e^{-j\omega n+j\omega(\frac{N-1}{2})} - e^{-j\omega(-n)-j\omega(N-1)+j\omega(\frac{N-1}{2})} \right] \right] e^{-j\omega(\frac{N-1}{2})} \\ &= \left[ \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[ e^{j\omega(\frac{N-1}{2}-n)} - e^{-j\omega(\frac{N-1}{2}-n)} \right] \right] e^{-j\omega(\frac{N-1}{2})} \end{aligned}$$

because

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$H(\omega) = \left[ \sum_{n=0}^{\frac{N-3}{2}} h(n) 2j \sin \left[ \omega \left( \frac{N-1}{2} - n \right) \right] \right] e^{-j\omega(\frac{N-1}{2})}$$

The operator  $j$  can be written as  $e^{(j\pi)/2}$

$$\begin{aligned} \therefore H(\omega) &= \left[ \sum_{n=0}^{\frac{N-3}{2}} 2h(n) e^{\frac{j\pi}{2}} \sin \left[ \omega \left( \frac{N-1}{2} - n \right) \right] \right] e^{-j\omega(\frac{N-1}{2})} \\ &= \left[ \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \sin \left[ \omega \left( \frac{N-1}{2} - n \right) \right] \right] e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} \end{aligned}$$

$$\text{Let } k = \frac{N-1}{2} - n \implies n = \frac{N-1}{2} - k$$

$$\text{When } n = 0; \implies k = \frac{N-1}{2}$$

$$\text{When } n = \frac{N-3}{2}; \implies k = \frac{N-1}{2} - \frac{N-3}{2} = 1$$

$$H(\omega) = \left[ \sum_{k=0}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - k\right) \sin \omega k \right] e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))}$$

On replacing  $k$  by  $n$  we get,

$$H(\omega) = \left[ \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}\right) \sin \omega n \right] e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} \quad (4.63)$$

Equation (4.63) is the frequency response of linear phase FIR filter when impulse response is anti-symmetric and  $N$  is odd. The magnitude function is given by Eq. (4.64). Magnitude function

$$|H(\omega)| = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}\right) \sin \omega n \quad (4.64)$$

The phase function is given by Eq. (4.65)

$$\angle H(\omega) = \frac{\pi}{2} - \omega \left( \frac{N-1}{2} \right) = \beta - \alpha \omega \quad (4.65)$$

where  $\beta = \frac{\pi}{2}$  and  $\alpha = \frac{N-1}{2}$ .

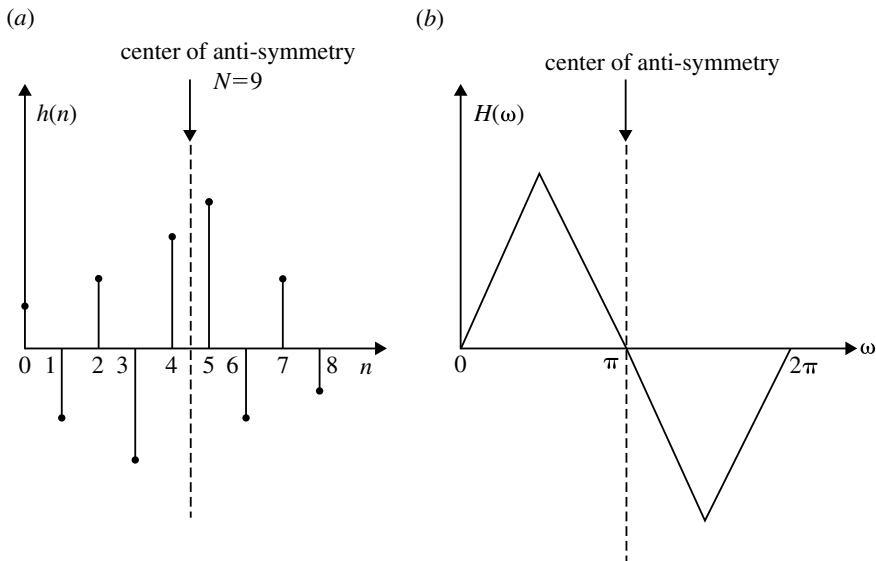
The sketch of symmetrical impulse response when  $N = 8$  and its corresponding magnitude response is shown in Fig. 4.23a, b respectively.

#### 4.4.2.4 Frequency Response of Linear Phase FIR Filters When Impulse Response Is Anti-symmetric When $N$ Is Even

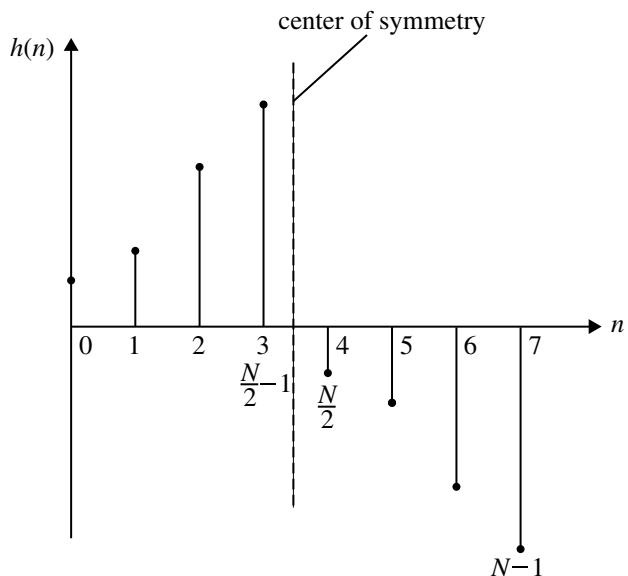
Let impulse response anti-symmetric with  $N$  is even is shown in Fig. 4.24.

The Fourier transform of  $h(n)$  for, impulse response of  $N$  samples

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$



**Fig. 4.23** **a** Anti-symmetric impulse response for  $N = 9$ . **b** Magnitude function of  $H(\omega)$



**Fig. 4.24** Anti-symmetric impulse response for  $N = \text{even}$

The impulse response is asymmetric in between  $n = \frac{N}{2} - 1$  and  $n = \frac{N}{2}$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n} \quad (4.66)$$

Let  $m = N - 1 - n \implies n = N - 1 - m$

When  $n = \frac{N}{2}; \implies m = N - 1 - \frac{N}{2} = \frac{N}{2} - 1$

When  $n = N - 1; \implies m = N - 1 - (N - 1) = 0$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N - 1 - m)e^{-j\omega(N-1-m)}$$

On replacing  $m$  by  $n$  we get

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N - 1 - n)e^{-j\omega(N-1-n)}$$

For anti-symmetric  $h(N - 1 - n) = -h(n)$

$$\begin{aligned} H(\omega) &= \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega n} - \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega(N-1)+j\omega n} \\ &= e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega(\frac{N-1}{2})-j\omega n} - \sum_{n=0}^{\frac{N}{2}-1} h(n)e^{-j\omega(\frac{N-1}{2})+j\omega n} \right] \\ &= e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[ e^{j\omega(\frac{N-1}{2}-n)} - e^{-j\omega(\frac{N-1}{2}-n)} \right] \right] \\ &= e^{-j\omega(\frac{N-1}{2})} \left[ \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[ 2j \sin \left( \frac{N-1}{2} - n \right) \omega \right] \right] \\ &= e^{-j\omega(\frac{N-1}{2})} e^{j\pi/2} \left[ \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \sin \left( \frac{N-1}{2} - n \right) \omega \right] \end{aligned}$$

$$= e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} \left[ \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \sin\left(\frac{N-1}{2} - n\right) \omega \right]$$

$$= e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} \left[ \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \sin\left(\frac{N}{2} - n - \frac{1}{2}\right) \omega \right]$$

Substitute  $k = \frac{N}{2} - n \implies n = \frac{N}{2} - k$

$$\text{When } n = 0; \implies k = \frac{N}{2}$$

$$\text{When } n = \frac{N}{2} - 1; \implies k = \frac{N}{2} - \frac{N}{2} + 1 = 1$$

$$H(\omega) = e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} \left[ \sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - k\right) \sin\left(k - \frac{1}{2}\right) \omega \right]$$

Replacing  $k$  by  $n$  we get,

$$H(\omega) = e^{j(\frac{\pi}{2} - \omega(\frac{N-1}{2}))} \left[ \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \sin\left(n - \frac{1}{2}\right) \omega \right] \quad (4.67)$$

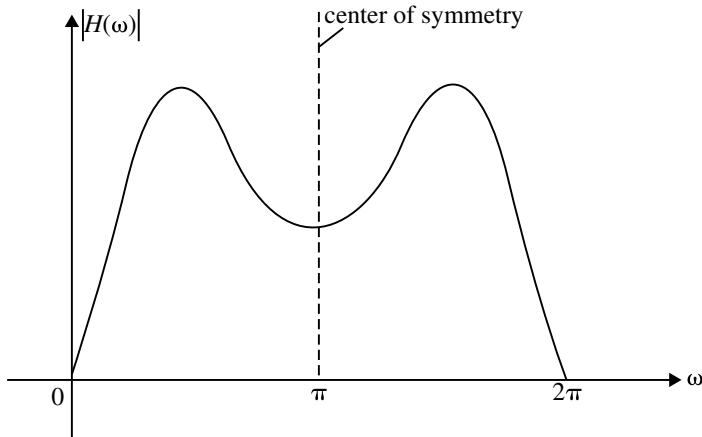
Equation (4.67) represents frequency response of linear phase FIR filter with impulse response as anti-symmetric when  $N$  is even. The magnitude response is given by Eq. (4.68) and is shown in Fig. 4.25.

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \sin\left(n - \frac{1}{2}\right) \omega \quad (4.68)$$

The phase response is given by

$$\begin{aligned} \angle H(e^{j\omega}) &= \frac{\pi}{2} - \omega \left( \frac{N-1}{2} \right) \\ \angle H(\omega) &= \beta - \omega\alpha \end{aligned} \quad (4.69)$$

where  $\alpha = \frac{N-1}{2}$  and  $\beta = \frac{\pi}{2}$ .



**Fig. 4.25** Magnitude function of  $H(e^{j\omega})$

## 4.5 Design Techniques for Linear Phase FIR Filters

Some of the well-known method of design techniques for linear phase FIR filters are:

1. Fourier series method.
2. Window method.
3. Frequency sampling method.

### 4.5.1 Fourier Series Method of FIR Filter Design

Fourier series analysis exists only for periodic function. That is, any periodic function can be expressed as a linear combination of complex exponentials. The frequency response of a digital filter is periodic, with period equal to the sampling frequency. Therefore, the desired frequency response of an FIR digital filter can be represented by Fourier series as

$$H_d(\omega)|_{\omega=\omega T} = H_d(\omega T) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega nT} \quad (4.70)$$

where the Fourier series coefficients  $h_d(n)$  are the desired impulse response and are given by

$$h_d(n) = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} H_d(\omega T) e^{j\omega nT} d\omega \quad (4.71)$$

where

$f_s$  = sampling frequency in Hz.

$\omega_s = 2\pi f_s$  = sampling frequency in rad/s.

$T = \frac{1}{f_s}$  = sampling period in sec.

The impulse response  $h_d(n)$  has infinite number of samples. For FIR filters, we truncate this infinite impulse response to a finite duration sequence of length  $N$ , where  $N$  is odd.

$$\therefore h(n) = h_d(n); \quad |n| \leq \left( \frac{N-1}{2} \right) \quad (4.72)$$

On taking  $z$ -transform of Eq. (4.72) we get

$$H(z) = \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} h(n) z^{-n} \quad (4.73)$$

The transfer function of Eq. (4.73) represents non-causal filter. Hence, the transfer function is multiplied by  $z^{-\frac{(N-1)}{2}}$ . This modification does not affect the amplitude response of the filter.

$$\begin{aligned} H(z) &= z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} h(n) z^{-n} \\ &= z^{-\left(\frac{N-1}{2}\right)} \left[ \sum_{n=-\frac{(N-1)}{2}}^{-1} h(n) z^{-n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) z^{-n} \right] \end{aligned} \quad (4.74)$$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[ \sum_{n=1}^{\frac{N-1}{2}} h(-n)z^n + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)z^{-n} \right]$$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[ h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)[z^n + z^{-n}] \right] \quad \because h(n) = h(-n) \quad (4.75)$$

The abrupt truncation of the Fourier series results in oscillations in the passband and stopband. These oscillations are due to the slow convergence of the Fourier series at the points of discontinuity. This effect is known as “Gibbs phenomenon.” This oscillation can be reduced by multiplying the desired impulse response coefficients by an appropriate window function.

The specifications of lowpass, highpass, bandpass and bandstop filters design by Fourier series are given below:

$$\text{Lowpass, } H_d(\omega) = \begin{cases} 1, & -\omega_c \leq \omega \leq \omega_c \\ 0, & -\frac{\omega_s}{2} \leq \omega \leq -\omega_c \text{ and } \omega_c < \omega \leq \frac{\omega_s}{2} \end{cases} \quad (4.76)$$

$$\text{Highpass, } H_d(\omega) = \begin{cases} 1, & -\frac{\omega_s}{2} \leq \omega \leq -\omega_c \text{ and } \omega_c \leq \omega \leq \frac{\omega_s}{2} \\ 0, & -\omega_c \leq \omega \leq \omega_c \end{cases} \quad (4.77)$$

Bandpass,  $H_d(\omega)$

$$= \begin{cases} 1, & -\omega_{c_2} \leq \omega \leq -\omega_{c_1} \text{ and } \omega_{c_1} \leq \omega \leq \omega_{c_2} \\ 0, & -\frac{\omega_s}{2} \leq \omega_c < -\omega_{c_2}, -\omega_{c_1} \leq \omega \leq \omega_{c_1}, \text{ and } \omega_{c_2} < \omega \leq \frac{\omega_s}{2} \end{cases} \quad (4.78)$$

Bandstop,  $H_d(\omega)$

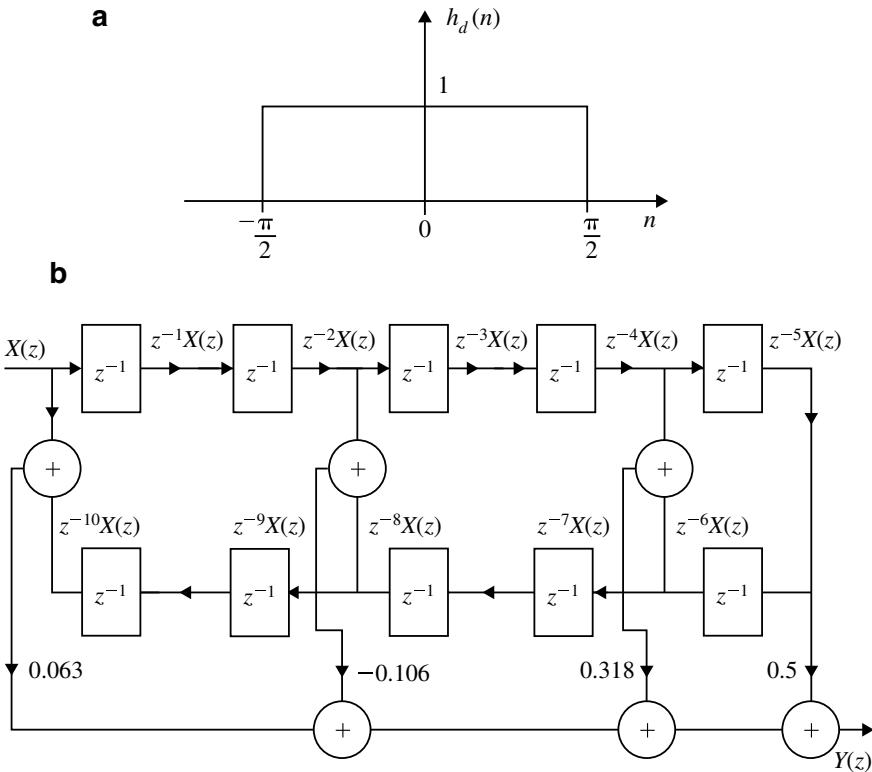
$$= \begin{cases} 1, & -\frac{\omega_s}{2} \leq \omega_c < -\omega_{c_2}, -\omega_{c_1} \leq \omega \leq \omega_{c_1}, \text{ and } \omega_{c_2} \leq \omega \leq \frac{\omega_s}{2} \\ 0, & -\omega_{c_2} < \omega < -\omega_{c_1} \text{ and } \omega_{c_1} < \omega < \omega_{c_2} \end{cases} \quad (4.79)$$

### Example 4.9

Design an ideal lowpass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & \text{for } -\frac{\pi}{2} \leq |\omega| \leq \frac{\pi}{2} \\ 0, & \text{for } \frac{\pi}{2} \leq |\omega| \leq \pi \end{cases}$$

Find the values of  $h(n)$  for  $N = 11$ . Find  $H(z)$ .



**Fig. 4.26** a Impulse response of LPF for Example 4.9. b Structure realization for Example 4.9

**Solution**  $h_d(n)$  is shown in Fig. 4.26a

$$\begin{aligned}
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi j n} [e^{j\omega n}]_{-\pi/2}^{\pi/2} \\
 &= \frac{1}{\pi n} \left[ \frac{e^{j(\pi/2)n} - e^{-j(\pi/2)n}}{2j} \right] \\
 &= \frac{\sin(\pi/2)n}{\pi n}, \quad -\infty \leq n \leq 11
 \end{aligned}$$

Here  $n = 11$

$$\therefore h(n) = \begin{cases} \frac{\sin(\pi/2)n}{\pi n}, & \text{for } |n| \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

For  $n = 0$

$$\begin{aligned} h_d(n) &= \lim_{n \rightarrow 0} \frac{\sin(\pi/2)n}{\pi n} \\ &= \frac{1}{2} \lim_{n \rightarrow 0} \frac{\sin(\pi/2)n}{(\pi/2)n} \quad \left[ \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right] \\ &= \frac{1}{2} \end{aligned}$$

From the frequency response,  $H_d(e^{j\omega})$  we find that  $\alpha = 0$ , and therefore, filter coefficients are symmetrical about  $n = 0$ , i.e.,  $h(n) = h(-n)$ .

$$\text{For } n = 1 : \quad h(1) = h(-1) = \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} = 0.3183$$

$$\text{For } n = 2 : \quad h(2) = h(-2) = \frac{\sin 2(\pi/2)}{2\pi} = 0$$

$$\text{For } n = 3 : \quad h(3) = h(-3) = \frac{\sin 3(\pi/2)}{3\pi} = -0.106$$

$$\text{For } n = 4 : \quad h(4) = h(-4) = \frac{\sin 2\pi}{4\pi} = 0$$

$$\begin{aligned} \text{For } n = 5 : \quad h(5) &= h(-5) = \frac{\sin 5(\pi/2)}{5\pi} \\ &= 0.06366 \end{aligned}$$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[ \sum_{n=1}^{\frac{N-1}{2}} h(n)[z^n + z^{-n}] + h(0) \right]$$

$$= z^{-5} \left[ h(0) + \sum_{n=1}^5 h(n)[z^n + z^{-n}] \right]$$

$$= z^{-5} [0.5 + h(1)(z + z^{-1}) + h(2)(z^2 + z^{-2}) + h(3)(z^3 + z^{-3}) \\ + h(4)(z^4 + z^{-4}) + h(5)(z^5 + z^{-5})]$$

$$H(z) = 0.5z^{-5} + 0.3183(z^{-4} + z^{-6}) + 0 - 0.106(z^{-2} + z^{-8}) \\ + 0.063(z^{-10} + 1)$$

$$Y(z) = 0.5z^{-5}X(z) + 0.3183(z^{-4} + z^{-6})X(z) - 0.106[X(z)z^{-2} + X(z)z^{-8}] \\ + 0.063[X(z)z^{-10} + X(z)]$$

The structure realization is shown in Fig. 4.26b.

### Example 4.10

Design an ideal highpass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & \text{for } -\frac{\pi}{4} \leq |\omega| \leq \pi \\ 0, & \text{for } |\omega| < \frac{\pi}{4} \end{cases}$$

Find the values of  $h(n)$  for  $N = 11$ . Find  $H(z)$ . Plot the magnitude response.

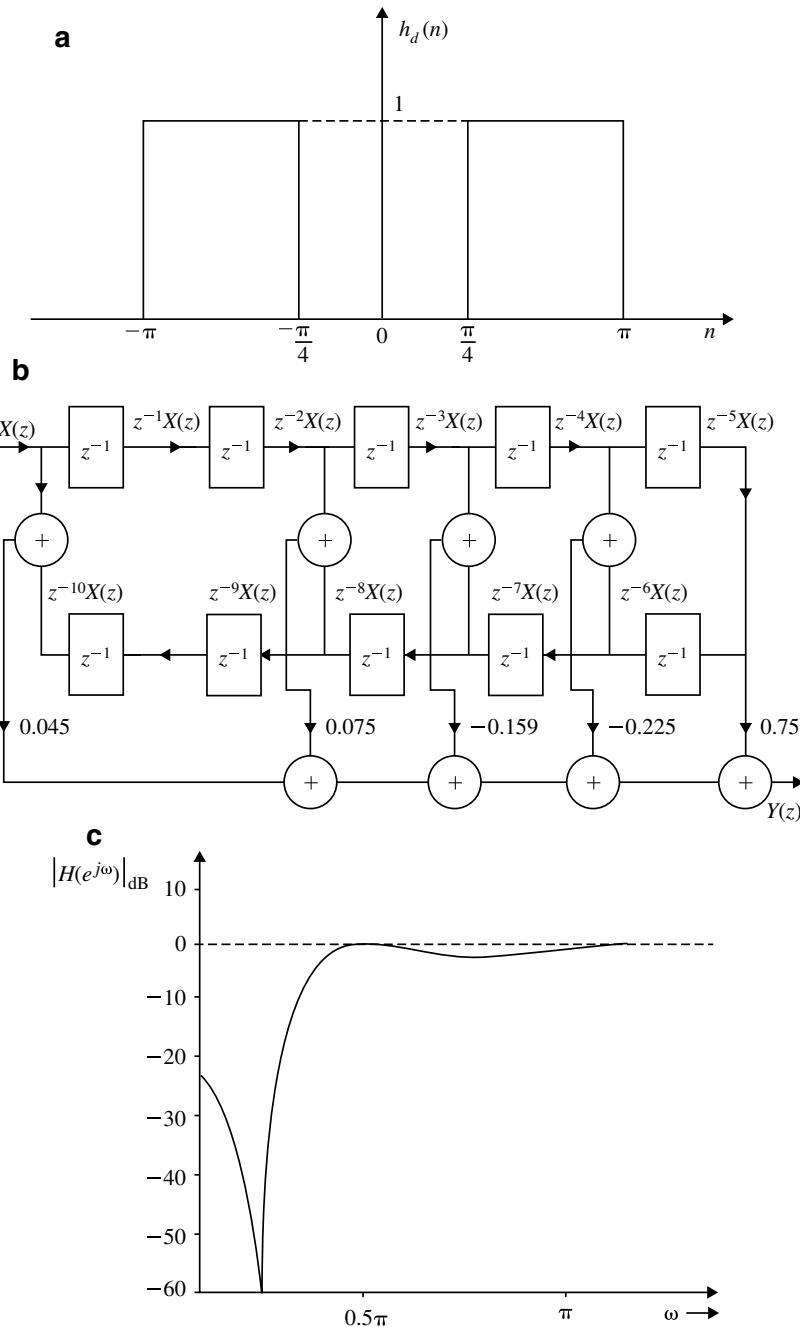
(Anna University, December, 2006)

**Solution**  $h_d(n)$  is shown in Fig. 4.27a

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \left[ \int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega \right] \\ &= \frac{1}{2\pi j n} \left[ e^{j\omega n} \Big|_{-\pi}^{-\pi/4} + e^{j\omega n} \Big|_{\pi/4}^{\pi} \right] \\ &= \frac{1}{\pi n (2j)} [e^{-j(\pi/4)n} - e^{-j\pi n} + e^{j\pi n} - e^{j(\pi/4)n}] \\ h_d(n) &= \begin{cases} \frac{1}{\pi n} [\sin \pi n - \sin(\pi/4)n], & \text{for } |n| \leq 5 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

For  $n = 0$

$$\begin{aligned} h(n) &= \lim_{n \rightarrow 0} \frac{\sin \pi n}{\pi n} - \lim_{n \rightarrow 0} \frac{\sin(\pi/4)n}{4(\pi n/4)} \\ &= 1 - \frac{1}{4} = 0.75 \end{aligned}$$



**Fig. 4.27** **a** Impulse response for Example 4.10. **b** Structure realization for Example 4.10. **c** Magnitude response for the Example 4.10

$$\begin{aligned}
 \text{For } n = 1 : \quad h(1) &= h(-1) = \frac{\sin \pi - \sin(\pi/4)}{\pi} \\
 &= -0.225 \\
 \text{For } n = 2 : \quad h(2) &= h(-2) = \frac{\sin 2\pi - \sin(2\pi/4)}{2\pi} \\
 &= -0.159 \\
 \text{For } n = 3 : \quad h(3) &= h(-3) = \frac{\sin 3\pi - \sin(3\pi/4)}{3\pi} \\
 &= -0.075 \\
 \text{For } n = 4 : \quad h(4) &= h(-4) = \frac{\sin 4\pi - \sin \pi}{4\pi} \\
 &= 0 \\
 \text{For } n = 5 : \quad h(5) &= h(-5) = \frac{\sin 5\pi - \sin(5\pi/4)}{5\pi} \\
 &= 0.045
 \end{aligned}$$

$$\begin{aligned}
 H(z) &= z^{-5} \left[ \sum_{n=1}^N h(n)(z^n + z^{-n}) + h(0) \right] \\
 &= z^{-5} [h(0) + h(1)(z + z^{-1}) + h(2)(z^2 + z^{-2}) + h(3)(z^3 + z^{-3}) \\
 &\quad + h(4)(z^4 + z^{-4}) + h(5)(z^5 + z^{-5})] \\
 H(z) &= 0.75z^{-5} - 0.225(z^{-4} + z^{-6}) - 0.159(z^{-3} + z^{-7}) - 0.075(z^{-2} + z^{-8}) \\
 &\quad + 0.045(1 + z^{-10}) \\
 Y(z) &= 0.75z^{-5}X(z) - 0.225[X(z)z^{-4} + X(z)z^{-6}] - 0.159[X(z)z^{-3} + X(z)z^{-7}] \\
 &\quad - 0.075[X(z)z^{-2} + X(z)z^{-8}] + 0.045[X(z) + X(z)z^{-10}]
 \end{aligned}$$

The structure realization FIR highpass filter is shown in Fig. 4.27b.

### Magnitude Response

The frequency response of FIR filter when impulse response is symmetric and  $N$  is odd.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega n$$

where

$$\begin{aligned}
 a(0) &= h\left(\frac{N-1}{2}\right) = h(5) = 0.75 \\
 a(n) &= 2h\left(\frac{N-1}{2} - n\right) \\
 a(1) &= 2h(5-1) = 2h(4) = -0.45 \\
 a(2) &= 2h(5-2) = 2h(3) = -0.318 \\
 a(3) &= 2h(5-3) = 2h(2) = -0.15 \\
 a(4) &= 2h(5-4) = 2h(1) = 0 \\
 a(5) &= 2h(5-5) = 2h(0) = 0.09
 \end{aligned}$$

$$\begin{aligned}
 H(e^{j\omega}) &= a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega \\
 &\quad + a(5)\cos 5\omega \\
 &= 0.75 - 0.45 \cos\omega - 0.318 \cos 2\omega - 0.15 \cos 3\omega + 0.09 \cos 5\omega
 \end{aligned}$$

The magnitude response is shown in Fig. 4.27c.

$\omega$ (deg/s)	0	10	30	50	80	100	120	140	160	170
$H(e^{j\omega})$	-0.08	-0.066	0.122	0.61	1.11	0.98	0.94	1.26	1.01	0.96
$ H(e^{j\omega}) _{\text{dB}}$	-22	-23.62	-18.2	-4.2	0.95	-0.132	-0.537	2	0.16	-0.31

### Example 4.11

Design an ideal bandpass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & \text{for } \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

Find the values of  $h(n)$  for  $N = 11$ . Find  $H(z)$  and also plot the frequency response.

(Anna University, December, 2004 and May, 2005)

**Solution** The impulse response  $h_d(n)$  is shown in Fig. 4.28a.

$$\begin{aligned}
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \left[ \int_{-(3\pi/4)}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{(3\pi)/4} e^{j\omega n} d\omega \right] \\
 &= \frac{1}{2\pi j n} [e^{(-j\pi n)/4} - e^{(j3\pi n)/4} + e^{(j3\pi n)/4} - e^{(j\pi n)/4}] \\
 h_d(n) &= \begin{cases} \frac{1}{\pi n} [\sin((3\pi)/4)n - \sin(\pi/4)n], & \text{for } |n| \leq 5 \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

For  $n = 0$

$$\begin{aligned}
 h_d(0) &= \lim_{n \rightarrow 0} \frac{\sin(3\pi/4)n}{\frac{4}{3} \left( \frac{\pi n^3}{4} \right)} - \lim_{n \rightarrow 0} \frac{\sin(\pi/4)n}{4 \left( \frac{\pi n}{4} \right)} \\
 &= \frac{3}{4} - \frac{1}{4} = 0.5
 \end{aligned}$$

$$\text{For } n = 1 : \quad h(1) = h(-1) = \frac{\sin \frac{3\pi}{4} - \sin \frac{\pi}{4}}{\pi} = 0$$

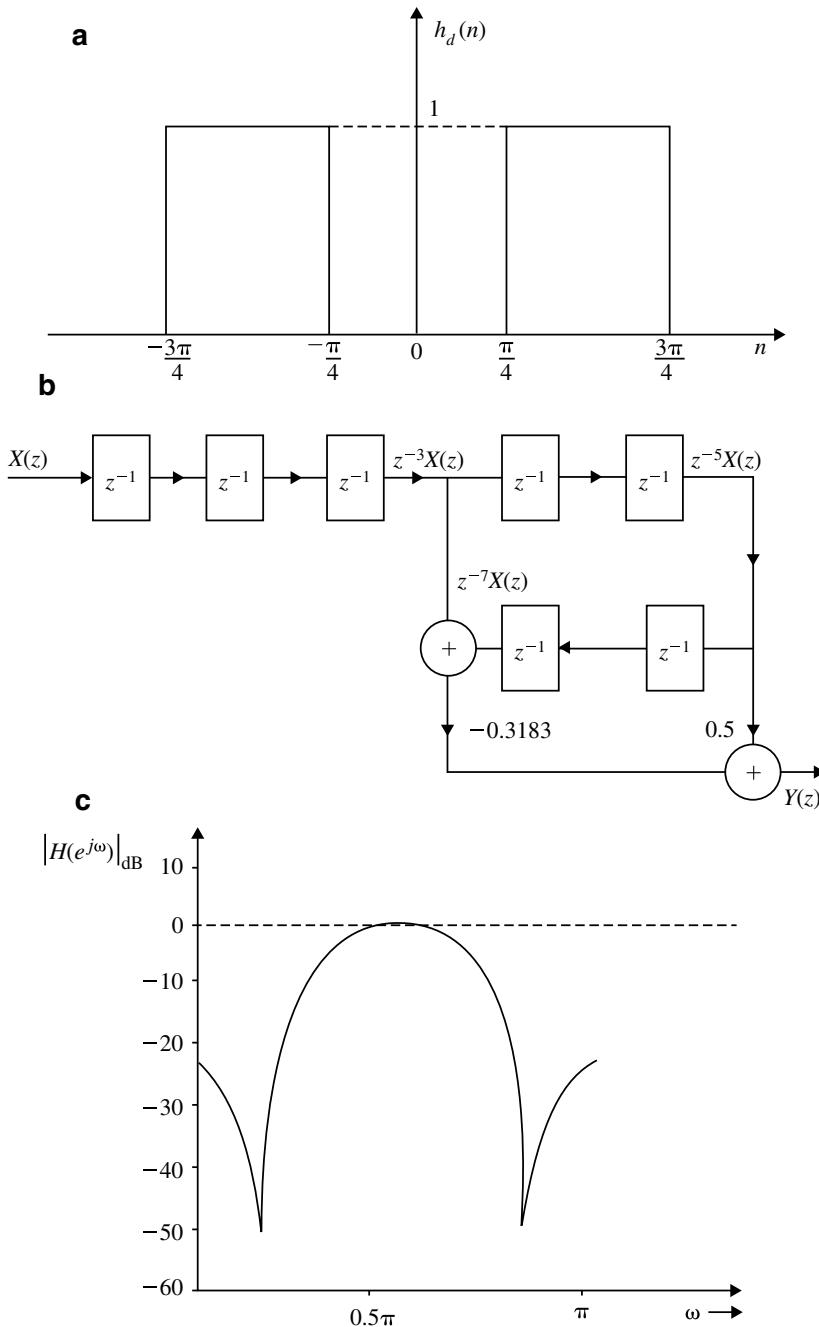
$$\text{For } n = 2 : \quad h(2) = h(-2) = \frac{\sin \frac{3\pi}{2} - \sin \frac{\pi}{2}}{2\pi} = -\frac{2}{2\pi} = -0.3183$$

$$\text{For } n = 3 : \quad h(3) = h(-3) = \frac{\sin \frac{9\pi}{4} - \sin \frac{3\pi}{4}}{3\pi} = 0$$

$$\text{For } n = 4 : \quad h(4) = h(-4) = \frac{\sin 3\pi - \sin \pi}{4\pi} = 0$$

$$\text{For } n = 5 : \quad h(5) = h(-5) = \frac{\sin \frac{15\pi}{4} - \sin \frac{5\pi}{4}}{5\pi} = 0$$

$$\begin{aligned}
 H(z) &= z^{-5} \left[ \sum_{n=1}^5 h(n)(z^n + z^{-n}) + h(0) \right] \\
 &= 0.5z^{-5} - 0.3183(z^{-3} + z^{-7})
 \end{aligned}$$



**Fig. 4.28** **a** Impulse response for the Example 4.11. **b** Structure realization for Example 4.11. **c** Frequency response of bandpass filter of Example 4.11

The filter coefficients of the causal filters are

$$\begin{aligned} h(0) &= h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0 \\ h(3) &= h(7) = -0.3183 \\ h(5) &= 0.5 \end{aligned}$$

The filter structure is shown in Fig. 4.28b.

$$\begin{aligned} |H(e^{j\omega})| &= \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos \omega n + h\left(\frac{N-1}{2}\right) \\ &= \sum_{n=1}^{\frac{N-1}{2}} a(n) \cos \omega n + a(0) \end{aligned}$$

where

$$\begin{aligned} a(0) &= h\left(\frac{N-1}{2}\right) = h(5) = 0.5 \\ a(n) &= 2h\left(\frac{N-1}{2} - n\right) \\ a(1) &= 2h(5-1) = 2h(4) = 0 \\ a(2) &= 2h(5-2) = 2h(3) = -0.6366 \\ a(3) &= 2h(5-3) = 2h(2) = 0 \\ a(4) &= 2h(5-4) = 2h(1) = 0 \\ a(5) &= 2h(5-5) = 2h(0) = 0 \end{aligned}$$

$$|H(e^{j\omega})| = 0.5 - 0.6366 \cos 2\omega$$

$\omega$ (deg/s)	0	20	45	60	90	120	135	180
$H(e^{j\omega})$	-0.1366	0.012	0.5	0.818	1.1366	0.818	0.5	-0.1366
$ H(e^{j\omega}) _{dB}$	-17.3	-38.17	-6.02	-1.74	1.11	-1.74	-6.02	-17.32

The magnitude response is shown in Fig. 4.28c.

**Example 4.12**

Design an ideal band reject filter with a desired frequency response

$$H_d(e^{j\omega}) = 1 \text{ for } |\omega| \leq \frac{\pi}{3} \text{ and } |\omega| \geq \frac{2\pi}{3}$$

Find the values of  $h(n)$  for  $N = 11$ . Find  $H(z)$ .

**Solution** The impulse response  $h_d(n)$  is shown in Fig. 4.29a

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \left[ \int_{-\pi}^{-2\pi/3} e^{j\omega n} d\omega + \int_{-(\pi/3)}^{\pi/3} e^{j\omega n} d\omega + \int_{2\pi/3}^{\pi} e^{j\omega n} d\omega \right] \\ &= \frac{1}{2\pi j n} [e^{-j2\pi n/3} - e^{-j\pi n} + e^{j\pi n/3} - e^{-j\pi n/3} + e^{j\pi n} - e^{j2\pi n/3}] \\ h_d(n) &= \begin{cases} \frac{1}{\pi n} [\sin \pi n + \sin \frac{\pi}{3} n - \sin \frac{2\pi}{3} n], & |n| \leq 5 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The filter coefficients are symmetrical about  $n = 0$  satisfying the condition  $h(n) = h(-n)$ . For  $n = 0$

$$\begin{aligned} h_d(0) &= \lim_{n \rightarrow 0} \left[ \frac{\sin \pi n}{\pi n} + \frac{\sin(\pi/3)n}{\pi n} - \frac{\sin(2\pi/3)n}{\pi n} \right] \\ &= \left[ 1 + \frac{1}{3} - \frac{2}{3} \right] = 0.667 \end{aligned}$$

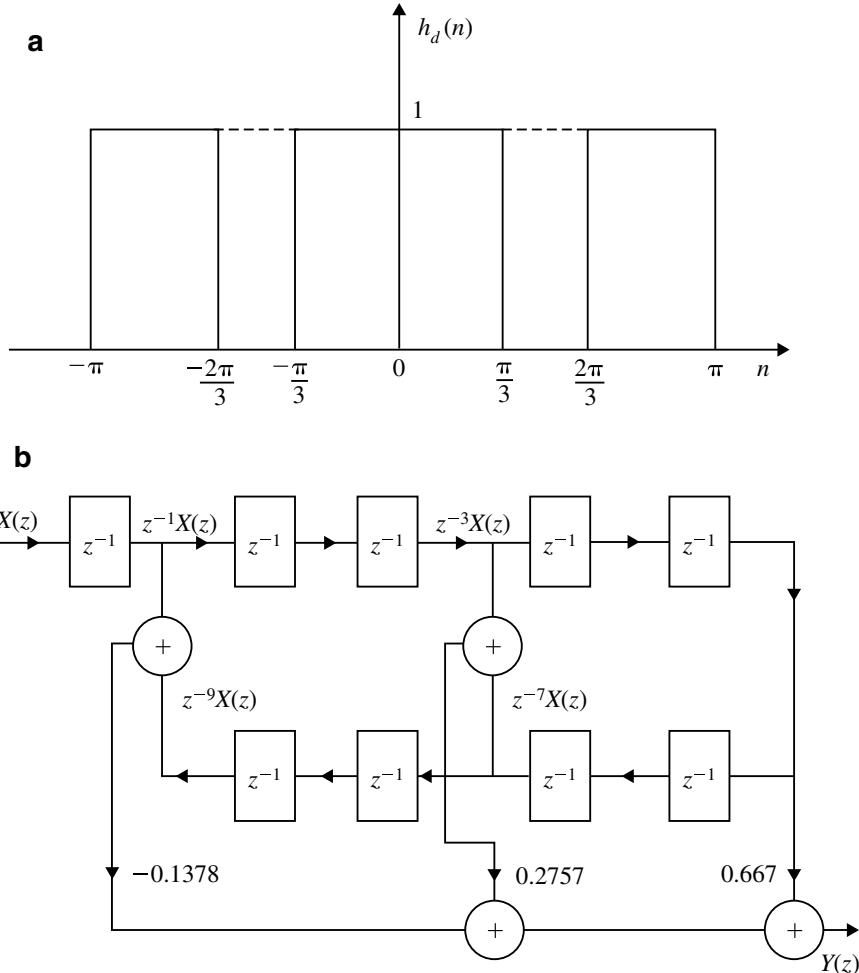
$$\text{For } n = 1 : \quad h(1) = h(-1) = \frac{\sin \pi + \sin \frac{\pi}{3} - \sin \frac{2\pi}{3}}{\pi} = 0$$

$$\text{For } n = 2 : \quad h(2) = h(-2) = \frac{\sin 2\pi + \sin \frac{2\pi}{3} - \sin \frac{4\pi}{3}}{2\pi} = 0.2757$$

$$\text{For } n = 3 : \quad h(3) = h(-3) = \frac{\sin 3\pi + \sin \pi - \sin 2\pi}{3\pi} = 0$$

$$\text{For } n = 4 : \quad h(4) = h(-4) = \frac{\sin 4\pi + \sin \frac{4\pi}{3} - \sin \frac{8\pi}{3}}{4\pi} = -0.1378$$

$$\text{For } n = 5 : \quad h(5) = h(-5) = \frac{\sin 5\pi + \sin \frac{5\pi}{3} - \sin \frac{10\pi}{3}}{5\pi} = 0$$



**Fig. 4.29** **a** Impulse response for Example 4.12. **b** Structure realization for Example 4.12

$$\begin{aligned}
 H(z) &= z^{-5} \left[ \sum_{n=1}^5 h(n)(z^n + z^{-n}) + h(0) \right] \\
 &= 0.667z^{-5} + 0.2757(z^{-3} + z^{-7}) - 0.1378(z^{-1} + z^{-9}) \\
 Y(z) &= 0.667z^{-5}X(z) + 0.2757[X(z)z^{-3} + X(z)z^{-7}] \\
 &\quad - 0.1378[X(z)z^{-1} + X(z)z^{-9}]
 \end{aligned}$$

The structure realization is shown in Fig. 4.29b.

### 4.5.2 Window Method

#### 4.5.2.1 Rectangular Window

Rectangular window function,

$$W_R(n) = \begin{cases} 1; & 0 \leq n \leq N - 1 \\ 0; & \text{otherwise} \end{cases} \quad (4.80)$$

(or)

$$W_R(n) = \begin{cases} 1; & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases} \quad (4.81)$$

The spectrum of the Rectangular window is given by

$$\begin{aligned} W_R(\omega) &= \sum_{n=-((N-1)/2)}^{\frac{N-1}{2}} e^{-j\omega n} = e^{j\omega \frac{(N-1)}{2}} + \cdots + e^{j\omega} + 1 + e^{-j\omega} \\ &\quad + e^{-2j\omega} + \cdots + e^{-j\omega \left(\frac{N-1}{2}\right)} \\ &= e^{j\omega \frac{(N-1)}{2}} [1 + e^{-j\omega} + \cdots + e^{-j\omega(N-1)}] \\ &= e^{j\omega \frac{(N-1)}{2}} \left[ \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \right] \quad \left( \because 1 + x + x^2 + \cdots + x^{N-1} = \frac{1 - x^N}{1 - x} \right) \end{aligned} \quad (4.82)$$

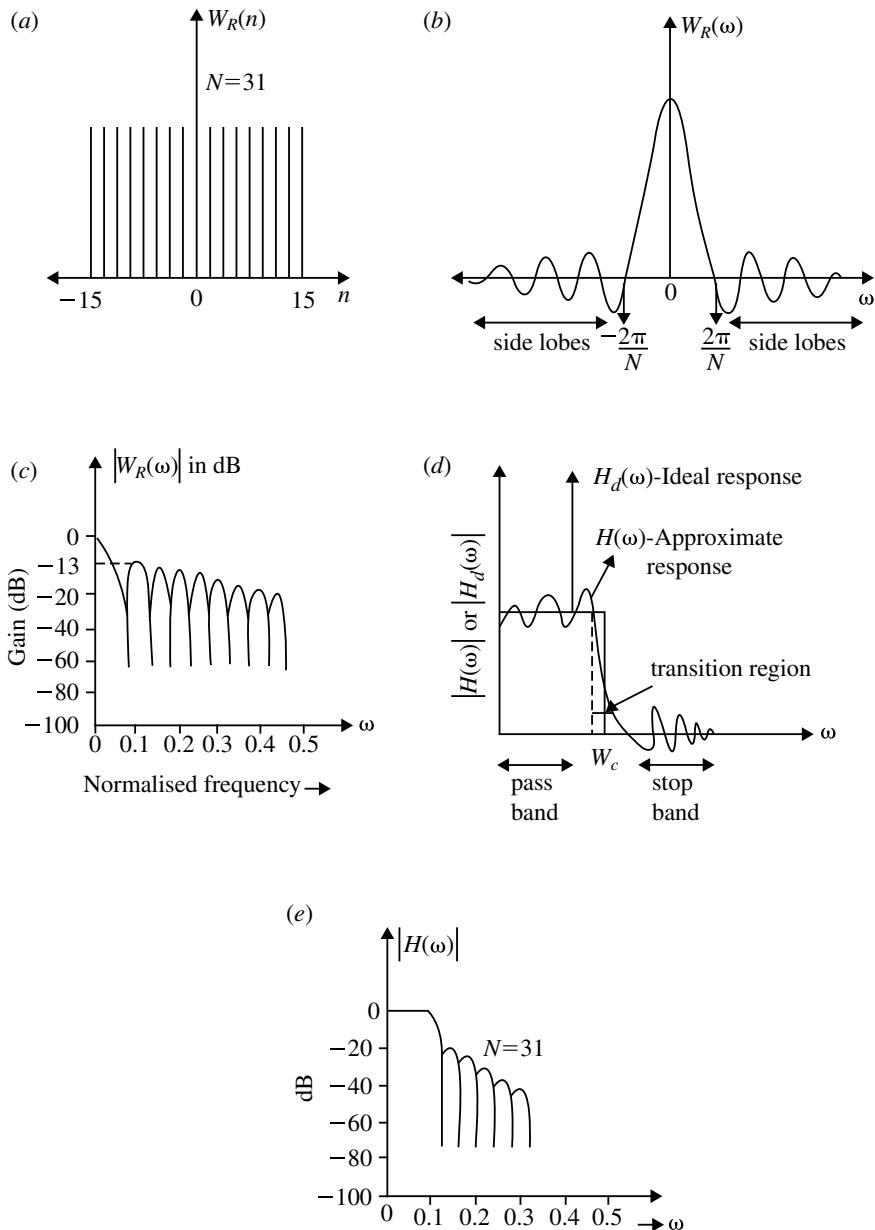
$$\begin{aligned} &= \frac{e^{\frac{j\omega N}{2}} (1 - e^{-j\omega N})}{e^{\frac{j\omega}{2}} (1 - e^{-j\omega})} = \frac{e^{\frac{j\omega N}{2}} - e^{-\frac{j\omega N}{2}}}{e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}} \\ &= \frac{\sin(\omega N/2)}{\sin(\omega/2)} \end{aligned} \quad (4.83)$$

The window spectrum for  $N = 31$  is shown in Fig. 4.30. The spectrum of  $W_R(\omega)$  has two important features and they are:

1. Width of mainlobe =  $\frac{4\pi}{N}$
2. Peak sidelobe magnitude (dB) =  $-13$  dB

The magnitude response  $|H(\omega)|$  of the lowpass filter designed using Rectangular window is shown in Fig. 4.30. The approximated filter response differs from the ideal desired response, i.e., in the passband a series of overshoots and undershoots occur. In the stopband the FIR filter has a nonzero response. This can be explained in terms of the features of the window spectrum.

The width of the transition region is related to the width of the mainlobe of  $W_R(\omega)$ . Since the mainlobe width of  $W_R(\omega)$  is equal to  $\frac{4\pi}{N}$ , the size of this transition region can be reduced to any desired size by increasing the size  $N$  of the window sequence.



**Fig. 4.30** Rectangular window sequence and its frequency response. **a** Rectangular window sequence. **b** Magnitude response of Rectangular window. **c** Log-magnitude response of Rectangular window. **d** Magnitude response of LPF approximated using Rectangular window and **e** Log-magnitude response of FIR LPF designed using Rectangular window

The increase in  $N$  also increases the number of computations necessary to implement the FIR filter.

In the passband the sidelobe effect of the  $W_R(\omega)$  appears as both overshoots and undershoots in the desired response. In the stopband these effects appear as a nonzero response. These sidelobe effects do not diminish significantly but remain constant as the duration of Rectangular window is increased.

It is observed that whatever be the number of elements of  $h_d(n)$  included in the  $h(n)$ , the magnitudes of the overshoot and leakage will not change significantly. This result is known as Gibb's phenomenon. To reduce these sidelobe effects, we must consider alternate window sequences having spectrum exhibiting smaller sidelobes. We can observe that the sidelobes of the window spectrum represents the contribution of the high frequency components. For the Rectangular window, these high frequency components are due to the sharp transitions from 0 to 1 and 1 to 0 at the edges of window sequence. Hence the amplitude of these high frequency components i.e., sidelobe levels can be reduced by replacing these sharp transitions by more gradually ones. So we go for raised cosine window sequences.

### Design Procedure

**Step 1** Choose the desired frequency response of the filter,  $H_d(\omega)$ .

**Step 2** Take inverse Fourier transform of  $H_d(\omega)$  to obtain the desired impulse response,  $h_d(n)$ .

**Step 3** Choose the window sequence  $W(n)$  and multiply  $h_d(n)$  by  $W(n)$  [i.e.,  $h(n) = W(n) \times h_d(n)$ ] to convert the infinite impulse response to finite impulse response.

$$W(n) = \begin{cases} 1, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

**Step 4** The transfer function of the filter  $H(z)$  is obtained by taking  $z$ -transform of  $h(n)$ .

**Step 5** Realize the filter using suitable realization method (either direct form (or) linear phase realization).

### Example 4.13

Design a LPF using Rectangular window by taking nine samples of  $W(n)$  and with cutoff frequency of 1.2 rad/s?

### Solution

**Step 1 Choose the Desired Frequency Response**

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\omega_c \leq \omega \leq \omega_c \quad [\text{for LPF}] \\ 0, & -\pi \leq \omega \leq -\omega_c \quad \text{and} \quad \omega_c \leq \omega \leq \pi \end{cases}$$

**Step 2 To Find  $H_d(n)$** 

Take inverse Fourier transform of  $H_d(\omega)$ . That is,

$$\begin{aligned}
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)} d\omega \\
 &= \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\omega_c}^{\omega_c} \\
 &= \frac{1}{2j\pi(n-\alpha)} [e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}] \\
 &= \frac{1}{\pi(n-\alpha)} [\sin \omega_c(n-\alpha)] \\
 \therefore h_d(n) &= \frac{1}{\pi(n-\alpha)} [\sin \omega_c(n-\alpha)]
 \end{aligned}$$

**Step 3 Conversion of Infinite to Finite Sequence**

$$h(n) = W(n) \times h_d(n)$$

Here, for Rectangular window,

$$\begin{aligned}
 W(n) &= \begin{cases} 1, & \text{for } 0 \leq n \leq (N-1) \\ 0, & \text{for otherwise} \end{cases} \\
 \therefore h(n) &= \begin{cases} \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}; & 0 \leq n \leq (N-1) \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

where

$$\omega_c = 1.2 \text{ rad/s},$$

$$n = 0 \text{ to } 8 \quad [\because 9 \text{ samples}]$$

$$\alpha = \frac{N+1}{2} = \frac{9-1}{2} = 4$$

Therefore

$$\begin{aligned} h(0) &= \frac{\sin 1.2(0 - 4)}{\pi(0 - 4)} = -0.079 \\ h(1) &= \frac{\sin 1.2(1 - 4)}{\pi(1 - 4)} = -0.0469 \\ h(2) &= \frac{\sin 1.2(2 - 4)}{\pi(2 - 4)} = 0.107 \\ h(3) &= \frac{\sin 1.2(3 - 4)}{\pi(3 - 4)} = 0.296 \\ h(4) &= \frac{\sin 1.2(4 - 4)}{\pi(4 - 4)} = 0 \end{aligned}$$

$$\text{Apply "L" Hospitals rule } \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\cos A\theta \cdot A}{A\theta} = A$$

$$\therefore h(4) = \frac{\sin 1.2(n - 4)}{\pi(n - 4)} = \frac{1.2}{\pi} = 0.3819$$

$$h(5) = \frac{\sin 1.2(5 - 4)}{\pi(5 - 4)} = 0.296$$

$$h(6) = \frac{\sin 1.2(6 - 4)}{\pi(6 - 4)} = 0.107$$

$$h(7) = \frac{\sin 1.2(7 - 4)}{\pi(7 - 4)} = -0.0469$$

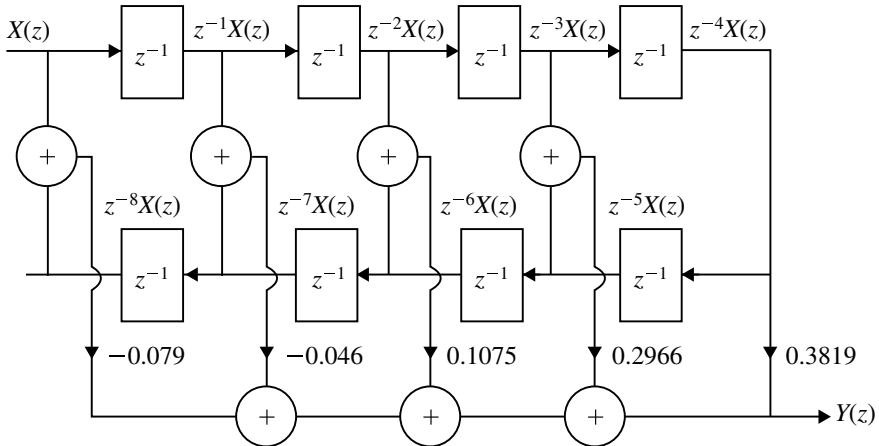
$$h(8) = \frac{\sin 1.2(8 - 4)}{\pi(8 - 4)} = -0.079.$$

#### Step 4 To Find $H(z)$

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} \\ &\quad + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} \\ &= -0.079 - 0.0469z^{-1} + 0.107z^{-2} + 0.296z^{-3} + 0.3819z^{-4} \\ &\quad + 0.296z^{-5} + 0.107z^{-6} - 0.0469z^{-7} - 0.079z^{-8} \\ &= -0.079(1 + z^{-8}) - 0.0469(z^{-1} + z^{-7}) \\ &\quad + 0.107(z^{-2} + z^{-6}) + 0.296(z^{-3} + z^{-5}) + 0.3819z^{-4} \end{aligned}$$

#### Step 5 Linear Phase Realization

The structure realization is shown in Fig. 4.31.



**Fig. 4.31** Linear phase realization of  $H(z)$  for Example 4.13

#### 4.5.2.2 Raised Cosine Windows

The raised cosine windows are smoother at the ends, but closer to one at the middle. The smoother ends and the broader middle section produce less distortion of  $h_d(n)$  around  $n = 0$ .

The window function is of the form

$$W_\alpha(n) = \begin{cases} \alpha + (1 - \alpha) \cos\left(\frac{2\pi n}{N-1}\right), & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0, & \text{otherwise} \end{cases} \quad (4.84)$$

If  $\alpha=0.5$ , the window is called as Hanning Window. If  $\alpha=0.54$ , the window is called as Hamming window. The window spectrum for raised cosine window is given by

$$W_\alpha(\omega) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} \left[ \alpha + (1 - \alpha) \cos\left(\frac{2\pi n}{N-1}\right) \right] e^{-j\omega n} \quad (4.85)$$

$$\begin{aligned} &= \underbrace{\alpha \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\omega n}}_X + \underbrace{\left(\frac{1-\alpha}{2}\right) \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\left(\omega - \frac{2\pi}{N-1}\right)n}}_Y \\ &\quad + \underbrace{\left(\frac{1-\alpha}{2}\right) \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\left(\omega + \frac{2\pi}{N-1}\right)n}}_Z \end{aligned} \quad (4.86)$$

$$\begin{aligned}
X &= \alpha \left[ e^{j\omega(\frac{N-1}{2})} + \dots + e^{j\omega} + 1 + e^{-j\omega} + \dots + e^{-j\omega(\frac{N-1}{2})} \right] \\
&= \alpha \left[ \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \right] e^{j\omega(\frac{N-1}{2})} \\
&= \alpha \left[ \frac{e^{\frac{j\omega N}{2}} - e^{-\frac{j\omega N}{2}}}{e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}} \right] \\
X &= \alpha \left[ \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}} \right]
\end{aligned} \tag{4.87}$$

$$\begin{aligned}
Y &= \frac{1 - \alpha}{2} \left[ e^{j(\omega - \frac{2\pi}{N-1})} \left( \frac{N-1}{2} \right) + \dots + e^{j(\omega - \frac{2\pi}{N-1})} \right. \\
&\quad \left. + 1 + e^{-j(\omega - \frac{2\pi}{N-1})} + \dots + e^{-j(\omega - \frac{2\pi}{N-1})} \left( \frac{N-1}{2} \right) \right] \\
&= \frac{1 - \alpha}{2} \left[ \frac{e^{j(\omega - \frac{2\pi}{N-1})} \left( \frac{N-1}{2} \right) \left[ 1 - e^{-j(\omega - \frac{2\pi}{N-1})N} \right]}{e^{j(\frac{\omega}{2} - \frac{\pi}{N-1})} \left[ 1 - e^{-j(\omega - \frac{2\pi}{N-1})} \right]} \right] \\
&= \frac{1 - \alpha}{2} \left[ \frac{e^{j(\frac{\omega N}{2} - \frac{\pi N}{N-1})} - e^{-j(\frac{\omega N}{2} - \frac{\pi N}{N-1})}}{e^{j(\frac{\omega}{2} - \frac{\pi}{N-1})} - e^{-j(\frac{\omega}{2} - \frac{\pi}{N-1})}} \right] \\
&= \frac{1 - \alpha}{2} \left[ \frac{\sin \left( \frac{\omega N}{2} - \frac{\pi N}{N-1} \right)}{\sin \left( \frac{\omega}{2} - \frac{\pi}{N-1} \right)} \right]
\end{aligned} \tag{4.88}$$

Similarly

$$z = \frac{1 - \alpha}{2} \left[ \frac{\sin \left( \frac{\omega N}{2} + \frac{\pi N}{N-1} \right)}{\sin \left( \frac{\omega}{2} + \frac{\pi}{N-1} \right)} \right] \tag{4.89}$$

Substituting Eqs. (4.87) to (4.89) in Eq. (4.85), we get

$$\begin{aligned}
W_\alpha(\omega) &= \alpha \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}} + \frac{1 - \alpha}{2} \left[ \frac{\sin \left( \frac{\omega N}{2} - \frac{\pi N}{N-1} \right)}{\sin \left( \frac{\omega}{2} - \frac{\pi}{N-1} \right)} \right] \\
&\quad + \frac{1 - \alpha}{2} \left[ \frac{\sin \left( \frac{\omega N}{2} + \frac{\pi N}{N-1} \right)}{\sin \left( \frac{\omega}{2} + \frac{\pi}{N-1} \right)} \right]
\end{aligned}
\tag{4.90}$$

### 4.5.2.3 Hanning Window

The Hanning window sequence can be obtained by substituting  $\alpha = 0.5$  in Eq. (4.84)

$$W_C(n) = \begin{cases} 0.5 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right); & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases} \quad (4.91)$$

(or)

$$W_C(n) = \begin{cases} 0.5 - 0.5 \cos\left(\frac{2n\pi}{N-1}\right); & 0 \leq n \leq N-1 \\ 0; & \text{otherwise} \end{cases} \quad (4.92)$$

Frequency response of the Hanning window is given by

$$W_C(\omega) = 0.5 \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin(\omega/2)} + 0.25 \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.25 \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

(4.93)

$$\text{Width of mainlobe} = \frac{8\pi}{N}$$

$$\text{Peak sidelobe magnitude (dB)} = -31 \text{ dB}$$

In the log-magnitude response of  $W_C(\omega)$  the magnitude of the first sidelobe is  $-31$  dB, an improvement of  $6$  dB over triangular window. Mainlobe width is same as in triangular window. But the magnitude of sidelobes is reduced, as shown in Fig. 4.32. Hence the Hanning window is preferable to triangular window since the magnitude responses of lowpass filter have improved stopband characteristics.

### Design Procedure

**Step 1** Choose the desired frequency response of the filter [ $H_d(\omega)$ ].

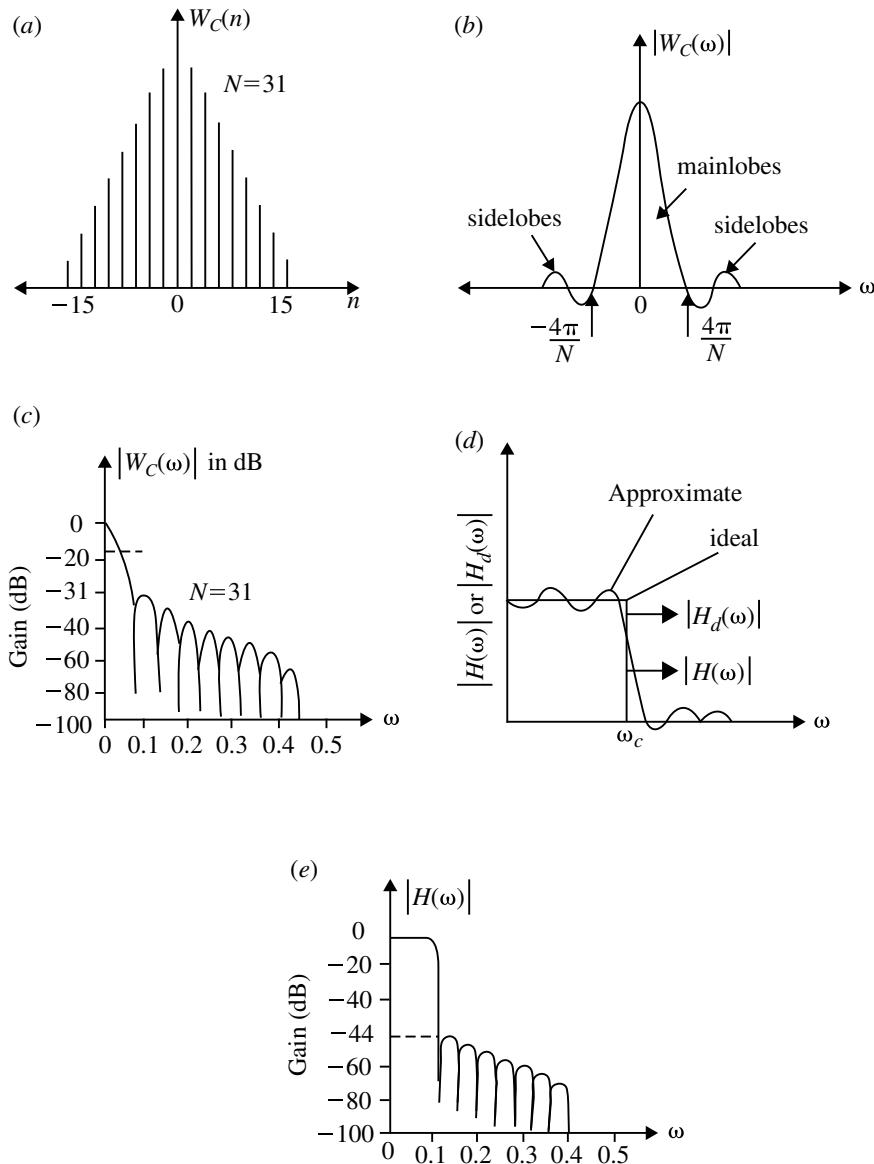
**Step 2** Take inverse Fourier transform of  $H_d(\omega)$  to obtain the desired impulse response  $h_d(n)$ .

**Step 3** Choose a window sequence  $W(n)$ .

$$W(n) = \begin{cases} 0.5 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

Multiply  $h_d(n)$  by  $W(n)$  to convert the infinite impulse response to a finite impulse response  $h(n)$ . That is

$$h(n) = h_d(n) \times W(n)$$



**Fig. 4.32** Hanning window sequence and its frequency response. **a** Rectangular window sequence. **b** Magnitude response of Hanning window. **c** Log-magnitude response of Hanning window. **d** Magnitude response of LPF approximated using Hanning window and **e** Log-magnitude response of FIR filter using Hanning window

**Step 4** The transfer function of the filter, i.e.,  $H(z)$  is obtained by taking  $z$ -transform of  $h(n)$ .

**Step 5** Realize the filter using suitable realization method (use direct form or linear phase).

---

### Example 4.14

Design a BPF to pass frequencies in the range 1–2 rad/s using Hanning window, with  $N = 5$ .

---

#### Solution

**Step 1** Choose  $H_d(\omega)$  such that

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\omega_{c_2} \leq \omega \leq -\omega_{c_1} \text{ and } \omega_{c_1} \leq \omega \leq \omega_{c_2} \\ 0, & \text{otherwise} \end{cases}$$

**Step 2** To Find  $h_d(n)$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \left[ \int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{-j\omega\alpha} e^{j\omega n} d\omega \right] \\ &= \frac{1}{2\pi} \left[ \int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega(n-\alpha)} d\omega + \int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega(n-\alpha)} d\omega \right] \\ &= \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\omega_{c_2}}^{-\omega_{c_1}} + \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{\omega_{c_1}}^{\omega_{c_2}} \\ &= \frac{1}{2\pi j(n-\alpha)} [e^{-j\omega_{c_1}(n-\alpha)} - e^{-j\omega_{c_2}(n-\alpha)} + e^{j\omega_{c_2}(n-\alpha)} - e^{j\omega_{c_1}(n-\alpha)}] \\ &= \frac{1}{2\pi j(n-\alpha)} [(e^{j\omega_{c_2}(n-\alpha)} - e^{-j\omega_{c_2}(n-\alpha)}) - (e^{j\omega_{c_1}(n-\alpha)} - e^{-j\omega_{c_1}(n-\alpha)})] \\ &= \frac{1}{2\pi j(n-\alpha)} \times 2j [\sin \omega_{c_2}(n-\alpha) - \sin \omega_{c_1}(n-\alpha)] \\ &= \frac{1}{\pi(n-\alpha)} [\sin \omega_{c_2}(n-\alpha) - \sin \omega_{c_1}(n-\alpha)] \end{aligned}$$

Given that  $\omega_{c_1} = 1$  rad/s and  $\omega_{c_2} = 2$  rad/s,  $\alpha = \frac{N-1}{2} \rightarrow \alpha = 2$ .

**Step 3** To Find  $h(n)$

$$\begin{aligned} W(n) &= 0.5 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) \\ h(n) &= h_d(n) \times W(n) \\ &= \frac{1}{\pi(n-2)} [\sin 2(n-2) - \sin(n-2)] \left[ 0.5 - 0.5 \cos\left(\frac{2\pi n}{4}\right) \right] \end{aligned}$$

$$h(0) = 0$$

$$h(1) = 0.0108$$

$$h(2) = 0.318 \quad [\text{Applying L' Hospital's rule}]$$

$$h(3) = 0.0108$$

$$h(4) = 0$$

**Step 4** Take  $z$ -transform

$$H(z) = \sum_{n=0}^4 h(n)z^{-n} = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}$$

$$H(z) = 0.0108[z^{-1} + z^{-3}] + 0.318z^{-2}$$

**Step 5** Draw the realization structure. This is shown in Fig. 4.33.

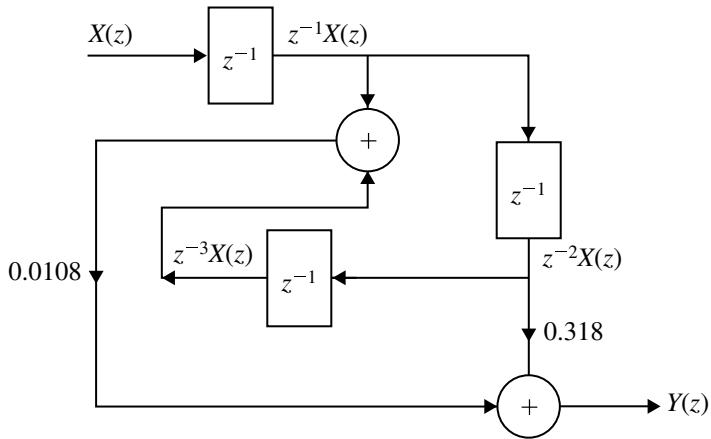
#### 4.5.2.4 Hamming Window

The Hamming window sequence can be obtained by substituting  $\alpha = 0.54$  in Eq. (4.84)

$$W_H(n) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right); & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases} \quad (4.94)$$

(or)

$$W_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2n\pi}{N-1}\right); & 0 \leq n \leq N-1 \\ 0; & \text{otherwise} \end{cases} \quad (4.95)$$



**Fig. 4.33** Linear phase realization of  $H(z)$  for Example 4.14

Frequency response,

$$W_H(\omega) = 0.54 \frac{\sin \frac{\omega N}{2}}{\sin(\omega/2)} + 0.23 \frac{\sin \left( \frac{\omega N}{2} - \frac{\pi N}{N-1} \right)}{\sin \left( \frac{\omega}{2} - \frac{\pi}{N-1} \right)} + 0.23 \frac{\sin \left( \frac{\omega N}{2} + \frac{\pi N}{N-1} \right)}{\sin \left( \frac{\omega}{2} + \frac{\pi}{N-1} \right)} \quad (4.96)$$

$$\text{Width of mainlobe} = \frac{8\pi}{N}$$

$$\text{Peak sidelobe magnitude (dB)} = -41 \text{ dB}$$

The Hamming window and its frequency response are shown in Fig. 4.34. The magnitude of the first sidelobe has been reduced to  $-41$  dB, an improvement of  $10$  dB compared to the Hanning window. But sidelobe magnitude at high frequencies is almost constant.

### Design Procedure

**Step 1** Choose the desired frequency response of the filter, i.e.,  $H_d(\omega)$ .

**Step 2** Take the inverse Fourier transform of  $H_d(\omega)$  to obtain the desired impulse response  $h_d(n)$ .

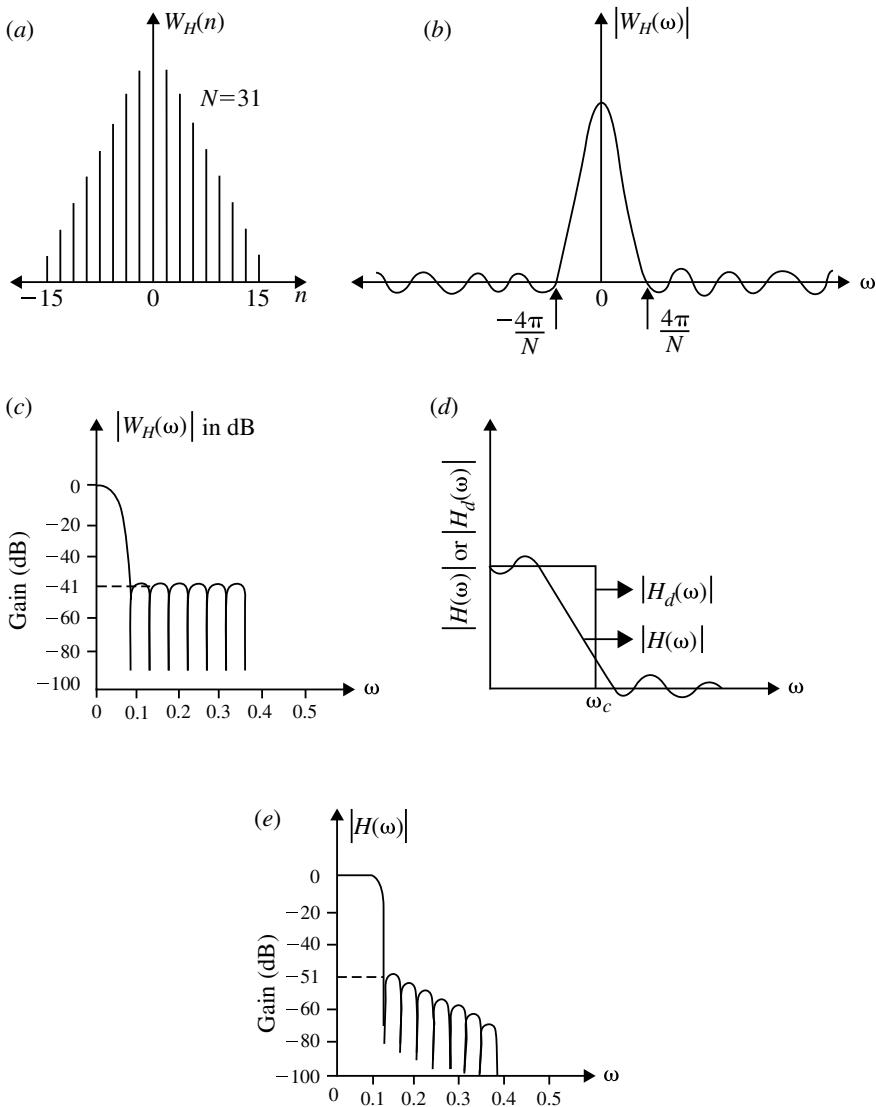
**Step 3** Choose a window sequence (here) Hamming window sequence with

$$W(n) = \begin{cases} 0.54 - 0.46 \cos \left( \frac{2\pi n}{N-1} \right); & 0 \leq n \leq N-1 \\ 0; & \text{otherwise} \end{cases}$$

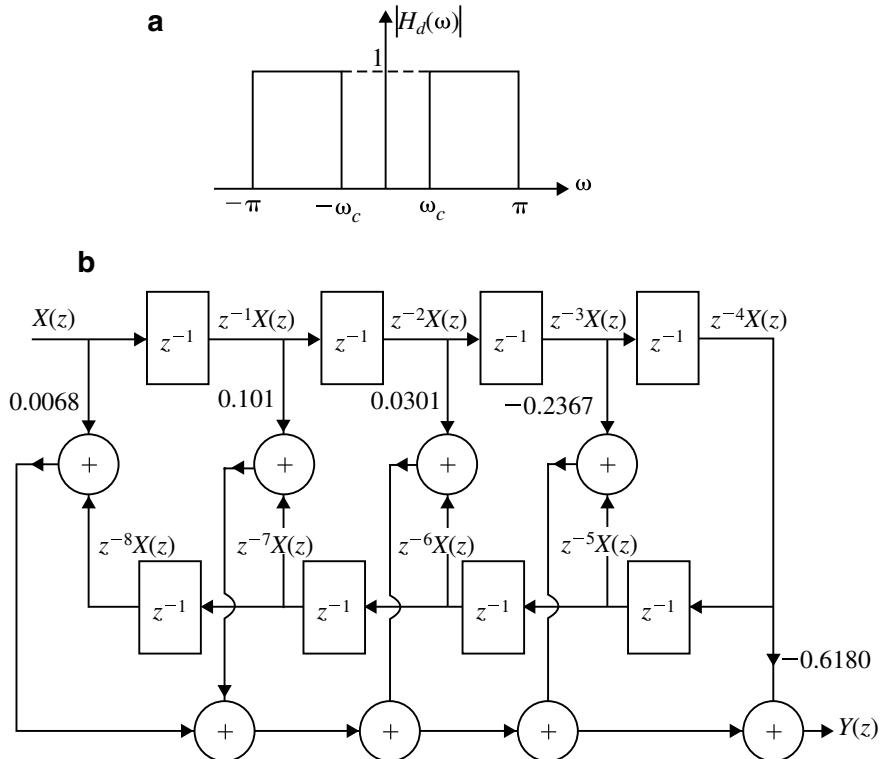
Now multiply  $h_d(n)$  and  $W(n)$  to get  $h(n)$

$$h(n) = h_d(n) \times W(n)$$

This step converts the infinite impulse response to finite impulse response  $h(n)$ .



**Fig. 4.34** Hamming window sequence and its frequency response. **a** Hamming window sequence. **b** Magnitude response of Hamming window. **c** Log-magnitude response of Hamming window. **d** Magnitude response of LPF approximated using Hamming window and **e** Log-magnitude response of FIR filter using Hamming window



**Fig. 4.35** a Hamming window characteristic. b Linear phase realization of  $H(z)$  for Example 4.15

**Step 4** The transfer function of the filter,  $H(z)$  is obtained by taking the  $z$ -transform of  $h(n)$ .

**Step 5** Realize the filter using suitable realization method (use Direct Form or Linear Phase).

### Example 4.15

Design a HPF with Hamming window. The cutoff frequency is  $1.2 \text{ rad s}^{-1}$   $N = 9$ .

#### Solution

**Step 1** Choose  $H_d(\omega)$ : The Hamming window characteristic is shown in Fig. 4.35a.

$$H(e^{j\omega}) = H_d(\omega) = \begin{cases} e^{-j\omega\alpha}; & -\pi \leq \omega \leq -\omega_c \quad \text{and} \quad \omega_c \leq \omega \leq \pi \\ 0; & \text{otherwise} \end{cases}$$

**Step 2** To Find  $h_d(n)$

$$\begin{aligned}
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \left[ \int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)} d\omega + \int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)} d\omega \right] \\
 &= \frac{1}{2\pi} \left[ \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\pi}^{-\omega_c} + \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{\omega_c}^{\pi} \right] \\
 &= \frac{1}{j2\pi(n-\alpha)} [e^{-j\omega_c(n-\alpha)} - e^{-j\pi(n-\alpha)} + e^{j\pi(n-\alpha)} - e^{j\omega_c(n-\alpha)}] \\
 &= \frac{1}{j2\pi(n-\alpha)} (2j) [\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)] \\
 &= \frac{1}{\pi(n-\alpha)} [\sin\pi(n-\alpha) - \sin\omega_c(n-\alpha)]
 \end{aligned}$$

**Step 3** To Find  $h(n)$

$$\begin{aligned}
 W(n) &= \left[ 0.54 - 0.46 \cos \left( \frac{2\pi n}{N-1} \right) \right] \\
 h(n) &= h_d(n) \times W(n) \\
 &= \frac{[\sin\pi(n-\alpha) - \sin\omega_c(n-\alpha)]}{\pi(n-\alpha)} \left[ 0.54 - 0.46 \cos \left( \frac{2\pi n}{N-1} \right) \right]
 \end{aligned}$$

where  $\alpha = \frac{N-1}{2} = 4$

$$h(n) = \frac{[\sin\pi(n-4) - \sin 1.2(n-4)]}{\pi(n-4)} \left[ 0.54 - 0.46 \cos \left( \frac{n\pi}{4} \right) \right]$$

$$h(0) = 6 \times 10^{-3} = 0.0063$$

$$h(1) = 0.0101$$

$$h(2) = 0.0581$$

$$h(3) = -0.2367$$

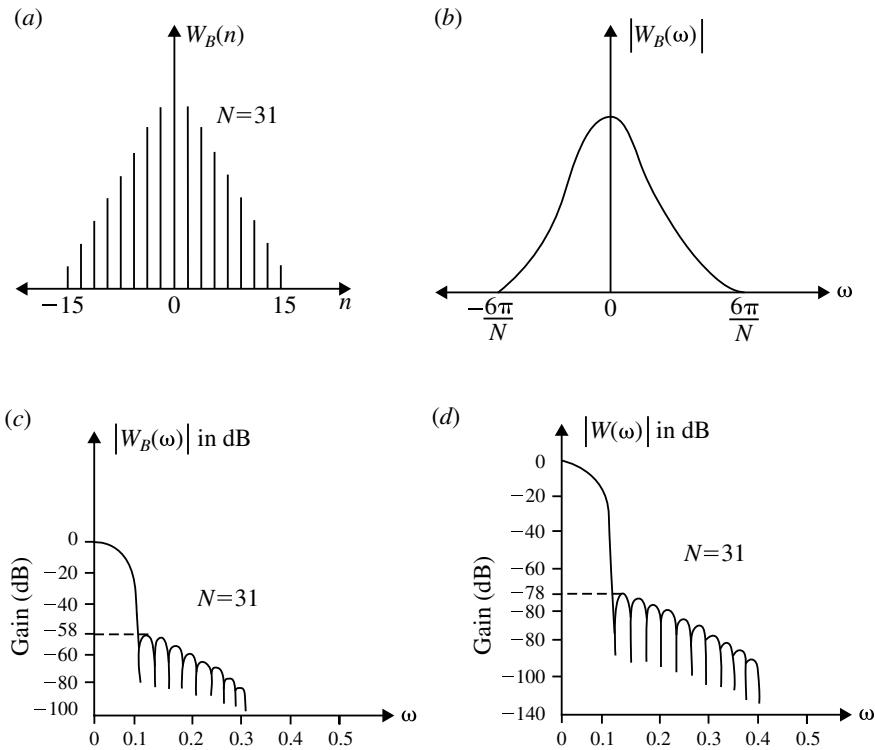
$$h(4) = -0.6180 \quad (\text{Applying L' Hospital rule})$$

$$h(5) = -0.2567$$

$$h(6) = 0.0581$$

$$h(7) = 0.0101$$

$$h(8) = 6.34 \times 10^{-3}$$



**Fig. 4.36** Blackman window sequence and its frequency response. **a** Blackman window sequence. **b** Magnitude response of Blackman window. **c** Log-magnitude response of Blackman window and **d** Log-magnitude response of FIR LPF approximated using Blackman window

**Step 4** Take  $z$ -transform:

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

$$\begin{aligned} H(z) = & 0.0063[1 + z^{-8}] + 0.101[z^{-1} + z^{-7}] + 0.0581[z^{-2} + z^{-6}] \\ & -0.2367[z^{-3} + z^{-5}] + (-0.6180)z^{-4} \end{aligned}$$

**Step 5** Draw the realization structure. The structure is realized as shown in Fig. 4.35b.

### 4.5.2.5 Blackman Window

The Blackman window function is another type of cosine window and is given by the following equation:

$$W_B(n) = \begin{cases} 0.42 + 0.5 \cos \frac{2n\pi}{N-1} + 0.08 \cos \frac{4n\pi}{N-1}; & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases} \quad (4.97)$$

(or)

$$W_B(n) = \begin{cases} 0.42 - 0.5 \cos \frac{2n\pi}{N-1} + 0.08 \cos \frac{4n\pi}{N-1}; & 0 \leq n \leq N-1 \\ 0; & \text{otherwise} \end{cases} \quad (4.98)$$

$$\text{Width of mainlobe} = \frac{12\pi}{N}$$

$$\text{Peak sidelobe magnitude (dB)} = -58 \text{ dB}$$

The frequency response and impulse response characteristics of Blackmann window are shown in Fig. 4.36.

The frequency response of Blackman window is shown in Fig. 4.36. The magnitude of the first sidelobe is  $-58$  dB and the sidelobe magnitude decreases with frequency. This desirable feature is achieved at the expense of increased mainlobe width.

### Design Procedure

1. Choose the desired frequency response  $H_d(\omega)$ . Take inverse Fourier transform of  $H_d(\omega)$  to obtain the desired impulse response  $h_d(n)$ .
2. Choose a window sequence  $W(n)$  and multiply  $h_d(n)$  and  $W(n)$  to convert the infinite impulse response to finite impulse response  $h(n)$ .
3. The transfer function  $H(z)$  is obtained by taking  $z$ -transform of  $h(n)$ .
4. Realize the filter using suitable realization method.

### Example 4.16

Design an ideal bandpass filter using Blackman window to pass frequencies in the range  $1.2\text{--}1.7$  rad/s with  $N = 7$ .

### Solution

**Step 1** Choose desired frequency response

$$\begin{aligned}
H_d(e^{j\omega}) &= \begin{cases} e^{-j\omega\alpha}, & -\omega_{c_2} \leq \omega \leq \omega_{c_1} \text{ and } \omega_{c_1} \leq \omega \leq \omega_{c_2} \\ 0, & \text{else} \end{cases} \\
\therefore H_d(e^{j\omega}) &= \begin{cases} e^{-j\omega\alpha}, & -1.7 \leq \omega \leq -1.2 \text{ and } 1.2 \leq \omega \leq 1.7 \\ 0, & \text{else} \end{cases} \\
h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\
h_d(n) &= \frac{1}{2\pi} \int_{-1.7}^{-1.2} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{1.2}^{1.7} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-1.7}^{-1.2} + \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{1.2}^{1.7} \\
&= \frac{1}{2\pi} \left[ \frac{e^{-j1.7(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-1.2j(n-\alpha)}}{j(n-\alpha)} + \frac{e^{1.7j(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j1.2(n-\alpha)}}{j(n-\alpha)} \right] \\
&= \frac{1}{2\pi} \left[ \frac{2j \sin 1.7(n-\alpha) - 2j \sin 1.2(n-\alpha)}{j(n-\alpha)} \right] \\
&= \frac{1}{\pi(n-\alpha)} [\sin 1.7(n-\alpha) - \sin 1.2(n-\alpha)]
\end{aligned}$$

where  $\alpha = \frac{N-1}{2} = 3$

$$\begin{aligned}
\therefore h_d(n) &= \frac{1}{\pi(n-3)} [\sin 1.7(n-3) - \sin 1.2(n-3)] \\
W(n) &= \begin{cases} 0.42 - 0.5 \cos\left(\frac{2\pi n}{6}\right) + 0.08 \cos\left(\frac{4\pi n}{6}\right); & 0 \leq n \leq 6 \\ 0; & \text{otherwise} \end{cases}
\end{aligned}$$

## Step 2

$$\begin{aligned}
h(n) &= W(n)h_d(n) \quad 0 \leq n \leq N-1 \\
&= \left[ 0.42 - 0.5 \cos\left(\frac{2\pi n}{6}\right) + 0.08 \cos\left(\frac{4\pi n}{6}\right) \right] \\
&\quad \times \frac{1}{\pi(n-3)} [\sin 1.7(n-3) - \sin 1.2(n-3)]; \quad 0 \leq n \leq 6
\end{aligned}$$

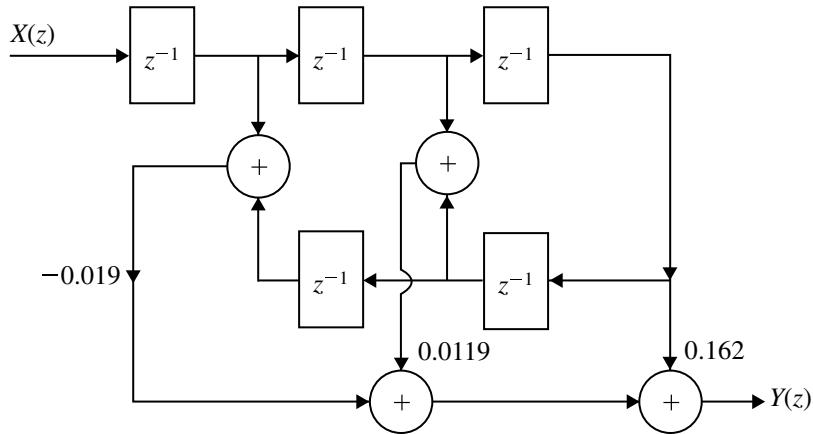
$$h(0) = 0$$

$$h(1) = -0.019$$

$$h(2) = 0.0119$$

$$h(3) = \text{Apply L'Hospital's Rule}$$

$$h(4) = 0.0119$$



**Fig. 4.37** Linear phase realization of  $H(z)$  for Example 4.16

$$h(5) = -0.019$$

$$h(6) = 0$$

$$\begin{aligned} h(3) &= \left( \left[ 0.42 - 0.5 \sin\left(\frac{2\pi \times 3}{6}\right) - 0.08 \sin\left(\frac{4\pi \times 3}{6}\right) \right] \right. \\ &\quad \times [\sin 1.7(n-3) - \sin 1.2(n-3)] \Big/ \pi(n-3) \\ &= \left[ 0.42 - 0.5 \cos\left(\frac{2\pi \times 3}{6}\right) + 0.08 \cos\left(\frac{4\pi \times 3}{6}\right) \right] \\ &\quad \times \left[ \frac{\sin 1.7(n-3)}{1.7(n-3)} \cdot 1 \cdot 7 - \frac{\sin 1.2(n-3)}{1.2(n-3)} \cdot 1 \cdot 2 \right] \Big) \Big/ \pi \\ &= \frac{1 \times (0.5)}{\pi} = 0.162 \end{aligned}$$

### Step 3

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} \\ &\quad + h(5)z^{-5} + h(6)z^{-6} \\ H(z) &= -0.019z^{-1} + 0.0119z^{-2} + 0.162z^{-3} + 0.0119z^{-4} - 0.019z^{-5} \\ &= -0.019(z^{-1} + z^{-5}) + 0.0119(z^{-2} + z^{-4}) + 0.162z^{-3} \end{aligned}$$

### Step 4 Realization Structure

The structure realization is shown in Fig. 4.37.

#### 4.5.2.6 Kaiser Window

Kaiser window function is given in the following form:

$$W_K(n) = \begin{cases} \frac{I_0\left(\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right)}{I_0(\alpha)}; & -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases} \quad (4.99)$$

or

$$W_K(n) = \begin{cases} \frac{I_0\left(\alpha\sqrt{\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2}\right)}{I_0(\alpha\frac{N-1}{2})}; & 0 \leq n \leq \frac{N-1}{2} \\ 0; & \text{otherwise} \end{cases} \quad (4.100)$$

The parameter  $\alpha$  is an independent variable that can be varied to control the sidelobe levels with respect to the mainlobe peak. The modified Bessel function of the first kind  $I_0(x)$  is given by

$$I_0(x) = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{k!} \left( \frac{x}{2} \right)^k \right]^2 = 1 + \sum_{k=1}^{\infty} \left( \frac{(0.5)^2}{(k!)^2} \right)^k \quad (4.101)$$

$$= 1 + \frac{(0.25x^2)^2}{(k!)^2} = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \dots \quad (4.102)$$

This equation can be used to compute  $I_0(\alpha)$  and  $I_0\left(\alpha\sqrt{\left(\frac{2n}{N-1}\right)^2}\right)$ . Figure 4.38 shows the Kaiser window sequence and its frequency response. The width of the mainlobe can be adjusted by varying the length  $N$  of the window sequence.

#### Design Procedure

- (i) Determine “ $\delta$ ” using the formula

$$\delta = \min(\delta_p, \delta_s)$$

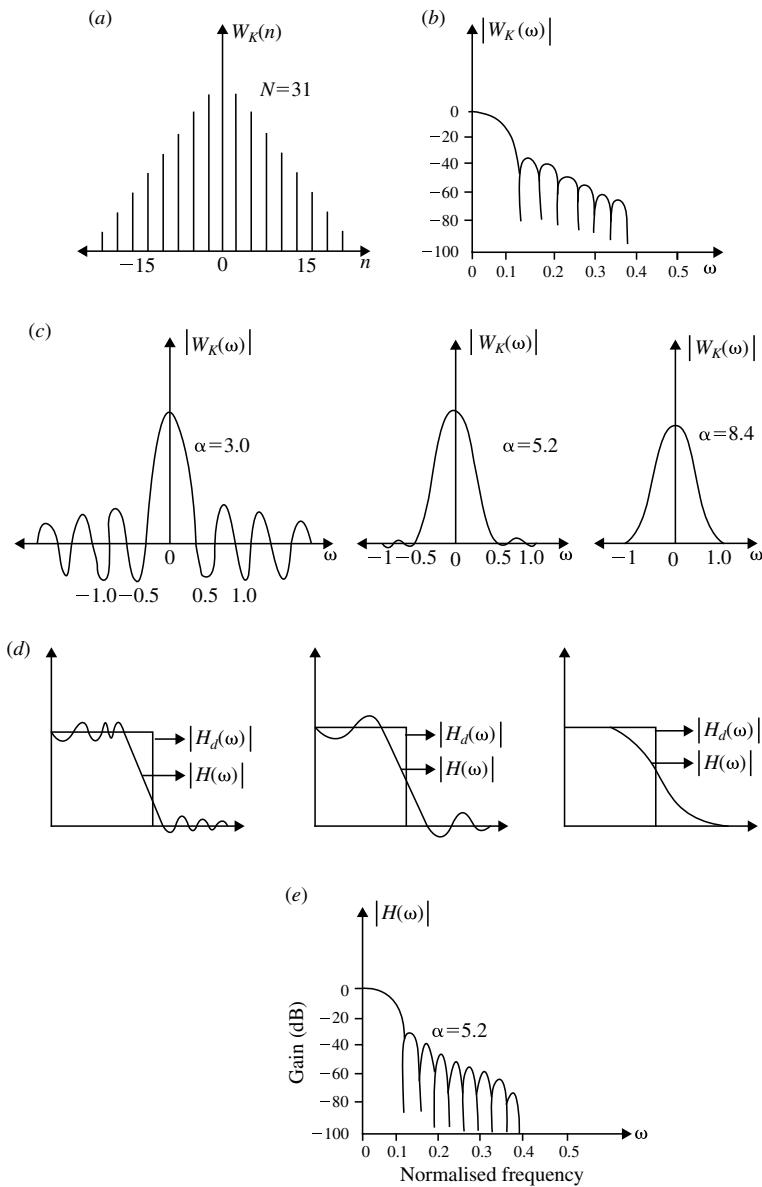
where

$$\delta_p = \frac{10^{0.05A_p} - 1}{10^{0.05A_p} + 1}$$

$$\delta_s = 10^{-0.05A_s}$$

- (ii) Find the actual stopband attenuation using “ $\delta$ ”

$$A'_s = -20 \log_{10} \delta$$



**Fig. 4.38** Kaiser window sequence and its frequency response. **a** Kaiser window sequence. **b** Log-magnitude response of Kaiser window. **c** Magnitude frequency response of Kaiser window for different values of  $\alpha$  (when  $N = 31$ ). **d** Magnitude response of Kaiser window for different values of  $\alpha$  (when  $N = 31$ ) and **e** Log-magnitude response of FIR LPF designed using Kaiser window

**Table 4.1** Comparison of window function in frequency domain characteristics

Type of window	Approximate width of mainlobe	Peak sidelobe magnitude (dB)
Rectangular	$4\pi/N$	-13
Hanning	$8\pi/N$	-31
Hamming	$8\pi/N$	-41
Blackman	$12\pi/N$	-58

Comparison of window function in frequency domain characteristics is shown in Table 4.1.

- (iii) Determine Kaiser window design parameter using the following equation:

$$\alpha_k = \begin{cases} 0, & A'_s \leq 21 \\ 0.5842(A'_s - 21)^{0.4} + 0.07886(A'_s - 21), & 21 \leq A'_s \leq 50 \\ 0.1102(A_s - 8.7), & A'_s > 50 \end{cases}$$

- (iv) Determine the design parameter “D” using the following equation:

$$D = \begin{cases} 0.9222, & A'_s \leq 21 \\ \frac{A'_s - 7.95}{14.36}, & A'_s > 21 \end{cases}$$

- (v) Determine the order of filter using the following equation:

$$N \geq \frac{FD}{\Delta_f} + 1$$

$$\Delta_f = f_s - f_p$$

where  $D$  = design parameter and  $F$  = sampling frequency.

- (vi) Select the desired frequency response  $H_d(\omega)$

$$\omega_c = \frac{2\pi f_c}{F} = \frac{2\pi(1/2)(f_p + f_s)}{F}$$

where  $f_c = (1/2)(f_p + f_s)$ .

- (vii) Find  $h_d(n)$  using inverse Fourier transform

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

(viii) Find the window function using  $\omega(n)$  equation. Bessel function

$$I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \frac{(0.25x^2)^4}{(4!)^2} + \frac{(0.25x^2)^5}{(5!)^2} + \frac{(0.25x^2)^6}{(6!)^2}$$

$$\omega(n) = \frac{I_0\left[\alpha_k \sqrt{\left(\frac{N-1}{2}\right)^2 \left(n - \frac{N-1}{2}\right)^2}\right]}{I_0\left[\alpha_k \left(\frac{N-1}{2}\right)\right]}; \quad 0 \leq n \leq N-1$$

$$\omega(n) = \frac{I_0\left[\alpha \sqrt{1 - \left(\frac{2n}{N-1}\right)^2}\right]}{I_0[\alpha]}; \quad -\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2}; \text{ and } |n| \leq \frac{N-1}{2}$$

(ix) Find  $h(n)$  using  $h(n) = h_d(n)\omega(n)$ .

(x) Find  $H(z)$  using  $z$ -transform

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

(xi) Draw suitable realization.

### Example 4.17

Design a FIR lowpass filter using Kaiser window filter the following specifications.

Passband cutoff frequency = 150 Hz

Stopband cutoff frequency = 250 Hz

Passband ripple = 0.1 dB

Stopband attenuation = 40 dB

Sampling frequency = 1000 Hz

### Solution Given

$$f_p = 150 \text{ Hz}$$

$$f_s = 250 \text{ Hz}$$

$$A_p = 0.1 \text{ dB}$$

$$A_s = 40 \text{ dB}$$

$$F = 1000 \text{ Hz}$$

**Step 1** Find  $\delta$  using the following equation:

$$\begin{aligned}\delta_p &= \frac{10^{0.05A_p} - 1}{10^{0.05A_p} + 1} = \frac{10^{0.05 \times 0.1} - 1}{10^{0.05 \times 0.1} + 1} = 5.76 \times 10^{-3} \\ &= 0.00576 \\ \delta_s &= 10^{-0.05A_s} = 10^{-0.05 \times 40} = 0.01 \\ \delta &= \min(\delta_p, \delta_s) \\ \therefore \delta &= 0.00576\end{aligned}$$

**Step 2** Find actual stopband attenuation,  $A'_s$  using the following equation:

$$\begin{aligned}A'_s &= -20 \log_{10} \delta \\ &= -20 \log_{10}(0.00576) \\ \therefore A'_s &= 44.8 \text{ dB}\end{aligned}$$

**Step 3** Find the Kaiser window parameter,  $\alpha_k$  using the following equation:

$$\begin{aligned}\alpha_k &= 0.5842(A'_s - 21)^{0.4} + 0.07886(A'_s - 21) \\ \alpha_k &= 0.5842(44.8 - 21)^{0.4} + 0.07886(44.8 - 21) \\ \therefore \alpha_k &= 3.953\end{aligned}$$

**Step 4** Find the design parameter “ $D$ ” (because  $A'_s > 21$ ) using the following equation:

$$\begin{aligned}D &= \frac{A'_s - 7.95}{14.36} = \frac{44.8 - 7.95}{14.36} \\ \therefore D &= 2.566\end{aligned}$$

**Step 5** Find the order of the filter,  $N$  using the following equation:

$$\begin{aligned}N &\geq \frac{F \cdot D}{\Delta f} + 1 = \frac{F \cdot D}{f_s - f_p} + 1 \\ &= \frac{1000 \times 2.566}{(250 - 150)} + 1 \\ &= 26.66 \\ \therefore N &= 27\end{aligned}$$

**Step 6** To find cutoff frequency,  $\omega_c$ , using the following equation is used

$$\begin{aligned}
 f_c &= \frac{1}{2}(f_s + f_p) \\
 &= \frac{1}{2}(250 + 150) = 200 \text{ Hz} \\
 \omega_c &= \frac{2\pi f_c}{F} \quad (\text{normalized by } F) \\
 \omega_c &= \frac{2\pi \times 200}{1000} = 1.2566 \\
 \therefore \omega_c &= 1.2566
 \end{aligned}$$

**Step 7** To find  $h_d(n)$ , the following procedure is followed.

For a lowpass filter, the frequency response is given by

$$\begin{aligned}
 H_d(e^{j\omega}) &= \begin{cases} e^{-j\omega\alpha}, & -\omega_c \leq \omega \leq \omega_c \\ 0, & \text{else} \end{cases} \\
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\
 h_d(n) &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega \\
 &\text{as } \omega_c = 1.2566 \text{ (from Step 6)} \\
 h_d(n) &= \frac{1}{2\pi} \int_{-1.2566}^{1.2566} e^{j\omega(n-\alpha)} d\omega \\
 h_d(n) &= \frac{1}{2\pi} \left. \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right|_{-1.2566}^{1.2566} \\
 h_d(n) &= \frac{1}{2\pi} \left[ \frac{e^{j1.2566(n-\alpha)} - e^{-j1.2566(n-\alpha)}}{j(n-\alpha)} \right] \\
 h_d(n) &= \frac{\sin 1.2566(n-\alpha)}{\pi(n-\alpha)}
 \end{aligned}$$

because

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\text{As } \alpha = \frac{N-1}{2} = \frac{27-1}{2} = 13$$

$$h_d(n) = \frac{\sin 1.2566(n-13)}{\pi(n-13)} \quad 0 \leq n \leq N-1 \text{ and } 0 \leq n \leq 26$$

$$\begin{aligned}
h_d(0) &= -0.01438 & h_d(7) &= 0.050 \\
h_d(1) &= 0.0156 & h_d(8) &= -1.1797 \times 10^{-5} \\
h_d(2) &= 0.0275 & h_d(9) &= -0.075 \\
h_d(3) &= -1.1797 \times 10^{-5} & h_d(10) &= -0.0623 \\
h_d(4) &= -0.033 & h_d(11) &= 0.09355 \\
h_d(5) &= -0.0233 & h_d(12) &= 0.3027 \\
h_d(6) &= 0.0267 & h_d(13) &= \frac{1.2566}{\pi} = 0.3999
\end{aligned}$$

**Step 8** To find the window function  $W(n)$ . As  $N$  is very large

$$\begin{aligned}
W(n) &= \frac{I_0 \left[ \alpha_k \sqrt{1 - \left( \frac{2n}{N-1} \right)^2} \right]}{I_0[\alpha_k]} \quad 0 \leq n \leq N-1; \quad 0 \leq n \leq 26 \\
W(n) &= \frac{I_0 \left[ 3.953 \sqrt{1 - \left( \frac{2n}{26} \right)^2} \right]}{I_0[3.953]} \quad [\because N = 27] \\
I_0[x] &= 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \cdots + \frac{(0.25x^2)^6}{(6!)^2}
\end{aligned}$$

$$\begin{aligned}
W(0) &= \frac{I_0[3.953]}{I_0[3.953]} = \frac{10.824}{10.824} = 1 \\
W(1) &= \frac{I_0[3.938]}{10.824} = 0.99 \\
W(2) &= \frac{I_0[3.9]}{10.824} = 0.9582 \\
W(3) &= \frac{I_0[3.843]}{10.824} = 0.9125 \\
W(4) &= \frac{I_0[3.758]}{10.824} = 0.8485 \\
W(5) &= \frac{I_0[3.646]}{10.824} = 0.77 \\
W(6) &= \frac{I_0[3.504]}{10.824} = 0.684 \\
W(7) &= \frac{I_0[3.328]}{10.824} = 0.5905 \\
W(8) &= \frac{I_0[3.113]}{10.824} = 0.4955
\end{aligned}$$

$$\begin{aligned}
 W(9) &= \frac{I_0[2.85]}{10.824} = 0.3998 \\
 W(10) &= \frac{I_0[2.5239]}{10.824} = 0.30967 \\
 W(11) &= \frac{I_0[2.105]}{10.824} = 0.22689 \\
 W(12) &= \frac{I_0[1.519]}{10.824} = 0.1539 \\
 W(13) &= \frac{I_0[0]}{10.824} = 0.09242
 \end{aligned}$$

**Step 9** To Find  $h(n)$

$$h(n) = h_d(n)W(n) \quad 0 \leq n \leq 26$$

$$\begin{array}{ll}
 h(0) = -0.01438 & h(7) = 0.029526 \\
 h(1) = 0.015444 & h(8) = -5.8419 \times 10^{-6} \\
 h(2) = 0.02635 & h(9) = -0.02998 \\
 h(3) = -1.0758 \times 10^{-5} & h(10) = -0.01929 \\
 h(4) = -0.028 & h(11) = 0.02122 \\
 h(5) = -0.0179 & h(12) = 0.04658 \\
 h(6) = 0.01826 & h(13) = 0.036958
 \end{array}$$

Applying the linear phase condition of an FIR filter, that is,  $h(n) = h(N - 1 - n)$ , we get

$$\begin{array}{ll}
 h(0) = h(26) & h(7) = h(19) \\
 h(1) = h(25) & h(8) = h(18) \\
 h(2) = h(24) & h(9) = h(17) \\
 h(3) = h(23) & h(10) = h(16) \\
 h(4) = h(22) & h(11) = h(15) \\
 h(5) = h(21) & h(12) = h(14) \\
 h(6) = h(20) & h(13) = h(13)
 \end{array}$$

**Step 10** To Find  $H(z)$

$$\begin{aligned}
 H(z) = & -0.01438(1+z^{-26}) + 0.01544(z^{-1}+z^{-25}) + 0.02635(z^{-2}+z^{-24}) \\
 & -1.075 \times 10^{-5}(z^{-3}+z^{-23}) - 0.0028(z^{-4}+z^{-22}) \\
 & -0.0179(z^{-5}+z^{-21}) + 0.01826(z^{-6}+z^{-20}) \\
 & + 0.029525(z^{-7}+z^{-19}) - 5.8419 \times 10^{-6}(z^{-8}+z^{-18})
 \end{aligned}$$

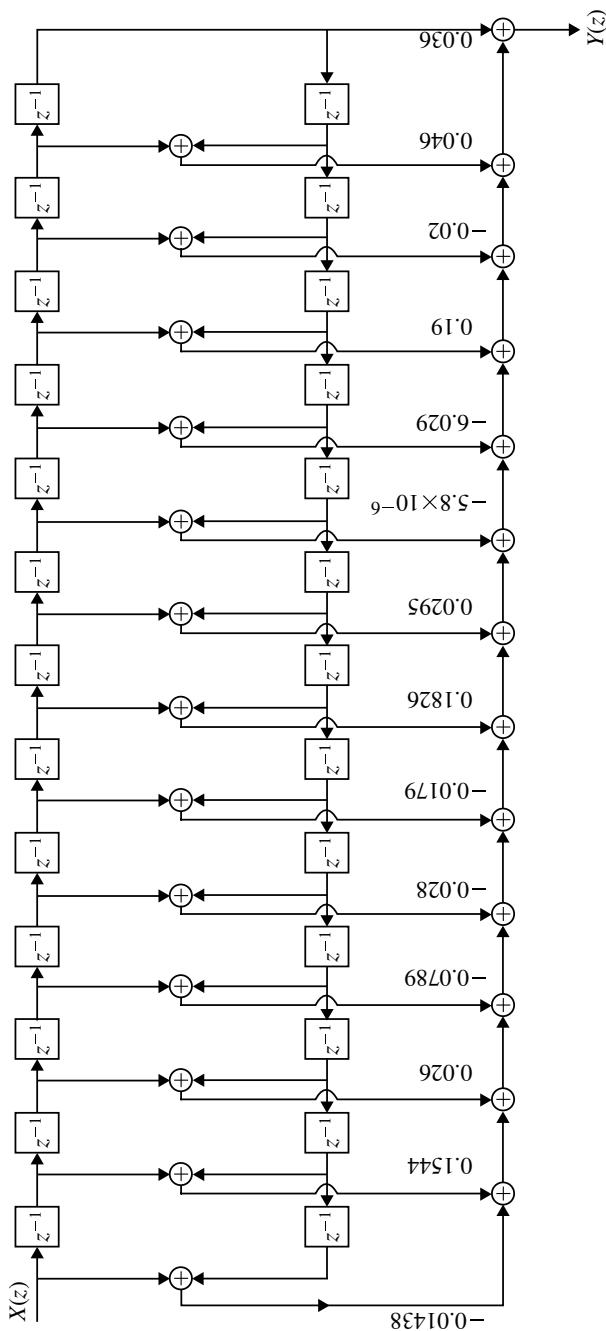


Fig. 4.39 Linear phase realization of  $H(z)$  for Example 4.17

$$\begin{aligned} & -0.02998(z^{-9} + z^{-17}) - 0.01929(z^{-10} + z^{-16}) \\ & + 0.02122(z^{-11} + z^{-15}) + 0.04658(z^{-12} + z^{-14}) + 0.036958(z^{-13}) \end{aligned}$$

**Step 11** Draw the realization structure. This is shown in Fig. 4.39.

### 4.5.3 Frequency Sampling Method

In this method the ideal frequency response is sampled at sufficient number of points.

Let  $h(n)$  be the filter coefficients of an FIR filter and  $H(k)$  be the DFT of  $h(n)$

$$\begin{aligned} H(k) &= \sum_{k=0}^{N-1} h(n)e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \dots, N-1 \\ h(n) &= \frac{1}{N} \sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi kn}{N}}, \quad n = 0, 1, \dots, N-1 \end{aligned}$$

The DFT samples  $H(k)$  for an FIR sequence can be regarded as samples of the filter  $z$ -transform evaluated at  $N$ -points equally spaced around the unit circle.

$$H(k) = H(z) \Big|_{z=e^{j\frac{2\pi k}{N}}}$$

The transfer function  $H(z)$  of an FIR filter is given by

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\ H(z) &= \sum_{n=0}^{N-1} \left[ \frac{1}{N} \sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi kn}{N}} \right] z^{-n} \\ &= \sum_{k=0}^{N-1} \frac{H(k)}{N} \sum_{n=0}^{N-1} H(k) \left( e^{j\frac{2\pi k}{N}} z^{-1} \right)^n \\ &= \sum_{k=0}^{N-1} \frac{H(k)}{N} \frac{1 - \left( e^{j\frac{2\pi k}{N}} z^{-1} \right)^N}{1 - e^{j\frac{2\pi k}{N}} z^{-1}} \\ &= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j\frac{2\pi k}{N}} z^{-1}} \end{aligned}$$

Using the following identity we get

$$H(e^{j\omega}) \Big|_{\omega=\frac{2\pi k}{N}} = H(e^{j\frac{2\pi kn}{N}}) = H(k)$$

i.e.,  $H(k)$  is the  $k^{\text{th}}$  DFT component obtained by sampling the frequency response  $H(e^{j\omega})$ . As such this approach for designing FIR filter is called the frequency sampling. There are two design procedures.

1. Type-1 design procedure.
2. Type-2 design procedure.

### Type-1 Design Procedure

**Step 1** Choose the desired frequency response  $H_d(\omega)$ .

**Step 2** Sample  $H_d(\omega)$  at “ $N$ ” points by taking  $\omega = \frac{2\pi k}{N}$  and generate the sequence

$$\tilde{H}(k) = H_d(\omega) \Big|_{\omega=\frac{2\pi k}{N}}, \quad k = 0 \text{ to } N - 1 \quad (4.103)$$

**Step 3** Compute  $h(n)$  using the following equations.

$$\text{When } N \text{ is odd } h(n) = \frac{1}{N} \left\{ \tilde{H}(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \operatorname{Re} \left[ \tilde{H}(k) e^{j\frac{2\pi kn}{N}} \right] \right\} \quad (4.104)$$

$$\text{When } N \text{ is even } h(n) = \frac{1}{N} \left\{ \tilde{H}(0) + 2 \sum_{k=1}^{(N/2)-1} \operatorname{Re} \left[ \tilde{H}(k) e^{j\frac{2\pi kn}{N}} \right] \right\} \quad (4.105)$$

**Step 4** Take  $z$ -transform of  $h(n)$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \quad (4.106)$$

**Step 5** Draw the realization structure.

### Type-2 Design Procedure

**Step 1** Choose the desired frequency response  $H_d(\omega)$ .

**Step 2** Sample  $H_d(\omega)$  at “ $N$ ” points by taking  $\omega = \frac{\pi(2k+1)}{N}$  and generate the sequence, i.e.,

$$\tilde{H}(k) = H_d(\omega) \Big|_{\omega=\frac{\pi(2k+1)}{N}}, \quad k = 0, 1, \dots, N - 1 \quad (4.107)$$

**Step 3** Compute  $h(n)$  using the following equations

$$\text{When } N \text{ is odd, } h(n) = \frac{2}{N} \left\{ \sum_{k=0}^{\frac{N-3}{2}} \operatorname{Re} \left[ \tilde{H}(k) e^{j n \pi (2k+1)} \right] \right\} \quad (4.108)$$

$$\text{When } N \text{ is even, } h(n) = \frac{2}{N} \left\{ \sum_{k=0}^{(N/2)-1} \operatorname{Re} \left[ \tilde{H}(k) e^{j n \pi (2k+1)} \right] \right\} \quad (4.109)$$

**Step 4** Take  $z$ -transform of  $h(n)$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \quad (4.110)$$

**Step 5** Draw the realization structure.

---

### Example 4.18

Determine the filters coefficient of a linear phase FIR filter of length  $N = 15$ , which has a symmetric unit sample response and a frequency response that satisfies the following conditions.

$$H_r \left( \frac{2\pi k}{15} \right) = \begin{cases} 1; & k = 0, 1, 2, 3 \\ 0.4; & k = 4 \\ 0; & k = 5, 6, 7 \end{cases}$$


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(Anna University, May, 2007)

#### Solution

##### Step 1

$$\tilde{H}(k) = H_d(\omega) \Big/ \omega = \frac{2\pi k}{15} = \begin{cases} 1e^{-j\omega\alpha}; & k = 0, 1, 2, 3 \\ 0.4e^{-j\omega\alpha}; & k = 4 \\ 0; & k = 5, 6, 7 \end{cases}$$

$$\alpha = \frac{N-1}{2} = 7$$

$$\tilde{H}(k) = \begin{cases} e^{-j\frac{2\pi k}{15}\cdot\alpha}; & k = 0, 1, 2, 3 \\ 0.4e^{-j\frac{2\pi k}{15}\alpha}; & k = 4 \\ 0; & k = 5, 6, 7 \end{cases}$$

**Step 2** When

$$\begin{aligned}
 k = 0, \quad \tilde{H}(0) &= 1 \\
 k = 1, \quad \tilde{H}(1) &= e^{-j \frac{2\pi}{15}(1) \times 7} = e^{-j \frac{14\pi}{15}} \\
 k = 2, \quad \tilde{H}(2) &= e^{-j \frac{2\pi}{15}(2) \times 7} = e^{-j \frac{28\pi}{15}} \\
 k = 3, \quad \tilde{H}(3) &= e^{-j \frac{2\pi}{15}(3) \times 7} = e^{-j \frac{42\pi}{15}} \\
 k = 4, \quad \tilde{H}(4) &= 0.4e^{-j \frac{2\pi}{15}(4) \times 7} = 0.4e^{-j \frac{56\pi}{15}} \\
 k = 5, \quad \tilde{H}(5) &= 0 \\
 k = 6, \quad \tilde{H}(6) &= 0 \\
 k = 7, \quad \tilde{H}(7) &= 0
 \end{aligned}$$

**Step 3** To Find  $h(n)$ . Here  $N = 15$  (odd)

$$\begin{aligned}
 \therefore h(n) &= \frac{1}{N} \left[ \tilde{H}(0) + 2 \sum_{k=1}^7 \operatorname{Re} \left[ \tilde{H}(k) e^{j \frac{2\pi kn}{N}} \right] \right] \\
 h(n) &= \frac{1}{15} \left[ 1 + 2 \sum_{k=1}^3 \operatorname{Re} \left[ e^{-j7 \times \frac{2\pi k}{15}} \times e^{j \frac{2\pi nk}{15}} \right] + 2 \operatorname{Re} \left[ 0.4 \times e^{-j \frac{56\pi}{15}} \times e^{j \frac{2\pi nk}{15}} \right] \right] \\
 &= \frac{1}{15} \left[ 1 + 2 \sum_{k=1}^3 \cos \frac{2\pi k}{15} (n - 7) + 0.8 \cos \frac{8\pi}{15} (n - 7) \right] \\
 &= \frac{1}{15} + \frac{2}{15} \cos \frac{2\pi}{15} (n - 7) + \frac{2}{15} \cos \frac{4\pi}{15} (n - 7) + \frac{2}{15} \cos \frac{6\pi}{15} (n - 7) \\
 &\quad + \frac{0.8}{15} \cos \frac{8\pi}{15} (n - 7)
 \end{aligned}$$

$$\begin{array}{ll}
 h(0) = -0.71155 & h(8) = 4.778 \\
 h(1) = 0.575 & h(9) = 0.459 \\
 h(2) = 0.973 & h(10) = -1.6015 \\
 h(3) = -0.499 & h(11) = -0.499 \\
 h(4) = -1.6015 & h(12) = 0.973 \\
 h(5) = 0.459 & h(13) = 0.575 \\
 h(6) = 4.778 & h(14) = -0.71155 \\
 h(7) = 7.053 &
 \end{array}$$

**Example 4.19**

A lowpass filter is to be designed with the following desired frequency response

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\pi/4 \leq \omega \leq \pi/4 \\ 0; & \pi/4 \leq |\omega| \leq \pi \end{cases}$$

Determine the filter coefficients  $h_d(n)$  if window function

$$\omega(n) = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

(Anna University, December, 2007)

**Solution** Given

$$\begin{aligned} H_d(e^{j\omega}) &= \begin{cases} e^{-j2\omega}, & -\pi/4 \leq \omega \leq \pi/4 \\ 0; & \pi/4 \leq |\omega| \leq \pi \end{cases} \\ h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j2\omega} \cdot e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega(n-2)} d\omega \\ &= \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-2)}}{j(n-2)} \right]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{2\pi} \left[ \frac{e^{j(\pi/4)(n-2)} - e^{-j(\pi/4)(n-2)}}{j(n-2)} \right] \\ &= \frac{1}{2\pi} \left[ \frac{\sin(\pi/4)(n-2)}{(n-2)} \right] \quad n \neq 2 \end{aligned}$$

From the given frequency response it is clear that

$$N - 1 = 4 \Rightarrow N = 5 \Rightarrow \alpha = \frac{N - 1}{2} = 2$$

if  $n = 2$ . Apply L' Hospital's rule.

$$h_d(n) = \lim_{n \rightarrow 2} \frac{\cos(\pi/4)(n-2) - (\pi/4)}{\pi(-1)}$$

$$h_d(2) = 1 \cdot \frac{\pi/4}{\pi} = \frac{1}{4}$$

$$\therefore h_d(n) = \begin{cases} \frac{\sin(\pi/4)(n-2)}{\pi(n-2)}, & n \neq 2 \\ \frac{1}{4}, & n = 2 \end{cases}$$

To determine finite impulse response  $h(n)$

$$h(n) = h_d(n)\omega(n) \quad \left[ \because \omega(n) = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{else} \end{cases} \right]$$

$$h(n) = \begin{cases} h_d(n), & 0 \leq n \leq 4 \\ 0, & \text{else} \end{cases}$$

Therefore, filter coefficients are

$$h(0) = \frac{\sin(\pi/4)(-2)}{\pi(n-2)} = 0.159$$

$$h(1) = \frac{\sin(\pi/4)(-1)}{\pi(-1)} = 0.225$$

$$h(2) = \frac{1}{4}$$

$$h(3) = \frac{\sin(\pi/4)(1)}{\pi(1)} = 0.225$$

$$h(4) = \frac{\sin(\pi/4)(2)}{\pi(2)} = 0.159$$

### Example 4.20

A filter is to be designed with the following desired frequency response

$$H_d(e^{j\omega}) = \begin{cases} 0, & -\pi/4 \leq \omega \leq \pi/4 \\ e^{-j2\omega}; & \pi/4 \leq |\omega| \leq \pi \end{cases}$$

Determine the filter coefficients  $h_d(n)$  using Hanning window with  $N = 5$ .

**Solution Given**

$$\begin{aligned}
 H_d(e^{j\omega}) &= \begin{cases} 0, & -\pi/4 \leq \omega \leq \pi/4 \\ e^{-j2\omega}, & \pi/4 \leq |\omega| \leq \pi \end{cases} \\
 h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \left[ \int_{-\pi}^{-\pi/4} e^{-j2\omega} \cdot e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{-j2\omega} \cdot e^{j\omega n} d\omega \right] \\
 &= \frac{1}{2\pi} \left[ \int_{-\pi}^{-\pi/4} e^{j\omega(n-2)} d\omega + \int_{\pi/4}^{\pi} e^{j\omega(n-2)} d\omega \right] \\
 &= \frac{1}{2\pi} \left[ \left[ \frac{e^{j\omega(n-2)}}{j(n-2)} \right]_{-\pi}^{-(\pi/4)} + \left[ \frac{e^{j\omega(n-2)}}{j(n-2)} \right]_{\pi/4}^{\pi} \right] \\
 &= \frac{1}{2\pi} \left[ \frac{e^{-j(\pi/4)(n-2)} - e^{-j\pi(n-2)} + e^{j\pi(n-2)} - e^{j(\pi/4)(n-2)}}{j(n-2)} \right] \\
 h_d(n) &= \left[ \frac{\sin \pi(n-2) - \sin(\pi/4)(n-2)}{\pi(n-2)} \right] \quad n \neq 2
 \end{aligned}$$

if  $n = 2$ . Apply L' Hospital's rule.

$$\begin{aligned}
 h_d(n) &= \lim_{n \rightarrow 2} \frac{\cos \pi(n-2)(-\pi) - \cos(\pi/4)(n-2)[-(\pi/4)]}{\pi(-1)} \\
 h_d(2) &= \frac{\pi - (\pi/4)}{\pi} = \frac{(3\pi/4)}{4} = \frac{3}{4} \\
 \therefore h_d(n) &= \begin{cases} \frac{\sin \pi(n-2) - \sin(\pi/4)(n-2)}{\pi(n-2)}, & n \neq 2 \\ \frac{3}{4}, & n = 2 \end{cases}
 \end{aligned}$$

To determine  $h(n)$ ,

$$h(n) = h_d(n)\omega(n)$$

where

$$\omega_{\text{Han}}(n) = \begin{cases} 0.5 - 0.5 \cos \frac{2\pi n}{N-1}, & 0 < n < N-1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore

$$\omega_{\text{Han}}(n) = \begin{cases} 0.5 - 0.5 \cos \frac{2\pi n}{4}, & 0 < n < 4 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the filter coefficients are,

$$\begin{array}{lll} h_d(0) = -0.159 & \omega(0) = 0 & h(0) = 0 \\ h_d(1) = -0.225 & \omega(1) = 0.5 & h(1) = -0.1125 \\ h_d(2) = \frac{3}{4} & \omega(2) = 1 & h(2) = \frac{3}{4} \\ h_d(3) = -0.225 & \omega(3) = 0.5 & h(3) = -0.1125 \\ h_d(4) = -0.159 & \omega(4) = 0 & h(4) = 0 \end{array}$$


---

### Example 4.21

The desired response of a LPF is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & -3\pi/4 \leq \omega \leq 3\pi/4 \\ 0; & 3\pi/4 \leq |\omega| \leq \pi \end{cases}$$

Determine the frequency response of the filter for  $N = 7$  using Hamming window.

(Anna University, December, 2003)

**Solution** Given

$$\begin{aligned} H_d(e^{j\omega}) &= \begin{cases} e^{-j3\omega} & -3\pi/4 \leq \omega \leq 3\pi/4 \\ 0, & 3\pi/4 \leq |\omega| \leq \pi \end{cases} \\ h_d(n) &= \frac{1}{2\pi} \int_{-3\pi/4}^{3\pi/4} e^{-j3\omega} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \left[ \int_{-3\pi/4}^{3\pi/4} e^{j\omega(n-3)} d\omega \right] \\ &= \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-3)}}{j(n-3)} \right]_{-3\pi/4}^{3\pi/4} \\ &= \frac{1}{2\pi} \left[ \frac{e^{j(3\pi/4)(n-3)} - e^{-j(3\pi/4)(n-3)}}{j(n-3)} \right] \\ h_d(n) &= \left[ \frac{\sin(3\pi/4)(n-3)}{\pi(n-3)} \right] \quad n \neq 3 \end{aligned}$$

For  $n = 3$ , apply L'Hospital's rule.

$$\begin{aligned} h_d(n) &= \lim_{n \rightarrow 3} \frac{\cos(3\pi/4)(n-3)[(3\pi/4)-1]}{\pi(-1)} \\ h_d(3) &= 1 \cdot \frac{(3\pi/4)}{\pi} = \frac{3}{4} \\ \therefore h_d(n) &= \begin{cases} \frac{\sin(3\pi/4)(n-3)}{\pi(n-3)}, & n \neq 3 \\ \frac{3}{4}, & n = 3 \end{cases} \end{aligned}$$

To determine  $h(n)$ ,

$$h(n) = h_d(n)\omega(n) \quad \text{where } \omega_{\text{Ham}}(n) = \begin{cases} 0.54 - 0.46 \cos \frac{2\pi n}{6}, & 0 < n < 6 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, filter coefficients are

$$\begin{array}{lll} h_d(0) = 0.0750 & \omega(0) = 0.08 & h(0) = 0.006 \\ h_d(1) = -0.1592 & \omega(1) = 0.31 & h(1) = -0.0494 \\ h_d(2) = 0.2251 & \omega(2) = 0.77 & h(2) = 0.1733 \\ h_d(3) = 0.75 & \omega(3) = 1 & h(3) = 0.75 \\ h_d(4) = 0.2251 & \omega(4) = 0.77 & h(4) = 0.1733 \\ h_d(5) = -0.1592 & \omega(5) = 0.31 & h(5) = -0.0494 \\ h_d(6) = 0.0750 & \omega(6) = 0.08 & h(6) = 0.006 \end{array}$$

The transfer function of the system is,

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\ &= h(0)z^{-0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} \\ &\quad + h(6)z^{-6} \\ &= h(0)[1 + z^{-6}] + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} + z^{-4}] + h(3)z^{-3} \\ &= 0.006[1 + z^{-6}] - 0.0494[z^{-1} + z^{-5}] + 0.1733[z^{-2} + z^{-4}] + 0.75z^{-3} \end{aligned}$$

The frequency response of the filter is,

$$\begin{aligned} H(e^{j\omega}) &= \left[ h\left[\frac{N-1}{2}\right] + 2 \sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \cos \omega n \right] e^{-j\omega\left(\frac{N-1}{2}\right)} \\ &= \left[ h(3) + 2 \sum_{n=1}^3 h(3-n) \cos \omega n \right] e^{-j3\omega} \end{aligned}$$

$$= (h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega)e^{-j3\omega}$$

$$H(e^{j\omega}) = (0.75 + 0.3466 \cos\omega - 0.0988 \cos 2\omega + 0.012 \cos 3\omega)e^{-j3\omega}$$

Magnitude response

$$H(e^{j\omega}) = 0.75 + 0.3466 \cos\omega - 0.0988 \cos 2\omega + 0.012 \cos 3\omega$$

Phase response

$$\angle H(e^{j\omega}) = -3\omega$$

### Example 4.22

A lowpass filter should have the frequency response given below. Find filter coefficients  $h_d(n)$ . Also determine  $\tau$  if  $h_d(n) = h_d(-n)$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau}, & -\omega_c \leq \omega \leq \omega_c \\ 0; & \omega_c \leq |\omega| \leq \pi \end{cases}$$

**Solution** Given

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & -\omega_c \leq \omega \leq \omega_c \\ 0; & \omega_c \leq |\omega| \leq \pi \end{cases}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)} d\omega$$

$$= \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-\tau)}}{j(n-\tau)} \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} \left[ \frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{j(n-\tau)} \right]$$

$$h_d(n) = \left[ \frac{\sin \omega_c(n-\tau)}{\pi(n-\tau)} \right] \quad n \neq \tau$$

if  $n = \tau$ ; apply L'Hospital's rule

$$h_d(n) = \lim_{n \rightarrow \tau} \frac{\cos \omega_c(n-\tau)(-\omega_c)}{\pi(-1)}$$

$$h_d(\tau) = \frac{\omega_c}{\pi}$$

$$\therefore h_d(n) = \begin{cases} \frac{\sin \omega_c(n-\tau)}{\pi(n-\tau)}, & n \neq \tau \\ \frac{\omega_c}{\pi}, & n = \tau \end{cases}$$

To determine  $\tau$ , given

$$h(n) = h_d(-n)$$

$$h_d(-n) = \frac{\sin \omega_c(n-\tau)}{\pi(n-\tau)} \Rightarrow \frac{-\sin \omega_c(n+\tau)}{-\pi(n+\tau)} = \frac{\sin \omega_c(n+\tau)}{\pi(n+\tau)}$$

$$= \frac{\sin \omega_c(n-\tau)}{\pi(n-\tau)} = \frac{\sin \omega_c(n+\tau)}{\pi(n+\tau)}$$

This exists only if

$$(n - \tau) = n + \tau$$

To achieve this  $\tau$  should be 0.

$$\tau = 0$$

### Example 4.23

A LPF has the desired response as given below:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & 0 \leq \omega \leq \pi/2 \\ 0; & \pi/2 \leq \omega \leq \pi \end{cases}$$

Determine filter coefficients  $h(n)$  for  $N = 7$  using type 1 frequency sampling technique.

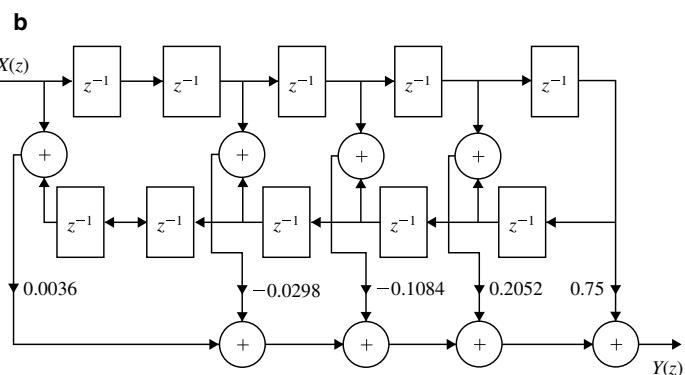
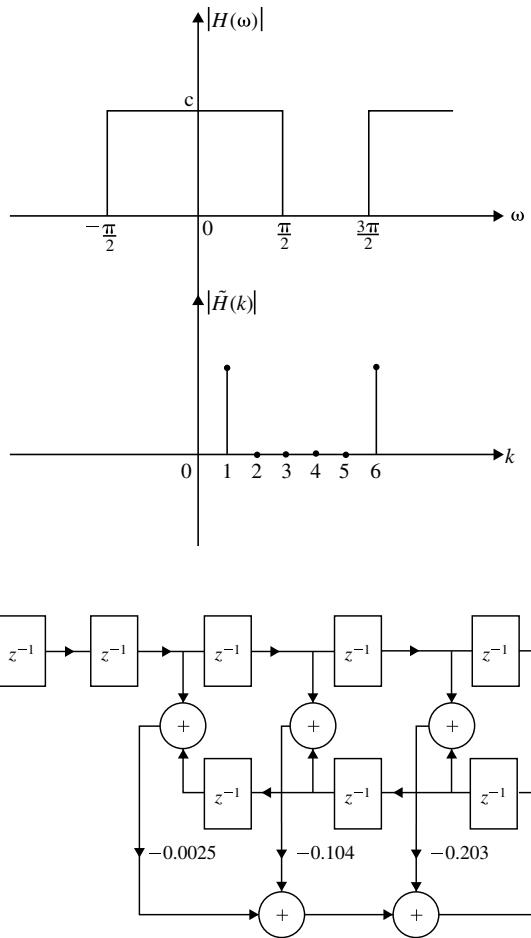
**Solution** Given

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & 0 \leq \omega \leq \pi/2 \\ 0, & \pi/2 \leq \omega \leq \pi \end{cases}$$

$$\tilde{H}(k) = H_d(\omega) \Big|_{\omega=\omega_k=\frac{2\pi k}{N}} \quad k = 0, 1, \dots, N-1$$

$$= H_d(\omega) \Big|_{\omega_k=\frac{2\pi k}{N}} \quad k = 0, 1, \dots, 6$$

The magnitude response is shown in Fig. 4.40.



**Fig. 4.40** Magnitude response for Example 4.23. **a** Structure realization for Example 4.24. **b** Structure realization for Example 4.24

$$\omega_k = \frac{2\pi k}{7}$$

where  $k = 0 \quad \omega_0 = 0$   
 $k = 1 \quad \omega_1 = \frac{2\pi}{7}$   
 $k = 2 \quad \omega_2 = \frac{4\pi}{7}$   
 $k = 3 \quad \omega_3 = \frac{6\pi}{7}$   
 $k = 4 \quad \omega_4 = \frac{8\pi}{7}$   
 $k = 5 \quad \omega_5 = \frac{10\pi}{7}$   
 $k = 6 \quad \omega_6 = \frac{12\pi}{7} \rightarrow \omega_k > \frac{3\pi}{2}$

$$\begin{aligned}\tilde{H}(k) &= \begin{cases} e^{-j3\omega_k}, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \\ e^{-j3\omega_k}, & k = 6 \end{cases} \\ &= \begin{cases} e^{-j6\pi k/7}, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \quad \therefore N \text{ is odd} \\ e^{-j6\pi k/7}, & k = 6 \end{cases} \\ h(n) &= \frac{1}{N} \left[ \tilde{H}(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \operatorname{Re} \left[ \tilde{H}(k) e^{j \frac{2\pi kn}{N}} \right] \right] \\ &= \frac{1}{7} \left[ \tilde{H}(0) + 2 \sum_{k=1}^3 \operatorname{Re} \left[ \tilde{H}(k) e^{j \frac{2\pi kn}{7}} \right] \right] \\ &= \frac{1}{7} \left[ \tilde{H}(0) + 2 \sum_{k=1}^3 \operatorname{Re} \left[ e^{j \frac{-j6\pi k}{7}} \cdot e^{j \frac{2\pi kn}{7}} \right] \right] \\ &= \frac{1}{7} \left[ 1 + 2 \sum_{k=1} \operatorname{Re} \left[ e^{j \frac{2\pi k}{7} (n-3)} \right] \right] \\ &= \frac{1}{7} \left[ 1 + 2 \sum_{k=1} \cos \frac{2\pi k}{7} (n-3) \right] \\ h(n) &= \frac{1}{7} \left[ 1 + 2 \cos \frac{2\pi}{7} (n-3) \right]\end{aligned}$$

The filter coefficients are

$$\begin{aligned} h(0) &= -0.11456 \\ h(1) &= 0.07927 \\ h(2) &= 0.321 \\ h(3) &= 0.4285 \\ h(4) &= h(2) = 0.321 \\ h(5) &= h(1) = 0.07927 \\ h(6) &= h(0) = -0.1145 \end{aligned}$$

### Example 4.24

Design a highpass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & -\pi/4 \leq \omega \leq \pi \\ 0; & |\omega| \leq \pi/4 \end{cases}$$

using (i) Hanning and (ii) Hamming window with  $N = 11$ .

### Solution

#### Hanning Window

##### Step 1

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & -\pi/4 \leq \omega \leq -\pi, \pi/4 \leq \omega \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

##### Step 2

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \left[ \int_{-\pi}^{-\pi/4} e^{-j\omega\alpha} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega \right] \\ &= \frac{1}{2\pi} \left[ \int_{-\pi}^{-\pi/4} e^{j\omega(n-\alpha)} d\omega + \int_{\pi/4}^{\pi} e^{j\omega(n-\alpha)} d\omega \right] \\ &= \frac{1}{2\pi} \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\pi}^{-\pi/4} + \left[ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{\pi/4}^{\pi} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[ \frac{e^{-j(\pi/4)(n-\alpha)} - e^{-j\pi(n-\alpha)} + e^{j\pi(n-\alpha)} - e^{j(\pi/4)(n-\alpha)}}{j(n-\alpha)} \right] \\
&= \frac{1}{2\pi j(n-\alpha)} 2j \left[ \sin \pi(n-\alpha) - \sin \frac{\pi}{4}(n-\alpha) \right] \\
h_d(n) &= \frac{\sin \pi(n-\alpha) - \sin \frac{\pi}{4}(n-\alpha)}{\pi(n-\alpha)}, \quad n \neq \alpha
\end{aligned}$$

When  $n = \alpha$ ; apply L' hospital's rule.

$$\begin{aligned}
h_d(n) &= \lim_{n \rightarrow \alpha} \frac{\cos \pi(n-\alpha)\pi - \cos(\pi/4)(n-\alpha)\pi/4}{\pi} \\
&= \frac{\pi - (\pi/4)}{\pi} = 0.75 \\
h_d(n) &= \begin{cases} \frac{\sin \pi(n-\alpha) - \sin(\pi/4)(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ 0.75, & n = \alpha \end{cases}
\end{aligned}$$

### Step 3

$$\begin{aligned}
h(d) &= h_d(n) + \omega_H(n) \\
\omega_H(n) &= \begin{cases} 0.5 - 0.5 \cos \left( \frac{2\pi n}{N-1} \right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \\
\omega_H(n) &= \begin{cases} 0.5 - 0.5 \cos \left( \frac{\pi n}{5} \right), & 0 \leq n \leq 10 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$\begin{array}{ll}
\omega_H(0) = 0 & \omega_H(6) = 0.9045 \\
\omega_H(1) = 0.0954 & \omega_H(7) = 0.6545 \\
\omega_H(2) = 0.3454 & \omega_H(8) = 0.3454 \\
\omega_H(3) = 0.6545 & \omega_H(9) = 0.0954 \\
\omega_H(4) = 0.9045 & \omega_H(10) = 0 \\
\omega_H(5) = 1 &
\end{array}$$

$$\begin{aligned}
\alpha &= \frac{N-1}{2} = 5 \\
h_d(n) &= \begin{cases} \frac{\sin \pi(n-5) - \sin(\pi/4)(n-5)}{\pi(n-5)}, & n \neq 5 \\ 0.75, & n = 5 \end{cases} \\
h_d(0) &= \frac{\sin \pi(-5) - \sin(\pi/4)(-5)}{\pi(-5)} = 0.0450 \\
h_d(1) &= 0 \quad h_d(6) = -0.225
\end{aligned}$$

$$\begin{array}{ll}
 h_d(2) = -0.075 & h_d(6) = -0.159 \\
 h_d(3) = -0.159 & h_d(6) = -0.075 \\
 h_d(4) = -0.225 & h_d(6) = 0 \\
 h_d(5) = 0.75 & h_d(6) = 0.0450
 \end{array}$$

$$h(n) = h_d(n)\omega_H(n)$$

$$\begin{aligned}
 h(0) &= 0.045 \times 0 = 0 \\
 h(1) &= 0 \times 0.0954 = 0 \\
 h(2) &= -0.075 \times 0.3454 = -0.0259 \\
 h(3) &= -0.159 \times 0.6545 = -0.104 \\
 h(4) &= -0.225 \times 0.9045 = -0.203 \\
 h(5) &= 0.75 \times 1 = 0.75 \\
 h(6) &= -0.225 \times 0.9045 = -0.203 \\
 h(7) &= -0.159 \times 0.6545 = -0.104 \\
 h(8) &= -0.075 \times 0.3454 = -0.0259 \\
 h(9) &= 0 \times 0.954 = 0 \\
 h(10) &= 0.045 \times 0 = 0
 \end{aligned}$$

#### Step 4

$$\begin{aligned}
 H(z) &= \sum_{n=0}^{10} h(n)z^{-n} \\
 H(z) &= -0.0259[z^{-2} + z^{-8}] - 0.104[z^{-3} + z^{-7}] - 0.203[z^{-4} + z^{-6}] \\
 &\quad + 0.75z^{-5}
 \end{aligned}$$

**Step 5** The structure realization is shown in Fig. 4.40a.

#### Hamming Window

#### Step 1

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & -\pi/4 \leq \omega \leq -\pi, \pi/4 \leq \omega \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

**Step 2**

$$h_d(n) = \begin{cases} \frac{\sin(n-\alpha) - \sin(\pi/4)(n-\alpha)}{\pi(n-\alpha)}, & n \neq \alpha \\ 0.75, & n = \alpha \end{cases}$$

**Step 3**

$$h(d) = h_d(n)\omega_H(n)$$

$$\omega_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0.54 - 0.46 \cos\left(\frac{\pi n}{5}\right), & 0 \leq n \leq 10 \end{cases}$$

\$\omega_H(0) = 0.08\$	\$\omega_H(6) = 0.9121\$
\$\omega_H(1) = 0.1678\$	\$\omega_H(7) = 0.6821\$
\$\omega_H(2) = 0.3978\$	\$\omega_H(8) = 0.3978\$
\$\omega_H(3) = 0.6821\$	\$\omega_H(9) = 0.1678\$
\$\omega_H(4) = 0.9121\$	\$\omega_H(10) = 0.08\$
\$\omega_H(5) = 1\$	

$$h(n) = h_d(n)\omega_H(n) \quad [\because h_d(n) \text{ same as above problem}]$$

$$\begin{aligned} h(0) &= 0.045 \times 0.08 = 0.0036 \\ h(1) &= 0 \times 0.1678 = 0 \\ h(2) &= -0.075 \times 0.3978 = -0.0298 \\ h(3) &= -0.159 \times 0.6821 = -0.1084 \\ h(4) &= -0.225 \times 0.9121 = -0.2052 \\ h(5) &= 0.75 \times 1 = 0.75 \\ h(6) &= -0.255 \times 0.9121 = -0.2052 \\ h(7) &= -0.159 \times 0.6821 = -0.1084 \\ h(8) &= -0.075 \times 0.3978 = -0.0298 \\ h(9) &= 0 \times 0.1678 = 0 \\ h(10) &= 0.045 \times 0.08 = 0.0036 \end{aligned}$$

**Step 4**

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} = \sum_{n=0}^{10} h(n)z^{-n}$$

$$H(z) = 0.0036[1 + z^{-10}] - 0.0298[z^{-2} + z^{-8}] - 0.1084[z^{-3} + z^{-7}]$$

$$-0.2052[z^{-4} + z^{-6}] + 0.75z^{-5}$$

**Step 5** The structure realization is shown in Fig. 4.40b.

### Example 4.25

Design an ideal differentiator with frequency response

$$H(e^{j\omega}) = j\omega, \quad -\pi \leq \omega \leq \pi$$

using (i) Rectangular and (ii) Hamming window with  $N = 8$ . For both cases plot magnitude response.

(Anna University, December, 2006)

#### Solution

##### Rectangular Window

##### Step 1

$$\begin{aligned} H(e^{j\omega}) &= j\omega = e^{j(\pi/2)}\omega \\ H(e^{j\omega}) &= e^{j(\pi/2)}e^{-j\alpha\omega}H(e^{j\omega}), \quad \alpha = 0 \\ H_d(e^{j\omega}) &= j\omega e^{-j\omega\alpha} \quad \text{for } -\pi \leq \omega \leq \pi, \quad N = 8 \end{aligned}$$

##### Step 2

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{-j\omega\alpha} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} j \int_{-\pi}^{\pi} \omega e^{j\omega(n-\alpha)} d\omega \\ &= \frac{1}{2\pi} \left[ \frac{\omega e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\pi}^{\pi} - \left[ \frac{e^{j\omega(n-\alpha)}}{[j(n-\alpha)]^2} \right]_{-\pi}^{\pi} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi} \left[ \frac{\omega e^{j\omega(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j\omega(n-\alpha)}}{[j(n-\alpha)]^2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left[ \frac{\pi e^{j\pi(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j\pi(n-\alpha)}}{j^2(n-\alpha)^2} - \frac{(-\pi)e^{-j\pi(n-\alpha)}}{j(n-\alpha)} + \frac{e^{-j\pi(n-\alpha)}}{j(n-\alpha)^2} \right] \\ &= \frac{1}{2\pi} \left[ \frac{\pi 2 \cos \pi(n-\alpha)}{j(n-\alpha)} + \frac{2j \sin \pi(n-\alpha)}{(n-\alpha)^2} \right] \end{aligned}$$

$$h_d(n) = \frac{\cos \pi(n - \alpha)}{n - \alpha} - \frac{\sin \pi(n - \alpha)}{\pi(n - \alpha)^2} \quad \text{where } N = \text{even}$$

$$h_d(n) = \frac{-\sin \pi(n - \alpha)}{\pi(n - \alpha)^2}$$

**Step 3**

$$h(n) = h_d(n)\omega_H(n)$$

$$\omega_H(n) = 1$$

$$h(n) = h_d(n)$$

$h_d(n)$  is anti-symmetry. So

$$h_d(n) = -h_d(N - 1 - n)$$

where

$$\alpha = (N - 1)/2 = \frac{7}{2}.$$

$$h_d(0) = -h_d(7) = \frac{-\sin(-7/2)\pi}{\pi(-7/2)} = -0.026$$

$$h_d(1) = -h_d(6) = 0.0509$$

$$h_d(2) = -h_d(5) = -0.1415$$

$$h_d(3) = -h_d(4) = 1.27$$

$$h(n) = h_d(n)$$

$$h_d(0) = -h_d(7) = -0.026$$

$$h_d(1) = -h_d(6) = 0.0509$$

$$h_d(2) = -h_d(5) = -0.1415$$

$$h_d(3) = -h_d(4) = 1.27$$

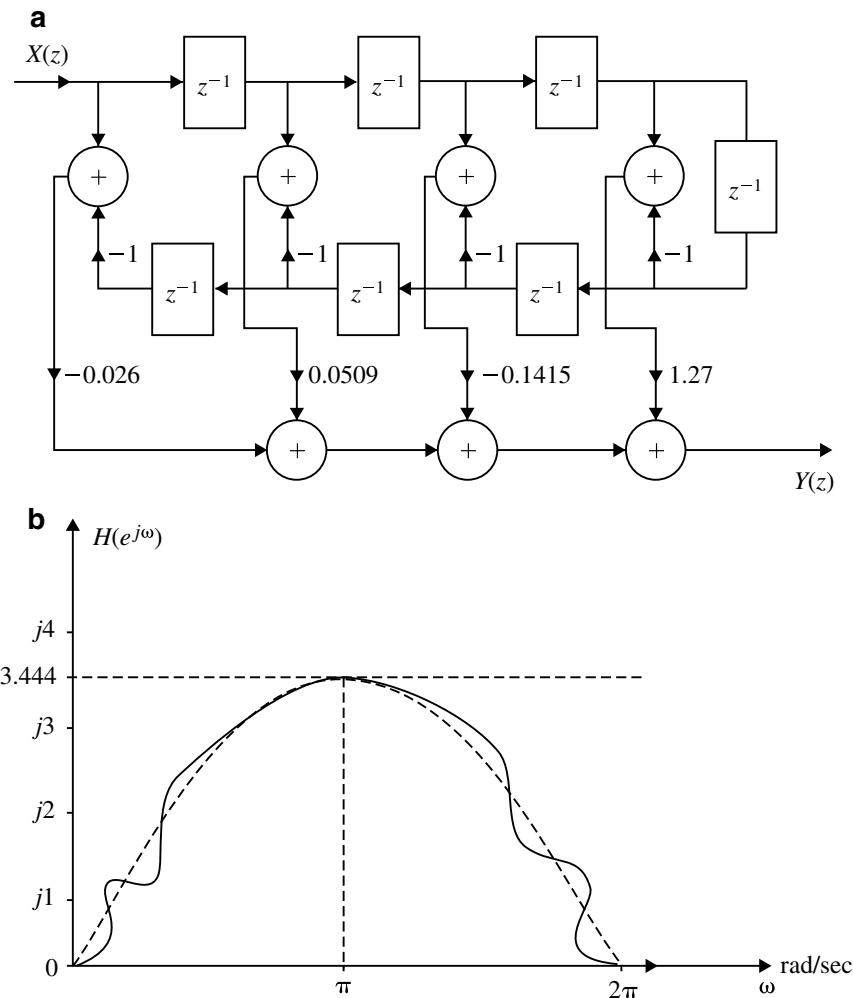
**Step 4**

$$H(z) = \sum_{n=0}^7 h(n)z^{-n}$$

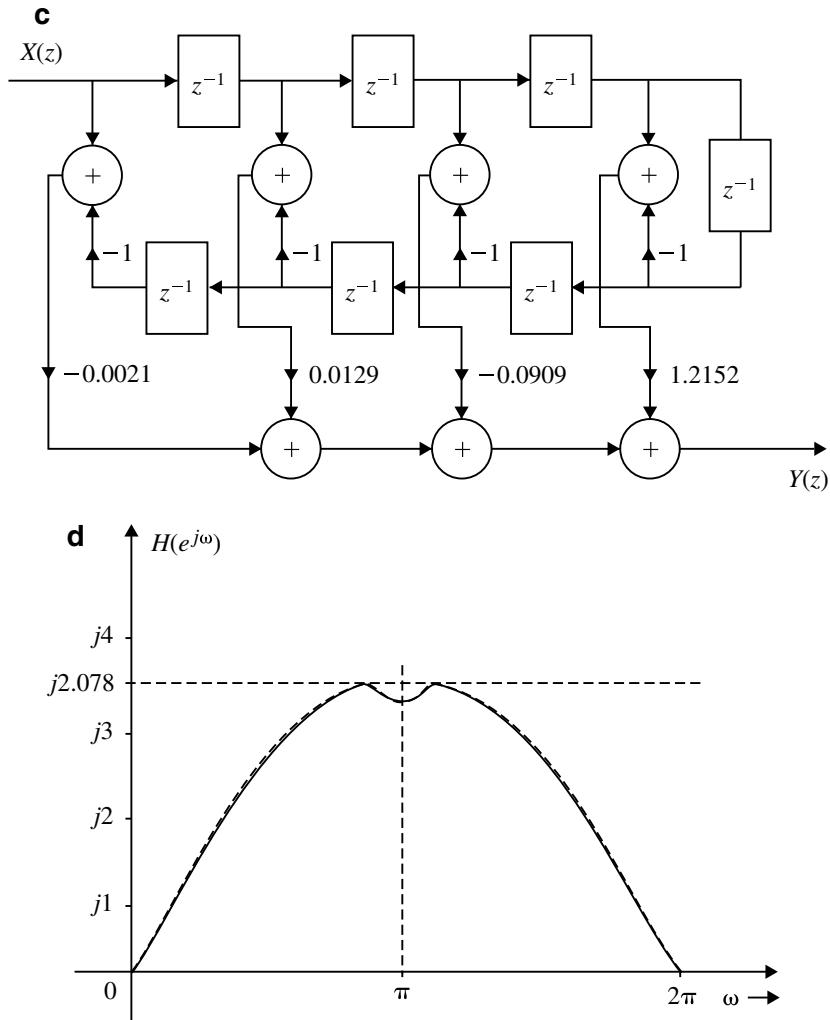
$$= -0.026[1 - z^{-7}] + 0.0509[z^{-1} - z^{-6}] - 0.1415[z^{-2} - z^{-5}]$$

$$+ 1.27[z^{-3} - z^{-4}]$$

**Step 5** The structure realized as shown in Fig. 4.41a.



**Fig. 4.41** **a** Structure realization (Rectangular window) for Example 4.25. **b** Frequency response plot. **c** Structure realization (Hamming window) for Example 4.25 (d)

**Fig. 4.41** (continued)

Frequency response of ideal differentiator for anti-symmetric impulse response and  $N = \text{even}$  is,

$$H(e^{j\omega}) = \sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left( n - \frac{1}{2} \right)$$

where

$$d(n) = \sum_{n=1}^{\frac{N}{2}} 2h \left( \frac{N}{2} - n \right)$$

$$d(1) = 2h\left(\frac{N}{2} - 1\right) = 2h(4 - 1) = 2h(3) = 2.54$$

$$d(2) = 2h(4 - 2) = 2h(2) = -0.283$$

$$d(3) = 2h(4 - 3) = 2h(1) = 0.1010$$

$$d(4) = 2h(0) = -0.52$$

$$H(e^{j\omega}) = d(1) \sin \frac{\omega}{2} + d(2) \sin \frac{3\omega}{2} + d(3) \sin \frac{5\omega}{2} + d(4) \sin \frac{7\omega}{2}$$

$$= 2.54 \sin \frac{\omega}{2} - 0.283 \sin \frac{3\omega}{2} + 0.1010 \sin \frac{5\omega}{2} - 0.52 \sin \frac{7\omega}{2}$$

$$H(e^{j\omega}) = j \left( 2.54 \sin \frac{\omega}{2} - 0.283 \sin \frac{3\omega}{2} + 0.1010 \sin \frac{5\omega}{2} - 0.52 \sin \frac{7\omega}{2} \right)$$

$\omega$ deg/s	0	20	40	60	80	100	180	240	270	360
$H(e^{j\omega})$	0	j0.5839	j0.3889	j1.2975	j0.5795	j1.9945	j3.444	j1.662	j1.89	0

The frequency response plot is shown in Fig. 4.41b.

### Hamming Window

**Steps 1 and 2** Same as Rectangular window.

#### Step 3

$$\omega_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0.54 - 0.46 \left(\cos \frac{\pi n}{5}\right), & 0 \leq n \leq N \end{cases}$$

$$\omega_H(0) = \omega_H(7) = 0.08$$

$$\omega_H(1) = \omega_H(6) = 0.2532$$

$$\omega_H(2) = \omega_H(5) = 0.6424$$

$$\omega_H(3) = \omega_H(4) = 0.954$$

$h_d(n)$  same as above.

$$h(0) = -h(7) = h_d(0)\omega_H(0) = -0.0021$$

$$h(1) = -h(6) = h_d(1)\omega_H(1) = 0.0129$$

$$h(2) = -h(5) = h_d(2)\omega_H(2) = -0.0909$$

$$h(3) = -h(4) = h_d(3)\omega_H(3) = 1.2152$$

#### Step 4

$$H(z) = -0.0021[1 - z^{-7}] + 0.0129[z^{-1} - z^{-6}] - 0.0909[z^{-2} - z^{-5}] + 1.2152[z^{-3} - z^{-4}]$$

**Step 5**

The structure realization is shown in Fig. 4.41c.

Frequency response

$$\bar{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N}{2}} d(n) \sin\left(n - \frac{1}{2}\right)\omega$$

where

$$\begin{aligned} d(n) &= \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \\ &= \sum_{n=1}^4 2h(4-n) \end{aligned}$$

$$d(1) = 2h(4-1) = 2h(3) = 2.4304$$

$$d(2) = 2h(4-2) = 2h(2) = -0.1818$$

$$d(3) = 2h(4-3) = 2h(1) = 0.0258$$

$$d(4) = 2h(0) = -0.0042$$

$$\begin{aligned} \bar{H}(e^{j\omega}) &= d(1) \sin \frac{\omega}{2} + d(2) \sin \frac{3\omega}{2} + d(3) \sin \frac{5\omega}{2} + d(4) \sin \frac{7\omega}{2} \\ &= 2.4304 \sin \frac{\omega}{2} - 0.1818 \sin \frac{3\omega}{2} + 0.0258 \sin \frac{5\omega}{2} - 0.0042 \sin \frac{7\omega}{2} \end{aligned}$$

$$\begin{aligned} H(e^{j\omega}) &= j\bar{H}(e^{j\omega}) \\ &= j \left( 2.4304 \sin \frac{\omega}{2} - 0.1818 \sin \frac{3\omega}{2} + 0.0258 \sin \frac{5\omega}{2} - 0.0042 \sin \frac{7\omega}{2} \right) \end{aligned}$$

$\omega$ deg/s	0	45	90	120	180	240	270	315	360
$ H(e^{j\omega}) $	0	$j0.7875$	$j1.568$	$j2.078$	$j2.6422$	$j2.078$	$j1.568$	$j0.7875$	0

**Example 4.26**

Design an ideal Hilbert transformer having frequency response

$$H(e^{j\omega}) = \begin{cases} j, & -\pi \leq \omega \leq 0 \\ -j, & 0 \leq \omega \leq \pi \end{cases}$$

using (i) Rectangular window and (ii) Blackman window with  $N = 11$ .

**Solution****Step 1**

$$H(e^{j\omega}) = \begin{cases} j, & -\pi \leq \omega \leq 0 \\ -j, & 0 \leq \omega \leq \pi \end{cases}$$

**Step 2**

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \left[ \int_{-\pi}^0 j e^{j\omega n} d\omega + \int_0^\pi -j e^{j\omega n} d\omega \right] \\ &= \frac{j}{2\pi} \left[ \left[ \frac{e^{j\omega n}}{jn} \right]_{-\pi}^0 - \left[ \frac{e^{j\omega n}}{jn} \right]_0^\pi \right] \\ &= \frac{j}{2\pi} \frac{1}{jn} [[e^0 - e^{-j\pi n} - e^{j\pi n} + e^0]] \\ &= \frac{1}{2\pi n} [2 - e^{j\pi n} + e^{-j\pi n}] \\ &= \frac{1}{2\pi n} [2 - 2 \cos \pi n] \\ &= \frac{1 - \cos \pi n}{\pi n} = \begin{cases} \frac{2 \sin^2(\frac{\pi n}{2})}{\pi n}, & n \neq 0 \\ 0, & n = 0 \end{cases} \\ h_d(n) &= \frac{1 - \cos \pi n}{\pi n} \quad \therefore h_d(n) \text{ is antisymmetry} \\ h_d(0) &= \frac{1 - \cos 0}{\pi 0} = 0 \\ h_d(1) &= \frac{1 - \cos \pi}{\pi} = \frac{2}{\pi} = -h_d(-1) \\ h_d(2) &= 0 = -h_d(-2) \\ h_d(3) &= \frac{2}{3\pi} = -h_d(-3) \\ h_d(4) &= 0 = -h_d(-4) \\ h_d(5) &= \frac{2}{5\pi} = -h_d(-5) \end{aligned}$$

**Step 3**

$$h(n) = h_d(n)\omega_H(n) \quad (\text{where } \omega_h(n) = 1)$$

$$h(n) = h_d(n)$$

$$h(0) = 0$$

$$h(1) = -h(-1) = \frac{2}{\pi}$$

$$\begin{aligned}
 h(2) &= -h(-2) = 0 \\
 h(3) &= -h(-3) = \frac{2}{3\pi} \\
 h(4) &= -h(-4) = 0 \\
 h(5) &= -h(-5) = \frac{2}{5\pi}
 \end{aligned}$$

**Step 4**

$$\begin{aligned}
 H(Z) &= z^{-5} \sum_{n=-5}^5 h(n)z^{-n} \\
 &= z^{-5} \left[ \frac{2}{\pi}[z - z^{-1}] + \frac{2}{3\pi}[z^3 - z^{-3}] + \frac{2}{5\pi}[z^5 - z^{-5}] \right] \\
 &= \frac{2}{\pi}[z^{-4} - z^{-6}] + \frac{2}{3\pi}[z^{-2} - z^{-8}] + \frac{2}{5\pi}[1 - z^{-10}]
 \end{aligned}$$

**Step 5** The structure realization is shown in Fig. 4.42a

**Blackman Window**

**Step 1 and 2** Same as Rectangular window.

**Step 3**

$$\begin{aligned}
 \omega_B(n) &= 0.42 + 0.5 \cos \frac{\pi n}{5} + 0.08 \cos \frac{2\pi n}{5} \quad \text{for } -5 \leq n \leq 5 \\
 \omega_B(0) &= 1 \\
 \omega_B(1) &= \omega_B(-1) = 0.849
 \end{aligned}$$

$$\begin{aligned}
 \omega_B(2) &= \omega_B(-2) = 0.509 \\
 \omega_B(3) &= \omega_B(-3) = 0.2 \\
 \omega_B(4) &= \omega_B(-4) = 0.4 \\
 \omega_B(5) &= \omega_B(-5) = 0
 \end{aligned}$$

$$h(n) = h_d(n)\omega_B(n)$$

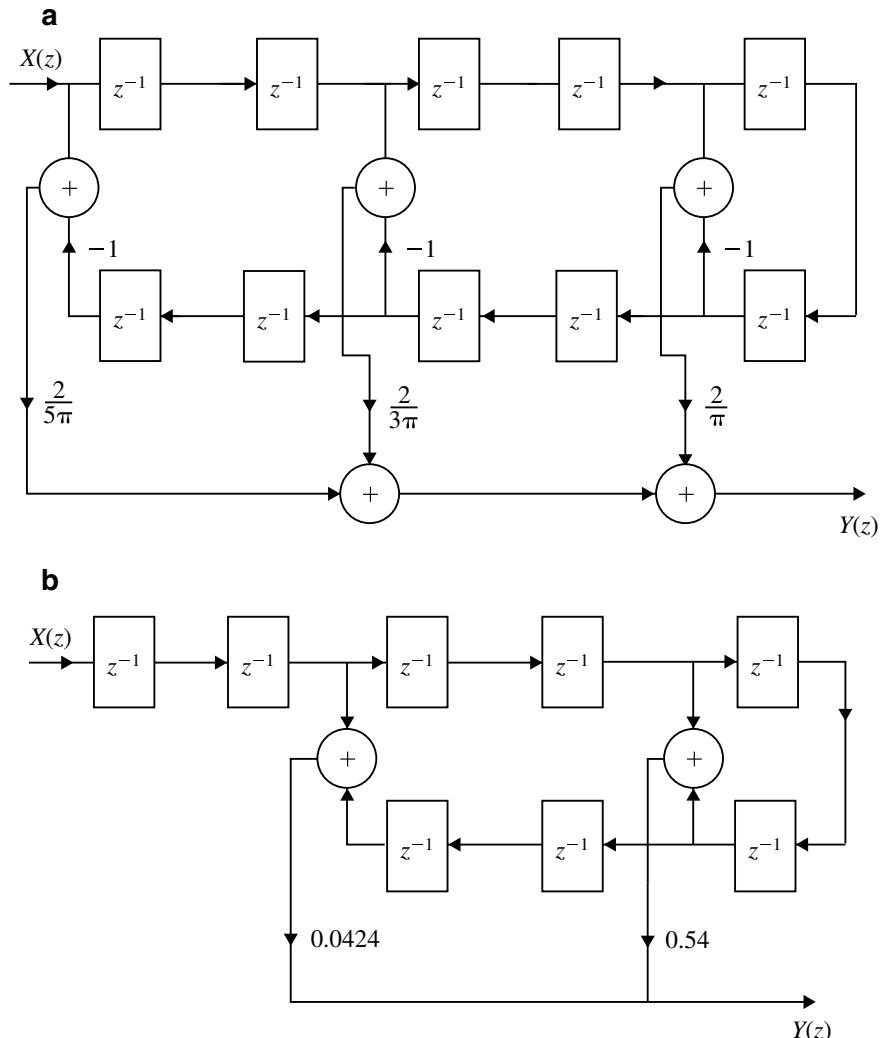
$$h(0) = 0$$

$$h(1) = -h(-1) = \left(\frac{2}{\pi}\right) 0.849 = 0.5405$$

$$h(2) = -h(-2) = 0 = 0$$

$$h(3) = -h(-3) = \left(\frac{2}{3\pi}\right) 0.2 = 0.0423$$

$$h(4) = -h(-4) = 0 = 0$$



**Fig. 4.42** **a** Structure realization (Rectangular window) for Example 4.26. **b** Structure realization (Blackmann window) for Example 4.26

$$h(5) = -h(-5) = \left(\frac{2}{5\pi}\right)0 = 0$$

**Step 4**

$$\begin{aligned} H(z) &= z^{-5} \sum_{n=-5}^5 h(n)z^{-n} \\ &= z^{-5}[0.54(z - z^{-1}) + 0.0424(z^3 - z^{-3})] \\ &= 0.54[z^{-4} - z^{-6}] + 0.0424[z^{-2} - z^{-8}] \end{aligned}$$

**Step 5**

The structure realization is shown in Fig. 4.42b

**Example 4.27**

Using a Rectangular window technique design a lowpass filter with passband gain of unity, cutoff frequency of 100 Hz and working at a sampling frequency of 5 kHz. The length of the impulse response should be 7.

(Anna University, May, 2007)

**Solution** Given  $f_c = 1000$  Hz,  $F = 5000$  Hz

$$H_d(e^{j\omega}) = \begin{cases} 1, & -\omega_c \leq \omega \leq \omega_c \\ 0, & -\pi \leq \omega \leq -\omega_c \text{ and } \omega_c \leq \omega \leq \pi \end{cases}$$

where

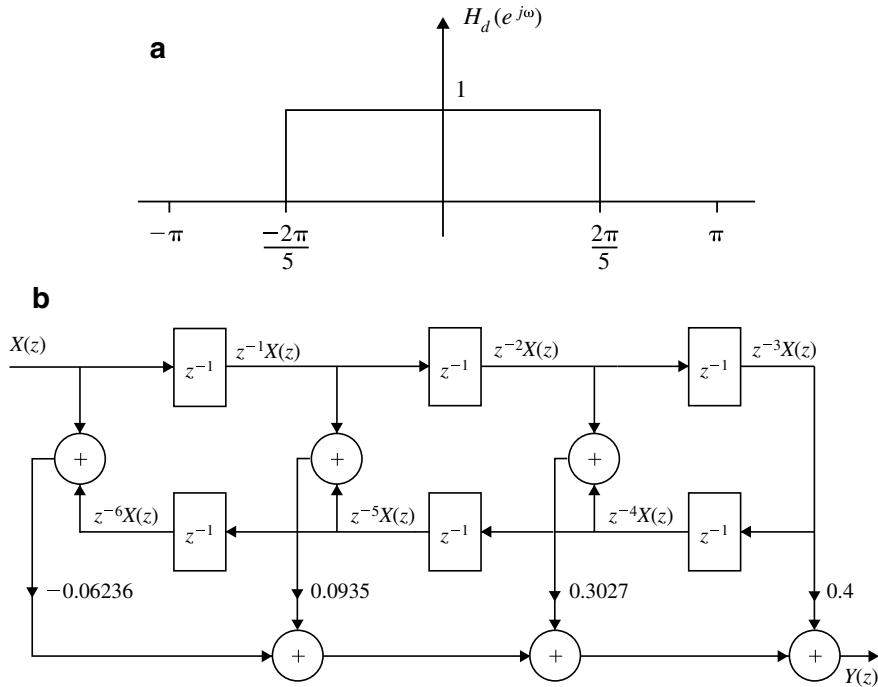
$$\begin{aligned} \omega_c &= 2\pi f_c \\ &= 2\pi \times 1000.T \end{aligned}$$

$$\boxed{\omega_c = \frac{2000\pi}{5000} = \frac{2\pi}{5}}$$

The desired frequency response of the LPF is shown in Fig. 4.43a.

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} d\omega$$



**Fig. 4.43** a Frequency response of for Example 4.27. b Linear phase realization of FIR filter for Example 4.27

$$\begin{aligned}
 &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \int_{-\frac{2\pi}{5}}^{\frac{2\pi}{5}} e^{j\omega n} d\omega \\
 &= \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{-\frac{2\pi}{5}}^{\frac{2\pi}{5}} \\
 &= \frac{1}{n\pi} \left[ \frac{e^{\frac{j2\pi n}{5}} - e^{-\frac{j2\pi n}{5}}}{2j} \right] \\
 h_d(n) &= \frac{\sin \frac{2\pi n}{5}}{n\pi}, -\infty \leq n \leq \infty
 \end{aligned}$$

The Rectangular window for  $N = 7$  is given by

$$\omega_R(n) = \begin{cases} 1, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

The filter coefficient

$$h(n) = h_d(n) \times \omega_R(n)$$

$$h(n) = \begin{cases} h_d(n), & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

For  $n = 0$ ,

$$h(0) = h_d(0) = \lim_{n \rightarrow 0} \frac{\sin \frac{2\pi n}{5}}{n\pi}$$

$$= \lim_{n \rightarrow 0} \frac{2}{5} \frac{\sin \frac{2\pi n}{5}}{\frac{2\pi n}{5}}$$

$$= \lim_{n \rightarrow 0} \frac{2}{5}$$

$$h(0) = \frac{2}{5}$$

For  $n = 1$ ;

$$h(1) = h(-1) = \frac{\sin \frac{2\pi}{5}}{\pi} = 0.3027$$

For  $n = 2$ ;

$$h(2) = h(-2) = \frac{\sin \frac{4\pi}{5}}{2\pi} = 0.0935$$

For  $n = 3$ ;

$$h(3) = h(-3) = \frac{\sin \frac{6\pi}{5}}{3\pi} = -0.06236$$

The filter coefficients of realizable filter are:

$$h(0) = h(6) = -0.06236$$

$$h(1) = h(5) = 0.0935$$

$$h(2) = h(4) = 0.3027$$

$$h(3) = 0.4$$

It is symmetric about  $\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$ . The transfer function of realizable filter is

$$\begin{aligned}
 H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\
 &= \sum_{n=0}^6 h(n)z^{-n} \\
 &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} \\
 &\quad + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} \\
 H(z) &= \frac{Y(z)}{X(z)} \\
 &= h(0)[1 + z^{-6}] + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} \\
 &\quad + z^{-4}] + h(3)z^{-3} \\
 Y(z) &= -0.06236[1 + z^{-6}]X(z) + 0.0935[z^{-1} + z^{-5}]X(z) \\
 &\quad + 0.3027[z^{-2} + z^{-4}]X(z) + 0.4z^{-3}X(z)
 \end{aligned}$$

Linear phase realization of the filter is shown in Fig. 4.43b.

### Example 4.28

Design a bandpass filter which approximates the ideal filter with cutoff frequencies at 0.2 and 0.3 rad/s. The filter order is  $N = 7$ . Use Hamming window.

(Anna University, December, 2007)

**Solution** Given  $\omega_{c1} = 0.2$  rad/s,  $\omega_{c2} = 0.3$  rad/s,  $N = 7$ .

**Step 1** Choose  $H_d(\omega)$  ideal filter such that

$$H_d(\omega) = \begin{cases} 1, & -\omega_{c2} \leq \omega \leq -\omega_{c1} \text{ and } \omega_{c1} \leq \omega \leq -\omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

**Step 2** To Find  $h_d(n)$

$$\begin{aligned}
 h_d(n) &= \frac{1}{2\pi} \left[ \int_{-\omega_{c2}}^{-\omega_{c1}} H_d(\omega) e^{j\omega n} d\omega + \int_{\omega_{c1}}^{\omega_{c2}} H_d(\omega) e^{j\omega n} d\omega \right] \\
 &= \frac{1}{2\pi} \left[ \int_{-0.3}^{-0.2} e^{j\omega n} d\omega + \int_{0.2}^{0.3} e^{j\omega n} d\omega \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[ \left[ \frac{e^{j\omega n}}{jn} \right]_{-0.3}^{-0.2} + \left[ \frac{e^{j\omega n}}{jn} \right]_{0.2}^{0.3} \right] \\
&= \frac{1}{2n\pi j} [e^{-j0.2n} - e^{-j0.3n} + e^{j0.3n} - e^{j0.2n}] \\
&= \frac{1}{n\pi} \left[ \frac{e^{j0.3n} - e^{-j0.3n}}{2j} - \frac{e^{j0.2n} - e^{-j0.2n}}{2j} \right] \\
&= \frac{1}{n\pi} [\sin 0.3n - \sin 0.2n] \\
h_d(n) &= \frac{1}{n\pi} [\sin 0.3n - \sin 0.2n]
\end{aligned}$$

**Step 3** To Find  $h(n)$

$$h(n) = h_d(n) \times \omega_H(n)$$

Hamming window function is,

$$\omega_H(n) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & -\frac{(N-1)}{2} \leq n \leq \frac{(N-1)}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_H(0) = 0.54 + 0.46 = 1$$

$$\omega_H(1) = \omega_H(-1) = 0.54 + 0.46 \cos \frac{2\pi}{6} = 0.77$$

$$\omega_H(2) = \omega_H(-2) = 0.54 + 0.46 \cos \frac{4\pi}{6} = 0.31$$

$$\omega_H(3) = \omega_H(-3) = 0.54 + 0.46 \cos \frac{6\pi}{6} = 0.08$$

$$h_d(n) = \frac{1}{n\pi} [\sin 0.3n - \sin 0.2n], \quad -3 \leq n \leq 3$$

$$\begin{aligned}
h_d(0) &= \lim_{n \rightarrow 0} \frac{1}{\pi} \left[ 0.3 \frac{\sin 0.3n}{0.3n} - 0.2 \frac{\sin 0.2n}{0.2n} \right] \\
&= \frac{0.3 - 0.2}{\pi} = 0.03183
\end{aligned}$$

$$h_d(1) = h_d(-1) = \frac{1}{\pi} [\sin 0.3 - \sin 0.2] = 0.0308$$

$$h_d(2) = h_d(-2) = \frac{1}{2\pi} [\sin 0.6 - \sin 0.4] = 0.0278$$

$$h_d(3) = h_d(-3) = \frac{1}{3\pi} [\sin 0.9 - \sin 0.6] = 0.0232$$

$$h(n) = h_d(n) \times \omega_H(n)$$

$$h(0) = h_d(0) \times \omega_H(0) = 0.03183$$

$$h(1) = h(-1) = h_d(+1) \times \omega_H(+1) = 0.02926$$

$$h(2) = h(-2) = h_d(2) \times \omega_H(2) = 0.0086$$

$$h(3) = h(-3) = h_d(3) \times \omega_H(3) = 0.01856$$

**Step 4** To Find  $H(z)$

$$\begin{aligned} H(z) &= z^{-\left(\frac{N-1}{2}\right)} \left[ \sum_{n=1}^{\frac{N-1}{2}} h(n)[z^n + z^{-n}] + h(0) \right] \\ &= z^{-3} \left[ h(0) + \sum_{n=1}^3 h(n)[z^n + z^{-n}] \right] \\ &= z^{-3} [h(0) + h(1)[z^1 + z^{-1}] + h(2)[z^2 + z^{-2}] + h(3)[z^3 + z^{-3}]] \end{aligned}$$

$$H(z) = [h(0)z^{-3} + h(1)[z^{-2} + z^{-4}] + h(2)[z^{-1} + z^{-5}] \\ + h(3)[1 + z^{-6}]]$$

$$H(z) = 0.03183z^{-3} + 0.02926[z^{-2} + z^{-4}] + 0.0086[z^{-1} + z^{-5}] \\ + 0.01856[1 + z^{-6}]$$

**Step 5** Realization of the filter

$$\begin{aligned} Y(z) &= h(0)z^{-3}X(z) + h(1)[z^{-2} + z^{-4}]X(z) \\ &\quad + h(2)[z^{-1} + z^{-5}]X(z) \\ &\quad + h(3)[1 + z^{-6}]X(z) \end{aligned}$$

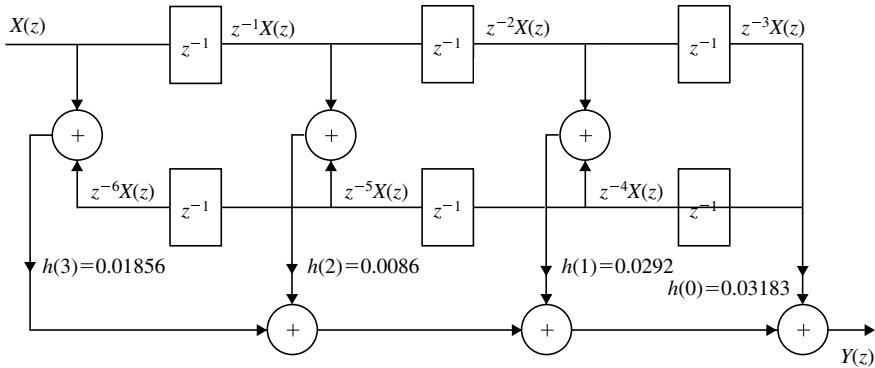
FIR filter is realized as shown in Fig. 4.44.

### Example 4.29

Obtain linear phase structure with minimum member of multipliers for the system described by the equation.

$$y(n) = x(n) + \frac{1}{2}x(n-1) - \frac{1}{4}x(n-2) + \frac{1}{2}x(n-3) + x(n-4)$$

(Anna University, December, 2007)



**Fig. 4.44** Realization of FIR filter for Example 4.28

**Solution** Given

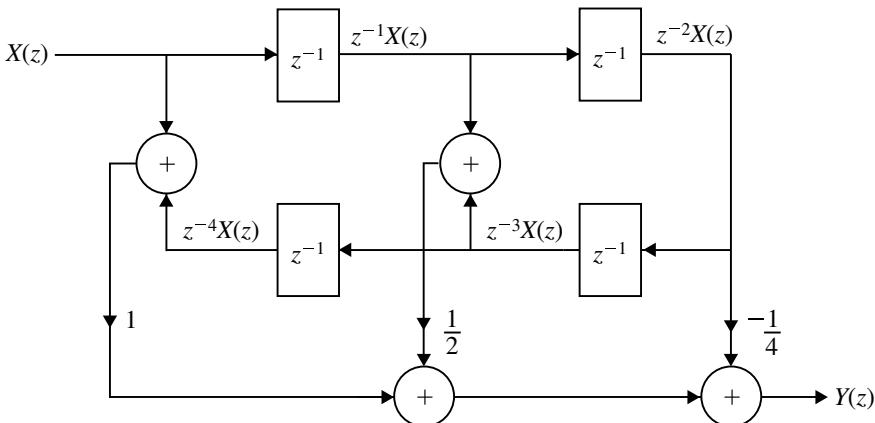
$$y(n) = x(n) + \frac{1}{2}x(n-1) - \frac{1}{4}x(n-2) + \frac{1}{2}x(n-3) + x(n-4)$$

Taking  $z$ -transform on both sides, we get

$$\begin{aligned} Y(z) &= X(z) + \frac{1}{2}z^{-1}X(z) - \frac{1}{4}z^{-2}X(z) + \frac{1}{2}z^{-3}X(z) + z^{-4}X(z) \\ &= X(z) \left[ 1 + \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3} + z^{-4} \right] \end{aligned}$$

$$\frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3} + z^{-4}$$

$$H(z) = 1 + \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3} + z^{-4}$$



**Fig. 4.45** Linear phase realization for the Example 4.29

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h(n)z^{-n} \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + \dots \end{aligned}$$

Comparing these equations, we get impulse response.

$$h(0) = 1, \quad h(1) = \frac{1}{2}, \quad h(2) = \frac{-1}{4}, \quad h(3) = \frac{1}{2}, \quad h(4) = 1$$

$$h(n) = \left\{1, \frac{1}{2}, \frac{-1}{4}, \frac{1}{2}, 1\right\} \text{ and } N = 5$$

Here  $h(n)$  satisfies the condition  $h(n) = h(N - 1 - n)$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = (1 + z^{-4}) + \frac{1}{2}[z^{-1} + z^{-3}] - \frac{1}{4}z^{-2} \\ Y(z) &= [X(z) + z^{-4}X(z)] + \frac{1}{2}[z^{-1}X(z) + z^{-3}X(z)] - \frac{1}{4}z^{-2}X(z) \end{aligned}$$

$Y(z)$  is realized as shown in Fig. 4.45.

### Example 4.30

Design a linear phase FIR digital filter for the given specifications using Hamming window of length  $M = 7$ .

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & \text{for } |\omega| \leq \frac{\pi}{6} \\ 0, & \text{for } \frac{\pi}{6} \leq |\omega| \leq \pi \end{cases}$$

(Anna University, May, 2007)

**Solution** Given

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & |\omega| \leq \frac{\pi}{6} \\ 0, & \frac{\pi}{6} \leq |\omega| \leq \pi \end{cases}$$

The frequency response having a term  $e^{-j\omega(\frac{N-1}{2})}$ , gives  $h(n)$  which is symmetrical about  $\alpha = \frac{N-1}{2} = 3$ , i.e., we get a causal sequence.

$$h_d(n) = \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} e^{-j\omega n} d\omega$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} e^{j(n-3)\omega} d\omega \\
&= \frac{1}{2\pi} \left[ \frac{e^{j(n-3)\omega}}{j(n-3)} \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
&= \frac{1}{\pi(n-3)} \left[ \frac{e^{j\frac{\pi}{6}(n-3)} - e^{-j\frac{\pi}{6}(n-3)}}{2j} \right] \\
h_d(n) &= \frac{\sin \frac{\pi}{6}(n-3)}{\pi(n-3)}, \quad 0 \leq n \leq N-1, \quad 0 \leq n \leq 6 \\
h_d(0) &= h_d(6) = \frac{\sin(\frac{\pi}{2})}{3\pi} = 0.1061 \\
h_d(1) &= h_d(5) = \frac{\sin(\frac{\pi}{3})}{2\pi} = 0.1378 \\
h_d(2) &= h_d(4) = \frac{\sin(\frac{\pi}{6})}{\pi} = 0.1591 \\
h_d(3) &= \lim_{n \rightarrow 3} \frac{\sin \frac{\pi}{6}(n-3)}{6 \cdot \frac{\pi}{6}(n-3)} = \frac{1}{6} = 0.1666
\end{aligned}$$

Hamming window function is,

$$\omega_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{6}\right), & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_H(0) = 0.54 - 0.46 = 0.08$$

$$\omega_H(1) = 0.54 - 0.46 \cos\left(\frac{2\pi}{6}\right) = 0.31$$

$$\omega_H(2) = 0.54 - 0.46 \cos\left(\frac{4\pi}{6}\right) = 0.77$$

$$\omega_H(3) = 0.54 - 0.46 \cos\left(\frac{6\pi}{6}\right) = 1$$

$$\omega_H(4) = \omega_H(2) = 0.77$$

$$\omega_H(5) = \omega_H(1) = 0.31$$

$$\omega_H(6) = \omega_H(0) = 0.08$$

$$h(n) = h_d(n) \times \omega_H(n)$$

$$h(0) = h_d(0) \times \omega_H(0)$$

$$h(0) = 0.0048$$

$$h(6) = h(0) = 0.0048$$

$$h(1) = h(5) = h_d(1) \times \omega_H(1) = 0.04271$$

$$h(2) = h(4) = h_d(2) \times \omega_H(2) = 0.1225$$

$$h(3) = h_d(3) \times \omega_H(3) = 0.16667$$

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} \\ &= h(0)[1 + z^{-6}] + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} + z^{-4}] + h(3)z^{-3} \end{aligned}$$

$$\begin{aligned} H(z) &= 0.0048[1 + z^{-6}] + 0.04271[z^{-1} + z^{-5}] \\ &\quad + 0.1225[z^{-2} + z^{-4}] + 0.16667z^{-3} \end{aligned}$$

Realization of the filter is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= 0.0048[1 + z^{-6}] + 0.04271[z^{-1} + z^{-5}] + 0.1225[z^{-2} + z^{-4}] \\ &\quad + 0.16667z^{-3} \\ Y(z) &= 0.0048[X(z) + z^{-6}X(z)] + 0.04271[z^{-1}X(z) + z^{-5}X(z)] \\ &\quad + 0.1225[z^{-2}X(z) + z^{-4}X(z)] + 0.16667z^{-3}X(z) \end{aligned}$$

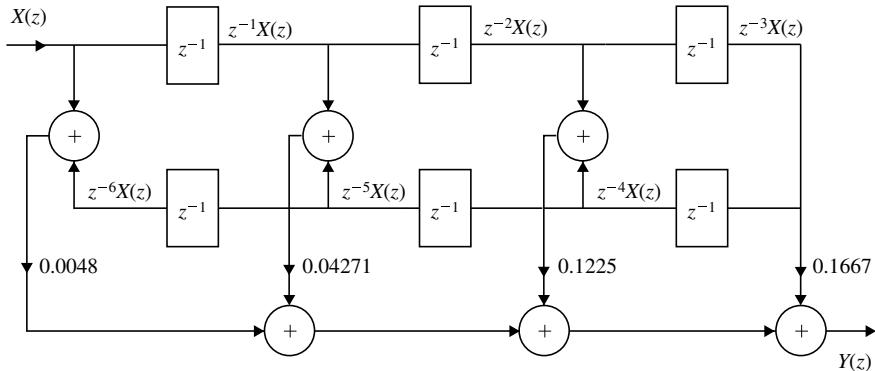
The structure of the filter is shown in Fig. 4.46.

### Example 4.31

Design and implement linear phase FIR filter of length  $N = 15$  which has following unit sample sequence:

$$H(k) = \begin{cases} 1, & k = 0, 1, 2, 3 \\ 0, & k = 4, 5, 6, 7 \end{cases}$$

(Anna University, May 2007)



**Fig. 4.46** FIR filter structure Example 4.30

**Solution** Linear phase FIR filter is symmetric about

$$\alpha = \frac{N-1}{2} = \frac{15-1}{2} = 7$$

$$|H(k)| = \begin{cases} 1, & \text{for } 0 \leq k \leq 3 \text{ and } 11 \leq k \leq 14 \\ 0, & \text{for } 4 \leq k \leq 10 \end{cases}$$

$$H(k) = \begin{cases} e^{-j\omega\alpha}, & k = 0, 1, 2, 3 \\ 0, & k = 4, 5, 6, 7 \end{cases}$$

$$H(k) = \begin{cases} e^{\frac{-j2\pi ka}{15}}, & k = 0, 1, 2, 3 \\ 0, & k = 4, 5, 6, 7 \end{cases}$$

$$H(k) = \begin{cases} e^{\frac{-j14\pi k}{15}}, & k = 0, 1, 2, 3 \\ 0, & 4 \leq k \leq 10 \\ e^{\frac{-j14\pi(k-15)}{15}}, & 11 \leq k \leq 14 \end{cases}$$

$$H(k) = \begin{cases} e^{\frac{j14\pi k}{15}}, & k = 0, 1, 2, 3 \\ 0, & k = 4, 5, 6, 7 \end{cases}$$

$$h(n) = \frac{1}{N} \left[ H(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \operatorname{Re}(H(k) e^{\frac{j2\pi nk}{15}}) \right]$$

$$= \frac{1}{15} \left[ 1 + 2 \sum_{k=1}^7 \operatorname{Re}[e^{\frac{j14\pi k}{15}} \cdot e^{\frac{j2\pi nk}{15}}] \right]$$

$$= \frac{1}{15} \left[ 1 + 2 \sum_{k=1}^3 \cos \frac{2\pi k(7-n)}{15} \right]$$

$$h(n) = \frac{1}{15} \left[ 1 + 2 \cos \frac{2\pi(7-n)}{15} + 2 \cos \frac{4\pi(7-n)}{15} + 2 \cos \frac{6\pi(7-n)}{15} \right]$$

$$h(0) = h(14) = \frac{1}{15} \left[ 1 + 2 \cos \frac{14\pi}{15} + 2 \cos \frac{28\pi}{15} + 2 \cos \frac{42\pi}{15} \right] = -0.05$$

$$h(1) = h(13) = \frac{1}{15} \left[ 1 + 2 \cos \frac{12\pi}{15} + 2 \cos \frac{24\pi}{15} + 2 \cos \frac{36\pi}{15} \right] = 0.041$$

$$h(2) = h(12) = \frac{1}{15} \left[ 1 + 2 \cos \frac{10\pi}{15} + 2 \cos \frac{20\pi}{15} + 2 \cos \frac{30\pi}{15} \right] = 0.0666$$

$$h(3) = h(11) = -0.0365$$

$$h(4) = h(10) = -0.1078$$

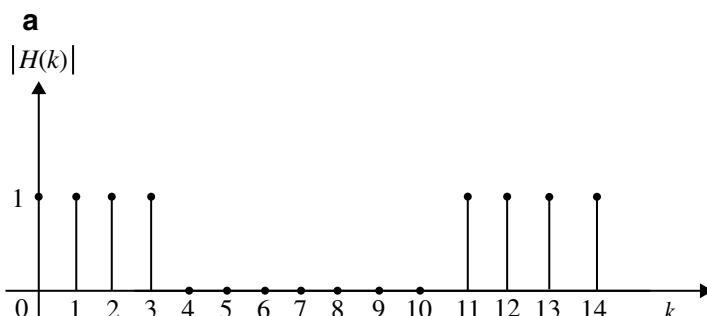
$$h(5) = h(9) = 0.034$$

$$h(6) = h(8) = 0.3188$$

$$h(7) = 0.466$$

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\ &= h(0)z^{-1} + h(1)z^{-2} + \dots \\ &= h(0)[1 + z^{-14}] + h(1)[z^{-1} + z^{-13}] + h(2)[z^{-2} + z^{-12}] \\ &\quad + h(3)[z^{-3} + z^{-11}] + h(4)[z^{-4} + z^{-10}] + h(5)[z^{-5} + z^{-9}] \\ &\quad + h(6)[z^{-6} + z^{-8}] + h(7)z^{-7} \\ H(z) &= -0.05[1 + z^{-14}] + 0.041[z^{-1} + z^{-13}] + 0.0666[z^{-2} + z^{-12}] \\ &\quad - 0.0365[z^{-3} + z^{-11}] - 0.1078[z^{-4} + z^{-10}] + 0.034[z^{-5} + z^{-9}] \\ &\quad + 0.3188[z^{-6} + z^{-8}] + 0.466z^{-7} \end{aligned}$$

The structure is shown in Fig. 4.47.



**Fig. 4.47** **a** Ideal magnitude response for Example 4.31. **b** Structure realization for Example 4.31

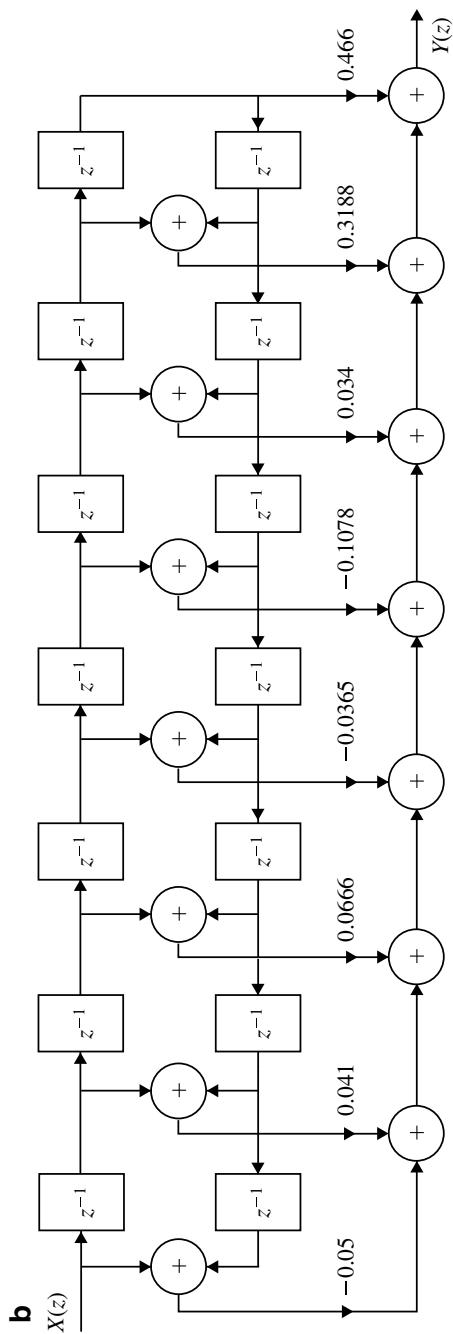


Fig. 4.47 (continued)

**Example 4.32**

Design a FIR filter approximating the ideal frequency response.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega}, & \text{for } |\omega| \leq \frac{\pi}{6} \\ 0, & \text{for } \frac{\pi}{6} \leq |\omega| \leq \pi \end{cases}$$

Determine the filter coefficients for  $N = 13$ .

(Anna University, December, 2005)

**Solution** We know that,

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} e^{-j\alpha\omega n} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} e^{j(n-\alpha)\omega} d\omega \\ &= \frac{1}{2} \left[ \frac{e^{j(n-\alpha)\omega}}{j(n-\alpha)} \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\ &= \frac{1}{\pi(n-\alpha)} \left[ \frac{e^{j(n-\alpha)\frac{\pi}{6}}}{2j} - \frac{e^{-j(n-\alpha)\frac{\pi}{6}}}{2j} \right] \\ &= \frac{\sin(n-\alpha)\frac{\pi}{6}}{\pi(n-\alpha)} \end{aligned}$$

From the frequency response we can find that the filter coefficients are symmetrical about  $\alpha = \frac{N-1}{2} = 6$ , satisfying  $h(n) = h(N-1-n)$

$$h_n = \begin{cases} h_d(n), & \text{for } 0 \leq n \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

$$h_n = \begin{cases} \frac{\sin \frac{\pi}{6}(n-\alpha)}{\pi(n-\alpha)}, & 0 \leq n \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
h(n) &= h(N - 1 - n) \\
h(n) &= h(13 - 1 - 0) = h(12) = 0 \\
h(1) &= h(11) = 0.0318 \\
h(2) &= h(10) = 0.0689 \\
h(3) &= h(9) = 0.106 \\
h(4) &= h(8) = 0.1379 \\
h(5) &= h(7) = 0.159 \\
h(6) &= \lim_{n \rightarrow 6} \frac{\sin \frac{\pi}{6}(n-6)}{6 \frac{\pi}{6}(n-6)} = \frac{1}{6} = 0.1667 \\
H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\
&= \sum_{n=0}^{12} h(n)z^{-n} \\
&= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + \cdots + h(12)z^{-12} \\
&= h(0)[1 + z^{-12}] + h(1)[z^{-1} + z^{-11}] + h(2)[z^{-2} + z^{-10}] \\
&\quad + h(3)[z^{-3} + z^{-9}] + h(4)[z^{-4} + z^{-8}] + h(5)[z^{-5} + z^{-7}] + h(6)z^{-6}
\end{aligned}$$

$$\begin{aligned}
H(z) = & 0.0318[z^{-1} + z^{-11}] + 0.0689[z^{-2} + z^{-10}] \\
& + 0.106[z^{-3} + z^{-9}] + 0.1379[z^{-4} + z^{-8}] \\
& + 0.159[z^{-5} + z^{-7}] + 0.1667z^{-6}
\end{aligned}$$

The structure realization is shown in Fig. 4.48.

### Example 4.33

Design a filter with

$$H_d(e^{-j\omega}) = \begin{cases} e^{-j3\omega}, & -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \leq \pi \end{cases}$$

(Anna University, May, 2004)

**Solution** Given

$$H_d(e^{-j\omega}) = e^{-j3\omega}$$

$$\alpha = 3 = \frac{N - 1}{2}$$

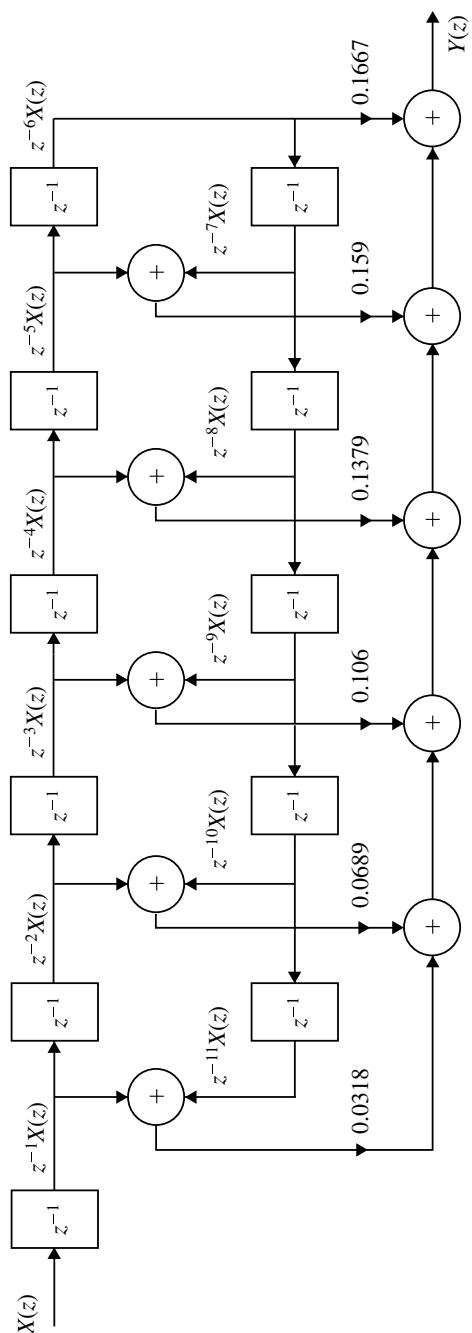


Fig. 4.48 Structure realization for Example 4.32

$$N = 7$$

The frequency response is having a term  $e^{-j\omega(\frac{N-1}{2})}$  which gives  $h(n)$  symmetrical about  $n = \frac{N-1}{2} = 3$ . Therefore, we get causal sequence.

$$\begin{aligned} h_d(n) &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j3\omega} e^{j\omega n} d\omega \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j(n-3)\omega} d\omega \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[ \frac{e^{j(n-3)\omega}}{j(n-3)} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ h_d(n) &= \frac{\sin \frac{\pi}{4}(n-3)}{\pi(n-3)} \end{aligned}$$

For  $N = 7$ , the filter coefficients are,

$$\begin{aligned} h(n) &= h_d(n) \\ h(0) &= h(6) = 0.075 \\ h(1) &= h(5) = 0.159 \\ h(2) &= h(4) = 0.22 \\ h(3) &= 0.25 \end{aligned}$$

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} = \sum_{n=0}^6 h(n)z^{-n} \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} \\ &= h(0)[1 + z^{-6}] + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} + z^{-4}] + h(3)z^{-3} \end{aligned}$$

$$H(z) = 0.075[1 + z^{-6}] + 0.159[z^{-1} + z^{-5}] + 0.22[z^{-2} + z^{-4}] + 0.25z^{-3}$$

**Example 4.34**

Design a filter with

$$H_d(e^{-j\omega}) = \begin{cases} e^{-j3\omega}, & -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4} \\ 0, & \frac{\pi}{4} \leq |\omega| \leq \pi \end{cases}$$

using a Hamming window with  $N = 7$ .

(Anna University, December, 2004)

**Solution** Given

$$H_d(e^{-j\omega}) = \begin{cases} e^{-j3\omega}, & -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4} \\ 0, & \frac{\pi}{4} \leq -|\omega| \leq \pi \end{cases}$$

$\alpha = 3$ ;  $\frac{N-1}{2} = 3$ ;  $N = 7$ . Impulse response  $h(n)$  is symmetric about  $\alpha = 3$ . Since the frequency response is having a term  $e^{-j\omega\alpha}$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j3\omega} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{j(n-3)\omega} d\omega \\ h_d(n) &= \frac{\sin \frac{\pi}{4}(n-3)}{\pi(n-3)} \end{aligned}$$

$$h_d(0) = h_d(6) = 0.075$$

$$h_d(1) = h_d(5) = 0.159$$

$$h_d(2) = h_d(4) = 0.22$$

$$h_d(3) = \lim_{n \rightarrow 3} \frac{\sin \frac{\pi}{4}(n-3)}{4 \cdot \frac{\pi}{4}(n-3)}$$

$$h_d(3) = \frac{1}{4} = 0.25$$

Hamming window function is,

$$\omega_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{6}\right), & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\omega_H(0) = 0.08$$

$$\omega_H(1) = 0.31$$

$$\omega_H(2) = 0.77$$

$$\omega_H(3) = 1$$

$$\omega_H(4) = 0.77$$

$$\omega_H(5) = 0.31$$

$$\omega_H(6) = 0.08$$

The filter coefficients are,

$$h(n) = h_d(n) \times \omega_H(n)$$

$$h(0) = h(6)$$

$$= h_d(0) \times \omega_H(0)$$

$$= 0.006$$

$$h(1) = h(5)$$

$$= h_d(0) \times \omega_H(1)$$

$$= 0.049$$

$$h(2) = h(4)$$

$$= h_d(0) \times \omega_H(2)$$

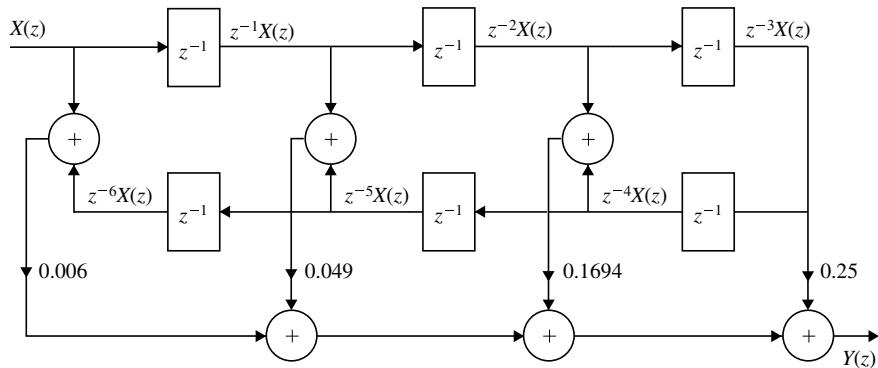
$$= 0.1694$$

$$h(3) = h_d(3)\omega_H(3)$$

$$= 0.25$$

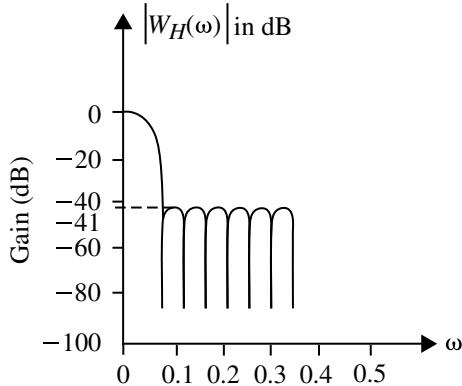
The transfer function of the filter is,

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h(n)z^{-n} \\ &= \sum_{n=0}^6 h(n)z^{-n} \end{aligned}$$



**Fig. 4.49** Structure realization for Example 4.34

Structure realization filter for Example 4.34 is shown in Fig. 4.49.



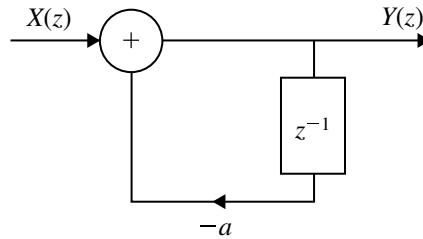
**Fig. 4.50** Log-magnitude spectrum of Hamming window

Log-magnitude spectrum of Hamming window for question 26 is shown in Fig. 4.50.

$$\begin{aligned}
 &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} \\
 &\quad + h(6)z^{-6} \\
 &= h(0)[1 + z^{-6}] + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} + z^{-4}] + h(3)z^{-3}
 \end{aligned}$$

$$\begin{aligned}
 H(z) = & 0.006[1 + z^{-6}] + 0.049[z^{-1} + z^{-5}] \\
 & + 0.1694[z^{-2} + z^{-4}] + 0.25z^{-3}
 \end{aligned}$$

The structure realized as shown in Figs. 4.49, 4.50 and 4.51.



**Fig. 4.51** Direct form realization for problem 49

Direct form realization for problem 49 is shown in Fig. 4.51.

### Example 4.35

A lowpass filter is required to be designed with the desired frequency response.

$$H_d(\omega) = \begin{cases} e^{-j2\omega}, & -0.25\pi \leq \omega \leq 0.25\pi \\ 0, & 0.25\pi \leq \omega \leq \pi \end{cases}$$

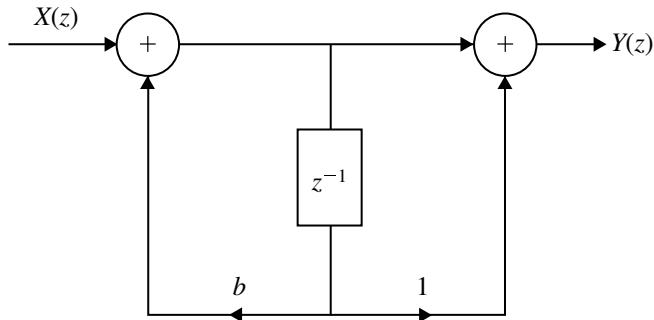
Obtain the filter coefficients,  $h(n)$  using Hamming window function. Also find the frequency response  $H(\omega)$  of the designed filter.

(Anna University, April, 2005)

**Solution** Given

$$H_d(\omega) = \begin{cases} e^{-j2\omega}, & -0.25\pi \leq \omega \leq 0.25\pi \\ 0, & 0.25\pi \leq \omega \leq \pi \end{cases}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-0.25\pi}^{0.25\pi} e^{-j2\omega} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-0.25\pi}^{0.25\pi} e^{j(n-2)\omega} d\omega \\ &= \frac{1}{2\pi} \left[ \frac{e^{j(n-2)\omega}}{j(n-2)} \right]_{-0.25\pi}^{0.25\pi} \\ &= \frac{1}{\pi(n-2)} \left[ \frac{e^{j0.25\pi(n-2)} - e^{-j0.25\pi(n-2)}}{2j} \right] \end{aligned}$$



**Fig. 4.52** A structure for problem 50

$$h_d(n) = \frac{1}{\pi(n-2)} \cdot \sin 0.25\pi(n-2)$$

From the given frequency it is clear that

$$\frac{N-1}{2} = 2 \rightarrow N-1 = 4 \rightarrow N = 5$$

$$\alpha = \frac{N-1}{2} = 2$$

If  $n = 2$ ,

$$\begin{aligned} h_d(n) &= \lim_{n \rightarrow 2} \frac{\sin(\frac{\pi}{4})(n-2)}{4 \cdot \frac{\pi}{4}(n-2)} \\ &= \frac{1}{4} \end{aligned}$$

$$H_d = \begin{cases} \frac{\sin 0.25(n-2)}{\pi(n-2)}, & n \neq 2 \\ \frac{1}{4}, & n = 2 \end{cases}$$

$$\begin{aligned} h_d(0) &= h_d(4) = \frac{\sin 0.25(-2)}{\pi(-2)} = 0.159 \\ h_d(1) &= h_d(3) = \frac{\sin 0.25(-1)}{\pi(-1)} = 0.225 \\ h_d(2) &= \frac{1}{4} = 0.25 \end{aligned}$$

### Hamming window

$$\begin{aligned}\omega_H(n) &= \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{4}\right), & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

$$\omega_H(0) = 0.54 - 0.46 = 0.08$$

$$\omega_H(1) = 0.54$$

$$\omega_H(2) = 1$$

$$\omega_H(3) = 0.54$$

$$\omega_H(4) = 0.54 - 0.46 = 0.08$$

$$h(n) = h_d(n) \times \omega_H(n)$$

$$h(0) = h(4) = h_d(0) \times \omega_H(0) = 0.0127$$

$$h(1) = h(3) = h_d(1) \times \omega_H(1) = 0.1215$$

$$h(2) = h_d(2) \times \omega_H(2) = 0.25$$

Filter coefficients are,

$h(0) = h(4) = 0.0127$
$h(1) = h(3) = 0.1215$
$h(2) = 0.25$

### Summary

- Digital filters are classified as FIR and IIR filters. They have their own merits and demerits. The choice of the filter is made depending upon the application.
- The difference equation which describes FIR filter is non-recursive; the output response is a function only of past and present values of the input. When FIR is implemented in this form, it is always stable.
- FIR filters have exactly linear phase characteristic which has unique properties and helps in the design and applications.
- The design of FIR filters are divided into five independent stages. They are:
  - (a) Filter specifications.
  - (b) Coefficient calculation.
  - (c) Realization.
  - (d) Analysis of errors.
  - (e) Implementation.
- FIR filters are designed usually by the following methods among others:

- (a) Frequency sampling method.
  - (b) Fourier series method.
  - (c) Design using windows.
- The following windowing techniques are generally used to design FIR filters:
- (a) Rectangular window.
  - (b) Raised cosine window.
  - (c) Hanning window.
  - (d) Hamming window.
  - (e) Blackman window.
  - (f) Kaiser window.
- There are different types of structures for realizing FIR filter system. The following structures are described in this chapter:
- (a) Direct form realization.
  - (b) Cascade form realization.
  - (c) Linear phase realization.
  - (d) Lattice structure realization.

### Short Questions and Answers

#### 1. What are FIR Filters?

The specifications of the desired filter will be given in terms of ideal frequency response  $H_d(\omega)$ . The impulse response  $h_d(n)$  of desired filter can be obtained by inverse Fourier transform of  $H_d(\omega)$ , which consists of infinite number of samples. The filters designed by selecting finite number of samples of impulse response are called FIR filters.

#### 2. What are advantages of FIR filters?

The advantages of FIR filters are:

- Linear phase FIR filters can be easily designed.
- Efficient realizations of FIR filter exist as both recursive and non-recursive structures.
- FIR filters realized non-recursively are always stable.
- The round off noise can be made small in non-recursive realization of FIR filter.

#### 3. What are disadvantages of FIR filters?

The disadvantages of FIR filters are:

- The duration of impulse response should be large to realize sharp cutoff filters.
- The non-integral delay can lead to problems in some signal processing applications.

**4. What are the conditions to be satisfied for constant phase delay in linear phase FIR filters?**

The conditions for constant phase delay are:

$$\text{Phase delay, } \alpha = \frac{N-1}{2} \text{ (i.e., phase delay is constant)}$$

Impulse response,  $h(n) = h(N-1-n)$  (i.e., impulse response is symmetric)

**5. What are the possible types of impulse response for linear phase FIR filter?**

There are four types of impulse response for linear phase FIR filter.

- Symmetric impulse response for  $N$  is odd.
- Symmetric impulse response for  $N$  is even.
- Anti-symmetric impulse response for  $N$  is odd.
- Anti-symmetric impulse response for  $N$  is even.

**6. Write the magnitude and phase function of FIR filter when impulse response is symmetric and  $N$  is odd.**

$$\text{Magnitude function, } |H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{N-1} 2h\left(\frac{N-1}{2} - n\right) \cos \omega n$$

$$\text{Phase function, } \angle H(\omega) = -\alpha\omega$$

**7. Write the magnitude and phase function of FIR filter when impulse response is symmetric and  $N$  is even.**

$$\text{Magnitude function, } |H(\omega)| = \sum_{n=1}^{N/2} 2h\left(\frac{N-1}{N} - n\right) \cos \left(\omega\left(n - \frac{1}{2}\right)\right)$$

$$\text{Phase function, } \angle H(\omega) = -\alpha\omega$$

**8. Write the magnitude and phase function of FIR filter when impulse response is anti-symmetric and  $N$  is odd.**

$$\text{Magnitude function, } |H(\omega)| = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{N} - n\right) \sin \omega n$$

$$\text{Phase function, } \angle H(\omega) = \beta - \alpha\omega$$

**9. Write the magnitude and phase function of FIR filter when impulse response is anti-symmetric and  $N$  is even**

$$\text{Magnitude function, } |H(\omega)| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right) \sin \left(\omega\left(n - \frac{1}{2}\right)\right)$$

Phase function,  $\angle H(\omega) = \beta - \alpha\omega$

**10. List the well-known design techniques for linear phase FIR filter.**

There are three well-known methods of design techniques for linear phase FIR filters. They are:

- Fourier series method and window method.
- Frequency sampling method.
- Optimal filter design method.

**11. What is Gibb's phenomenon (or Gibb's oscillation)?**

In FIR filter design by Fourier series method (or Rectangular window method) the finite duration impulse response is truncated to finite duration impulse response. The abrupt truncation of impulse response introduces oscillations in the passband and stopband. This effect is known as Gibb's phenomenon (or Gibb's oscillation).

**12. Write the procedure for designing FIR filter using windows?**

- Choose the desired frequency response of the filter  $H_d(\omega)$ .
- Take inverse Fourier transform of  $H_d(\omega)$  to obtain the desired impulse response  $h_d(n)$ .
- Choose a window sequence  $\omega(n)$  and multiply  $h_d(n)$  by  $\omega(n)$  to convert the infinite duration impulse response  $h(n)$ .
- The transfer function  $H(z)$  of the filter is obtained by taking  $z$ -transform of  $h(n)$ .

**13. Write the procedure for FIR filters designed by frequency sampling method.**

- Choose the desired frequency response  $H_d(\omega)$ .
- Take  $N$ -samples of  $H_d(\omega)$  to generate the sequence  $H(k)$ .
- Take inverse DFT of  $H(k)$  to get the impulse response  $h(n)$ .
- The transfer function  $H(z)$  of the filter is obtained by taking  $z$ -transform of impulse response.

**14. What is the drawback in FIR filter design using windows and frequency sampling method? How is it overcome?**

The FIR filter designs by window and frequency sampling method do not have precise control over the critical frequencies such as  $\omega_p$ , and  $\omega_s$ .

This drawback can be overcome by designing FIR filter using Chebyshev approximation technique. In this technique an error function is used to approximate the ideal frequency response, in order to satisfy the desired specifications.

**15. Write the characteristics features of Rectangular window.**

- The mainlobe width is equal to  $4\pi/N$ .
- The maximum sidelobe magnitude is  $-13$  dB.
- The sidelobe magnitude does not decrease significantly with increasing  $\omega$ .

**16. List the features of FIR filter designed using Rectangular window.**

- The width of the transition region is related to the width of the mainlobe of window spectrum.
- Gibb's oscillations are noticed in the passband and stopband.
- The attenuation in the stopband is constant and cannot be varied.

**17. How the transition width of the FIR filter can be reduced in design using windows?**

In FIR filter design using windows, the width of the transition region is related to the width of the mainlobe in window spectrum. If the mainlobe width is narrow then the transition region in FIR filter will be small. In general the width of mainlobe is  $x\pi/N$ , where  $x = 4$  or  $8$  or  $12$  and  $N$  is the length of the window sequence used for designing the filter. Hence, the width of mainlobe can be reduced by increasing the value of  $N$ , which in turn reduces the width of the transition region in the FIR filter.

**18. Why Gibb's oscillations are developed in Rectangular window and how can it be eliminated or reduced?**

The Gibb's oscillations in Rectangular window are due to sharp transition from  $1$  to  $0$  at the edges of window sequence. These oscillations can be eliminated or reduced by replacing the sharp transition by gradual transition. This is the motivation for development of triangular and cosine windows.

**19. List the characteristics of FIR filters designed using windows.**

- The width of transition band depends on the type of window.
- The width of transition band can be made narrow by increasing the value of  $N$  where  $N$  is the length of window sequence.
- The attenuation in the stopband is fixed for a given window, except in case of Kaiser window where it is variable.

**20. Write the frequency response of Hanning window.**

Frequency response of Hanning window is,

$$W_C(\omega) = 0.5 \frac{\sin \omega(N/2)}{\sin \omega/2} + 0.25 \frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))} \\ + 0.25 \frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}$$

**21. Write the frequency response of Hamming window.**

Frequency response of Hamming window is,

$$W_H(\omega) = 0.54 \frac{\sin \omega(N/2)}{\sin \omega/2} + 0.23 \frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))} + 0.23 \frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}$$

**22. Give the equation for Hanning window function.**

Hanning window function is,

$$W_C(n) = \begin{cases} 0.5 - 0.5 \cos(2\pi n/(N-1)); & \text{for } 0 \leq n \leq (N-1) \\ 0; & \text{else} \end{cases}$$

**23. List the features of Hanning window spectrum.**

- The mainlobe width is equal to  $8\pi/N$ .
- The maximum sidelobe magnitude is  $-31$  dB.
- The sidelobe magnitude decreases with increasing  $\omega$ .

**24. Compare the Rectangular window and Hanning window.**

	<b>Rectangular window</b>	<b>Hanning Window</b>
1	The mainlobe width is equal to $4\pi/N$	The mainlobe width is equal to $8\pi/N$
2	The maximum sidelobe magnitude in window spectrum is $-13$ dB	The maximum sidelobe magnitude in window spectrum is $-31$ dB
3	In window spectrum, the sidelobe magnitude slightly decreases with increasing $\omega$	In window spectrum, the sidelobe magnitude decreases with increasing $\omega$
4	In FIR filter designed using Rectangular window, the minimum stop-band attenuation is $22$ dB	In FIR filter designed using Hanning window, the minimum stop-band attenuation is $44$ dB

**25. Write the equation for Hamming window function.**

Hanning window function is,

$$W_H(n) = \begin{cases} 0.54 - 0.46 \cos(2\pi n/(N-1)); & \text{for } 0 \leq n \leq (N-1) \\ 0; & \text{else} \end{cases}$$

**26. Sketch the log-magnitude spectrum of Hamming window.**

**27. Compare the Rectangular window and Hamming window.**

	<b>Rectangular window</b>	<b>Hamming window</b>
1	The mainlobe width is equal to $4\pi/N$	The mainlobe width is equal to $8\pi/N$
2	The maximum sidelobe magnitude in window spectrum is $-13$ dB	The maximum sidelobe magnitude in window spectrum is $-41$ dB
3	In window spectrum, the sidelobe magnitude slightly decreases with increasing $\omega$	In window spectrum, the sidelobe magnitude remains constant
4	In FIR filter designed using Rectangular window, the minimum stopband attenuation is $22$ dB	In FIR filter designed using Hamming window, the minimum stopband attenuation is $51$ dB

**28. List the features of Hamming window spectrum.**

- The mainlobe width is equal to  $8\pi/N$ .
- The maximum sidelobe magnitude is  $-41$  dB.
- The sidelobe magnitude remains constant for increasing  $\omega$ .

**29. Compare the Hanning window and Hamming window.**

	<b>Hanning window</b>	<b>Hamming window</b>
1	The mainlobe width in window spectrum is equal to $8\pi/N$	The mainlobe width in window spectrum is equal to $8\pi/N$
2	The maximum sidelobe magnitude in window spectrum is $-31$ dB	The maximum sidelobe magnitude in window spectrum is $-41$ dB
3	In window spectrum, the sidelobe magnitude decreases with increasing $\omega$	In window spectrum, the sidelobe magnitude remains constant. Here, the increased sidelobe attenuation is achieved at the expense of constant attenuation at high frequencies
4	In FIR filter designed using Hanning window, the minimum stopband attenuation is $44$ dB	In FIR filter designed using Hamming window, the minimum stopband attenuation is $51$ dB

### 30. Compare Hamming window and Blackman window?

	<b>Hamming window</b>	<b>Blackman window</b>
1	The mainlobe width in window spectrum is equal to $8\pi/N$	The mainlobe width in window spectrum is equal to $12\pi/N$
2	The maximum sidelobe magnitude in window spectrum is $-41$ dB	The maximum sidelobe magnitude in window spectrum is $-58$ dB
3	The higher value of sidelobe attenuation is achieved at the expense of constant attenuation at high frequencies	The higher value of sidelobe attenuation is achieved at the expense of increased mainlobe width
4	In window spectrum, the sidelobe magnitude remains constant with increasing $\omega$	In window spectrum, the sidelobe magnitude decreases rapidly with increasing $\omega$
5	In FIR filter designed using Hamming window, the minimum stopband attenuation is $51$ dB	In FIR filter designed using Blackman window, the minimum stopband attenuation is $78$ dB

### 31. List the features of Blackman window spectrum.

- The mainlobe width is  $12\pi/N$ .
- The maximum sidelobe magnitude is  $-58$  dB.
- The sidelobe magnitude decreases with increasing  $\omega$ .
- The sidelobe attenuation in Blackman window is the highest among windows, which is achieved at the expense of increased mainlobe width. However, the mainlobe width can be reduced by increasing the value of  $N$ .

### 32. Write the expression for Kaiser window function.

Kaiser window function is,

$$W_k(n) = \begin{cases} \frac{I_0(\beta)}{I_0(\alpha)}; & \text{for } -(N-1)/2 \leq n \leq (N-1)/2 \\ 0; & \text{else.} \end{cases}$$

where  $\beta = \alpha \left( 1 - \left( \frac{2n}{(N-1)} \right)^2 \right)^{0.5}$

$$I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \dots$$

The series of  $I_0(x)$  is used to compute  $I_0(\beta)$  and  $I_0(\alpha)$  and are computed for any desired accuracy. Usually 25 terms of the series are sufficient for most practical purposes.

### 33. List the desirable features of Kaiser window spectrum.

- The width of mainlobe and the peak sidelobe are variable.
- The parameter  $\alpha$  in the Kaiser window function is an independent variable that can be varied to control the sidelobe levels with respect to mainlobe peak.

- The width of the mainlobe in window spectrum can be varied by varying the length  $N$  of the window sequence.

#### 34. Compare the Hamming window and Kaiser window.

	<b>Hamming Window</b>	<b>Kaiser Window</b>
1.	The width of mainlobe in window spectrum is $8\pi/N$ .	The width of mainlobe in window spectrum depends on the values of $\alpha$ and $N$
2.	The maximum sidelobe magnitude in windows spectrum is fixed at -41 dB.	The maximum sidelobe magnitude with respect to peak of mainlobe is variable using the parameter $\alpha$
3.	In window spectrum, the sidelobe magnitude remains constant with increasing $\omega$ .	In window spectrum, the sidelobe magnitude decreases with increasing $\omega$
4	In FIR filter designed using Hamming window, the minimum stopband attenuation is 51 dB	In FIR filter designed using Kaiser window, the minimum stopband attenuation is variable and depends on the value of $\alpha$

#### 35. Define an IIR filter.

The filter designed by considering all the infinite samples of impulse response is called IIR filter. The impulse response is obtained by taking inverse Fourier transform of ideal frequency response.

#### 36. Compare IIR and FIR filter.

	<b>IIR Filter</b>	<b>FIR Filter</b>
1.	All infinite samples of impulse response are considered.	Only $N$ samples of impulse response are considered
2.	The impulse response cannot be directly converted to digital filter transfer function.	The impulse response can be directly converted to digital filter transfer function
3.	The design involves design of analog filter and then transforming analog filter to digital filter.	The digital filter can be directly designed to achieve the desired specification
4.	The specifications include the desired characteristics for magnitude response only.	The specifications include the desired characteristics for both magnitude and phase response
5.	Linear phase characteristics cannot be achieved	Linear phase filters can be easily designed

#### 37. What are the properties that are maintained same in the transformation of analog to digital filters? (Or mention two properties that an analog filter should have for effective transformation).

The analog filters should be stable and causal for effective transformation to digital filters. While transforming the analog filters to digital filters, these two properties (i.e., stability and causality) are maintained same, which means that the transformed digital filter should also be stable and physically realizable.

#### 38. What is the condition to be satisfied by linear phase FIR filter?

(Anna University, 2007)

For an FIR filter to have linear phase, its phase response should satisfy the following equations

$$\begin{aligned}\theta(\omega) &= -\alpha\omega \\ \theta(\omega) &= \beta - \alpha\omega\end{aligned}$$

If the first equation is satisfied, the filter will have constant group and constant phase delay responses. For this the impulse response of the filter must have positive symmetry. If the second equation is satisfied, the filter will have constant group delay. In this case, the impulse response of the filter will have negative symmetry.

**39. In the design of FIR filter, how is Kaiser window different from other windows?**

In other windows the width of the mainlobe and the attenuation of the sidelobes depend only upon the length  $N$  of the filter and they cannot be controlled independently whereas in Kaiser window it is possible to control the length of the filter and the transition width of the mainlobe by introducing additional parameter.

**40. Show that the filter with  $h(n) = \{-1, 0, 1\}$  is a linear phase filter.**

$$\begin{aligned}H(e^{j\omega}) &= -e^{-j\omega}2 \sin \omega \\ \theta(\omega) &= \pi - \omega\end{aligned}$$

Hence, it has linear phase.

**41. What is the desirable characteristic of the frequency response of window function?**

The desirable characteristics of the frequency response of window function are:

- (a) The central lobe of the frequency response of the window should be narrow and contain maximum energy.
- (b) The sidelobes should decrease in energy as  $\omega$  tends  $\pi$ .
- (c) The highest sidelobe level should be small.

**42. Why is linear phase is important in Digital Signal Processing applications?**

The linear phase does not alter the shape of the signal and hence the output response is not distorted. Hence, digital filters with linear phase are designed.

**43. List out the well-known design techniques for linear phase FIR filter?**

The following techniques are used to design linear phase FIR filter.

- (a) Frequency sampling method.
- (b) Windows method.
- (c) Optimal or minimax design.

**44. What are the disadvantages of FIR filter?**

The disadvantages of FIR filter are:

- (a) Memory requirement and execution time are very high.

- (b) FIR filters require narrow transition band which increases the cost as it requires large arithmetic operations and hardware components such as multipliers, adders and time delay elements.

**45. What is frequency sampling?**

The filter coefficients  $h(n)$  can be obtained as the inverse DFT of the frequency samples,

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j(\frac{2\pi}{N})nk}$$

where,  $H(k) = 0, 1, \dots, (N - 1)$  are samples of the ideal frequency response. For linear phase, with positive symmetrical impulse response, the above equation is written as

$$h(n) = \frac{1}{N} \left[ \sum_{k=1}^{\frac{N}{2}-1} 2|H(k)| \cos \frac{2\pi k(n - \alpha)}{N} + H(0) \right]$$

where  $\alpha = (N - 1)/2$ . The resulting filter will have a frequency response which is exactly same as the original response at the sampling instants. This is called frequency sampling.

**46. For what type of filters frequency sampling method is suitable?**

The frequency sampling method is very suitable for the design of non-recursive FIR frequency selective filters such as lowpass, highpass and bandpass filters and also for filters with arbitrary frequency response.

**47. What is meant by FIR filter and why is it stable?**

Consider the following difference equation and the transfer function of a digital filter which are described as given below:

$$y(n) = \sum_{n=0}^{N-1} h(k)x(n - k)$$

$$H(z) = \sum_{n=0}^{N-1} h(k)z^{-k}$$

where  $h(n)$  are the coefficients of the filter. When the digital filter is implemented in this form, it is called FIR filters. The poles of  $H(z)$  all fall at the origin of the  $z$ -plane, and therefore FIR filters are always stable.

**48. Find the number of delays required in a direct form I structure that realizes a second-order transfer function?**

Four delay elements are required to realize a second-order system in direct form-I structure.

**49. The unit sample response of an FIR filter is**

$$h(n) = a^n \{u(n) - u(n-2)\}$$

**Draw the direct form realization of this system.**

$$H(z) = \frac{1}{(1 + az^{-1})}$$

50. A first-order filter structure is shown in Fig. 4.52. Find the transfer function?

$$H(z) = \frac{(1 + z^{-1})}{(1 - bz^{-1})}$$

**Long Answer Type Questions**

1. Describe the design of FIR filter using frequency sampling technique.

(Anna University, December, 2007)

2. A bandpass FIR filter of length 7 is required. It is to have lower and upper cutoff frequencies of 3 kHz and 5 kHz respectively and is intended to be used with a sampling frequency of 24 kHz. Determine the filter coefficients using Hanning window. Consider the filter to be causal.

(Anna University June, 2007)

3. Compare the Hamming window and Blackman window.

(Anna University, December, 2007)

4. Explain the polyphase decomposition for FIR filter structure.

(Anna University, December, 2007)

5. What is the principle of designing FIR filter using frequency sampling method?

(Anna University, December, 2009)

6. Design a second-order band reject filter with  $\omega_1$  and  $\omega_2$  as cutoff frequency and sampling interval as  $T$ .

(Anna University, May, 2009)

7. List the three well-known methods of design techniques for FIR filters and explain any one.

(Anna University, December, 2006)

8. Design an ideal lowpass filter for a cutoff frequency  $\omega_c = 0.4$  rad using (8-tap) window design method.

(Anna University, June, 2006)

9. Determine the filter coefficients  $h(n)$  of length  $M = 15$  obtained by sampling its frequency response as

$$H\left[\left(\frac{2\pi}{15}\right)k\right] = \begin{cases} 1, & k = 0, 1, 2, 3, 4 \\ 0.4, & k = 5 \\ 0, & k = 6, 7 \end{cases}$$

using Rectangular window.

(Anna University, June, 2006)

10. Design a digital filter with

$$H_d(e^{j\omega}) = \begin{cases} 1, & 2 \leq |\omega| \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

using Hamming window with  $N = 7$ . Draw the frequency response.

(Anna University, June, 2006)

11. List the various steps in designing FIR filters.

(Anna University, June, 2006)

12. The desired frequency response of a desired filter is

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & -\frac{\pi}{4} \leq \omega \leq \frac{\pi}{4} \\ 0, & \frac{\pi}{4} \leq |\omega| \leq \pi \end{cases}$$

Determine the filter coefficients if the window function is defined as

$$\omega(x) = \begin{cases} 1, & 0 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$$

(Anna University, June, 2006)

13. Illustrate the steps involved in the design of linear phase FIR filter by the frequency sampling method.  
 14. An FIR filter is given by

$$y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Find the lattice structure coefficients.

(Anna University, May, 2004)

15. What are the issues in designing FIR filter using window method?

(Anna University, May, 2004)

16. Mention the advantages and disadvantages of FIR and IIR filters.

(Anna University, May, 2004)

17. Design a digital lowpass FIR filter of length 11 with cutoff frequency of 0.5 kHz and sampling rate 2 kHz using Hamming window.

(Anna University, May, 2004)

18. Derive the frequency response of a linear phase FIR filter with symmetric impulse response.

19. Explain the design procedure for designing FIR filter using window function.

20. Explain the concept of optimum equiripple filter.

21. Design a nine-tap linear phase filter having the ideal response.

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{6} \\ 0, & \frac{\pi}{6} \leq |\omega| \leq \frac{\pi}{3} \\ 1, & \frac{\pi}{3} \leq |\omega| \leq \pi \end{cases}$$

using Hamming window and draw the realization for the same.

22. Design an FIR digital filter to approximate an ideal lowpass filter with passband gain of unity, cutoff frequency of 850 kHz and working at a sampling frequency of  $f_s = 5$  kHz. The length of the impulse response should be 5. Use a Rectangular window.

(Anna University, December, 2003)

23. The desired response of a lowpass filter is,

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & \frac{-3\pi}{4} \leq \omega \leq \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} \leq |\omega| \leq \pi \end{cases}$$

Determine the frequency response of the filter for  $M = 7$  using a Hamming window.

(Anna University, December, 2003)

# Chapter 5

## Finite Word Length Effects



### Learning Objectives

After completing this chapter, you should be able to:

- ✖ Provide an understanding of the errors that arise in practical DSP systems due to quantization and use of finite word length arithmetic.
- ✖ Study the effects of errors on signal quality.
- ✖ Develop the techniques to combat the errors.
- ✖ Enhance the skill in the design of DSP systems.

### 5.1 Introduction

When digital systems are implemented either in hardware or in software, the filter coefficients are stored in binary registers. These registers can accommodate only a finite number of bits, and hence, the filter coefficients have to be truncated or rounded off in order to fit into these registers. Truncation or rounding of the data results in degradation of system performance. Also, in digital processing systems, a continuous time input signal is sampled and quantized in order to get the digital signal. The process of quantization introduces an error in the signal which is called round off noise. This makes the system nonlinear and leads to limit cycle behavior.

In general the effects due to finite precision representation of numbers in a digital system are commonly referred to as finite word length effects. Some of the finite word length effects in digital filters are:

1. Errors due to quantization of input data by A/D converter.
2. Errors due to quantization of filter coefficients.
3. Errors due to rounding the product in multiplication.
4. Errors due to overflow in addition.
5. Limit cycles.

The effects of these errors introduced by signal processing depend on a number of factors which include the type of arithmetic used, the quality of the input signal and the DSP algorithm implemented. These are discussed in this chapter.

## 5.2 Representation of Numbers in Digital System

The basic operations involved in Digital Signal Processing are multiplications, additions and delays. They are often carried out using either fixed point or floating point arithmetic. Block floating point arithmetic combines the benefits of the above two operations. Fixed point arithmetic is the most widely used arithmetic in DSP because it is very fast and less expensive when implemented. However, it is limited in the range of numbers that can be represented. Further it is susceptible to problems of overflow which may occur when the result of an addition exceeds the permissible number range. To prevent this the operands are scaled. However, this degrades the performance of DSP systems which reduces the signal to noise ratio.

Floating point arithmetic is preferred where the magnitude of the variables or system coefficients vary widely and eliminate overflow problem. Further, floating point processing simplifies programming. However, floating point arithmetic is more expensive and often slower. While fixed point digital signal processors with large word lengths are extensively used in DSP techniques where wide dynamic range and high precision are required the floating point processing provides a simpler and more natural way of achieving these requirements. The applications of floating point arithmetic include real-time parameter equalization of digital audio signals, spectrum analysis in radar and sonar, seismology, biomedicine. The above two arithmetic operations are discussed below.

### 5.2.1 Fixed Point Representation

In fixed point representation, the bits allotted for integer part and fraction part are fixed, and so the position of binary point is also fixed.

#### 5.2.1.1 Positive Binary Fraction Number

In fixed point representation, there is only one unique way of representing positive binary number as given by the following equation:

$$\text{Positive binary fraction number, } N_p = \sum_{i=0}^B d_i 2^{-i} \quad (5.1)$$

where  $d_i = i$ th digit of the number and  $B =$  number of fractional digits.

### 5.2.1.2 Negative Binary Fraction Number

In fixed point representation, there are three different formats for representing negative binary numbers. They are:

1. Sign-magnitude format.
2. One's complement format.
3. Two's complement format.

#### 1. Sign-magnitude Format

Except the sign bit all other digits of the negative of a given number are same as that of its positive representation. The sign bit is 0 for positive number and 1 for negative number.

$$\text{Positive binary fraction number, } N_p = (0 \times 2^0) + \sum_{i=1}^B d_i 2^{-i} \quad (5.2)$$

$$\text{Negative binary fraction number, } N_n = (1 \times 2^0) + \sum_{i=1}^B d_i 2^{-i} \quad (5.3)$$

For example,

$$\begin{aligned} +0.125_{10} &\longrightarrow 0.001_2 \\ -0.125_{10} &\longrightarrow 1.001_2 \end{aligned}$$

#### 2. One's Complement Format

- In one's complement format, the positive number is same as that of sign-magnitude format.
- The negative of the given number is obtained by bit by bit complement of its positive representation

$$\text{Complement of } d_i = \bar{d}_i = (1 - d_i) \quad (5.4)$$

Negative binary fraction number in one's complement is,

$$N_{1c} = (1 \times 2^0) + \sum_{i=1}^B (1 - d_i) 2^{-i} \quad (5.5)$$

For example,

$$\begin{aligned} +0.125_{10} &\longrightarrow 0.001_2 \\ -0.125_{10} &\longrightarrow 1.110_2 \end{aligned}$$

### 3. Two's Complement Format

- In two's complement format, the positive number is same as that of the given magnitude format.
- The negative of the given number is obtained by taking one's complement of its positive representation and then adding one to the least significant bit.

Negative binary fraction number in two's complement is,

$$N_{2c} = (1 \times 2^0) + \sum_{i=1}^B (1 - d_i)2^{-i} + (1 \times 2^{-B}) \quad (5.6)$$

For example,

$$\begin{aligned} +0.125_{10} &\longrightarrow 0.001_2 \\ -0.125_{10} &\longrightarrow 1.111_2 \end{aligned}$$

### Disadvantage of Fixed Point Representation

It is impossible to represent too large and too small numbers by fixed point representation. Therefore, the range of numbers that can be represented in fixed point method for a given binary word size is less compared in floating point representation.

### Example 5.1

Convert the decimal number 25.625 to binary form.

#### Solution

##### Integer part conversion

$$\begin{array}{r} \cancel{2} \underline{25} \\ \cancel{2} \underline{12} - 1 \quad \uparrow \text{(LSB)} \\ \cancel{2} \underline{6} - 0 \quad \uparrow \\ \cancel{2} \underline{3} - 0 \quad \uparrow \\ \cancel{1} \underline{1} - 1 \quad \uparrow \\ \rightarrow \quad \rightarrow \rightarrow \\ (\text{MSB}) \end{array}$$

##### Fractional part conversion

Integer	
$0.625 \times 2 = 1.25$	1 (MSB)
$0.25 \times 2 = 0.5$	0 ↓
$0.5 \times 2 = 1.0$	1 (LSB)

Therefore,

$$(25.625)_{10} = (11001.101)_2$$

**Example 5.2**

Convert the following decimal numbers into binary. (a)  $(20.675)_{10}$  and (b)  $(120.75)_{10}$ .

**Solution**

(a)  $(20.675)_{10} \Rightarrow$  integer part conversion by successive division

$$\begin{array}{r} \cancel{2} \ 20 \\ \cancel{2} \ \cancel{10} - 0 \quad \uparrow \text{(LSB)} \\ \cancel{2} \ \cancel{5} - 0 \quad \uparrow \\ \cancel{2} \ \cancel{2} - 1 \quad \uparrow \\ \cancel{1} - 0 \quad \uparrow \\ \rightarrow \quad \rightarrow \rightarrow \\ (\text{MSB}) \end{array}$$

$$(10100)_2$$

Fractional part conversion by successive multiplication

$$\begin{array}{ll} 0.675 \times 2 = 1.35 & 1 \quad (\text{MSB}) \\ 0.35 \times 2 = 0.7 & 0 \quad \downarrow \\ 0.7 \times 2 = 1.4 & 1 \quad \downarrow \\ 0.4 \times 2 = 0.8 & 0 \quad \downarrow \\ 0.8 \times 2 = 1.6 & 1 \quad \downarrow \\ 0.6 \times 2 = 1.2 & 1 \quad \downarrow \\ 0.2 \times 2 = 0.4 & 0 \quad (\text{LSB}) \end{array}$$

$$(.1010110\ldots)_2$$

Binary equivalent of  $(20.675)_{10}$  is

$(10100.1010110\ldots)_2$

(b)  $(120.75)_{10} \Rightarrow$  integer part conversion

$$\begin{array}{r}
 \cancel{2} \underline{120} \\
 \cancel{2} \underline{60} - 0 \quad \uparrow \text{(LSB)} \\
 \cancel{2} \underline{30} - 0 \quad \uparrow \\
 \cancel{2} \underline{15} - 0 \quad \uparrow \\
 \cancel{2} \underline{7} - 1 \quad \uparrow \\
 \cancel{2} \underline{3} - 1 \quad \uparrow \\
 \cancel{2} \underline{1} - 1 \quad \uparrow \\
 \rightarrow \quad \rightarrow \rightarrow \\
 \text{(MSB)}
 \end{array}$$

$$(1111000)_2$$

fractional part conversion

$$\begin{array}{r}
 0.75 \times 2 = 1.5 \quad 1 \quad \text{(MSB)} \\
 \downarrow \\
 0.5 \times 2 = 1.0 \quad 1 \quad \text{(LSB)}
 \end{array}$$

$$(0.11)_2$$

Binary equivalent of  $(120.75)_{10}$  is

$$\boxed{(1111000.11)_2}$$

### Example 5.3

Represent the following number in fixed point representation.

$$(a) +0.375_{10} \quad (b) -0.75_{10}$$

### Solution

$$(a) +(0.375)_{10}$$

In fixed point representation

$$+0.375_{10} = (0.011)_2$$

(b)  $-(0.75)_{10}$

In fixed point representation there are three different formats for representing negative number:

- (i) Sign magnitude:  $-(0.75)_{10} = 1.110.$
- (ii) One's complement:  $-(0.75)_{10} = 1.001.$
- (iii) Two's complement:  $-(0.75)_{10} = 1.010.$

### Example 5.4

Represent the following numbers in sign-magnitude form.

$$(a) +8.25_{10} \quad (b) -8.25_{10}$$

### Solution

$$+(8.25)_{10} = 01000.010$$

↑  
sign bit

$$-(8.25)_{10} = 11000.010$$

↑  
sign bit

### Example 5.5

Represent the following numbers in one complement form.

$$(a) -0.375_{10} \quad (b) -0.0625_{10}$$

### Solution

$$(a) -0.375_{10} = (1.1001111)_2$$

$$0.375_{10} = (0.0110000)_2$$

Complementing each bit we get  $-0.375_{10} = (1.1001111)_2.$

$$(b) -0.0625_{10} = (1.1110111)_2$$

$$0.0625_{10} = (0.0001000)_2$$

Complementing each bit we get  $-0.0625_{10} = (1.1110111)_2$ .

---

### Example 5.6

Represent the following numbers in two's complement form.

$$(a) -0.125_{10} \quad (b) -0.25_{10}$$


---

#### Solution

$$(a) -0.125_{10} \text{ where } +0.125_{10} = 0.001_2$$

One's complement  $\Rightarrow 1.110_2$

Two's complement  $\Rightarrow 1.111_2$

$$(b) -0.25_{10} \text{ where } +0.25_{10} = 0.010_2$$

One's complement  $\Rightarrow 1.101_2$

Two's complement  $\Rightarrow 1.110_2$

### 5.2.2 Floating Point Representation

In floating point representation, the binary point can be shifted to desired position so that bits in the integer part and fraction part of a number can be varied. In general, the floating point number can be represented as

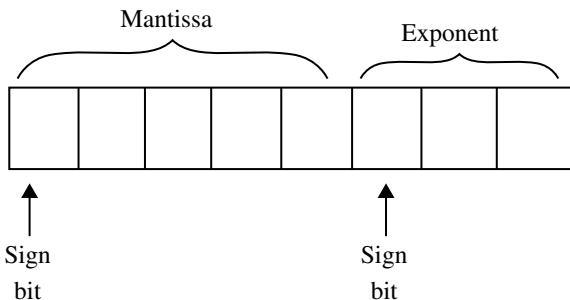
$$N_f = M \times 2^E \quad (5.7)$$

where  $M$  = mantissa and  $E$  = exponent.

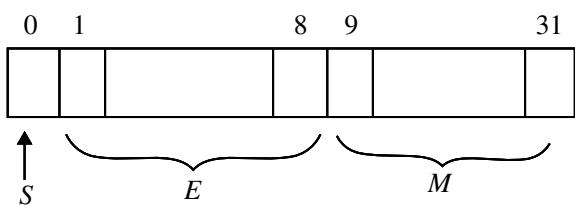
#### Mantissa

- It will be in binary fraction format and in the range of  $0.5 \leq M \leq 1$ .
- If mantissa is characterized by 5 bits in which the left most one bit is used for representing sign and other 4 bits are used to represent a binary fraction number.

**Fig. 5.1** Floating point number representation



**Fig. 5.2** IEEE-754 format for 32 bit floating point number



### Exponent

- It is either a positive or negative integer.
- If it is characterized by 3 bits out of which the left most one bit is used to represent sign and the other two bits are used to represent a positive or negative binary integer number. The representation of floating point number is shown in Fig. 5.1.

The range of number that can be represented by floating point format is from  $\pm(2^{-4} \times 2^{-3})$  to  $\pm((2 - 2^{-4}) \times 2^{-3})$ .

Here 4 in  $2^{-4}$  represents the 4 bits allotted for fractional binary number in mantissa, and the 3 in  $2^{-3}$  or  $2^{+3}$  represents the maximum size of integer.

The IEEE-754 standard for 32 bit single precision floating point number is given by

$$N_f = (-1)^S \times 2^{E-127} \times M \quad (5.8)$$

where

$S = 1$  bit field for sign of number.

$E = \varepsilon$  bit field for exponent.

$M = 23$  bit field for mantissa.

IEEE-754 format for 32 bit floating point number is shown in Fig. 5.2.

**Example 5.7**

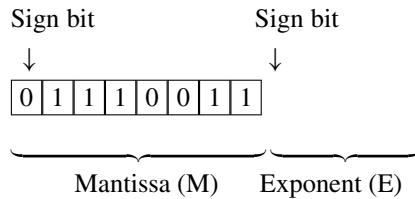
Represent the following numbers in floating point representation with five bits for mantissa and three bits for exponent.

$$(a) +7_{10} \quad (b) +0.25_{10} \quad (c) -7_{10} \quad (d) -0.25_{10}$$

**Solution**

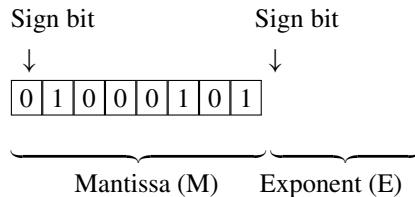
(a)

$$+7_{10} = +111_2 = 0.1110 \times 2^{+3} = 0.1110 \times 2^{+11_2}$$



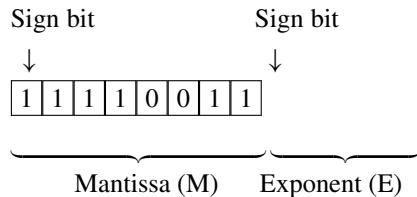
(b)

$$+0.25_{10} = +0.01_2 = 0.0100 \times 2^0 = 0.1000 \times 2^{-1} = 0.1000 \times 2^{-01_2}$$



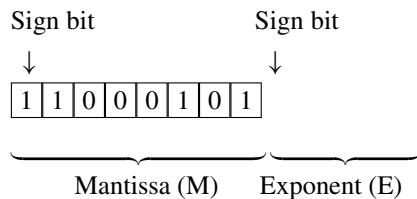
(c)

$$-7_{10} = -111_2 = 1.1110 \times 2^{+3} = 1.1110 \times 2^{+11_2}$$



(d)

$$-0.25_{10} = -0.01_2 = 1.0100 \times 2^0 = 1.1000 \times 2^{-1} = 1.1000 \times 2^{-01_2}$$



### Example 5.8

Add  $+0.125_{10}$  and  $-0.375_{10}$  in one's complement and two's complement forms.

#### Solution

##### One's Complement Form

$$\begin{array}{r}
 +0.125_{10} \Rightarrow 0.001_2 \\
 -0.375_{10}, \text{(in one's complement)} \Rightarrow 1.100_2 \\
 \\ 
 \hline
 (+) \\
 1.101_2
 \end{array}$$

Since carry is zero the sum is negative. The sum can be converted to decimal as

$$1.101_2 \xrightarrow{\text{extract sign bit}} 0.101_2 \xrightarrow{\substack{\text{complement the} \\ \text{fractional part}}} -0.010_2 \xrightarrow{\text{Convert to decimal}} -0.25_{10}$$

**Two's Complement Form**

$$\begin{array}{r}
 +0.125_{10} \Rightarrow 0.001_2 \\
 -0.375_{10} \text{ (in two's complement)} \Rightarrow 1.101_2 \\
 \hline
 (+) & \hline \\
 & 1.110_2
 \end{array}$$

Since carry is zero the sum is negative

$$1.110_2 \xrightarrow{\text{extract sign bit}} 0.110_2 \xrightarrow{\text{two's complement}} -0.010_2 \xrightarrow{\text{Convert to decimal}} -0.25_{10}$$


---

**Example 5.9**

Add  $-0.125_{10}$  and  $+0.375_{10}$  in one's complement and two's complement forms.

**Solution****One's Complement Form**

$$\begin{array}{r}
 -0.125_{10} \text{ (in one's complement)} \Rightarrow 1.110_2 \\
 +0.375_{10} \Rightarrow 0.011_2 \\
 \hline
 (+) & \hline \\
 & 1.001 \\
 & \quad 1 \\
 \hline
 & 0.010
 \end{array}$$

Since carry is one, the sum is positive

$$0.010_2 \xrightarrow{\text{Covert to decimal}} +.25_{10}$$

**Two's Complement Form**

$$\begin{array}{r}
 -0.125_{10} \text{ (in two's complement)} \Rightarrow 1.111_2 \\
 +0.375_{10} \Rightarrow 0.011_2 \\
 \hline
 (+) & \hline \\
 & 10.010
 \end{array}$$

Discard the carry, since the sum is positive.

$$0.010_2 \xrightarrow{\text{convert to decimal}} +0.25_{10}$$


---

### Example 5.10

Multiply  $+0.25_{10}$  and  $+5_{10}$  in fixed point format.

#### Solution

$$+0.25_{10} = 0.010_2$$

$$+5_{10} = 101_2$$

$$\begin{array}{r} 0.010_2 \\ \times 101_2 \\ \hline \end{array}$$

$$\begin{array}{r} 0010 \\ 0000 \\ 0010 \\ \hline \end{array}$$

$$\begin{array}{r} 001010_2 \\ \hline \end{array}$$

$$001.010_2 \xrightarrow{\text{covert to decimal}} 1.25_{10}$$


---

### Example 5.11

Add  $+0.125_{10}$  and  $+5_{10}$  in floating point format.

#### Solution

$$\begin{aligned} +5_{10} &= +101_2 = 0.101000 \times 2^{+3_{10}} = 0.101000 \times 2^{11_2} \\ &= 0.101000 \times 2^{011_2} \end{aligned}$$

$$\begin{aligned}
 +0.125_{10} &= +.001_2 = 0.100000 \times 2^{-2_{10}} \xrightarrow{\text{unnormalized}} 0.000001 \times 2^{+3_{10}} \\
 &= 0.000001 \times 2^{011_2} \\
 +5_{10} &\Rightarrow 0.101000 \times 2^{011_2} \\
 +0.125_{10} &\Rightarrow 0.000001 \times 2^{011_2} \\
 \\ 
 &\hline \\
 &0.101001 \times 2^{011_2} \\
 \\ 
 &0.101001 \times 2^{+3_2} = 101.001 \times 2^0 = 101.001_2 = 5.125_{10}
 \end{aligned}$$

**Example 5.12**

Multiply  $+0.125_{10}$  and  $+5_{10}$  in floating point format.

**Solution**

$$\begin{aligned}
 +0.125_{10} &= 0.100000 \times 2^{-2_{10}} \\
 +5_{10} &= 0.101000 \times 2^{+3_{10}} \\
 +0.125_{10} \times +5_{10} &\Rightarrow (0.100000 \times 0.101000) \times 2^{-2+3} \\
 &\Rightarrow 0.010100 \times 2^1
 \end{aligned}$$

$$0.010100 \times 2^1 \xrightarrow{\text{normalized}} 0.10100 \times 2^0 = 0.625_{10}$$

**5.3 Methods of Quantization**

The process of converting a discrete time continuous amplitude signal into a digital signal by expressing each sample value as a finite number of digits is called **quantization**. The error introduced in representing the continuous valued signal by a finite set of discrete level is called “**quantization error or quantization noise**.” The quantization error is a sequence which is defined as the difference between the quantized value and the actual sample value. The actual values of the samples of  $x(n)$  cannot be processed by DSP or a digital computer since it is very difficult to store and manipulate all the samples. To eliminate the excess digits that occur due to quantization either discard them (truncation) or discard them by rounding the resulting number (rounding).

Thus, there are two methods of quantization employed in digital system. They are: (1) truncation and (2) rounding. They are discussed below.

### 5.3.1 Truncation

Truncation is the process of reducing the size of binary numbers by discarding all bits less significant than the least significant bit that is retained. In the truncation of a binary numbers to  $b$  bits, all the less significant bits beyond  $b^{\text{th}}$  bit are discarded. The quantization steps are marked on  $y$ -axis, and the range of unquantized numbers is marked on  $x$ -axis.

1. Any positive unquantized number in the range  $0 \leq N \leq (1 \times 2^{-b})$  will be assigned the quantization step  $(0 \times 2^{-b})$ .
2. Any positive unquantized number in the range  $(1 \times 2^{-b}) \leq N \leq (2 \times 2^{-b})$  will be assigned the quantization step  $(1 \times 2^{-b})$  and so on.

#### Example 5.13

Perform the quantization of  $0.875_{10}$  to 2 bit by truncation.

#### Solution

$$0.875_{10} \xrightarrow[\text{to binary}]{\text{convert}} 0.1110_2 \xrightarrow{\text{truncate to 2 bits}} 0.11_2 \xrightarrow[\text{to decimal}]{\text{convert}} 0.75_{10}$$

#### 5.3.1.1 Fixed Point Number System

In fixed point number system, the effect of truncation on positive numbers is same in all the three representations. The error due to truncation of negative number depends on the type of representation of the number. Let  $N$  = unquantized fixed point binary numbers and  $N_t$  = fixed point binary number quantized by truncation. The quantization error due to truncation is

$$\text{Truncation error, } e_t = N_t - N \quad (5.9)$$

The range of errors in truncation of fixed point numbers in different types of representation is shown in Fig. 5.3 and tabulated in Table 5.1.

The truncation of a positive number results in a number that is smaller than the unquantized number; hence, truncation error is always negative.

For the truncation of negative numbers represented in sign magnitude and one's complement format, the error is always positive because truncation basically reduces the magnitude of the numbers.

In the two's complement representation the negative of a number is obtained by subtracting the corresponding positive number from 2. Therefore, the effect of truncation on a negative number is to increase the magnitude of the negative number and so the truncation error is always negative.

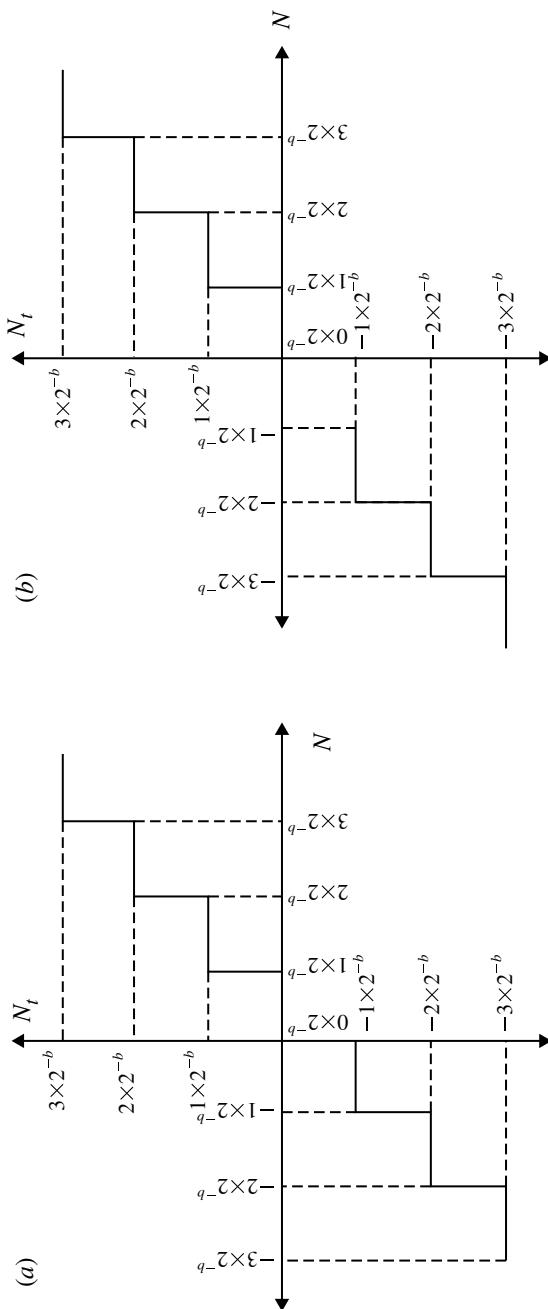
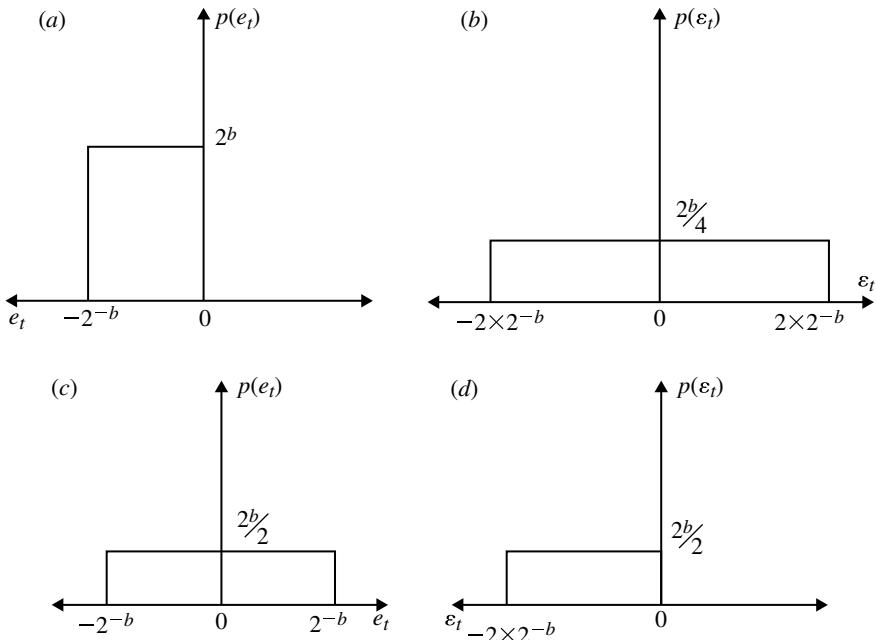


Fig. 5.3 Truncation characteristics of **a** two's complement, **b** Sign magnitude and one's complement

**Table 5.1** Range of error in truncation of fixed point numbers

Numbers and its representation	Range of error when truncated to $b$ bits
Positive numbers	$0 \geq e > -2^{-b}$
Sign-magnitude negative number	$0 \leq e < -2^{-b}$
One's complement negative number	$0 \leq e < -2^{-b}$
Two's complement negative number	$0 \geq e > -2^{-b}$

**Fig. 5.4** Quantization noise probability density function for truncations. **a** Fixed point two's complement, **b** Floating point when mantissa in two's complement, **c** Fixed point one's complement or sign magnitude and **d** Floating point when mantissa is one's complement or in sign magnitude

### 5.3.1.2 Floating Point Number System

In floating point representation, the mantissa of the number alone is truncated. The truncation error in a floating point number is proportional to the number being quantized.

Let  $N_f$  = unquantized floating point binary number and  $N_{tf}$  = truncated floating point binary number. Now

$$N_{tf} = N_f + N_f \varepsilon_t \quad (5.10)$$

**Table 5.2** Range of error in truncation of floating point number

Numbers and its representation	Range of error when truncated to $b$ bits
Two's complement positive mantissa	$0 \geq \varepsilon_t > -2^{-b} \times 2$
Two's complement negative mantissa	$0 \leq \varepsilon_t < 2^{-b} \times 2$
One's complement positive and negative mantissa	$0 \leq \varepsilon_t < -2 \times 2^{-b}$
Sign-magnitude positive and negative mantissa	$0 \leq \varepsilon_t < -2 \times 2^{-b}$

where  $\varepsilon_t$  is the relative error due to truncation of floating point number. Relative error due to truncation is,

$$\varepsilon_t = \frac{N_{tf} - N_f}{N_f} \quad (5.11)$$

In truncation of binary number, the range of error is known, but the probability of obtaining an error within the range is not known. Hence, it is assumed that the errors occur uniformly throughout the interval. The range of error in truncation of floating point number is shown in Table 5.2, and the corresponding quantization noise probability density function is shown in Fig. 5.4.

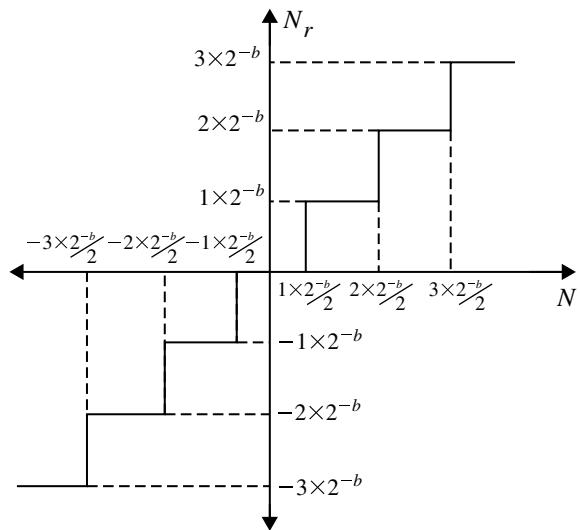
### 5.3.2 Rounding

Rounding is the process of reducing the size of a binary number to finite word size of  $b$  bits such that the rounded  $b$  bit number is closest to the original unquantized number. The rounding process consists of truncation and addition. In rounding of a number to  $b$  bits, first the unquantized number is truncated to  $b$  bits by retaining the most significant  $b$  bits. Then a zero or one is added to LSB of the truncated number depending on the bit that is next to the least significant bit that is retained. If the bit next to the least significant bit that is retained is zero, then zero is added to the least significant bit of the truncated number. If the bit next to the least significant bit that is retained is one, then one is added to the least significant bit of the truncated number. The input-output characteristics of the quantizer used for rounding are shown in Fig. 5.5.

The quantization steps are marked on  $y$ -axis, and the range of unquantized numbers are marked on  $x$ -axis.

1. Any positive unquantized number in the range  $1 \times \frac{2^{-b}}{2} \leq N < 2 \times \frac{2^{-b}}{2}$  will be assigned the quantization step  $1 \times 2^{-b}$ .
2. Any positive unquantized number in the range  $2 \times \frac{2^{-b}}{2} \leq N < 3 \times \frac{2^{-b}}{2}$  will be assigned the quantization step  $2 \times 2^{-b}$  and so on.

**Fig. 5.5** Input-output characteristics of the quantizer used for rounding



### Example 5.14

Perform the quantization of  $0.0625_{10}$  to 3 bit by rounding.

#### Solution

$$0.0625_{10} \xrightarrow[\text{to binary}]{\text{convert}} 0.0001_2 \xrightarrow{\text{rounded to 3 bits}} 0.001_2 \xrightarrow[\text{to decimal}]{\text{convert}} 0.125_{10}$$

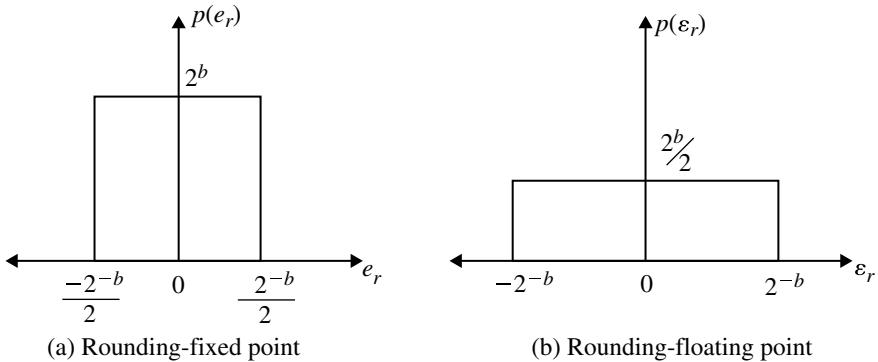
#### Fixed Point Number

Let  $N$  = unquantized fixed point binary number and  $N_r$  = fixed point binary number quantized by rounding. The quantization error in fixed point number due to rounding is defined as

$$\text{Rounding error, } e_r = N_r - N \quad (5.12)$$

The range of error due to rounding for all the three formats of fixed point representation is same. In fixed point representation the range of error made by rounding a number to  $b$  bits is

$$\frac{-2^{-b}}{2} \leq e_r \leq \frac{2^{-b}}{2} \quad (5.13)$$



**Fig. 5.6** Quantization noise probability density functions for rounding

# Floating Point Number

Let  $N_f$  = unquantized floating point binary number and  $N_{rf}$  = rounded floating point binary number. Now

$$N_{rf} = N_f + N_f \varepsilon_r \quad (5.14)$$

where  $\varepsilon_r$  is the relative error due to rounding of a floating point number.

$$\therefore \text{Relative error due to rounding, } \varepsilon_r = \frac{N_{rf} - N_f}{N_f} \quad (5.15)$$

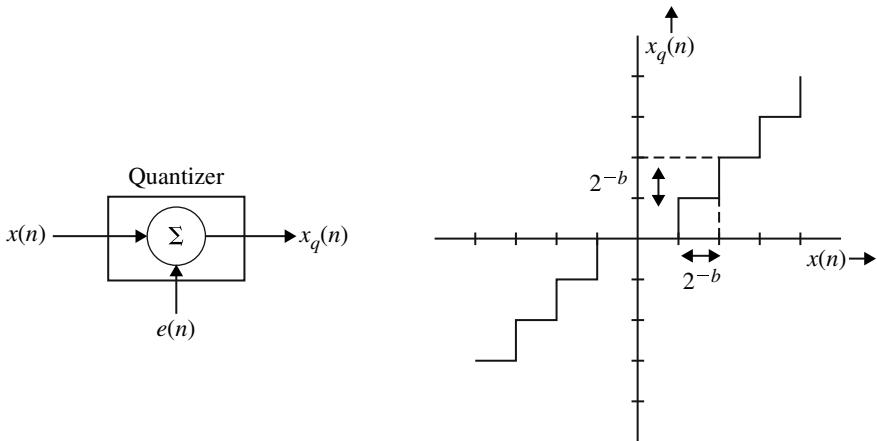
The range of error by rounding a number in floating point representation to  $b$  bits is,

$$-2^{-b} < \varepsilon_r < 2^{-b}. \quad (5.16)$$

The probability density function for rounding fixed point and floating numbers is shown in Fig. 5.6.

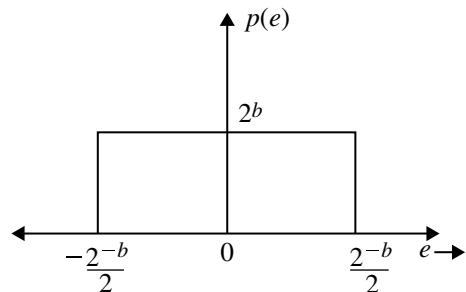
## 5.4 Quantization of Input Data by Analog to Digital Converter

The process of analog to digital conversion involves: (i) sampling the continuous time signal at a rate much greater than Nyquist rate and (ii) quantizing the amplitude of the sampled signal into a set of discrete amplitude levels. The input–output characteristics of a uniform quantizers are shown in Fig. 5.7. This quantizer rounds the sampled signal to the nearest quantized output level. The difference between the quantized signal amplitude  $x_q(n)$  and the actual signal amplitude  $x(n)$  is called the quantization error  $e(n)$ . That is



**Fig. 5.7** Two's complement number quantization

**Fig. 5.8** Probability density function for quantization round off error in A/D conversion



$$e(n) = x_q(n) - x(n)$$

Since, rounding is involved in the process of quantization the range of values for the quantization error is

$$-\frac{2^{-b}}{2} \leq e(n) \leq \frac{2^{-b}}{2} \quad (5.17)$$

The quantization error is assumed to be uniformly distributed over  $-\frac{2^{-b}}{2} \leq e(n) \leq \frac{2^{-b}}{2}$ . It is also assumed that this quantization noise  $e(n)$  is a stationary white noise sequence  $x(n)$  which traverses several quantization levels between two successive samples.

In the process of quantization, the samples value is rounded off to the nearest quantization level. The probability density function for the quantization round off error in A/D conversion is shown in Fig. 5.8.

It can be noted from the Fig. 5.8 that the quantization error is uniformly distributed and the mean value of error is zero. The power of the quantization noise, which is nothing but variance ( $\sigma_e^2$ ), is given by

$$\sigma_e^2 = E[e^2(n)] - E^2[e(n)] \quad (5.18)$$

$$= E[e^2(n)] \quad (5.19)$$

Therefore, mean value of error is zero, i.e.,  $E[e(n)] = 0$ .

Quantization step size is expressed as,

$$q = \frac{R}{2^b} \quad (\text{for two's complement}) \quad (5.20)$$

where  $R$  = range of analog signal to be quantized. Usually the analog signal is scaled such that the magnitude of quantized signal is less or equal to one. In such case the range of analog signal to be quantized is  $-1$  to  $1$ , therefore,  $R = 2$ .

$$\text{Quantization step size } q = \frac{2}{2^b} = 2 \cdot 2^{-b} \quad (5.21)$$

The quantization error for rounding will be in the range of  $-q/2$  to  $+q/2$

$$\therefore \text{Variance of error signal } \sigma_e^2 = \frac{1}{\frac{q}{2} - (-q/2)} \int_{-q/2}^{q/2} e^2 de \quad (5.22)$$

$$\begin{aligned} &= \frac{1}{q} \left[ \frac{e^3}{3} \right]_{-(q/2)}^{q/2} = \frac{1}{3q} \left[ \frac{q^3}{8} + \frac{q^3}{8} \right] \\ &= \frac{1}{3q} \cdot \frac{2q^3}{8} = \frac{q^2}{12} \end{aligned} \quad (5.23)$$

$$\sigma_e^2 = \frac{1}{12} \left( \frac{R}{2^{-b}} \right)^2$$

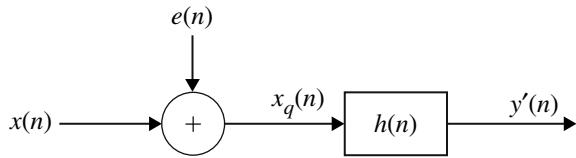
(5.24)

The variance of error signal is also called steady-state noise power due to input quantization.

#### 5.4.1 Output Noise Power Due to the Quantization Error Signal

After converting the continuous time signal into digital signal, let us assume that this quantized signal is applied as an input to a digital system with impulse response  $h(n)$ .

**Fig. 5.9** Representation of quantization noise in digital system



The quantized input signal of a digital system can be represented as a sum of unquantized signal  $x(n)$  and error signal  $e(n)$  as shown in Fig. 5.9

$$y'(n) = x_q(n) * h(n) \quad (5.25)$$

$$= [x(n) + e(n)] * h(n)$$

$$y'(n) = [x(n) * h(n)] + [e(n) * h(n)] \quad (5.26)$$

$$y'(n) = y(n) + e(n) \quad (5.27)$$

where

$y(n) = x(n) * h(n)$  is the output due to input signal

$\epsilon(n) = e(n) * h(n)$  is the output due to error signal

Variance of the signal  $\epsilon(n)$  is called the output noise power or steady output noise power.

Output noise power (or) steady-state output noise power due to quantization errors is given by the following equation:

$$\sigma_{e0}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) \quad (5.28)$$

The summation of  $h^2(n)$  can be evaluated using Parseval's theorem

$$\sigma_{e0}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)H(z^{-1})z^{-1} dz \quad (5.29)$$

The closed contour integration can be evaluated using residue theorem of  $z$ -transform

$$\sigma_{e0}^2 = \sigma_e^2 \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=p_i}$$

(5.30)

where  $p_1, p_2, \dots, p_n$  are poles of  $H(z)H(z^{-1})z^{-1}$ . Since the closed contour integration is around the unit circle  $|z| = 1$ , only the residue of the poles that is inside the unit circle is considered.

---

### Example 5.15

The output of an A/D converter is applied to a digital filter whose system function is,

$$H(z) = \frac{z(0.5)}{z - 0.5}$$

Find the output noise power from the digital filter, when the input signal is quantized to have eight bits.

---

**Solution** Given  $b = 8$  (assuming sign bit is included). Let  $R = 2$

$$\text{Quantization step size } q = \frac{R}{2^b} = \frac{2}{2^8} = 2^{2-b} = 2 \times 2^{-8} = 2^{-7}.$$

The input quantization noise power is obtained using Eq. 5.22

$$\sigma_e^2 = \frac{q^2}{12} = \frac{(2^{-7})^2}{12} = \frac{2^{-14}}{12} = 5.086 \times 10^{-6}$$

The output noise power is given by

$$\begin{aligned} \sigma_{e0}^2 &= \frac{\sigma_e^2}{2\pi j} \oint_c H(z)H(z^{-1})z^{-1} dz \\ &= \sigma_e^2 \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=p_i} \\ H(z)H(z^{-1})z^{-1} &= \frac{0.5z}{z - 0.5} \cdot \frac{0.5z^{-1}}{z^{-1} - 0.5} \cdot z^{-1} \\ &= \frac{0.25z^{-1}}{z^{-1}(z - 0.5)(1 - 0.5z)} \\ &= \frac{0.25}{(z - 0.5)(1 - 0.5z)} \quad [\text{poles are at } z = 0.5, z = 2] \end{aligned}$$

$\text{Res}[H(z)H(z^{-1})z^{-1}]$  due to pole  $z = 0.5$  alone is to be considered.

$$\begin{aligned}\text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=0.5} &= \frac{(z - 0.5)0.25}{(z - 0.5)(1 - 0.5z)} \Big|_{z=0.5} \\ &\quad [\because z = 0.5 \text{ pole lies inside the unit circle}] \\ &= \frac{1}{3}\end{aligned}$$

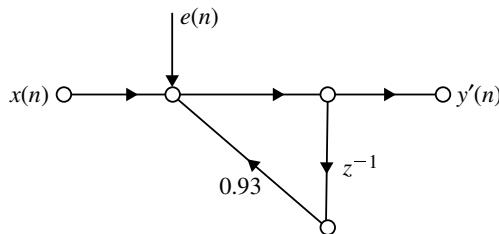
Therefore, the output noise power is

$$\begin{aligned}\sigma_{e0}^2 &= \sigma_e^2 \times \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=0.5} \\ &= 5.086 \times 10^{-6} \times \frac{1}{3}\end{aligned}$$

$$\boxed{\sigma_{e0}^2 = 1.6954 \times 10^{-6}}.$$

### Example 5.16

For the recursive filter shown in figure, the input  $x(n)$  has a peak value of 10V, represented by 6 bits. Compute the variance of output due to A/D conversion process



**Solution** Given  $R = 10V$ ,  $b = 6$  bits.

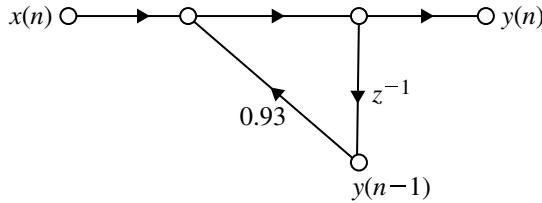
Note: If in case,  $R$  value is not mentioned in the problem assume  $R = 2V$ .

$$\begin{aligned}\text{Quantization step size, } q &= \frac{R}{2^b} \\ \therefore q &= \frac{10}{2^6} = 0.15625\end{aligned}$$

The variance of the error signal is  $\sigma_e^2$ .

$$\therefore \sigma_e^2 = \frac{q^2}{12} = \frac{(0.15625)^2}{12} = 2.0345 \times 10^{-3}$$

The given recursive (first order) filter can be redrawn as follows:



The difference equation of the system is,

$$y(n) = 0.93y(n - 1) + x(n)$$

Taking  $z$ -transform for the above equation we get

$$\begin{aligned} Z\{y(n)\} &= 0.93Z\{y(n - 1)\} + Z\{x(n)\} \\ Y(z) &= 0.93z^{-1}Y(z) + X(z) \quad [:\ Z\{f(n - 1)\} = z^{-1}F(z)] \\ X(z) &= Y(z)(1 - 0.93z^{-1}) \\ \frac{Y(z)}{X(z)} &= \frac{1}{(1 - 0.93z^{-1})} = H(z) \end{aligned}$$

The steady-state output noise power due to quantization is,

$$\begin{aligned} \sigma_{e_0}^2 &= \sigma_e^2 \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=p_i} \\ H(z) &= \frac{1}{1 - 0.93z^{-1}} = \frac{z}{z - 0.93} \\ H(z^{-1}) &= \frac{z^{-1}}{z^{-1} - 0.93} \quad [\text{Replace } z \text{ by } z^{-1} \text{ in } H(z) \text{ to get } H(z^{-1})] \\ \therefore H(z)H(z^{-1})z^{-1} &= \frac{z}{z - 0.93} \times \frac{z^{-1}}{z^{-1} - 0.93} \times z^{-1} \\ &= \frac{z^{-1}}{(z - 0.93)(z^{-1} - 0.93)} \\ &= \frac{1}{(z - 0.93)(1 - 0.93z)} \end{aligned}$$

Here  $H(z)$  has only one pole at  $p_i = 0.93$ .

$$\begin{aligned} \therefore \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=p_i} \\ &= \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=0.93} \\ &\quad [\text{because if } p_i \text{ is the pole } \text{Res}[H(z)] = (s - p_i) \times H(z)|_{z=p_i}] \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^N &= \frac{(z - 0.93)z^{-1}}{(z - 0.93)(z^{-1} - 0.93)} \Big|_{z=0.93} \\ &= \frac{(0.93)^{-1}}{((0.93)^{-1} - 0.93)} = 7.4019 \end{aligned}$$

Therefore, the output noise power (or variance) due to A/D conversion process is,

$$\begin{aligned} \sigma_{e_0}^2 &= \sigma_e^2 \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=0.93} \\ &= 2.0345 \times 10^{-3} \times 7.4019 \end{aligned}$$

$$\boxed{\sigma_{e_0}^2 = 0.0151}$$

### Example 5.17

The input to the system

$$y(n) = 0.999y(n-1) + x(n)$$

is applied to an ADC. What is the power produced by the quantization noise at the output of the filter if the input is quantized to (a) 8 bits and (b) 16 bits?

(Anna University, May, 2007)

**Solution** Given

$$y(n) = 0.999y(n-1) + x(n)$$

Taking  $z$ -transform on the both sides we get

$$Y(z) = 0.999z^{-1}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.999z^{-1}}$$

The quantization noise power at the output of the digital filter is

$$\sigma_{e_0}^2 = \sigma_e^2 \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=p_i}$$

$$H(z)H(z^{-1})z^{-1} = \left( \frac{1}{1 - 0.999z^{-1}} \right) \left( \frac{1}{1 - 0.999z} \right) z^{-1}$$

$$= \frac{z^{-1}}{z^{-1}(z - 0.999)(1 - 0.999z)}$$

$$= \frac{z^{-1}}{(z - 0.999)(1 - 0.999z)}$$

Here  $H(z)$  has only one pole at  $p = 0.999$  which lies inside the unit circle. Therefore

$$\sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}] \Big|_{z=p_i} = \text{Res}[H(z)H(z^{-1})z^{-1}] \Big|_{z=0.999}$$

$$= (z - 0.999) \frac{1}{[z - 0.999][1 - 0.999z]} \Big|_{z=0.999}$$

$$= \frac{1}{(1 - 0.999z)} \Big|_{z=0.999}$$

$$= \frac{1}{(1 - 0.999)^2}$$

$$= 500.25$$

Therefore

$$\sigma_{e_0}^2 = \sigma_e^2(500.25)$$

$$= \frac{q^2}{12}(500.25) = \frac{\left(\frac{R}{2^b}\right)^2}{12}(500.25)$$

Let  $R = 2V$

(a) Given  $b = 8$  bits (including sign bit)

$$\sigma_{e_0}^2 = \frac{\left(\frac{2}{2^8}\right)^2}{12}[500.25]$$

$$= \frac{2^{-14}}{12}(500.25) = 2.544 \times 10^{-3}$$

(b) Given  $b = 16$  bits

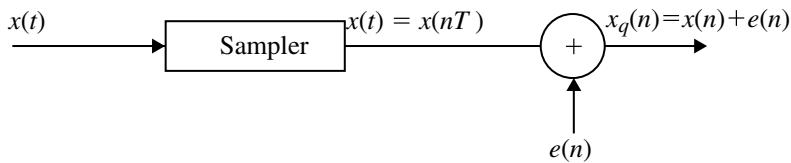
$$\begin{aligned}\sigma_{e_0}^2 &= \frac{\left(\frac{2}{2^{16}}\right)^2}{12}[500.25] \\ &= \frac{2^{-30}}{12}(500.25) = 3.882 \times 10^{-8}\end{aligned}$$

### Example 5.18

Consider  $(b + 1)$  bits (including sign bit) bipolar A/D converter. Obtain an expression for signal to quantization noise ratio. State the assumption made.

(Anna University, May 2007)

**Solution** The quantization noise model of A/D converter is,



The A/D converter output is the sum of the input signal  $x(n)$  and the error signal  $e(n)$ . If the rounding is used for quantization then the quantization error is

$$e(n) = x_q(n) - x(n) \text{ is bounded by } \frac{-q}{2} \leq e(n) \leq \frac{q}{2}$$

In most cases, we can assume that the A/D conversion error  $e(n)$  has the following properties:

- (i) The error sequence  $e(n)$  is a sample sequence of a stationary random process.
- (ii) The error signal is uncorrected with  $x(n)$  and other signal in the system.
- (iii) The error is a white noise process with uniform amplitude probability distribution over the range of quantization error.

The variance of  $e(n)$  is given by

$$\begin{aligned}\sigma_e^2 &= E[e^2(n)] - E^2[e(n)] \\ \sigma_e^2 &= E[e^2(n)] \\ &= \frac{1}{q} \int_{-q/2}^{q/2} e^2(n) de = \frac{1}{q} \left[ \frac{e^3}{3} \right]_{-q/2}^{q/2}\end{aligned}$$

$$= \frac{1}{3q} \times \left[ \frac{q^3}{8} + \frac{q^3}{8} \right] = \frac{q^2}{12}$$

$$\sigma_e^2 = \frac{\left(\frac{2}{2^{b+1}}\right)^2}{12}$$

$$\boxed{\sigma_e^2 = \frac{2^{-2b}}{12}}$$

where  $b$  is number of bits (excluding sign bit).  $\sigma_e^2$  is also known as the steady-state noise power due to input quantization.

If the input signal is  $x(n)$  and its variance is  $\sigma_x^2$ , then the ratio of signal power to noise power which is known as signal to noise ratio for rounding is

$$\boxed{\text{SNR} = \frac{\sigma_x^2}{\sigma_e^2} = \frac{\sigma_x^2}{\frac{2^{-2b}}{12}} = 12(2^{2b}\sigma_x^2)}$$

Signal to noise ratio in dB is expressed as

$$\text{SNR (dB)} = 10 \log_{10} \frac{\sigma_x^2}{\sigma_e^2} = 10 \log_{10}(12 2^{2b}\sigma_x^2)$$

$$\boxed{\text{SNR (dB)} = 6.02b + 10.79 + 10 \log_{10} \sigma_x^2}$$

SNR increases approximately by 6dB for each bit added to register length.

## 5.5 Quantization of Filter Coefficients

In the design of a digital filter the coefficients are evaluated with infinite precision. But they are limited by the word length of the register used to store the coefficients. Usually the filter coefficients are quantized to the word size of the register used to store them either by truncation or by rounding.

The location of poles and zeros of the digital filters directly depends on the value of filter coefficients. The quantization of the filter coefficients will modify the value of poles and zeros and so the location of the poles and zeros will be shifted from the desired location. This will create deviation in the frequency response of the system. Hence we obtain a filter having a frequency response that is different from the frequency response of the filter with unquantized coefficients. The sensitivity of the filter frequency response characteristics to quantization of the filter coefficients

is minimized by realizing the filter having a large number of poles and zeros as an interconnection of second-order section. **Therefore, the coefficient quantization has less effect in cascade realization when compared to other realizations.**

---

### Example 5.19

For the second-order IIR filter, the system function is,

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.45z^{-1})}$$

Study the effect of shift in pole location with 3 bit coefficient representation in direct and cascade forms.

---

### Solution

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.45z^{-1})} = \frac{z^2}{(z - 0.5)(z - 0.45)}$$

Original poles of  $H(z) \Rightarrow p_1 = 0.5$  and  $p_2 = 0.45$ .

#### Case (i) Direct Form

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.45z^{-1})} = \frac{1}{(1 - 0.95z^{-1} + 0.225z^{-2})}$$

#### Quantization of coefficient by truncation

$$\begin{array}{c} .95_{10} \xrightarrow{\text{Convert to binary}} .1111_2 \xrightarrow{\text{Truncate to 3 bits}} .111_2 \xrightarrow{\text{Convert to decimal}} .815_{10} \\ .225_{10} \xrightarrow{\text{Convert to binary}} .0011_2 \xrightarrow{\text{Truncate to 3 bits}} .001_2 \xrightarrow{\text{Convert to decimal}} .125_{10} \end{array}$$

$$H(z) = \frac{1}{1 - 0.875z^{-1} + 0.125z^{-2}}$$

$$H(z) = \frac{1}{(1 - 0.695z^{-1})(1 - 0.179z^{-1})}$$

The poles are at  $p_1 = 0.695$  and  $p_2 = 0.179$ .

### Case (ii) Cascade Form

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.45z^{-1})}$$

#### Quantization by truncation

$$\begin{array}{ccccccc} .5_{10} & \xrightarrow{\text{Convert to binary}} & .1000_2 & \xrightarrow{\text{Truncate to 3 bits}} & .100_2 & \xrightarrow{\text{Convert to decimal}} & .5_{10} \\ .45_{10} & \xrightarrow{\text{Convert to binary}} & .0111_2 & \xrightarrow{\text{Truncate to 3 bits}} & .011_2 & \xrightarrow{\text{Convert to decimal}} & .375_{10} \end{array}$$

$$H(z) = \frac{1}{1 - 0.52z^{-1}} \times \frac{1}{1 - 0.375z^{-1}}$$

The poles are  $p_1 = 0.5$  and  $p_2 = 0.375$ .

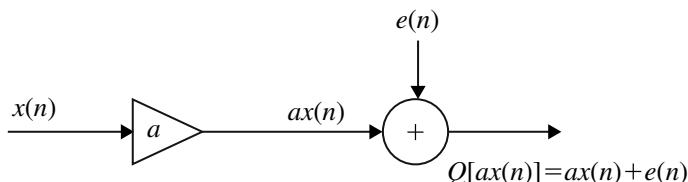
*Conclusion:*

- From direct form, we can see that the quantized poles deviate very much from the original poles.
- From cascade form, we can see that one pole is exactly the same while the other pole is very close to the original pole.

## 5.6 Product Quantization Error

In fixed point arithmetic the product of two  $b$  bit numbers results in number of  $2b$  bits length. If the word length of the register used to store the result is  $b$  bit, then it is necessary to quantize the product to  $b$  bits, which produce an error known as **product quantization error or product round off noise**. In realization structures of digital system, multipliers are used to multiply the signal by constants.

The model for fixed point round off noise following a multiplication is shown in Fig. 5.10.



**Fig. 5.10** Fixed point product round off noise model

The multiplication is modeled as an infinite precision multipliers followed by an adder where round off noise is added to the product so that overall result equals some quantization level. The round off noise sample is a zero mean random variable with a variance  $(2^{-2b}/3)$ , where  $b$  is the number of bits used to represent the variables.

In general the following assumptions are made regarding the statistical independence of the various noise sources in the digital filter.

1. Any two different samples from the same noise source are uncorrelated.
2. Any two different noise source, when considered as random processes, are uncorrelated.
3. Each noise source is uncorrelated with the input sequence.

The product quantization noise model for first-order and second-order system is shown in Fig. 5.11. The product quantization noise models for IIR using cascade are shown in Fig. 5.12.

In each noise model there are a number of noise sources. The output noise variance due to each source is computed separately by considering one noise source at a time. The total output noise variance is given by sum of the output noise variance at all the noise sources. For each noise source, the noise transfer function (NTF) has to be determined by treating the noise source as input and the output being the output of the system. NTF for noise sources  $e_{a11}(n)$  in Fig. 5.12 =  $H_1(z)$  and NTF for noise sources  $e_{a12}(n)$  in Fig. 5.12 =  $H_2(z)$ .

Let  $e_k(n)$  be the error signal from  $k^{\text{th}}$  noise source,  $h_k(n)$  the impulse response for  $k^{\text{th}}$  noise source and  $T_k(n)$  the noise transfer function (NTF) for  $k^{\text{th}}$  noise source.

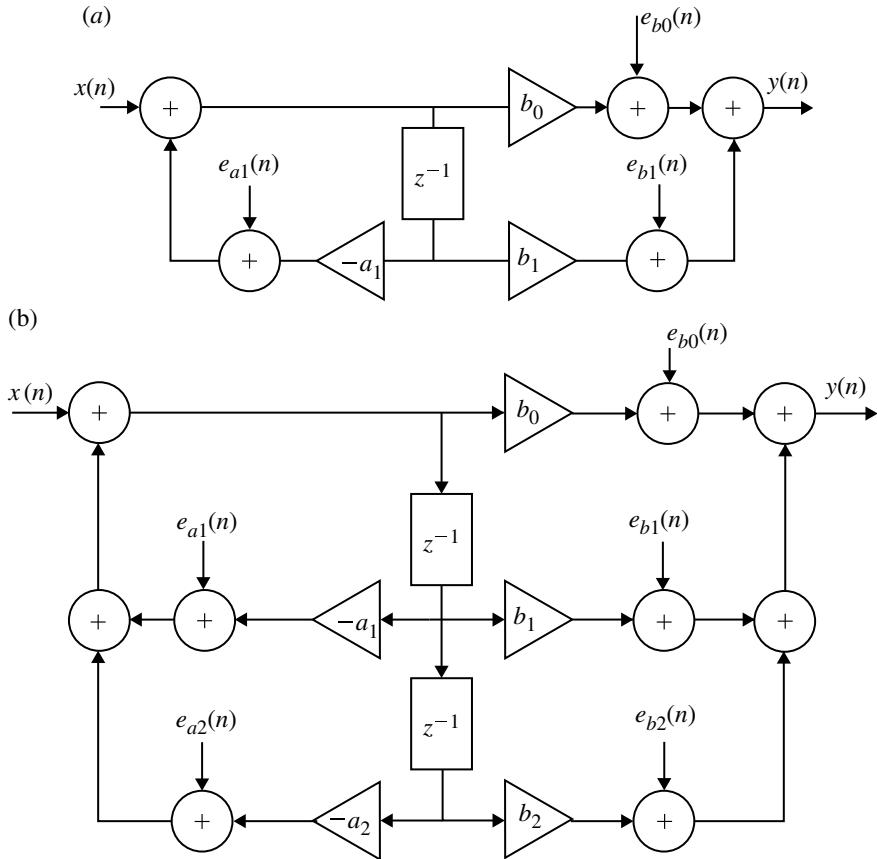
$$\text{Variance of } k^{\text{th}} \text{ noise source } \sigma_{ek}^2 = \frac{q^2}{12} = \frac{2^{-2b}}{3} \quad [\because R = 2]$$

Output noise variance due to  $k^{\text{th}}$  noise source is,

$$\sigma_{e0k}^2 = \sigma_{ek}^2 \sum_{n=0}^{\infty} h_k^2(n)$$

$$\begin{aligned} \sigma_{e0k}^2 &= \sigma_{ek}^2 \frac{1}{2k_j} \oint T_k(z) T_k(z^{-1}) z^{-1} dz \\ &= \sigma_{ek}^2 \sum_{i=1}^n \text{Res} [T_k(z) T_k(z^{-1}) z^{-1}]_{z=p_i} \end{aligned} \quad (5.31)$$

where  $p_1, p_2, p_3, \dots, p_n$  are poles of  $T_k(z) T_k(z^{-1}) z^{-1}$ . Let the number of noise sources in digital system be  $M$ .



**Fig. 5.11** Product quantization noise models of IIR systems for direct form realization. **a** First-order direct form-II and **b** Second-order direct form-II

Therefore, total output noise variance due to product quantization error is,

$$\sigma_{e0}^2 = \sum_{k=1}^M \sigma_{e0k}^2 \quad (5.32)$$

$$\boxed{\sigma_{e0}^2 = \sigma_{e01}^2 + \sigma_{e02}^2 + \dots + \sigma_{e0M}^2}$$

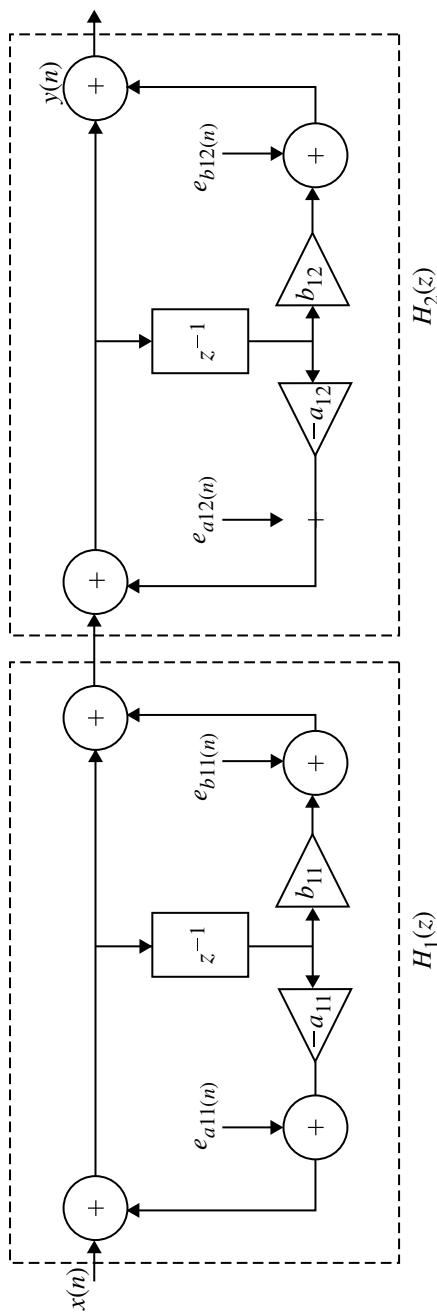


Fig. 5.12 Cascading of two first-order sections

**Example 5.20**

In the IIR system given below the products are rounded to 4 bits (including sign bits). The system function is

$$H(z) = \frac{1}{(1 - 0.35z^{-1})(1 - 0.62z^{-1})}$$

Find the output round off noise power in **a** direct form realization and **b** cascade form realization.

**Solution****(a) Direct Form Realization**

$$H(z) = \frac{1}{(1 - 0.35z^{-1})(1 - 0.62z^{-1})}$$

$$H(z) = \frac{1}{(1 - 0.97z^{-1} + 0.217z^{-2})}$$

Direct form realization of  $H(z)$  is shown in Fig. 5.13.

The variance of the error signal is,

$$\sigma_e^2 = \frac{q^2}{12}; \quad q = \frac{R}{2^b}$$

Here  $R$  is not given. So take  $R = 2V$  and  $b = 4$  bits

$$\therefore q = \frac{R}{2^b} = \frac{2}{2^4} = \frac{1}{2^3} = \frac{1}{8}$$

$$\sigma_e^2 = \frac{(1/8)^2}{12} = \frac{q^2}{12}$$

$$\boxed{\sigma_e^2 = 1.3021 \times 10^{-3}}$$

Output noise power due to the noise signal  $e_1(n)$  is,

$$\sigma_{e_{01}}^2 = \sigma_e^2 \sum_{i=1}^N \text{Res}[T_1(z)T_1(z^{-1})z^{-1}]|_{z=p_i}$$

Here

$$T_1(z) = H(z) = \frac{z^2}{(z - 0.35)(z - 0.62)}$$

Therefore

$$\begin{aligned} T_1(z)T_1(z^{-1})z^{-1} &= \frac{z^2}{(z - 0.35)(z - 0.62)} \times \frac{z^{-2}}{(z^{-1} - 0.35)(z^{-1} - 0.62)} \times z^{-1} \\ &= \frac{z^{-1}}{(z - 0.35)(z - 0.62)(z^{-1} - 0.35)(z^{-1} - 0.62)} \end{aligned}$$

The poles of  $H(z)$  are  $p_1 = 0.35$  and  $p_2 = 0.62$ .

$$\begin{aligned} \text{Res}[T(z)T(z^{-1})z^{-1}]|_{z=0.35} &= (z - 0.35) \frac{z^{-1}}{(z - 0.35)(z - 0.62)(z^{-1} - 0.35)(z^{-1} - 0.62)} \Big|_{z=0.35} \\ &= -1.8867 \\ \text{Res}[T_1(z)T_1(z^{-1})z^{-1}]|_{z=0.62} &= (z - 0.62) \frac{z^{-1}}{(z - 0.35)(z - 0.62)(z^{-1} - 0.35)(z^{-1} - 0.62)} \Big|_{z=0.62} \\ &= 4.7640 \\ \sum_{i=1}^N \text{Res}[T_1(z)T_1(z^{-1})z^{-1}] &= \text{Res}[T_1(z)T_1(z^{-1})z^{-1}]|_{z=0.35} + \text{Res}[T_1(z)T_1(z^{-1})z^{-1}]|_{z=0.62} \\ &= -1.8867 + 4.7640 \\ &= 2.8773. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma_{e_{01}}^2 &= \sigma_e^2 \sum_{i=1}^N \text{Res}[T_1(z)T_1(z^{-1})z^{-1}]|_{z=p_i} \\ &= 1.3021 \times 10^{-3} \times 2.8733 \\ \sigma_{e_{01}}^2 &= 3.7465 \times 10^{-3} \end{aligned}$$

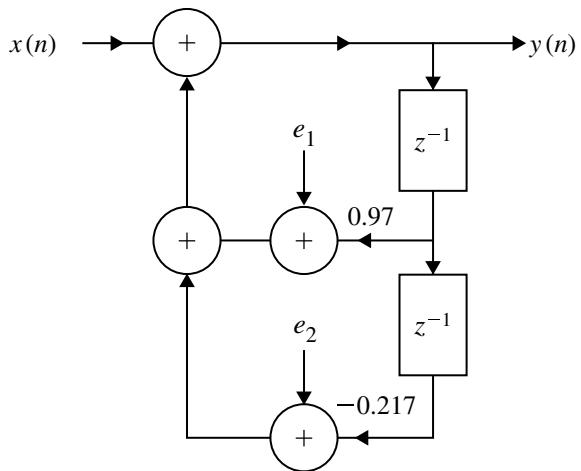
Here the output noise due to error source  $e_2(n)$  is same as that of  $e_1(n)$ , i.e.,

$$\begin{aligned} e_2(n)'s \text{ noise power} &= \text{noise power of } e_1(n) \\ \sigma_{e_{01}}^2 &= \sigma_{e_{02}}^2 \\ &= 3.7465 \times 10^{-3} \end{aligned}$$

Total output noise power due to all the noise sources is,

$$\sigma_{e_0}^2 = \sigma_{e_{01}}^2 + \sigma_{e_{02}}^2$$

**Fig. 5.13** Direct form realization of  $H(z)$  for Example 5.20



$$\sigma_{e_0}^2 = 7.493 \times 10^{-3}$$

### (b) Cascade Realization

Given

$$H(z) = \frac{1}{(1 - 0.35z^{-1})(1 - 0.62z^{-1})}$$

Let  $H(z) = H_1(z)H_2(z)$ , i.e.,

$$H_1(z) = \frac{1}{(1 - 0.35z^{-1})} \quad \text{and} \quad H_2(z) = \frac{1}{(1 - 0.62z^{-1})}$$

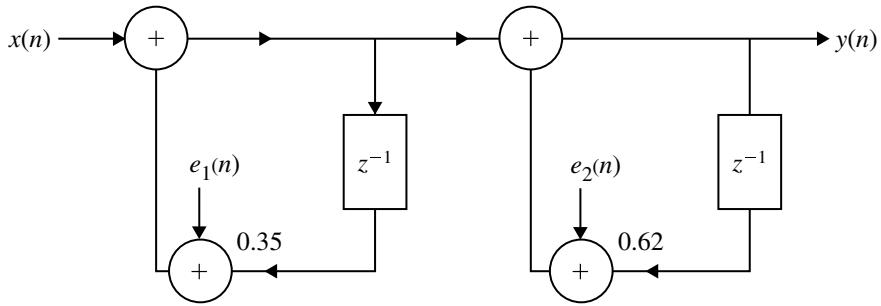
**Case (i)**  $H(z) = H_1(z)H_2(z)$

The cascade form realization of  $H(z)$  is shown in Fig. 5.14.

The order of cascading is  $H_1(z)H_2(z)$ . Output noise power due to error signal  $e_1(n)$  is

$$\sigma_{e_{01}}^2 = \sigma_e^2 \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=p_i} = 3.7465 \times 10^{-3}$$

[Refer Direct Form where  $H(z) = T(z)$ ].



**Fig. 5.14** Cascade form realization of  $H(z)$  for Example 5.20

Output noise power due to the error signal  $e_2(n)$  is

$$\sigma_{e_{02}}^2 = \sigma_e^2 \sum_{i=1}^N \text{Res}[T_2(z)T_2(z^{-1})z^{-1}]|_{z=p_i}$$

Here  $T_2(z) = H_2(z)$

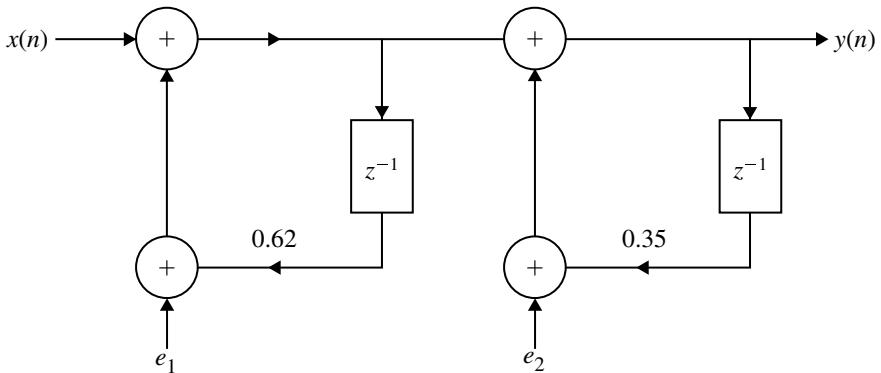
$$\begin{aligned} T_2(z)T_2(z^{-1})(z^{-1}) &= \frac{z^{-1}}{(z - 0.62)(z^{-1} - 0.62)} \\ \sum_{i=1}^N \text{Res}[T_2(z)T_2(z^{-1})z^{-1}]|_{z=p_i} &= \left. \frac{z^{-1} \times (z - 0.62)}{(z - 0.62)(z^{-1} - 0.62)} \right|_{z=0.62} \\ &= 1.6244 \\ \sigma_{e_{02}}^2 &= \sigma_e^2 \times \sum_{i=1}^N \text{Res}[T_2(z)T_2(z^{-1})z^{-1}] \\ &= 1.3021 \times 10^{-3} \times 1.6244 \\ &= 2.1151 \times 10^{-3} \end{aligned}$$

Output noise power

$$\begin{aligned} \sigma_{e_0}^2 &= \sigma_{e_{01}}^2 + \sigma_{e_{02}}^2 \\ &= 3.7465 \times 10^{-3} + 2.1151 \times 10^{-3} \end{aligned}$$

$$\boxed{\sigma_{e_0}^2 = 5.8616 \times 10^{-3}}$$

**Case (ii)** The order of cascading is  $H(z) = H_2(z)H_1(z)$  and is shown in Fig. 5.15.



**Fig. 5.15** Cascade form realization of  $H(z)$  for Example 5.20

The output noise power due to error source  $e_1$  is,

$$\sigma_{e_{01}}^2 = 3.7465 \times 10^{-3}$$

[Same as in Direct Form because  $T(z) = H(z)$ ].

The output noise power due to error source  $e_2(n)$  is,

$$\sigma_{e_{02}}^2 = \sigma_e^2 \sum_{i=1}^N \text{Res}[T_3(z)T_3(z^{-1})]_{z=p_i}$$

Here  $T_3(z) = H_1(z)$

$$\begin{aligned} \text{Res}[T_3(z)T_3(z^{-1})z^{-1}]|_{z=p_i} &= \frac{(z - 0.35)z^{-1}}{(z - 0.35)(z^{-1} - 0.35)} \Big|_{z=0.35} \\ &= 1.1396 \\ \sigma_{e_{02}}^2 &= 1.1396 \times 1.3021 \times 10^{-3} \\ &= 1.4839 \times 10^{-3} \end{aligned}$$

Total output noise power

$$\begin{aligned} \sigma_{e_0}^2 &= \sigma_{e_{01}}^2 + \sigma_{e_{02}}^2 \\ &= 3.7465 \times 10^{-3} + 1.4839 \times 10^{-3} \end{aligned}$$

$\sigma_{e_0}^2 = 5.2304 \times 10^{-3}$

*Conclusion:* Thus, in cascade form realization, the product noise round off power is less in case (ii) when compared to case (i) and also direct form realization.

## 5.7 Limit Cycles in Recursive System

### 5.7.1 Zero-Input Limit Cycles

In recursive systems, when the input is zero or some nonzero constant value, the nonlinearities due to finite precision arithmetic operation may cause periodic oscillations, in the output. During periodic oscillations, the output  $y(n)$  of a system will oscillate between a finite positive and negative value for increasing  $n$  or the output will become constant for increasing  $n$ . Such oscillations are called limit cycles. If the system output enters a limit cycle, it will continue to remain in limit cycle even when the input is made zero. Hence, these limit cycles are also called zero-input limit cycles.

Consider the following difference equation of first-order system with one pole only.

$$y(n) = ay(n - 1) + x(n) \quad (5.33)$$

The system has one product  $ay(n - 1)$ . If the product is quantized to finite word length then the response  $y(n)$  will deviate from actual value. Let  $y'(n)$  be the response of the system when the product is quantized.

$$y'(n) = Q[ay'(n - 1)] + x(n) \quad (5.34)$$

Let  $y'(n) = 0$ , for  $n < 0$  and  $a = \frac{1}{2}$

$$x(n) = \begin{cases} 0.875, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

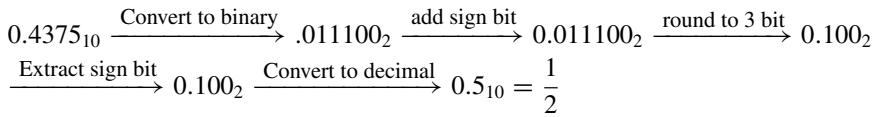
Let the product be quantized to three bit (excluding sign bit) binary by rounding when  $n = 0$ ,

$$\begin{aligned} y'(n) &= Q[ay'(n - 1)] + x(n) \\ y'(0) &= Q\left[\frac{1}{2}y'(-1)\right] + x(0) \\ &= Q\left[\frac{1}{2} \times 0\right] + 0.875 = 0.875_{10} = \frac{7}{8} \end{aligned}$$

When  $n = 1$ ,

$$\begin{aligned} y'(1) &= Q[ay'(0)] + x(1) \\ &= Q\left[\frac{1}{2} \times 0.875\right] + 0 \end{aligned}$$

$$= Q\left[\frac{1}{2} \times \frac{7}{8}\right] = Q\left[\frac{7}{16}\right] = Q[0.4375]$$



When  $n = 2$ ,

$$\begin{aligned} y'(2) &= Q[ay'(1)] + x(2) \\ &= Q\left[\frac{1}{2} \times \frac{1}{2}\right] + 0 \\ &= Q\left[\frac{1}{4}\right] = Q[0.25] \end{aligned}$$

$$0.25_{10} \xrightarrow{\text{Convert to binary}} 0.01000_2 \xrightarrow{\text{round to 3 bit}} 0.010_2 \xrightarrow{\text{Convert to decimal}} 0.25_{10} = \frac{1}{4}$$

When  $n = 3$ ,

$$\begin{aligned} y'(3) &= Q[ay'(2)] + x(3) \\ &= Q\left[\frac{1}{2} \times \frac{1}{4}\right] + 0 \\ &= Q\left[\frac{1}{8}\right] = Q[0.125] \end{aligned}$$

$$0.125_{10} \xrightarrow{\text{Convert to binary}} 0.00100_2 \xrightarrow{\text{round to 3 bit}} 0.001_2 \xrightarrow{\text{Convert to decimal}} 0.25_{10} = \frac{1}{8}$$

When  $n = 4$ ,

$$\begin{aligned} y'(4) &= Q[ay'(3)] + x(4) \\ &= Q\left[\frac{1}{2} \times \frac{1}{8}\right] + 0 = Q\left[\frac{1}{16}\right] = Q[0.0625] \end{aligned}$$

$$0.0625_{10} \xrightarrow{\text{Convert to binary}} 0.000100_2 \xrightarrow{\text{round to 3 bit}} 0.001_2 \xrightarrow{\text{Convert to decimal}} 0.125_{10} = \frac{1}{8}$$

**Table 5.3** Limit cycle in recursive system

$n$	$x(n)$	$y(n - 1)$	$ay(n - 1)$	$Q[ay(n - 1)]$	$y'(n) = Q[ay(n - 1)] + x(n)$
0	0.875	0	0	0.000	7/8
1	0	7/8	-7/16	1.100	-1/2
2	0	-1/2	1/4	0.010	1/4
3	0	1/4	-1/8	-0.001	-1/8
4	0	-1/8	1/16	0.001	1/8
5	0	1/8	-1/16	1.001	-1/8
6	0	-1/8	1/16	0.001	1/8

The Limit cycle in recursive system is shown in Table 5.3.

When  $n = 5$ ,

$$\begin{aligned} y'(5) &= Q[ay'(4)] + x(5) \\ &= Q\left[\frac{1}{2} \times \frac{1}{8}\right] + 0 = Q\left[\frac{1}{16}\right] = 0.125 = \frac{1}{8} \end{aligned}$$

For all values of  $n$ , where  $n \geq 3$ , the  $y'(n) = 1/8 = 0.001$ . Hence, the output becomes constant for  $n \geq 3$ . Also for  $n \geq 3$ , the input  $x(n)$  is zero. Therefore, the system enters a limit cycle even though the input becomes zero for  $n \geq 3$ .

When  $a = -1/2$  we can see from Table 5.3 that the output oscillates between  $+0.125$  and  $-0.125$ .

### Dead Band

In a limit cycle the amplitudes of the output are confined to a range of values, which is called the dead band of the filter.

For a first-order system described by the equation,  $y(n) = ay(n - 1) + x(n)$ , the dead band is given by

$$\text{Dead band} = \pm \frac{2^{-b}}{1 - |a|} = \left[ \frac{-2^{-b}}{1 - |a|}, +\frac{2^{-b}}{1 - |a|} \right] \quad (5.35)$$

where  $b$  = number of bits (including sign bits) used to represent the product. For a second-order system described by the difference equation  $y(n) = a_1y(n - 1) + a_2y(n - 2) + x(n)$ , the dead band in

$$\text{Dead band} = \pm \frac{2^{-b}}{1 - |a_2|} = \left[ \frac{-2^{-b}}{1 - |a_2|}, +\frac{2^{-b}}{1 - |a_2|} \right] \quad (5.36)$$

The following example illustrates the method of finding dead band.

**Example 5.21**

Explain the characteristics of a limit cycle oscillation with respect to the system described by the equation

$$y(n) = 0.95y(n - 1) + x(n).$$

Determine the dead band of the filter.

(Anna University, December, 2006)

**Solution** Given that

$$y(n) = 0.95y(n - 1) + x(n)$$

The recursive realization of the given system involves the product  $0.95y(n - 1)$ . Let  $y'(n)$  be the response of system when the product is quantized by rounding.

$$\therefore y'(n) = Q[0.95y(n - 1)] + x(n)$$

where  $Q$  is quantization. Let us consider 5 bit sign-magnitude binary representation with 4 bit for magnitude and 1 bit for sign. Let

$$y'(n) = 0 \text{ for } n < 0$$

and

$$x(n) = \begin{cases} 0.75, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

When  $n = 0$ ,

$$\begin{aligned} y'[0] &= Q[0.95y(-1)] + x[0] \\ &= Q[0.95 \times 0] + 0.75 \\ &= Q[0] + 0.75 \\ y'[0] &= 0.75_{10} = 0.1100_2 \end{aligned}$$

$$\begin{array}{rcl} +.75 & \xrightarrow{\text{Convert to binary}} & .11000_2 & \xrightarrow{\text{add sign bit}} & 0.11000_2 & \xrightarrow{\text{round to 4 bit}} 0.1100_2 \\ & & & & & \\ +0.1100_2 & \xrightarrow{\text{Convert to decimal}} & +.75_{10} & & & \end{array}$$

When  $n = 1$ ,

$$y'[n] = y'[1] = Q[0.95y'(n - 1)] + x[n]$$

$$\begin{aligned}
 &= Q[0.95y'(0)] + x[1] \\
 &= Q[0.95 \times 0.75] + 0 \\
 &= Q[0.7125] \\
 y'[1] &= 0.6875_{10} = 0.1011_2
 \end{aligned}$$

$$\begin{aligned}
 Q[0.7125] \implies & \\
 +.7125 &\xrightarrow{\text{Convert to binary}} .10110_2 \xrightarrow{\text{add sign bit}} 0.10110_2 \xrightarrow{\text{round to 4 bit}} 0.1011_2 \\
 +0.1011_2 &\xrightarrow{\text{Convert to decimal}} +.6875_{10}
 \end{aligned}$$

When  $n = 2$ ,

$$\begin{aligned}
 y'[2] &= Q[0.95y'(1)] + x[2] \\
 &= Q[0.95 \times 0.6875] + 0 \\
 &= Q[0.653125] \\
 y'[2] &= 0.625 \\
 &= 0.01010_2
 \end{aligned}$$

$$\begin{aligned}
 Q[0.653125_{10}] \implies & \\
 .653125_{10} &\xrightarrow{\text{Convert to binary}} .10100_2 \xrightarrow{\text{add sign bit}} 0.10100_2 \xrightarrow{\text{round to 4 bit}} 0.1010_2 \\
 +0.1010_2 &\xrightarrow{\text{Convert to decimal}} +.625_{10}
 \end{aligned}$$

When  $n = 3$ ,

$$\begin{aligned}
 y'[3] &= Q[0.95y'(2)] + x[3] \\
 &= Q[0.95 \times 0.625] + 0 \\
 &= Q[0.59375] \\
 y'[3] &= 0.625_{10} \\
 &= 0.1010_2
 \end{aligned}$$

$$\begin{aligned}
 Q[0.59375] \implies & \\
 .59375 &\xrightarrow{\text{Convert to binary}} .10011_2 \xrightarrow{\text{add sign bit}} 0.10011_2 \xrightarrow{\text{round to 4 bit}} 0.1010_2 \\
 +0.1010_2 &\xrightarrow{\text{Convert to decimal}} +0.625_{10}
 \end{aligned}$$

Thus,  $y'(2) = y'(3)$ , and hence for all values of  $n \geq 2$ ,  $y'(n)$  will remain as 0.625. Therefore, the system enters into the limit cycle when  $n = 2$ .

For the first-order system with only one pole, dead band is given by

$$\text{Dead band} = \pm \frac{2^{-b}}{1 - |a|}$$

where  $b$  is number of bit in binary representation and  $|a| = |0.95|$

$$\begin{aligned}\text{Dead band} &= \pm \frac{2^{-5}}{1 - 0.95} = \pm 0.625 \\ &= [-0.625, +0.625]\end{aligned}$$


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### Example 5.22

An IIR causal filter has the system function

$$H(z) = \frac{z}{z - 0.97}$$

Assume that the input signal is zero valued and the computed output signal values are rounded to one decimal place. Show that under those stated conditions, the filter output exhibits dead band effect. What is the dead band range?

(Anna University, May, 2007)

**Solution** Given

$$\begin{aligned}H(z) &= \frac{z}{z - 0.97} \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{z}{z - 0.97} \\ &= \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.97z^{-1}} \\ X(z) &= Y(z) - 0.97z^{-1}Y(z)\end{aligned}$$

Taking inverse  $z$ -transform on both sides we get

$$\begin{aligned}y(n) - 0.97y(n-1) &= x(n) \\ y(n) &= 0.97y(n-1) + x(n)\end{aligned}$$

Let  $y'(n)$  be the response of the system when the product is quantized by rounding

$$y'(n) = Q[0.97y'(n-1)] + x(n)$$

For a causal filter

$$y(n) = 0, \quad \text{for } n < 0$$

Let

$$x(n) = \begin{cases} 11, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

When  $n = 0$ ,

$$\begin{aligned} y'(0) &= Q[0.97y'(n-1)] + x(n) \\ &= 0 + 11 = 11 \end{aligned}$$

When  $n = 1$ ,

$$\begin{aligned} y'(1) &= Q[0.97y(0)] + x(1) \\ &= Q[0.97 \times 11] + 0 \\ &= Q[10.67] \end{aligned}$$

$$Q[10.67] \Rightarrow$$

$$(10.67)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1010.101)_2 \xrightarrow[\text{decimal place}]{\text{rounded to the}} (1010.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (10.5)_{10}$$

$$y'(1) = Q[10.67] = 10.5$$

When  $n = 2$ ,

$$\begin{aligned} y'(2) &= Q[0.97y'(1)] \\ &= Q[0.97 \times 10.5] \\ &= Q[10.185] \end{aligned}$$

$$Q[10.185] \Rightarrow$$

$$(10.185)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1010.001)_2 \xrightarrow[\text{one decimal}]{\text{rounded to the}} (1010)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (10)_{10}$$

$$y'(2) = Q[10.185] = 10$$

When  $n = 3$ ,

$$\begin{aligned}y'(3) &= Q[0.97y'(2)] \\&= Q[0.97 \times 10] \\&= Q[9.7]\end{aligned}$$

$$\begin{aligned}Q[9.7] &\Rightarrow (9.7)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1001.101)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1001.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (9.5)_{10} \\y'(3) &= Q[9.7] = 9.5\end{aligned}$$

When  $n = 4$ ,

$$\begin{aligned}y'(4) &= Q[0.97y'(3)] \\&= Q[0.97 \times 9.5] \\&= Q[9.215]\end{aligned}$$

$$\begin{aligned}Q[9.215] &\Rightarrow (9.215)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1001.001)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1001)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (9)_{10} \\y'(4) &= Q[9.215] = 9\end{aligned}$$

When  $n = 5$ ,

$$\begin{aligned}y'(5) &= Q[0.97y'(4)] \\&= Q[0.97 \times 9] \\&= Q[8.73]\end{aligned}$$

$$\begin{aligned}Q[8.73] &\Rightarrow \\(8.73)_{10} &\xrightarrow[\text{to binary}]{\text{convert}} (1001.101)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1000.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (8.5)_{10} \\y'(5) &= Q[8.73] = 8.5\end{aligned}$$

When  $n = 6$ ,

$$\begin{aligned}y'(6) &= Q[0.97y'(5)] \\&= Q[0.97 \times 8.5] \\&= Q[8.245]\end{aligned}$$

$$Q[8.245] \Rightarrow (8.245)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1000.001)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1000)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (8)_{10}$$

$$y'(6) = Q[8.245] = 8$$

When  $n = 7$ ,

$$\begin{aligned} y'(7) &= Q[0.97y'(6)] \\ &= Q[0.97 \times 8] \\ &= Q[7.76] \end{aligned}$$

$$Q[7.76] \Rightarrow (7.76)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (0111.110)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1000)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (8)_{10}$$

$$y'(7) = Q[7.76] = 8$$

Thus,  $y'(7) = y'(6)$  and hence for all values of  $n \geq 6$ ,  $y'(n)$  will remain as 8. Therefore, the system enters into limit cycle when  $n = 6$

$$\begin{aligned} \text{Dead band} &= \pm \frac{2^{-b}}{1 - |\alpha|} \\ &= \pm \frac{2^{-2}}{1 - 0.97} \\ \text{Dead band} &= \pm 8.333 \end{aligned}$$

Thus, the dead band interval is  $[-8.333, 8.333]$ .

### Example 5.23

A causal filter is defined by the difference equation

$$y(n) = x(n) - 0.9y(n - 1)$$

The unit sample response  $h(n)$  is computed such that the computed values are rounded to one decimal place. Show that the filter exhibits dead band effect. Determine the dead band range.

(Anna University, May, 2007)

**Solution** Given

$$y(n) = x(n) - 0.9y(n - 1)$$

For causal system  $y(n) = 0$ , for  $n < 0$ . Consider the input

$$x(n) = \begin{cases} 12, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Let  $y'(n)$  be the response of the filter when the product is quantized by rounding.

$$y'(n) = -Q[0.9y'(n-1)] + x(n)$$

When  $n = 0$ ,

$$\begin{aligned} y'(0) &= -Q[0.9y'(-1)] + x(0) \\ &= 0 + 12 \\ &= 12 \end{aligned}$$

When  $n = 1$ ,

$$\begin{aligned} y'(1) &= -Q[0.9y'(0)] + x(1) \\ &= -Q[0.9 \times 12] + 0 \\ &= -Q[10.8] \end{aligned}$$

$$\begin{aligned} Q[10.8] &\Rightarrow (10.8)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1010.110)_2 \xrightarrow[\text{one decimal}]{\text{rounded to the}} (1011)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (11)_{10} \\ y'(1) &= -Q[10.185] = -11 \end{aligned}$$

When  $n = 2$ ,

$$\begin{aligned} y'(2) &= -Q[0.9y'(1)] \\ &= -Q[0.9 \times (-11)] \\ &= -Q[-9.9] \end{aligned}$$

$$\begin{aligned} Q[-9.9] &\Rightarrow -(9.9)_{10} \xrightarrow[\text{(two's complement)}]{\text{convert to binary}} (0110.001)_2 \xrightarrow[\text{decimal place}]{\text{rounded to one}} (0110)_2 \\ &\quad \xrightarrow[\text{complement}]{\text{take two's}} (1010)_2 \xrightarrow[\text{to decimal}]{\text{convert}} -(10)_{10} \\ y'(2) &= -Q[9.9] = -(-10) = 10 \end{aligned}$$

When  $n = 3$ ,

$$\begin{aligned}y'(3) &= -Q[0.9y'(2)] \\&= -Q[0.9 \times 10] \\&= -Q[9] = -9\end{aligned}$$

When  $n = 4$ ,

$$\begin{aligned}y'(4) &= -Q[0.9y'(3)] \\&= -Q[0.9 \times (-9)] \\&= -Q[-8.1]\end{aligned}$$

$$\begin{aligned}Q[-8.1] &\Rightarrow -(8.1)_{10} \xrightarrow[\text{(two's complement)}]{\text{convert to binary}} (1000.000)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1000)_2 \\&\quad \xrightarrow[\text{complement}]{\text{take two's}} -(1000)_2 \xrightarrow[\text{to decimal}]{\text{convert}} -(8)_{10} \\y'(4) &= -Q[-8.1] = -(-8) = 8\end{aligned}$$

When  $n = 5$ ,

$$\begin{aligned}y'(5) &= -Q[0.9y'(4)] \\&= -Q[0.9 \times 8] \\&= -Q[7.2]\end{aligned}$$

$$\begin{aligned}Q[7.2] &\Rightarrow (7.2)_{10} \xrightarrow{\text{to binary}} (0111.001)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0111)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (7)_{10} \\y'(5) &= -Q[7.2] = -7\end{aligned}$$

When  $n = 6$ ,

$$\begin{aligned}y'(6) &= -Q[0.9y'(5)] \\&= -Q[0.9 \times (-7)] \\&= -Q[-6.3]\end{aligned}$$

$$Q[-6.3] \Rightarrow -(6.3)_{10} \xrightarrow[\text{(two's complement)}]{\text{convert to binary}} (1001.110)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1010)_2$$

$$\begin{array}{c} \xrightarrow{\text{take two's}} -(0110)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (-6)_{10} \\ \text{complement} \\ y'(6) = -Q[-6.3] = -(-6) = 6 \end{array}$$

When  $n = 7$ ,

$$\begin{aligned} y'(7) &= -Q[0.9y'(6)] \\ &= -Q[0.9 \times 6] \\ &= -Q[5.4] \end{aligned}$$

$$\begin{array}{c} Q[5.4] \Rightarrow (5.4)_{10} \xrightarrow{\substack{\text{convert} \\ \text{to binary}}} (0101.011)_2 \xrightarrow{\substack{\text{rounded to} \\ \text{one decimal}}} (0101)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (5.5)_{10} \\ y'(7) = -Q[5.4] = -5.5 \end{array}$$

When  $n = 8$ ,

$$\begin{aligned} y'(8) &= -Q[0.9y'(7)] \\ &= -Q[0.9 \times (-5.5)] \\ &= -Q[-4.95] \end{aligned}$$

$$\begin{array}{c} Q[-4.95] \Rightarrow -(4.95)_{10} \xrightarrow{\substack{\text{convert to binary} \\ (\text{two's complement})}} (1011.001)_2 \xrightarrow{\substack{\text{rounded to} \\ \text{one decimal}}} (1011)_2 \\ \xrightarrow{\substack{\text{take two's} \\ \text{complement}}} -(0101)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (-5)_{10} \\ y'(8) = -Q[-4.95] = -(-5) = 5 \end{array}$$

When  $n = 9$ ,

$$\begin{aligned} y'(9) &= -Q[0.9y'(8)] \\ &= -Q[0.9 \times 5] \\ &= -Q[4.5] \end{aligned}$$

$$\begin{array}{c} Q[4.5] \Rightarrow (4.5)_{10} \xrightarrow{\substack{\text{convert} \\ \text{to binary}}} (0100.10)_2 \xrightarrow{\substack{\text{rounded to} \\ \text{one decimal}}} (0100.1)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (4.5)_{10} \\ y'(9) = -Q[4.5] = -4.5 \end{array}$$

When  $n = 10$ ,

$$y'(10) = -Q[0.9y'(9)]$$

$$\begin{aligned}
 &= -Q[0.9 \times (-4.5)] \\
 &= -Q[-4.05]
 \end{aligned}$$

$$\begin{aligned}
 Q[-4.05] &\Rightarrow -(4.05)_{10} \xrightarrow[\text{(two's complement)}]{\text{convert to binary}} (1100.000)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1100)_2 \\
 &\quad \xrightarrow[\text{complement}]{\text{take two's}} -(0100)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (-4)_{10} \\
 y'(10) &= -Q[-4.05] = -(-4) = 4
 \end{aligned}$$

When  $n = 11$ ,

$$\begin{aligned}
 y'(11) &= -Q[0.9y'(10)] \\
 &= -Q[0.9 \times 4] \\
 &= -Q[3.6]
 \end{aligned}$$

$$\begin{aligned}
 Q[3.6] &\Rightarrow (3.6)_{10} \xrightarrow{\text{convert}} (0011.100)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0011.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (3.5)_{10} \\
 y'(11) &= -Q[3.6] = -3.5
 \end{aligned}$$

When  $n = 12$ ,

$$\begin{aligned}
 y'(12) &= -Q[0.9y'(11)] \\
 &= -Q[0.9 \times 3.5] \\
 &= -Q[-3.15]
 \end{aligned}$$

$$\begin{aligned}
 Q[-3.15] &\Rightarrow -(3.15)_{10} \xrightarrow[\text{(two's complement)}]{\text{convert to binary}} (1100.111)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1101)_2 \\
 &\quad \xrightarrow[\text{complement}]{\text{take two's}} -(0011)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (-3)_{10} \\
 y'(12) &= -Q[-3.15] = -(-3) = 3
 \end{aligned}$$

When  $n = 13$ ,

$$\begin{aligned}
 y'(13) &= -Q[0.9y'(12)] \\
 &= -Q[0.9 \times 3] \\
 &= -Q[2.7]
 \end{aligned}$$

$$Q[2.7] \Rightarrow (2.7)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (0010.101)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0010.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (2.5)_{10}$$

$$y'(13) = -Q[2.7] = -2.5$$

When  $n = 14$ ,

$$\begin{aligned} y'(14) &= -Q[0.9y'(13)] \\ &= -Q[0.9 \times (-2.5)] \\ &= -Q[-2.25] \end{aligned}$$

$$Q[-2.25] \Rightarrow -(2.25)_{10} \xrightarrow[\text{(two's complement)}]{\text{convert to binary}} (1101.10)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (1101.1)_2$$

$$\xrightarrow[\text{complement}]{\text{take two's}} -(0010.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} -(2.5)_{10}$$

$$y'(14) = -Q[-2.25] = -(-2.5) = 2.5$$

For all values of  $n \geq 13$ ,  $y'(n)$  will remain in the same magnitude as 2.5 with alternative sign. Therefore, the system enters into limit cycle when  $n = 13$

$$\begin{aligned} \text{Dead band} &= \pm \frac{2^{-b}}{1 - |\alpha|} = \pm \frac{2^{-2}}{1 - 0.9} \\ \text{Dead band} &= \pm \frac{0.25}{1 - 0.9} \\ &= \pm 2.5 \end{aligned}$$

Thus the dead band interval is  $[-2.5, 2.5]$ .

### Unit Impulse Response

Given

$$y(n) = -0.9y(n-1) + x(n)$$

Taking  $z$ -transform on the both sides, we get

$$\begin{aligned} Y(z) &= -0.9z^{-1}Y(z) + X(z) \\ Y(z)[1 + 0.9z^{-1}] &= Y(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{1 + 0.9z^{-1}} \end{aligned}$$

The unit impulse response  $h(n)$  is

$$h(n) = z^{-1}[H(z)]$$

$$h(n) = (-0.9)^n u(n)$$

### Example 5.24

A digital system is characterized by the difference equation

$$y(n) = 0.9y(n - 1) + x(n)$$

Determine the dead band of the system when  $x(n) = 0$  and  $y(-1) = 12$ .

*(Anna University, May, 2004)*

**Solution** Given

$$y(n) = -0.9y(n - 1) + x(n)$$

where  $x(n) = 0$ ,  $y(-1) = 12$ . Let  $y'(n)$  be the response of the filter when the product is quantized.

$$y'(n) = Q[0.9y'(n - 1)] + x(n)$$

When  $n = 0$ ,

$$\begin{aligned} y'(0) &= Q[0.9y'(-1)] \\ &= Q[0.9 \times 12] \\ &= Q[10.8] \end{aligned}$$

$$Q[10.8] \Rightarrow (10.8)_{10} \xrightarrow[\text{to binary}]{} (1010.110)_2 \xrightarrow[\text{one decimal}]{\text{rounded to the}} (1011)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (11)_{10}$$

$$y'(0) = Q[10.8] = 1$$

When  $n = 1$ ,

$$\begin{aligned} y'(1) &= Q[0.9y'(0)] \\ &= Q[0.9 \times 1] \\ &= Q[9.9] \end{aligned}$$

$$Q[9.9] \Rightarrow (9.9)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1001.111)_2 \xrightarrow[\text{one decimal}]{\text{rounded to the}} (1010)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (10)_{10}$$

$$y'(1) = Q[9.9] = 10$$

When  $n = 2$ ,

$$\begin{aligned} y'(2) &= Q[0.9y'(1)] \\ &= Q[0.9 \times 10] \\ &= Q[9] = 9 \end{aligned}$$

When  $n = 3$ ,

$$\begin{aligned} y'(3) &= Q[0.9y'(2)] \\ &= Q[0.9 \times 9] \\ &= Q[8.1] \end{aligned}$$

$$Q[8.1] \Rightarrow (8.1)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (1000.000)_2 \xrightarrow[\text{one decimal}]{\text{rounded to the}} (1000)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (8)_{10}$$

$$y'(3) = Q[8.1] = 8$$

When  $n = 4$ ,

$$\begin{aligned} y'(4) &= Q[0.9y'(3)] \\ &= Q[0.9 \times 8] \\ &= Q[7.2] \end{aligned}$$

$$Q[7.2] \Rightarrow (7.2)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (0111.000)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0111)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (7)_{10}$$

$$y'(4) = Q[7.2] = 7$$

When  $n = 5$ ,

$$\begin{aligned} y'(5) &= Q[0.9y'(4)] \\ &= Q[0.9 \times 7] \\ &= Q[6.3] \end{aligned}$$

$$\begin{aligned} Q[6.3] &\Rightarrow (6.3)_{10} \xrightarrow{\substack{\text{convert} \\ \text{to binary}}} (0110.010)_2 \xrightarrow{\substack{\text{rounded to} \\ \text{one decimal}}} (0110.1)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (6.5)_{10} \\ y'(5) &= Q[6.3] = 6.5 \end{aligned}$$

When  $n = 6$ ,

$$\begin{aligned} y'(6) &= Q[0.9y'(5)] \\ &= Q[0.9 \times 6.5] \\ &= Q[5.85] \end{aligned}$$

$$\begin{aligned} Q[5.85] &\Rightarrow (5.85)_{10} \xrightarrow{\substack{\text{convert} \\ \text{to binary}}} (0101.110)_2 \xrightarrow{\substack{\text{rounded to} \\ \text{one decimal}}} (0110)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (6)_{10} \\ y'(6) &= Q[5.85] = 6 \end{aligned}$$

When  $n = 7$ ,

$$\begin{aligned} y'(7) &= Q[0.9y'(6)] \\ &= Q[0.9 \times 6] \\ &= Q[5.4] \end{aligned}$$

$$\begin{aligned} Q[5.4] &\Rightarrow (5.4)_{10} \xrightarrow{\substack{\text{convert} \\ \text{to binary}}} (0101.011)_2 \xrightarrow{\substack{\text{rounded to} \\ \text{one decimal}}} (0101.1)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (5.5)_{10} \\ y'(7) &= Q[5.4] = 5.5 \end{aligned}$$

When  $n = 8$ ,

$$\begin{aligned} y'(8) &= Q[0.9y'(7)] \\ &= Q[0.9 \times 5.5] \\ &= Q[4.95] \end{aligned}$$

$$\begin{aligned} Q[4.95] &\Rightarrow (4.95)_{10} \xrightarrow{\substack{\text{convert to} \\ \text{binary}}} (0100.111)_2 \xrightarrow{\substack{\text{rounded to} \\ \text{one decimal}}} (0101)_2 \xrightarrow{\substack{\text{convert} \\ \text{to decimal}}} (5)_{10} \\ y'(8) &= Q[4.95] = 5 \end{aligned}$$

When  $n = 9$ ,

$$y'(9) = Q[0.9y'(8)]$$

$$\begin{aligned} &= Q[0.9 \times 5] \\ &= Q[4.5] \end{aligned}$$

$$Q[4.5] \Rightarrow (4.5)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (0100.10)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0100.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (4.5)_{10}$$

$$y'(9) = Q[4.5] = 4.5$$

When  $n = 10$ ,

$$\begin{aligned} y'(10) &= Q[0.9y'(9)] \\ &= Q[0.9 \times 4.5] \\ &= Q[4.05] \end{aligned}$$

$$Q[4.05] \Rightarrow (4.05)_{10} \xrightarrow[\text{binary}]{\text{convert to}} (0100.000)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0100)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (4)_{10}$$

$$y'(10) = Q[4.05] = 4$$

When  $n = 11$ ,

$$\begin{aligned} y'(11) &= Q[0.9y'(10)] \\ &= Q[0.9 \times 4] \\ &= Q[3.6] \end{aligned}$$

$$Q[3.6] \Rightarrow (3.6)_{10} \xrightarrow[\text{binary}]{\text{convert}} (0011.100)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0011.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (3.5)_{10}$$

$$y'(11) = Q[3.6] = 3.5$$

When  $n = 12$ ,

$$\begin{aligned} y'(12) &= Q[0.9y'(11)] \\ &= Q[0.9 \times 3.5] \\ &= Q[3.15] \end{aligned}$$

$$Q[3.15] \Rightarrow (3.15)_{10} \xrightarrow[\text{binary}]{\text{convert to}} (0011.001)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0011)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (3)_{10}$$

$$y'(12) = Q[3.15] = 3$$

When  $n = 13$ ,

$$\begin{aligned}y'(13) &= Q[0.9y'(12)] \\&= Q[0.9 \times 3] \\&= Q[2.7]\end{aligned}$$

$$Q[2.7] \Rightarrow (2.7)_{10} \xrightarrow[\text{to binary}]{\text{convert}} (0010.101)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0010.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (2.5)_{10}$$

$$y'(13) = Q[2.7] = 2.5$$

When  $n = 14$ ,

$$\begin{aligned}y'(14) &= Q[0.9y'(13)] \\&= Q[0.9 \times 2.5] \\&= Q[2.25]\end{aligned}$$

$$Q[2.25] \Rightarrow (2.25)_{10} \xrightarrow[\text{binary}]{\text{convert to}} (0010.010)_2 \xrightarrow[\text{one decimal}]{\text{rounded to}} (0010.1)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (2.5)_{10}$$

$$y'(14) = Q[2.25] = 2.5$$

For all values of  $n \geq 13$ ,  $y'(n)$  will remain with the same magnitude as 2.5. Therefore, the system enters into limit cycle when  $n = 13$

$$\begin{aligned}\text{Dead band} &= \pm \frac{2^{-b}}{1 - |\alpha|} \\&= \pm \frac{2^{-2}}{1 - 0.9} \\&= \pm \frac{0.25}{0.1} \\&= \pm 2.5\end{aligned}$$

Thus, the dead band interval is  $[-2.5, 2.5]$ .

**Example 5.25**

Determine the dead band of the following filter if 8 bits are used for representation

$$y(n) = 0.2y(n - 1) + 0.5y(n - 2) + x(n)$$

(Anna University, December, 2003)

**Solution** Given  $b = 8$  bits (including sign bit)

$$y(n) = 0.2y(n - 1) + 0.5y(n - 2) + x(n)$$

Consider the second-order system, which is described by the following difference equation.

$$y(n) = a_1y(n - 1) + a_2y(n - 2) + x(n)$$

Comparing these two equations we get  $a_1 = 0.2$  and  $a_2 = 0.5$ .

For the second-order system, dead band is given by

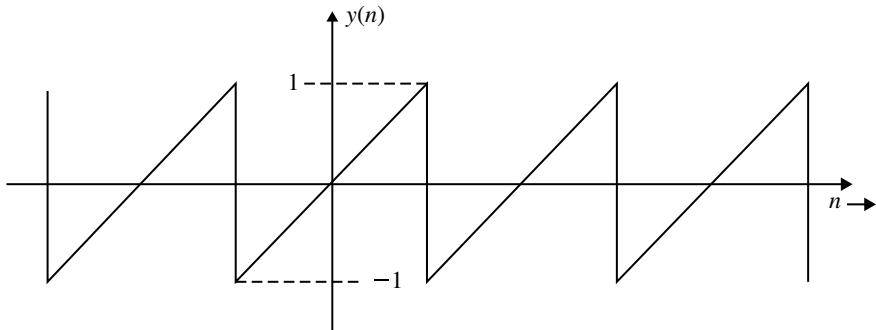
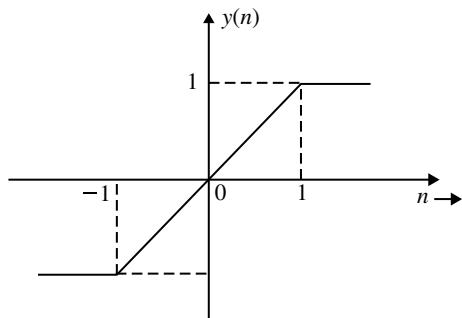
$$\begin{aligned}\text{Dead band} &= \pm \frac{2^{-b}}{1 - |a_2|} \\ &= \pm \frac{2^{-8}}{1 - |0.5|}\end{aligned}$$

$\boxed{\text{Dead band} = \pm 0.0078125}$

Thus, the dead band has the interval of  $[-0.0078125, +0.0078125]$ .

### 5.7.2 Overflow Limit Cycle Oscillation

In fixed point addition of two binary numbers the overflow occurs when the sum exceeds the finite word length of the register used to store the sum. The overflow in addition may lead to oscillation in the output which is referred to as an overflow limit cycle. An overflow in addition of two or more binary numbers occurs when the sum exceeds the dynamic range of number system. Let us consider two positive numbers  $n_1$  and  $n_2$  which are represented in sign magnitudes.

**Fig. 5.16** Transfer characteristics of an adder**Fig. 5.17** Characteristics of saturation adder

$$n_1 = \frac{3}{8} \rightarrow 0.011$$

$$n_2 = \frac{6}{8} \rightarrow 0.110$$

$$n_1 + n_2 = 1.001 \rightarrow -\frac{1}{8} \text{ in sign magnitude}$$

In the above example, when two positive numbers are added the sum is wrongly interpreted as a negative number. The transfer characteristics of an adder is shown in Fig. 5.16, where  $n$  is the input to the adder and  $y(n)$  is the corresponding output. The overflow oscillations can be eliminated if saturation arithmetic is performed. The characteristics of saturation adder is shown in Fig. 5.17. In saturation arithmetic, when an overflow is sensed, the output is set equal to maximum allowable value and when an underflow is sensed, the output (sum) is set equal to minimum allowable value. The saturation arithmetic introduces nonlinearity in the adder, and the signal distortion due to this nonlinearity is small if the saturation occurs infrequently.

## 5.8 Scaling to Prevent Overflow

Saturation arithmetic eliminates limit cycle due to overflow, but it causes undesirable signal distortion due to nonlinearity of the clipper. In order to limit the amount of nonlinear distortion, it is important to scale the input signal to the adder such that the overflow becomes a rare event. Let

$$x(n) = \text{Input to the system}$$

$$h_m(n) = \text{Impulse response between the input and output of node } -m$$

$$y_m(n) = \text{Response of the system at node } -m$$

To scale the input signal so that

$$\sum_{n=-\infty}^{\infty} |y_m(n)|^2 \leq S^2 \sum_{n=-\infty}^{\infty} |x(n)|^2$$

where  $S$  is the scaling factor, using Parseval's and residue theorems the expression for scaling factor is given by

$$\begin{aligned} S^2 &= \frac{1}{\sum_{n=-\infty}^{\infty} |h_m(n)|^2} = \frac{1}{2\pi j \oint_c S(z)S(z^{-1})z^{-1}dz} \\ S^2 &= \frac{1}{\sum_{i=1}^N \text{Res}[S(z)S(z^{-1})z^{-1}dz]|_{z=p_i}} \end{aligned} \quad (5.37)$$

where  $S(z)$  is the transfer function seen between the input to system and output of summing node  $-m$ . For example, consider the second-order system which is shown in Fig. 5.18.

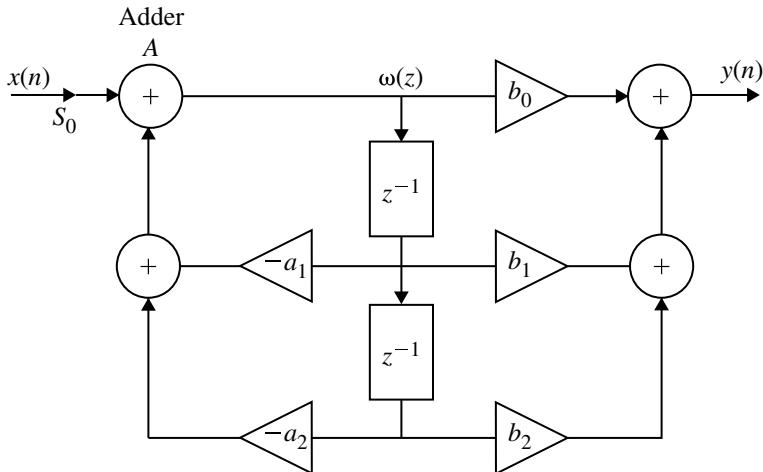
The transfer function between the input to the system and output of adder  $A$  is given by

$$S(z) = \frac{W(z)}{X(z)}$$

The output signal of adder  $A$  is

$$\begin{aligned} W(z) &= X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \\ X(z) &= W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) \end{aligned}$$

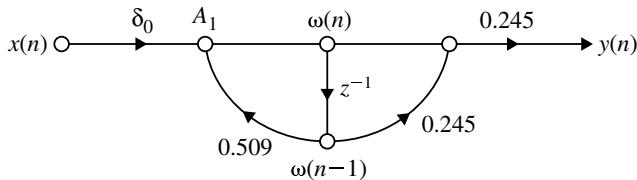
$$\therefore S(z) = \frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$



**Fig. 5.18** Second-order system with input in sealed by  $S_0$

### Example 5.26

For digital network shown in figure, find  $H(z)$  and scale factor so to avoid overflow in register  $A_1$ .



**Solution** In figure, the output signal at node  $A_1$ , is  $\omega(n)$ . At node  $A_1$ , the following equation is written.

$$\omega(n) = \delta_0 x(n) + 0.509(\omega(n-1))$$

On taking  $z$ -transform we get,

$$\begin{aligned} W(z) &= \delta_0 X(z) + 0.509 z^{-1} W(z) \\ W(z) - 0.509 z^{-1} W(z) &= \delta_0 X(z) \\ W(z)[1 - 0.509 z^{-1}] &= \delta_0 X(z) \\ \frac{W(z)}{X(z)} &= \frac{\delta_0}{1 - 0.509 z^{-1}} \end{aligned}$$

At the output node the following equation is written.

$$\begin{aligned} y(n) &= 0.245(\omega(n) + 0.245\omega(n-1)) \\ y(n) &= 0.245\omega(n) + 0.06\omega(n-1) \end{aligned}$$

On taking  $z$ -transform we get,

$$\begin{aligned} Y(z) &= 0.245W(z) + 0.06W(z)z^{-1} \\ Y(z) &= (0.245 + 0.06z^{-1})W(z) \\ \therefore \frac{Y(z)}{W(z)} &= (0.245 + 0.06z^{-1}) \end{aligned}$$

The transfer function of the system  $\frac{Y(z)}{X(z)}$  is obtained by multiplying  $W(z)/X(z)$  and  $Y(z)/W(z)$

$$\begin{aligned} \therefore \text{Transfer function of the system } \frac{Y(z)}{X(z)} &= \frac{Y(z)}{W(z)} \times \frac{W(z)}{X(z)} \\ &= \frac{\delta_0[0.245 + 0.06z^{-1}]}{[1 - 0.509z^{-1}]} \end{aligned}$$

The scaling factor  $\delta_0$  is given by

$$\delta_0^2 = \frac{1}{\frac{1}{2\pi j} \oint_c \delta(z)\delta(z^{-1})z^{-1}dz}$$

where  $\delta(z) \rightarrow$  transfer function seen between system input and output of register  $A_1$ .

$$\delta(z) = \frac{W(z)}{X(z)}$$

When there is no scaling multiplier, we get

$$\begin{aligned} \delta(z) &= \frac{1}{1 - 0.509z^{-1}} = \frac{z}{z - 0.509} \\ \therefore \delta(z)\delta(z^{-1})z^{-1} &= \frac{z^{-1}}{(z - 0.509)(z^{-1} - 0.509)} \\ \frac{1}{2\pi j} \oint_c \delta(z)\delta(z^{-1})z^{-1}dz &= \sum_{i=1}^N \text{Res}[\delta(z)\delta(z^{-1})z^{-1}]_{z=p_i} \end{aligned}$$

We have only one pole at  $z = 0.509$

$$\begin{aligned}
 &= (z - 0.509) \frac{z^{-1}}{(z - 0.509)(z^{-1} - 0.509)} \Big|_{z=0.509} \\
 &= \frac{z^{-1}}{z^{-1} - 0.509} \Big|_{z=0.509} \\
 &= \frac{0.509^{-1}}{0.509^{-1} - 0.509} = 1.3497 \\
 \therefore \delta_0^2 &= \frac{1}{1.3497} = 0.7409
 \end{aligned}$$

Scaling factor

$$\boxed{\delta_0 = \sqrt{0.7409} = 0.8608.}$$

### Example 5.27

The output signal of an A/D converter is passed through a first-order lowpass filter, with transfer function given by

$$H(z) = \frac{(1-a)z}{z-a}, \quad 0 < a < 1$$

Find the steady-state output noise power due to quantization at the output of the digital filter.

**Solution** Using Eqs. 5.29 and 5.30 we get,

$$\begin{aligned}
 \sigma_{e0}^2 &= \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z) H(z^{-1}) z^{-1} dz \\
 &= \sigma_e^2 \sum_{i=1}^N [\text{Res}(H(z) H(z^{-1}) z^{-1} dz)]|_{z=p_i}
 \end{aligned}$$

Given,

$$\begin{aligned}
 H(z) &= \frac{(1-a)z}{z-a} \\
 H(z^{-1}) &= \frac{(1-a)z^{-1}}{z^{-1}-a}
 \end{aligned}$$

$$H(z)H(z^{-1})z^{-1} = \frac{(1-a)z(1-a)z^{-1} \cdot z^{-1}}{(z-a)(z^{-1}-a)} = \frac{(1-a)^2 z^{-1}}{(z-a)(z^{-1}-a)}$$

There are two poles at  $z = a, z = 1/a$  out of which the pole at  $z = a$  lies inside of the unit circle since  $0 < a < 1$ . Therefore, residue due to pole  $z = a$  is given as,

$$\begin{aligned}\text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=a} &= (z-a) \frac{(1-a)^2 z^{-1}}{(z-a)(z^{-1}-a)} \Big|_{z=a} \\ &= \frac{(1-a)^2 a^{-1}}{a^{-1}-a} \\ &= \frac{(1-a)^2}{(1-a^2)}\end{aligned}$$

Therefore, steady-state output noise power is

$$\begin{aligned}\sigma_{e0}^2 &= \sigma_e^2 \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=a} \\ &= \sigma_e^2 \left[ \frac{(1-a)^2}{1-a^2} \right] \\ &= \sigma_e^2 \left[ \frac{1-a}{1+a} \right]\end{aligned}$$

where  $\sigma_e^2 = (q^2/12)$ , if  $R = 1$

$$\boxed{\sigma_e^2 = \left( \frac{2^{-2b}}{12} \right)}.$$

### Example 5.28

Find the steady-state variance of the noise in the output due to quantization of input for the first-order filter described by the following difference equation.

$$y(n) = 0.5y(n-1) + x(n)$$

(Anna University, May, 2007)

**Solution Given**

$$y(n) = 0.5y(n-1) + x(n)$$

Taking  $z$ -transform on both sides we have

$$\begin{aligned} Y(z) &= 0.5z^{-1}Y(z) + X(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.5z^{-1}} = \frac{z}{z - 0.5} \\ H(z^{-1}) &= \frac{z^{-1}}{z^{-1} - 0.5} \end{aligned}$$

The steady-state output noise power as given by Equation (5.30) is,

$$\begin{aligned} \sigma_{e0}^2 &= \sigma_e^2 \sum_{i=1}^N \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=p_i} \\ H(z)H(z^{-1})z^{-1} &= \frac{z \cdot z^{-1} \cdot z^{-1}}{(z - 0.5)(z^{-1} - 0.5)} \\ &= \frac{z^{-1}}{(z - 0.5)(z^{-1} - 0.5)} \end{aligned}$$

The pole  $z = 0.5$  only lies inside of the unit circle, and therefore

$$\begin{aligned} \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=0.5} &= (z - 0.5) \frac{z^{-1}}{(z - 0.5)(z^{-1} - 0.5)} \Big|_{z=0.5} \\ &= \frac{(0.5)^{-1}}{(0.5)^{-1} - 0.5} \\ &= \frac{1}{1 - (0.5)^2} = \frac{1}{1 - 0.25} = 1.333 \end{aligned}$$

The steady-state output noise variance (power) is

$$\sigma_{e0}^2 = \sigma_e^2 \text{Res}[H(z)H(z^{-1})z^{-1}]|_{z=0.5}$$

$\boxed{\sigma_{e0}^2 = \sigma_e^2 1.333.}$

### Example 5.29

Consider the transfer function

$$H(z) = H_1(z)H_2(z)$$

where

$$H_1(z) = \frac{1}{1 - 0.5z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1}{1 - 0.6z^{-1}}$$

Find the output round off noise power.

(Anna University, May, 2007)

**Solution** The round off noise model for  $H(z) = H_1(z)H_2(z)$  is shown in Fig. 5.19. From the realization, the noise transfer function seen by noise source  $e_1(n)$  is written as,

$$T_1(z) = H(z) = H_1(z)H_2(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.6z^{-1})}$$

The noise transfer function seen by  $e_2(n)$  is written as,

$$T_2(z) = H_2(z) = \frac{1}{1 - 0.6z^{-1}}$$

The steady-state output noise power due to  $e_1(n)$  as given by Equation (5.30) is,

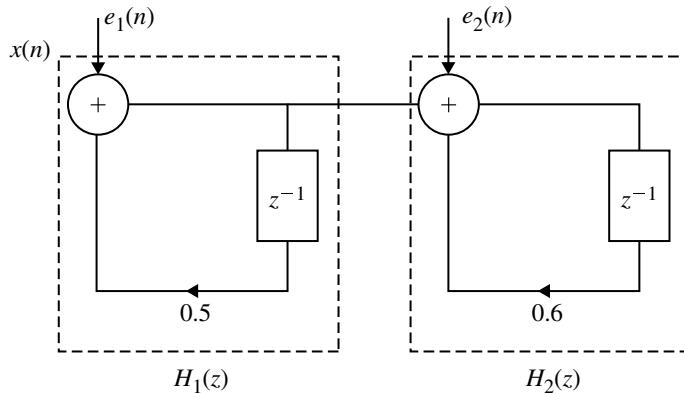


Fig. 5.19 Round off noise model for Example 5.29

$$\begin{aligned}\sigma_{e0_1}^2 &= \sigma_e^2 \sum_{i=1}^N [\text{Res}(T_1(z)T_1(z^{-1})z^{-1})]|_{z=p_i} \\ &= \sigma_e^2 \text{Res} \left[ \frac{z^{-1}}{(1-0.5z^{-1})(1-0.6z^{-1})(1-0.5z)(1-0.6)} \right]_{z=0.5,0.6,(1/0.5),(1/0.6)}\end{aligned}$$

Only two poles at  $z = 0.5$  and  $z = 0.6$  lie inside of unit circle

$$\begin{aligned}\sigma_{e0_1}^2 &= \sigma_e^2 \left[ \sum \text{Res}(T_1(z)T_1(z^{-1})z^{-1}) \right] \Big|_{z=0.5 \text{ and } z=0.6} \\ &= \sigma_e^2 \left[ (z-0.5) \frac{z}{(z-0.5)(z-0.6)(1-0.5z)(1-0.6)} \Big|_{z=0.5} \right. \\ &\quad \left. + (z-0.6) \frac{z}{(z-0.5)(z-0.6)(1-0.5z)(1-0.6)} \Big|_{z=0.6} \right] \\ &= \sigma_e^2 \left[ \frac{0.5}{(0.5-0.6)(1-.25)(1-0.3)} + \frac{0.6}{(0.6-0.5)(1-0.3)(1-0.36)} \right] \\ &= \sigma_e^2 [-9.5238 + 13.3928]\end{aligned}$$

$$\boxed{\sigma_{e0_1}^2 = \sigma_e^2 [3.8690]}$$

The steady-state output noise power due to  $e_2(n)$  is,

$$\begin{aligned}\sigma_{e0_2}^2 &= \sigma_e^2 \sum_{i=1}^N \text{Res}[(T_2(z)T_2(z^{-1})z^{-1})]|_{z=p_i} \\ &= \sigma_e^2 \sum_{i=1}^N \text{Res}[(H_2(z)H_2(z^{-1})z^{-1})]|_{z=p_i} \\ &= \sigma_e^2 \sum \text{Res} \left[ \frac{1}{1-0.6z^{-1}} \cdot \frac{1}{1-0.6z} \cdot z^{-1} \right]_{z=0.6 \text{ and } z=(1/0.6)}\end{aligned}$$

The residue at pole  $z = (1/0.6)$  is zero, since that pole lies outside of the unit circle

$$\begin{aligned}\sigma_{e0_2}^2 &= \sigma_e^2 \text{Res} \left[ \frac{1}{(z-0.6)(1-0.6z)} \right]_{z=0.6} \\ &= \sigma_e^2 \left[ (z-0.6) \frac{1}{(z-0.6)(1-0.6z)} \right]_{z=0.6}\end{aligned}$$

$$\boxed{\sigma_{e0_2}^2 = \sigma_e^2 [1.5626]}$$

The total steady-state output noise power is,

$$\begin{aligned}\sigma_{e0}^2 &= \sigma_{e0_1}^2 + \sigma_{e0_2}^2 \\ &= \sigma_e^2 [3.8690 + 1.5626] \\ &= \sigma_e^2 [5.4315]\end{aligned}$$

For example for  $b = 4$  bits (including sign bit)

$$\begin{aligned}\sigma_e^2 &= \frac{q^2}{12} = \frac{(R/2^b)^2}{12} \\ \sigma_e^2 &= \frac{(2/2^4)^2}{12} = \frac{2^{-6}}{12} = 1.302 \times 10^{-3} \\ \sigma_{e0}^2 &= 1.302 \times 10^{-3} \times 5.4315 = 7.0718 \times 10^{-3}\end{aligned}$$

The total round off noise power is,

$$\boxed{\sigma_{e0}^2 = 7.0718 \times 10^{-3}}$$


---

### Example 5.30

Draw the quantization noise model for a second-order system

$$H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}$$

and find the steady-state output noise variance.

(Anna University, May, 2005)

**Solution** Given

$$H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}$$

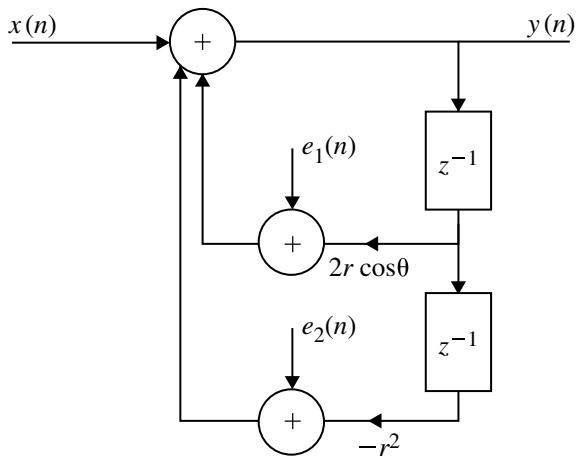
The quantization noise model is shown in Fig. 5.20.

$$\sigma_{e0}^2 = \sigma_{e0_1}^2 + \sigma_{e0_2}^2$$

Both noise sources see the same transfer function

$$\begin{aligned}H(z) &= \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}} \\ \sigma_{e0_1}^2 &= \sigma_{e0_2}^2\end{aligned}$$

**Fig. 5.20** Quantization noise model for Example 5.30



The impulse response of the transfer function is given by

$$h(n) = r^n \frac{\sin(n+1)\theta}{\sin \theta} u(n)$$

(Please refer  $z$ -transform table)

$$\begin{aligned}\sigma_{e0_1}^2 &= \sigma_{e0_2}^2 = \sigma_e^2 \sum_{n=-\infty}^{\infty} h^2(n) \\ \sigma_{e0}^2 &= 2\sigma_{e0_1}^2 \\ &= 2\sigma_e^2 \sum_{n=0}^{\infty} \frac{r^{2n} \sin^2(n+1)\theta}{\sin^2 \theta} \\ &= 2\sigma_e^2 \frac{1}{2 \sin^2 \theta} \sum_{n=0}^{\infty} r^{2n} [1 - \cos 2(n+1)\theta] \quad [\because \cos 2\theta = 1 - 2 \sin^2 \theta] \\ &= 2\sigma_e^2 \frac{1}{2 \sin^2 \theta} \left[ \sum_{n=0}^{\infty} r^{2n} - \sum_{n=0}^{\infty} r^{2n} \cos 2(n+1)\theta \right] \\ &= 2\sigma_e^2 \frac{1}{2 \sin^2 \theta} \left[ \frac{1}{1-r^2} - \frac{1}{2} \left[ \sum_{n=0}^{\infty} r^{2n} e^{j(2n+1)\theta} + \sum_{n=0}^{\infty} r^{2n} e^{-j(2n+1)\theta} \right] \right] \\ &= 2\sigma_e^2 \cdot \frac{1}{2 \sin^2 \theta} \left[ \frac{1}{1-r^2} - \frac{1}{2} \left[ \frac{e^{2j\theta}}{1-r^2 e^{2j\theta}} + \frac{e^{-2j\theta}}{1-r^2 e^{-2j\theta}} \right] \right]\end{aligned}$$

$$\begin{aligned}\sigma_{e0}^2 &= 2\sigma_e^2 \cdot \frac{1}{2 \sin^2 \theta} \left[ \frac{1}{1-r^2} - \frac{\cos 2\theta - r^2}{1-2r^2 \cos 2\theta + r^4} \right] \\ &= 2\sigma_e^2 \cdot \frac{1}{2 \sin^2 \theta} \left[ \frac{(1+r^2)(1-\cos 2\theta)}{(1-r^2)(1-2r^2 \cos 2\theta + r^4)} \right] \\ &= 2\sigma_e^2 \cdot \frac{1}{2 \sin^2 \theta} \left[ \frac{(1+r^2)2 \sin^2 \theta}{(1-r^2)(1-2r^2 \cos 2\theta + r^4)} \right]\end{aligned}$$

$$\boxed{\sigma_{e0}^2 = 2\sigma_e^2 \left[ \frac{1+r^2}{(1-r^2)(1-2r^2 \cos 2\theta + r^4)} \right]}$$

where

$$\sigma_e^2 = \left( \frac{q^2}{12} \right) = \frac{(R/2b)^2}{12}.$$

### Example 5.31

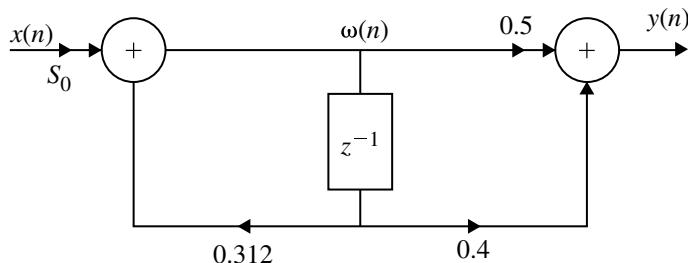
Given

$$H(z) = \frac{0.5 + 0.4z^{-1}}{1 - 0.312z^{-1}}$$

is the transfer function of a digital filter. Find the scaling factor  $S_0$  to avoid overflow in adder of the digital filter shown in Fig. 5.21.

**Solution** From Eq. 5.37, the scaling factor is written as,

$$S_0^2 = \frac{1}{\sum_{i=1}^N \text{Res}(S(z)S(z^{-1})z^{-1})} \Big|_{z=p_i}$$



**Fig. 5.21** Realization of transfer function for Example 5.31

where

$$\begin{aligned}
 S(z) &= \frac{W(z)}{X(z)} \\
 \omega(n) &= x(n) + 0.312\omega(n-1) \\
 W(z) &= X(z) + 0.312z^{-1}W(z) \\
 S(z) &= \frac{W(z)}{X(z)} = \frac{1}{1 - 0.312z^{-1}} \\
 S_0^2 &= \frac{1}{\sum \text{Res} \left[ \frac{z^{-1}}{(1-0.312z^{-1})(1-0.312z)} \right]} \\
 &= \frac{1}{\sum \text{Res} \left[ \frac{1}{(z-0.312)(1-0.312z)} \right]} \\
 &= \frac{1}{\text{Res} \left[ \frac{1}{(z-0.312)(1-0.312z)} \right] \Big|_{z=0.312}} \\
 &= \frac{1}{(z-0.312) \left[ \frac{1}{(z-0.312)(1-0.312z)} \right]_{z=0.312}} \\
 S_0^2 &= \frac{1}{1.1078}
 \end{aligned}$$

$$S_0 = 0.9501.$$

## Summary

- The performance of a DSP system is limited by the number of bits used in the implementation. The common sources of errors are due to input quantization, coefficient quantization, product round off and addition overflow.
- The ADC quantization noise is reduced by increasing the number of ADC bits or by using multirate techniques.
- When an IIR digital filter is implemented, errors arise in representing the filter coefficients. These errors can be reduced to acceptable level by using more bits. However, this increases the cost.
- Addition of two large numbers of a similar sign may produce an overflow which results in excess of permissible word length. This occurs at the output of adders and can be prevented by scaling the inputs to the adders in such a way that the outputs are kept low. However, this reduces the signal to noise ratio and increases the cost.
- In digital filter, the product of two variables requires very long bits. For recursive filters, if it is not reduced, subsequent computations will cause the number of bits to grow without limit. Truncation or rounding is used to quantize the products to the permissible word length. When the products are quantized, round off errors

occur and they may lead to oscillations in the output even when there is no input. Further the signal to noise ratio is also small. The round off noise can be reduced or minimized by passing through subsequent sections of a cascade structure where it is alternated.

- Reduction in signal to noise ratio due to round off error can be offset by the use of error spectral shaping scheme. However this scheme increases the number of multiplications and additions but computationally more efficient.

### Short Questions and Answers

**1. What is meant by finite word length effects in digital filters?**

The fundamental operations in digital filters are multiplication and addition. When these operations are performed in a digital system, the input data as well as the product and sum (output data) have to be represented in finite word length, which depends on the size (length) of the register used to store the data. In digital computation the input and output data (sum and product) are quantized by rounding or truncation to convert them into finite word size. This creates error (in noise) in the output or creates oscillations (limit cycles) in the output. These effects due to finite precision representation of numbers in digital system are called as finite word length effects.

**2. List some of the finite word length effects in digital filters.**

1. Errors due to quantization of the input data.
2. Errors due to quantization of the filter coefficients.
3. Errors due to rounding the product in multiplications.
4. Limit cycles due to product quantization and overflow in addition.

**3. Explain the fixed point representation of binary numbers.**

In fixed point representation of binary numbers in a given word size, the bits allotted for integer part and fraction part of the numbers are fixed, and therefore the position of binary point is fixed. The most significant bit is used to represent the sign of the number. This is shown in Fig. 5.22.

**4. What are the different formats of fixed point representation?**

In fixed point representation, there are three different formats for representing binary numbers:

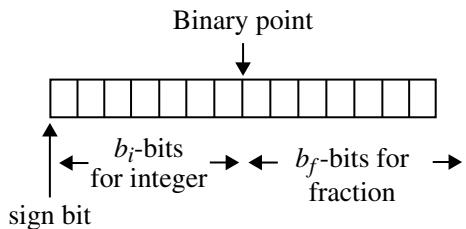
1. Sign-magnitude format.
2. One's complement format.
3. Two's complement format.

In all the three formats, the positive number is same, but they differ only in representing negative numbers.

**5. Explain the floating point representation of binary numbers.**

The floating numbers will have a mantissa part and exponent part. In a given word size the bits allotted for mantissa and exponent are fixed. The mantissa is used to represent a binary fraction number, and the exponent is a positive

**Fig. 5.22** Fixed point representation of binary numbers (question 3)



**Fig. 5.23** IEEE standard format for 32 bit floating point (question 6)

0	1	8	9	31
<i>S</i>	<i>E</i>			<i>M</i>

or negative binary integer. The value of the exponent can be adjusted to move the position of the binary point in mantissa. Hence, this representation is called floating point. The floating point number is expressed as

$$\text{Floating point number, } N_f = M \times 2^E$$

where  $M$  = mantissa and  $E$  = exponent.

#### 6. Give the IEEE-754 standard format for 32 bit floating point numbers.

The IEEE-754 standard for 32 bit single precision floating point number is given by floating point numbers,  $N_f = (-1) \times 2^{E-127} \times M$ . This is shown in Fig. 5.23.

$S = 1$  bit field for sign of number.

$E = 8$  bit field for exponent.

$M = 23$  bit field for mantissa.

#### 7. What are the types of arithmetic used in digital computers?

The floating point arithmetic and two's complement arithmetic are the two types of arithmetic employed in digital systems.

#### 8. Compare the fixed point and floating point number representations.

	Fixed point representation	Floating point representation
1.	In a $b$ bit binary the range of numbers represented is less when compared to floating point representation	In a $b$ bit binary the range of the numbers represented is large when compared to fixed point representation
2.	The position of binary point is fixed	The position of binary point is variable
3.	The resolution is uniform throughout the range	The resolution is variable

**9. Compare the fixed point and floating point arithmetic.**

	Fixed point arithmetic	Floating point arithmetic
1.	The accuracy of the result is less due to smaller dynamic range	The accuracy of the result will be higher due to larger dynamic range
2.	Speed of processing is high	Speed of processing is low
3.	Hardware implementation is cheaper	Hardware implementation is costlier
4.	Fixed point arithmetic can be used for real-time computation	Floating point arithmetic cannot be used for real-time computation
5.	Quantization error occurs only in multiplication	Quantization error occurs in both multiplication and addition

**10. What are the two types of quantization employed in digital system?**

The two types of quantization in digital system are truncation and rounding.

**11. What is Truncation?**

The truncation is the process of reducing the size of binary number by discarding all bits less significant than the least significant bit that is retained. (In truncation of a binary number to  $b$  bits all the less significant bits beyond  $b^{\text{th}}$  bit are discarded).

**12. Sketch the characteristics of the quantizer used for truncation.**

**13. What is rounding?**

Rounding is the process of reducing the size of a binary number to finite word sizes of  $b$  bits such that, the rounded  $b$  bit number is closest to the original unquantized number.

**14. Sketch the noise probability density function for rounding.**

**15. What are the errors generated by A/D process?**

The A/D process generates two types of errors. They are quantization error and saturation error. The quantization error is due to representation of the sampled signal by a fixed number of digital level (quantization levels). The saturation error occurs when the analog signal exceeds the dynamic range of A/D converter.

**16. What is quantization step size?**

In digital systems, the numbers are represented in binary. With  $b$  bit binary we can generate  $2^b$  different binary codes. Any range of analog value to be represented in binary should be divided into  $2^b$  levels with equal increment. The  $2^b$  levels are called quantization levels, and the increment in each level is called quantization step size. If  $R$  is the range of analog signal then,

$$\text{Quantization step size, } q = \frac{R}{2^b} \quad (5.38)$$

**17. How the input quantization noise is represented in LTI system?**

The quantized input signal of a digital system can be represented as a sum of unquantized signal  $x(n)$  and error signal  $e(n)$  as shown in Fig. 5.26.

**18. What is steady-state output noise due to input quantization?**

The input signal to digital system can be considered as a sum of unquantized signal and error signal due to input quantization. The response of the system can be expressed as a summation of response due to unquantized input and error signal. The response of the system due to error signal is given by convolution of error signal and impulse response. The variance of the response of the system for error signal is called steady-state output noise power.

**19. How the digital filter is affected by quantization of filter coefficients?**

The quantization of filter coefficients will modify the values of poles and zeros and so the location of poles and zeros will be shifted from the desired location. This will create deviation in the frequency response of the system. Hence, the resultant filter will have a frequency response different from that of the filter with unquantized coefficients.

**20. What is meant by product quantization error?**

In digital computation, the output of the multiplier *i.e.*, the products are quantized to the finite word length in order to store them in registers and to be used in subsequent calculation. The error due to the quantization of the output of multiplier is referred to as product quantization error.

**21. Why rounding is preferred for quantizing the product?**

In digital system the product quantization is performed by rounding due to the following desirable characteristics of rounding.

- (i) The rounding error is independent of the type of arithmetic.
- (ii) The mean value of rounding error signal is zero.
- (iii) The variance of the rounding error signal is least.

**22. Define noise transfer function (NTF).**

The noise transfer function (NTF) is defined as the transfer function from the noise source to the filter output. The NTF depends on the structure of the digital network.

**23. Draw the statistical model of the fixed point product quantization.**

The multiplier is considered as an infinite precision multiplier. Using an adder the error signal is added to the output of the multiplier so that the output of the adder is equal to the quantized product.

**24. Draw the product quantization noise model of second-order IIR system.****25. Draw the product quantization noise model of second-order IIR system with two first-order section in cascade?****26. What are limit cycles?**

In recursive systems when the input is zero or some nonzero constant value, the nonlinearities due to finite precision arithmetic operations may cause periodic oscillations in the output. These oscillations are called limit cycles.

**27. What are two types of limit cycles?**

The two types of limit cycles are zero input limit cycle and overflow limit cycle.

**28. What is zero Input limit cycle?**

In recursive system, the product quantization may create periodic oscillations in the output. These oscillations are called limit cycles. If the system output enters a limit cycle, it will continue to remain in limit cycle even when the input is made zero. Hence, these limit cycles are also called zero-input limit cycles.

**29. What is dead band?**

In a limit cycle the amplitudes of the output are confined to a range of values, which is called dead band of a filter.

**30. How the system output can be brought out of the limit cycle?**

The system output can be brought out of limit cycle by applying an input of large magnitude, which is sufficient to drive the system out of limit cycle.

**31. Draw the transfer characteristics of two's complement adder?**

**32. What is saturation arithmetic?**

In saturation arithmetic when the result of arithmetic operations exceeds the dynamic range of number system, then the result is set to maximum or minimum possible value. If the upper limit is exceeded, then the result is set to maximum possible value. If the lower limit is exceeded, then the result is set to minimum possible value.

**33. What is overflow limit cycle?**

In fixed point addition the overflow occurs when the sum exceeds the finite word length of the register used to store the sum. The overflow in addition may lead to oscillations in the output which is called overflow limit cycle.

**34. How overflow limit cycles can be eliminated?**

The overflow limit cycles can be eliminated either by using saturation arithmetic or by scaling the input signal to the adder.

**35. What is the drawback in saturation arithmetic?**

The saturation arithmetic introduces nonlinearity in the adder which leads to signal distortion.

**36. Give the rounding errors for fixed and floating print arithmetic.**

**For fixed point arithmetic**

$$\text{Rounding error, } e_r = N_r - N$$

where  $N_r$  is the fixed point binary number quantized by rounding and  $N$  is the unquantized fixed point binary number.

The range of error due to rounding is

$$-\frac{2^{-b}}{2} \leq e_r \leq \frac{2^{-b}}{2}$$

**For floating point arithmetic**

$$\text{Rounding error, } E_r = \frac{N_{rf} - N_f}{N_f}$$

where  $N_{rf}$  is rounded floating point binary number and  $N_f$  unquantized floating point binary number.

The range of error due to rounding is

$$-2^{-b} \leq e_r \leq 2^{-b}$$

- 37. What is the steady-state noise power at the output of an LTI system due to the quantization at the input to  $L$  bits?**

$$\text{Quantization step } (q) = \frac{R}{2^{-b}} = \frac{R}{2^{-L}}$$

$$\text{Steady-state input noise power } \sigma_e^2 = \frac{q^2}{12} = \frac{(R/2^{-L})^2}{12}$$

$$\text{Steady-state noise power at the output is } \sigma_{e0}^2 = \sigma_e^2 \sum_{i=1}^N \text{Re}[H(z)H(z^{-1})z^{-1}]|_{z=p_i}$$

- 38. What are the three quantization errors due to finite word length, registers in digital filters?**

1. Input quantization error.
2. Coefficient quantization error.
3. Product quantization error.

- 39. Write the two's complement of the following (a) +7 and (b) -7**

$$(a) +7_{10} = (0111)_2$$

$$(b) -7_{10} = \text{one's complement} \Rightarrow 1000$$

$$= \text{two's complement} \Rightarrow 1001$$

$$-7_{10} = (1001)_2$$

- 40. What is the steady-state noise power due to quantization if the number of bits is  $b$ ?**

Steady-state noise power

$$\sigma_e^2 = \frac{2^{-2b}}{12}$$

where  $b$  is the number of bits excluding sign bit.

- 41. Why rounding is preferred to truncation in realizing digital filter?**

Rounding is preferred to truncation due to the following desirable characteristics of rounding.

1. The rounding error is independent of the type of arithmetic.
2. The mean value of rounding error signal is zero.
3. The variance of rounding error signal is least.

42. Express the fraction  $(-7/32)$  in signed magnitude and two's complement notation using 6 bits?

$$\begin{aligned} \left( \frac{-7}{32} \right)_{10} &= -(0.21875)_{10} \\ \text{signed magnitude: } \left( \frac{-7}{32} \right)_{10} &= (1.0011)_2 \\ \text{two's complement: } \left( \frac{-7}{32} \right)_{10} &= (1.11001)_2 \end{aligned}$$

43. Identify the various factors which degrade the performance of the digital filter implementation when finite word length is used.

1. Error due to quantization of the input data.
2. Error due to quantization of the filter coefficients.
3. Error due to rounding the product in multiplication.
4. Limit cycles due to product quantization and overflow in addition.

44. Express the fraction  $(7/8)$  and  $(-7/8)$  in sign magnitude and two's complement and one's complement.

$$\begin{aligned} \text{Fraction: } \left( \frac{7}{8} \right)_{10} &= (0.111)_2 \text{ in sign magnitude} \\ &= (0.111)_2 \text{ is one's and two's complement} \\ \text{Fraction: } \left( -\frac{7}{8} \right)_{10} &= (1.111)_2 \text{ in sign magnitude} \\ &= (1.000)_2 \text{ in one's complement} \\ &= (1.001)_2 \text{ in two's complement} \end{aligned}$$

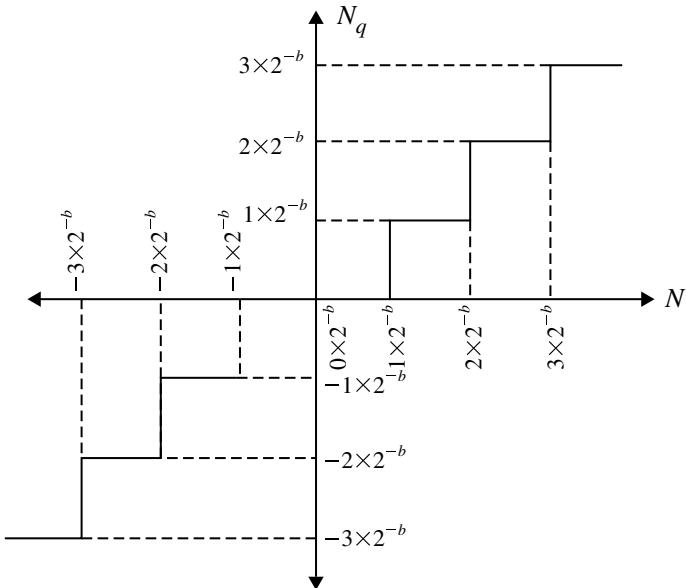
45. What are the different quantization methods?

The common methods of quantization are (1) truncation and (2) rounding.

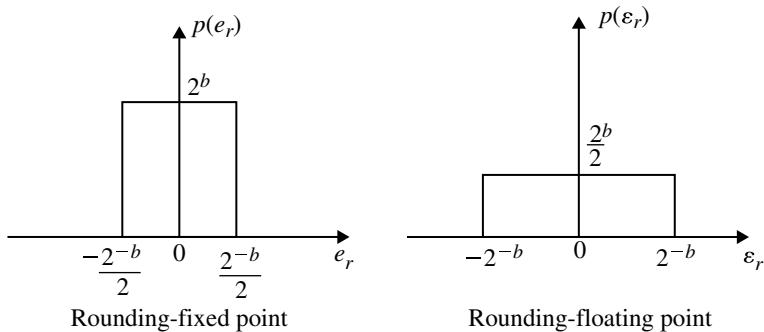
46. Plot the truncation error for sign magnitude and two's complement numbers.
47. Give the expression for signal to quantization noise ratio and calculate the improvement with an increase of 2 bits to the existing bit.

$$\text{Signal to noise ratio (SNR)} = 6b - 1.24\text{dB}$$

With an increase of 2 bits, increase in SNR is approximately 12 dB (Figs. 5.24, 5.25, 5.26, 5.27, 5.28, 5.29, 5.30, 5.31, 5.32).

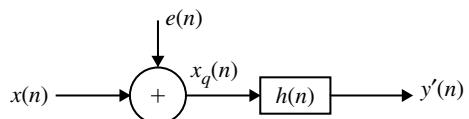


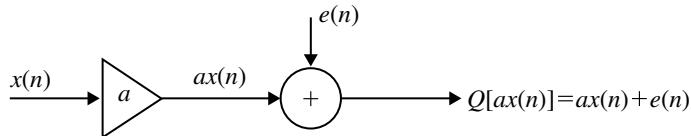
**Fig. 5.24** Characteristic of quantizer used for truncation (for question 12)



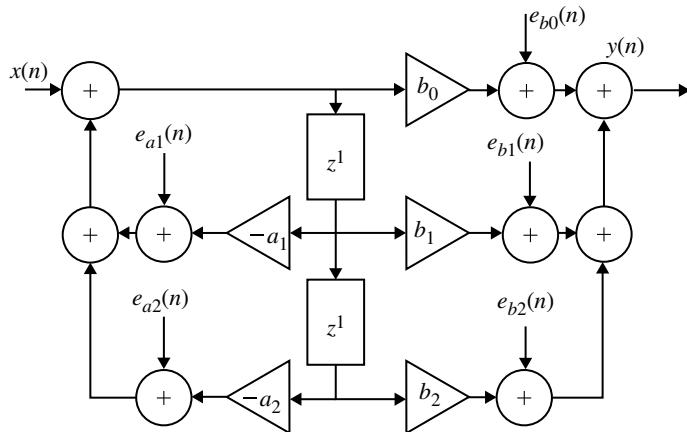
**Fig. 5.25** Noise probability of density function for rounding (for question 14)

**Fig. 5.26** Representation of input quantization noise (for question 17)

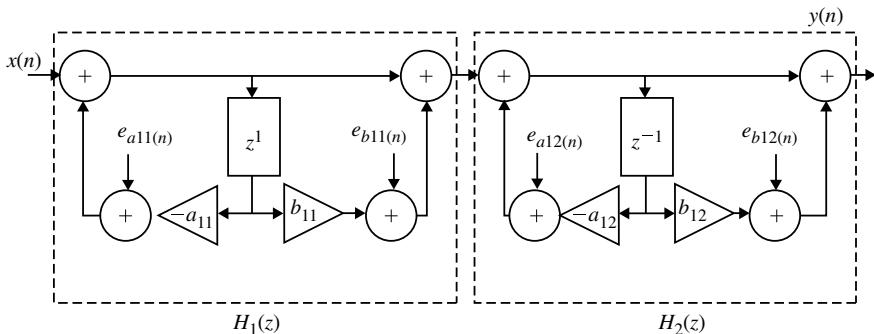




**Fig. 5.27** Statistical model of fixed point product quantization (question 23)



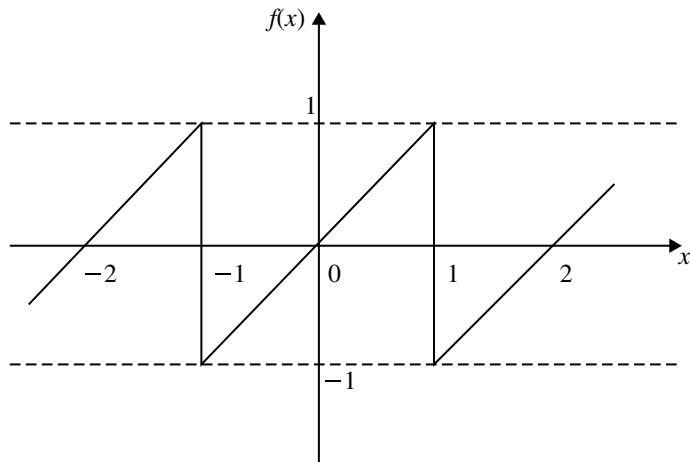
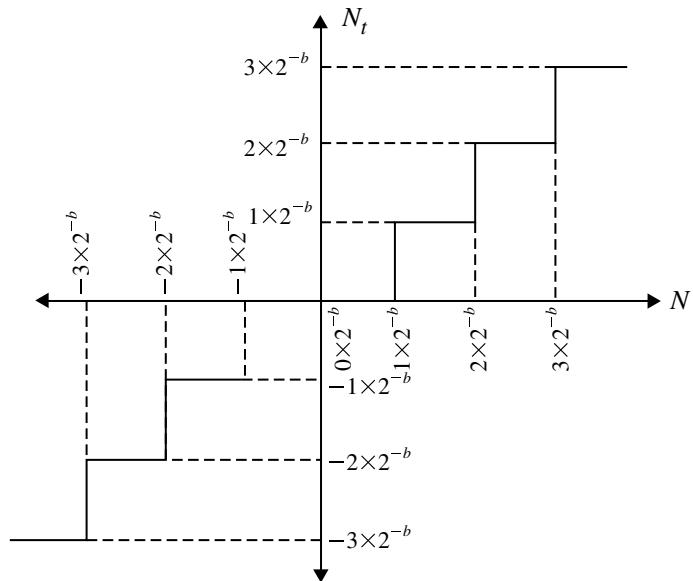
**Fig. 5.28** Product quantization noise model of second-order IIR system (question 24)

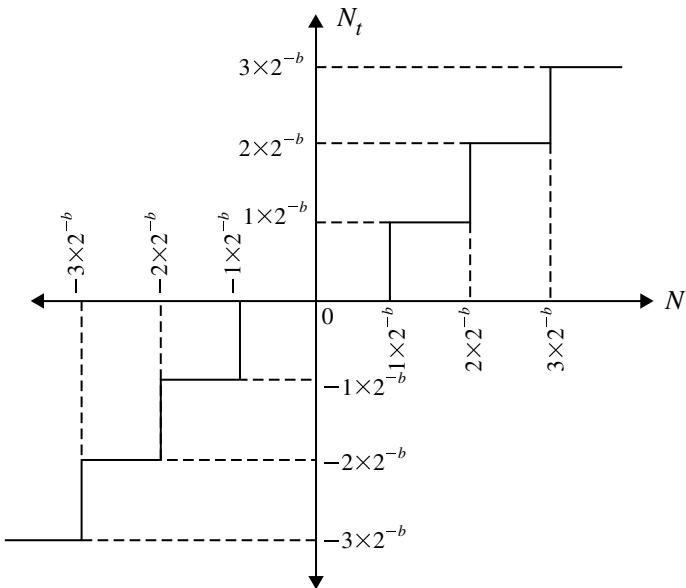


**Fig. 5.29** product quantization noise model of second-order IIR system (question 25)

### Long Answer Type Questions

1. Explain coefficient quantization effects in direct form realization of IIR filter.
2. Explain about fixed point and floating point representations.
3. Derive the steady-state noise power at the output of an LTI system due to quantization at the input.

**Fig. 5.30** Transfer characteristics of two's complementary adder (question 31)**Fig. 5.31** Two's complement with truncation error (question 46)



**Fig. 5.32** Sign complement

4. Explain the aliasing effect in the down sampling process if the original spectrum is not band limited to  $\omega = \pi/M$ .
5. Prove that the up sampler is time varying system.
6. Illustrate the characteristics of a limit cycle oscillation of a typical first-order system.
7. Enumerate with suitable examples the truncation and rounding errors.
8. The output of an A/D converter is applied to a digital filter with the system function

$$H(z) = \frac{0.53}{(z - 0.5)}$$

Find the output noise power from the digital filter when the input signal is quantized to have eight bits.

**Ans:**

$$\sigma_{e0}^2 = 1.675 \times 10^{-6}$$

9. What do you mean by down sampling?
10. Obtain the spectrum (expression) of the down sampled signal.
11. Plot the spectra of any signal  $x(n)$  and its down sampled version.
12. Discuss on efficient transversal structure for decimator and interpolator.

13. Consider the truncation of negative fraction numbers represented in  $(\beta + 1)$  bit fixed point binary form including sign bit. Let  $(\beta - b)$  bits be truncated. Obtain the range of truncation error for signed magnitude, two's complements and one's complement representations of the negative numbers.

**Ans:**

$$\text{Sign magnitude: } = 0 \leq e < -2^{-(\beta-b)}$$

$$\text{One's complement: } = 0 \leq e < -2^{-(\beta-b)}$$

$$\text{Two's complement: } = 0 \geq e > -2^{-(\beta-b)}$$

14. An 8 bit ADC feeds a DSP system characterized by the following transfer function

$$H(z) = \frac{1}{(z + 0.5)}$$

Estimate the steady-state quantization noise power at the output of the system.

**Ans:**

$$\boxed{\sigma_{e0}^2 = 6.7813 \times 10^{-6}}$$

15. The coefficients of a system defined by

$$H(z) = \frac{1}{(1 - 0.4z^{-1})(1 - 0.55z^{-1})}$$

are represented in a number system with a sign bit and 3 data bits using signed magnitude representation and truncation. Determine the new pole locations for direct realization and for cascade realization of first-order systems.

**Ans:**

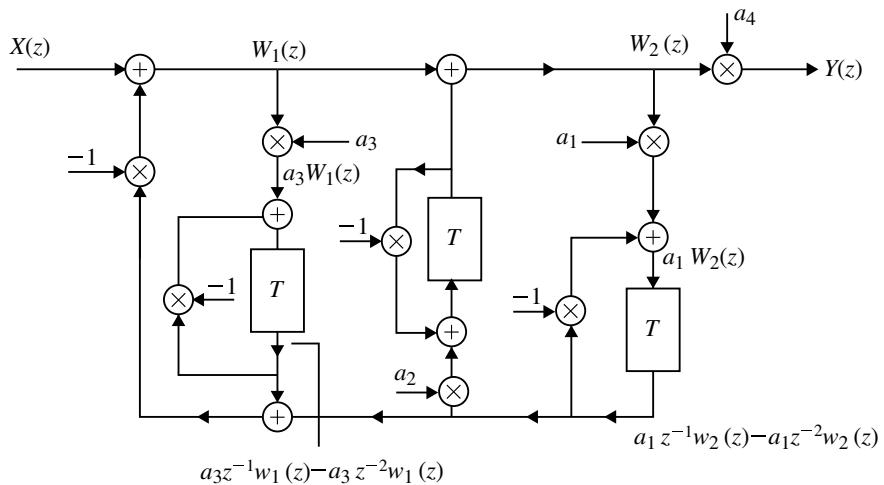
Direct form realization = New pole locations are  $z_1 = 0.695$ ,  $z_2 = 0.1798$

Cascade form realization = New pole locations are  $z_1 = 0.5$ ,  $z_2 = 0.375$

16. Draw the product quantization noise model of second-order IIR system.  
 17. Find the output round off noise power for the system having transfer function

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.4z^{-1})}$$

Which is realized in cascade form. Assume word length is 4 bits.



**Fig. 5.33** Determination of  $H(z)$  (question 19)

**Ans:**

$$\sigma_{e0}^2 = 4.648 \times 10^{-3}$$

18. Explain signal scaling to prevent overflow limit cycle in the second-order digital filter implementation.
19. Determine  $H(z) = \frac{Y(z)}{X(z)}$  for Fig. 5.33.

**Ans:**

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= a_4 \left/ \left[ 1 + (a_1 + a_1 a_2 + a_3)z^{-1} - (-a_1 + 2a_1 a_2 + a_1 a_2 a_3 - a_3)z^{-2} \right. \right. \\ &\quad \left. \left. + (a_1 a_2 - a_1 a_2 a_3)z^{-3} - (3a_1 a_2 a_3)z^{-4} + (a_1 a_2 a_3)z^{-5} \right] \right. \end{aligned}$$

20. For the given T.F.  $H(z) = H_1(z)H_2(z)$  where

$$H_1(z) = \frac{1}{(1 - 0.5z^{-1})} \quad \text{and} \quad H_2(z) = \frac{1}{(1 - 0.4z^{-1})},$$

find the output round off noise power. Calculate the value if  $b = 3$  (excluding sign bit). What is the round off noise power if the system realized is in direct form?

**Ans:**

$$\sigma_{e0}^2 = 6.1968 \times 10^{-3}$$

21. Find the effect of coefficient quantization on pole location of the given second-order IIR system when it is realized in direct form-I and in cascade form. Assume a word length of 4 bits through truncation.
22. Explain the effects of word length in FIR digital filters.
23. Describe briefly about limit cycle oscillations in recursive systems.
24. Determine the variance of the round off noise at the output of the two cascade realization of the filter with system function.

$$H(z) = H_1(z)H_2(z)$$

$$H_1(z) = \frac{1}{(1 - 0.5z^{-1})}$$

$$H_2(z) = \frac{1}{(1 - 0.25z^{-1})}$$

**Ans:**

$$\sigma_{e0}^2 = 2.8953\sigma_e^2, \quad \sigma_e^2 = \frac{2^{-2b}}{12}$$

25. Explain in detail about finite word length effect in the digital filter design.
26. Explain fixed point representation of binary numbers.

# Chapter 6

## Multi-rate Digital Signal Processing



### Learning Objectives

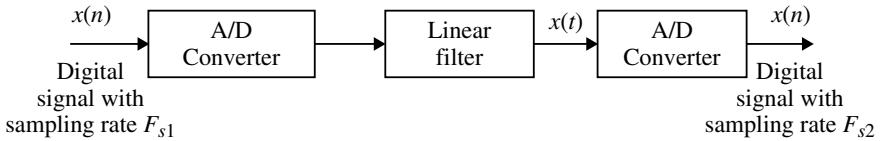
After completing this chapter, you should be able to:

- ☒ understand the concept of multi-rate digital signal processing and its application.
- ☒ understand decimation by integer factors (downsampling).
- ☒ understand interpolation by integer factors (sampling rate increase-upsampling).
- ☒ represent the spectrum of down and upsampled signals.
- ☒ implement polyphase structure of decimator and interpolator.
- ☒ understand the various applications of MDSP.

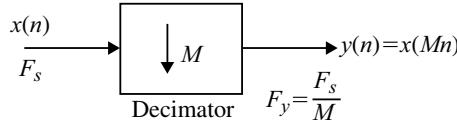
### 6.1 Introduction

Modern digital systems require to process data at more than one sampling rate. Systems that use single sampling rate from A/D converter to D/A converter are called single rate systems. The discrete systems that process data at more than one sampling rate are known as multi-rate systems and the processing of signals by these systems is called multi-rate digital signal processing (MDSP). In many practical applications such as digital audio and video, different sampling rates are used. This is achieved using an upsampler and downampler. In MDSP, this is achieved by interpolation and decimation. The process of decimation and interpolation are the fundamental operations in DSP. The sampling frequency is increased or decreased without any undesirable effects of errors due to quantization and aliasing.

In multi-rate signal processing, the sampling rate conversion is generally done by two methods. In the first method, sampling rate conversion is achieved by D/A and A/D converters as shown in Fig. 6.1a.



(a) Sampling rate conversion using D/A and A/D converters.



(b) Block diagram of a decimator.

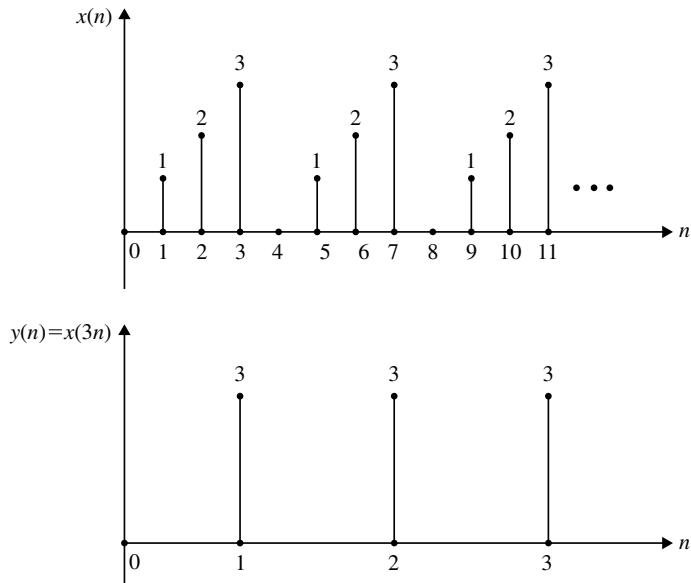
**Fig. 6.1** **a** Sampling rate conversion using D/A and A/D converters. **b** Block diagram of a decimator

In this method, the digital signal is passed through a D/A converter, filtered if necessary and then re-sampled at the desired sampling rate using A/D converter. In the second method, the sampling rate is achieved through interpolator and decimator depending upon the sampling rate requirement. The fundamental operations of interpolation and decimation of an MDSP are discussed in this chapter.

## 6.2 Advantages and Applications of Multi-rate Signal Processing

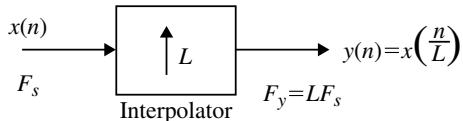
Advantages and applications of multi-rate signal processing are many. They are listed below:

1. In speech processing, the use of MDSP reduces the transmission rate of speech data. The original speech is reconstructed from the low bit rate representation.
2. In data acquisition and storage systems, signals of different bandwidths which require different sampling frequencies are efficiently handled without anti-aliasing analog filters.
3. MDSP efficiently implements DSP functions such as implementation of narrow band digital FIR filters with less computational requirements.
4. It is used in the compact disk player and simplifies the use of D/A conversion process without loss of quality of sound.
5. It is used in the acquisition of high quality data, high resolution spectral analysis and design and implementation of narrow band digital filters.
6. MDSP is widely used in antenna systems, communication systems and radar systems.



**Fig. 6.2** Downsampling the signal for  $M = 3$ , illustrated

**Fig. 6.3** Block diagram representation of an upsampler



The block diagram of an upsampler is shown in Fig. 6.3.

### 6.3 Downsampling (Decimator)

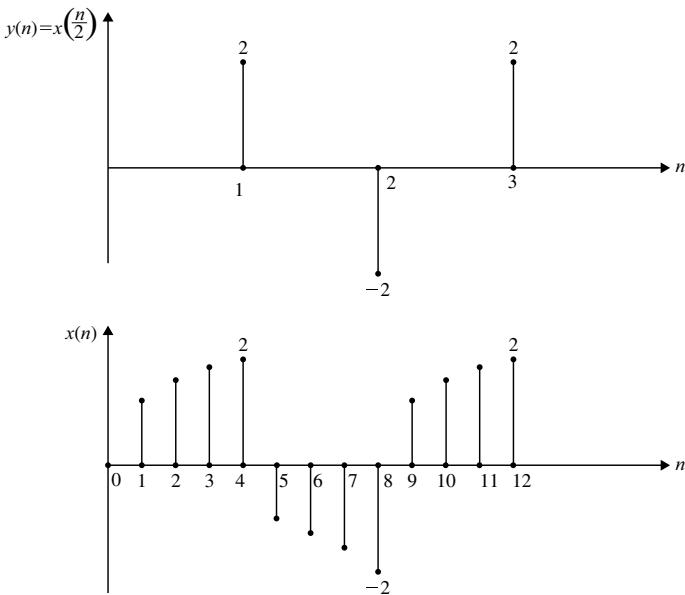
Downsampling of a signal is the process of decimating a signal  $x(n)$  by an integer factor  $M$  and this is represented in Fig. 6.1b. The sampling rate compressor is represented by a down arrow along with the decimation factor  $M$ . The output

$$y(n) = x(Mn)$$

Here, the sampling rate reduction is achieved by discarding  $(M - 1)$  samples for every  $M$  samples. Figure 6.2 shows the downsampling for  $M = 3$  (Fig. 6.3).

### 6.4 Upsampling (Interpolator)

Upsampling is the process of increasing the sampling frequency of the signal. Upsampling is also called sampled rate expander or interpolator. The interpolator is repre-



**Fig. 6.4** Illustration of upsampled DT signal for  $L = 4$

sented by an upward arrow along with the interpolation factor  $L$ . While the input sampling frequency  $F_s$  is changed as  $F_s/M$  in downsampling, it is changed as  $F_sL$  by upsampling. The upsampling of  $x(n)$  is illustrated in Fig. 6.4 for  $L = 4$ .

For upsampling, for each sample of  $x(n)$ , the interpolator includes  $(L - 1)$  zero valued samples to form the new signal at a rate of  $LF_s$ .

It is noted that for downsampling an anti-aliasing filter is to be connected before it is applied to a decimator. By this, the input  $x(n)$  is bandlimited to less than  $F_s/2M$ . Similarly when upsampling is done, the signal  $x(n)$  after passing through the interpolator (sample rate expander) it should be passed through anti-imaging filter. This is necessary to remove image frequencies created by the rate increase of  $L$  output samples.

## 6.5 Sampling Rate Conversion by Non-integer Factors Represented by Rational Number

Some applications of multi-rate digital signal processing require sampling rate by non-integer factor. When two different digital audio systems have different sampling rates, the transfer of data from one storage system to another require non-integer factor. For example, when the data from a compact disk at a rate of 44.1 kHz is transferred to a digital audio tape at 48 kHz, it requires data rate upsampling by a factor 48/44.1, which is a non-integer. However, this non-integer factor can be achieved

by the rational numbers  $L$  and  $M$  which are integers. Here, the sampling frequency change is achieved by an interpolator as shown in Fig. 6.5a with interpolation factor  $L$ . The output of the interpolator is passed through a LPF and the upsampling frequency is obtained as  $LF_s$ . This signal is passed through a downampler with a decimation factor  $M$ . It is necessary to pass the upsampled signal to pass through the LPF before it is applied to the downampler to avoid removal of some of the desired frequency components. Since the two LPF filters  $h_0(k)$  and  $h_D(k)$  which form part of the interpolator and downampler, respectively, are connected in cascade and are operated at same frequency rate they can be replaced by a single LPF  $h(k)$  as shown in Fig. 6.5b. Further, it should be noted that the interpolation process should precede decimation process to preserve the spectral characteristics of  $x(n)$ .

The sampling rate conversion by non-integer factors as applied to CD and digital audio tape can be explained as follows. Let us choose  $L = 160$ . The CD data rate is increased to  $LF_s = 160 \times 44.1 = 7056$  Hz. By choosing the decimation factor  $M = 147$ , the data rate of DAT becomes  $7056 \div 147 = 48$  kHz. Here,  $L$  and  $M$  are integers. Further, the ratios  $\frac{L}{M} = \frac{160}{147}$  is rational. It is to be noted that, in general if  $M < L$ , the operation is a decimation process and  $M > 1$ , it is integer interpolation process. The mathematical proof of the above concept is derived as shown below:

Let  $H(\omega_v)$  be the frequency response function of the lowpass filter  $h(k)$  which can be characterized as given below:

$$H(\omega_v) = \begin{cases} 1, & 0 \leq |\omega_v| \leq \min\left(\frac{\pi}{M}, \frac{\pi}{L}\right) \\ 0, & \text{otherwise} \end{cases} \quad (6.1)$$

The output of the upsample is given by the following sequence:

$$V(l) = \begin{cases} x\left(\frac{l}{L}\right), & l = 0, \pm L \pm 2L : \pm \dots \\ 0, & \text{otherwise} \end{cases} \quad (6.2)$$

The output of the LPF can be written by its convolution as

$$\begin{aligned} \omega(L) &= \sum_{k=-\infty}^{\infty} h(l-k)v(k) \\ &= \sum_{k=-\infty}^{\infty} h(l-kL)x(k) \end{aligned} \quad (6.3)$$

In the above equation  $h(k)$  is the impulse response of the LPF. The output  $y(m)$  is obtained by downsampling  $\omega(l)$  and the following equation is written.

$$\begin{aligned} y(m) &= \omega(mM) \\ &= \sum_{k=-\infty}^{\infty} h(mM - kL)x(k) \end{aligned} \quad (6.4)$$

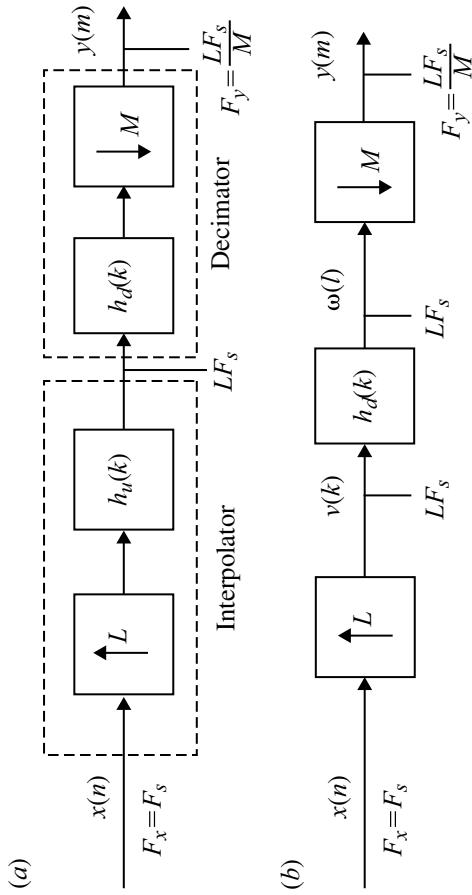


Fig. 6.5 Sampling rate conversion integer factors

Let

$$k = \left\lfloor \frac{mM}{L} \right\rfloor - n \quad (6.5)$$

Substituting Eq. (6.5) in (6.4) we get,

$$y(m) = \sum_{n=-\infty}^{\infty} x \left( \left\lfloor \frac{mM}{L} - n \right\rfloor \right) h \left( mM - \left\lfloor \frac{mM}{L} \right\rfloor L + nL \right) \quad (6.6)$$

Substituting the following

$$\begin{aligned} mM - \left\lfloor \frac{mM}{L} \right\rfloor L &= mM \text{ modulo } L \\ &= (mM)_I \end{aligned}$$

in Eq. (6.6) we get,

$$y(m) = \sum_{n=-\infty}^{\infty} x \left( \left\lfloor \frac{mM}{L} - n \right\rfloor \right) h(nL + (mM)_I) \quad (6.7)$$

Equation (6.7) shows that the output  $y(m)$  is obtained by passing through the time invariant filter with the following impulse response.

$$g(n, m) = h(nL + (mM)_I) \quad -\infty < m, n < \infty \quad (6.8)$$

In Eq. (6.8)  $h(k)$  is the impulse response of the time invariant LPF whose sampling rate is  $LF_s$ . Equation (6.8), for any integer  $k$ , can be written as follows:

$$\begin{aligned} g(n, m + kL) &= h(nL + (mM + kML)_I) \\ &= h(nL + (mM)_I) \\ &= g(n, m) \end{aligned} \quad (6.9)$$

This shows that  $g(n, m)$  is periodic with period  $L$ . The frequency response of the LPF can be written as

$$\begin{aligned} V(\omega_v) &= H(\omega_v)X(\omega_v L) \\ V(\omega_v) &= \begin{cases} LX(\omega_v L), & 0 \leq \omega_v \leq \min\left(\frac{\pi}{M}, \frac{\pi}{L}\right) \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The spectrum of the output sequence is expressed as

$$Y(\omega_y) = \frac{1}{L} \sum_{k=0}^{M-1} V\left(\frac{\omega_y - 2\pi k}{L}\right) \quad (6.10)$$

Since the linear filter prevents aliasing, Eq. (6.10) is written as,

$$Y(\omega_y) = \begin{cases} \frac{L}{M} X\left(\frac{\omega_y}{M}\right), & 0 \leq |\omega_y| \leq \min\left(\pi, \frac{\pi M}{L}\right) \\ 0, & \text{otherwise} \end{cases} \quad (6.11)$$

From Eq. (6.11) it is evident that the sampling rate conversion can be achieved by a factor  $\frac{L}{M}$  by first increasing the sampling rate by  $L$  accomplished by inserting  $(L - 1)$  zeros between successive values of the input sequence  $x(n)$ , followed by linear filtering of the resulting sequence to eliminate unwanted images of  $X(\omega)$  and finally by downsampling the filtered signal by a factor  $M$ .

### Example 6.1

A three stage decimator is used to reduce the sampling rate from 3072 kHz. The decimation factors are 16, 8 and 4. Draw the block diagram and indicate the sampling rate at the output of each stages.

**Solution** To retain all the desired frequency components of the signal to be down-sampled, it is necessary that the signal is passed through lowpass filter which is shown in Fig. 6.6, for three stages.

At the end of stage 1, the sampling rate is

$$F_{s1} = \frac{3072}{16} = 192 \text{ kHz}$$

At the end of the second stage the frequency sampling rate is

$$F_{s2} = \frac{192}{8} = 24 \text{ kHz}$$

At the end of the third stage the frequency sampling rate is

$$F_{s3} = \frac{24}{4} = 6 \text{ kHz}$$

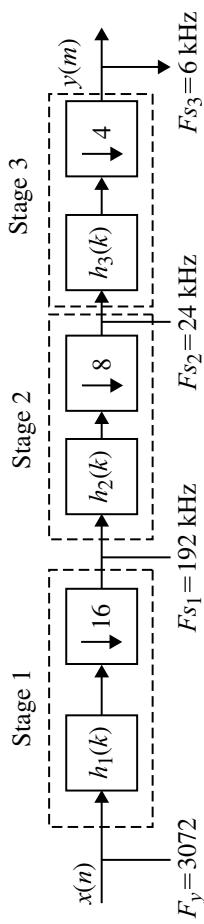


Fig. 6.6 Sampling rate reduction in three stages

## 6.6 Characteristics of Filter and Downampler

As already stated while a signal is downsampled, it is necessary to have anti-aliasing filter. By this the signal is band limited before it is applied to the downampler. The input sequence  $x(n)$  is passed through a lowpass filter whose impulse response is  $h(n)$  and the frequency response  $H_D(\omega)$  which satisfies the following condition:

$$H_D(\omega) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{M} \\ 0, & \text{otherwise} \end{cases} \quad (6.12)$$

Equation (6.12) implies that the spectrum  $X(\omega)$  is eliminated by the filter in the range  $\frac{\pi}{M} < \omega < \pi$  and allows the components in the range  $|\omega| \leq \frac{\pi}{M}$  for further processing. Now consider Fig. 6.7.

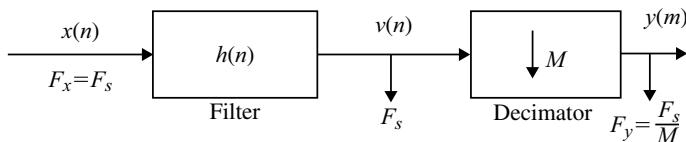
The output of LPF sequence  $v(n)$  is obtained using convolution as

$$v(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \quad (6.13)$$

The sequence  $v(n)$  is applied to the decimator and the output of the decimator is given by

$$\begin{aligned} y(m) &= v(mM) \\ &= \sum_{k=0}^{\infty} h(k)x(mM - k) \end{aligned} \quad (6.14)$$

It is well-known that the lowpass filter is linear and time invariant. However, the decimator even though has linear characteristics, is time varying. This is also true in the case of an interpolator. The mathematical proof of the above statement is given in the section to follow.



**Fig. 6.7** Representation of a decimator with filter

## 6.7 Linearity and Time Invariancy of Decimator and Interpolator

### 6.7.1 Linearity of Decimator

A discrete time system is linear if the weighted sum of the output is equal to the output due to the weighted sum of the input. The input-output of a decimator is written as

$$\begin{aligned} y(m) &= x(nM) \\ y_1(m) &= x_1(nM) \\ y_2(m) &= x_2(nM) \end{aligned} \tag{6.15}$$

The weighted sum of the output is,

$$\begin{aligned} y_3(m) &= a_1y_1(m) + a_2y_2(m) \\ &= a_1x_1(nM) + a_2x_2(nM) \end{aligned}$$

The output due to the weighted sum of the input is

$$\begin{aligned} y_4(m) &= a_1x_1(nM) + a_2x_2(nM) \\ \therefore y_3(m) &= y_4(m) \end{aligned}$$

The weighted sum of the output is equal to the output due to the weighted sum of the input. Hence, the system with a downsample is linear.

### 6.7.2 Linearity of an Interpolator

The input-output of an interpolator is written as

$$\begin{aligned} y(m) &= x\left(\frac{n}{L}\right) \\ y_1(m) &= x_1\left(\frac{n}{L}\right) \\ y_2(m) &= x_2\left(\frac{n}{L}\right) \end{aligned} \tag{6.16}$$

The weighted sum of the output is

$$\begin{aligned} y_3(m) &= a_1y_1(m) + a_2y_2(m) \\ &= a_1x_1\left(\frac{n}{L}\right) + a_2x_2\left(\frac{n}{L}\right) \end{aligned}$$

The output due to the weighted sum of the input is

$$y_4(m) = a_1x_1\left(\frac{n}{L}\right) + a_2x_2\left(\frac{n}{L}\right)$$

The weighted sum of the output is equal to the output due to the weighted sum of the input. Hence, the system with interpolator is linear.

### 6.7.3 Time Invariancy of a Decimator

A discrete time system is said to be time invariant if the delayed output  $y(n, n_0)$  where  $n_0$  is the delay, is equal to the output  $y(n - n_0)$  which is due to the delayed input.

Consider the decimator with the following equation:

$$y(n) = x(nM)$$

The delayed output is,

$$\begin{aligned} y(n - n_0) &= x((n - n_0)M) \\ &= x(nM - n_0M) \end{aligned}$$

The output due to the delayed input is

$$\begin{aligned} y(n, n_0) &= x(nM - n_0) \\ y(n, n_0) &\neq y(n - n_0) \end{aligned}$$

Hence, the decimator is a time varying system. However, it can be made time invariant if the delay  $n_0$  is made multiple of the decimation factor  $M$ .

### 6.7.4 Time Invariancy of an Interpolator

Let us consider the interpolator with the following equation:

$$y(n) = x\left(\frac{n}{L}\right)$$

The delayed output is

$$y(n, n_0) = x\left(\frac{n - n_0}{L}\right)$$

The output due to the delayed input is

$$\begin{aligned} y(n - n_0) &= x\left(\frac{n}{L} - n_0\right) \\ y(n, n_0) &\neq y(n - n_0) \end{aligned}$$

Hence, the interpolator is a time varying system.

## 6.8 Spectrum of Downsampled Signal

Consider Fig. 6.7 where the downsampler together with the LPF is connected. The frequency response of the output sequence  $y(m)$  with that of the input sequence  $x(n)$  is obtained as given below. The sequence  $v(n)$  is defined as follows:

$$\bar{v}(n) = \begin{cases} v(n), & n = 0, \pm M, \pm 2M, \dots \\ 0, & \text{otherwise} \end{cases} \quad (6.17)$$

$\bar{v}(n)$  is interpreted as a sequence obtained by multiplying  $v(n)$  by train of impulses  $p(n)$  with period  $M$ .  $p(n)$  can be expressed in discrete Fourier series as

$$p(n) = \frac{1}{M} \sum_{k=0}^{M-1} e^{j2\pi kn/M} \quad (6.18)$$

Also,

$$\bar{v}(n) = v(n)p(n) \quad (6.19)$$

$$\begin{aligned} y(m) &= \bar{v}(mM) \\ &= v(mM)p(mM) \\ &= v(mM) \end{aligned} \quad (6.20)$$

The  $z$ -transform of the output sequence is,

$$\begin{aligned} Y(z) &= \sum_{m=-\infty}^{\infty} y(m)z^{-m} \\ &= \sum_{m=-\infty}^{\infty} \bar{v}(mM)z^{-m} \end{aligned} \quad (6.21)$$

Since  $\bar{v}(m) = 0$  except at multiples of  $M$ , the above equation is written as:

$$Y(z) = \sum_{m=-\infty}^{\infty} \bar{v}(m) z^{-\frac{m}{M}}$$

Using Eqs. (6.18), (6.19) and (6.21) we get,

$$\begin{aligned} Y(z) &= \sum_{m=-\infty}^{\infty} v(m) \left[ \frac{1}{M} \sum_{k=0}^{M-1} e^{j2\pi mk/M} \right] z^{-\frac{m}{M}} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{m=-\infty}^{\infty} v(m) (e^{-j2\pi k/M} z^{-\frac{1}{M}})^{-m} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} V(e^{-j2\pi k/M} z^{\frac{1}{M}}) \end{aligned}$$

Making use of the property that  $V(z) = H_D(z)X(z)$  the above equation is written as

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_D(e^{-j2\pi k/M} z^{\frac{1}{M}}) X(e^{-j2\pi k/M} z^{\frac{1}{M}}) \quad (6.22)$$

The spectrum of the output  $y(m)$  is obtained by evaluating  $Y(z)$  in the unit circle. Denoting the frequency variable of  $y(m)$  by  $\omega_y$ , Eq. (6.22) becomes

$$Y(\omega_y) = \frac{1}{M} \sum_{k=0}^{M-1} H_D \frac{(\omega_y - 2\pi k)}{M} X \frac{(\omega_y - 2\pi k)}{M} \quad (6.23)$$

where  $\omega_y$  is expressed in rad/s.

Also

$$\begin{aligned} \omega_y &= 2\pi F_y \\ \omega_y &= M\omega_x \end{aligned}$$

If aliasing is eliminated by proper design of LPF, Eq. (6.23) is written as:

$$Y(\omega_y) = \frac{1}{M} H_D \left( \frac{\omega_y}{M} \right) X \left( \frac{\omega_y}{M} \right) \quad (6.24)$$

For  $0 \leq |\omega_y| \leq \pi$ , the above equation is written as:

$$\boxed{Y(\omega_y) = \frac{1}{M} X\left(\frac{\omega_y}{M}\right)} \quad (6.25)$$

The spectra of  $x(n)$ ,  $h(n)$ ,  $v(n)$  and  $y(m)$  are shown in Fig. 6.8.

## 6.9 Effect of Aliasing in Downsampling

The signal  $x(n)$  is passed through a lowpass filter before it is passed through a decimator. The function of this anti-aliasing filter is to prevent aliasing of the sampled signal. Aliasing refers to distortion of the signal spectrum due to low sampling rate. Now consider the downampler without the LPF in Fig. 6.7. From Eq. (6.22), the  $z$ -transform of the output is obtained as

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{-j2\pi k/M} z^{\frac{1}{M}}) \quad (6.26)$$

The frequency spectrum of the above equation is obtained by substituting  $z = e^{j\omega}$

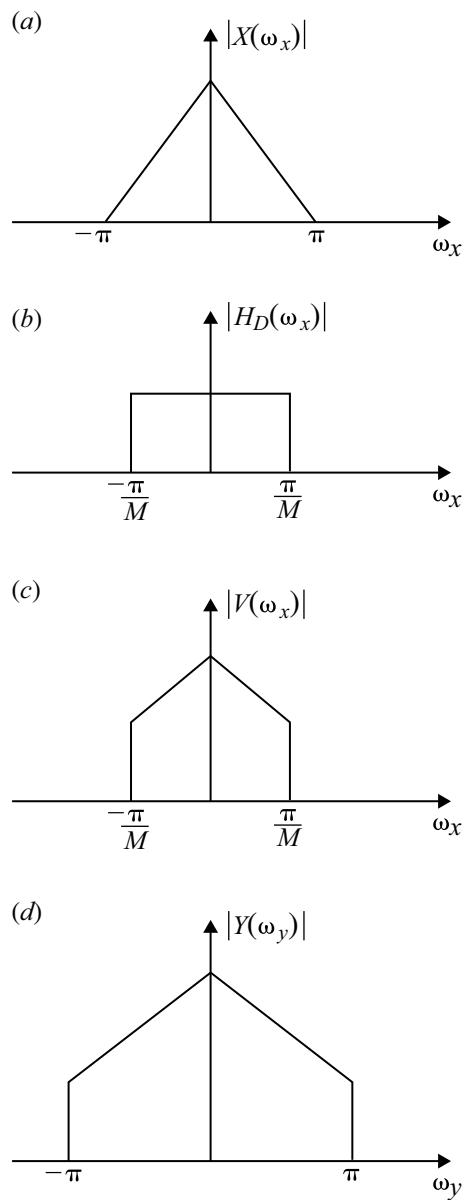
$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{\frac{-j2\pi k}{M}} e^{\frac{j\omega}{M}}) \quad (6.27)$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(e^{\frac{j(\omega-2\pi k)}{M}}) \quad (6.28)$$

The plots of  $|X(e^{j\omega})|$  and  $|Y(e^{j\omega})|$  with respect to  $\omega_x$  and the plot of  $|Y(e^{j\omega})|$  with respect to  $\omega_y$  are shown in Fig. 6.9. The spectrum of  $X(e^{j\omega})$  is a periodic one with period  $2\pi$ . The plot of  $|Y(e^{j\omega})|$  with respect to  $\omega_x$  is nothing but the stretched version of  $|X(e^{j\omega})|$  and scaled by a factor  $\frac{1}{M}$ . In the interval 0 to  $2\pi$ , there will be  $(M - 1)$  equally spaced replica of  $X(e^{j\omega})$  with period  $\frac{2\pi}{M}$ . This is shown in Fig. 6.9b. Using the relation  $F_y = MF_x$ , the plot of  $|Y(e^{j\omega})|$  drawn for  $\omega_y$  is shown in Fig. 6.9c.

Now consider the signal  $x(n)$  whose frequency is greater than  $\pm \frac{\pi}{M}$ . The frequency spectra of  $|X(e^{j\omega})|$  and  $|Y(e^{j\omega})|$  are plotted as shown in Fig. 6.10a, b, respectively. In Fig. 6.10b the aliasing effect is shown. Because of aliasing signal distortion in  $y(m)$  will take place and hence a lowpass filter is to be connected before the signal  $x(n)$  is passed through the downampler. This will limit the input signal to the downampler to  $\pm \frac{\pi}{M}$  which avoids aliasing and hence signal distortion.

**Fig. 6.8** Frequency spectra of decimated signal



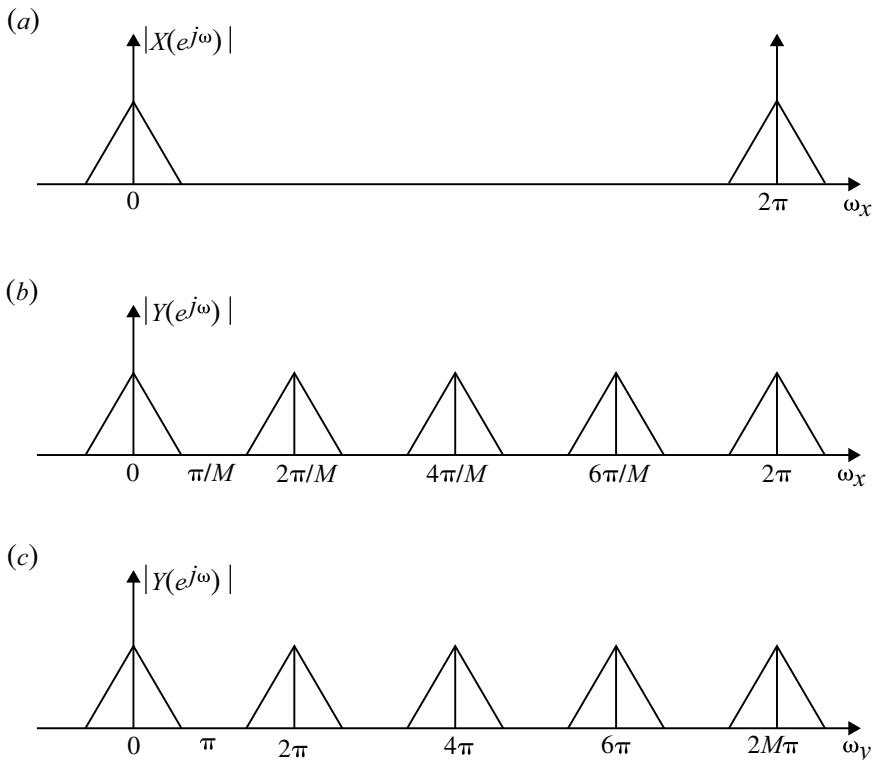
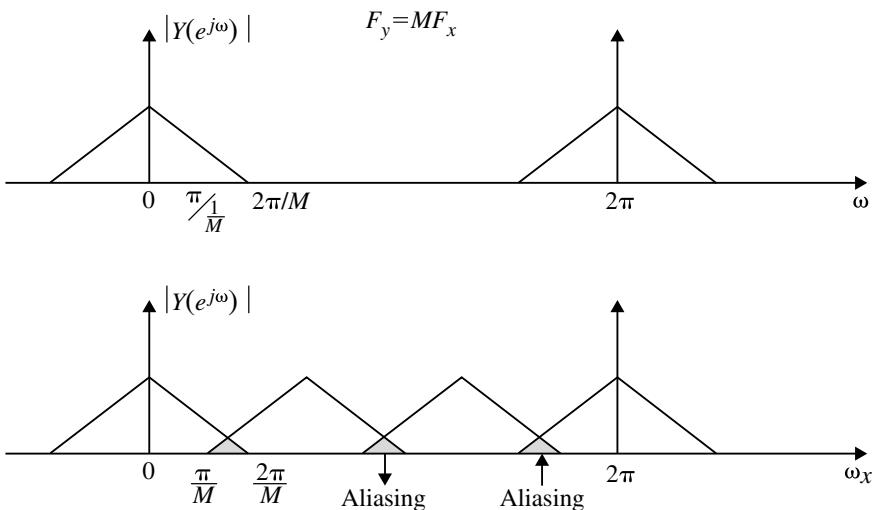
Fig. 6.9 Magnitude of  $X(e^{j\omega})$  and  $Y(e^{j\omega})$ 

Fig. 6.10 Aliasing effect in decimator

## 6.10 Spectrum of Upsampling Signal

The increase in sampling rate is achieved by an interpolator or an upsampler. The output of the interpolator is passed through the LPF to avoid removal of some of the desired frequency components of the sampled signal. Let  $v(m)$  be the sequence obtained from the interpolator.  $v(m)$  is obtained by adding  $(L - 1)$  zeros between successive values of  $x(n)$ . If  $F_s$  is the sampling rate of  $x(n)$ , then the sampling rate of  $v(m)$  is  $LF_s$  which is same as the sampling rate of  $y(m)$ . The interpolator system is shown in Fig. 6.11a.  $v(m)$  is characterized by the following equation:

$$v(m) = \begin{cases} x\left(\frac{m}{L}\right), & m = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases} \quad (6.29)$$

The  $z$ -transform of the above equation is written as:

$$\begin{aligned} V(z) &= \sum_{m=-\infty}^{\infty} v(m)z^{-m} \\ &= \sum_{m=-\infty}^{\infty} x(m)z^{-mL} \\ &= X(z^L) \end{aligned} \quad (6.30)$$

The frequency spectrum of  $v(m)$  is obtained by evaluating  $V(z)$  on the unit circle ( $z = e^{j\omega}$ ) in the  $z$ -plane. Thus,

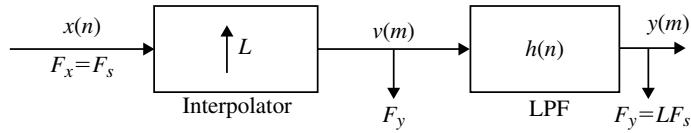
$$\begin{aligned} V(e^{j\omega}) &= X(e^{j\omega y L}) \\ \text{or } V(\omega_y) &= X(\omega_y L) \end{aligned} \quad (6.31)$$

where  $\omega_y = 2\pi F_y$  and  $L\omega_y = \omega_x$ .

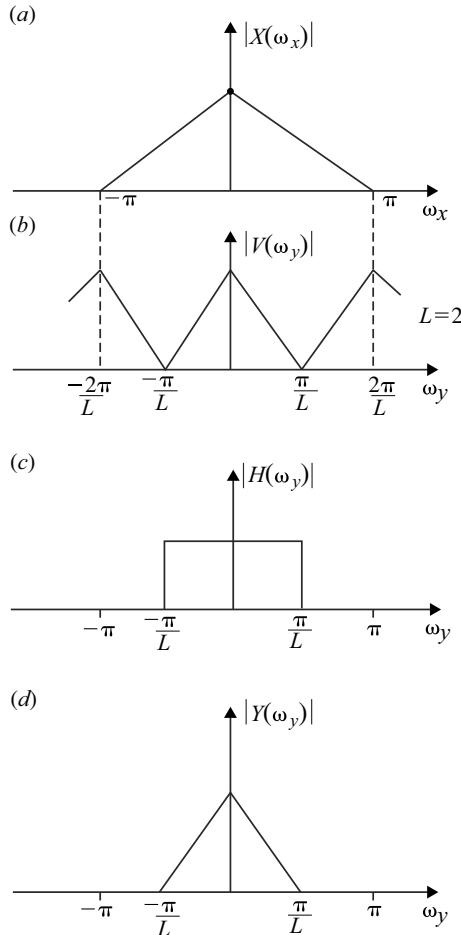
The spectra of  $|X(\omega_x)|$ ,  $|V(\omega_y)|$ ,  $|H(\omega_y)|$  and  $|Y(\omega_y)|$  are shown in Fig. 6.11b. From Fig. 6.11b it is observed that  $(L - 1)$  zero samples are added between successive values of  $x(n)$ .

### 6.10.1 Anti-imaging Filter

In the interpolation process when the sampling rate is increased by a factor  $L$ , the sampling frequency is also increased by the same factor. For each sample of the input signal  $x(n)$ , the interpolator inserts  $(L - 1)$  zero valued samples to form the new signal  $v(m)$  at the rate  $LF_s$  where  $F_s$  is the sampling frequency of  $x(n)$ . However  $v(m)$  contains images created by the rate increase and hence it becomes necessary to remove these image frequencies by passing through the LPF and the output  $y(m)$  is



(A) Interpolator with a filter.

(B) Frequency spectra of  $x(n)$ ,  $v(n)$ ,  $h(n)$  and  $y(m)$ .**Fig. 6.11** **a** Interpolator with a filter. **b** Frequency spectra of  $x(n)$ ,  $v(n)$ ,  $h(n)$  and  $y(m)$

obtained which has a sampling rate  $L F_s$ . The insertion of  $(L - 1)$  zero valued samples spreads the energy of each signal sample over  $L$  output samples. This interpolation process is called anti-image filtering which is shown in Fig. 6.11a.

### Example 6.2

Plot the spectra of any signal  $x(n)$  and its downsampled version for a sampling rate  $M = 2$ .

**Solution** From Eq. (6.28) the frequency response of a downampler is given by

$$Y(e^{j\omega}) = \sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

For  $M = 2$ , the above equation is written as:

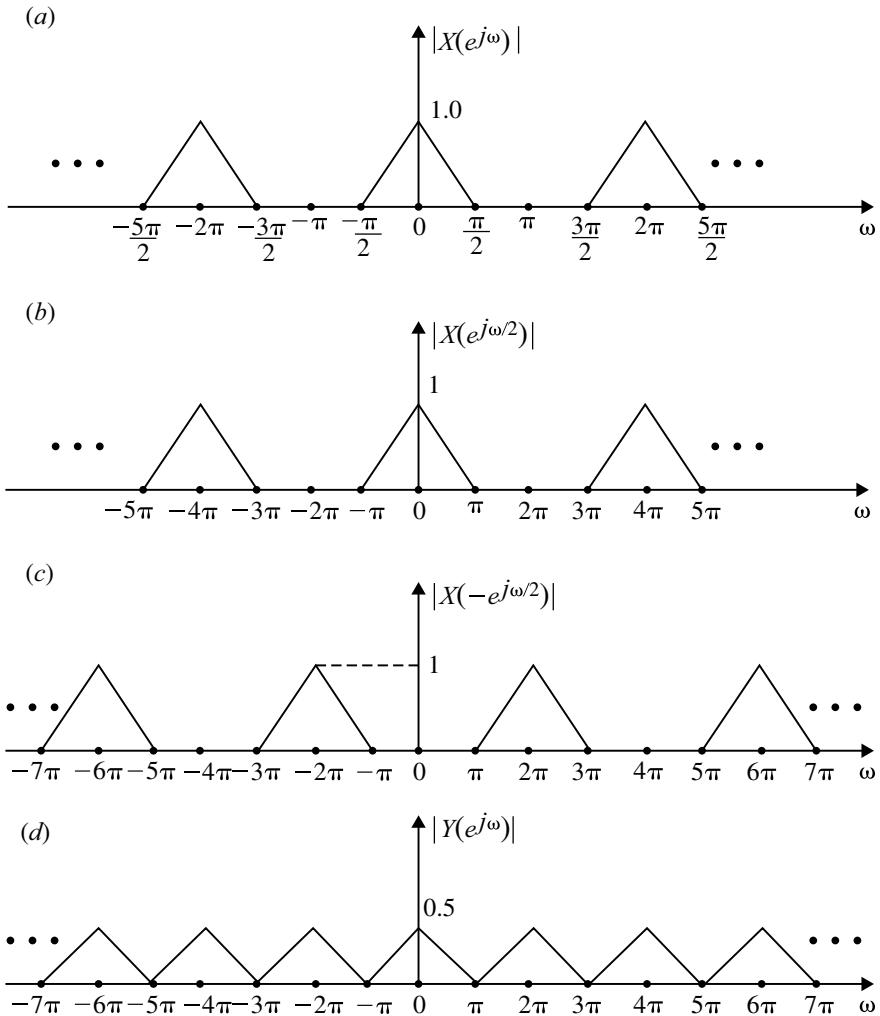
$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2} \sum_{k=0}^1 X(e^{j(\omega-2\pi k)/2}) \\ &= \frac{1}{2}[X(e^{j\omega/2}) + X(e^{j(\omega-2\pi)/2})] \\ &= \frac{1}{2}[X(e^{j\omega/2}) + X(e^{j\omega/2}e^{-j\pi})] \\ &= \frac{1}{2}[X(e^{j\omega/2}) + X(-e^{j\omega/2})] \end{aligned}$$

The plots of  $X(e^{j\omega})$ ,  $X(e^{j\omega/2})$ ,  $X(-e^{j\omega/2})$  and  $Y(e^{j\omega})$  are shown in Fig. 6.12.

The spectrum  $|X(e^{j\omega})|$  of an arbitrary sequence  $x(n)$  is shown in Fig. 6.12a. The plot of  $|X(e^{j\omega/2})|$  is shown in Fig. 6.12b whose frequency is expanded by a factor 2 and its maximum amplitude remains the same as 1. The plot of  $|X(-e^{j\omega/2})|$  is nothing but the spectrum of  $|X(e^{j\omega/2})|$  shifted by  $2\pi$  and is shown in Fig. 6.12c.  $|Y(e^{j\omega})|$  is obtained by adding Fig. 6.12b, c with the amplitude being divided by the factor 2 and is shown in Fig. 6.12d. It is evident from Fig. 6.12d, that if the signal is band limited to  $\frac{\pi}{2}$  there is no overlapping in  $Y(e^{j\omega})$  and there won't be any aliasing. If the signal frequency is greater than  $\frac{\pi}{2}$ , aliasing occurs. To avoid aliasing, in general, the signal before downsampled, should be band limited to  $\frac{\pi}{M}$ .

### Example 6.3

Draw the spectra of any signal  $x(n)$  and its upsampled version for a sampling rate of  $L = 3$ .

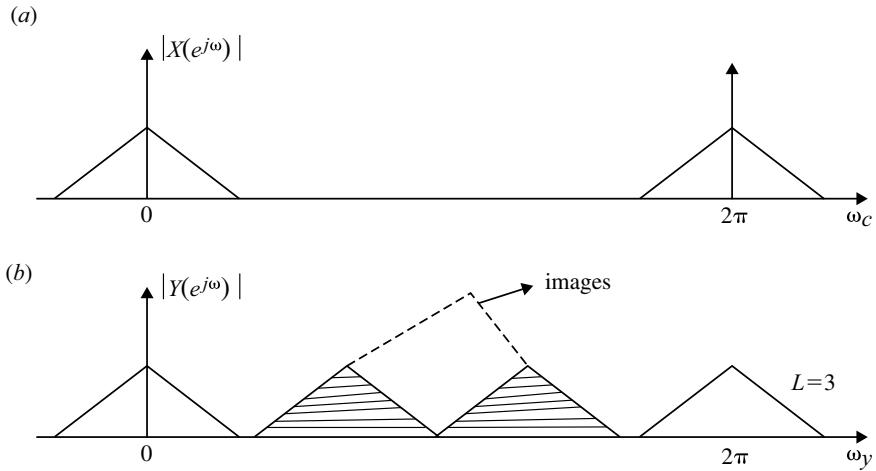
**Fig. 6.12** Spectrum of  $x(n)$ 

**Solution** Recall Eq. (6.31) which gives the frequency response for any input  $x(n)$  of an upsampler.

$$Y(e^{j\omega}) = X(e^{j\omega L})$$

For  $L = 3$ , the above equation is written as:

$$Y(e^{j\omega}) = X(e^{j3\omega})$$



**Fig. 6.13** Magnitude spectrum of  $X(e^{j\omega})$  and  $Y(e^{j\omega})$

$X(e^{j3\omega})$  implies that  $X(e^{j\omega})$  is repeated 3 times. The plot of  $X(e^{j\omega})$  is plotted as shown in Fig. 6.13a for any arbitrary signal  $x(n)$  which has a periodicity  $2\pi$ . In the frequency interval  $0 < \omega_y < 2\pi$ , there should be  $L - 1 = 3 - 1 = 2$  images of  $X(e^{j\omega})$  which has the same spectrum as  $x(n)$  with appropriate scaling factor. This is represented in Fig. 6.13.

## 6.11 Efficient Transversal Structure for Decimator

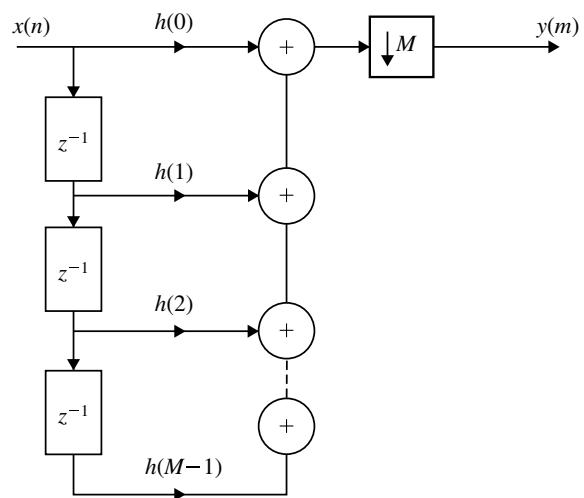
The decimator consists of an anti-aliasing filter and a downampler. The anti-aliasing filter can be realized using a direct form structure as shown in Fig. 6.14. In this configuration, the filter is operating at a high sampling rate  $F_s$  while only one output out of every  $M$  output samples is required and such a decimator is inefficient. Now consider the decimator shown in Fig. 6.15.

In this structure all the multiplications and additions are performed at lower sampling rate  $\frac{F_s}{M}$  and works more efficiently.

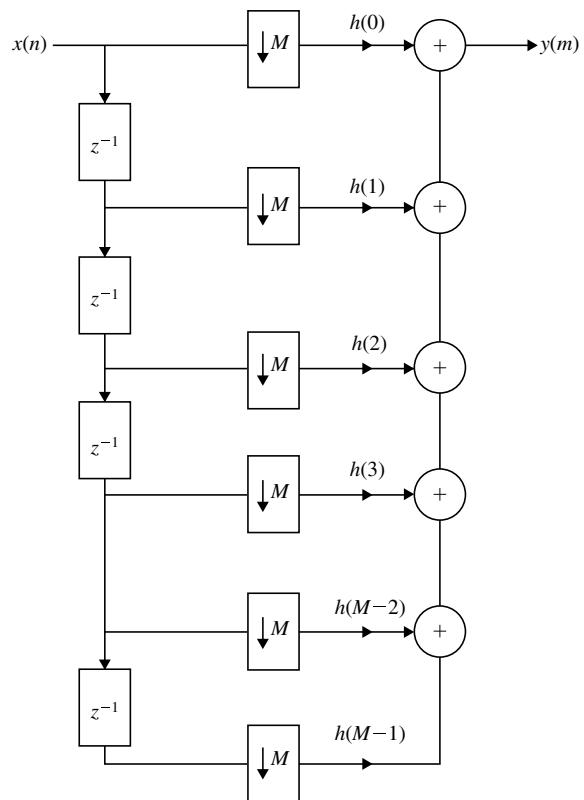
## 6.12 Efficient Transversal Structure for Interpolator

The interpolator inserts  $(L - 1)$  zeros between samples of the sequence  $x(n)$  and filtered. FIR filter is normally used for filtering. The upsampler connected to the transposed direct form FIR filter is shown in Fig. 6.16. If the sampling rate of  $x(n)$  is  $F_s$ , the input sampling rate at the input of FIR filter is  $LF_s$  and hence filter computa-

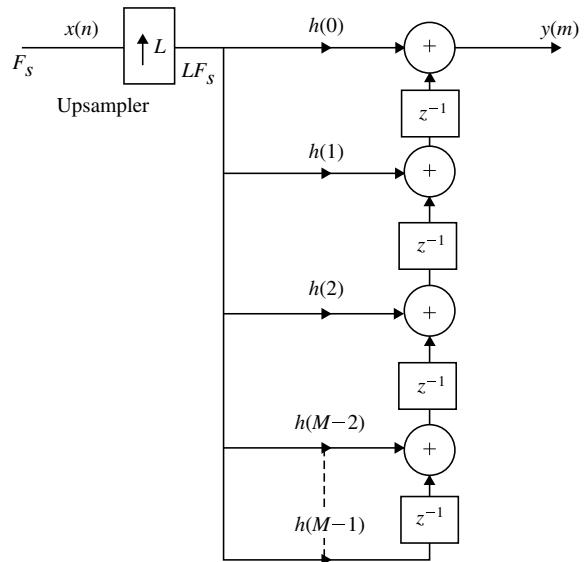
**Fig. 6.14** Direct form realization of decimator  
inefficient



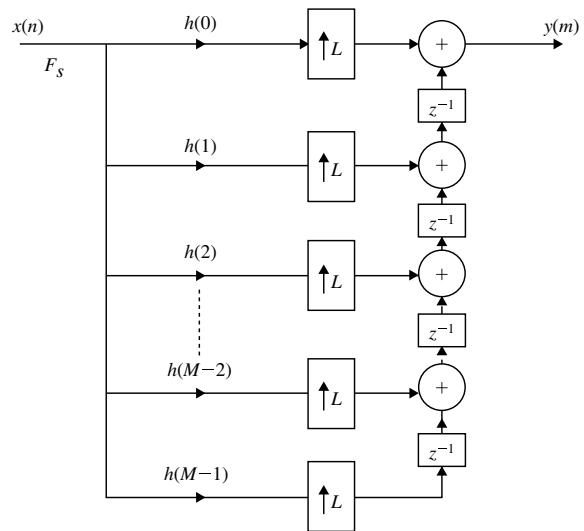
**Fig. 6.15** Efficient realization for decimator



**Fig. 6.16** Transposed direct form realization for interpolator



**Fig. 6.17** Efficient realization for interpolator



tations are to be performed at this high sampling rate and the interpolator becomes an inefficient one. The interpolator is made more efficient by putting the upsampler within the filter as shown in Fig. 6.17. This enables that all the filter multiplications and additions are performed at the low sampling rate  $F_s$  and the interpolator becomes more efficient.

## 6.13 Identities

The sampling rate decimation and interpolation are represented in block diagram. The block diagrams can be modified and represented in convenient form using identities. These identities (equivalence) are given below without proof in Fig. 6.18.

## 6.14 Polyphase Filter Structure of a Decimator

Efficient implementation of sampling rate conversion was discussed in Sections 6.11 and 6.12 for upsampling and downsampling, respectively, in a single stage. The computational requirements can be further decreased using multi-stage design.

Additional reduction in the computational complexity is possible by realizing the FIR filters using the polyphase decomposition which is described below:

### 6.14.1 The Polyphase Decomposition

Consider an FIR with an impulse response having  $N$ -coefficients. The system function is written as:

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} \quad (6.32)$$

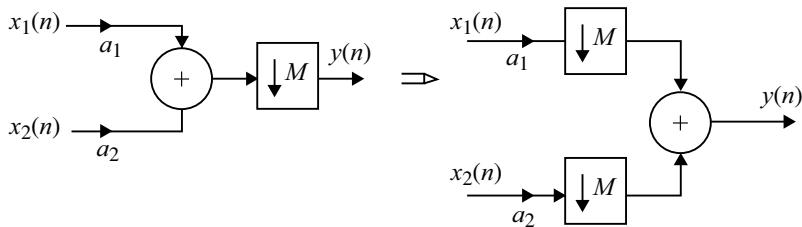
For  $N = 7$ , the above equation is written as:

$$\begin{aligned} H(z) &= \sum_{n=0}^6 h(n)z^{-n} \\ &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} \\ &\quad + h(5)z^{-5} + h(6)z^{-6} \\ &= (h(0) + h(2)z^{-2} + h(4)z^{-4} + h(6)z^{-6}) + \\ &\quad z^{-1}(h(1) + h(3)z^{-2} + h(5)z^{-4}) \\ &= P_0(z^2) + z^{-1}P_1(z^2) \end{aligned} \quad (6.33)$$

where

$$\begin{aligned} P_0(z^2) &= h(0) + h(2)z^{-1} + h(4)z^{-2} + h(6)z^{-4} \\ P_1(z^2) &= h(1) + h(3)z^{-1} + h(5)z^{-2} \end{aligned}$$

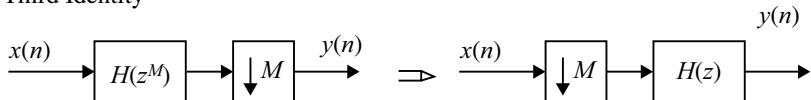
(a) First Identity



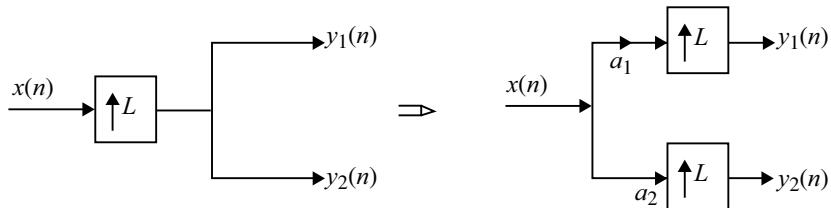
(b) Second Identity



(c) Third Identity



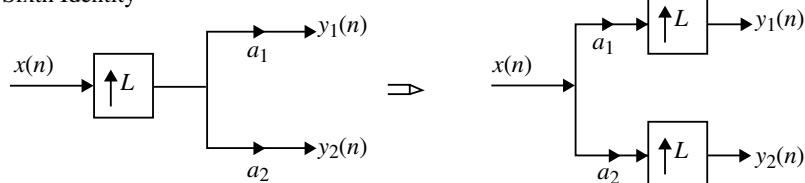
(d) Fourth Identity



(e) Fifth Identity



(f) Sixth Identity

**Fig. 6.18** Identities of downsampling and upsampling

The decomposition of  $H(z)$  as given in Eq. (6.33) is known as polyphase decomposition of the system function with two branches. In general for  $M$  branches  $H(z)$  can be decomposed as

$$H(z) = \sum_{m=0}^{N-1} z^{-m} P_m(z^M) \quad (6.34)$$

where

$$P_m(z^m) = \sum_{n=0}^{(N+1)/M} h(Mn + m)z^{-n} \quad 0 \leq m \leq M - 1$$

The  $z$ -transform of infinite duration sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \quad (6.35)$$

If  $H(z)$  is decomposed into  $M$ -branches we get,

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^m) \quad (6.36)$$

where

$$\begin{aligned} P_m(z) &= \sum_{l=-\infty}^{\infty} h(lM + m)z^{-l} \\ H(z) &= \sum_{m=0}^{M-1} \sum_{l=-\infty}^{\infty} h(lM + m)z^{-Ml} \\ &= \sum_{m=0}^{M-1} \sum_{l=-\infty}^{\infty} h(lM + m)z^{-(lM+m)} \end{aligned}$$

Let  $h(lM + m) = P_m(l)$

$$\begin{aligned}
\frac{Y(z)}{X(z)} &= H(z) = \sum_{m=0}^{M-1} \sum_{l=-\infty}^{\infty} P_m(l) z^{-(lM+m)} \\
Y(z) &= \sum_{m=0}^{M-1} \sum_{l=-\infty}^{\infty} P_m(l) z^{-(lM+m)} X(z) \\
&= \sum_{m=0}^{M-1} \sum_{l=-\infty}^{\infty} P_m(l) Z[x(n - (lM + m))]
\end{aligned}$$

Taking inverse  $z$ -transform on both sides we get,

$$y(n) = \sum_{m=0}^{M-1} \sum_{l=-\infty}^{\infty} P_m(l) x(n - (lM + m)) \quad (6.37)$$

Let  $x_m(l) = x(lM - m)$

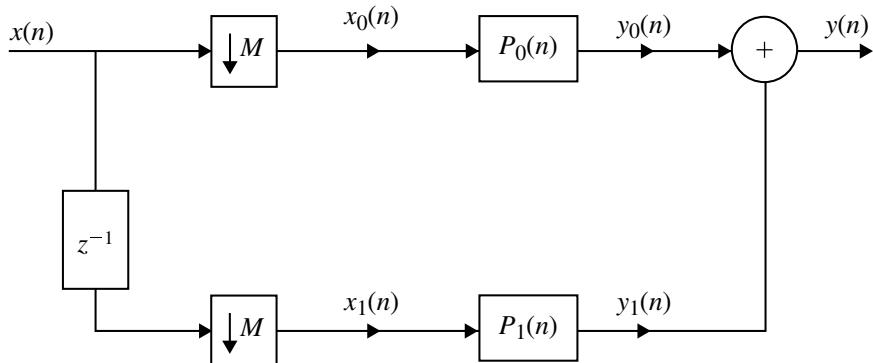
$$\begin{aligned}
y(n) &= \sum_{m=0}^{M-1} \sum_{l=-\infty}^{\infty} P_m(l) x(n - l) \\
&= \sum_{m=0}^{M-1} P_m(n) * x_m(n)
\end{aligned} \quad (6.38)$$

$$y(n) = \sum_{m=0}^{M-1} y_m(n) \quad (6.39)$$

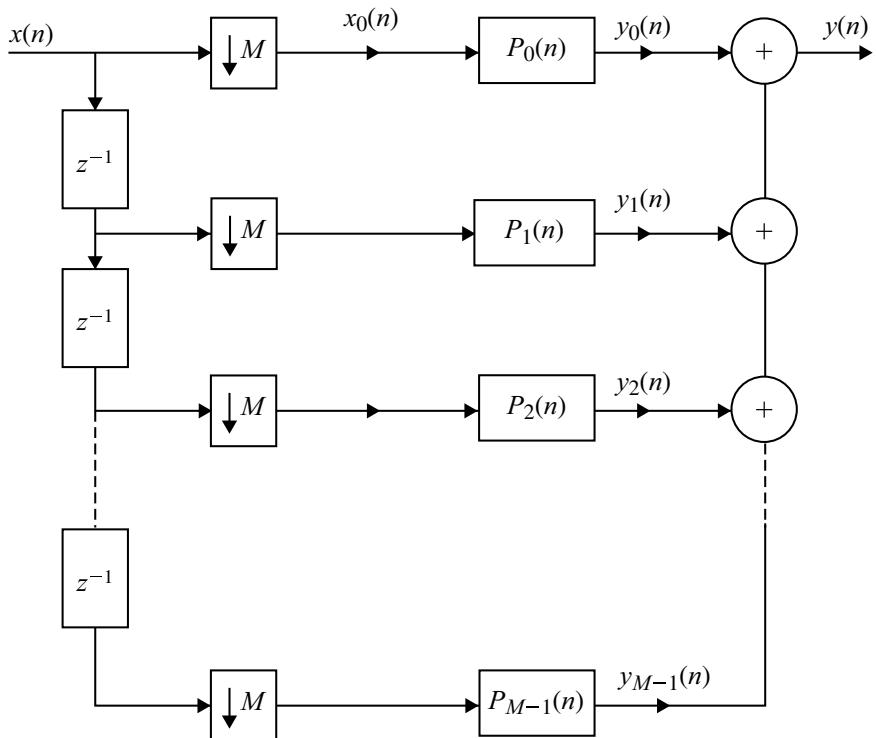
where  $y_m(n) = P_m(n) * x_m(n)$  which is known as polyphase convolution. For example for  $M = 2$ , Eq. (6.38) becomes

$$\begin{aligned}
y(n) &= \sum_{m=0}^1 P_m(l) * x_m(n) \\
y(n) &= P_0(n) * x_0(n) + P_1(n) * x_1(n)
\end{aligned}$$

In Eq. (6.38),  $x_m(n)$  is obtained first by delaying  $x(n)$  by  $m$  units and de-sampling by a factor  $M$ .  $y_m(n)$  is obtained by convolving  $x_m(n)$  with  $P_m(n)$ . The structure of a polyphase decimator with two branches is shown in Fig. 6.19a and the polyphase structure of a  $M$  branch decimator is shown in Fig. 6.19b.

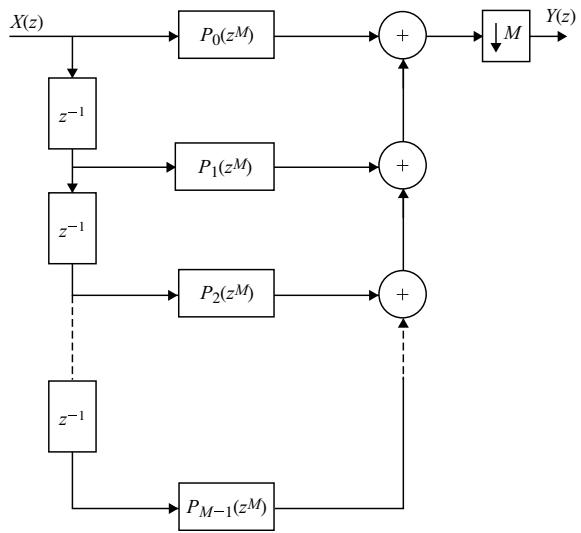


(a) Polyphase structure of a two branch decimator.

(b) Polyphase structure of  $M$  branch decimator.

**Fig. 6.19** **a** Polyphase structure of a two branch decimator. **b** Polyphase structure of  $M$  branch decimator

**Fig. 6.20** Polyphase structure of  $M$  branch decimator using  $z$ -transform



### 6.14.2 Polyphase Structure of a Decimator Using $z$ -Transform

Consider Eq. (6.36) which gives the FIR filter function.

$$H(z) = \sum_{m=0}^{N-1} z^{-m} P_m(z^M)$$

We have  $M$  sub-filters  $P_0(z)$ ,  $P_1(z)$ , ...,  $P_{M-1}(z)$  of FIR filters and can be represented as shown in Fig. 6.20.

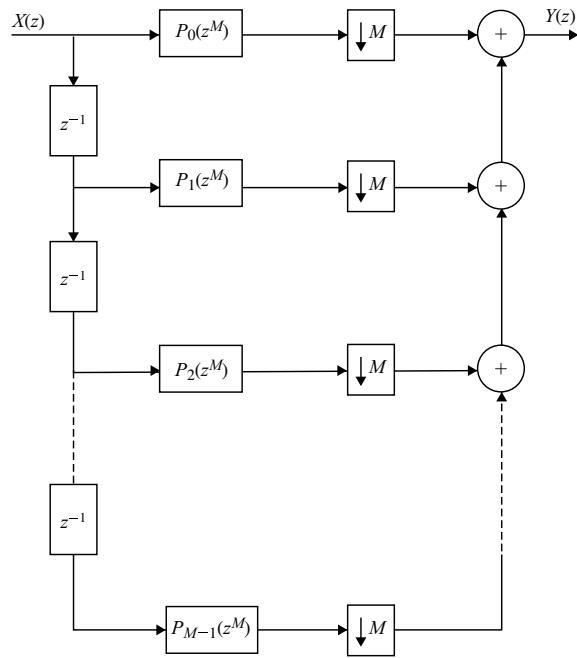
Using first identity Fig. 6.20 can be represented as shown in Fig. 6.21 and using third identity Fig. 6.20 can be represented as shown in Fig. 6.22.

### 6.14.3 Polyphase Structure of an Interpolator

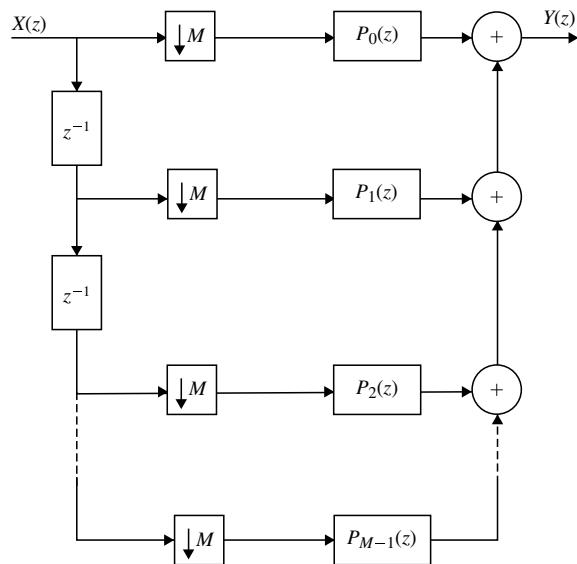
The polyphase structure of an interpolator can be obtained by transposing the structure of a decimator which is shown in Fig. 6.20. Here, the polyphase components of impulse responses are given by the following equation (refer to Eq. (6.37)).

$$P_m(l) = h(lL + m), \quad m = 0, 1, 2, \dots (L - 1)$$

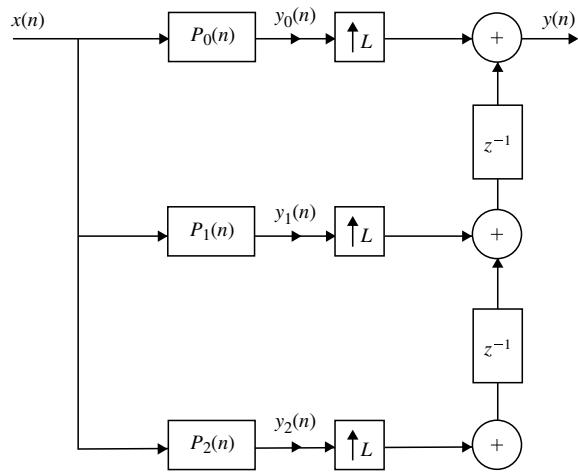
**Fig. 6.21**  $z$ -transform representation of polyphase decimation



**Fig. 6.22**  $z$ -transform representation of polyphase decimation



**Fig. 6.23** Polyphase structure of a three branch interpolator



the output of  $L$  sub-filters are represented as

$$y_m(n) = x(n) \times P_m(n)$$

The overall output of the interpolator is (refer Eq. (6.39))

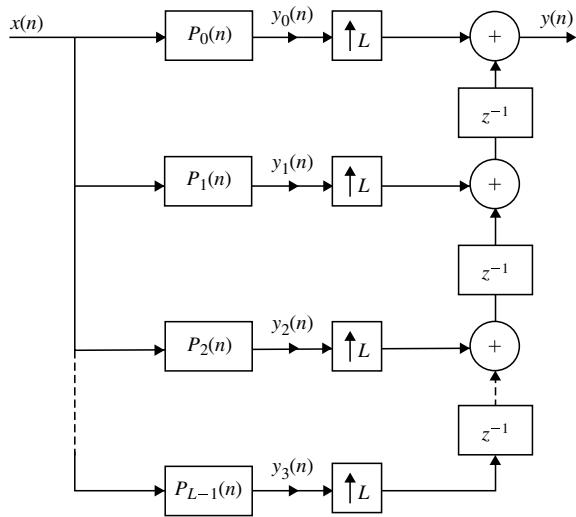
$$y(n) = \sum_{m=0}^{(L-1)} y_m(n)$$

For  $L = 3$

$$\begin{aligned} y(n) &= \sum_{m=0}^2 y_m(n) \\ &= y_0(n) + y_1(n) + y_2(n) \\ &= x(n) \times P_0(n) + x(n) \times P_1(n) + x(n) \times P_2(n) \end{aligned}$$

The polyphase structure of an interpolator with three branch is shown in Fig. 6.23. In general, when the sampling rate increases to  $L$ , the polyphase structure with  $L$  branch is shown in Fig. 6.24.

**Fig. 6.24** Polyphase structure of a  $L$  branch interpolator



#### 6.14.4 Polyphase Structure of an Interpolator Using $z$ -Transform

From Eq. (6.34), the transfer function of an interpolator is written by replacing  $M$  by  $L$

$$\begin{aligned} H(z) &= \sum_{m=0}^{L-1} z^{-m} P_m(z^L) \\ &= P_0(z) + z^{-1} P_1(z^L) + z^{-2} P_2(z^L) + \dots \\ &\quad + z^{-(L-1)} P_{L-1}(z^L) \end{aligned} \tag{6.40}$$

The interpolator with the system function  $H(z)$  is shown in Fig. 6.25. Using fourth and sixth identities an efficient polyphase, structure of the interpolator is realized as shown in Fig. 6.26.

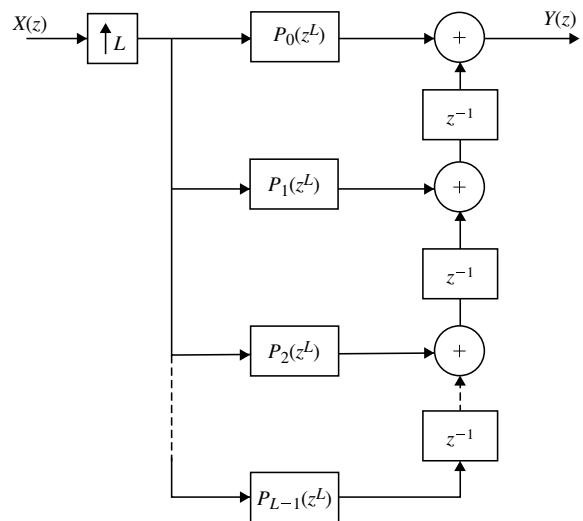
The structure for Example 6.4 is shown in Fig. 6.27.

#### Example 6.4

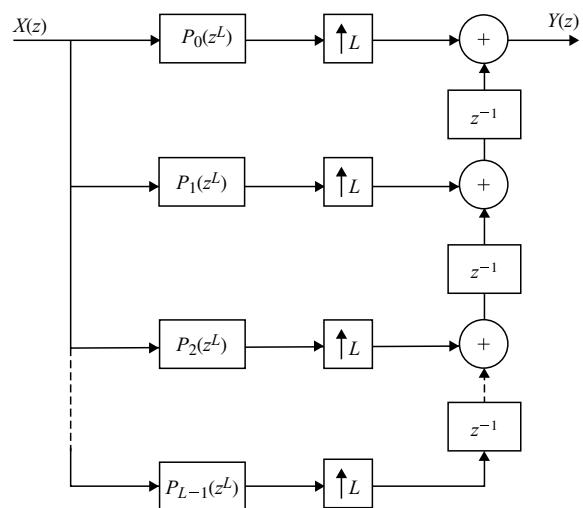
For the structure shown in Fig. 6.27, find the relationship between  $x(n)$  and  $y(n)$ .

(Anna University, April, 2005)

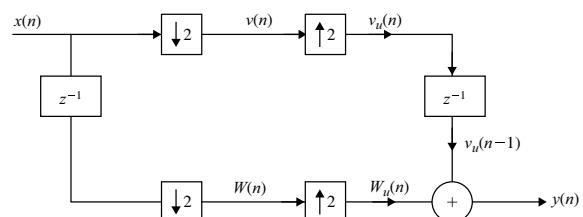
**Fig. 6.25** Polyphase structure of interpolator ( $z$ -transform representation)



**Fig. 6.26** Efficient polyphase  $z$ -transform representation of interpolator



**Fig. 6.27** Structure for Example 6.4



**Solution** The relations between various signal variables and the input are tabulated below:

$n$	0	1	2	3	4	5	6
$x(n)$	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$
$v(n)$	$x(0)$	$x(2)$	$x(4)$	$x(6)$	$x(8)$	$x(10)$	$x(12)$
$w(n)$	$x(-1)$	$x(1)$	$x(3)$	$x(5)$	$x(7)$	$x(9)$	$x(11)$
$v_u(n)$	$x(0)$	0	$x(2)$	0	$x(4)$	0	$x(6)$
$w_u(n)$	$x(-1)$	0	$x(1)$	0	$x(3)$	0	$x(5)$
$v_u(n - 1)$	0	$x(0)$	0	$x(2)$	0	$x(4)$	0
$y(n)$	$x(-1)$	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$

$$\begin{aligned} y(n) &= w_n(n) + v_n(n - 1) \\ &= x(-1) + x(0) + x(1) + x(2) + x(3) + \dots \end{aligned}$$

$$y(n) = x(n - 1)$$

## 6.15 Polyphase Decomposition of IIR Transfer Function

The polyphase decomposition of IIR transfer function  $H(z)$  is not that straight forward as that of FIR transfer function explained above. One approach is to decompose

$$H(z) = \frac{P(z)}{D(z)}$$

into  $P'(z)/D'(z^M)$  by multiplying the denominator  $D(z)$  and the numerator  $P(z)$  by an appropriate polynomial. The  $M$ -branch polyphase decomposition is done for  $P'(z)$ . This is illustrated by the following example.

### Example 6.5

Consider the following IIR transfer function:

$$H(z) = \frac{(1 - 3z^{-1})}{(1 + 4z^{-1})}$$

Obtain a two band decomposition with  $M = 2$ .

**Solution**

$$H(z) = \frac{(1 - 3z^{-1})}{(1 + 4z^{-1})}$$

Multiplying the numerator and the denominator by  $(1 - 4z^{-1})$  we get

$$\begin{aligned} H(z) &= \frac{(1 - 3z^{-1})(1 - 4z^{-1})}{(1 + 4z^{-1})(1 - 4z^{-1})} \\ &= \frac{(1 - 7z^{-1} + 12z^{-2})}{(1 - 16z^{-2})} \\ &= \frac{(1 + 12z^{-2})}{(1 - 16z^{-2})} + z^{-1} \frac{(-7)}{(1 - 16z^{-2})} \\ &= E_0(z^2) + E_1(z^2) \end{aligned}$$

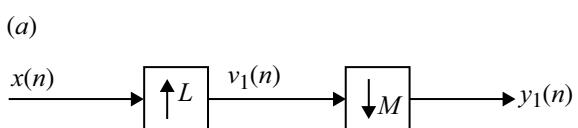
where

$$\begin{aligned} E_0(z) &= \frac{(1 + 12z^{-1})}{(1 - 16z^{-1})} \\ E_1(z) &= \frac{-7}{(1 - 16z^{-1})} \end{aligned}$$

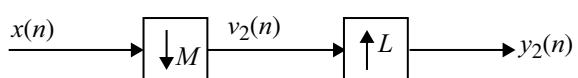
## 6.16 Cascading of Upsampler and Downampler

The sampling rate alteration devices change the sampling rate of signal by an integer. If fractional change in the sampling rate is required, it is possible to achieve this by cascading an upsampler with downampler without any change in the input-output relation. Consider the cascade structure shown in Fig. 6.28a. The following equations are written.

**Fig. 6.28** Cascading of up and downsamplers



(b)



$$V_1(z) = X(z^2) \quad (6.41)$$

$$Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} V_1(z^{1/M} W_M^{-k}) \quad (6.42)$$

where  $W_M = e^{-j2\pi/M}$ .

Combining the above two equations we get

$$Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{L/M} W_M^{-kL}) \quad (6.43)$$

Now consider Fig. 6.28b. The following equations are written:

$$V_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k}) \quad (6.44)$$

$$Y_2(z) = V_2(z^L) \quad (6.45)$$

Combining Eqs. (6.44) and (6.45) we get

$$Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k}) \quad (6.46)$$

For  $Y_1(z) = Y_2(z)$ , the following conditions is to be satisfied.

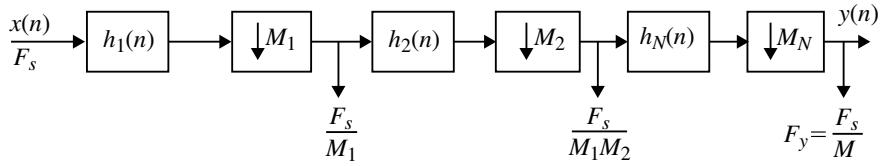
$$\sum_{k=0}^{M-1} X(z^{L/M} W_M^{-kL}) = \sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k}) \quad (6.47)$$

Equation (6.47) is valid iff  $M$  and  $L$  do not have a common factor which is an integer greater than one.

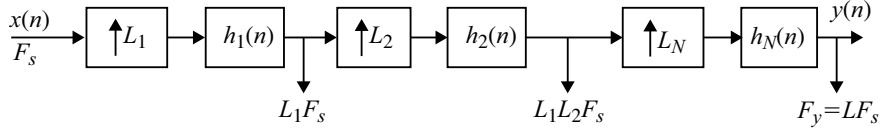
## 6.17 Multi-stage Rating of Sampling Rate Conversion

Single stage sampling rate conversion is inefficient if the decimation factor  $M$  or interpolation factor  $L$  are very much greater than unity. If  $M \gg 1$  or  $L \gg 1$ , sampling rate conversion is done in multi-stage. The multi-stage conversion where  $M \gg 1$  is represented in Fig. 6.29. The decimation factor  $M$  is expressed as a product of positive integers as given below:

$$M = \prod_{k=1}^N M_k \quad (6.48)$$



**Fig. 6.29** Multi-stage implementation of sampling rate conversion for  $M \gg 1$



**Fig. 6.30** Multi-stage implementation of sampling rate conversion for  $L \gg 1$

Each decimator is implemented and cascaded to get  $N$  stages as shown in Figure 6.29. The final sampling rate achieved is

$$F_y = \frac{F_s}{M}$$

where  $M = M_1 M_2 \dots M_N$ .

Similar to multi-stage downsampling multi-stage upsampling is done in  $N$  stages. Here, the final sampling rate achieved is

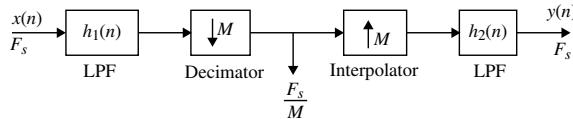
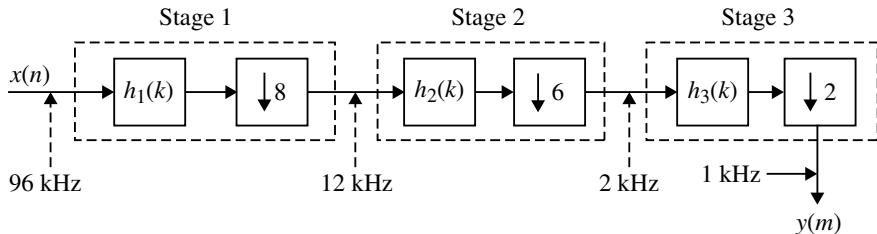
$$F_y = L F_s.$$

For  $L \gg 1$ , this is represented in Fig. 6.30. The interpolation factor is expressed as

$$L = \prod_{k=1}^N L_k$$

## 6.18 Implementation of Narrow Band Lowpass Filter

A narrow band filter is identified with a narrow pass band and a narrow transition band. It requires a very large number of filter coefficients and therefore finite word length effect occurs in such a digital filter design. Further it requires more number of computations and memory locations. To overcome these problems, multi-rate signal processing is applied using decimator and interpolator, which are connected in cascade as shown in Figure 6.31. The sampling rate of the sequence  $x(n)$  is reduced by a factor  $M$  and lowpass filtering is performed. The sampling rate of the output

**Fig. 6.31** Implementation of narrow band LPF**Fig. 6.32** Three stage decimator for Example 6.6

sequence  $y(n)$  is increased by a factor  $M$ . Thus, the sampling rates of  $x(n)$  and  $y(n)$  are the same which is  $F_s$ . The filters in decimator  $h_1(n)$  and interpolator  $h_2(n)$  are chosen to be identical with pass band ripple  $\delta_{p/2}$  and stop band ripple  $\delta_{s/2}$  to get the desired specification of a narrow band LPF.

### Example 6.6

The block diagram of a three stage decimator (Fig. 6.32) which is used to reduce the sampling range from 96 kHz to 1 kHz is given. Assuming decimation factors of 8, 6, 2, indicate the sampling rate at the output of each of the three stages.

### Solution

- At the first stage sampling rate is reduced by a factor of 8 from 96 kHz to 12 kHz.
- At the second stage sampling rate is further reduced by a factor of 6 from 12 kHz to 2 kHz.
- At the third stage the sampling rate is reduced by a factor of 2 from 2 kHz to 1 kHz.

### Example 6.7

Assume that the decimator in Example 6.6 satisfies the following overall specifications.

Input sampling frequency  $F = 96 \text{ kHz}$

Decimator factor  $M = 96$

Pass band ripple = 0.01 dB

Stop band ripple = 60 dB

Frequency band of interest =  $0 - 450 \text{ kHz}$

Determine band edge frequency for the decimating filter at each stage.

---

**Solution** The pass band edge frequency of each of three decimating filters is the same (namely 455 Hz) to preserve the frequency band of interest.

The stop band edge frequencies are different to exploit the differences in the sampling rate. The bandstop frequencies are given by the relation as,

$$f_{si} = F_i - \frac{F_s}{2M}, \quad 1, 2, 3$$

Stage One:

$$\begin{aligned} f_{s1} &= 12 - \frac{96}{2 \times 96} \\ &= 11.5 \text{ kHz} \end{aligned}$$

Thus, band edge frequencies are 0, 450, 11.5, 48 kHz.

Stage Two:

$$\begin{aligned} f_{s2} &= 2 - \frac{96}{2 \times 96} \\ &= 1.5 \text{ kHz} \end{aligned}$$

Thus, band edge frequencies are 0, 450 kHz, 1.5 kHz, 6 kHz.

Stage Three:

$$\begin{aligned} f_{s3} &= 1 - \frac{96}{2 \times 96} \\ &= 0.5 \text{ kHz} \end{aligned}$$

Thus, band edge frequencies are 0, 450 kHz, 0.5 kHz, 1 kHz.

---

### Example 6.8

Assume that the input and output sampling rate of a decimator are 96 kHz and 1 kHz, respectively:

- (i) Write down the overall decimator factor.
  - (ii) Write down all the possible sets of integer decimation factor (written in descending order only) assuming two stages of decimation.
  - (iii) Repeat (ii) but assuming three stages of decimation.
  - (iv) Repeat (iii) but assuming four stages of decimation.
- 

**Solution**

- (i) Overall decimator factor is  $96/1$  or  $8 \times 6 \times 2 = 96$ .
- (ii)

$$\begin{aligned} & 48 \times 2 \\ & 24 \times 4 \\ & 12 \times 8 \end{aligned}$$

- (iii)

$$\begin{aligned} & 24 \times 2 \times 2 \\ & 6 \times 4 \times 4 \\ (iv) \quad & 3 \times 2 \times 4 \times 4 \end{aligned}$$

**Example 6.9**

For the decimator in Example 6.6, calculate the total number of multiplications per seconds (MPs) and to storage requirement.

---

**Solution**

$$\begin{aligned} \text{MPs} &= \sum_{i=1}^I N_i F_i \\ \text{TSR} &= \sum_{i=1}^I N_i \end{aligned}$$

where  $N_i$  is number of filter coefficients for “ $i$ ” stages.

$$\begin{aligned} \text{MPs} &= N_1 \times 12 \times 10^3 + N_2 \times 2 \times 10^3 + N_3 \times 1 \times 10^3 \\ \text{TSR} &= N_1 + N_2 + N_3 \end{aligned}$$

## 6.19 Adaptive Filters

In Chaps. 3 and 4 IIR and FIR filters design was discussed which requires the knowledge of second order statistics of the signals. However in applications such as channel equalization, echo cancelation and system modeling to mention a few these statistics cannot be specified a prior. In such applications digital filters with adjustable coefficients are designed. These filters have self-adjusting characteristics and therefore they are called as ADAPTIVE FILTERS. These filters are widely used in communication systems, control systems, adaptive antenna systems, digital communication receivers, adaptive noise canceling technique, etc. An adaptive filter has the property that it automatically adjusts or modifies its frequency response characteristics according to the changes in the input signal characteristics. These filters are also used when there is spectral overlap between the signal and noise is unknown. In these cases the conventional filters would lead to distortion of the desired signal.

### 6.19.1 Concepts of Adaptive Filtering

An adaptive filter consists of two distinct parts and they are:

1. A digital filter with adjustable coefficients.
2. An adaptive algorithm which is used to adjust or modify the coefficients.

In almost all adaptive systems, FIR digital filter is used since it has simple structure and stability is guaranteed. Only in limited applications, IIR or lattice filter is used. Adaptive algorithms commonly used are:

1. Least Mean Square (LMS) algorithm.
2. Recursive Least Square (RLS) algorithm.

There are many configurations of adaptive filter. Some of them include the following:

1. Adaptive noise canceler.
2. Adaptive self-tuning filter.
3. Adaptive line enhancer.
4. System modeling.
5. Linear combiner.

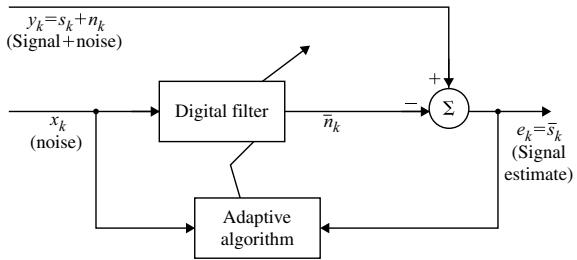
The above configurations are described below.

### 6.19.2 Adaptive Noise Canceller

The block diagram of an adaptive filter as a noise canceler is shown in Fig. 6.33.

The input signals  $x_k$  and  $y_k$  are simultaneously applied.  $x_k$  is the noise  $y_k$  contains the desired signal  $s_k$  polluted with noise  $n_k$ . The noise  $x_k$  is processed by the digital

**Fig. 6.33** Block diagram of an adaptive filter as a noise canceler



filter and gives out the estimate  $\bar{n}_k$  of  $n_k$ . An estimate of the desired signal is obtained by subtracting the digital filter output  $\bar{n}_k$  from  $y_k$ . From Fig. 6.33 the following equation is written.

$$\bar{s}_k = y_k - \bar{n}_k = s_k + n_k - \bar{n}_k \quad (6.49)$$

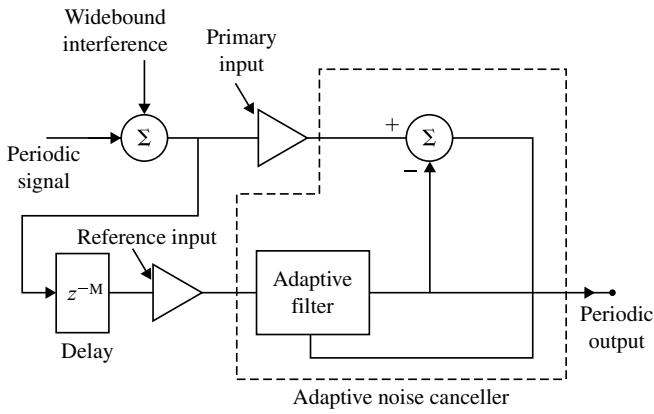
From Eq. (6.49), it is evident that optimum estimate of the desired signal  $s_k$  is obtained by producing an optimum estimate of the noise in the polluted signal. This is achieved by feeding the signal estimate  $\bar{s}_k$  to the adaptive filter which adjusts the digital filter coefficients. By using suitable adaptive algorithm, the noise in  $\bar{s}_k$  is minimized. Thus the output signal  $\bar{s}_k$  is used as an estimate of the desired signal  $s_k$  and also as an error signal which is used to adjust the filter coefficients. The configurations of adaptive self-tuning filter, adaptive line enhancer, adaptive system modeling and adaptive linear combiner are shown in Fig. 6.34a–d, respectively.

### 6.19.3 Main Components of the Adaptive Filter

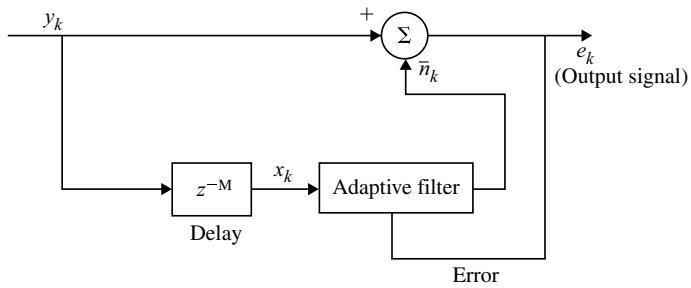
In most of the systems, the digital filter shown in Fig. 6.33 is used. The output can be obtained as

$$\bar{n}_k = \sum_{i=0}^{N-1} w_k(i)x_{k-i} \quad (6.50)$$

where  $w_k(i)$ ,  $i = 0, 1, \dots$ , are the adjustable filter coefficients or the weights, and  $x_k(i)$  and  $\bar{n}_k$  are the input and output of the filter. Fig. 6.35 shows the single input–signal output system.

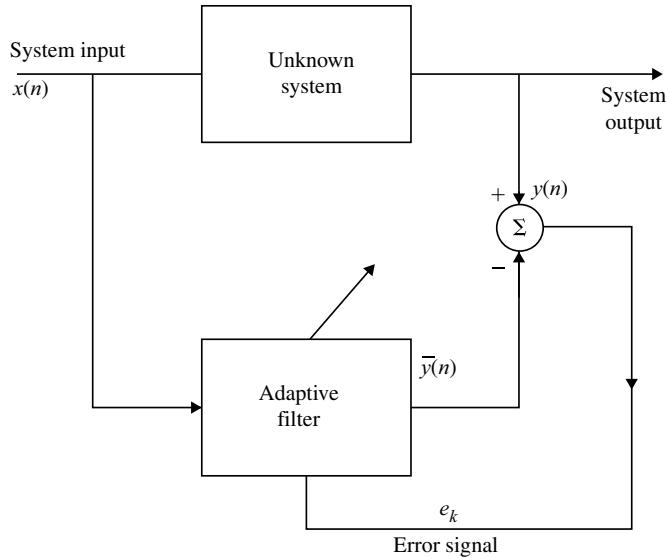


(a) Adaptive self tuning filter.

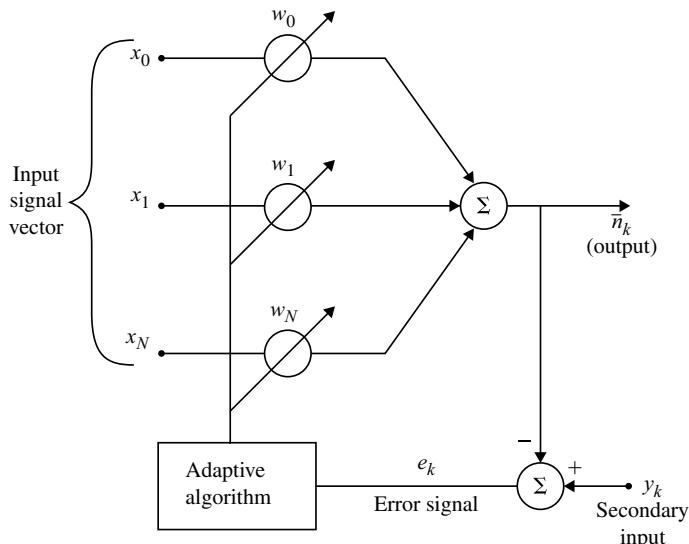


(b) Adaptive line enhancer.

**Fig. 6.34** **a** Adaptive self-tuning filter. **b** Adaptive line enhancer. **c** System modeling. **d** Adaptive linear combiner

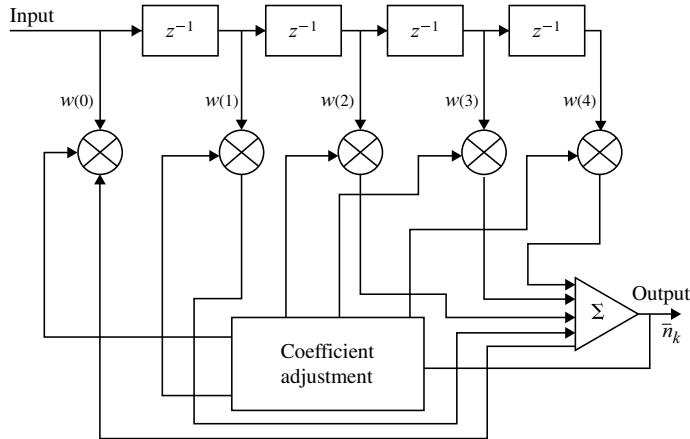


(c) System modeling.



(d) Adaptive linear combiner.

**Fig. 6.34** (continued)



**Fig. 6.35** Direct form adaptive FIR filter

### 6.19.4 Adaptive Algorithms

Adaptive algorithms are used to adjust or modify the coefficients of digital filter so that the error signal  $e_k$  is minimized to meet certain constraints. The most commonly used algorithms in adaptive filters are:

1. Least Mean Square (LMS) algorithm
2. Recursive Least Squares (RLS) algorithm
3. Kalman Filter algorithm.

LMS algorithm is the most efficient algorithm in terms of computation and storage requirements. It also does not suffer from numerical instability problem which is inherently present in other two algorithms. However, RLS algorithm has superior convergence properties. In the section to follow LMS and RMS algorithms are described.

#### 6.19.4.1 Least Mean Square (LMS) Algorithm

In LMS algorithm the coefficients of the digital filter are adjusted from sample to sample in such a way as to minimize the Mean Square Error (MSE). The weight vectors are updated from sample to sample as given below:

$$W_{k+1} = W_r - \mu \nabla_k \quad (6.51)$$

where  $W_k$  and  $\nabla_k$  are the weights and true gradient vectors, respectively, at the  $k$ th instant and  $\mu$  controls the stability and rate of convergence. The LMS algorithm aims at getting the digital filter weight  $W_k$ . The LMS algorithm for updating the weights

from sample to sample as suggested by Widow-Hopf is given by

$$W_{k+1} = W_r + 2\mu e_k X_k \quad (6.52)$$

where

$$e_k = y_k - W_x^T x_k \quad (6.53)$$

where  $x_k$  is the input vector. The weights obtained by the LMS algorithm are only the estimates which gradually improve with time as the weights are adjusted and the filter learns the characteristics of the signals and finally the weights converge with the following condition.

$$0 < \mu < \frac{1}{\lambda_{\max}} \quad (6.54)$$

where  $\lambda_{\max}$  is the maximum eigen value of the input data matrix. The computational procedure for the LMS algorithm is summarized below.

1. Initially, set each weight  $w_k(i) = 0, 1, \dots, N - 1$  to an arbitrary fixed value.  
For subsequent sampling instant  $k$ , carry out the following steps.
2. Compute filter output

$$\bar{n}_k = \sum_{i=0}^{N-1} w_k(i) x_{k-i}$$

3. Compute the error estimate

$$e_k = y_k - \bar{n}_k$$

4. Update the next filter weights

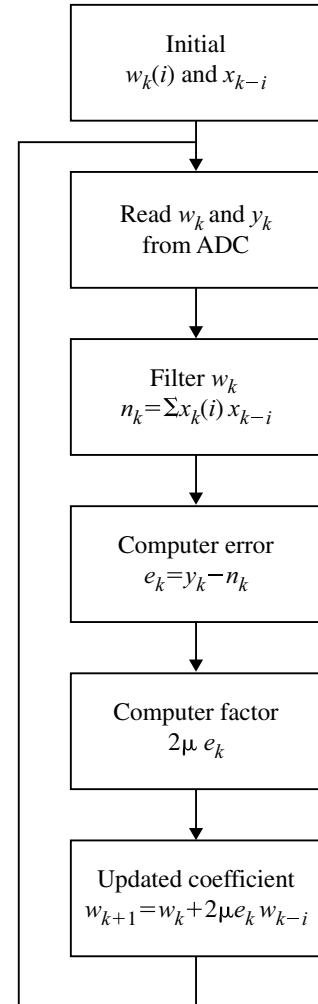
$$w_{k+1}(i) = w_k(i) + 2\mu e_k x_{k-i}$$

The LMS algorithm requires approximately  $2N + 1$  multiplications and  $2N + 1$  addition for each new set of input and output samples. Modern DSPs are very much suited for direct implementation of LMS algorithm. The simplicity and ease of implementation make the LMS algorithm of first choice in many real time systems. The flow chart of LMS algorithm is given in Fig. 6.36.

#### 6.19.4.2 Recursive Least Squares Algorithm

The major advantage of the LMS algorithm is its simplicity in computation. However, if the eigenvalues of the auto correlation matrix has a large spread, convergence becomes very slow which is a major disadvantage of LMS algorithm. Further the

**Fig. 6.36** Flow chart for LMS adaptive filter

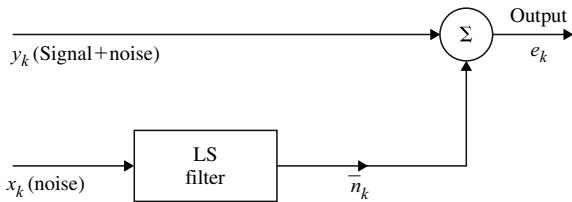


algorithm has a single adjustable parameter for controlling the convergence rate. To obtain the faster convergence more than one adjustable parameters are required. In recursive least squares algorithm for each eigenvalue one parameter is chosen and convergence becomes faster. However the algorithm becomes more complex.

The RLS algorithm is based on the least squares method which is illustrated in Fig. 6.36. From Fig. 6.36, the following equation connecting the input and output is written (Fig. 6.37):

$$y_k = \sum_{i=0}^{n-1} w(i)x_k(i) + e_k \quad (6.55)$$

**Fig. 6.37** Illustrations of RLS



where  $e_k$  is the error,  $w(i)$  is the weight of the  $i$ th input and  $y_k$  is the primary signal which contains noise. In the least square method, the objective is to estimate  $w(o)$  to  $w(n - 1)$  given  $x_k(i)$  and  $y_k$ . For the filter weight  $w(i)$ , the optimum estimates are given as

$$W_m = [X_m^T X_m]^{-1} X_m^T Y_m \quad (6.56)$$

where  $W_m$ ,  $X_m$  and  $Y_m$  are given by

$$Y_m = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \dots \\ y_{m-1} \end{bmatrix}; \quad X_m = \begin{bmatrix} X^T(0) \\ X^T(1) \\ X^T(2) \\ \dots \\ X^T(m-1) \end{bmatrix}; \quad W_m = \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ \dots \\ w(n-1) \end{bmatrix}$$

The filter output is obtained as

$$\bar{n}_k = \sum_{i=0}^{n-1} \bar{w}(i)x_{k-i}, \quad k = 1, 2, \dots, m \quad (6.57)$$

In Eq. (6.57), the estimates of  $W_m$  is obtained using recursive method. Here, the estimates of  $W_m$  are updated for each new set of data acquired. A suitable RLS algorithm is obtained by exponentially weighting the data to remove gradually the effects of old data on  $W_m$ . Thus the following equation is written.

$$W_k = W_{k-1} + G_k e_k \quad (6.58)$$

$$P_k = \frac{1}{\gamma} [P_{k-1} - G_k X^T P_{k-1}] \quad (6.59)$$

where

$$G_k = \frac{P_{k-1} X(k)}{\alpha_k} \quad (6.60)$$

$$e_k = y_k - X^T(k) W_{k-1} \quad (6.61)$$

$$\alpha_k = \gamma + X^T(k) P_{k-1} X(k) \quad (6.62)$$

The RLS algorithm is represented in Fig. 6.37.

In Eq. (6.59)  $P_k$  represents recursive way of computing  $[X_k^T X_k]^{-1}$ .  $k$  represents that the quantities are obtained at each sample when  $k$  is varied and  $\gamma$  is called the forgetting factor.

The main limitation of RLS algorithm is its sensitivity to computer round off errors which leads to instability.

### Summary

- Multi-rate signal processing has many applications in which the given sampling rate is converted into another signal with a different sampling rate.
- The two basic sampling rate alteration devices are the upsampler and the down-sampler.
- To down-sample a signal it is essential to pass the signal through an anti-aliasing LPF to band limit the signal before it is applied to the decimator.
- If an interpolator is used to up sample a signal, it is essential that the output of the signal from the interpolator is passed through a LPF to avoid removal of some of the desired frequency components of the sampled signal.
- The up sampler and downsample possess the property of linearity and time variancy.
- If  $F_s$  is the sampling rate of the sequence  $x(n)$ , the down-sampled signal will have the sampling rate  $\frac{F_s}{M}$  and the up sampled signal will have the sampling rate  $L F_s$ .
- If a signal  $x(n)$  is down-sampled by a factor  $M$  and then up sampled by a factor  $L$ , then the sampling rate conversion is  $L/M$  which is a rational factor.
- Sampling rate conversion can be more efficiently done using polyphase decomposition. This reduces computational complexity.

### Short Questions and Answers

#### 1. What do you understand by down-sampling?

Downsampling is the process of creating an output sequence  $y(n)$  from the input sequence  $x(n)$  using a downsample with a sampling factor of  $M$  where  $M$  is a positive integer. The down-sampling operation is implemented by keeping every  $M$ th sample of the input sequence  $x(n)$  and removing  $(M - 1)$  in between samples. The input–output are related by the following equation

$$y(n) = x(nM)$$

#### 2. What do you understand by upsampling?

Upsampling is the process of sampling rate alteration in which the sampling rate of the output sequence is  $L$  times larger than that of the input sequence where  $L$  is a positive integer, which is called upsampling factor. Here,  $(L - 1)$  equidistant zero valued samples between two consequential samples of the input sequence are inserted. The input–output are related by the following equation.

$$y(n) = \begin{cases} x\left(\frac{n}{L}\right), & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases}$$

### 3. What is a Decimator?

The decimator consists of two blocks. The input sequence is applied to a lowpass filter followed by down sampler. The LPF is used to avoid aliasing. When both these blocks are connected in cascade the unit is called decimator.

### 4. What is an Interpolator?

The process of up sampling is done by an up sampler and a LPF which are connected in cascade. The LPF removes unwanted images from the signal. The upsampler together with LPF is called an Interpolator.

### 5. What is sampling rate conversion?

The process of converting a signal from a given sampling rate to a different sampling rate is called sampling rate conversion.

### 6. What is a multi-rate systems?

Discrete time system with unequal sampling rates at various parts of the system are called multi-rate systems.

### 7. What is aliasing?

When the signal  $x(n)$  is downsampled, the plot of the frequency response of the output  $Y(e^{j\omega})$  has an overlap and the original shape of the input  $X(e^{j\omega})$  is lost. This is aliasing. If  $X(e^{j\omega})$  is zero for  $|\omega| \geq \frac{\pi}{2}$  there is no aliasing. To avoid aliasing the signal is first sent through anti-aliasing LPF and then the band limited signal is downsampled.

### 8. What is the necessary condition to be satisfied for connecting up-sampler and down sampler in cascade?

The necessary condition for connecting the up-sampler and down sampler in cascade is that the decimation and interpolation factor  $M$  and  $L$  are relatively prime. This implies that  $M$  and  $L$  do not have a common factor that is an integer which is greater than one.

## Long Answer Type Questions

1. Explain the need for multi-stage implementation of sampling rate conversion.
2. Explain clearly the downsampling and upsampling in multi-rate signal processing.
3. Explain sampling rate reduction by an integer factor  $M$  and derive input-output relationship in both time and frequency domains.
4. Explain sampling rate increase by an integer factor  $L$  and derive input-output relationship in both time and frequency domains.
5. Propose a scheme for a sampling rate conversion by a rational factor  $L/M$  where  $L$  is the upsampling integer factor and  $M$  is the downsampling integer factor.
6. Prove that decimator and interpolators are linear and time varying.
7. Derive the spectrum of the output of a decimator.

8. Derive the spectrum of the output of a interpolator.
9. Explain the aliasing effect in the downsampling process if the original spectrum is not band limited to  $|\omega| = \pi/M$ .
10. Explain the identities used in multi-rate processing.
11. Discuss the efficient transversal structure for decimator and interpolator.
12. Explain the polyphase decomposition for FIR filter structure.
13. Explain filter design and implementation (structure) for sampling rate conversion system.

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