CHAPTER

2

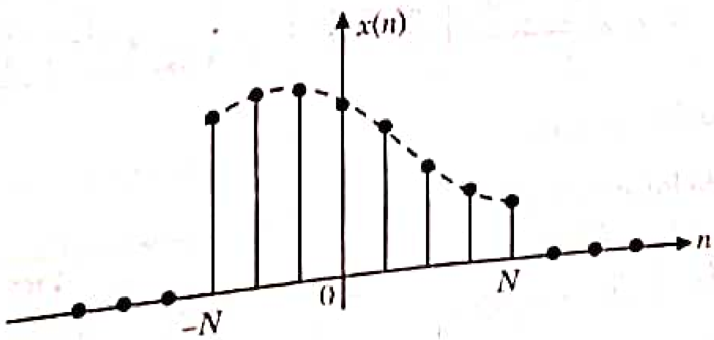
Review of DTFT

SYLLABUS

Discrete-Time Fourier Transform (DTFT), Existense of DTFT, Properties of DTFT, Discrete-Time LTI Systems Characterized by Linear Constant-Coefficient Difference Equations.

Inside this Chapter

- Introduction
- The Discrete-Time Fourier Transform: Representation of Aperiodic Signals
- Properties of Discrete-Time Fourier Transform (DTFT)
- Discrete-Time LTI Systems Characterized by Linear Constant-Coefficient Difference Equations



2.1 INTRODUCTION

As we know that broadly there are two types of signals namely continuous-time signals and discrete-time signals. A signal (continuous-time or discrete-time) may be analysed with the help of Fourier series and Fourier transform. Fourier series is used for periodic signals and Fourier transform is used for aperiodic signals. In general, Fourier transform may be used for both aperiodic and periodic signals.

When we analyse continuous-time signals with the help of Fourier Series and Fourier transform, then the Fourier series and Fourier transform are called continuous-time Fourier series and continuous-time Fourier transform respectively or simply Fourier series and Fourier transform. If discrete-time signals are analysed with the help of Fourier series and Fourier transform, then the Fourier series and Fourier transform are known as Discrete-time Fourier series (DTFS) and Discrete-time Fourier transform (DTFT) respectively.

In this chapter, we shall discuss Discrete-time Fourier transform (DTFT).

Like continuous-time Fourier transform, we shall begin our analysis with discrete-time aperiodic signals in order to develop a Fourier transform representation for discrete-time aperiodic signals and then extend this approach of discrete-time Fourier transform for discrete-time periodic signals.

2.2 THE DISCRETE-TIME FOURIER TRANSFORM : REPRESENTATION OF APERIODIC SIGNALS

In this section, we shall develop discrete-time Fourier transform for discrete-time aperiodic signals.

Let there be a discrete-time aperiodic (non-periodic) signal $x(n)$ as shown in figure 2.1.

We shall show that an aperiodic signal $x(n)$ may be expressed as a continuous sum (integral) of everlasting exponentials.

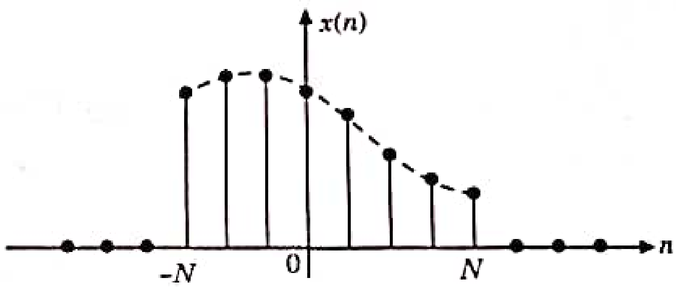


FIGURE 2.1.

To represent an aperiodic signal $x(n)$ (figure 2.1) by everlasting exponential signals, let us construct a new periodic signal $x_{N_0}(n)$ Formed by repeating the signal $x(n)$ every N_o units as shown in figure 2.2.

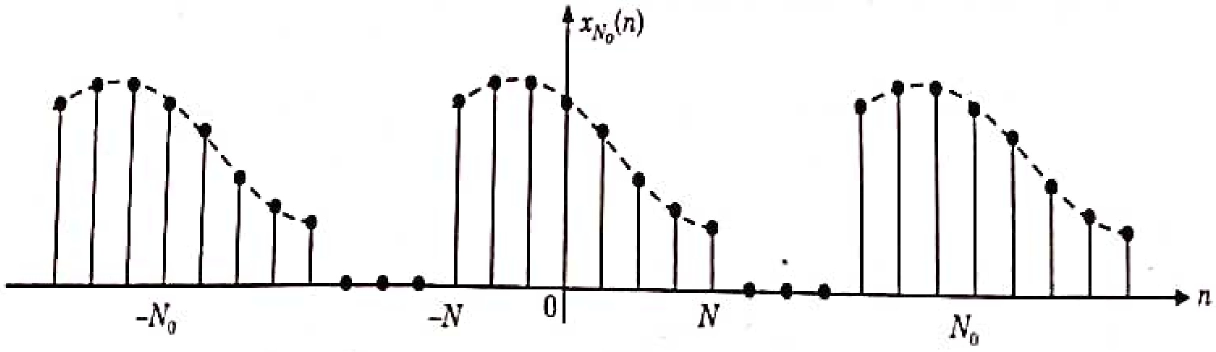


FIGURE 2.2 *Generation of a periodic signal by periodic extension of a signal $x(n)$.*

Here, the period N_o is made long enough to avoid overlap between the repeating cycles ($N_o \geq 2N + 1$).

This periodic signal $x_{N_0}(n)$ may be represented by an exponential Fourier series.

If we let $N_o \to \infty$, then the signal $x(n)$ repeats after an infinite interval, and hence

DO YOU KNOW?

The DTFT is always periodic with period one in the F domain or period 2π in the Ω domain.

$$\lim_{No \to \infty} x_{N_o}(n) = x(n)$$

Hence, the Fourier series representing $x_{N_o}(n)$ will also represent $x(n)$ in the limit $N_o \to \infty$. The exponential Fourier series for $x_{N_o}(n)$ is given as

$$x_{N_o}(n) = \sum_{r=<No>} D_r . e^{jr\omega_0 n}$$

$$\omega_0 = \frac{2\pi}{No} \quad ...(2.1)$$

where
$$D_r = \frac{1}{No} \sum_{n=-\infty}^{\infty} x(n)\, e^{-jr\omega_0 n} \quad ...(2.2)$$

Actually, the limits for the sum on the right-hand side of equation (2.1) must be from $-N$ to N. But because $x(n) = 0$ for $|n| > N$, hence it does not matter if the limits are taken from $-\infty$ to ∞.

To observe, how the nature of the spectrum changes as N_o increases, let us define $X(e^{j\omega})$ which is a continuous function of ω as

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \text{ s} \quad ...(2.3)$$

From this equation, we get

$$D_r = \frac{1}{No} X(j\omega_o) \quad ...(2.4)$$

Here,
$$\omega_0 = \frac{2\pi}{No}$$

This result shows that the Fourier coefficients D_r are $\frac{1}{No}$ times the samples of $X(j\omega)$ taken every ω_0 rad/sec. Hence, $\frac{1}{N_o}$ $X(j\omega)$ is the envelope for the coefficients D_r.

We now let $No \to \infty$ by doubling N_o repeatedly. Doubling No halves the fundamental frequency ω_0 and so the spacing between successive spectral components (or harmonics) is halved.

Hence, there are now twice as many components or samples in the spectrum. At the same time, by doubling N_o, the envelop of the coefficients D_r is halved as may be observed by equation (2.4).

Now, if we continue this process of doubling N_o repeatedly, the number of components doubles in each step and the spectrum progressively becomes denser while its magnitude D_r becomes smaller.

In the limit $N_o \to \infty$, the fundamental frequency $\omega_0 \to 0$ and $D_r \to 0$.

The separation between successive harmonics which is ω_0, is approaching zero (or infinite signal) and the spectrum becomes so dense that it appears **continuous.**

Let us observe what happens mathematically as the period $N_o \to \infty$.

According to equation (2.3)

$$X(j\omega_o) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jr\omega_0 n} \quad ...(2.5)$$

Using equations and we may express equation as

$$x_{N_o}(n) = \frac{1}{No} \sum_{r=<No>} X(r\omega_0)\, e^{jr\omega_0 n} \quad ...(2.6)$$

$$x_{N_o}(n) = \sum_{r=<N_o>} X(r\omega_0)\, e^{jr\omega_0 n} \left(\frac{\omega_0}{2\pi}\right) \qquad ...(2.7)$$

Now in the limit as $N_o \to \infty$, $\omega_o \to 0$ and $xN_o(n) \to x(n)$, we have

$$x(n) = \underset{\omega_o \to 0}{Lim} \sum_{r=<N_o>} \left[\frac{X(r\omega_0)\,\omega_o}{2\pi}\right] e^{jr\omega_o n} \qquad ...(2.8)$$

As $N_o \to 0$, ω_o becomes infinitesimal ($\omega_0 \to 0$). Due to this reason, we can replace ω_o with an infinitesimal notation $\Delta\omega$ as

$$\Delta\omega = \frac{2\pi}{\omega_0} \qquad ...(2.9)$$

Hence, equation (2.8) may be expressed as

$$x(n) = \underset{\Delta\omega \to 0}{Lim} \sum_{r=<N_o>} \left[\frac{X(r\Delta\omega)\,\Delta\omega}{2\pi}\right] e^{jr\Delta\omega n} \qquad ...(2.10)$$

or
$$x(n) = \underset{\Delta\omega \to 0}{Lim} \frac{1}{2\pi} \sum_{r=<N_o>} X(r\Delta\omega)\; e^{jr\Delta\omega n}\, \Delta\omega \qquad ...(2.11)$$

Here, the range $r = <N_o>$ implies the interval of N_o number of harmonics, which is $N_o\,\Delta\omega = 2\pi$ according to equation (2.9).

In this limit, the right-hand side of equation (2.11) becomes the integral as

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(j\omega)\, e^{j\omega n}\, d\omega \qquad ...(2.12)$$

Here, $\int_{2\pi}$ indicates integration over any continuous interval of 2π.

The spectrum $X(e^{j\omega})$ is given by equation (2.3)

i.e.,
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad ...(2.13)$$

The integral on the right-hand side of equation (2.12) is called as the **Fourier Integral**.

Hence, we have represented an aperiodic signal $x(n)$ by a Fourier integral rather than a Fourier series.

The function $X(e^{j\omega})$ acts as a spectral function which shows the relative amounts of various exponential components of $x(n)$.

Now, we call $X(e^{j\omega})$ as the **Discrete-Time Fourier Transform (DTFT)** of $x(n)$ and $x(n)$ is called as the **Inverse Discrete-Time Fourier Transform (IDTFT)** of $X(e^{j\omega})$.

Discrete-Time Fourier Transform (DTFT), $X(e^{j\omega})$ is periodic with period 2π. So any interval of length 2π is sufficient for the complete specification of the spectrum. Generally, we draw the spectrum in the fundamental interval $[-\pi, \pi]$.

It may be noted that all the spectral information contained in the fundamental interval is necessary for the complete description of the signal.

Therefore, the range of integration is always 2π.

2.2.1 The Discrete-Time Fourier Transform : Few Points

As discussed in last article

(i) The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a discrete-time signal $x(n)$ is expressed as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \quad ...(2.14)$$

or $$DTFT\, x(n) = X(e^{j\omega}) \quad ...(2.15)$$

And Inverse Discrete-Time Fourier Transform (IDTFT) is expressed as

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad ...(2.16)$$

or $$IDTFT\, X(e^{j\omega}) = x(n) \quad ...(2.17)$$

(ii) From equations (2.15) and (2.17), it is clear that $x(n)$ and $X(e^{j\omega})$ are a **Discrete-time Fourier transform pair.**

Symbolically, this may be expressed as

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega}) \quad ...(2.18)$$

(iii) The Discrete-time Fourier transform $X(e^{j\omega})$ is the frequency-domain representation of $x(n)$. The inverse discrete Fourier transform $x(n)$ (equation (2.16)) is called the **synthesis** equation. This synthesis equation indicates that an aperiodic signal $x(n)$ may be represented as a linear combination of complex exponentials infinitesimally close in frequency.

(iv) Like continuous-time Fourier transform, the frequency spectrum in discrete-time Fourier transform is also continuous in nature.

(v) There is a major difference between continuous-time Fourier transform (CTFT) and discrete-time Fourier transform (DTFT). The frequency spectrum is not periodic in continuous-time Fourier transform whereas in discrete-time Fourier transform, the frequency spectrum $X(e^{j\omega})$ is periodic with period 2π and the synthesis equation $x(n)$ involves an integration only over a frequency interval that produces distinct complex exponentials (or any interval of length 2π). This means that to synthesize $x(n)$, we need to use the spectrum over a frequency interval of only 2π, starting at any value of ω.

> **DO YOU KNOW?**
>
> The more a signal is localized in one domain (time or frequency), the less it is localized in the other domain.

2.2.2 Existence of Discrete-Time Fourier Transform (DTFT)

As discussed in last article, Discrete-time Fourier transform (DTFT) is expressed as

$$DTFT\, x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad ...(2.19)$$

Now since $\left|e^{-j\omega n}\right| = 1$

Then according to equation (2.19), the existence of DTFT $X(e^{j\omega})$ is guarnteed if $x(n)$ is absolutely summable. Mathematically,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \quad ...(2.20)$$

It may be noted that this condition is sufficient but it is not the necessary condition for the existence of $X(e^{j\omega})$.

EXAMPLE 2.1 **Find the Discrete-Time Fourier Transform (DTFT) of the discrete-time signal $x(n) = \gamma^n \cdot u(n)$ for $|\gamma| < 1$.** *(U.P. Tech. Tutorial Question Bank)*

Solution : Given that

$$x(n) = \gamma^n \cdot u(n) \qquad \text{for } |\gamma| < 1$$

We know that the DTFT is expressed as

$$DTFT\, x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

or
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \gamma^n\, u(n)\, e^{-j\omega n}$$

or
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \gamma^n\, e^{-j\omega n} \qquad [\because u(n) \text{ exists for positive values only}]$$

or
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (\gamma^n\, e^{-j\omega})^n \qquad ...(i)$$

It may be observed that equation (*i*) represents a geometric progression with a common ratio $\gamma e^{-j\omega}$.

Therefore, $X(e^{j\omega})$ may be written as

$$X(e^{j\omega}) = \frac{1}{1-\gamma e^{-j\omega}} \qquad \left[\text{provided that } \left|\gamma e^{-j\omega}\right| < 1\right]$$

But since $\qquad \left|e^{-j\omega}\right| = 1$

which implies that $|\gamma| < 1$.

But this condition is already given in question. Hence,

$$X(e^{j\omega}) = \frac{1}{1-\gamma e^{-j\omega}} \quad |\gamma| < 1 \qquad ..(ii)$$

If $|\gamma| > 1$, $X(e^{j\omega})$ will not converge.

Equation (*ii*) may be expressed as

$$X(e^{j\omega}) = \frac{1}{1-\gamma(\cos\omega - j\sin\omega)} \qquad [\because e^{j\theta} = \cos\theta + j\sin\theta]$$

or
$$X(e^{j\omega}) = \frac{1}{1-\gamma\cos\omega + j\gamma\sin\omega}$$

Therefore, magnitude spectrum is given by

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-\gamma\cos\omega)^2 + (\gamma\sin\omega)^2}} = \frac{1}{\sqrt{1+\gamma^2 - 2\gamma\cos\gamma}}$$

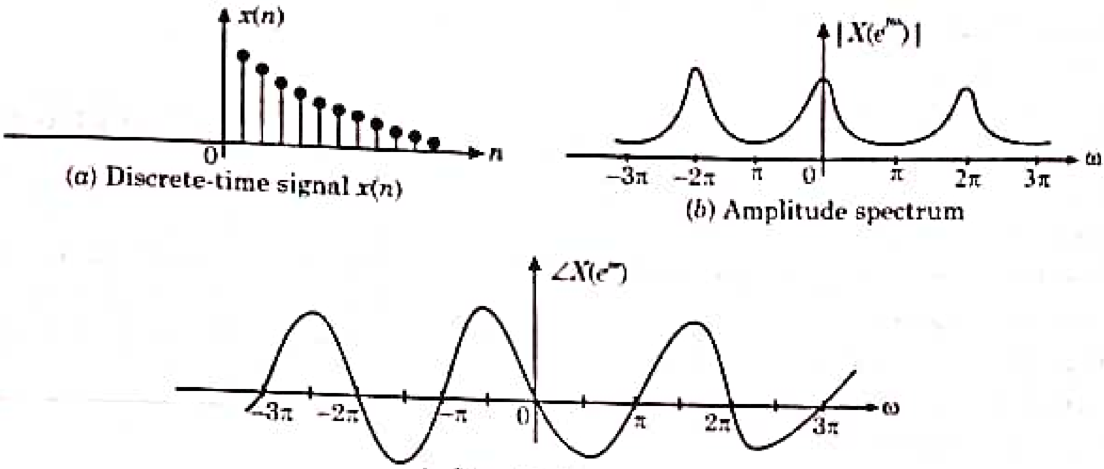
and phase spectrum

$$\angle X(e^{j\omega}) = -\tan^{-1}\left[\frac{\gamma\sin\omega}{1-\gamma\cos\omega}\right]$$

Figure 2.3 (*a*) shows $x(n) = \gamma^n u(n)$. Figure 2.3 (*b*) and (*c*) show the frequency spectrum of $x(n)$. Figure 2.3 (*b*) shows the magnitude (amplitude) spectrum and figure 2.3 (*c*) shows the phase spectrum.

It may be observed that the frequency spectrums are continuous and periodic functions of ω with the period 2π.

It may also be noted that amplitude spectrum $|X(e^{j\omega})|$ is an even function and the phase spectrum $\angle X(e^{j\omega})$ is an odd function of ω.



(a) Discrete-time signal $x(n)$

(b) Amplitude spectrum

(c) Phase spectrum

FIGURE 2.3

EXAMPLE 2.2 Figure 2.4 shows the discrete-time signal $x(n) = \delta(n)$. Find its discrete-time Fourier transform (DTFT).

Solution : We know that *DTFT* is expressed as

$$DTFT\, x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Substituting value of $x(n)$, we have

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n)e^{-j\omega n}$$

Using the shifting property of impulse function, we have

$$X(e^{j\omega}) = \left|e^{-j\omega n}\right|_{\text{at } n=0} = 1$$

Therefore, $\delta(n) \xleftrightarrow{DTFT} 1$

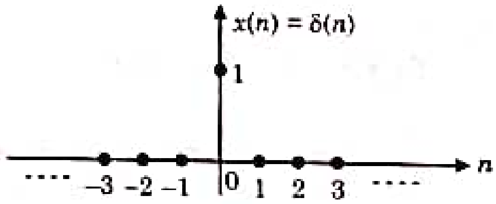


FIGURE 2.4.

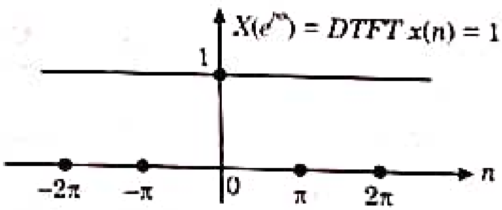


FIGURE 2.5 *DTFT of δ(n).*

2.3 PROPERTIES OF DISCRETE-TIME FOURIER TRANSFORM (DTFT)

In this article, we shall discuss about the various properties of Fourier transform of discrete-time signals. Fourier transform of discrete-time signals is also referred to as discrete-time Fourier transform (DTFT). These properties are often useful in reducing the complexity in evaluation of the Discrete-Time Fourier Transform (DTFT) and inverse Discrete-time Fourier Transform (IDTFF). Also, we shall observe some of the similarities and differences between CTFT and DTFT. The derivation of DTFT properties is essentially identical to its continuous-time counterpart, *i.e.* CTFT.

Here, we shall use one notation similar to that used for CTFT to indicate the pairing of a signal and its Fourier transform, *i.e.*,

$$X(e^{j\omega}) = \text{DTFT}\,[x(n)]$$

or

$$x(n) = \text{Inverse DTFT}\,[X(e^{j\omega})]$$

or

$$x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

In this section, we shall discuss following properties of the DTFT, in detail :

1. Periodicity of the DTFT
2. Linearity of the DTFT
3. Time shifting
4. Frequency shifting
5. Multiplication by n: Frequency Differentiation
6. Complex conjugation and conjugate symmetry
7. Time reversal of a discrete-time sequence
8. The convolution property for DTFT
9. Scaling property
10. Multiplication in time domain
11. Parseval's Theorem
12. Energy spectrum density
13. Duality

DO YOU KNOW?

Given the Fourier transform of a sequence, it is possible to use Fourier transform properties to determine whether a particular sequence has a number of different properties.

2.3.1 Periodicity

The Discrete-Time Fourier Transform is always periodic in ω with period 2π. Mathematically,

$$X[e^{j(\omega+2\pi k)}] = X(e^{j\omega})$$

It may be noted that this is opposite to the continuous-time Fourier transform (CTFT) which, in general, not periodic.

Proof : We know that

$$DTFT\, x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Substituting $\omega = \omega + 2\pi k$ in above equation, we get

$$X[e^{j(\omega+2\pi k)}] = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}.e^{-j2\pi kn} \quad ...(2.21)$$

Now, $\quad e^{-j2\pi kn} = \cos(2\pi kn) - j\sin(2\pi kn)$

It may be noted that in above equation both k and n are integers.

Thus $\quad \cos(2\pi kn) = 1$ always and

$\quad \sin(2\pi kn) = 0$ always

Hence, $\quad e^{-j2\pi kn} = 1$

Therefore, equation (2.21) becomes

$$X[e^{j(\omega+2\pi k)}] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

or $\quad X[e^{j(\omega+2\pi k)}] = X(e^{j\omega})$

or $\quad X[e^{j(\omega+2\pi k)}] =$ DTFT of $x(n)$ **Hence Proved.**

2.3.2 Linearity

According to this property, the discrete-time Fourier transform (DTFT) is linear. Mathematically,

If $\quad x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

and $\quad x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

Then, according to this property, we have

$$z(n) = ax_1(n) + bx_2(n) \xleftrightarrow{DTFT} Z(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

where a and b are constants.

Proof : We know that

$$\text{DTFT } z(n) = Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Substituting $z(n) = ax_1(n) + bx_2(n)$ *i.e.*, a linear combination of two inputs in above equation, we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} [ax_1(n)+bx_2(n)]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} ax_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2(n)\, e^{-j\omega n}$$

or
$$Z(e^{j\omega}) = a \sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j\omega n} = a\, X_1(e^{j\omega}) + b\, X_2(e^{j\omega})$$

Thus, the outputs are also linearly related. This is superposition principle. **Hence Proved.**

2.3.3 Time-Shifting Property

(U.P. Tech., Tutorial Question Bank)

This property states that if a discrete-time signal is shifted in the time-domain by n_0 samples, its magnitude spectrum remains unchanged. However, the phase spectrum is changed by an amount $-\omega n_0$.

Mathematically,

If
$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

Then $x(n - n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} \cdot X(e^{j\omega})$; where n_0 is an integer.

Proof : We know that, $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

or
$$DTFT\, x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Therefore,

$$DTFT\, x(n - n_0) = \sum_{n=-\infty}^{\infty} x(n - n_0)\, e^{-j\omega n}$$

Putting $n - n_0 = m$, so that $n = m + n_0$, we get

$$DTFT\, x(n - n_0) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega(m + n_0)}$$

or
$$DTFT\, x(n - n_0) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m} \cdot e^{-j\omega n_0}$$

or
$$DTFT \cdot x(n - n_0) = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m} = e^{-j\omega n_0}\, X(e^{j\omega})$$

or $$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

Hence, according to above equation, time-shifting property states that delaying a discrete-time signal by n_0 units does not change its amplitude spectrum. The phase-spectrum, however, is changed by $-\omega n_0$.

This added phase is thus a linear function of ω with a slop $-n_0$.

This means that the time-delay in a discrete-time signal causes a linear phase shift in its spectrum.

2.3.4 Frequency-Shifting Property

This property states that multiplication of a sequence $x(n)$ by $e^{j\omega_0 n}$ is equivalent to a frequency translation of the spectrum $X(e^{j\omega})$ by ω_0. Since the spectrum $X(e^{j\omega})$ is periodic, the shift ω_0 applies to the spectrum of the signal in every period.

Mathematically, we have

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

Then $$e^{j\omega_0 n}\, x(n) \xleftrightarrow{DTFT} X[e^{j(\omega-\omega_0)}]$$

Proof : We know that

$$DTFT\; x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

then $$DTFT\; e^{j\omega_0 n}\, x(n) = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n} \cdot e^{-j\omega n}$$

or $$DTFT\; e^{j\omega_0 n} \cdot x(n) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n}$$

or $$DTFT\; e^{j\omega_0 n} \cdot x(n) = X[e^{j(\omega-\omega_0)}]$$

or $$e^{j\omega_0 n} \cdot x(n) \xleftrightarrow{DTFT} X[e^{j(\omega-\omega_0)}]$$

Figure 2.6 illustrates the frequency shifting properly

DO YOU KNOW?

As a consequence of the periodicity and frequency shifting properties of the discrete-time Fourier transform (DTFT), there exists a special relationship between ideal low-pass and ideal highpass discrete-time filters.

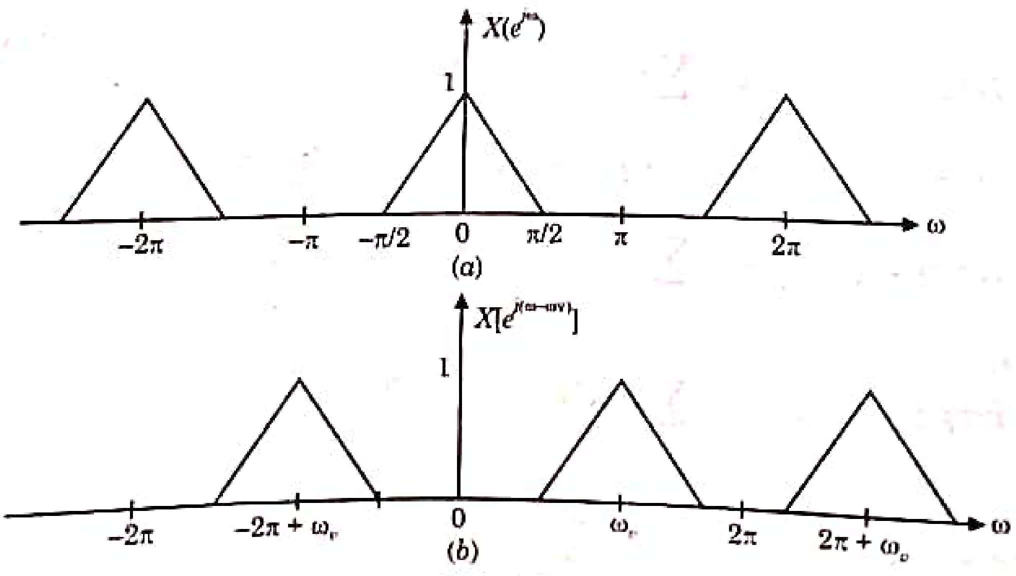


FIGURE 2.6 *Illustration of frequency shifting property*

2.3.5 Multiplication by n : Frequency Differentiation

This property states that

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$n\,x(n) \xleftrightarrow{DTFT} j\,\frac{dX(e^{j\omega})}{d\omega}$$

Proof : We know that

$$DTFT\,x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n}$$

Now $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} \frac{d}{d\omega}\left[x(n)\,e^{-j\omega n}\right]$$

or $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jn\,x(n)\,e^{-j\omega n}$$

or $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jn\,x(n)\,e^{-j\omega n} = DTFT\,[-jn\,x(n)]$$

or $$-jn\,x(n) \xleftrightarrow{DTFT} \frac{dX(e^{j\omega})}{d\omega}$$

or $$nx(n) \xleftrightarrow{DTFT} j\,\frac{dX(e^{j\omega})}{d\omega}$$ **Hence Proved.**

2.3.6 Complex Conjugation and Conjugate Symmetry

This property states that we can obtain complex conjugation of a complex discrete-time signal $x(n)$ by reversing the sign of the imaginary part of the complex signal, $x(n)$.

Mathematically,

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$x^*(n) \xleftrightarrow{DTFT} X^*(e^{-j\omega})$$

Now, if the discrete-time signal $x(n)$ is a real valued signal, then its DTFT $X(e^{j\omega})$ will be conjugate symmetric. This means that

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \qquad [\text{if } x(n) \text{ is real}]$$

> **Note :** *From above, it is clear that $Re[X(e^{j\omega})]$ is an even function of ω and $I_m\,[X(e^{j\omega})]$ is an odd function of ω. In the same way, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle $\underline{\angle X(e^{j\omega})}$ is an odd function.*

2.3.7 Time Reversal

According to this property, if a discrete-time signal is folded about the origin in time, its magnitude spectrum remains unchanged, however, the phase spectrum undergoes a change in sign. Mathematically, we have

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})$$

Proof : Let there be a discrete-time signal $x(n)$ with $DTFT\ X(e^{j\omega})$, i.e.,

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

Let there be another discrete-time signal $y(n)$ with $DTFT\ Y(e^{j\omega})$, i.e.,

$$y(n) \xleftrightarrow{DTFT} Y(e^{j\omega}) \qquad \text{such that } y(n) = x(-n)$$

Now,
$$Y(e^{j\omega}) = DTFT\ y(n) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

Substituting $m = -n$ into above expression, we get

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-m)\omega}$$

or
$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{j\omega m} = X(e^{-j\omega}) = DTFT\,[x(-n)]$$

Therefore, $x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})$ **Hence Proved.**

2.3.8 Convolution Property

(U.P. Tech., Tutorial Question Bank)

In third chapter, we have discussed the importance of CTFT with respect to its effect on the convolution operation and its application in dealing with continuous-time LTI systems. In this section, we shall discuss the importance of DTFT with respect to its effect on the convolution operation and analysis of discrete-time LTI systems.

The convolution property states that

If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

and $y(n) \xleftrightarrow{DTFT} Y(e^{j\omega})$

then $z(n) = x(n) \otimes y(n) \xleftrightarrow{DTFT} Z(e^{j\omega}) = X(e^{j\omega})\ Y(e^{j\omega})$

Proof : We know that

$$DTFT\ z(n) = Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Substituting
$$z(n) = x(n) \otimes y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

in above equation, we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, y(n-k) \right]$$

Changing the order of summations we get

$$Z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, e^{-j\omega n}$$

Again, substituting $n - k = m$, the above equation becomes

$$Z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega(m+k)} = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega m}\, e^{-j\omega k}$$

or
$$Z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega k} \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega m} = X(e^{j\omega})\, Y(e^{j\omega})$$

Thus convolution of the two sequences is equivalent to multiplication of their spectrums.

Now, let us apply DTFT for representing and analyzing discrete-time LTI systems, particularly, when $x(n)$, $h(n)$ and $y(n)$ are the input, impulse response and the output, respectively of a discrete-time LTI system as shown in figure 2.7.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow x(n)$$

FIGURE 2.7 *Discrete-time LTI system*

Now, here, the output $y(n)$ can be determined by convolving $x(n)$ and $h(n)$, i.e.,

$$y(n) = x(n) \otimes h(n) \qquad ...(2.22)$$

Using DTFT, the above equation takes the form

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) \qquad ...(2.23)$$

DO YOU KNOW?

Convolution and multiplication of functions are dual operations in the time and frequency domains.

where $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ are the DTFTs of $x(n)$, $h(n)$ and $y(n)$ respectively.

Now, combining equations (2.22) and (2.23), we have

$$y(n) = x(n) \otimes h(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}) \qquad ...(2.24)$$

where $H(e^{j\omega})$ is the discrete-time Fourier transform (DTFT) of the impulse response $h(n)$ of the discrete-time LTI system. **It is also known as the frequency response of the discrete-time LTI system.**

EXAMPLE 2.3 Determine the frequency response of a discrete-time LTI system with impulse response $h(n) = \delta(n - n_0)$. Also determine the output for this system.

Solution : We know that the frequency response of a discrete-time LTI system is equal to the DTFT of the impulse response $h(n)$ of the system. The frequency response is determined as

$$H(e^{j\omega}) = \text{DTFT}\,[h(n)] = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

or
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n - n_0)\, e^{-j\omega n} = e^{-j\omega n_0} \,\star \qquad ...(i)$$

Now, according to the convolution property of DTFT, we have

$$y(n) = h(n) \otimes x(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

or
$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega}) \qquad ...(ii)$$

Substituting equation (*i*) in equation (*ii*), we have

$$Y(e^{j\omega}) = e^{-j\omega n_0}\, X(e^{j\omega}) \qquad ...(iii)$$

The output $y(n)$ of above discrete-time LTI system is determined by taking the inverse DTFT of equation (*iii*).

Thus, taking inverse DTFT, we get

$$y(n) = \text{Inverse DTFT}\,[Y(e^{j\omega})] = \text{Inverse DTFT}\,[e^{-j\omega n_0}\, X(e^{j\omega})]$$

or
$$y(n) = x(n - n_0)$$

*Hence, it may be noted that in this example, the output $y(n)$ is equal to the shifted version of the input $x(n)$ by a constant time n_0. The frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ is purely time-shifted and has unity magnitude at all the frequencies. Its phase characteristics are equal to $-\omega n_0$ i.e., **it is linear with frequency.***

2.3.9 Scaling

Let the discrete-time sequence be scaled as

$$y(n) = x(pn) \text{ for } p\text{-integer}$$

In this case, information in $x(n)$ is discarded. Then scaling property has no meaning with such sequences. Hence, this property is applicable only to those sequences for which,

$$x(n) = 0 \text{ for } \frac{n}{p} \neq \text{integer}$$

then $x(pn) \neq 0$ for all n values.

Then data will not be discarded. The discarded data due to scaling will be zeros.

Thus, the scaling property may be expressed as

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$y(n) = x(pn) \xleftrightarrow{DTFT} Y(e^{j\omega}) = X\left(\frac{\omega}{p}\right)$$

Proof : We know that

$$\text{DTFT } y(n) = Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

or $$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(pn)\, e^{-j\omega n} \qquad \text{...(2.25)}$$

Let us substitute $pn = m$

Now, since n has the range of $-\infty$ to ∞, m will also have the same range.

then above equation (2.25) becomes

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m/p} = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\left(\frac{\omega}{p}\right)m} = X\left(e^{j\frac{\omega}{p}}\right)$$

Thus, expanding in time-domain is equivalent to compressing in frequency domain.

Hence Proved.

2.3.10 Multiplication in Time-Domain

This property states that the multiplication of two time-domain sequences is equivalent to the convolution of their Discrete-time Fourier transforms.

Mathematically,

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

and $$y(n) \xleftrightarrow{DTFT} Y(e^{j\omega})$$

then $$z(n) = x(n)\, y(n) \xleftrightarrow{DTFT} Z(e^{j\omega}) = \frac{1}{2\pi}[X(e^{j\omega}) \otimes Y(e^{j\omega})]$$

Proof : We know that

$$\text{DTFT } z(n) = Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Substituting $z(n) = x(n)\,y(n)$ in above equation, we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, y(n)\, e^{-j\omega n} \qquad \text{...(2.26)}$$

Also, inverse DTFT is expressed as

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})\, e^{j\lambda} d\lambda$$

Here, we have used separate frequency variable λ. Substituting the above expression of $x(n)$ in equation (2.26), we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})\, e^{j\lambda n}\, d\lambda . y(n)\, e^{-j\omega n}$$

Now, interchanging the order of summation and integration, we get

$$Z(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda}) \sum_{n=-\infty}^{\infty} y(n)\, e^{j\lambda n}\, e^{-j\omega n}\, d\lambda = \frac{1}{12}\int_{-\pi}^{\pi} X(e^{j\lambda}) \left[\sum_{n=-\infty}^{\infty} y(n)\, e^{-j(\omega-\lambda)n}\right] d\lambda$$

The term in brackets is equal to $Y[e^{j(\omega-\lambda)}]$.

Thus, above equation becomes

$$Z(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})\, Y[e^{j(\omega-\lambda)}]\, d\lambda$$

But, above equation represents convolution of $X(e^{j\omega})$ and $Y(e^{j\omega})$, *i.e.*,

$$Z(e^{j\omega}) = \frac{1}{2\pi}[X(e^{j\omega}) \otimes Y(e^{j\omega})]$$

Thus multiplication of the sequences in time-domain is equivalent to convolution of their spectrums. **Hence Proved**

2.3.11. Parseval's Theorem

Parseval's theorem states that the total energy of a discrete-time signal $x(n)$ may be determined by the knowledge of its Discrete-Time Fourier Transform (*DTFT*), $X(e^{j\omega})$.

Mathematically,

If $\quad x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

Then, according to Parseval's theorem, the energy E of discrete-time signal $x(n)$ is expressed as

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

Proof : We know that the energy of a discrete-time signal $x(n)$ is expressed as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Since $\quad |x(n)| = x(n)\, x^*(n)$

Therefore,
$$E = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n) \qquad ...(2.27)$$

We know that the inverse DTFT of $x^*(n)$ is expressed as

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})e^{-j\omega n} d\omega$$

Thus, replacing $x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega$, in equation (2.27), we get

$$E = \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega\right]$$

Integrating the order of integration and summation, we get

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right] d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, X(e^{j\omega})\, d\omega$$

But we know that $X^*(e^{j\omega})\, X(e^{j\omega}) = |X(e^{j\omega})|^2$

Therefore, we have

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

Thus, energy 'E' of the discrete-time signal $x(n)$ is

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega \qquad ...(2.28)$$

This is the Parseval's theorem for discrete-time aperiodic signals with finite energy which states that energy of a discrete-time signal may also be obtained with the help of DTFT.

Hence Proved

2.3.12 Energy Density Spectrum of Discrete-Time Aperiodic Signals

We know that energy of a discrete-time signal $x(n)$ is expressed as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad ...(2.29)$$

According to Parseval's theorem, this energy may also be expressed in terms of discrete-time Fourier transform as under :

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

Substituting $|X(e^{j\omega})|^2 = \psi(e^{j\omega})$ in above expression, we get

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(e^{j\omega})\, d\omega$$

Hence, it may be observed that the quantity

$$\psi(e^{j\omega}) = |X(e^{j\omega})|^2$$

represents the distribution of energy as a function of frequency and so it is known as the **energy density spectrum of discrete-time signal $x(n)$.**

It may also be noted that $\psi(e^{j\omega})$ does not contain any phase information.
Now, if $x(n)$ is real, then we have

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$

or equivalently $$|X(e^{-j\omega})| = |X(e^{j\omega})|$$

This is called even symmetry.

Now, since $$\psi(e^{j\omega}) = |X(e^{j\omega})|^2$$

Therefore, it follows that

$$\psi(e^{-j\omega}) = \psi(e^{j\omega}) \quad \text{(even symmetry)}$$

Hence Proved.

2.3.13 Duality

In case of continuous-time Fourier transform, a duality or symmetry may be observed between the analysis equation and the synthesis equation. However for discrete-time Fourier transform, no duality exists between the analysis equation and the synthesis equation. But there exists a duality in the discrete-time Fourier series. In addition to this, there exists a duality between the discrete-time Fourier transform and the continuous-time Fourier series.

DO YOU KNOW?

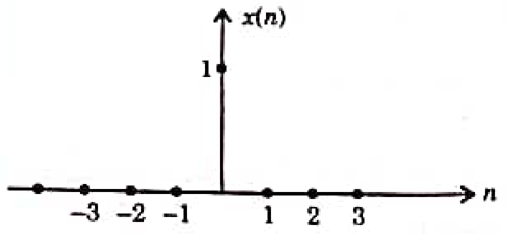
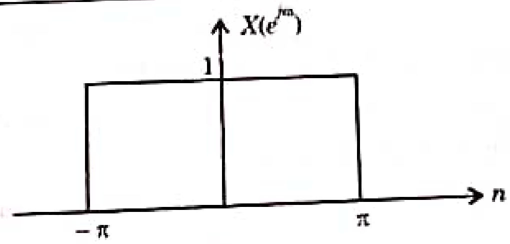
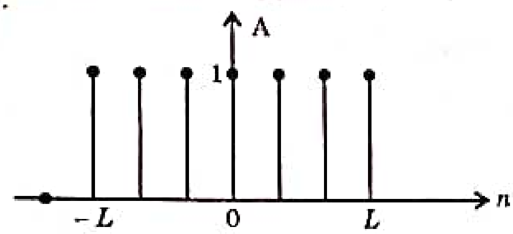
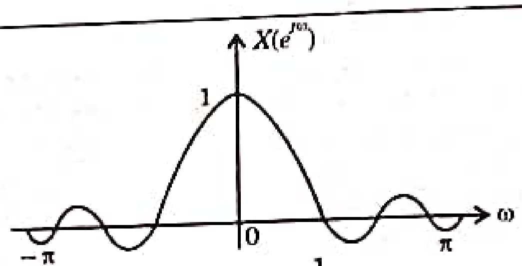
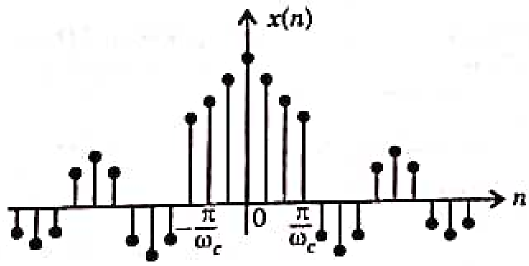
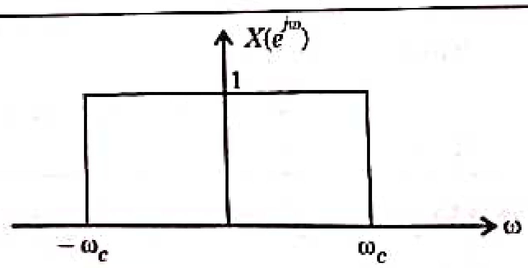
The duality between the discrete-time Fourier transform synthesis equation and the continuous-time Fourier series analysis equation may be exploited to determine the discrete-time Fourier transform of the sequence :

$$x(n) = \frac{\sin(\pi n/2)}{\pi n}$$

TABLE 2.1 *Properties of Discrete-Time Fourier Transform*

S. No.	Name of Property	Time-Domain Expression	Frequency-Domain Expression
1.	Notation	$x(n)$	$X(e^{j\omega})$
2.	Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$
3.	Time-shifting	$x(n - n_0)$	$e^{-j\omega n_0} X(e^{j\omega})$
4.	Frequency-shifting	$e^{j\omega_0 n} x(n)$	$X(e^{j(\omega - \omega_0)})$
5.	Frequency-differentiation	$n\,x(n)$	$j\frac{dX(e^{j\omega})}{d\omega}$
6.	Conjugation	$x^*(n)$	$X^*(e^{-j\omega})$
7.	Time-reversal	$x(-n)$	$X(e^{-j\omega})$
8.	Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$
9.	Multiplication	$x_1(n)\,x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(e^{j\omega-\lambda})d\lambda$
10.	Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(e^{j\omega+\omega_0}) + \frac{1}{2}X(e^{j\omega-\omega_0})$
11.	Scaling	$x(pn)$	$X\left(\frac{\omega}{p}\right)$
12.	Convolution	$x(n) \otimes y(n)$	$X(e^{j\omega})\,Y(e^{j\omega})$

TABLE 2.2 *Few Useful Discrete-Time Fourier Transform Pairs*

S.No.	Discrete-Time Signal $x[n]$	Discrete-time Fourier Transform
1.	 $x(n) = \delta(n)$	 $X(e^{j\omega}) = 1$
2.	 $x(n) = \begin{cases} A & \lvert n \rvert \le L \\ 0 & \lvert n \rvert \ge L \end{cases}$	 $X(e^{j\omega}) = A\dfrac{\sin\left(L+\frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}$
3.	 $x(n) = \begin{cases} \dfrac{\omega_c}{\pi} & \text{for } n = 0 \\ \dfrac{\sin \omega_c n}{\pi n} & \text{for } n \neq 0 \end{cases}$	 $X(e^{j\omega}) = \begin{cases} 1 & \text{for } \lvert\omega\rvert < \omega_c \\ 0 & \text{for } \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$
4.	$x(n) = \begin{cases} a^n & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$	$X(e^{j\omega}) = \dfrac{1}{1 - ae^{-j\omega}}$

2.4 DISCRETE-TIME LTI SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

We know that the general form of a linear constant-coefficient difference equation for a discrete-time LTI system can be written as under :

$$\sum_{k=0}^{N} A_k y(n-k) = \sum_{k=0}^{M} B_k x(n-k) \qquad \text{...(2.30)}$$

This difference equation is generally known as Nth order difference equation.

Here, A_k and B_k are the constant coefficients, $x(n)$ and $y(n)$ are the inputs and output of the discrete-time LTI system, respectively.

Here, we shall determine the frequency response $H(e^{j\omega})$ for an LTI system described by equation (2.30). In determine the frequency response $H(e^{j\omega})$ of a discrete-time LTI system, we shall take advantage of various important properties of the DTFT.

There are two related methods for determining the frequency response $H(e^{j\omega})$ of the discrete-time LTI system.

In the first-method, we explicity use the fact that complex exponential functions are eigenfunctions of LTI systems. Specifically, if $x(n) = e^{j\omega n}$ is the input to a discrete-time LTI systems then the output of this system is given by the multiplication of $H(e^{j\omega})$ and $e^{j\omega n}$.

Now, substituting $x(n) = e^{j\omega n}$ and $y(n) = H(e^{j\omega})$ in equation (2.30), we obtain

$$\sum_{k=0}^{N} A_k\, y(n-k) = \sum_{k=0}^{M} B_k\, x(n-k)$$

or $$\sum_{k=0}^{N} A_k\, H(e^{j\omega})\, e^{j\omega(n-k)} = \sum_{k=0}^{M} B_k\, e^{j\omega(n-k)}$$

or $$H(e^{j\omega}) \sum_{k=0}^{N} A_k\, e^{j\omega n} e^{-j\omega k} = \sum_{k=0}^{M} B_k e^{j\omega n}\, e^{-j\omega k}$$

or $$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} B_k\, e^{-j\omega k}}{\sum_{k=0}^{N} A_k\, e^{-j\omega k}} \qquad \text{...(2.31)}$$

In the second-method, we make use of the convolution, linearity and time-shifting properties of DTFT. Let $X(e^{j\omega})$, $Y(e^{j\omega})$ be the DTFT of the input $x(n)$, output $y(n)$ and impulse response $h(n)$ respectively. Figure 2.11, shows a discrete-time LTI system.

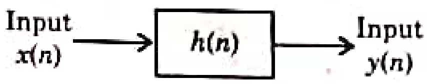


FIGURE 2.8 *Discrete-time LTI system.*

Using the convolution property of the DTFT, we have

$$y(n) = x(n) \otimes h(n) \xleftrightarrow{DTFT} Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}) \qquad \text{...(2.32)}$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

or $$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad \text{...(2.33)}$$

Now, taking the DTFT of both sides of equation (2.30) and using the properties of linearity and time-shifting of DTFT, we obtain

$$\sum_{k=0}^{N} A_k y(n-k) = \sum_{k=0}^{M} B_k\, x(n-k) \qquad \text{...(2.34)}$$

$$DTFT\left[\sum_{k=0}^{N} A_k y(n-k)\right] = DTFT\left[\sum_{k=0}^{M} B_k x(n-k)\right]$$

or $$\sum_{k=0}^{N} A_k DTFT\,[y(n-k)] = \sum_{k=0}^{M} B_k DTFT\,[x(n-k)]$$

or
$$\sum_{k=0}^{N} A_k e^{-jk\omega_0} Y(e^{j\omega}) = \sum_{k=0}^{M} B_k e^{-jk\omega_0} X(e^{j\omega})$$

or
$$Y(e^{j\omega}) \sum_{k=0}^{N} A_k e^{-jk\omega_0} = X(e^{j\omega}) \sum_{k=0}^{M} B_k e^{-jk\omega_0}$$

or
$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} B_k e^{-jk\omega}}{\sum_{k=0}^{N} A_k e^{-jk\omega}} \qquad ...(2.35)$$

From equations (2.33) and (2.35) we get

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} B_k e^{-jk\omega}}{\sum_{k=0}^{N} A_k e^{-jk\omega}} \qquad ...(2.36)$$

DO YOU KNOW?

If in LTI system is stable, then, its impulse response is absolutely summable. Therefore, the frequency response always converges for stable systems.

Note : *It may be observed from equation (2.36) that frequency response $H(e^{j\omega})$ is a ratio of two polynomials in the variable $e^{-j\omega}$. The coefficient of the numerator polynomial are the same as the ones that appear on the RHS of equation (2.30) ones that appear on the LHS of equation (2.30). Therefore, frequency response of an LTI system described by equation (2.30) can be written by inspection.*

SUMMARY

1. Broadly, there are two types of signals namely continuous-time signals and discrete-time signals. A signal (continuous-time or discrete-time) may be analysed with the help of Fourier Series and Fourier Transform. Fourier Series is used for periodic signals and Fourier transform is used for aperiodic signals. In general, Fourier transform may be used for both aperiodic and periodic signals.
2. When we analyse continuous-time signals with the help of Fourier Series and Fourier Transform, then the Fourier series and Fourier Transform are called continuous-time Fourier Series and continuous-time Fourier Transform respectively or simply Fourier Series and Fourier Transform.
3. If discrete-time signals are analysed with the help of Fourier Series and Fourier Transform, then the Fourier Series and Fourier Transform are known as Discrete-time Fourier Series (DTFS) and Discrete-time Fourier Transform (DTFT) respectively.
4. Like continuous-time Fourier transform, we shall begin our analysis with discrete-time aperiodic signals in order to develop a Fourier-Transform representation for discrete-time aperiodic signals and then extend this approach of discrete-time Fourier Transform for discrete-time periodic signals.
5. The function $X(e^{j\omega})$ acts as a spectral function which shows the relative amounts of various exponential components of $x(n)$.
6. We call $X(e^{j\omega})$ as the Discrete-Time Fourier Transform (DTFT) of $x(n)$ and $x(n)$ is called as the Inverse Discrete-Time Fourier Transform (IDTFT) of $X(e^{j\omega})$.
7. Discrete-Time Fourier Transform (DTFT), $X(e^{j\omega})$ is periodic with period 2π. So any interval of length 2π is sufficient for the complete specification of the spectrum. Generally, we draw the spectrum in the fundamental interval $[-\pi, \pi]$.
8. All the spectral information contained in the fundamental interval is necessary for the complete description of the signal.

9. Inverse Discrete-Time Fourier Transform (*IDTFT*) is expressed as

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

or $$IDTFT\; X(e^{j\omega}) = x(n)$$

10. Symbolically, Discrete-time Fourier transform pair may be expressed as

$$x(n) \xrightarrow{DTFT} X(e^{j\omega})$$

11. The Discrete-time Fourier transform $X(e^{j\omega})$ is the frequency-domain representation of $x(n)$. The inverse discrete Fourier transform $x(n)$ is called the synthesis equation. This synthesis equation indicates that an aperiodic signal $x(n)$ may be represented as a linear combination of complex exponentials infinitesimally close in frequency.
12. Like continuous-time Fourier transform, the frequency spectrum in discrete-time Fourier transform is also continuous in nature.
13. There is a major difference between continuous-time Fourier transform (CTFT) and discrete-time Fourier transform (DTFT). The frequency spectrum is not periodic in continuous-time Fourier transform whereas in discrete-time Fourier transform, the frequency spectrum $X(e^{j\omega})$ is periodic with period 2π and the synthesis equation $x(n)$ involves an integration only over a frequency interval that produces distinct complex exponentials (or any interval of length 2π). This means that to synthesize $x(n)$, we need to use the spectrum over a frequency interval of only 2π, starting at any value of ω.
14. The Discrete-Time Fourier Transform is always periodic in ω with period 2π.

SHORT QUESTIONS WITH ANSWERS

Q.1. Define Fourier Transform of a sequence.

Ans. The Fourier Transform of a discrete time signal $x(n)$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Q.2. Write down the sufficient condition for the existence of Discrete Time Fourier Transform (DTFT)?

Ans. For a sequence $x(n)$, the sufficient condition is as below :

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

Q.3. State Parseval's theorem for discrete time signals.

Ans. Parseval's theorem for discrete-time signals is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 . d\omega$$

Q.4. What is DTFT pair ?

Ans. The Fourier transform pair of discrete time signals is given by

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}).e^{j\omega n} d\omega \;; \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n).e^{-j\omega n}$$

Q.5. State the properties of Fourier Transform of a discrete-time aperiodic sequence.

Ans. (*i*) The Fourier spectrum of an aperiodic sequence is continuous.

(*ii*) The Fourier spectrum is periodic with period 2π.

REVIEW QUESTIONS

1. Define Discrete-time Fourier transform and inverse discrete-time Fourier transform.
2. Explain the difference between continuous-time Fourier transform and discrete-tiem Fourier transform.
3. Explain the periodicity of discrete-time Fourier transform.
4. Explain the linearity property of discrete-time Fourier transform.
5. State and prove time-shifting property of discrete-time Fourier transform.
6. State and prove frequency-shifting property of discrete-time Fourier transform.
7. State and prove frequency differentiation property of the discrete-time Fourier transform.
8. State and prove frequency differentiation property of the discrete-time Fourier transform.
9. Explain the conjugation property of DTFT.

NUMERICAL PROBLEMS

1. Find the response of a disctete-time LTI system with impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

for input $$x(n) = \left(\frac{3}{n}\right)^n u(n)$$

Use DTFT analysis and synthesis equations. [Ans. $y(n) = 3\left(\frac{3}{4}\right)^n u(n) - 2\left(\frac{1}{2}\right)^n u(n)$]

2. The difference equation of a causal discrete-time LTI system is given as

$$y(n) = -\frac{1}{2} y(n-1) + x(n)$$

(a) Find frequency response $H(e^{j\omega})$ for the system
(b) Find output response to the input given as

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$ [Ans. (a) $H(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$ (b) $\frac{1}{2}\left(\frac{1}{2}\right)^n u(n) + \frac{1}{2}\left(-\frac{1}{2}\right)^n u(n)$]

3. Two discrete-time LTI systems are cascaded. These two systems have frequency response

$$H_1(e^{j\omega}) = \frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}} \qquad H_2(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}}$$

(a) Find the difference equation describing the overall system
(b) Find the impulse response of the over all system

[Ans. (a) $y(n) + \frac{1}{8} y(n-3) = 2s(n) - s(n-1)$

(b) $h(n) = \frac{4}{3}\left(-\frac{1}{2}\right)^n u(n) + \frac{1+\sqrt{3}}{3}\left(\frac{1}{2}ae^{j/120}\right)^n u(n) + \frac{1-j\sqrt{3}}{3}\left(\frac{1}{2}e^{-j/120}\right)^n u(n)$]

CHAPTER

3

Review of z-Transform

SYLLABUS

z-transform: Basic fundamentals Region of Convergence (ROC), Properties of z-transform, the inverse z-transform, Block-diagram representation for discrete-time LTI systems.

Inside this Chapter

- Introduction
- The z-Transform
- Region of Convergence (ROC)
- Properties of z-Transform
- The Inverse z-Transform
- Applications of z-Transform in the Analysis of Discrete-Time Linear-Time Invariant (LTI) Systems
- Block Diagram Representation for Discrete-Time LTI Systems

CHAPTER

4

Discrete Fourier Transform

SYLLABUS

Discrete Fourier Transform: Frequency Domain Sampling: The Discrete Fourier Transform Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals. The Discrete Fourier Transform (DFT). The DFT as a linear Transformation. Ralationship of the DFT to Other Transforms, Properties of the DFT, Periodicity, Linearity, and Symmetry Properties. Multiplication of two DFTs and Circular Convolution. Additional DFT Properties. Frequency analysis of signals using the DFT.

Inside this Chapter

- Introduction
- Frequency Domain Sampling
- The Discrete Fourier Transform (DFT)
- The DFT as a Linear Transformation
- Relationship of the DFT to Other Transforms
- Relationship of DFT to the Fourier Series Coefficients of a Periodic Sequence
- Relationship of DFT to the Fourier Transform of an Aperiodic Sequence
- Relationship of the DFT to the z-transform
- Relationship to the Fourier Series Coefficients of a Continuous-time Signal
- Properties of Discrete Fourier Transfer (DFT)
- Additional DFT Properties
- Frequency Analysis of Signals Using the DFT
- Properties of Discrete Fourier Transfer (DFT)
- Additional DFT Properties
- Frequency Analysis of Signals Using the DFT

4.1 INTRODUCTION

In previous chapters, we have discussed the representation of sequences and Linear shift Invariant (LSIV) systems in terms of the Fourier transforms and z-transforms. If the sequences are of finite duration, it forms a special case and we shall see that it is possible to develop an alternative Fourier representation, which we refer to as Discrete Fourier Transform (DFT).

As a matter of fact, the frequency analysis of discrete-time signals is usually and conveniently performed on a digital signal processor. Now, this digital signal processor may be a general purpose digital computer or a specially designed digital hardware. We know that the Fouier transform of a discrete time signal $x(n)$ is called Discrete-Time Fourier Transform (DTFT) and it is denoted by $X(e^{j\omega})$. We also know that DTFT $X(e^{j\omega})$ is a continuous function of frequency ω. Therefore, this type of representation is not a computationally convenient representation of a discrete-time signal $x(n)$. Thus, taking one step further, we represent a sequence by samples of its continuous spectrum. This type of frequency-domain representation of a signal is known as **Discrete-Fourier Transform (DFT)**. It is very powerful tool for frequency analysis of discrete-time signals.

It may be noted that the Discrete-Fourier Transform (DFT) is itself a sequence rather than a function of a continuous variable and it corresponds to equally spaced frequency samples of Discrete-Time Fouier Transform (DTFT) of a signal. Also, Fourier series representation of the periodic sequence corresponds to the Discrete Fourier Transform (DFT) of the finite length sequence.

In short, we can say that DFT is used for transforming discrete-time sequence x(n) of finite length into discrete-frequency sequence X(k) of finite length. This means that by using DFT, the discrete-time sequence x(n) is transformed into corresponding discrete-frequency sequence X(k).

Further, in the design of a DSP system, two fundamental tasks are involved. These are :

(*i*) analysis of the input signal, and

(*ii*) design of a processing system to provide the desired output.

In this perspective, the discrete Fourier transform (DFT) and Fast Fourier Transform (FFT) are very important mathematical tools to carry out these types of tasks. Both these transforms can be used to analyse a two-dimensional signal.

4.2 FREQUENCY DOMAIN SAMPLING: THE DISCRETE FOURIER TRANSFORM *(U.P. Tech., Tutorial Question Bank)*

Before we discuss discrete Fourier transform (DFT) in detail, let us first introduce the sampling of the Fourier transform of an aperiodic discrete-time sequence. This means that we shall establish the relationship between the sampled Fourier transform and the discrete Fourier transform (DFT).

4.2.1 Frequency-Domain Sampling

According to the definition of DTFT, we write

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

We know that $X(\omega)$ is Fourier transform of discrete time signal $x(n)$.. The range of 'ω' is from 0 to 2π or $-\pi$ to π. Hence, it is not possible to compute $X(\omega)$ on dgital computer. Because in above expression, the range of summation is from $-\infty$ to $+\infty$. However, if we make this range finite then it is possible to do these calculations on digital computer.

When a Fourier transform is calculated only at discrete points then it is called as discrete fourier transform (DFT).

Now, if we have aperiodic time domain signal then discrete time Fourier transform (DTFT) is obtained. But, DTFT is continous in nature and its range is from $-\infty$ to $+\infty$. Then a finite range sequence is obtained by extracting a particular protion from such infinite sequence.

Since, $X(\omega)$ is a continuous-time signal, a discrete-time signal is obtained by sampling $X(\omega)$. A particular sequence which is extracted from infinite sequence is called is **windowed sequence. A** windowed signal is considered as periodic signal. We can obtain periodic extension of this signal. This periodic extension in frequency domain is called is **Discrete Fourier Transform (DFT).** From this original sequence, $x(n)$ is obtained by performing inverse process which is know as **Inverse Discrete Fourier Transform (IDFT).**

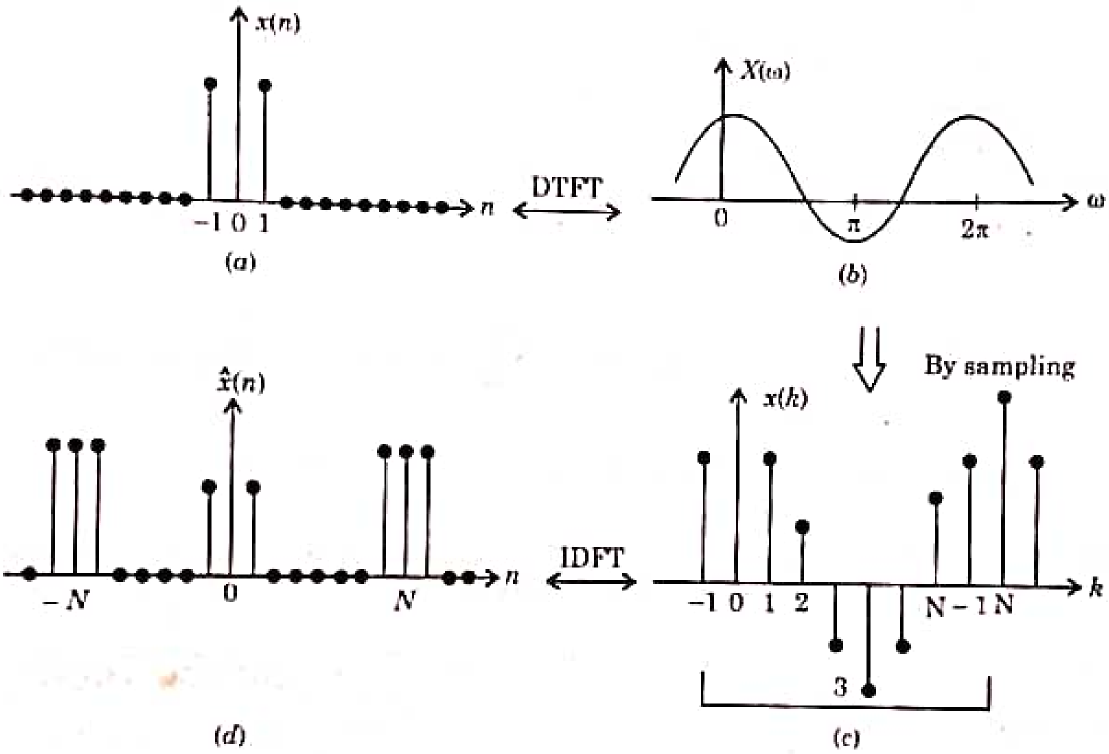


FIGURE 4.1.

This process is explained graphically as shown in figure 4.1. Figure 4.1(a) shows discrete time signal $x(n)$. By taking DTFT of $x(n)$, $X(\omega)$ is obtained as shown in figure 4.1(b). The sampled version of $X(\omega)$ is denoted by $X(k)$ which is called as DFT. It is shown in figure 4.1(c). By performing IDFT, original signal is obtained. It is denoted by $\hat{x}(n)$. It is shown in figure 4.1(d). It is periodic extension of sequence $x(n)$.

Here, N denotes the number of samples of input sequence and the number of frequency points in the DFT output.

4.2.2 Reconstruction of Discrete-Time Signals

We know that aperiodic finite-energy signals have continuous spectra. Let us consider an aperiodic discrete-time signal $x(n)$ with Fourier transform as under:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad \text{...(4.1)}$$

Now, let us assume that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. Because, $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are essential. For simplicity, let us take N equidistant samples in the interval $0 \le \omega < 2\pi$ with a spacing $\delta\omega = \frac{2\pi}{N}$ as depicted in figure 4.2.

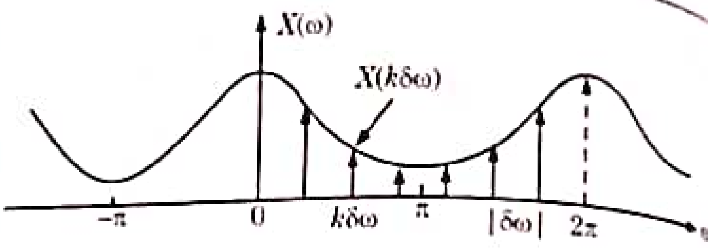


FIGURE 4.2 *Frequency-domain sampling of the Fourier transform.*

First, let us consider the selection of N, the number of samples in the frequency domain. If we evaluate equation (4.1) at

$$\omega = \frac{2\pi k}{N}, \text{ we get}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1 \qquad \ldots(4.2)$$

The above summation may be subdivided into an infinite numbers of summations, where each sum consists of N terms.

Hence, we have

$$X\left(\frac{2\pi}{N}k\right) = \ldots\ldots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \ldots\ldots$$

or

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N} \qquad \ldots(4.3)$$

In equation (4.3), if we change the index in the inner summation from n to $n - lN$ and interchange the order of the summation, we obtain the following result:

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N} \qquad \ldots(4.4)$$

for $\quad k = 0, 1, 2, \ldots N-1$

Also, the signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \qquad \ldots(4.5)$$

DO YOU KNOW?

The discrete Fourier Transform (DFT) is a tool for computing the spectra of discrete-time signals.

obtained by the periodic repetition of $x(n)$ every N samples, is clearly periodic with fundamental period N. As a result of this, it my be expanded in a Fourier series as under.

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{j2\pi kn/N}, \quad n = 0, 1, \ldots N-1 \qquad \ldots(4.6)$$

with the following Fourier coefficients:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1 \qquad \ldots(4.7)$$

Comparing equations (4.4) and (4.7), we get

$$c_k = \frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right), \quad k = 0, 1, \ldots N-1 \qquad \text{...(4.8)}$$

Hence, we have

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(e^{j\frac{2\pi}{N}k}\right) e^{j2\pi kn/N}, \quad n = 0, 1, \ldots N-1 \qquad \text{...(4.9)}$$

The relationship obtained in equation (4.9) provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$. But it does not mean that we can recover $X(\omega)$ or $x(n)$ from the samples. In fact, to achieve this, we need to consider the relationship between $x_p(n)$ and $x(n)$.

Now, since $x_p(n)$ is the periodic extension of $x(n)$ as given by the following expression:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \qquad \text{...(4.10)}$$

then, it is obvious that $x(n)$ can be recovered from $x_p(n)$ if there is no alising in the time-domain. This means that if $x(n)$ is time-limited to less than the period N of $x_p(n)$. Figure 4.3 illustrates this particular situation.

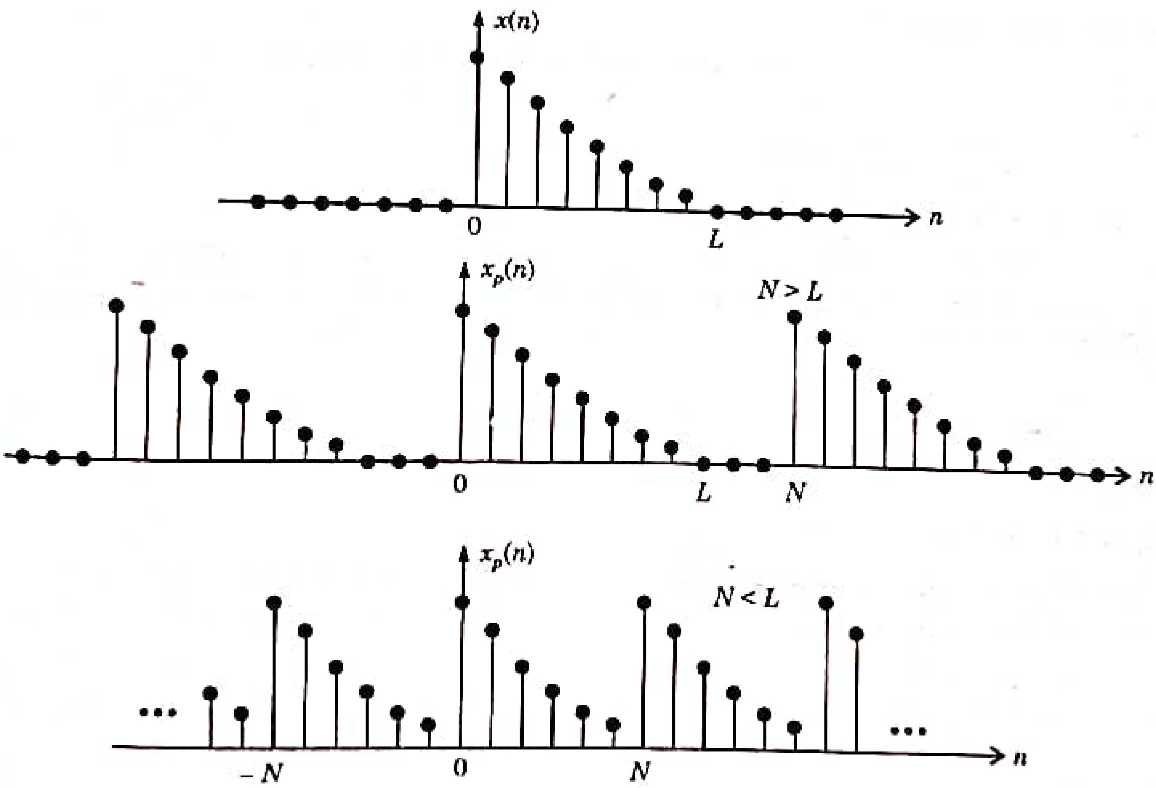


FIGURE 4.3 *Aperiodic sequence $x(n)$ of length L and its periodic extension for $n \geq L$ (no aliasing) and $N < L$ (aliasing).*

In this figure, we have considered a finite-duration sequence $x(n)$; This finite-duration sequence $x(n)$ is nonzero in the interval $0 \leq n \leq L-1$.

It may be observed that

when $N \geq L$, we have

$$x(n) = x_p(n) \quad \text{for} \quad 0 \leq n \leq N-1 \qquad \text{...(4.11)}$$

so that $x(n)$ can be recovered from $x_p(n)$.

On the other hand,

If $N < L$, then it is not possible to recover $x(n)$ from its periodic extension due to time-domain aliasing.

Therefore, we can conclude that the spectrum of an aperiodic discrete-time signal with finite duration L, can be exactly recovered from its samples at frequencies $w_k = \frac{2\pi k}{N}$, if $N \geq L$.

The procedure is to compute $x_p(n)$, $n = 0, 1, \ldots N-1$ with the help of equation (4.9), then, we have

$$x(n) = \begin{cases} x_p(n) & \text{for} \quad 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \quad \ldots(4.12)$$

and, finally, $X(\omega)$ may be computed with the help of equation (4.1)

4.3 THE DFT AS A LINEAR TRANSFORMATION

(i) Definition of DFT

It is a finite duration discrete frequency sequence which is obtained by sampling one period of Fourier transform. Sampling is done at 'N' equally spaced points over the period points over the period extending from $\omega = 0$ to $\omega = 2\pi$.

(ii) Mathematical Expressions

The DFT of discrete sequence $x(n)$ is denoted by $X(k)$. It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \quad \ldots(4.13)$$

Here, $k = 0, 1, 2 \ldots N-1$

Since, this summation is taken for N points; it is called as **N-point DFT.**

We can obtain discrete sequence $x(n)$ from its DFT. It is called as inverse discrete fourier transform (IDFT). It is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad \ldots(4.14)$$

Here, $n = 0, 1, 2, \ldots N-1$

This is called as **N-point IDFT.**

(iii) Twiddle Factor and its Importance

Now, we will define the new term W as,

$$W_N = e^{-j2\pi/N} \quad \ldots(4.15)$$

This is called as **twiddle factor.** Twiddle factor makes the computation of DFT a bit easy and fast.

Using twiddle factor, we can write equations of DFT and IDFT as under :

$$X(k) = \sum_{n=0}^{N-1} X(n)\ W_N^{kn} \quad \ldots(4.16)$$

Here, $n = 0, 1, 2 \ldots N-1$

and

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\ W_N^{-kn} \quad \ldots(4.17)$$

Here, $n = 0, 1, 2, \ldots N-1$

(iv) Linear Transformation

Let us view the DFT and IDFT as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively. Let us define an N-point vector x_N of frequency samples, and an $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1\cdots & 1 & \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{1(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

With these definitions, the N-point DFT may be expressed in matrix form as

$$X_N = W_N x_N$$

where W_N is the matrix of the linear transformation. We observe that W_N is a symmetric matrix. If we assume that the inverse of W_N exists, then the last expression can be inverted by premultiplying both sides by W_n^{-1}. Thus we obtain

$$x_N = W_N^{-1} X_N \qquad \text{...(4.18)}$$

But this is just an expression for the IDFT.

In fact, the IDFT can be expressed in matrix form as under

$$x_N = \frac{1}{N} W_N^* X_N \qquad \text{...(4.19)}$$

where W_N^* denotes the complex conjugate of the matrix W_N. Comparison of (4.19) with (4.18) leads us to conclude that

> **DO YOU KNOW?**
>
> When DFTs are used to process continuous-time signals by sampling, several potential error sources may be important. These are aliasing, spectral leakage, and picket-fence effect.

$$W_N^{-1} = \frac{1}{N} W_N^* \qquad \text{...(4.20)}$$

which, in turn, implies that

$$W_N W_N^* = N I_N \qquad \text{...(4.21)}$$

where I_N is an $N \times N$ identity matrix.

Therefore, the matrix W_N in the transformation is an orthogonal (unitary) matrix. Furthermore, its inverse exists and is given as W_N^*/N. In fact, the existence of the inverse of W_N was established previously from our derivation of the IDFT.

> **Note :** *The DFT and IDFT are computational tools that play a very important role in many digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimation, and linear filtering. The importance of the DFT and IDFT in such practical applications is due to a large extent on the existences of computationally efficient algorithms, known collectively as fast Fourier transform (FFF) algorithms, for computing the DFT and IDFT.*

4.4 COMPARISON OF DTFT AND DFT

(U.P. Tech., Tutorial Question Bank)

We know that the DTFT is discrete time Fourier transform and is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \text{...(4.22)}$$

$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}$

The range of ω is from $-\pi$ to π or 0 to 2π.

Now, we know that discrete Fourier transform (DFT) is obtained by sampling one cycle of Fourier transform. Also, DFT of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad ...(4.23)$$

Comparing equations (4.22) and (4.23), we can say that DFT is obtained from DTFT by substituting $\omega = \frac{2\pi k}{N}$.

Hence, $$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}}$$

Few Important Points

By comparing DFT with DTFT, we can write

(i) The continuous frequency spectrum $X(\omega)$ is replaced by discrete Fourier spectrum $X(k)$.

(ii) Infinite summation in DTFT is replaced by finite summation in DFT.

(iii) The continuous frequency variable is replaced by finite number of frequencies located at $\frac{2\pi k}{NT_s}$, where T_s is called as *sampling time*.

4.5 DISCRETE FOURIER TRANSFORM (DFT) OF SOME STANDARD SIGNALS

In this article, let us obtain DFT of few standard signals in the form of solved examples as follows :

EXAMPLE 4.1 Obtain DFT of unit impulse $\delta(n)$.

Solution : Here $x(n) = \delta(n)$ ...(i)

According to the definition of DFT, we have,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad ...(ii)$$

But $\delta(n) = 1$ only at $n = 0$.

Thus, equation *(ii)* becomes,

$$X(k) = \delta(0)e^0 = 1$$

Therefore, we can write

$$\delta(n) \xleftrightarrow{DFT} 1$$

This is the standard DFT pair.

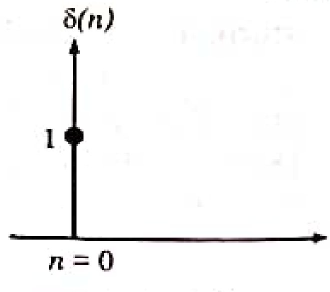


FIGURE 4.4.

EXAMPLE 4.2 Obtain DFT of delayed unit impulse $\delta(n - n_0)$.

Solution : We know that $\delta(n - n_0)$ indicates unit impulse delayed by 'n_0' samples.

Here, $x(n) = \delta(n - n_0)$ ...(i)

Now, we have, $$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad ...(ii)$$

But, $\delta(n - n_0) = 1$ only at $n = n_0$.

Thus, equation *(ii)* becomes,

$$X(k) = 1 \cdot e^{-j2\pi kn_0/N}$$

Hence $$\delta(n - n_0) \xleftrightarrow{DFT} e^{-j2\pi kn_0/N}$$

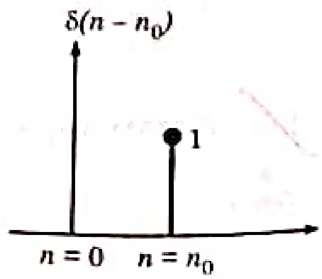


FIGURE 4.5.

Similarly, we can write,

$$\delta(n + n_0) \xleftrightarrow{DFT} e^{j2\pi k n_0/N}$$

EXAMPLE 4.3 Compute N-point DFT of the following exponential sequence :

$$x(n) = a^n u(n) \text{ for } 0 \le n \le N-1$$

Solution : According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(i)$$

Here, $x(n) = a^n u(n)$

The multiplication of a^n with $u(n)$ indicates that sequence is positive. Substituting $x(n) = a^n$ in equation (i), we obtain

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j2\pi kn/N}$$

or

$$X(k) = \sum_{n=0}^{N-1} \left(a e^{-j2\pi k/N}\right)^n \quad ...(ii)$$

Now, let us use the following standard summation expression:

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here, $N_1 = 0$, $N_2 = N - 1$ and $A = ae^{-j2\pi k/N}$

Therefore,

$$X(k) = \frac{\left(ae^{-j2\pi k/N}\right)^0 - \left(ae^{-j2\pi k/N}\right)^{N-1+1}}{1-ae^{-j2\pi k/N}} = \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j2\pi k/N}} \quad ...(iii)$$

Making use of Euler's identity to the numerator term, we shall have

$$e^{-j2\pi k} = \cos 2\pi k - j \sin 2\pi k$$

But, k is an integer

Therefore, $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$

or $e^{-j2\pi k} = 1 - j0 = 1$

or

$$X(k) = \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

Hence,

$$a^n u(n) \xleftrightarrow{DFT} \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

EXAMPLE 4.4 Compute the DFT of following window function:

$$w(n) = u(n) - u(n-N)$$

Solution : According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...(i)$$

The given equation is $x(n) = w(n) = 1$ for $0 \le n \le N-1$.

Let us assume some value of N.

Let $N = 4$, so we will get 4-point DFT.

Hence, $$X(k) = \sum_{n=0}^{3} 1.e^{-j2\pi kn/4} \quad ...(ii)$$

The range of k is from 0 to $N-1$. Therefore, in this case, k will vary from 0 to 3.

For $k = 0$, we have $X(0) = \sum_{n=0}^{3} 1.e^0 = \sum_{n=0}^{3} 1 = 1 + 1 + 1 + 1 = 4$

For $k = 1$, we have $X(1) = \sum_{n=0}^{3} e^{-j2\pi n/4}$

$\therefore \quad X(1) = e^0 + e^{-j2\pi/4} + e^{-j4\pi/4} + e^{-j6\pi/4}$

or $\quad X(1) = 1 + \left(\cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4}\right) + \left(\cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4}\right) + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right)$

or $\quad X(1) = 1 + (0 - j) + (-1 - 0) + (0 + j)$

or $\quad X(1) = 1 - j - 1 + j = 0$

For $k = 2$, we have, $X(2) = \sum_{n=0}^{3} e^{-j2\pi \times 2n/4} = \sum_{n=0}^{3} e^{-j\pi n}$

$\therefore \quad X(2) = e^0 + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi}$

or $\quad X(2) = 1 + (\cos \pi - j \sin \pi) + (\cos 2\pi - j \sin 2\pi) + (\cos 3\pi - j \sin 3\pi)$

or $\quad X(2) = 1 + (-1 - 0) + (1 - 0) + (-1 - 0)$
$= 1 - 1 + 1 - 1 = 0$

For $k = 3$, we have

$$X(3) = \sum_{n=0}^{3} e^{-j2\pi \times 3n/4} = \sum_{n=0}^{3} e^{-j6\pi n/4}$$

$\therefore \quad X(3) = e^0 + e^{-j6\pi/4} + e^{-j3\pi} + e^{-j9\pi/2}$

or $\quad X(3) = 1 + \left(\cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4}\right) + (\cos 3\pi - j \sin 3\pi) + \left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$

or $\quad X(3) = 1 + (0 + j) + (-1 - 0) + (0 - j) = 1 + j - 1 - j = 0$

or $\quad X(k) = \{4, 0, 0, 0\}$

Ans.

DO YOU KNOW?

An inverse DFT can be computed using a direct DFT algorithm by conjugating the frequency samples, taking the DFT of these conjugated samples, conjugating the output of the DFT operation, and dividing by N.

4.6 DETAILED EXPLANATION OF CYCLIC PROPERTY OF TWIDDLE FACTOR

(U.P. Tech., Sem. Exam., 2003-04)(05 marks)

The twiddle factor is denoted by W_N and is given by,

$$W_N = e^{-j2\pi/N} \quad ...(4.24)$$

Now, the discrete time sequence $x(n)$ can be denoted by x_N. Here, N stands for N point DFT. While in case of N point DFT, the range of n is from 0 to $N-1$.

Now, the sequence x_N can be represented in the matrix from as under:

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdot \\ \cdot\ \cdot \\ x(N-1) \end{bmatrix}_{N\times 1} \qquad ...(4.25)$$

This is a $N \times 1$ matrix and n varies from 0 to $N - 1$. Now, the DFT of $x(n)$ is denoted by $X(k)$. We have denoted $x(n)$ by x_N. Similarly, we can denote $X(k)$ by X_N. In the matrix form, X_k can be represented as under:

$$X_k = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdot \\ \cdot\ \cdot \\ X(N-1) \end{bmatrix}_{N\times 1} \qquad ...(4.26)$$

This is also $N \times 1$ matrix and k varies from 0 to $N - 1$. Recall the definition of DFT *i.e.*,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad (4.27)$$

We can also represent W_N^{kn} in the matrix form. Further, since k varies from 0 to $N - 1$ and n also varies from 0 to $N - 1$, therefore, we have

$$W_N^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{ccccc} n=0 & n=1 & n=2 & \cdots & n=N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}_{N\times N} \end{array} \qquad ...(4.28)$$

Note that each value is obtained by taking multiplication of k and n.

As an example, if $k = 2$, $n = 2$, then we get $W_N^{kn} = W_N^4$.

Thus, DFT can be represented in the matrix form as,

$$X_N = [W_N]x_N \qquad ...(4.29)$$

Similarly, IDFT can be represented in the matrix forms as,

$$x_N = \frac{1}{N}\left[W_N^*\right]X_N \qquad ...(4.30)$$

Here, W_N^* is complex conjugate of W_N.

Now, let us show that W_N possess the periodicity property. This means that after some period, the value of W_N repeats. Let us consider 8-point DFT, *i.e.*, $N = 8$.

We have, $\qquad W_N = e^{-\frac{j2\pi}{N}}$

Therefore, $\qquad W_N^{kn} = e^{-\frac{j2\pi}{N}\times kn}$

But, $N = 8$

Hence, $$W_8^{kn} = e^{-\frac{j2\pi}{8}\times kn} = e^{-\frac{j\pi}{4}\times kn} \quad ...(4.31)$$

Now, let us obtain W_8^{kn} by substituting different values of kn. This has been shown in Table 4.1.

Here, it may be noted that all calculations have been done by making use of Euler's identity. For example, when $kn = 1$, equation (4.47) becomes,

$$W_8^1 = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

TABLE 4.1.

S.No.	Value of kn	$W_8^{kn} = e^{-j\frac{\pi}{4}\times kn}$	Value of the phasor
1	0	$W_8^0 = e^0$	1
2	1	$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = e^{-j\frac{\pi}{4}}$	$0.707 - j\,0.707$
3	2	$W_8^2 = e^{-j\frac{\pi}{4}\times 2} = e^{-j\frac{\pi}{2}}$	$0 - j\,1$
4	3	$W_8^3 = e^{-j\frac{\pi}{4}\times 3} = e^{-j\frac{3\pi}{4}}$	$-0.707 - j\,0.707$
5	4	$W_8^4 = e^{-j\frac{\pi}{4}\times 4} = e^{-j\pi}$	-1
6	5	$W_8^5 = e^{-j\frac{\pi}{4}\times 5} = e^{-j\frac{5\pi}{4}}$	$-0.707 + j\,0.707$
7	6	$W_8^6 = e^{-j\frac{\pi}{4}\times 6} = e^{-j\frac{3\pi}{2}}$	$0 + j\,1$
8	7	$W_8^7 = e^{-j\frac{\pi}{4}\times 7} = e^{-j\frac{7\pi}{4}}$	$0.707 + j\,0.707$
9	8	$W_8^8 = e^{-j\frac{\pi}{4}\times 8} = e^{-j2\pi}$	1
10	9	$W_8^9 = e^{-j\frac{\pi}{4}\times 9} = e^{-j\frac{9\pi}{4}}$	$0.707 - j\,0.707$
11	10	$W_8^{10} = e^{-j\frac{\pi}{4}\times 10} = e^{-j\frac{5\pi}{2}}$	$0 - j\,1$
12	11	$W_8^{11} = e^{-j\frac{\pi}{4}\times 11} = e^{-j\frac{11\pi}{4}}$	$-0.707 - j\,0.707$

From table 4.1, it may be observed that the value of W_8^0 is same as W_8^8. Similarly, W_8^1 is same as W_8^9 and W_8^2 is same as W_8^{10}. Since, this is 8-point DFT (N = 8), after 8 points, the value repeats.

This means that

$$W_8^0 = W_8^8 = W_8^{16} \;....$$

$$W_8^1 = W_8^9 = W_8^{17} \ldots$$
$$W_8^2 = W_8^{10} = W_8^{18} \ldots$$
$$\vdots$$
$$W_8^7 = W_8^{15} = W_8^{23} \ldots$$

Twiddle Factor

This property of twiddle factor is called as ***periodicity property*** *or* ***cyclic property.***

Few Important Points

(i) In Table 4.1, every value of W_N^{kn} can be represented in terms of magnitude and angle.

For example, we have $W_8^0 = e^0$. We know that magnitude and angle can be expressed as, **magnitude** $e^{j\,\text{angle}}$. Thus, for $W_8^0 = 1.e^0$. Here, magnitude is 1 and angle is zero.

(ii) Similarly, we have $W_8^1 = 1 \,.\, e^{-j\frac{\pi}{4}}$, so magnitude is 1 and angle is $-\,.\frac{\pi}{4}$.

(iii) Likewise, we can write the magnitude and angle of each value. The cyclic property of twiddle factor has been illustrated in figure 4.6.

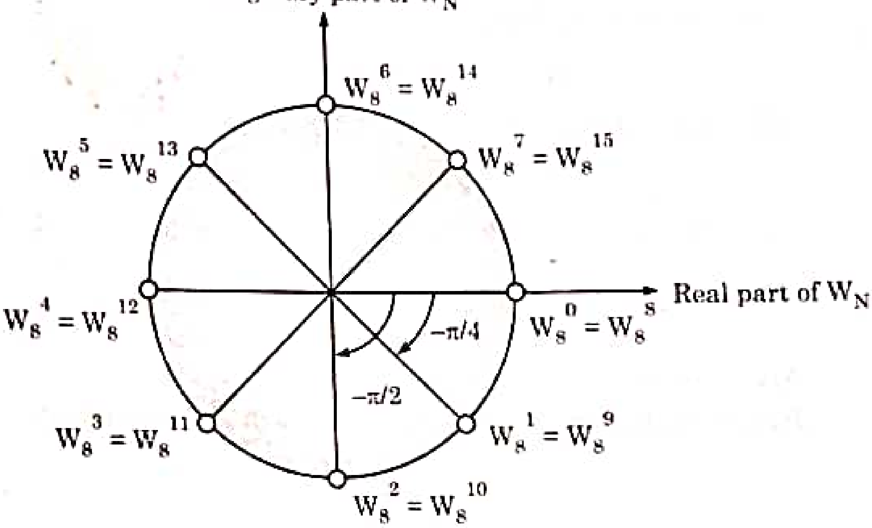


FIGURE 4.6 *Cyclic property of twiddle factor.*

(iv) In figure 4.6, we have drawn unit circle that means a circle having radius equal to 1. Every point is in the clockwise direction because we have negative angles. Here, we have considered 8-point DFT. Therefore, the circle is divided into 8 points. This spacing of DFT or the resolution of DFT is also called as the **bin spacing of DFT output.**

EXAMPLE 4.5 Compute 2-point and 4-point DFT of the following sequence:

$$x(n) = u(n) - u(n-2)$$

Sketch the magnitude of DFT in both the cases.

Solution : First, let us obtain the sequence $x(n)$. It has been represented as shown in figure 4.7.

Thus, from figure (4.7), we get

$$x(n) = \{1, 1\} \quad \ldots(i)$$

(i) **Determination of 2-point DFT**

For 2-point DFT, $N = 2$

We have, $W_N = e^{-j\frac{2\pi}{N}}$

so that $W_2 = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$

Hence, $W_2^{kn} = e^{-j\pi kn} \quad \ldots(ii)$

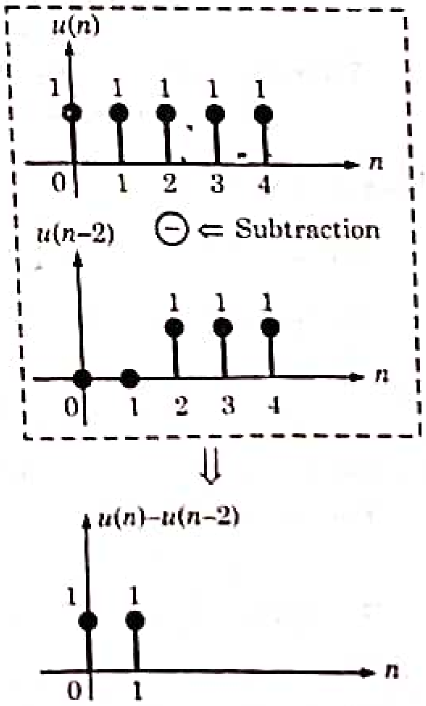


FIGURE 4.7 $x(n) = u(n) - u(n-2)$.

We know that n is from 0 to $N-1$. In this case, n is from 0 to 1. Similarly, k is from 0 to $N-1$. In this case, k is from 0 to 1.

Now, the matrix $W_N = W_2^{kn} = e^{-j\pi kn}$ can be written as under:

$$W_2^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \end{array}\begin{array}{c} \begin{array}{cc} n=0 & n=1 \end{array} \\ \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \end{array} \quad ...(iii)$$

According to equation (*ii*), we have

$$W_2^{kn} = e^{-j\pi kn}$$

For $kn = 0$, we have

$$W_2^0 = e^{-j\pi \times 0} = e^0 = 1$$

For $kn = 1$, we have

$$W_2^1 = e^{-j\pi \times 1} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

Substituting all these values in equation (*iii*), we shall get

$$W_2^{kn} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad ...(iv)$$

Also, given sequence is $x(n) = \{1, 1\}$.

In the matrix form, this sequence can be written as,

$$x_N = x(n) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad ...(v)$$

For above sequence, the DFT matrix is given by,

$$X_N = [W_N]x_N$$

Substituting values from equation (*iv*) and (*v*), we get

$$X_N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\times1)+(1\times1) \\ (1\times1)+(1\times-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Therefore, the 2-point DFT will be

$$X(k) = \{2, 0\} \quad ...(vi)$$

Magnitude plot

We know that magnitude $= \sqrt{(\text{Real part})^2 + (\text{Imaginary part})^2}$

In equation (*vi*), the imaginary part is zero.

Hence, the magnitude at $k = 0$ is 2 and magnitude at $k = 1$ is 0. This magnitude plot has been shown in figure 4.8.

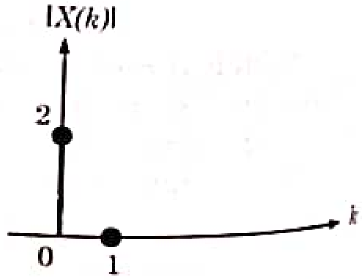


FIGURE 4.8 *Magnitude plot.*

(*ii*) Determination of 4-point DFT

For 4-point DFT, $N = 4$

We have, $W_N = W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$

Therefore, $W_N^{kn} = e^{-j\frac{\pi}{2}kn} \quad ...(vii)$

The range of k and n is from 0 to $N-1$, *i.e.*, 0 to 3.

Now, the matrix $W_N = W_4 = W_4^{kn}$ can be written as

$$[W_4] = W_4^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{c} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array} \quad ...(viii)$$

Making use of equation (*vii*), we obtain

$$W_4^0 = e^{-j\frac{\pi}{2}\times 0} = e^0 = 1$$

$$W_4^1 = e^{-j\frac{\pi}{2}\times 1} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = e^{-j\frac{\pi}{2}\times 2} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$W_4^3 = e^{-j\frac{\pi}{2}\times 3} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = +j$$

According to cyclic property of DFT, we know that

$$W_4^0 = W_4^4 = 1$$

$$W_4^1 = W_4^5 = W_4^9 = -j$$

$$W_4^2 = W_4^6 = W_4^{10} = -1$$

and $$W_4^3 = W_4^7 = W_4^{11} = +j$$

Substituting all these values in equation (*viii*), we shall get the matrix $[W_4]$ *i.e.*,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad ...(ix)$$

The given sequence is $x(n) = \{1, 1\}$. Since, the desirable length of this sequence is equal to 4, it is obtained by adding zeros at the end of sequence. This is called as **zero padding.**

Thus, with the help of zero padding, we get

$$x(n) = \{1, 1, 0, 0\}$$

Hence, $$x_N = x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad ...(x)$$

Now, the discrete Fourier Transform (DFT) is given by

$$X_N = [W_N]x_N$$

or $$X_N = X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+1+0+0 \\ 1-j+0+0 \\ 1-1+0+0 \\ 1+j+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

or $$X_4 = \{2,\ 1-j,\ 0,\ 1+j\}$$

The above DFT sequence can also be written as under:

$$X_4 = \{\underset{\substack{\uparrow \\ k=0}}{2+j\,0},\ 1-j,\ 0+j\,0,\ 1+j\}$$

Magnitude Plot

The magnitude at different values can be obtained as under:

For $k = 0$, we have

$$|X(k)| = \sqrt{(2)^2+(0)^2} = 2$$

For $k = 1$, we have

$$|X(k)| = \sqrt{(1)^2+(-1)^2} = \sqrt{2} = 1.414$$

For $k = 2$, we have

$$|X(k)| = \sqrt{0+0} = 0$$

For $k = 3$, we have

$$|X(k)| = \sqrt{(1)^2+(1)^2} = \sqrt{2} = 1.414$$

This magnitude plot has been shown in figure 4.9.

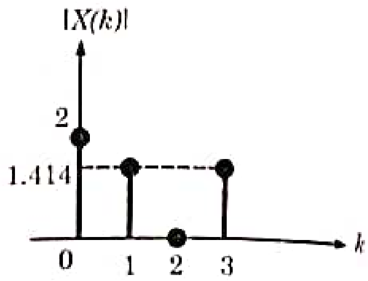


FIGURE 4.9 *Magnitude plot.*

EXAMPLE 4.6 Determine the discrete Fourier Transform (DFT) of four point sequence $x(n) = \{0, 1, 2, 3\}$

Solution. The 4-point DFT in the matrix form is given by,

$$X_4 = [W_4]\,.\,x(n)$$

Thus,
$$X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ 2j-2 \\ -2 \\ -2j-2 \end{bmatrix}$$

DO YOU KNOW?

Under suitable restrictions, the DFT closely approximates the spectrum of a continuous-time signal at a discrete set of frequencies.

Simplifying, we shall get

$$X_4 = \{\underset{\uparrow}{6},\ 2j-2,\ -2,\ -2j-2\} \quad \textbf{Ans.}$$

EXAMPLE 4.7 Compute the length-4 sequence from its DFT which is given by
$$X(k) = \{4,\ 1-j,\ -2,\ 1+j\}$$

(U.P. Tech., Sem. Exam., 2004-05)(05 marks)

Solution: We know that the IDFT in matrix form is expressed as

$$\text{IDFT} = x(n) = x_N = \frac{1}{N}\left[W_N^*\right].X_N \quad \ldots(i)$$

Here, X_N is the given DFT matrix. Also, '*' indicates complex conjugate. To obtain the complex conjugate, we have to change the sign of j term. For example, complex conjugate of $1 - j1$ is $1 + j1$.

Now, we have already obtained the matrix W_4 in previous examples. It is reproduced here *i.e.*,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad ...(ii)$$

Therefore, $$\left[W_4^*\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \quad ...(iii)$$

Given matrix of DFT is

$$X_N = X_4 = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} \quad ...(iv)$$

Substituting equations (*iii*) and (*iv*), and substituting $N = 4$ in equation (*i*), we shall have

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

or $$x_N = \frac{1}{4}\begin{bmatrix} 4+1-j-2+1+j \\ 4+j-j^2+2-j-j^2 \\ 4-1+j-2-1-j \\ 4-j+j^2+2+j+j^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4+2+1+1 \\ 4-4 \\ 4+2-2 \end{bmatrix} \quad (\because j^2 = -1)$$

Simplifying, we get

or $$x_N = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

or $x(n) = \{1, 2, 0, 1\}$ **Ans.**

EXAMPLE 4.8 Determine the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and check the validity of your answer by calculating its IDFT.

Solution: Let us compute 4 point DFT. We have already obtained the matrix for $[W_4]$ in previous example. It is reproduced here

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -j \\ 1 & +j & -1 & -j \end{bmatrix}$$

The given sequence is $x(n) = \{1, 1, 0, 0\}$

The DFT of this sequence has been computed in previous examples. It is $X_N = X(k) = \{2, 1 - j, 0, 1 + j\}$

Now, let us check this answer by using the expression for IDFT.

The IDFT is given by,

$$x(n) = \frac{1}{N}\left[W_N^*\right].X_N$$

Here, $$\left[W_N^*\right] = \left[W_4^*\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\begin{bmatrix} 2+1-j+1+j=4 \\ 2+j-j^2-j-j^2 \\ 2-2j^2=4 \\ \\ \end{bmatrix}$$

and $$X_N = X_4 = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

Therefore, we have

$$x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+1-j+0+1+j \\ 2+j+1+0-j+1 \\ 2-1+j+0-1-j \\ 2-j-1+0+j-1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

This means that $x(n) = \{1, 1, 0, 0\}$

But, this is same as the given sequence. Therefore, calculated DFT is correct.

Hence Proved

EXAMPLE 4.9 **If** $y(n) = \dfrac{[x(n) \quad x(-n)]}{2}$

Find $Y(k)$ **if** $X(k) = \{0.5, 2 + j, 3 + j2, j, 3, -j, 3 - j2, 2 - j\}$

Solution. We have,

$$\boxed{y(n) = \frac{[x(n) + x(-n)]}{2}} \quad \text{...(i)}$$

Taking DFT of both sides, we obtain

$$Y(k) = \frac{[X(k) + X(-k)]}{2} \quad \text{...(ii)}$$

Given $X(k) = \{0.5, 2 + j, 3 + j2, j, 3, -j, 3 - j2, 2 - j\}$

Therefore, $X(-k) = \{0.5, 2 - j, 3 - j2, -j, +3, j, 3 + j2, 2 + j\}$

Putting these values in equation (*ii*), we get,

$$Y(k) = \frac{1}{2}\{1, 4, 6, 0, 6, 0, 6, 4\}$$

or $Y(k) = \{0.5, 2, 3, 0, 6, 0, 3, 2\}$ **Ans.**

4.7 RELATIONSHIP OF THE DFT TO OTHER TRANSFORMS

In this section, we shall discuss the relationship of DFT with several other transforms.

4.8 RELATIONSHIP OF DFT TO THE FOURIER SERIES COEFFICIENTS OF A PERIODIC SEQUENCE

We know that the Fourier series of a periodic sequence $x_p(n)$ with fundamental period N is expressed as

$$x_p(n) = \sum_{k=0}^{N-1} c_k . e^{j2\pi nk/N} \quad , -\infty < n < \infty$$

where the Fourier series coefficients are expressed by

$$c_k = \frac{1}{N} = \sum_{n=0}^{N-1} x_p(n) . e^{-j2\pi nk/N} \quad , k = 0, 1 \;.... \; N-1$$

Now, by comparing the above two equations with that of DFT pair and defining a sequence $x(n)$ which is identical to $x_p(n)$ over a single period, we get

$$X(k) = N{\cdot}c_k$$

If a periodic sequence $x_p(n)$ is formed by periodically repeating $x(n)$ every N samples *i.e.*,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

The discrete-frequency domain representation is expressed as

$$X(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi k/N} = Nc_k, \; k = 1, 1 \;.... \; N-1$$

and the IDFT is

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k/N} \quad , -\infty < n < \infty$$

4.9 RELATIONSHIP OF DFT THE FOURIER TRANSFORM OF AN APERIODIC SEQUENCE

Let $x(n)$ be an aperiodic finite energy sequence.

The Discrete-time Fourier transform (DTFT) is expressed as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

If $X(e^{j\omega})$ is sampled at N equally spaced frequencies, then we have

$$w_k = \frac{2\pi k}{N} = 0, 1, 2, ..., N-1$$

then, we have

$$X(k) = X(e^{j\omega})\Big|_{at\ \omega = \frac{2\pi k}{N}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} \quad , k = 0, 1, ... \; N-1$$

The spectral components $\{X(k)\}$ correspond to the spectrum of a periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

If $\hat{x}(n)$ is a finite duraion sequence given by

$$\hat{x}(n) = \begin{cases} x_p(n) \; for \; 0 \le n \le N-1 \\ 0 \qquad \text{otherwise} \end{cases}$$

Then $\hat{x}(n)$ resembles the original sequence $x(n)$ only when the duration of the sequence $x(n)$, $L \le N$.

In this case,

$$x(n) = \hat{x}(n), \; 0 \le n \le N-1$$

Otherwise, if $L > N$, then there will be no resemblance.

4.10 RELATIONSHIP OF THE DFT TO THE Z-TRANSFORM

Let $X(z)$ be the z-transform for a sequence $x(n)$ which is expressed as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n}$$

with a ROC which includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle,

$$Z_k = e^{j2\pi k/N}, \; k = 0, 1, 2, ... N-1$$

then

$$X(k) = X(z)\big|_{at\; z = e^{j2\pi k/N}}, \quad k = 0, 1, ... N-1$$

or

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)e^{-2\pi nk/N}$$

> **DO YOU KNOW?**
>
> Applications of the DFT include filtering, spectrum analyzers, convolution (with applications to filtering and system identification), and processing of random signals through computation of such statistics as auto-correlation functions and power spectra.

Now, it may be noted that this is identical to the Fourier transform $X(e^{j\omega})$ evaluated at the N equally spaced frequencies *i.e.*,

$$\omega_k = 2\pi k/N, \; k = 0, 1, ... N-1.$$

If the sequence $x(n)$ has a finite duration of length N, then the z-transform is given as

$$X(z) = \sum_{n=0}^{N-1} x(n).z^{-n}$$

Now, substituting the IDFT relationship for $x(n)$, we obtain

$$X(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N}\right].z^{-n} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1}(e^{j2\pi k/N}z^{-1})^n$$

or

$$X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot\frac{1-z^{-N}}{1-e^{j2\pi k/N}z^{-1}} = \frac{1-z^{-N}}{N}\cdot\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j2\pi k/N}.z^{-1}}$$

This equation is identical to that of frequency sampling form.

Now, when this is evaluated over an unit circle, then we write

$$X(e^{j\omega}) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j(\omega - 2\pi k/N)}}$$

4.11 RELATIONSHIP TO THE FOURIER SERIES COEFFICIENTS OF A CONTINUOUS-TIME SIGNAL

Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier Series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t F_0}$$

where (c_k) are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain the discrete-time sequence

$$x(n) \equiv x_a(nT) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k n/N} \qquad \text{...(4.32)}$$

$$= \sum_{k=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} c_{k-lN}\right] e^{j2\pi k n/N}$$

It is clear that (4.48) is in the form of an IDFT formula, where

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \circ N\,\overline{c}_k \qquad \text{...(4.33)}$$

and
$$\overline{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN}$$

Hence, the $(\overline{c}_k)$ sequence is an alised version of the sequence (c_k).

4.12 PROPERTIES OF DISCRETE FOURIER TRANSFER (DFT)

The properties of Discrete Fourier Transform (DFT) are quite useful in the practical techniques for processing signals. At this point, it may be noted that most of the properties of the DFT and z-transform have some similarity since they have some relationship between them.

The properties of Discrete Fourier Transform (DFT) can be listed as under :

1. Periodicity
2. Linearity
3. Shifting property
4. Circular convolution

Now, let us discuss these properties one by one in the subsequent sub-sections.

4.12.1 Periodicity

(i) Statement

This property states that if a discrete-time signal is periodic then its DFT will also be periodic. Also, if a signal or sequence repeats its waveform after N number of samples then it is called a periodic signal or sequence and N is called the period of signal. Mathematically,

If $X(k)$ is an N-point DFT of $x(n)$ *i.e.*,

If $x(n) \xleftrightarrow[N]{DFT} X(k)$, then we have

$$x(n+N) = x(n) \text{ for all values of } n \qquad \text{...(4.34)}$$

$$X(k+N) = X(k) \text{ for all values of } k \qquad \text{...(4.35)}$$

(ii) Proof :

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(4.36)}$$

Replacing k by $k + N$, we obtain

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(k+N)n} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}\, W_N^{Nn} \qquad \text{...(4.37)}$$

We know that W_N is a twiddle factor and it is expressed as

$$W_N = e^{-j\frac{2\pi}{N}}$$

Also,

$$W_N^{Nn} = \left(e^{-j\frac{2\pi}{N}}\right)^{Nn} = e^{-j\frac{2\pi}{N}.Nn} = e^{-j2\pi n}$$

or

$$W_N^{Nn} = \cos 2\pi n - j \sin 2\pi n \qquad \text{...(4.38)}$$

Since, n is an integer, therefore, we have

$$\cos 2\pi n = 1 \quad \text{and} \quad \sin 2\pi n = 0$$

Hence,

$$W_N^{Nn} = 1 \qquad \text{...(4.39)}$$

Substituting this value in equation (4.37), we have

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(4.40)}$$

Comparing equations (4.36) and (4.39), we get

$X(k + N) = X(k)$ **Hence proved.**

4.12.2 Linearity

Linearity properties states that DFT of linear combination of two or more signals is equal to the sum of linear combinatin of DFT of individual signals. Let us consider that $X_1(k)$ and $X_2(k)$ are the N-points DFTs of $x_1(n)$ and $x_2(n)$ respectively, and a and b are arbitrary constants either real or complex-valued, then mathematically

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$a\, x_1(n) + b\, x_2(n) \xleftrightarrow[N]{DFT} a\, X_1(k) + b\, X_2(k)$

Here, a and b are some constants

Proof: According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(4.41)}$$

Here,

$$x(n) = a\, x_1(n) + b\, x_2(n)$$

Therefore,

$$X(k) = \sum_{n=0}^{N-1} [a\, x_1(n) + b\, x_2(n)]\, W_N^{kn} = \sum_{n=0}^{N-1} a\, x_1(n)\, W_N^{kn} + \sum_{n=0}^{N-1} b\, x_2(n)\, W_N^{kn}$$

Since, a and b are constants, therefore, we can take them out of the summation sign.

Hence,

$$X(k) = a\sum_{n=0}^{N-1} x_1(n)\, W_N^{kn} + b\sum_{n=0}^{N-1} x_2(n)\, W_N^{kn} \qquad \text{...(4.42)}$$

Comparing equation (6.57) with the definition of DFT, we obtain

$$X(k) = a\,X_1(k) + b\,x_2(k)$$

Hence Proved.

4.12.3 Circular Symmetries of a Sequence

We have discussed the periodicity property of discrete Fourier transform (DFT). Let us consider that input discrete time sequence is $x(n)$ then, the periodic sequence is denoted by $x_p(n)$. The period of $x_p(n)$ is N which means that after N the sequence $x(n)$ repeats itself.

Now, we can write the sequence $x_p(n)$ as under:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \qquad \text{...(4.43)}$$

Now, let us consider one example.

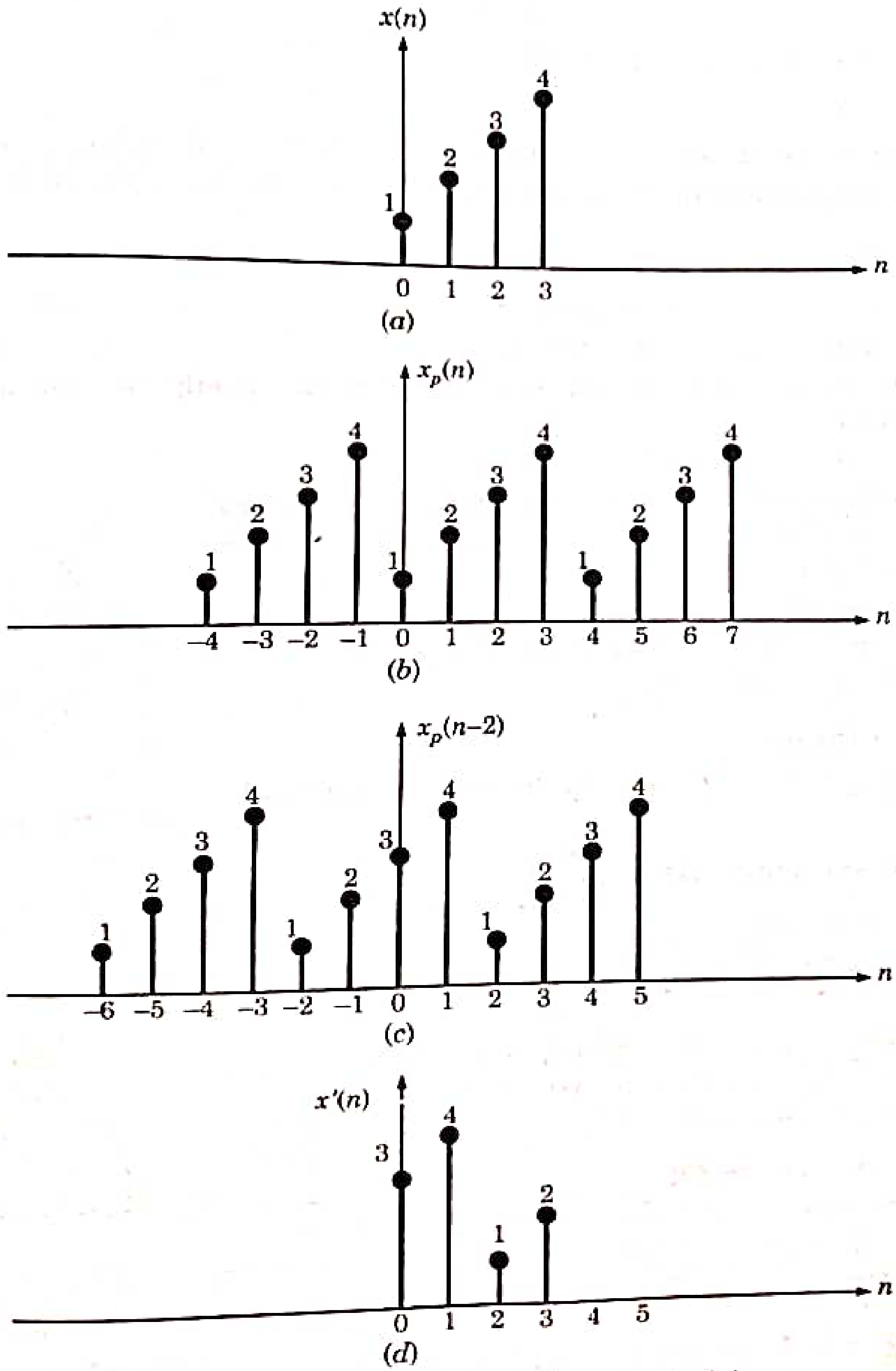


FIGURE 4.10 *Shifting of sequence x(n).*

Let $x(n) = \{1, 2, 3, 4\}$. This sequence is shown in figure 4.10(a). The periodic sequence $x_p(n)$ is shown in figure 4.10(b). We shall delay the periodic sequence $x_p(n)$ by two samples as shown in figure 4.10(c).This sequence is denoted by $x_p(n-2)$. Now, the original signal is present in the range $n = 0$ to $n = 3$. In the same range, we shall write the shifted signal as shown in figure 4.10 (d). This signal is denoted by $x'(n)$.

Now, from figure 4.10, we can write every sequence as under:

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\} \quad ...(4.44)$$

$$x_p(n) = \{... 1, 2, 3, 4, \underset{\uparrow}{1}, 2, 3, 4, 1, 2, 3, 4, ...\} \quad ...(4.45)$$

$$x_p(n-2) = \{... 1, 2, 3, 4, 1, 2, \underset{\uparrow}{3}, 4, 1, 2, 3, 4, ...\} \quad ...(4.46)$$

and

$$x'(n) = \{\underset{\uparrow}{3}, 4, 1, 2\} \quad ...(4.47)$$

Now, from equation (4.44) and (4.47), we can say that the sequence $x'(n)$ is obtained by **circularly shifting** sequence $x(n)$, by two samples. This means that **$x'(n)$ is related to $x(n)$ by circular shift.**

Notation

This relation of circular shift is denoted by,

$$x'(n) = x(n - k, \text{ modulo } N) \quad ...(4.48)$$

It means that divide $(n - k)$ by N and retain the remainder only. We can also use the short hand notation as under :

$$x'(n) = x((n - k))_N \quad ...(4.49)$$

Here, k indicates the number of samples by which $x(n)$ is delayed and N indicates N-point DFT. In the present example, the sequence $x(n)$ is delayed by two samples; thus $k = 2$. Because, there are four samples in $x(n)$, this is 4-point DFT. Hence, $N = 4$.

Now, for this example, equation (4.48) becomes,

$$x'(n) = x((n - 2))_4 \quad ...(4.50)$$

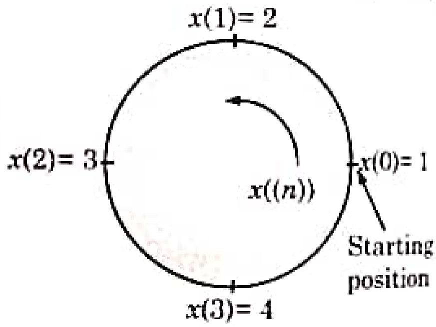


FIGURE 4.11 *$x((n))_4$ - The samples of $x(n)$ are plotted circularly anticlockwise.*

Graphical Representation

The circular shifting of a sequence can be plotted graphically as under:

(*i*) Circular plot of sequence $x(n)$

Here, we have considered,

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Circular plot of $x(n)$ is denoted by $x((n))_4$. This plot is obtained by writing the samples of $x(n)$ circularly anticlockwise. It is shown in figure 4.11.

(*ii*) Circular delay by one sample

To delay sequence $x(n)$ circularly by one sample, shift every sample circularly in anticlockwise direction by 1. This is shown in figure 4.12. This operation is denoted by $x((n - 1))$.

It may be noted that **delay by k samples means shift the sequence circularly in anticlockwise direction by k.**

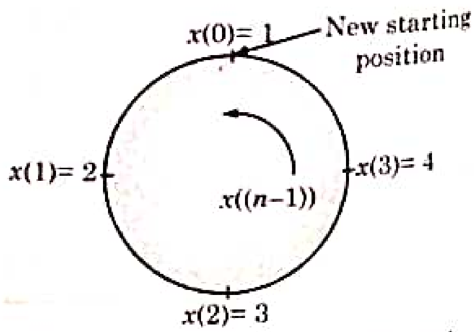


FIGURE 4.12 *$x((n - 1))$ shift every sample by 1 in anticlockwise direction.*

(iii) Circular advance by one sample

To advance sequence $x(n)$ circularly by one sample shift every sample circularly in clockwise direction by 1 sample. This sequence is denoted by $x((n+1))$. It is shown in figure 4.13.

It may be noted that **advance by k samples means shift the sequence circularly in clockwise direction by k.**

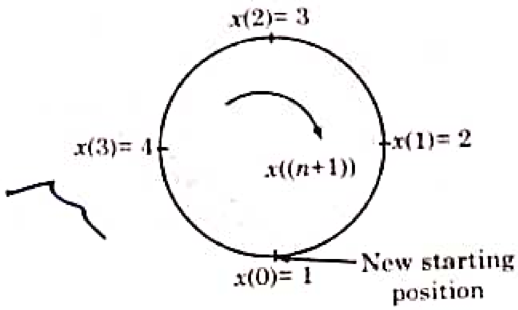


FIGURE 4.13 *$x((n+1))$ shift every sample by one in clockwise direction.*

(iv) Circularly folded sequence

A circularly folded sequence is denoted by $x((-n))$. We have plotted sequence $x((n))$ in anticlockwise direction. So, folded sequence $x((-n))$ is plotted in clockwise direction. It is shown in figure 4.14.

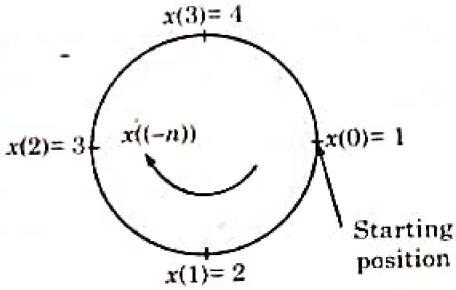


FIGURE 4.14 *$x((-n))$ samples are plotted circularly clockwise.*

It may be noted that **circular folding means plot the samples in clockwise direction.**

Now recall equation (4.61) it is,

$$x'(n) = x((n-2))_4$$

It indicates delay of sequence $x(n)$ by two samples. It is obtained by rotating samples of figure 4.16 in anticlockwise direction by two samples. This sequence is shown in figure 4.15.

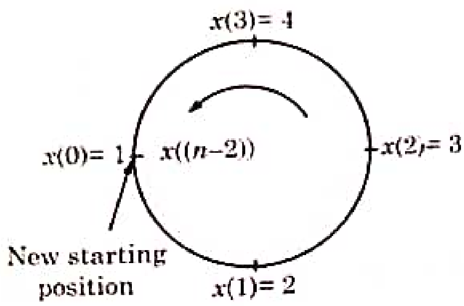


FIGURE 4.15 *Plot of $x((n-2))_4$*

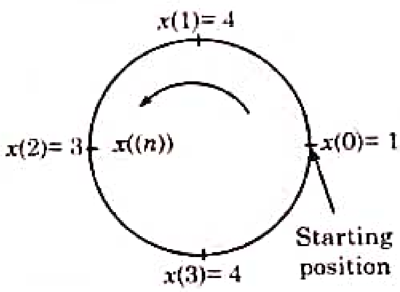


FIGURE 4.16 $x(n) = \{1, 4, 3, 4\}$

(v) Circularly even sequence

The N-point discrete time sequence is circularly even if it is symmetric about the point zero on the circle.

This means that

$$x(N-n) = x(n), \; 1 \le n \le N-1$$

Now, let us consider the sequence

$$x(n) = \{1, 4, 3, 4\}.$$

It has been plotted as shown in figure 4.16.

It may be noted that this sequence is symmetric about point zero on the circle. So, it is circularly even sequence. We can also verify it using mathematical equation.

The sequence is $x(n) = \{1, 4, 3, 4\}$

$\therefore \; x(0) = 1, \quad x(1) = 4, \quad x(2) = 3$ and $x(3) = 4$

We have the following condition for circularly even sequence:

> **DO YOU KNOW?**
>
> The use of window functions to minimize spectral leakage requires windows having transforms with narrow mainlobs and low side-lobs. Useful window functions having transforms with sidelobes lower than those of rectangular windows include triangular, Hanning, Hamming, and Kaiser-Bessel windows.

$$x(N-n) = x(n) \quad ...(4.51)$$

Here, $N = 4$.

Let us check this condition by substituting different values of n as under :

For $n = 1$, we have

$x(4-1) = x(1)$ this means $x(3) = x(1) = 4$

For $n = 2$, we have

$x(4-2) = x(2)$ this means $x(2) = x(2) = 3$

For $n = 3$, we have

$x(4-3) = x(3)$ this means $x(1) = x(3) = 4$.

Since for all values of n, equation (4.51) is satisfied, the given sequence is circularly even.

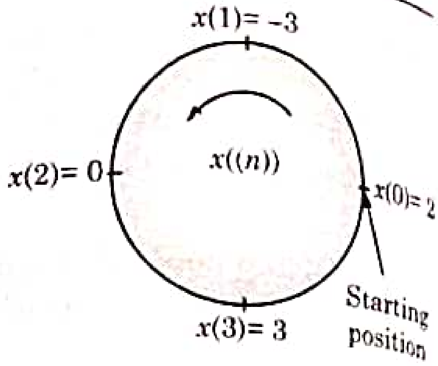


FIGURE 4.17 *Plot of* $x(n) = \{2, -3, 0, 3\}$.

(vi) Circularly odd sequence

A N-point sequence is called circularly odd if it is antisymmetric about point zero on the circle.

This means that

$$x(N-n) = -x(n), \quad 1 \le n \le N-1$$

Let us consider the sequence,

$$x(n) = \{2, -3, 0, 3\}$$

This sequence has been plotted as shown in figure 4.17.

Here, $x(0) = 2$, $x(1) = -3$, $x(2) = 0$ and $x(3) = 3$.

We have the following condition for circularly odd sequence:

$$x(N-n) = -x(n), \quad \text{for} \quad 1 \le n \le N-1 \quad ...(4.64)$$

For $n = 1$, we have

$x(4-1) = -x(1)$ this means $x(3) = -x(1)$

For $n = 2$, we have

$x(4-2) = -x(2)$ this means $x(2) = -x(2)$

For $n = 3$, we have

$x(4-3) = -x(3)$ this means $x(1) = -x(3)$

Thus, for all values of n, equation (4.64) is satisfied. Hence, the sequence is circularly odd.

Summary of Circular Properties

Table 4.2 shows summary of circular property.

TABLE 4.2

S.No.	Sequence	Expression	Explanation
1.	Input sequence	$x((n))$	Plot the samples of $x(n)$ in anti-clockwise direction. Anticlockwise means positive direction.
2.	Circular delay	$x((n-k))$	Shift sequence $x(n)$ in anticlock-wise direction by k samples.
3.	Circular advance	$x((n+k))$	Shift sequence $x(n)$ in clockwise direction by k samples.
4.	Circular folding	$x((-n))$	Plot the samples of $x(n)$ in clockwise direction. Clockwise means negative direction.
5.	Circularly even	$x(N-n) = x(n)$	Sequence is symmetric about the point zero on the circle.
6.	Circularly odd	$x(N-n) = -x(n)$	Sequence is antisymmetric about the point zero on the circle.

Bmp

4.12.4 Symmetry Properties of DFT

The symmetry properties of DFT can be derived in a similar way as we derived DTFT symmetry properties. We know that DFT of sequence $x(n)$ is denoted by $X(k)$. Now, if $x(n)$ and $X(k)$ are complex valued sequence then it can be represented as under :

$$x(n) = x_R(n) + jx_I(n), \quad 0 \le n \le N-1 \qquad ...(4.52)$$

and

$$X(k) = X_R(k) + jX_I(k), \quad 0 \le k \le N-1 \qquad ...(4.53)$$

Here, R stands for real part and I stands for imaginary part

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad ...(4.54)$$

Substituting equation (4.65) in equation (4.67), we have

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + jx_I(n)]\, e^{-j2\pi kn/N} \qquad ...(4.55)$$

But according to Euler's identity, we have

$$e^{-j2\pi kn/N} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)$$

Substituting this value in equation (4.68), we obtain

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + jx_I(n)]\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

or

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - jx_R(n)\sin\left(\frac{2\pi kn}{N}\right) + jx_I(n)\cdot\cos\left(\frac{2\pi kn}{N}\right) - j^2x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right]$$

Here, $j^2 = -1$ and writing summation for real and imaginary parts separately we get,

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right] - j\sum_{n=0}^{N-1} \left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(4.56)$$

Comparing equations (4.69) and (4.66), we can write

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(4.57)$$

and

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(4.58)$$

Equations (4.70) and (4.71) are obtained by using definition of DFT. Similarly, we can obtain real and imaginary parts of $x(n)$ using definition of IDFT.

Hence, we have

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(4.59)$$

and

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right] \qquad ...(4.60)$$

Now, let us consider different cases as under:

Case (*i*) : When $x(n)$ is real valued

Statement : If $x(n)$ is real valued then, we have

$$X(N-k) = X(-k) = X^*(k)$$

Proof

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(4.61)}$$

Replacing k by $N-k$, we have

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\, W_N^{(N-k)n}$$

or

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\, W_N^{Nn}\, W_N^{-kn} \qquad \text{...(4.62)}$$

Now we have,

twiddle factor $\quad W_N = e^{-\frac{j2\pi}{N}}$

Therefore $\quad W_N^{Nn} = \left(e^{-\frac{j2\pi}{N}}\right)^{Nn} = e^{-j2\pi n} = \cos 2\pi n - j \sin 2\pi n$

Since, n is an integer, $\cos 2\pi n = 1$ and $\sin 2\pi n = 0$

Therefore $\quad W_N^{Nn} = 1$

Thus, equation (4.62) becomes,

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn} \qquad \text{...(4.63)}$$

Comparing equation (4.63) with definition of DFT (equation (4.61)), we get

$$X(N-k) = X(-k) \qquad \text{...(4.64)}$$

Now, using equation (4.66), we can write

$$X^*(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn} \qquad \text{...(4.65)}$$

Thus, from equations (4.64) and (4.65), we get

$$X(N-k) = X(-k) = X^*(k)$$

Case (*ii*) : When $x(n)$ is real and even

Statement : When $x(n)$ is real and even which means,

$x(n) = x(N-n)$, then DFT becomes

$$X(k) = X_R(k)$$

Proof

Since imaginary part is zero, substituting $x_I(n) = 0$ in equation (4.57), we get

$$X_R(k) = \sum_{n=0}^{N-1} X_R(n) \cos\left(\frac{2\pi kn}{N}\right)$$

Similarly, IDFT can be written by substituting $X_I(k) = 0$ in equation (4.59) *i.e.*,

$$x_R(n) = \frac{1}{N} \sum_{n=0}^{N-1} X_R(k) \cos\left(\frac{2\pi kn}{N}\right)$$

Case (iii) : When $x(n)$ is real and odd

Statement : When $x(n)$ is real and odd which means,

$x(n) = -x(N-n)$ then the DFT becomes,

$$X(k) = -j\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Proof

Because, $x(n)$ is real, let us substitute $x_I(n) = 0$ in equation (4.56). Similarly, $x(n)$ is odd and 'cos' is even function so we can write, $\cos\left(\frac{2\pi kn}{N}\right) = 0$. Thus, first summation in equation (4.56) becomes zero. In the second summation of equation (4.69), substituting $x_I(n) = 0$, we shall have

$$X(k) = -j\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right)$$

Similarly, IDFT can be written as

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\sin\left(\frac{2\pi kn}{N}\right)$$

Case (iv) : When $x(n)$ is purely imaginary sequence

When x(n) is purely imaginary which means $x_R(n) = 0$ and $x(n) = jx_I(n)$ then substituting $x_R(n) = 0$ in equation (4.70), we get

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n)\sin\left(\frac{2\pi kn}{N}\right)$$

And substiting $x_R(n) = 0$ in equation (4.71), we obtain

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n)\cos\left(\frac{2\pi kn}{N}\right)$$

Symmetry properties can be summarized as shown in figure 4.18.

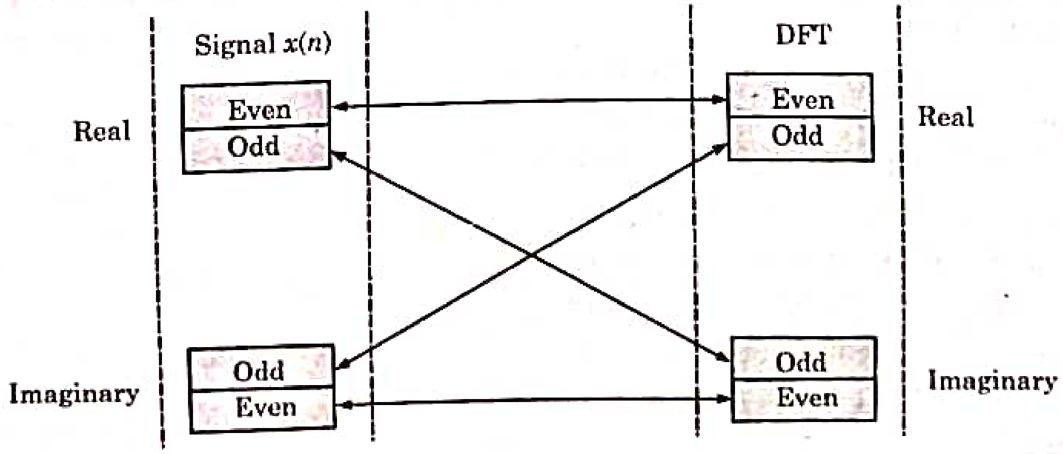


FIGURE 4.18 *Summary of symmetry property.*

Table 4.4 illustrates this summary.

TABLE 4.4

S.No.	N point sequence $x(n)$, $0 \le n \le N-1$	N point DFT
1.	$x^*(n)$	$X^*(N-k)$
2.	$x^*(N-n)$	$X^*(k)$
3.	$x_R[n]$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
4.	$jx_I[n]$	$X_{ce}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
5.	$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
6.	$x_{co}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$jX_I(k)$

EXAMPLE 4.10 The first five points of the 8 point DFT of a real valued sequence are {0.25, 0.125 − j0.3018, 0, 0.125 − j0.0518, 0}. Determine the remaining three points.

Solution : Given DFT points are :

$$X(0) = 0.25$$
$$X(1) = 0.125 - j0.3018$$
$$X(2) = 0$$
$$X(3) = 0.125 - j0.0518$$
$$X(4) = 0$$

Given sequence is a real valued sequence. According to the symmetry property we have,

$$X^*(k) = X(N-k)$$

or $$X(k) = X^*(N-k) \quad \text{...}(i)$$

This is 8 point DFT. Thus, $N = 8$

Therefore, $$X(k) = X^*(8-k) \quad \text{...}(ii)$$

Now, we want remaining three samples namely $X(5)$, $X(6)$ and $X(7)$. Substituting $k = 5$ in equation (*ii*), we have

$$X(5) = X^*(8-5) = X^*(3)$$

We have $X(3) = 0.125 - j0.0518$

Also $X^*(3) = 0.125 + j0.0518$

Hence, $X(5) = 0.125 + j0.0518$

Substiting $k = 6$ in equation (*ii*), we have

$$X(6) = X^*(8-6) = X^*(2)$$

We have $X(2) = 0$,

Thus $X^*(2) = 0$

Therefore, $X(6) = 0$

Similarly, substituting $k = 7$ in equation (*ii*), we obtain

$$X(7) = X^*(8-7) = X^*(1)$$

We have $\quad X(1) = 0.125 - j0.3018$

Thus, $\quad X(7) = 0.125 + j0.3018 \quad$ **Ans.**

EXAMPLE 4.11 **The first five DFT points of real and even sequence $x(n)$ of length eight are given below. Determine remaining three points. $X(k) = \{5, 1, 0, 2, 3, ...\}$**

Solution : Given DFT points are

$$X(0) = 5,\ X(1) = 1,\ X(2) = 0,\ X(3) = 2 \quad \text{and} \quad X(4) = 3.$$

According to symmetry property, we have

$$X^*(k) = X(N-k)$$

Also, $\quad X(k) = X^*(N-k)$

This is 8 point DFT.

Thus, $\quad N = 8$

Therefore, $\quad X(k) = X^*(8-k)$

Hence $\quad X(5) = X^*(8-5) = X^*(3)$

or $\quad X(5) = 2$

Now, $\quad X(6) = X^*(8-6) = X^*(2)$

so that $\quad X(6) = 0$

Also, $\quad X(7) = X^*(8-7) = X^*(1)$

or $\quad X(7) = 1$

4.12.5 Duality Property

Statement : If $\quad x(n) \xleftrightarrow[N]{DFT} X(k)$

then $\quad X(n) \xleftrightarrow[N]{DFT} Nx[((-k))_N]$

Proof

Let us consider a discrete time sequence $x(n)$. Its periodic extension is denoted by $x_p(n)$. Now, DFT of x(n) is $X(k)$ and the periodic expension of X(k) is denoted by $X_p(k)$.

This means that

$$x_p(n) = x((n))_N \qquad ...(4.66)$$

and

$$X_p(k) = X((k))_N \qquad ...(4.67)$$

Thus, we can write

$$x_p(n) \xleftrightarrow[N]{DFT} X_p(k) \qquad ...(4.68)$$

Now, let us define periodic sequence $x_{1p}(n) = X_p(n)$. One period of this sequence is a finite duration sequence $x_1(n) = x(n)$.

The discrete Fourier series coefficients of $x_{1p}(n)$ are denoted by $X_{1p}(k)$ and $X_{1p}(k) = Nx_p(-k)$. Thus, DFT of $x_1(n)$ which is denoted by $X_1(k)$ will be

$$X_1(k) = \begin{cases} Nx_p(-k) & \text{for} \quad 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} \qquad ...(4.69)$$

Equation (4.82) can also be written as,

$$X_1(k) = \begin{cases} Nx[((-k))]_N & \text{for} \quad 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} \qquad ...(4.70)$$

Now $X_1(k)$ is DFT of $x_1(n)$ and $x_1(n)$ is a finite duration sequence denoted by $X(n)$.

Therefore, $X(n) \xleftrightarrow[N]{DFT} Nx[((-k))_N]$

4.12.6 Multiplication of Two DFTs and Circular Convolution*

(U.P. Tech, Sem. Exam. 2006-07) (05 marks)

This property states that the multilication of two DFTs is equivalent to the circular convolution of their sequences in time domain.

Mathematicaly, we have

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$x_1(n) \; Ⓝ \; x_2(n) \xleftrightarrow[N]{DFT} X_1(k) \,.\, X_2(k) \qquad ...(4.71)$$

Here, Ⓝ indicates circular convolution.

Let the result of circular convolution of $x_1(n)$ and $x_2(n)$ be $y(m)$ then the circular convolution can also be expressed as,

$$y(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \quad m = 0, 1, ... N-1 \qquad ...(4.72)$$

Here, the term $x_2((m-n))_N$ indicates the circular convolution.

Proof

Let us consider two discrete time sequences $x_1(n)$ and $x_2(n)$.

The DFT of $x_1(n)$ can be expressed as under:

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi kn}{N}}, \; k = 0, 1, ... N-1 \qquad ...(4.73)$$

To avoid the confusion, let us write the DFT of $x_2(n)$ using different index of summation *i.e.*,

$$X_2(k) = \sum_{l=0}^{N-1} x_2(n) e^{\frac{-j2\pi kl}{N}}, \; k = 0, 1, ... N-1 \qquad ...(4.74)$$

It may be noted that in equation (4.74), instead of n we have used l.

Let us denote the multiplication of two DFTs $X_1(k)$ and $X_2(k)$ by $Y(k)$.

Therefore, $Y(k) = X_1(k) \,.\, X_2(k)$. ...(4.75)

Let IDFT of $Y(k)$ be $y(m)$. Then using definition of IDFT, we have

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\frac{j2\pi km}{N}} \qquad ...(4.76)$$

Substituting equation (4.75) in equation (4.76), we obtain

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) \,.\, X_2(k) \; e^{\frac{j2\pi km}{N}} \qquad ...(4.77)$$

Substituting the values of $X_1(k)$ and $X_2(k)$ from equations (4.73) and (4.74) in equation (4.77), we obtain

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-\frac{j2\pi kn}{N}}\right]\left[\sum_{l=0}^{N-1} x_2(l) e^{-\frac{j2\pi kl}{N}}\right] e^{\frac{j2\pi km}{N}} \qquad ...(4.78)$$

* State and prove the 'circular convolution' property of DFT.

(U.P. Tech. Sem. Exam. 2006-07) (05 marks)

Rearranging the summations and terms in equation (4.91), we get

$$y(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{\frac{-j2\pi kl}{N}} \cdot e^{\frac{-j2\pi kn}{N}} \cdot e^{\frac{j2\pi km}{N}}\right]$$

Therefore, we have

$$y(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{+j2\pi k(m-n-l)/N}\right] \Rightarrow N \qquad ...(4.79)$$

Let us consider the last term of equation (4.79). It may be written as under:

$$e^{j2\pi k(m-n-l)/N} = [e^{j2\pi(m-n-l)/N}]^k \qquad ...(4.80)$$

Now, let us use the following standard summation expression:

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & \text{for } a = 1 \\ \dfrac{1-a^N}{1-a} & \text{for } a \neq 1 \end{cases} \qquad ...(4.81)$$

Let, here, $\qquad a = e^{+j2\pi\frac{(m-n-l)}{N}}$ $\qquad ...(4.82)$

Now, according to equation (4.81), we shall consider two cases :

Case (*i*) : When $a = 1$

If $(m - n - l)$ is multiple of N which means,

$(m - n - l) = N, 2N, 3N, ...$ then equation (4.95) becomes,

$$a = e^{+j2\pi} = e^{+j2\pi(2)} = e^{+j2\pi(3)}... = 1$$

Thus, when $(m - n - l)$ is multiple of N (this means that $a = 1$), then according to equation (4.81), the third summation in equation (4.79) becomes equal to N.

Case (*ii*) : When $a \neq 1$

If $a \neq 1$, this means that if $m - n - l$ is not multiple of N then according to equation (4.81), we have

$$\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a} \qquad ...(4.83)$$

Substituting equation (4.82) in equation (4.83), we obtain

$$\sum_{k=0}^{N-1} [e^{+j2\pi(m-n-l)}]^k = \frac{1-e^{+j2\pi(m-n-l)}}{1-e^{\frac{+j2\pi(m-n-l)}{N}}} \qquad ...(4.84)$$

Here, m, n and l are integers.

Hence, $e^{+j2\pi(m-n-l)} = 1$ always. Therefore, R.H.S. of equation (4.84) becomes zero when $a \neq 1$. Therefore to get the result of equation (4.81), we have to consider the condition $a = 1$. This means that when $m - n - l$ is multiple of N. For this condition, we have the result of summation equals to N. Thus, equation (4.79) becomes,

$$y(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l) \cdot N$$

Therefore, we have

$$y(m) = \sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l) \qquad ...(4.85)$$

We have obtained equation (4.98) for the condition $(m - n - l)$ is multiple of N. This condition can be expressed as,

$$m - n - l = -pN \quad ...(4.86)$$

Here, p is an integer and an integer can be positive or negative. For simplicity, we have considered negative integer. Now, from equation (4.86), we obtain

$$l = m - n + pN \quad ...(4.87)$$

Substituting this value in equation (4.85), we get,

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m - n + pN) \quad ...(4.88)$$

Here, we have not considered the second summation of equation (4.85). Because this summation is in terms of l and exponential term is absent in equation (4.88).

Now, the term $x_2(m - n + pN)$ indicates a periodic sequence with period N. This is because p is an integer. This term also indicates that the periodic sequence is delayed by n samples. Further, we know that if a sequence is periodic and delayed then it can be expressed as,

$$x_2(m - n + pN) = x_2((m - n))_N \quad ...(4.89)$$

Here, the R.H.S. term indicates circular shifting of $x_2(n)$. Substituting this value in equation (4.88), we get

$$\boxed{y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m - n))_N} \quad m = 0, 1, \ldots N - 1 \quad ...(4.90)$$

This is the equation of circular convolution. Since in this equation, the sequence $x_2(n)$ is shifted circularly, this type of convolution is called as **Circular Convolution.**

EXAMPLE 4.12 Given the two sequences of length 4 as under:

$$x(n) = \{0, 1, 2, 3\}$$
$$h(n) = \{2, 1, 1, 2\}$$

Compute the circular convolution.

Solution : According to the definition of circular convolution, we have

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m - n))_N \quad ...(i)$$

Here, given sequences are $x(n)$ and $h(n)$. The length of sequence is 4 that means $N = 4$. Thus, equation (i) becomes,

$$y(m) = \sum_{n=0}^{3} x(n)h((m - n))_4 \quad ...(ii)$$

(i) We draw $x(n)$ and $h(n)$ as shown in figure 4.19(a) and (b).

It may be noted that $x(n)$ and $h(n)$ are plotted in anticlockwise direction.

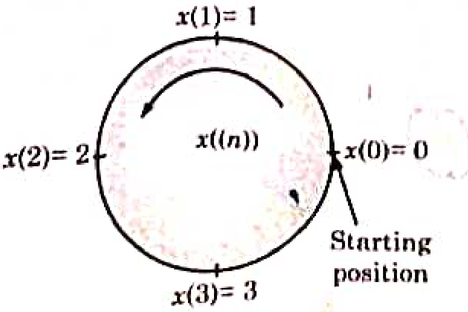


FIGURE 4.19 (a) $x(n) = \{0, 1, 2, 3\}$.

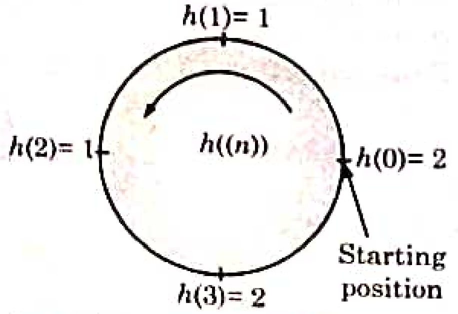


FIGURE 4.19 (b) $h(n) = \{2, 1, 1, 2\}$.

Now, let us calculate different values of $y(m)$ by putting $m = 0$ to $m = 3$ in equation (*ii*).

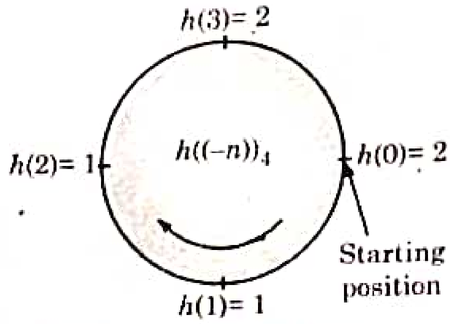


FIGURE 4.19 *(c) h(n) is plotted in clockwise direction.*

(ii) Calculation of $y(0)$

Substituting $m = 0$ in equation (*ii*), we get

$$y(0) = \sum_{n=0}^{3} x(n)h((-n))_4 \qquad \text{...}(iii)$$

Equation (*iii*) shows that we have to obtain the product of $x(n)$ and $h((-n))$, and then we have to take the summation of product elements. Using graphical method, this calculation is done as follows.

The sequence $h((-n))_4$ indicates circular folding of $h(n)$. This sequence is obtained by plotting $h(n)$ in a clockwise direction as shown in figure 4.19(*c*).

To do the calculations, we plot $x(n)$ and $h((-n))$ on two concentric circles as shown in figure 4.19(*d*). Also, $x(n)$ is plotted on the inner circle and $h((-n))$ is plotted on the outer circle.

Now, according to equation (*ii*), individual values of product $x(n)$ and $h((-n))$ are obtained by multiplying two sequences point by point. Then, $y(0)$ is obtained by adding all product terms.

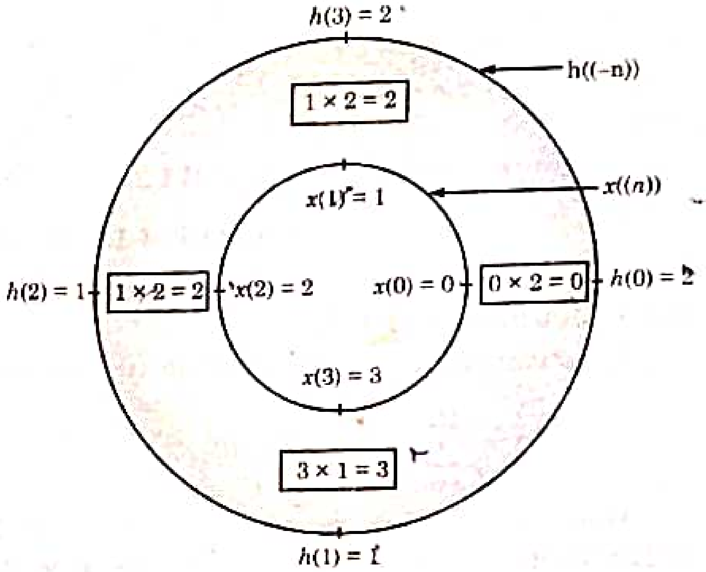


FIGURE 4.19 *(d)* $\sum_{n=0}^{3} x(n)h((-n))_4$.

Therefore, we have

$$y(0) = (0 \times 2) + (1 \times 2) + (1 \times 2) + (3 \times 1) = 0 + 2 + 2 + 3$$

or $$y(0) = 7$$

(iii) Calculation of $y(1)$

Substituting $m = 1$ in equation (*ii*), we have

$$y(1) = \sum_{n=0}^{3} x(n)h((1-n))_4 \qquad \text{...}(iv)$$

Here, $h((1-n))_4$ is same as $h((-n+1))_4$. This indicates delay of $h((-n))$ by 1 sample. This is obtained by shifting $h((-n))$ in anticlockwise direction by 1 sample, as shown in figure 4.19(*e*).

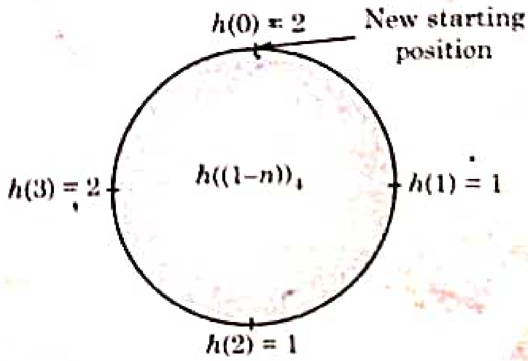


FIGURE 4.19 *(e).* $h((-n+1))_4$.

We have already drawn the sequence $x(n)$ as shown in figure 4.19(*a*). To do the calculations, according to equation (*iv*), two sequences $x(n)$ and $h((1-n))_4$ are plotted on two concentric circles as shown in figure 4.19(*f*). Also, $y(1)$ is obtained by adding the product of individual terms.

Therefore, we write

$$y(1) = (0 \times 1) + (3 \times 1) + (2 \times 2) + (1 \times 2)$$
$$= 0 + 3 + 4 + 2$$

or $$y(1) = 9$$

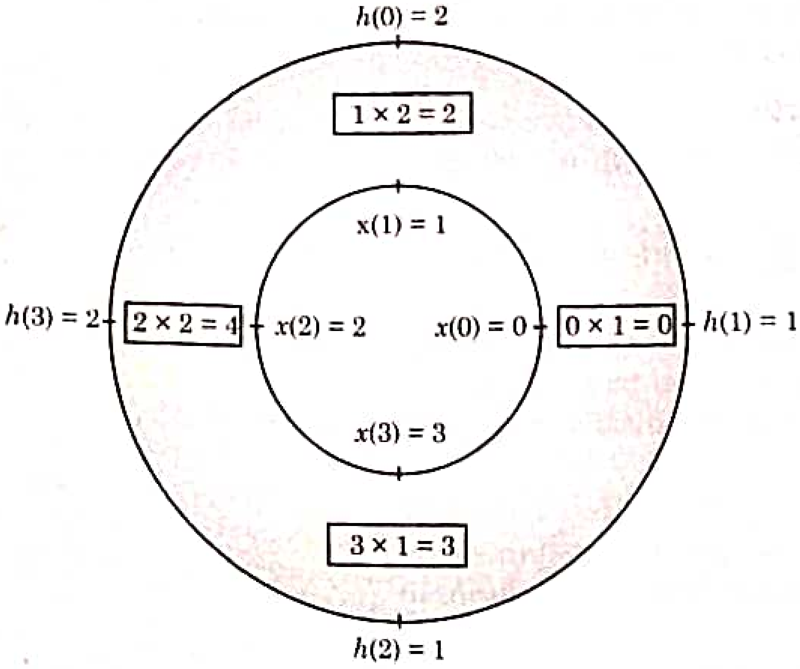


FIGURE 4.19 *(f)* $y(1) = \sum_{n=0}^{3} x(n)h((1-n))_4$.

(*iv*) Calculation of $y(2)$

Substituting $m = 2$ in equation (*ii*), we obtain,

$$y(2) = \sum_{n=0}^{3} x(n)h((2-n))_4 \qquad ...(v)$$

Here, $h((2-n))_4$ is same as $h((-n+2))_4$. It indicates delay of $h((-n))_4$ by 2 samples. It is obtained by shifting $h((-n))_4$ by two samples in anticlockwise direction as shown in figure 4.19(*g*).

According to equation (*iv*), the value of $y(2)$ is obtained adding individual product terms as shown in figure 4.19(*h*).

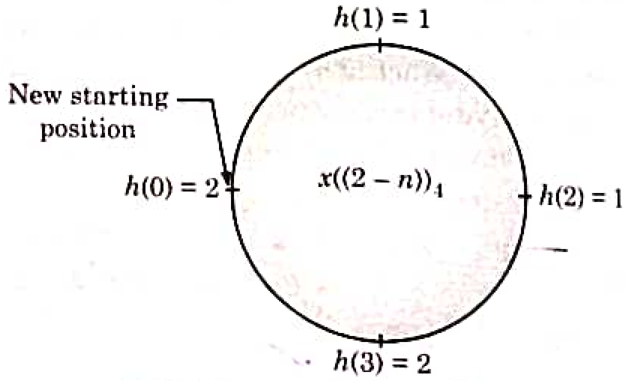


FIGURE 4.19 *(g)* $h((-n+2))_4$.

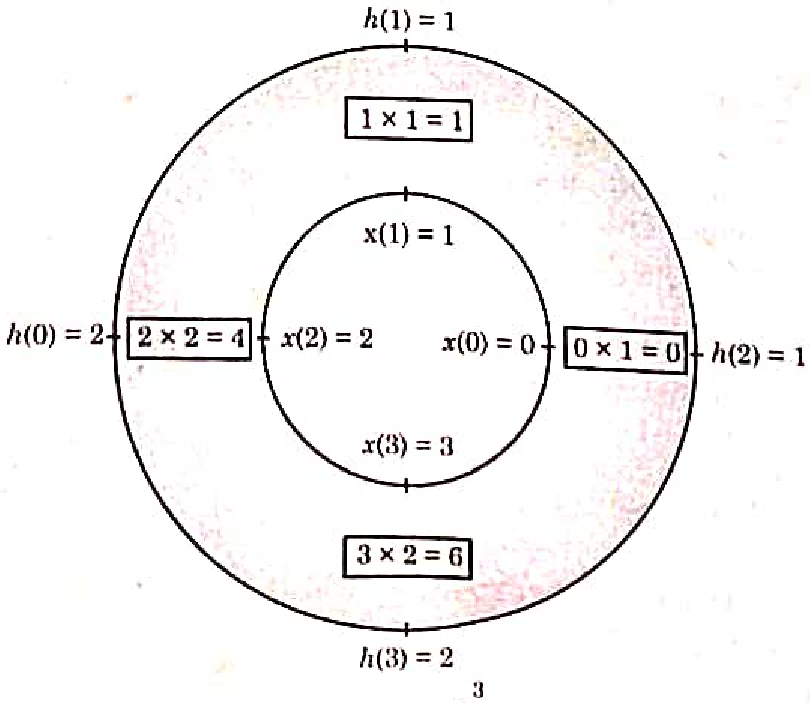


FIGURE 4.19 *(h)* $y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_4$.

Therefore, we have

$$y(2) = (0 \times 1) + (3 \times 2) + (2 \times 2) + (1 \times 1)$$

or $$y(2) = (0 + 6 + 4 + 1)$$

or $$y(2) = 11$$

Step V : Calculation of $y(3)$

Substituting $m = 3$ in equation (*ii*), we get

$$y(3) = \sum_{n=0}^{3} x(n)h((3 - n))_4 \quad ...(iv)$$

Here, $h((3 - n))_4$ is same as $h((-n + 3))_4$. It indicates delay of $h((-n))_4$ by 3 samples. It is obtained by shifting $h((-n))_4$ by 3 samples in anticlockwise direction as shown in figure 4.19(*i*).

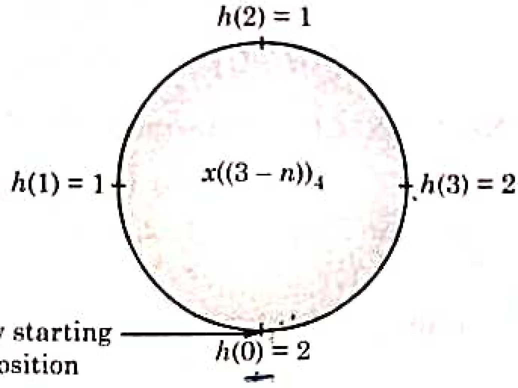


FIGURE 6.19 *(i). $h((-n + 3))_4$.*

According to equation (v), $y(3)$ is obtained by adding individual product terms as shown in figure 4.19(*j*) *i.e.*,

$$y(3) = (0 \times 2) + (3 \times 2) + (2 \times 1) + (1 \times 1)$$

or $$y(3) = 0 + 6 + 2 + 1$$

or $$y(3) = 9$$

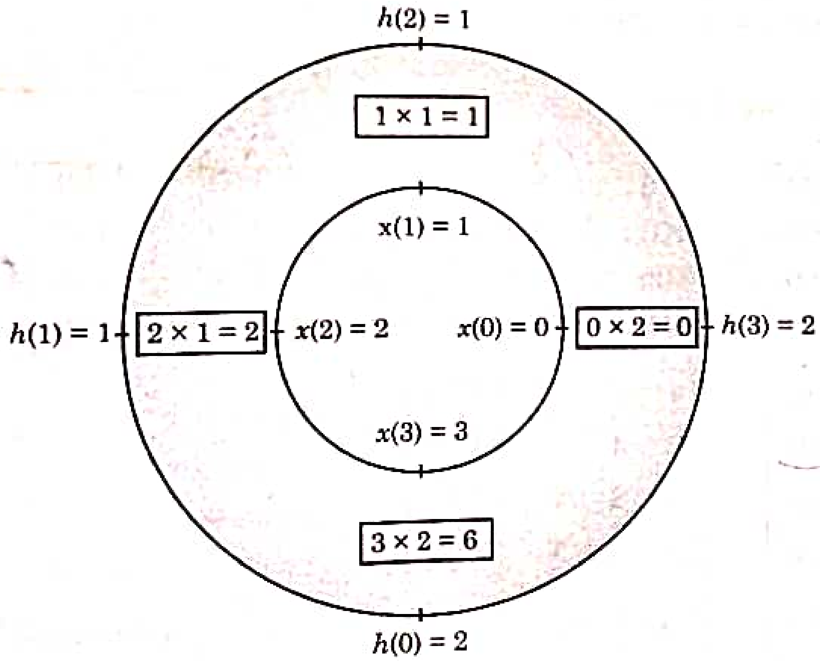


FIGURE 4.19 *(j). $y(3) = \sum_{n=0}^{3} x(n)h((3 - n))_4$*

Now, the resultant sequence $y(m)$ can be written as under:

$$y(m) = \{y(0), y(1), y(2), y(3)\}$$

or $$y(m) = \{7, 9, 11, 9\}.$$ **Ans.**

4.12.6.1. Circular Convolution Using Matrix Method

The graphical method which we have just discussed is quite tedious, especially when many samples are present. While the matrix method is more convenient. In the matrix method, one sequence is repeated via circular shifing of samples. It is represented as under :

we have $y(m) = x(n) \; (N) \; h(n) = h(n) \; (N) \; x(n)$

4.13.7 Praseval's Theorem

For complex-valued sequences $x(n)$ and $y(n)$, in general, if

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

and

$$y(n) \xleftrightarrow[N]{DFT} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k) \qquad x(n).y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k).Y^*(k) \qquad ...(4.99)$$

Proof:

We have
$$\sum_{n=0}^{N-1} x(n)y^*(n) = \tilde{r}_{xy}(0)$$

and
$$\tilde{r}_{xy}(l) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{R}_{xy}(k)e^{j2\pi kl/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)e^{j2\pi kl/N}$$

Hence (4.99) follows by evaluating the IDFT at $l = 0$.

The expression in (4.99) is the general form of Parseval's theorem. In the special case where $y(n) = x(n)$, equation (4.99) reduces to

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 \qquad ...(4.100)$$

which expresses the energy in the finite-duration sequence $x(n)$ in terms of the frequency components $|X(k)|$.

4.13.8 Summary of DFT Properties

Table 4.5 shows the summary of DFT properties.

TABLE 4.5

S. No.	Name of DFT Property	Expression in time domain	Expression in frequency domain
1.	Periodicity	$x(n) = x(n + N)$	$X(k) = X(k + n)$
2.	Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
3.	Time Reversal	$x(N - n)$	$X(N - k)$
4.	Circular time shift	$x((n - l))_N$	$X(k)e^{-j2\pi kl/N}$
5.	Circular frequency shift	$x(n)e^{j\pi 2ln/N}$	—
6.	Circular convolution	$x_1(n)$ Ⓝ $x_2(n)$	$X_1(k)X_2(k)$
7.	Circular correlation	$x(n)$ Ⓝ $y^*(-n)$	$X(k)\ Y^*(k)$
8.	Multiplication of two sequences	$x_1(n)\ x_2(n)$	$\frac{1}{N}X_1(k)$ Ⓝ $X_2(k)$
9.	Complex conjugate	$x^*(n)$	$X^*(N - k)$
10.	Parseval's theorem	$\sum_{n=0}^{N-1} x(n)\ y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)\ Y^*(k)$

4.14 LINEAR FILTERING TECHNIQUES BASED ON DFT : LINEAR CONVOLUTION USING DFT

As a matter of fact, linear filtering is same as linear convolution. In this article, we shall discuss how the linear convolution is obtained using DFT.

We know that the linear convolution of $x(n)$ and $h(n)$ is given by,

$$y(n) = x(n) \otimes h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

or

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad ...(4.101)$$

If we obtain the Fourier transform of $x(n)$ and $h(n)$ then we shall get $X(\omega)$ and $H(\omega)$. We know that convolution is equivalent to multiplication in the frequency domain. Therefore, by multiplying $X(\omega)$ and $H(\omega)$, we get $Y(\omega)$.

This means that

$$Y(\omega) = X(\omega) \cdot H(\omega) \qquad ...(4.102)$$

If we take inverse Fourier transform of equation (4.102) then we will get the sequenc $y(n)$. This sequence will be same as the linear convolution of $x(n)$ and $h(n)$. However, we cannot use Fourier transform to obtain linear convolution because of the following two reasons:

(*i*) In Fourier transform, ω is continuous function of frequency. Hence, the computation cannot be done on digital computers. Because for the digital signal processors, discrete-time signals in place of continuous-time signals are required.

(*ii*) If we use DFT, then the computation will be more efficient because of the avaiability of Fast Fourier Transform (FFT) algorithms. Therefore, we must use DFT to obtain the linear filtering operation.

DO YOU KNOW?

In applying the DFT to signal analysis, the sampling theorem and the desired resolution imose a constraint between total signal duration, T, signal band-width, W, and the number of DFT points, N.

This relationship is $\frac{1}{T} \geq \frac{2W}{N}$

In case of DFT, we know that the multiplication of two DFTs in frequency domain is equvalent to the circular convolution, *i.e.*,

$$X(k).H(k) = x(n) \; \text{Ⓝ} \; h(n) \qquad ...(4.103)$$

However, in this case, we want linear convolution (linear filtering) and not the circular convolution.

Earlier, we have studied that if we adjust the length of two sequences, x(n) and h(n), then, the same result can be obtained using linear convolution and circular convolution.

Let us consider an FIR filter having impulse response $h(n)$ as shown in figure 4.25.*

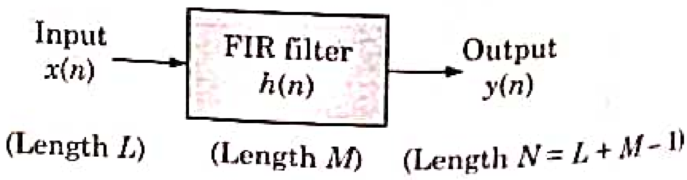


FIGURE 4.25 *FIR filter*

Here, $x(n)$ = Input sequence having length L

Therefore, $x(n) = \{0, 1, 2, \ldots, L-1\}$

$h(n)$ = Impulse response of filter having length M.

Therefore, $h(n) = \{0, 1, 2, \ldots, M-1\}$

Here, the linear convolution of $x(n)$ and $h(n)$ produces the output sequence $y(n)$ and the length of $y(n)$ is,

$$N = L + M - 1 \qquad ...(4.104)$$

* FIR filters will be discussed in details in chapter 7.

In this case, both the sequences $x(n)$ and $h(n)$ are finite. Hence linear convolution will be finite. Thus, equation (4.101) becomes,

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k) \qquad \text{...(4.105)}$$

Now, if we adjust the length of $x(n)$ and $h(n)$ equal to N and if we perform the circular convolution of $x(n)$ and $h(n)$ then, the result will be same as linear convolution. The length of $x(n)$ and $h(n)$ can be made equal to N by adding required number of zeros in $x(n)$ and $h(n)$. This is known as **zero padding**. This means that we have to increase the length of $x(n)$ by M points and length of $h(n)$ by L points to make the total length $N = L + M - 1$. Then, we can obtain the DFT of $x(n)$ and $h(n)$, that is, $X(k)$ and $H(k)$. The multiplication of these two DFTs yields sequence $Y(k)$, *i.e.*,

$$Y(k) = X(k) \,.\, H(k) \qquad \text{...(4.106)}$$

Now, by taking IDFT of $Y(k)$, the output sequence $y(n)$ may be obtained. Here, the linear filtering may be obtained using DFTs.

4.14.1 Evaluation of Linear Filtering using DFT

(*i*) First, we calculate the value of N using $N = L + M - 1$. Here, L represents number of samples in $x(n)$ and M represents number of samples in $h(n)$.

(*ii*) By adding zeros, we make the length of $x(n)$ and $h(n)$ equal to N.

(*iii*) We calculate DFT of $x(n)$ that means $X(k)$.

(*iv*) We calculate DFT of $h(n)$ that means $H(k)$.

(*v*) Then, we multiply $X(k)$ and $H(k)$ to get $Y(k) = X(k) \,.\, H(k)$.

(*vi*) Lastly, we obtain IDFT of $Y(k)$ that means $y(n)$.

EXAMPLE 4.22 Determine the response of FIR filter using DFT if Imp

$$x(n) = \{\underset{\uparrow}{1}, 2\} \text{ and } h(n) = \{\underset{\uparrow}{2}, 2\}$$

Solution: (*i*) Here, length of $x(n) = L = 2$

and length of $h(n) = M = 2$

Therefore, $\boxed{N = L + M - 1} = 2 + 2 - 1 = 3$

Hence, we have to calculate 3-point DFT. We shall compute standard 4-point DFT *i.e.*, with $N = 4$.

(*ii*) We will make length of $x(n)$ and $h(n)$ equal to 4 by adding zeros at the end.

Hence, $$x(n) = \{\underset{\uparrow}{1}, 2, 0, 0\} \qquad \text{...(i)}$$

and $$h(n) = \{\underset{\uparrow}{2}, 2, 0, 0\} \qquad \text{...(ii)}$$

(*iii*) Calculation of $X(k)$

Let us calculate DFT of $x(n)$, $X(k)$ using matrix method.

We have $$X(k) = W_N \cdot x_N \qquad \text{...(iii)}$$

Earlier, we have obtained the matrix for twiddle factor W_4. It is,

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \text{...(iv)}$$

We have input matrix

$$x_N = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad ...(v)$$

Substituting equations (*iv*) and (*v*) in equation (*iii*), we have

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2+0+0 \\ 1-2j+0+0 \\ 1-2+0+0 \\ 1+2j+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

Therefore, $X(k) = \{3,\ 1-2j,\ -1,\ 1+2j\}$ ...(*vi*)

(*iv*) Calculation of $H(k)$

We have $H(k) = W_N \cdot h_N$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+2+0+0 \\ 2-2j+0+0 \\ 2-2+0+0 \\ 2+2j+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2-2j \\ 0 \\ 2+2j \end{bmatrix}$$

or $H(k) = \{\underset{\uparrow}{4},\ 2-2j,\ 0,\ 2+2j\}$ ...(*vii*)

(*v*) Now, $Y(k) = X(k) \cdot H(k)$

Therefore, $Y(k) = \{\underset{\uparrow}{3},\ 1-2j,\ -1,\ 1+2j\} \cdot \{\underset{\uparrow}{4},\ 2-2j,\ 0,\ 2+2j\}$

Hence,

$$\begin{aligned} Y(0) &= X(0) \cdot H(0) = 3 \times 4 = 12 \\ Y(1) &= X(1) \cdot H(1) = (1-2j)(2-2j) \\ &= 2 - 2j - 4j + 4j^2 = 2 - 6j - 4 = -2 - 6j \\ Y(2) &= X(2) \cdot H(2) = (-1)(0) = 0 \\ Y(3) &= X(3) \cdot H(3) = (1+2j)(2+2j) \\ &= 2 + 2j + 4j + 4j^2 = 2 + 6j - 4 = -2 + 6j \end{aligned}$$

Thus, sequence $Y(k)$ will be

$$Y(k) = \{Y(0),\ Y(1),\ Y(2),\ Y(3)\}$$

Hence, $Y(k) = \{12,\ (-2-6j),\ 0,\ (-2+6j)\}$

(*vi*) Now, let us obtain $y(n)$ by taking IDFT of $Y(k)$.

We have, $y(n) = \frac{1}{N} W_N^* \, Y(k)$

Here, W_N^* is complex conjugate of W_N, which is obtained by changing the sign of j term. This means that

$$W_N^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Therefore, $y(n)$ will be given by

$$y(n) = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 12 \\ -2-6j \\ 0 \\ -2+6j \end{bmatrix}$$

$$y(n) = \frac{1}{4}\begin{bmatrix} 12-2-6j+0-2+6j \\ 12-2j+6+0+2j+6 \\ 12+2+6j+0+2-6j \\ 12+2j-6+0-2j-6 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 24 \\ 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 0 \end{bmatrix}$$

Therefore,

$$y(n) = \{\underset{\uparrow}{2}, 6, 4, 0\}$$

This is the required response of the filter.

Verification of Result

Let us verify the result by calculating linear convolution of $x(n)$ and $h(n)$. It has been shown in figure 4.26.

Hence, $y(n) = \{\underset{\uparrow}{2}, 6, 4\}$

Therefore, we conclude that the two results are same.

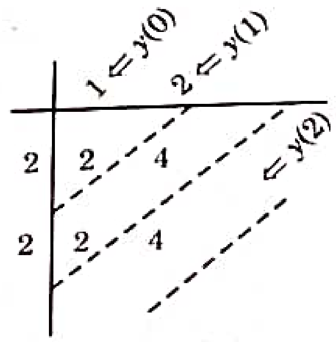


FIGURE 4.26 $x(n) \circledast h(n)$

4.14.2 Linear Filtering of Long Data Sequences

In real time applications, the input data sequence is very long. It is not possible to do the calculations for such long data sequences This is because, the digital computer has limited memory.

Hence, long input data sequence is broken into small length sequences (blocks). The computation of each block is done separately. Then all processed blocks are fitted one after other to get the final output.

There are two methods to obtain such computation. These are as under:

(i) Overlap Save method

(ii) Overlap Add method

4.14.2.1. *Overlap Save Method*

In this method, the filtering of each block is done separately. Let '*M*' be the impulse response of filter. Here, input data sequence is very long. This data sequence is divided into blocks. Let, N be the length of each block.

Forming different blocks

In this method, each block of length N is formed by taking L samples of the current segment and $M - 1$ samples from the previous segment.

Thus, $N = L + M - 1$.

For the first block, $x_1(n)$, there are no previous segments. So, this block is padded with $M - 1$ zeros. And, the first L samples are taken from the input sequence.

Therefore, $$x_1(n) = \{\underbrace{0, 0, 0, \ldots 0,}_{(M-1)\text{zeros}}\ \underbrace{x(0), x(1) \ldots x(L-1)}_{\text{First } L \text{ samples from input sequence } x(n)}\} \quad \ldots(4.107)$$

Let the long input sequence is,

$$x(n) = \{1, 2, 3, 4, 5, -1, 2, -3, 4, 5, 1, 2, 3, 4, 5, \ldots..\}$$

Let $M = 4$ and $L = 5$

So, $M - 1 = 3$ and $N = M + L - 1 = 4 + 5 - 1 = 8$

Thus, we can write the first sequence $x_1(n)$ as,

$$x_1(n) = \{\underbrace{0,0,0,}_{\text{Three zeros}} \quad \underbrace{1,2,3,4,5\}}_{\text{First five samples of } x(n)}$$

Now, $x_2(n)$ is formed by taking last $M - 1$ samples from $x_1(n)$ and next L samples from input sequence $x(n)$. Here, $M - 1 = 3$. Hence, last 3 samples of $x_1(n)$ are 3,4,5.

Therefore, $$x_2(n) = \{\underbrace{3,4,5,}_{(M-1)\text{ samples of } x_1(n)} \quad \underbrace{-1,2,-3,4,5\}}_{\text{Next } L \text{ samples of } x(n)}$$

Hence, general equation of $x_2(n)$ becomes,

$$x_2(n) = \{\underbrace{x(L-M+1)\ldots x(L-1),}_{(M-1)\text{ samples from } x_1(n)} \quad \underbrace{x(L), x(L+1)\ldots x(2L-1)\}}_{\text{Next } L \text{ samples of } x(n)} \qquad \ldots(4.108)$$

Similarly, $x_3(n)$ is formed by taking last $M - 1$ samples from $x_2(n)$ and next L samples from $x(n)$.

Therefore,

$$x_3(n) = \underbrace{\{x(2L-M+1\ldots\ldots\ldots x(2L-1),}_{(M-1)\text{ samples from } x_2(n)} \quad \underbrace{x(2L), x(2L+1)\ldots\ldots\ldots x(3L-1)\}}_{\text{Next L samples of } x(n)} \qquad \ldots(4.109)$$

Likewise the different blocks of length N are formed now, we have the length of each block equals to N and $N = L + M - 1$. We have assumed that the length of impulse response is M. This lenght must be same as the length of each block (N). This is done by adding $(L - 1)$ zeros in the sequence $h(n)$.

Therefore, $$h(n) = \{\underbrace{h(0), h(1)\ldots\ldots h(M-1),}_{\text{M samples of } h(n)} \quad \underbrace{0,0,0\ldots\ldots(L-1 \text{ zeros})\}}_{(L-1)\text{ zeros}} \qquad \ldots(4.110)$$

The N point DFT of $h(n)$ is $H(k)$. Similarly, N point DFT of each block is computed separately and it is stored in the memory. Let the DFT of m^{th} input block be $X_m(k)$. Now, the corresponding output is obtained by taking multiplication of $H(k)$ and $X_m(k)$. Let this output be $\hat{Y}_m(k)$.

$$\hat{Y}_m(k) = H(k).\ X_m(k) \qquad \ldots(4.111)$$

Now, the corresponding time domain sequence is obtained by taking IDFT of $\hat{Y}_m(k)$. Let this sequence be $y_m(n)$.

Hence, $$y_m(n) = \text{IDFT}\left\{\hat{Y}_m(k)\right\} \qquad \ldots(4.112)$$

Similarly, the time domain sequence corresponding to the DFT of each input block is obtained.

To avoid loss of data due to aliasing

Let us observe each input block $x_1(n)$, $x_2(n)$........ . Here, each input block consists of initial '$M - 1$' samples taken from the previous block. This means that there is overlap of sequences. While the last L samples of each block are the actual input samples.

Now, bacause of the circular shift of DFT and because of the overlapping of input data blocks; there is alising in the initial $M - 1$ samples in the corresponding output block. To avoid this. aliasing first $M - 1$ samples of $y_m(n)$ are discarded. This means that after computing time domain sequences, initial '$M - 1$' samples are discarded. Then each black is fitted one after other to obtain the final output. The overlap save method is described as shown in figure 4.27.

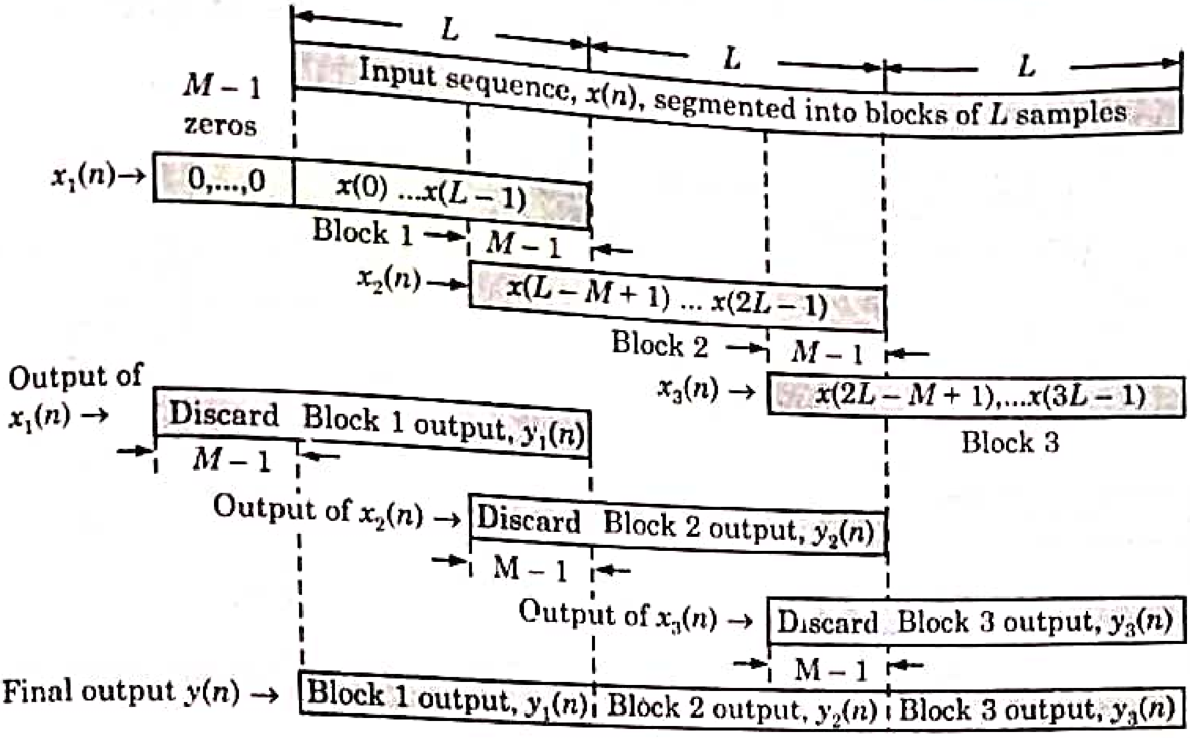


FIGURE 4.27 *Illustration of overlap save method.*

4.14.2.2. Overlap Add Method

In the overlap add method, the size of each input data block is N. Each input data block is formed by taking L samples from the input sequence and adding *M-1* zeros at the end of each input sequence.

Therefore, $$x_1(n) = \underbrace{\{x(0),\ldots\ldots x(L-1),}_{\text{First L samples of input sequence } x(n)} \quad \underbrace{0,0\ldots\ldots 0\}}_{(M-1)\text{ zeros}} \qquad \ldots(4.113)$$

Also, $x_2(n)$ is formed by taking next L samples from long input sequence x (n) and padding *M-1* zeros at the end.

Therefore, $$x_2(n) = \underbrace{\{x(L), x(L+1)\ldots\ldots x(2L-1),}_{\text{Next L samples of input sequence } x(n)} \quad \underbrace{0,0\ldots\ldots 0\}}_{(M-1)\text{ zeros}} \qquad \ldots(4.114)$$

Similarly, we have

$$x(3) = \underbrace{\{x(2L), x\,(2L+1)\ldots\ldots x(3L-1)}_{\text{Next L camples of input sequence } x(n)} \quad \underbrace{0,0\ldots\ldots 0\}}_{(M-1)\text{ zeros}} \qquad \ldots(4.115)$$

Impulse response of the filter is $h(n)$ and its lenght is M. This length must be again made equal to N. Therefore, *L-1* zeros are added to form the sequence $h(n)$.

Thus, we have

$$h(n) = \underbrace{\{h(0), h(1)\ldots\ldots h(M-1),}_{M \text{ samples of impulse sequence}} \quad \underbrace{0,0\ldots\ldots 0\}}_{\substack{(L-1)\text{ zeros to make} \\ \text{total length } = N}} \qquad \ldots(4.116)$$

Now, the DFT of each input data block is computed separately. Similarly, the DFT of $h(n)$ is computed that is, $H(k)$. Now, the output of m^{th} block is obtained by multiplying DFT of m^{th} input block by $\hat{H}(k)$.

Therefore, $$Y_m(k) = H(k).\ X_m(k) \qquad \ldots(4.117)$$

The time domain sequence $Y_m(n)$ is obtained by taking IDFT of $Y_m(k)$. Here, we are performing N point DFT. Hence, the length of each output is N. The different output sequences are as under :

$$y_1(n) = \{(y_1(0), y_1(1) \ldots y_1(L-1), y_1(L), y_1(L+1) \ldots y_1(L+M-1)\}$$
$$y_2(n) = \{y_2(0), y_2(1) \ldots y_2(L-1), y_2(L), y_2(L+1) \ldots y_2(L+M-1)\}$$

Similarly, all output sequences are obtained

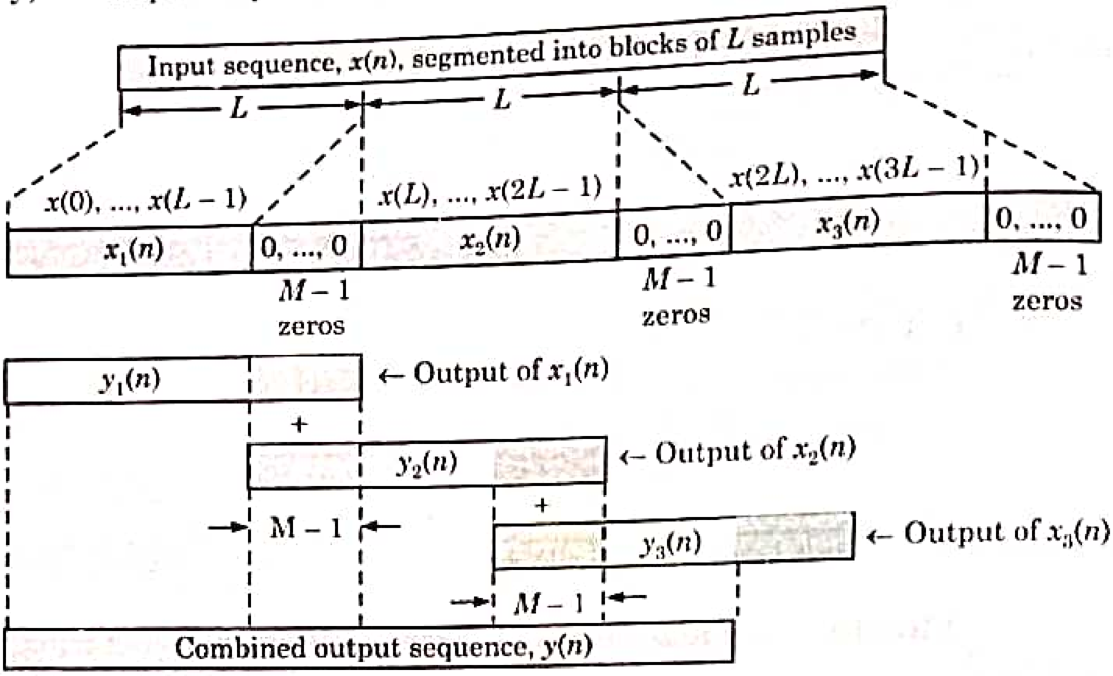


FIGURE 4.28 *Illustration of overlap add method.*

In this method, each input sequence is padded with M – 1 zeros at the end to make the length N = L + M – 1. So, last M – 1 samples of each output block must be overlapped and added to the first M – 1 samples of succeeding block as shown in figure 6.28. That is why, this method is called as overlap add method.

Here '$M - 1$' samples are not discarded because there is no aliasing effect.

4.15 FREQUENCY ANALYSIS OF SIGNALS USING DISCRETE FOURIER TRANSFORM (DFT)

The frequency analysis is also called as spectrum analysis. To do the frequency analysis, time domain signal should be first converted into the frequency domain.

If input signal is analog then first it is passed through the antialiasing filter. Then this signal is sampled at the rate f_s; where f_s is the sampling frequency. To avoid the aliasing, f_s should be greater than or equals to $2W$.. Here, W is the maximum frequency of input signal.

In practical cases, the time interval of signal is maintained at T_o and $T_o = LT$. Here, L = number of samples and T is the sampling interval.

Suppose we want to analyze the long input sequence $x(n)$. Now, to limit the time interval of sequence means this input sequence is multiplied by a rectangular window. A rectangular window is denoted by $W_R(n)$ and it is defined as,

$$W_R(n) = \begin{cases} 1 & \text{for } 0 \le n \le L-1 \\ 0 & \text{elsewhere} \end{cases}$$

This rectangular window is shown in figure 4.29.

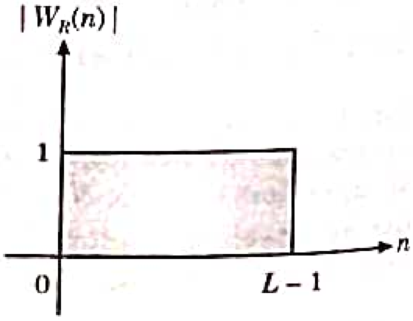


FIGURE 4.29 *Rectangular window*

Since, output of $W_R(n)$ is zero after the interval $L - 1$, multiplying $x(n)$ by $W_R(n)$ produces signal $\hat{x}(n)$ only in the range 0 to $L - 1$.

Therefore, $\dot{x}(n) = x(n)\, W_R(n)$ ...(4.118)

Then, by taking DFT of equation (6.118), we obtain

$$\dot{X}(k) = \sum_{n=0}^{L-1} [x(n)\, W_R(n)] e^{-j2\pi kn/N} \quad \text{...(4.119)}$$

Here, we are considering only '*L*' samples of input signal and not the complete input signal. The spectrum will be more accurate if the value of '*L*' is large.

Spectral Leakage

As shown in figure 4.30, the magnitude spectrum is not localized to a single frequency, but, it is spreadout over the entire frequency range. That means that the power of a signal is spreadout in entire frequency range. This leakage of the power is called as spectral leakage which is taking place because of '*windowing*' of input sequence.

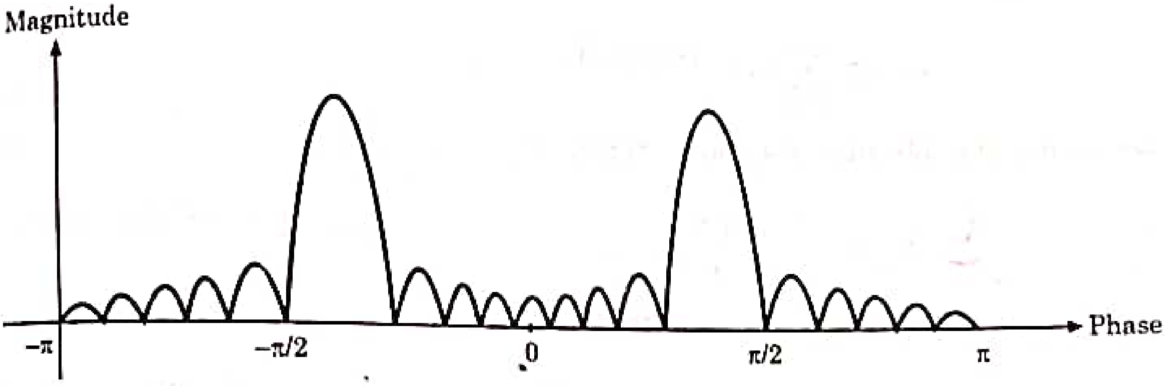


FIGURE 4.30 *Magnitude spectrum.*

4.15.1 Advantages and Limitations of Spectrum Analysis Using DFT

1. Advantages or Salient Features

(*i*) Fast processing of DFT can be done using FFT algorithms.

(*ii*) Estimation of power spectrum can be done.

(*iii*) Calculation of harmonics can also be done.

(*iv*) The resolution can be improved by increasing the number of samples in the calculation of DFT.

2. Limitations

(*i*) The frequency spectrum of entire input signal is not obtained because of windowing.

(*ii*) The leakage of power (*i.e.*, spectral leakage) takes place.

(*iii*) If we increase the number of samples to obtain the better accuracy, then the processing time is increased.

SOLVED EXAMPLES FROM PREVIOUS YEARS EXAMINATIONS

EXAMPLE 23 (*a*) Compute the N-point DFT of each of the following sequences :

(*i*) $x_1(n) = \delta(n - n_0)$, where $0 \le n_0 < N$

(*ii*) $x_2(n) = \alpha^n$, $0 \le n < N$ ***(UP Tech. Sem. Exam. 2006-07) (05 marks)***

Solution: (*i*) $x_1(n) = \delta(n - n_0)$ where $0 \le n_0 < N$

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

UNIT-2

EFFICIENT COMPUTATION OF DFT

SYLLABUS

Efficient Computation of the DFT: FFT Algorithms, Direct Computation of the DFT, Radix-2 FFT algorithms. Efficient computation of the DFT of two real sequences, Efficient computation of the DFT of a 2N-point real sequences, Gortezel Algorithm, Chirp z-transform algorithm.

Chapter 5: Efficient: Computation of DFT:
Fast Fourier Transform Algorithms

5.1 INTRODUCTION

As discussed in last chapter the Discrete Fourier Transform (DFT) plays a vital role in several applications of digital signal processing (DSP). These applications may include linear filtering, correlation analysis, and spectrum analysis. A major reason for its importance is the existence of efficient algorithms for computing the discrete Fourier Transform (DFT).

In previous chapter, we have studied how to obtain DFT of a sequence by using direct computation. Basically, the direct computation of DFT requires large number of computations. Hence, more processing time is required.

For the computation of N-point DFT, N^2 complex multiplications and $N^2 - N$ complex additions are required. If the value of N is large then the number of computations will go into lakhs. This proves inefficiency of direct DFT computation.

The primary objective of the present chapter is the description of computationally efficient algoirthms for evaluating the DFT. We shall be discussing two different approaches. One is a divide-and-conquer approach in which a DFT of size N, where N is a composite number, is reduced to the computation of smaller DFTs from which the larger DFT is computed. In particular, we present important computational algorithms, called **Fast Fourier Transform (FFT)** algorithms, for computing the DFT when the size N is a power of 2 and when it is power of 4. These FFT algorithms are very efficient in terms of computations. By using these algorithms, number of arthmetic operations involved in the computation of DFT are greatly reduced.

The second approach is based on the formulation of the DFT as a linear filtering operation on the data. This approach leads to two algorithms, the Goertzel algorithm and the chirp-z transform algorithms for computing the DFT via linear filtering of the data sequence.

5.2 EFFICIENT COMPUTATION OF THE DFT : FFT ALGORITHMS

Now, let us discuss few different methods for computing the DFT efficienly. Because of importance of the DFT in various digital signal processing applications, such as linear filtering, correlation analysis, and spectrum analysis its efficient computation is an important aspect which has received considerable attention by many mathematicians, engineers and applied scientists.

Basically, the computational problem for the DFT is to compute the sequence $\{X(k)\}$ of N complex-valued numbers given another sequence of data $\{x(n)\}$ of length N, according to the following expression :

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad 0 \le k \le N-1 \qquad \text{... (5.1)}$$

where $$W_N = e^{-j2\pi/N} \qquad \text{... (5.2)}$$

Usually, the data sequence $x(n)$ is also assumed to be complex valued.

Similarly, the IDFT becomes

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \qquad 0 \le n \le N-1 \qquad \text{... (5.3)}$$

Since, the DFT and IDFT involve basically the same type of computations, the efficient computational algorithms for the DFT applies equally well to the efficient computation of the IDFT.

It may be observed that for each value of k, direct computation of $X(k)$ involves N complex multiplications ($4N$ real multiplications) and $N - 1$ complex additions ($4N - 2$ real additions). As a result of this, to compute all N values of the DFT requires N^2 complex multiplications and $N^2 - N$ complex additions.

Direct computation of the DFT is basically inefficient primarily because it does not exploit the symmetry and periodicity properties of the phase factor W_N. In particular, these two properties are :

Symmetry property, $W_N^{k+(N/2)} = -W_N^k$ ... (5.4)

Periodicity property, $W_N^{k+N} = W_N^k$ ... (5.5)

The computationally efficient algorithms described in this section, known collectively as **Fast Fourier transform (FFT) algorithms** exploit these two basic properties of the phase factor.

5.3 DIRECT COMPUTATION OF THE DFT

For a complex-valued sequence $x(n)$ of N points, the DFT may be expressed as

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right] \quad \text{... (5.6)}$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right] \quad \text{... (5.7)}$$

Also, the direct computation of (5.6) and (5.7) requires the following :

(*i*) $2N^2$ evaluations of trigonometric functions.

(*ii*) $4N^2$ real multiplications.

(*iii*) $4N(N-1)$ real additions.

(*iv*) A number of indexing and addressing operations.

These operations are typical of DFT computational algorithms. The operations in items 2 and 3 result in the DFT values $X_R(k)$ and $X_I(k)$. The indexing and addressing operations are necessary to fetch the data $x(n)$, $0 \le n \le N-1$, and the phase factors and to store the results. The variety of DFT algorithms optimize each of these computational processes in a different way.

5.4 DIVIDE-AND-CONQUER APPROACH TO COMPUTATION OF THE DFT

The development of computationally efficient algorithms for the DFT is made possible if we adopt a divide-and-conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs. This basic approach leads to a firmly of computationally efficient algorithms known collectively as **FFT algorithms.**

To illustrate the basic notions, let us consider the computation of an N-point DFT, where N can be factored as a product of two integers, *i.e.*,

$$N = LM \quad \text{... (5.8)}$$

The assumption that N is not a prime number is not restrictive, since we can pad any sequence with zeros to ensure a factorization of the form (5.8).

Now the sequence $x(n)$, $0 \le n \le N-1$, can be stored in either a one-dimensional array indexed by n or as a two-dimensional array indexed by l and m, where $0 \le l \le L-1$ and $0 \le m \le M-1$ as illustrated in figure 5.1. Note that l is the row index and m is the column index. Thus, the sequence $x(n)$ can be stored in a rectangular array in a variety of ways, each of which depends on the mapping of index n to the indexes $(l\ m)$.

For example, suppose that we select the mapping

$$n' = Ml + m \quad \text{... (5.9)}$$

The leads to an arrangement in which the first row consists of the first M elements of $x(n)$, the second row consists of the next M elements of $x(n)$, and so on, as illustrated in figure 5.2. On the other hand, the mapping

$$n = l + mL \qquad \dots (5.10)$$

stores the first L elements of $x(n)$ in the firs column, the first L elements in the second column, and so on, as illustrated in figure 5.2.

A similar arrangement can be used to store the computed DFT values. In particular, the mapping is from the index k to a pair of indices (p, q), where $0 \le p \le L - 1$ and $0 \le q \le M - 1$. If we select the mapping

$$k = Mp + q$$

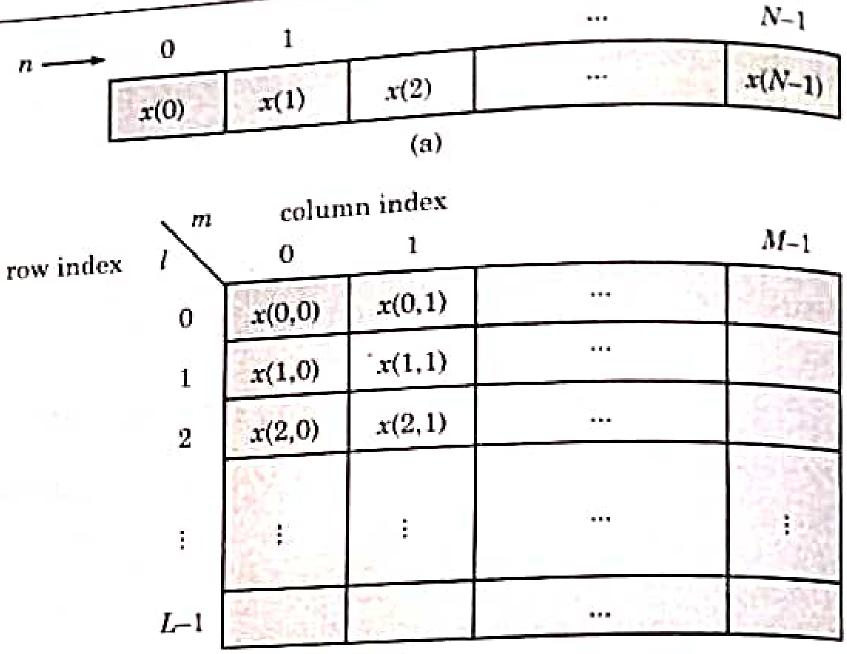


FIGURE 5.1 *Two dimensional data array for storing the sequence $x(n)$, $0 \le n \le N - 1$.*

The DFT is stored on a row-wise basis, where the first row contains the first M elements of the DFT $X(k)$, the second row contains the next set of M elements and so on. On the other hand, the mapping

$$k = qL + p \qquad \dots (5.12)$$

results in a column-wise storage of $X(k)$, where the first L elements are stored in the first column, the second set of L elements are stored in the second column, and so on.

Now, suppose that $x(n)$ is mapped into the rectangular array $x(l, m)$ and $X(k)$ is mapped into a corresponding rectangular array $X(p, q)$. Then, the DFT can be expressed as a double sum over the elements of the rectangular array multiplied by the corresponding phase factors. To be specific, let us adopt a column-wise mapping for $x(n)$ given by (5.10) and the row-wise mapping for the DFT given by (5.11). Then

$$X(p, q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l, m)\, W_N^{(M_p+q)(mL+l)} \qquad \dots (5.13)$$

But $W_N^{(M_p+q)(mL+l)} = W_N^{MLmq}\, W_N^{mq}\, W_N^{Mpl}\, W_N^{lq} \quad \dots (5.14)$

However, $\quad W_N^{Nmp} = 1,\; W_N^{mqL} = W_{N/L}^{mq} = W_M^{mq}$

and $\quad W_N^{Mpl} = W_{N/M}^{pl} = W_L^{pl}$

DO YOU KNOW?

There are many algorithms for efficiently computing the DFT. These are referred to collectively as the Fast Fourier Transform (FET).

With these simplifications, equation (5.13) can be expressed as

$$X(p, q) = \sum_{l=0}^{L-1} \left\{ W_N^{lq} \left[\sum_{n=0}^{M-1} x(l, m)\, W_M^{mq} \right] \right\} W_L^{lp} \qquad \dots (5.15)$$

The expression in (5.15) involves the computation of DFTs of length M and length L. To elaborate, let us subdivide the computation into three steps :

Row-wise $n = Ml + m$

$l \backslash m$	0	1	2		$M-1$
0	$x(0)$	$x(1)$	$x(1)$	...	$x(M-1)$
1	$x(M)$	$x(M+1)$	$x(M+2)$	...	$x(2M-1)$
2	$x(2M)$	$x(2M+1)$	$x(2M+2)$	...	$x(3M-1)$
	⋮	⋮	⋮	...	⋮
$L-1$	$x((L-1)M)$	$x((L-1)M+1)$	$x((L-1)M+2)$	...	$x((L M+1)$

Column-wise $n = l + mL$

$l \backslash m$	0	1	2		$M-1$
0	$x(0)$	$x(L)$	$x(2L)$	...	$x((M-1)L)$
1	$x(1)$	$x(L+1)$	$x(2L+2)$	...	$x((M-1)L+1)$
2	$x(2)$	$x(L+2)$	$x(2L+2)$	...	$x((M-1)L+2)$
	⋮	⋮	⋮	...	⋮
$L-1$	$x(L-1)$	$x(2L-1)$	$x(3L-1)$	...	$x(LM-1)$

FIGURE 5.2 *Two arrangements for the data arrays.*

1. First, we compute the M-point DFTs

$$F(l, q) \equiv \sum_{m=0}^{M-1} x(l, m)\, W_M^{mq}, \quad 0 \le q \le M-1 \qquad \text{... (5.16)}$$

for each of the rows $l = 0, 1, ..., L-1$.

2. Second, we compute a new rectangular array $G(l, q)$ defined as

$$G(l, q) = W_N^{lq}\ F(l, q) \quad 0 \le l \le L-1$$
$$0 \le q \le M-1$$

3. Finally, we compute the L-point DFTs

$$X(p, q) = \sum_{l=0}^{L-1} G(l, q)\, W_L^{lp} \qquad \text{... (5.18)}$$

for each column, $q = 0, 1, ..., M-1$ of the array $G(l, q)$

Therefore, the computational complexity is

Complex multiplications : $N(M + L + 1)$ ... (5.19)

Complex additions : $N(M + L - 2)$

where $N = ML$. Thus the number of multiplications has been reduced form N^2 to $N(M + L + 1)$ and the number of additions has been reduced from $N(N-1)$ to $N(M + L - 2)$.

For example, suppose that $N = 1000$ and we select $L = 2$ and $M = 500$. Then, instead of having to perform 10^6 complex multiplications via direct computation of the DFT, this approach leads to 503,000 complex multiplications. This represents a reduction by approximately a factor of 2. The number of additions is also reduced by about a factor of 2.

When N is a highly composite number, *i.e.*, N can be factored into a product of prime numbers of the form

$$N = r_1 r_2 \dots r_v \qquad \dots (5.20)$$

then the decomposition above can be repeated $(v - 1)$ more times. This procedure results in smaller DFTs, which, in turn, leads to a more efficient computational algorithm.

In effect, the first segmentation of the sequence $x(n)$ into a rectangular array of M columns with L elements in each column resulted in DFTs of sizes L and M. Further decomposition of the data in effect involves the segmentation of each row (or column) into smaller rectangular arrays which result in smaller DFTs. This procedure terminates when N is factored into its prime factors.

5.4.1 Summary of Algorithms

To summarize, the algorithm that we introduced involves the following computations.

Algorithm 1 :

1. Store the signal column-wise.
2. Compute the M-point DFT of each row.
3. Multiply the resulting array by the phase factors W_N^{lq}.
4. Compute the L-point DFT of each column.
5. Read the resulting array row-wise.

An additional algorithm with a similar computational structure can be obtained if the input signal is stored row-wise and the resulting transformation is column-wise. In this case, we select as

$$n = Ml + m \qquad \dots (5.21)$$
$$k = qL + p$$

This choice of indices leads to the formula for the DFT in the form

$$X(p, q) = \sum_{m=0}^{M-1} \sum_{l=0}^{L-1} x(l, m)\, W_N^{pm}\, W_L^{pl} W_M^{qm} \qquad \dots (5.22)$$

or $$X(p, q) = \sum_{m=0}^{M-1} W_M^{-mq} \left[\sum_{l=0}^{L-1} x(l, m)\, W_L^{lp} \right] W_N^{mp}$$

Thus we obtain a second algorithm.

Algorithm 2 :

1. Store the signal row-wise.
2. Compute the L-point DFT at each column.
3. Multiply the resulting array by the factors W_N^{pm}.
4. Compute the M-point DFT of each row.
5. Read the resulting array column-wise.

> **DO YOU KNOW?**
>
> The decimation-in-time and decimation-in-frequency FFT algorithms produce a DFT with approximately $2\,N\log_2 N$ real multiplies, whereas direct computation by means of the DFT sum requires about $4N(N - 1)$ real multiplies. For large N, this gives tremendous computational svaings in performing the transform by an FFT algorithm over a direct evaluation of the DFT.

The two algorithms given above have the same complexity. However, they differ in the arrangement of the computations. In the following sections, we exploit the divide-and-conquer approach to derive fast algorithms when the size of the DFT is restricted to be a power of 2 or a power of 4.

5.5 RADIX-2 FFT ALGORITHMS

(U.P. Tech, Tutorial Question Bank)

1. Background Concept

In previous article, we discussed two algorithms for efficient computation of the DFT based on the divide-and-conquer approach. However, such an approach is applicable when the number N of data points is not a prime. In particular, the approach is very efficient when N is highly composite, *i.e.*, when N can be factored as $N = r_1 r_2 r_3 \dots r_\upsilon$ where the $\{r_j\}$ are prime.

There is an important case in which $r_1 = r_2 = \dots = r_\upsilon \equiv r$, so that $N = r^\upsilon$. In such a case, the DFTs are of size r, so that the computation of the N-point DFT has a regular pattern. **The number r is called the radix of the FFT algorithm.**

2. Basic Definition

Now, let us discuss radix-2 algorithms. These algorithms are the most widely used FFT algorithms.

While calculating DFT, we have discussed that we always calculate N-point DFT. The number N can be factored as,

$$N = r_1, r_2, r_3 \dots\dots r_\upsilon \qquad \dots(5.23)$$

Here, every r is a prime,

Now if $\quad r_1 = r_2 = r_3 = \dots\dots = r_\upsilon = r$

then, we can write,

$$N = r^\upsilon \qquad \dots(5.24)$$

Here, r is called as radix (base) of FFT algorithm and υ indicates number of stages in FFT algorithm.

Now, radix means base and if its value is '2' then it is called as **radix-2 FFT algorithm.** Thus, when $r = 2$, equation (5.24) becomes,

$$N = 2^\upsilon \qquad \dots(5.25)$$

Thus, if we are computing 8 point DFT then $N = 8$

so $\quad 8 = 2^\upsilon$

or $\quad \upsilon = 3 \qquad \dots(5.26)$

Therefore, for 8 point DFT, there are three stages of FFT algorithm.

3. Types

While computing FFT, divide number of input samples by 2, till we reach minimum two samples. Based on this division, there are two algorithms as under :

(*i*) Radix-2 Decimation in time (DIT) algorithm,

(*ii*) Radix-2 Decimation in Frequency (DIF) algorithm.

4. Few Important Properties of Twiddle Factor

Before studying these algorithms, let us derive some important properties of twiddle factor W_N.

We know that the twiddle factor W_N is given by,

$$W_N = e^{-\frac{j2\pi}{N}} \qquad \dots(5.27)$$

1. $W_N^k = W_N^{k+N}$

Using equation (5.27), we have,

$$W_N^{k+N} = \left[e^{-\frac{j2\pi}{N}}\right]^{k+N} = e^{-\frac{j2\pi k}{N}} . e^{-j2\pi}$$

But, we know that $e^{-j2\pi} = \cos 2\pi - j \sin 2\pi = 1 - j0 = 1$

Therefore, $$W_N^{k+N} = e^{-\frac{j2\pi}{N}k}$$

or $$W_N^{k+N} = \left(e^{-j\frac{2\pi}{N}}\right)^k \quad \text{...(5.28)}$$

In equation (5.28), the bracket term is W_N

Hence, $$W_N^{k+N} = W_N^k \quad \text{...(5.29)}$$

Equation (5.29) indicates that twiddle factor is periodic.

2. $W_N^{k+\frac{N}{2}} = -W_N^k$

Using equation (5.28), we can write,

$$W_N^{k+\frac{N}{2}} = \left[e^{-\frac{j2\pi}{N}}\right]^{\left(k+\frac{N}{2}\right)} = e^{-\frac{j2\pi k}{N}} \cdot e^{-\frac{j2\pi}{N}\cdot\frac{N}{2}}$$

or $$W_N^{k+\frac{N}{2}} = e^{-\frac{j2\pi k}{N}} \cdot e^{-j\pi} \quad \text{...(5.30)}$$

But $$e^{-j\pi} = \cos \pi - j \sin \pi = -1 - 0 = -1$$

Therefore, we have

$$W_N^{k+\frac{N}{2}} = -e^{-j\frac{2\pi k}{N}}$$

or $$W_N^{k+\frac{N}{2}} = -\left(e^{-j\frac{2\pi}{N}}\right)^k \quad \text{...(5.31)}$$

In equation (5.31), the bracket term is, W_N.

Thus, we write $$W_N^{k+\frac{N}{2}} = -W_N^k \quad \text{...(5.32)}$$

Equation (5.32) indicates that twiddle factor is symmetric

3. $W_N^2 = W_{N/2}$

From equation (5.27), we have,

$$W_{N/2} = e^{-\frac{j2\pi}{N/2}} = e^{-\frac{j2\pi}{N}\cdot 2} = \left[e^{-\frac{j2\pi}{N}}\right]^2$$

Therefore, we have $W_{N/2} = W_N^2$

5.6 RADIX-2 DECIMATION IN TIME (DIT) ALGORITHM (DIT FFT)

(U.P. Tech, Sem. Exam., 2004-05)(10 marks)

Here, the word 'decimate' means to break into parts. Therefore, DIT indicates dividing (splitting) the sequence in time domain. The different stages of decimation are as under:

(i) First stage of decimation

Let $x(n)$ be the given input sequence containing N samples. Now, for decimation in time, we shall divide $x(n)$ into even and odd sequences *i.e.*,

$$x(n) = f_1(m) + f_2(m) \qquad ...(5.33)$$

Here, $f_1(m)$ is even sequence and $f_2(m)$ is odd sequence

Also, $$f_1(m) = x(2m), \quad m = 0, 1, ... \frac{N}{2} - 1 \qquad ...(5.34)$$

and $$f_2(m) = x(2m+1), \quad m = 0, 1, ..., \frac{N}{2} - 1 \qquad ...(5.35)$$

Input sequence $x(n)$ has N samples. Therefore, after decimation, $f_1(m)$ and $f_2(m)$ will consist of $\frac{N}{2}$ samples.

Now, according to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad ...(5.36)$$

Since, we have divided $x(n)$ into two parts, therefore, we can write separate summation for even and odd sequences as under :

$$X(k) = \sum_{n \text{ even}} x(n)\, W_N^{kn} + \sum_{n \text{ odd}} x(n)\, W_N^{kn} \qquad ...(5.37)$$

In the above expression given by equation 5.37 the first summation represents even sequence. So, we shall substitute $n = 2m$ in first summation. The second summation represents odd sequence, so, we shall substitute $n = (2m + 1)$ in second summation. Since, even and odd sequences contain $\frac{N}{2}$ samples each, the limits of summation will be from $m = 0$ to $\frac{N}{2} - 1$.

Therefore, we have

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m)\, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1)\, W_N^{k(2m+1)} \qquad ...(5.38)$$

But, $x(2m)$ is an even sequence, so it is $f_1(m)$ and $x(2m + 1)$ is an odd sequence, so it is $f_2(m)$.

Hence, $$X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\, W_N^{2km} \cdot W_N^{k}$$

or $$X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\left(W_N^2\right)^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\left(W_N^2\right)^{km} \qquad ...(5.39)$$

Now, we have $$W_N^2 = W_{N/2}$$

Therefore, $$X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\, W_{N/2}^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\, W_{N/2}^{km} \qquad ...(5.40)$$

Comparing each summation with the definition of DFT, we have

$$X(k) = F_1(k) + W_N^k\ F_2(k), \quad k = 0, 1, ..., N-1 \qquad ...(5.41)$$

Let us consider an example of 8 point DFT. This means that $N = 8$

Here, $F_1(k)$ is $\frac{N}{2}$ point DFT of $f_1(m)$ and $F_2(k)$ is $\frac{N}{2}$ point DFT of $f_2(m)$. That means $F_1(k)$ and $F_2(k)$ are 4-point DFTs.

Equation (5.41) indicates that $F_2(k)$ is multiplied by W_N^k and it is added with $F_1(k)$, to obtain (4 + 4) *i.e.* 8–point DFT. Graphically, equation (5.41) can be represented as shown in figure 5.3. Remember that in equation (5.41), k varies from 0 to $N-1$ (*i.e.* 0 to 7 for 8 point DFT).

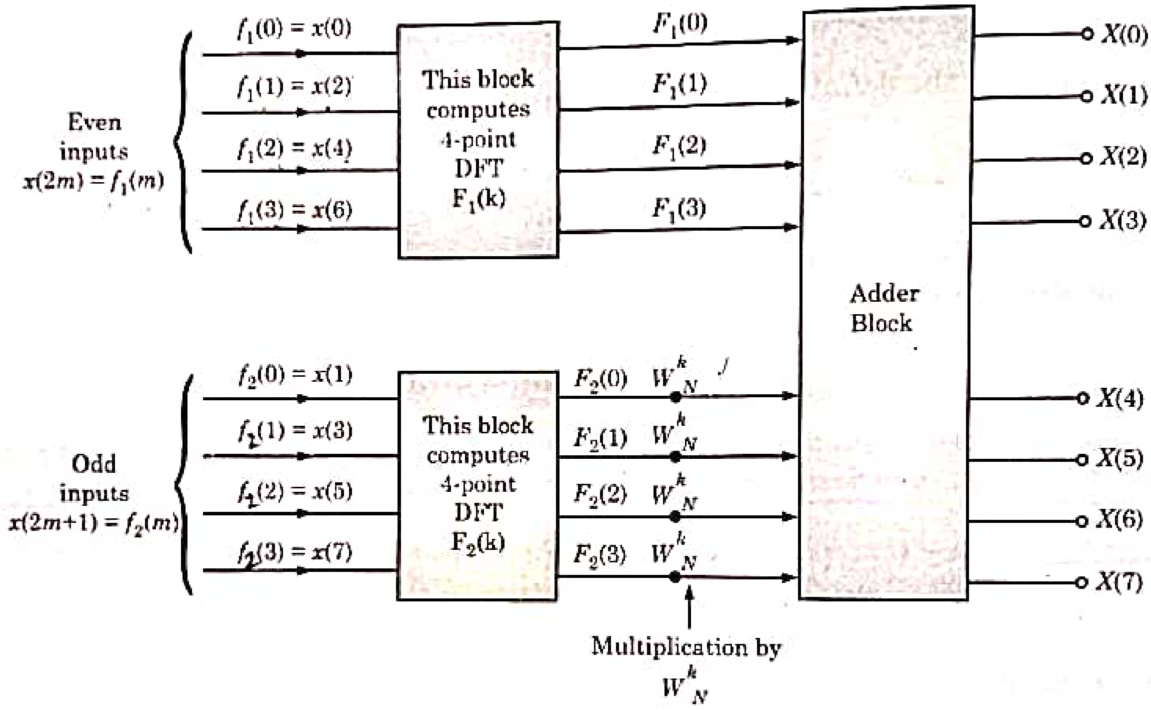


FIGURE 5.3 *Graphical representation of $X(k) = F_1(k) + W_N^k\ F_2(k)$*

Now, $F_1(k)$ and $F_2(k)$ are 4-point $\left(\frac{N}{2}\right)$ DFTs. They are periodic with period $\frac{N}{2}$.

Using periodicity property of DFT, we have.

$$F_1\left(k + \frac{N}{2}\right) = F_1(k) \qquad ...(5.42)$$

and

$$F_2\left(k + \frac{N}{2}\right) = F_2(k) \qquad ...(5.43)$$

Replacing k by $k + \frac{N}{2}$ in equation (5.41), we have

$$X\left(k + \frac{N}{2}\right) = F_1\left(k + \frac{N}{2}\right) + W_N^{k + \frac{N}{2}}\ F_2\left(k + \frac{N}{2}\right) \qquad ....(5.44)$$

Now, we have,

$$W_N^{k + \frac{N}{2}} = -\ W_N^k$$

Therefore, $X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) - W_N^k\ F_2\left(k+\frac{N}{2}\right)$ ...(5.45)

Using equations (5.42) and (5.43), we obtain,

$$X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k\ F_2(k) \quad ...(5.46)$$

Here, $X(k)$ is N point DFT. We can take $k = 0$ to $\frac{N}{2} - 1$, then, by using equations (5.41) and (5.46), we can obtain combined N-point DFT.

Hence, $X(k) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1, \ldots \frac{N}{2} - 1$ ✓ ...(5.47)

and $X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k\ F_2(k), \quad k = 0, 1, \ldots. \frac{N}{2} - 1$ ✓ ...(5.48)

We are considering an example of 8 point DFT ($N = 8$). So, in equations (5.47) and (5.48), k varies from 0 to 3. Now, substituting $k = 0$ to 3 in equations (5.47) and (5.48), we obtain

$$\left.\begin{aligned} X(0) &= F_1(0) + W_N^0\ F_2(0) \\ X(1) &= F_1(1) + W_N^1\ F_2(1) \\ X(2) &= F_1(2) + W_N^2\ F_2(2) \\ X(3) &= F_1(3) + W_N^3\ F_2(3) \end{aligned}\right\} \quad ...(5.49)$$

and

$$\left.\begin{aligned} X(0+4) = X(4) &= F_1(0) - W_N^0\ F_2(0) \\ X(1+4) = X(5) &= F_1(1) - W_N^1\ F_2(1) \\ X(2+4) = X(6) &= F_1(2) - W_N^2\ F_2(2) \\ X(3+4) = X(7) &= F_1(3) - W_N^3\ F_2(3) \end{aligned}\right\} \quad ....(5.50)$$

The graphical representation of first stage of decimation for 8 point DFT is as shown in figure 5.4.

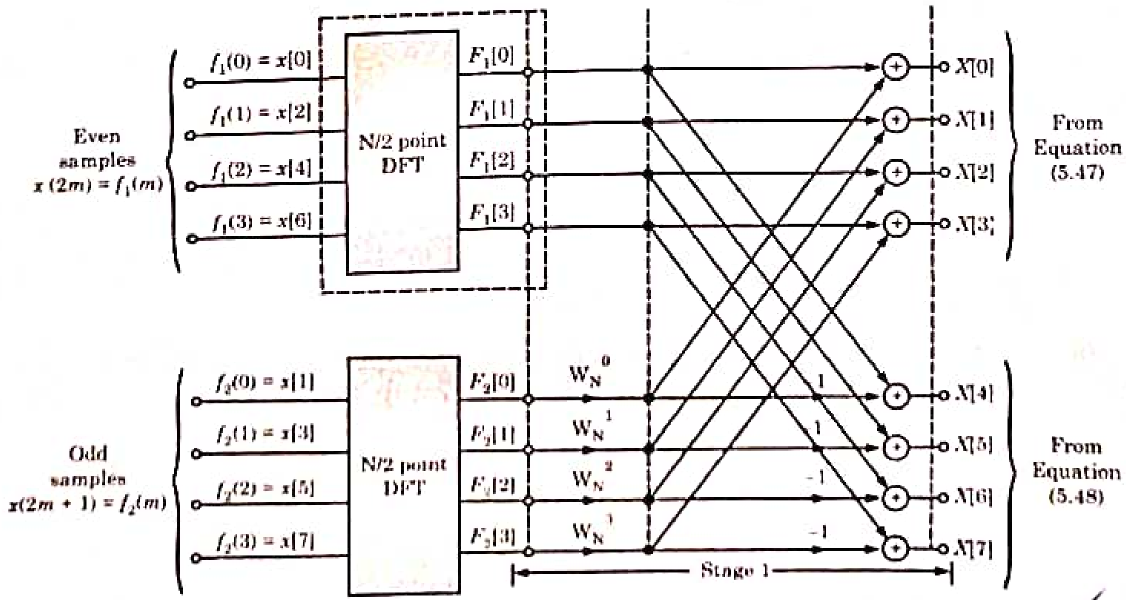


FIGURE 5.4 *First stage of decimation.*

Rakesh

In figure 5.4, input sequences are ...(5.51)

$$f_1(m) = x(2m) = \{x(0), x(2), x(4), x(6)\} \quad ...(5.52)$$

and $$f_2(m) = x(2m+1) = \{x(1), x(3), x(5), x(7)\}.$$

This means that each sequence consists of $\frac{N}{2}$ samples

(ii) Second stage of decimation

In the first stage of decimation, we obtained the sequences of length $\frac{N}{2}$. That means for 8-point DFT ($N = 8$), the length of each sequence is 4 as given by equations (5.51) and (5.52). We discussed that we have to continue this process till we get 2 point sequence.

We can further decimate $f_1(m)$ into even and odd samples

Let $g_{11}(n) = f_1(2m)$, which contains even samples and let $g_{12}(n) = f_1(2m+1)$, which contains odd samples of $f_1(m)$.

Note that here range of 'n' and 'm' is from 0 to $\frac{N}{4} - 1$.

Now, recall equations (5.47) and (5.48). We obtained sequences $X(k)$ and $X\left(k+\frac{N}{2}\right)$ from $F_1(k)$ and $F_2(k)$. The length of each sequence was $\frac{N}{2}$. Here, in the second stage of decimation, we are further dividing the sequences into even and odd parts. So similar to equations (5.47) and (5.48), we can write for $F_1(k)$ as under :

$$F_1(k) = G_{11}(k) + W_{N/2}^k\, G_{12}(k), \quad k = 0, 1, ... \frac{N}{4} - 1 \quad ...(5.53)$$

and $$F_1\left(k+\frac{N}{4}\right) = G_{11}(k) - W_{N/2}^k\, G_{12}(k), \quad k = 0, 1, ... \frac{N}{4} - 1 \quad ...(5.54)$$

Hence, for $N = 8$, we have the range of k from $k = 0$ to $k = 1$. Here, $G_{11}(k)$ is DFT of $g_{11}(n)$ and $G_{12}(k)$ is DFT of $g_{12}(n)$. Substituting the values of k in equation (5.53), we obtain

$$\left.\begin{aligned} F_1(0) &= G_{11}(0) + W_{N/2}^0\, G_{12}(0) \\ F_1(1) &= G_{11}(1) + W_{N/2}^1\, G_{12}(1) \end{aligned}\right\} \quad ...(5.55)$$

Similarly, from equation (5.54), we obtain

$$F_1\left(0+\frac{8}{4}\right) = F_1\, 2) = G_{11}(0) - W_{N/2}^0\, G_{12}(0) \quad ...(5.56)$$

$$F_1\left(1+\frac{8}{4}\right) = F_1(3) = G_{11}(1) - W_{N/2}^1\, G_{12}(1)$$

Here, the values of k are 0 and 1. This means that it is 2-point DFT. Also, equations (5.55) and (5.56) show that we can obtain 4-point DFT by combining two 2-points DFTs. The graphical representation has been shown in figure (5.5).

Now, we have

and $$\left.\begin{aligned} g_{11}(n) &= f_1(2m) = x(4n) = \{x(0), x(4)\} \\ g_{12}(n) &= f_1(2m+1) = x(4n+2) = \{x(2), x(6)\} \end{aligned}\right\} \quad ...(5.57)$$

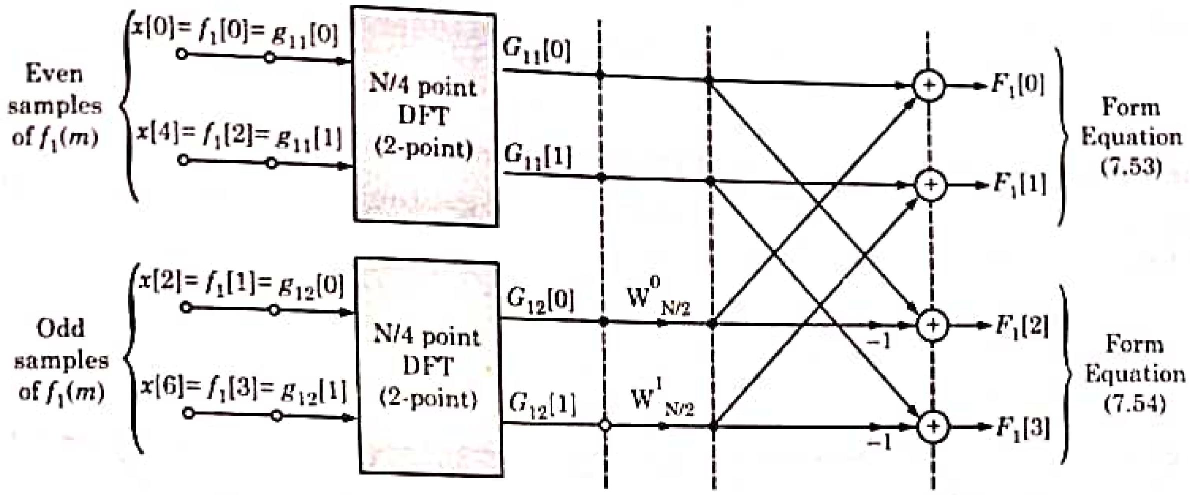


FIGURE 5.5 $F_1(k)$, $\frac{N}{2}$ *point DFT.*

Now, similar to equations (5.53) and (5.54), we can write equations for $F_2(k)$ as under:

$$F_2(k) = G_{21}(k) + W^k_{N/2}\, G_{22}(k), \quad k = 0, 1, \ldots, \frac{N}{4} - 1 \qquad \text{...(5.58)}$$

and $$F_2\left(k + \frac{N}{4}\right) = G_{21}(k) - W^k_{N/2}\, G_{22}(k), \quad k = 0, 1, \ldots, \frac{N}{4} - 1 \qquad \text{...(5.59)}$$

Here, $G_{21}(k)$ is DFT of $g_{21}(n)$ and $G_{22}(k)$ is DFT of $g_{22}(n)$. The values of k are 0 and 1. Substituting these values in equation (5.58), we shall have

$$F_2(0) = G_{21}(0) + W^0_{N/2}\, G_{22}(0) \qquad \text{...(5.60)}$$

$$F_2(1) = G_{21}(1) + W^1_{N/2}\, G_{22}(1)$$

Similarly, form equation (5.59), we have

$$\left.\begin{aligned} F_2\left(0+\frac{8}{4}\right) &= F_2(2) = G_{21}(0) - W^0_{N/2}\, G_{22}(0) \\ F_2\left(1+\frac{8}{4}\right) &= F_2(3) = G_{21}(1) - W^1_{N/2}\, G_{22}(1) \end{aligned}\right\} \qquad \text{...(5.61)}$$

The graphical representation of equations (5.60) and (5.61) has been shown in figure (5.6).

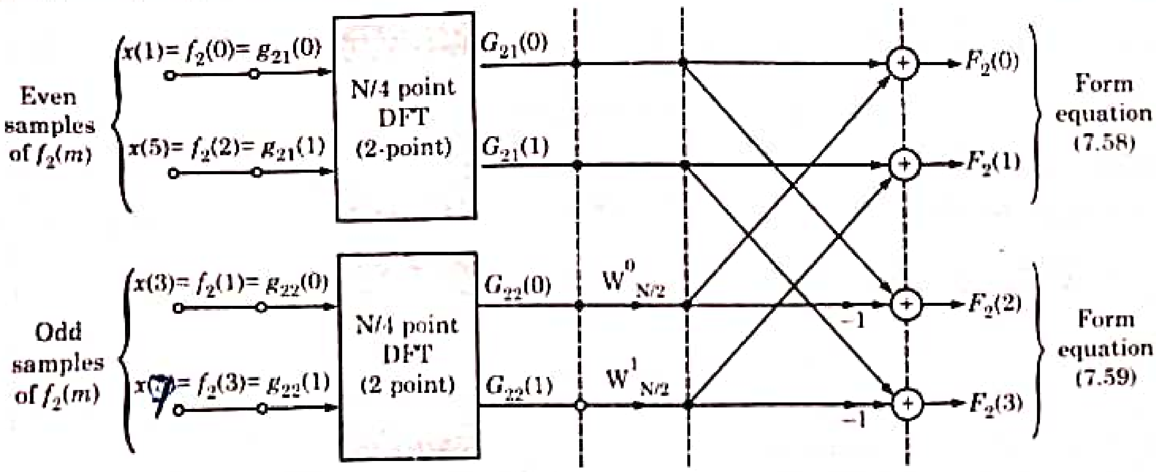


FIGURE 5.6 $F_2(k)$, $\frac{N}{2}$ *point DFT.*

Note that here, we have

$$\left.\begin{aligned} g_{21}(n) &= f_2(2n) = x(4n+1) = \{x(1), x(5)\} \\ g_{22}(n) &= f_2(2n+1) = x(4n+3) = \{x(3), x(7)\} \end{aligned}\right| \qquad ...(5.62)$$

(iii) Combination of first and second stage of decimation

Combining figure 5.5 and figure 5.6 in figure 5.4, we get the combination of first and second stage of decimation. It has been shown in figure 5.7.

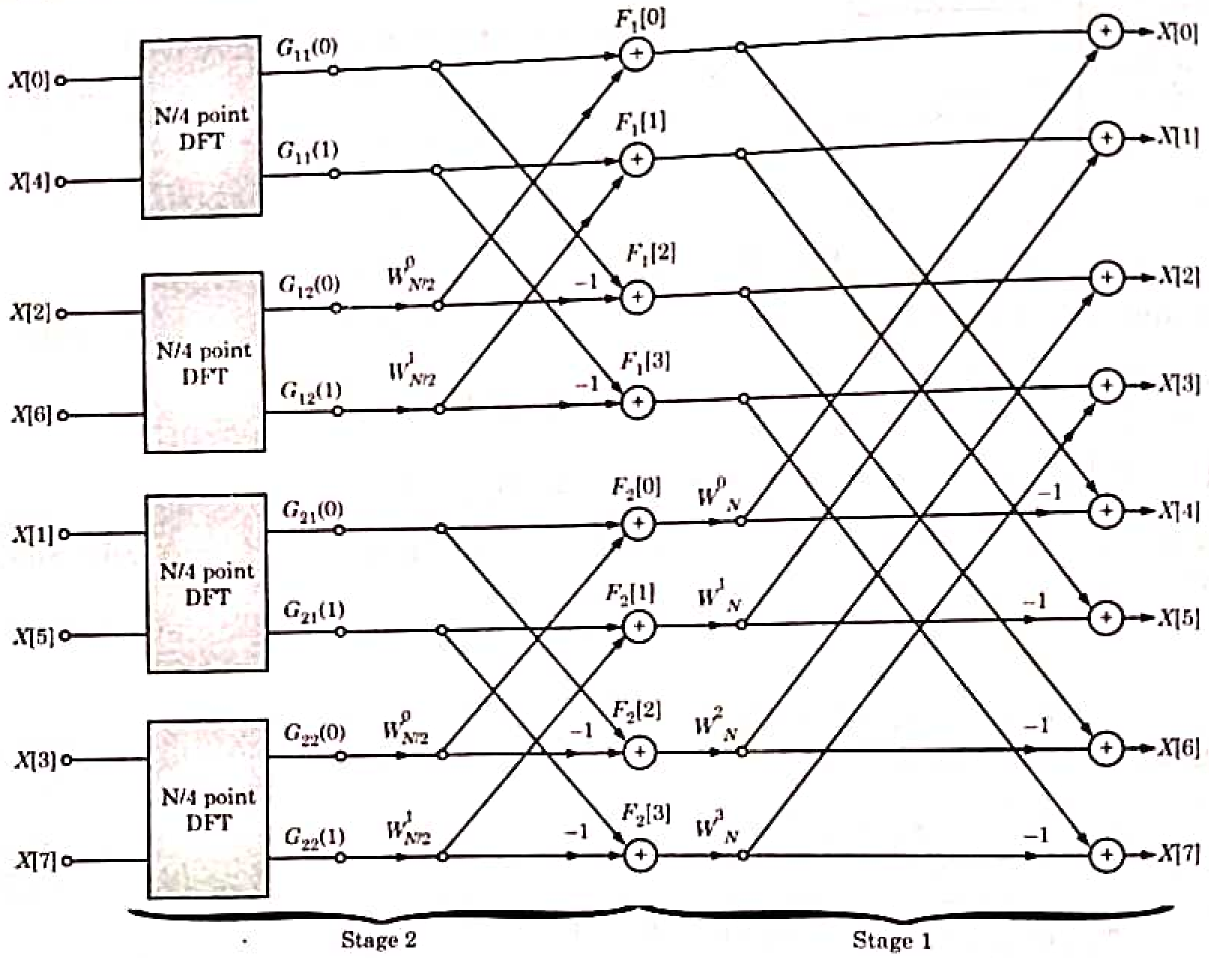


FIGURE 5.7 *Combination of first and second stage of decimation*

At this stage, we have $\frac{N}{4}$ that means 2 point sequences. So, further decimation is not possible. As shown in figure 5.7, we have to compute 2-point DFT.

(iv) Computation of 2-point DFT

According to the basic definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$k = 0, 1, ... N-1 \qquad ...(5.63)$$

Let us use equation (5.63) to compute 2-point DFT. From figure 5.8, let us consider the first block of 2-point DFT. It has been separately drawn as shown in figure 5.8.

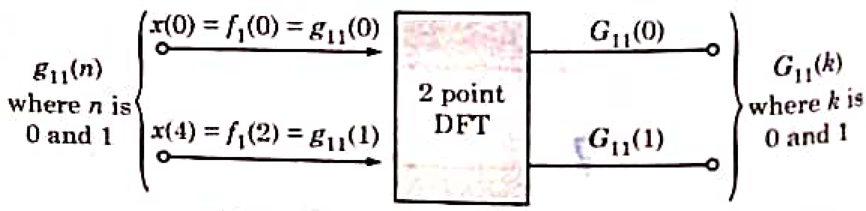


FIGURE 5.8 *Block of 2-point DFT.*

Here, input sequences are $g_{11}(0)$ and $g_{11}(1)$. We denote it by $g_{11}(n)$; where n varies from 0 to 1. Now, the output sequences are $G_{11}(0)$ and $G_{11}(1)$. We can denote it by $G_{11}(k)$, where k varies from 0 to 1. Here, $G_{11}(k)$ is DFT of $g_{11}(n)$.

Thus, for $G_{11}(k)$, we can write equation (5.63) as under:

$$G_{11}(k) = \sum_{n=0}^{1} g_{11}(n)\, W_2^{kn}, \quad k = 0, 1 \qquad \text{...(5.64)}$$

Note that this is 2 point DFT, so we have to substitute $N = 2$

Now, substituting values of k in equation (5.64), we obtain as under:

For $k = 0$, we have

$$G_{11}(0) = \sum_{n=0}^{1} g_{11}(n)\, W_2^0$$

But $\quad W_2^0 = 1$

Therefore, $\quad G_{11}(0) = \sum_{n=0}^{1} g_{11}(n)$

Expanding the summation, we shall have

$$G_{11}(0) = g_{11}(0) + g_{11}(1) \qquad \text{...(5.65)}$$

For, $k = 1$, we have

$$G_{11}(1) = \sum_{n=0}^{1} g_{11}(n)\, W_2^n$$

Expanding the summation, we get

$$G_{11}(1) = g_{11}(0)\, W_2^0 + g_{11}(1)\, W_2^1 \qquad \text{...(5.66)}$$

We have $\quad W_N = e^{-\frac{j2\pi}{N}}$

Therefore, $\quad W_2^1 = \left(e^{-\frac{j2\pi}{2}}\right)^1 = e^{-j\pi} = \cos\pi - j\sin\pi = -1 - j0$

Also, $\quad W_2^1 = -1$

and $\quad W_2^0 = 1$

Substituting these values in equation (5.66), we shall have

$$G_{11}(1) = g_{11}(0) - g_{11}(1) \qquad \text{...(5.67)}$$

Using equations (5.65) and (5.67), we can represent the computation of 2-point DFT as shown to figure 5.9. This structure looks like a butterfly. Hence, it is called as **FFT butterfly structure.**

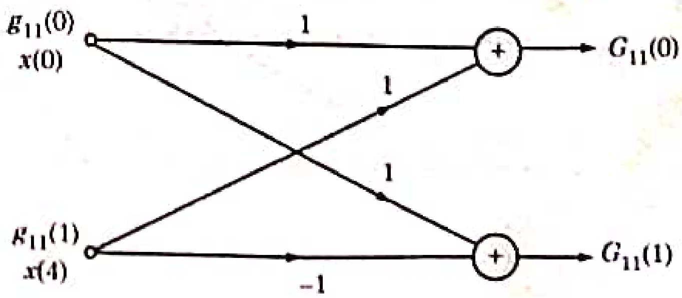


FIGURE 5.9 *FFT butterfly structure.*

Now, we know that $W_2^0 = 1$. Thus, we can modify equations (5.65) and (5.67) as under :

$$G_{11}(0) = g_{11}(0) + W_2^0\, g_{11}(1) \qquad ...(5.68)$$

and $G_{11}(1) = g_{11}(0) - W_2^0\, g_{11}(1)$ ...(5.69)

The modified butterfly structure is shown in figure 5.10.

Similarly, for other 2-point DFTs, we can draw the butterfly structure.

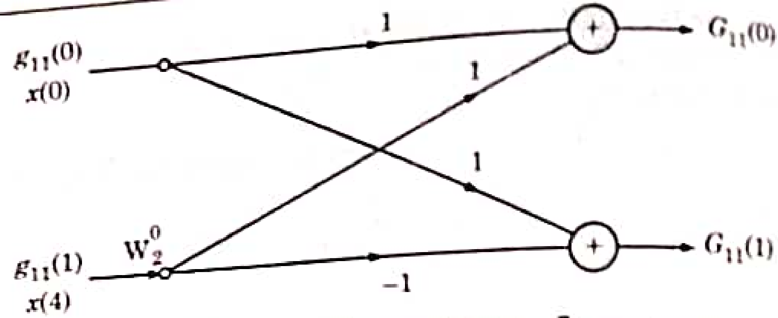


FIGURE 5.10 *Modified butterfly structure.*

(v) Total signal flow-graph for 8-point DIT FFT

The total signal flow graph is obtained by interconnecting all stages of decimation. In this case, it is obtained by interconnecting first and second stage of decimation. But, the starting block is the block used to compute 2-point DFT (butterfly structure). The total signal flow graph has been shown in figure 5.11.

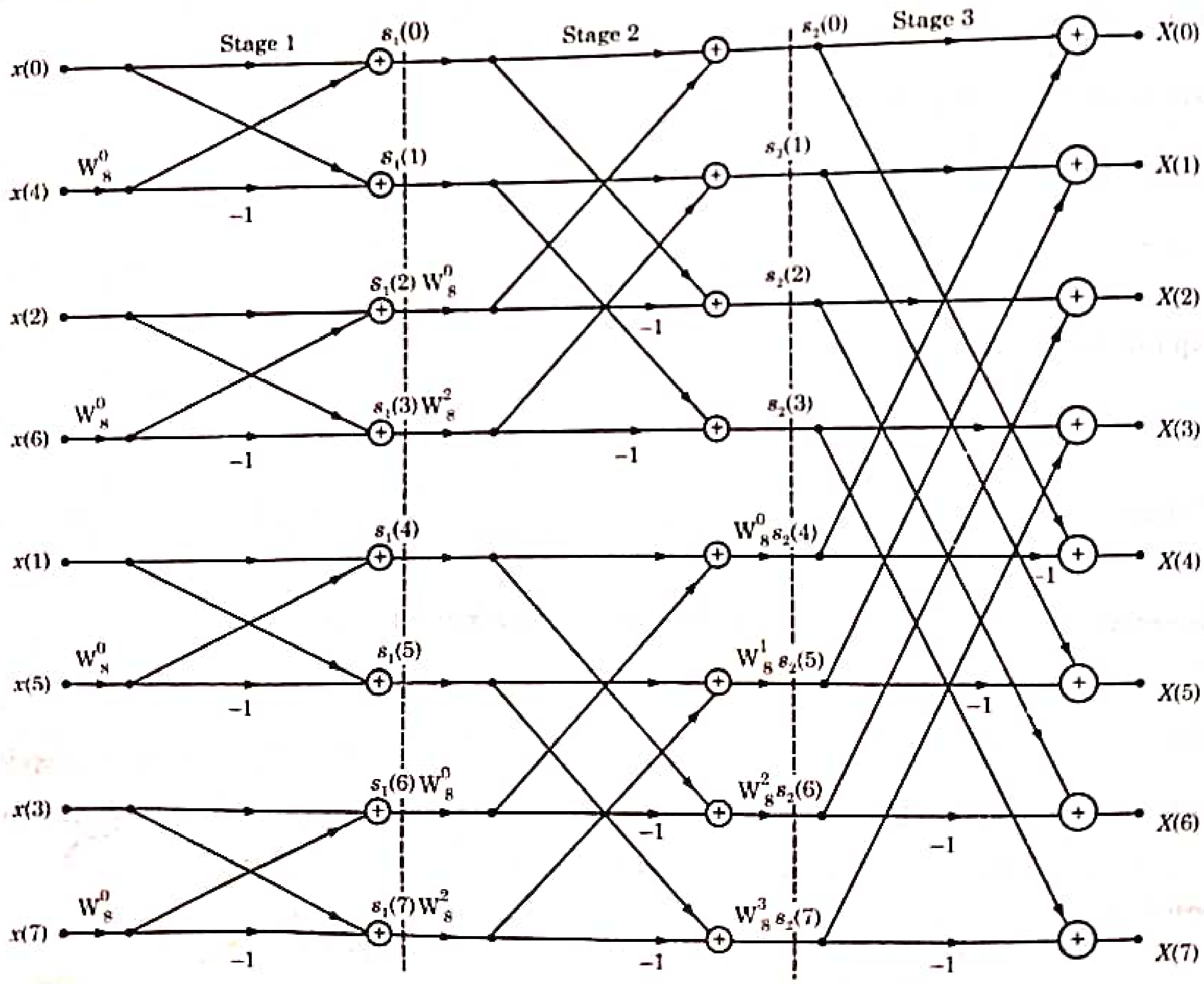


FIGURE 5.11 *Total signal flow graph for 8 point DIT FFT.*

EXAMPLE 5.1 Determine the 8-point DFT of the following sequence.

$$x(n) = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right\}.$$

Use in-place radix-2 deci-mation in time FFT algorithm.

Solution : This flow graph has been shown in figure 5.12.

Here, $s_1(n)$ represents output of stage 1 and $s_2(n)$ represents output of stage 2. The different values of twiddle factor are

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-j\frac{\pi}{2}} = -j$$

$$W_8^3 = -0.707 - j\,0.707$$

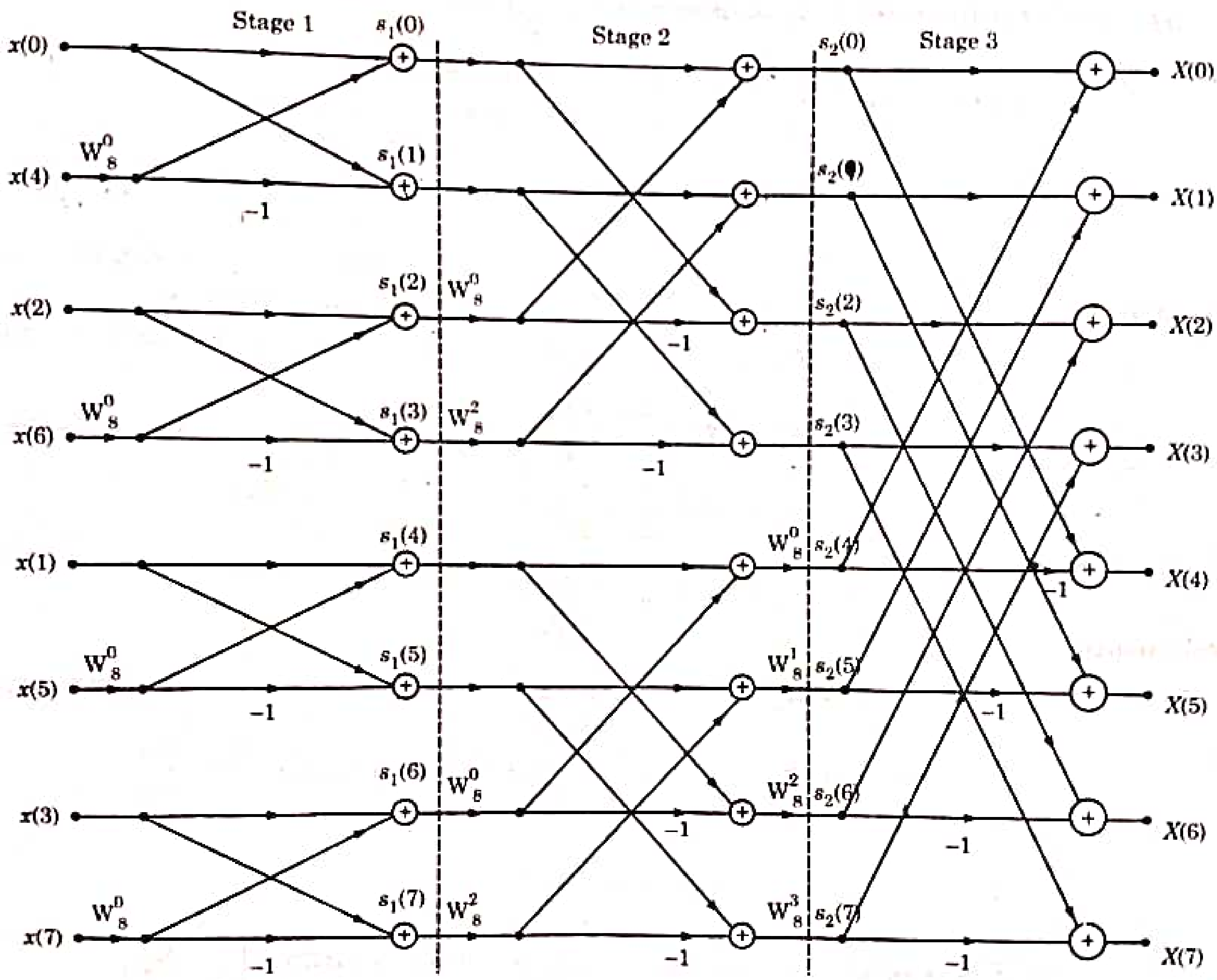


FIGURE 5.12.

Output of stage 1

$$s_1(0) = x(0) + W_8^0\, x(4) = \frac{1}{2} + 1\,(0) = \frac{1}{2}$$

$$s_1(1) = x(0) - W_8^0\, x(4) = \frac{1}{2} - 1\,(0) = \frac{1}{2}$$

$$s_1(2) = x(2) + W_8^0\, x(6) = \frac{1}{2} + 1\,(0) = \frac{1}{2}$$

$$s_1(3) = x(2) - W_8^0\, x(6) = \frac{1}{2} - 1\,(0) = \frac{1}{2}$$

$$s_1(4) = x(1) + W_8^0\ x(5) = \frac{1}{2} + 1.(0) = \frac{1}{2}$$

$$s_1(5) = x(1) - W_8^0\ x(5) = \frac{1}{2} - 1.(0) = \frac{1}{2}$$

$$s_1(6) = x(3) + W_8^0\ x(7) = \frac{1}{2} + 1.(0) = \frac{1}{2}$$

$$s_1(7) = x(3) - W_8^0\ x(7) = \frac{1}{2} - 1.(0) = \frac{1}{2}$$

Output of stage 2

$$s_2(0) = s_1(0) + W_8^0\ s_1(2) = \frac{1}{2} + 1.\left(\frac{1}{2}\right) = 1$$

$$s_2(1) = s_1(1) + W_8^2\ s_1(3) = \frac{1}{2} - j\frac{1}{2}$$

$$s_2(2) = s_1(0) - W_8^0\ s_1(2) = \frac{1}{2} - \frac{1}{2} = 0$$

$$s_2(3) = s_1(1) - W_8^2\ s_1(3) = \frac{1}{2} + j\frac{1}{2}$$

$$s_2(4) = s_1(4) + W_8^0\ s_1(6) = \frac{1}{2} + \frac{1}{2} = 1$$

$$s_2(5) = s_1(5) + W_8^2\ s_1(7) = \frac{1}{2} - j\frac{1}{2}$$

$$s_2(6) = s_1(4) - W_8^0\ s_1(6) = \frac{1}{2} - \frac{1}{2} = 0$$

$$s_2(7) = s_1(5) - W_8^2\ s_1(7) = \frac{1}{2} + j\frac{1}{2}$$

Final output

$$X(0) = s_2(0) + W_8^0\ s_2(4) = 1 + 1 = 2$$

$$X(1) = s_2(1) + W_8^1\ s_2(5) = \left(\frac{1}{2} - j\frac{1}{2}\right) + (0.707 - j\,0.707)\left(\frac{1}{2} - j\frac{1}{2}\right)$$

$$X(1) = 0.5 - j\,1.207$$

$$X(2) = s_2(2) + W_8^2\ s_2(6) = 0 + (-j)(0) = 0$$

$$X(3) = s_2(3) + W_8^3\ s_2(7) = \left(\frac{1}{2} + j\frac{1}{2}\right) + (-0.707 - j\,0.707)\left(\frac{1}{2} + j\frac{1}{2}\right)$$

$$X(3) = \left(\frac{1}{2} + j\frac{1}{2}\right) + (0 - j\,0.707) = 0.5 - j\,0.207$$

$$X(4) = s_2(0) - W_8^0\ s_2(4) = 1 - 1.1 = 0$$

$$X(5) = s_2(1) - W_8^1\ s_2(5)$$

$$X(5) = \left(\frac{1}{2} - j\frac{1}{2}\right) - (0.707 - j\,0.707)\left(\frac{1}{2} - j\frac{1}{2}\right)$$

or

$$X(5) = \left(\frac{1}{2} - j\frac{1}{2}\right) - (-0.707\,j)$$

Also, $X(5) = 0.5 + j\,0.207$

$$X(6) = s_2(2) - W_8^2\, s_2(6) = 0 + j\,.(0) = 0$$

$$X(7) = s_2(3) - W_8^3\; s_2(7) = \left(\frac{1}{2} + j\frac{1}{2}\right) - (-0.707 - j\,0.707)\left(\frac{1}{2} + j\frac{1}{2}\right)$$

$$X(7) = \left(\frac{1}{2} + j\frac{1}{2}\right) + 0.707\,j = 0.5 + j\,1.21$$

Thus, we have

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

or $X(k) = \{2, 0.5 - j\,1.207, 0, 0.5 - j\,0.207, 0, 0.5 + j\,0.207, 0, 0.5 + j\,1.21\}$ **Ans.**

EXAMPLE 5.2 Let $x(n)$ be a finite duration sequence of length 8 such that

$x(n) = (-1, 0, 2, 0 -4, 0, 2, 0)$

(i) Find $X(k)$ using DIT FFT flow graph.

(ii) Using the result in (i) and not otherwise find DFT of sequence

$x_1(n) = \{-1, 2 -4, +2\}$. Justify your answer.

(iii) Using result in (ii) find DFT of sequence

$x_2(n) = (-4, +2, -1, +2)$ *(U.P. Tech, Sem. Exam., 2002-03)(10 marks)*

Solution. (i) This flow graph is as shown in figure 5.13 (a).

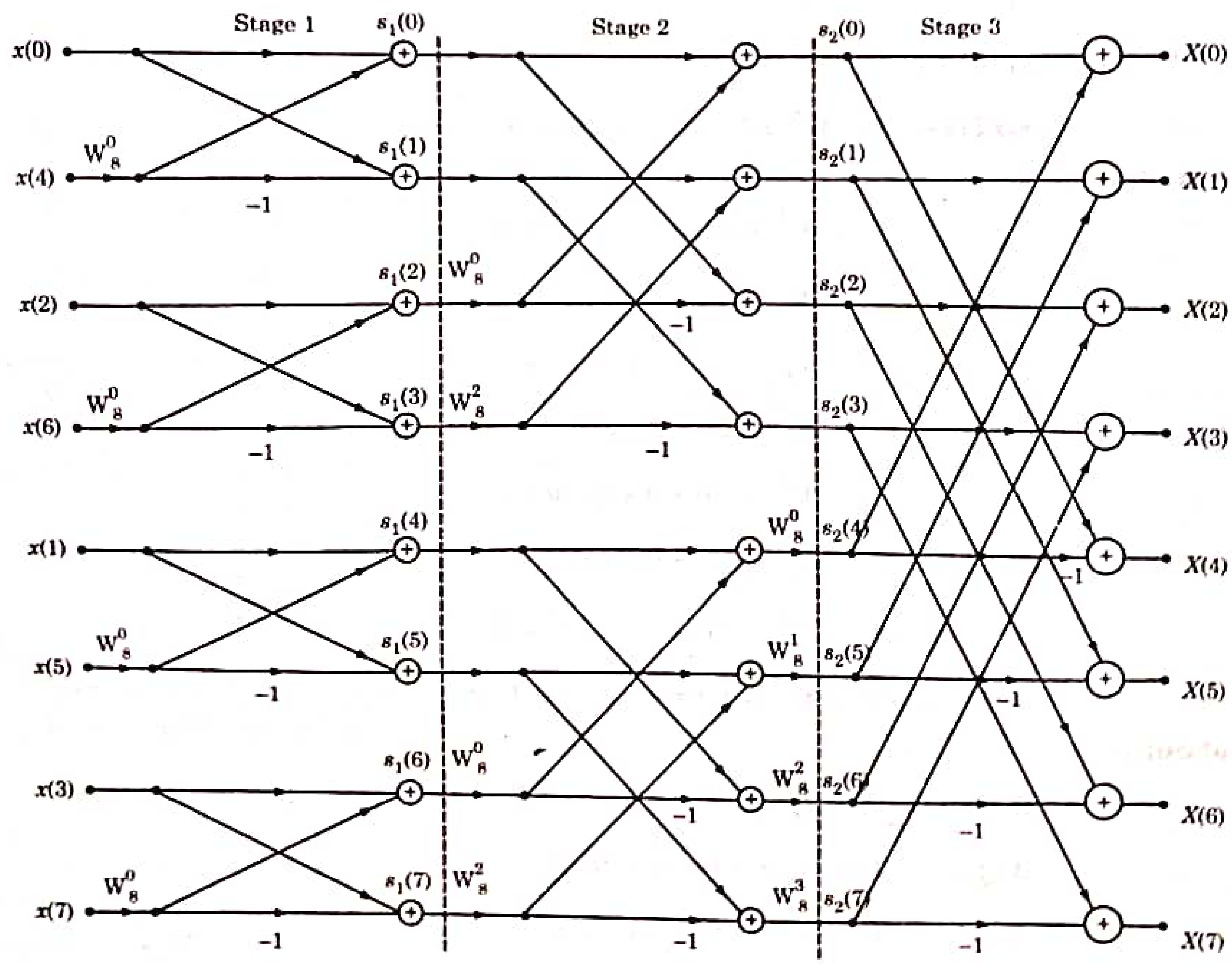


FIGURE 5.13 (a)

Here, $s_1(n)$ represents output of stage 1 and $s_2(n)$ represents output of stage 2. The different values of twiddle factor are as under :

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-j\frac{\pi}{2}} = -j$$

$$W_8^3 = e^{-\frac{j3\pi}{4}} = -0.707 - j\,0.707$$

Output of stage 1

$$s_1(0) = x(0) + W_8^0\, x(4) = -1 + 1\,.\,(-4) = -5$$

$$s_1(1) = x(0) - W_8^0\, x(4) = -1 - 1\,(-4) = 3$$

$$s_1(2) = x(2) + W_8^0\, x(6) = 2 + 1\,.\,(2) = 4$$

$$s_1(3) = x(2) - W_8^0\, x(6) = +2 - 1\,.\,(2) = 0$$

$$s_1(4) = x(1) + W_8^0\, x(5) = 0 + 1\,.\,(0) = 0$$

$$s_1(5) = x(1) - W_8^0\, x(5) = 0 - 1\,(0) = 0$$

$$s_1(6) = x(3) + W_8^0\, x(7) = 0$$

$$s_1(7) = x(3) - W_8^0\, x(7) = 0 - 1\,.\,(0) = 0$$

Output of stage 2

$$s_2(0) = s_1(0) + W_8^0\, s_1(2) = -5 + 1\,.\,(4) = -1$$

$$s_2(1) = s_1(1) + W_8^2\, s_1(3) = 3 - j\,(0) = 3$$

$$s_2(2) = s_1(0) - W_8^0\, s_1(2) = -5 - 1\,.\,(4) = -9$$

$$s_2(3) = s_1(1) - W_8^2\, s_1(3) = 3 + j\,(0) = 3$$

$$s_2(4) = s_1(4) + W_8^0\, s_1(6) = 0 + 1\,.\,(0) = 0$$

$$s_2(5) = s_1(5) + W_8^2\, s_1(7) = 0 - j\,(0) = 0$$

$$s_2(6) = s_1(4) - W_8^0\, s_1(6) = 0 - 1\,.\,(0) = 0$$

$$s_2(7) = s_1(5) - W_8^2\, s_1(7) = 0 + j\,(0) = 0$$

Final output

$$X(0) = s_2(0) + W_8^0\, s_2(4) = -1 + 1\,.\,(0) = -1$$

$$X(1) = s_2(1) + W_8^1\, s_2(5) = 3 + (0.707 - j\,0.707)\,.\,0 = 3$$

$$X(2) = s_2(2) + W_8^2\, s_2(6) = -9 - j\,(0) = -9$$

$$X(3) = s_2(3) + W_8^3\ s_2(7) = 3 + (-0.707 - j\,0.707)\,.\,0 = 3$$

$$X(4) = s_2(0) - W_8^0\ s_2(4) = -1 - 1\,.\,(0) = -1$$

$$X(5) = s_2(1) - W_8^1\ s_2(5) = 3 - 1\,.\,(0) = 3$$

$$X(6) = s_2(2) - W_8^2\ s_2(6) = -9 + j\,(0) = -9$$

$$X(7) = s_2(3) - W_8^3\ s_2(7) = 3 - (-0.707 - j\,0.707)\,.\,0 = 3$$

Therefore, we have

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

or $\quad X(k) = (-1, 3, -9, 3, -1, 3, -9, 3)$ **Ans.**

(*ii*) Let $x(n) = (a, b, c, d)$ and let its DFT be the sequence $X(k) = (A, B, C, D)$. If we add one zero after each sample in $x(n)$, then we will get the following sequence:

$$x_1(n) = (a, 0, b, 0, c, 0, d, 0)$$

This process is called **upsampling** process. Since, in this sequence, one zero is added after each sample, the entire DFT repeats one time. If we add two zeros after each sample then entire DFT will repeat two times.

Hence, DFT $[x_1(n)] = X_1(k) = (A, B, C, D, A, B, C, D)$

In part (*i*) for the sequence $x(n)$, we have

$$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$$

We have obtained the DFT

$$X(k) = (-1, 3, -9, 3, -1, 3, -9, 3)$$

We observe that first four DFT samples are repeated only once. This is because in $x(n)$, zero is added after each sample.

The given sequence is,

$$x_1(n) = \{-1, 2, -4, 2\}$$

Therefore, its DFT will be

$$X_1(k) = \{-1, 3, -9, 3\}$$ **Ans.**

Justification of answer

We shall prove the property of DFT used in this example. Let $x(n) = (a, b, c, d)$ and $X(k) = (A, B, C, D)$.

Let us consider the following sequence :

$$x_1(n) = (a, 0, b, 0, c, 0, d, 0)$$

According to the definition of DFT, we have

$$X_1(k) = \sum_{n=0}^{7} x_1(n)\,.\,W_8^{kn} \qquad ...(i)$$

Let us divide the sequence $x_1(n)$ into odd part and even part. Let $x_1(2n)$ represent even part and $x_1(2n+1)$ represent odd part.

Therefore $$X_1(k) = \sum_{n=0}^{3} x_1(2n)\ W_8^{2kn} + \sum_{n=0}^{3} x_1(2n+1)\ W_8^{(2n+1)\,.\,k} \qquad ...(ii)$$

We observe that in the first summation, n is replaced by $2n$ and in the second summation n is replaced by $(2n+1)$. But, in the second summation, $x_1(2n+1)$ represents odd samples of sequence $x_1(n)$ and all these samples are zero.

Hence, we have

$$X_1(k) = \sum_{n=0}^{3} x_1(2n)\ W_8^{2kn} \quad ...(iii)$$

Now, we have the property of twiddle factor

$$W_N^{2kn} = W_{N/2}^{kn}$$

or

$$W_8^{2kn} = W_4^{kn}$$

so that, we have $X_1(k) = \sum_{n=0}^{3} x_1(2n) \cdot W_4^{kn}$ ...(iv)

But, $x_1(2n)$ represents even samples of $x_1(n)$. This means that $x_1(2n) = x(n)$.

Therefore, $X_1(k) = \sum_{n=0}^{3} x(n)\ W_4^{kn} = X(k)$

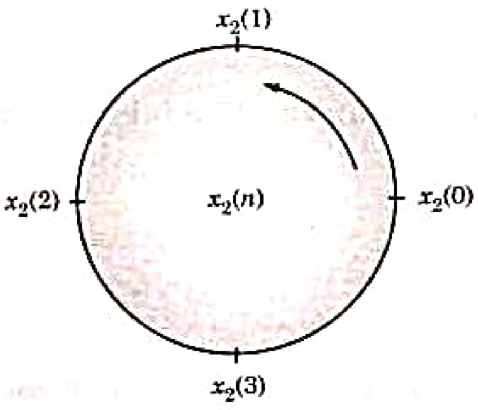
But, $x_1(n)$ is an eight point sequence.

Therefore, $X_1(k) = (A, B, C, D, A, B, C, D)$

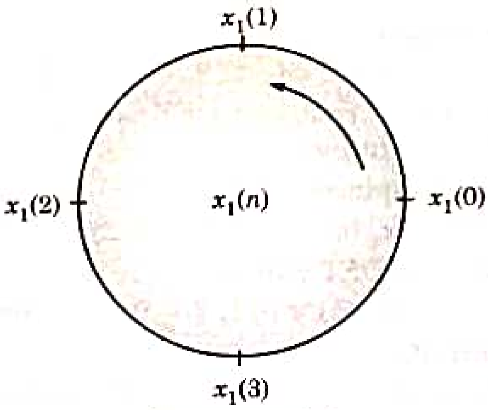
(*iii*) Here, $x_2(n) = (-4, 2, -1, 2)$

We have $x_1(n) = (-1, 2, -4, 2)$

Let us plot the sequences $x_1(n)$ and $x_2(n)$ as shown in figure 5.13 (*b*) and 5.13 (*c*).



Sequence $x_2(n)$
(*b*)



Sequence $x_1(n)$
(*c*)

FIGURE 5.13 *(b) and (c)*

From these diagrams, we can conclude that $x_2(n)$ is obtained by circularly routing $x_1(n)$ by 2 positions in anticlockwise direction. This means that $x_2(n)$ is obtained by deleying $x_1(n)$ by 2 positions.

Therefore, $x_2(n) = x_1((n-2))$

Now, according to circular time shifting property, we have

$$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\ W_N^{kl}$$

Thus, in this case, we can write

$$X_2(k) = X_1(k) \cdot W_4^{2k} = X_1(k) \cdot e^{-\frac{j\pi}{4}k}$$

or

$$X_2(k) = e^{-\frac{j\pi}{2}k} \cdot X_1(k)$$

We have $X_1(k) = (-1, 3, -9, 3)$

Let us calculate sequence $X_2(k)$ for difference values of k as under:

For $k = 0$ we have, $X_2(0) = e^0 . X_1(0) = -1$

For $k = 1$ we have, $X_2(1) = e^{-j\pi} X_1(1) = (\cos\pi - j\sin\pi) . 3 = -3$

For $k = 2$ we have, $X_2(2) = e^{-j\pi 2} X_1(2) = (\cos 2\pi - j\sin 2\pi) . (-9) = -9$

For $k = 3$ we have, $X_2(3) = e^{-j\pi 3} X_1(3) = (\cos 3\pi - j\sin 3\pi) . 3 = -3$

Therefore,

$X_2(k) = \{X_2(0), X_2(1), X_2(2), X_2(3)\}$

or $X_2(k) = [-1, -3, -9, -3]$

EXAMPLE 5.3 Derive DIT FFT flow graph for $N = 4$ and hence find DFT of $x(n) = \{1, 2, 3, 4\}$

Solution: (*i*) First stage of decimation

Let us write the following equations for first stages of decimation:

$$X(k) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots(i)$$

and $$X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k), \quad k = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots(ii)$$

Here, since, $N = 4$, therefore, we have

$$X(k) = F_1(k) + W_4^k F_2(k), \quad k = 0, 1 \qquad \ldots(iii)$$

and $$X(k+2) = F_1(k) - W_4^k F_2(k), \quad k = 0, 1 \qquad \ldots(iv)$$

Substituting values of k in equation (*iii*), we obtain

$$\left.\begin{aligned} X(0) &= F_1(0) + W_4^0 F_2(0) \\ \text{and} \quad X(1) &= F_1(1) + W_4^1 F_2(1) \end{aligned}\right\} \qquad \ldots(v)$$

Similarly, substituting values of k in equation (*iv*), we obtain

$$\left.\begin{aligned} X(2) &= F_1(1) - W_4^0 F_2(0) \\ \text{and} \quad X(3) &= F_1(1) - W_4^1 F_2(1) \end{aligned}\right\} \qquad \ldots(vi)$$

This signal flow graph has been shown in figure 5.14 (*a*).

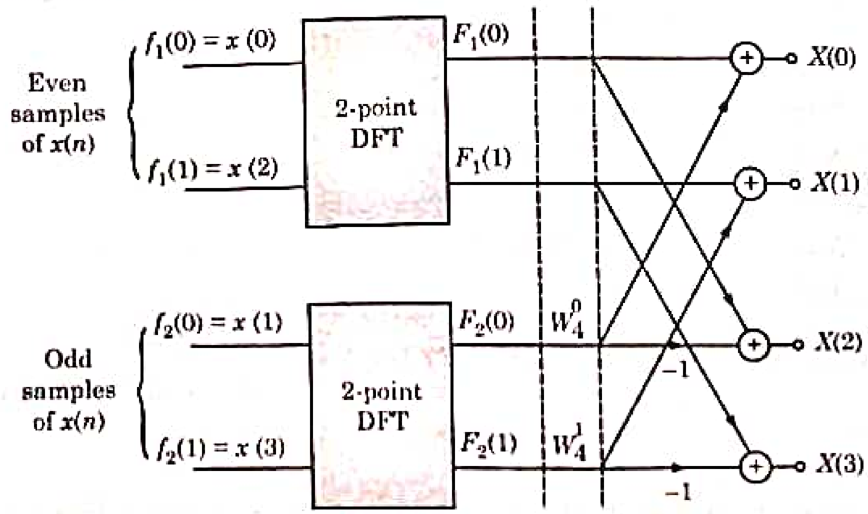


FIGURE 5.14 (*a*)

Now, let us replace each 2-point DFT by butterfly structure as shown in figure 5.14 (*b*).

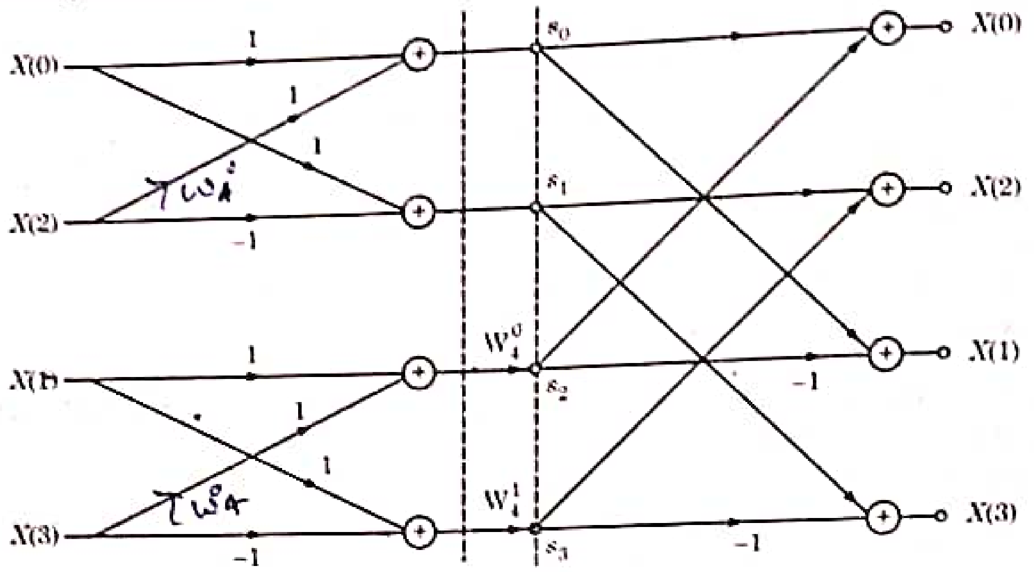


FIGURE 5.14 (*b*)

The given sequence is

$$x(n) = \{1, 2, 3, 4\}$$

The different values of twiddle factor are as under: $x(1) - W_4^0\, x(3)$

$$W_4^0 = 1$$

$$W_4^1 = e^{-\frac{j2\pi}{4}.1} = e^{-\frac{j\pi}{2}}$$

$$= \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j.$$

The output $s(n)$ will be

$$s_0 = x(0) + x(2) = 1 + 3 = 4$$

$$s_1 = x(0) - x(2) = 1 - 3 = -2$$

$$s_2 = [x(1) + x(3)]\ W_4^0 = 2 + 4 = 6$$

$$s_3 = [x(1) - x(3)]\ W_4^1 = (2 - 4).(-j) = 2j$$

The final output will be given by

$$X(0) = s_0 + s_2 = 4 + 6 = 10$$

$$X(1) = s_1 + s_3 = -2 + j2$$

$$X(2) = s_0 - s_2 = 4 - 6 = -2$$

$$X(3) = s_1 - s_3 = -2 - j2$$

Thus, we have

$$X(k) = \{X(0), X(1), X(2), X(3)\}$$

Therefore, $X(k) = \{10, -2 + j2, -2, -2 - j2\}$ **Ans.**

EXAMPLE 5.4 Draw flow diagram of DIT FFT for $N = 16$.

(U.P. Tech, Tutorial Question Bank)

Solution: (*i*) Here, $N = 16$, means that it is 16 point DFT.

(*ii*) Total number of stages = 4

(*iii*) The first stage of decimation using two 8-point DFT is shown in figure 5.15 (*a*).

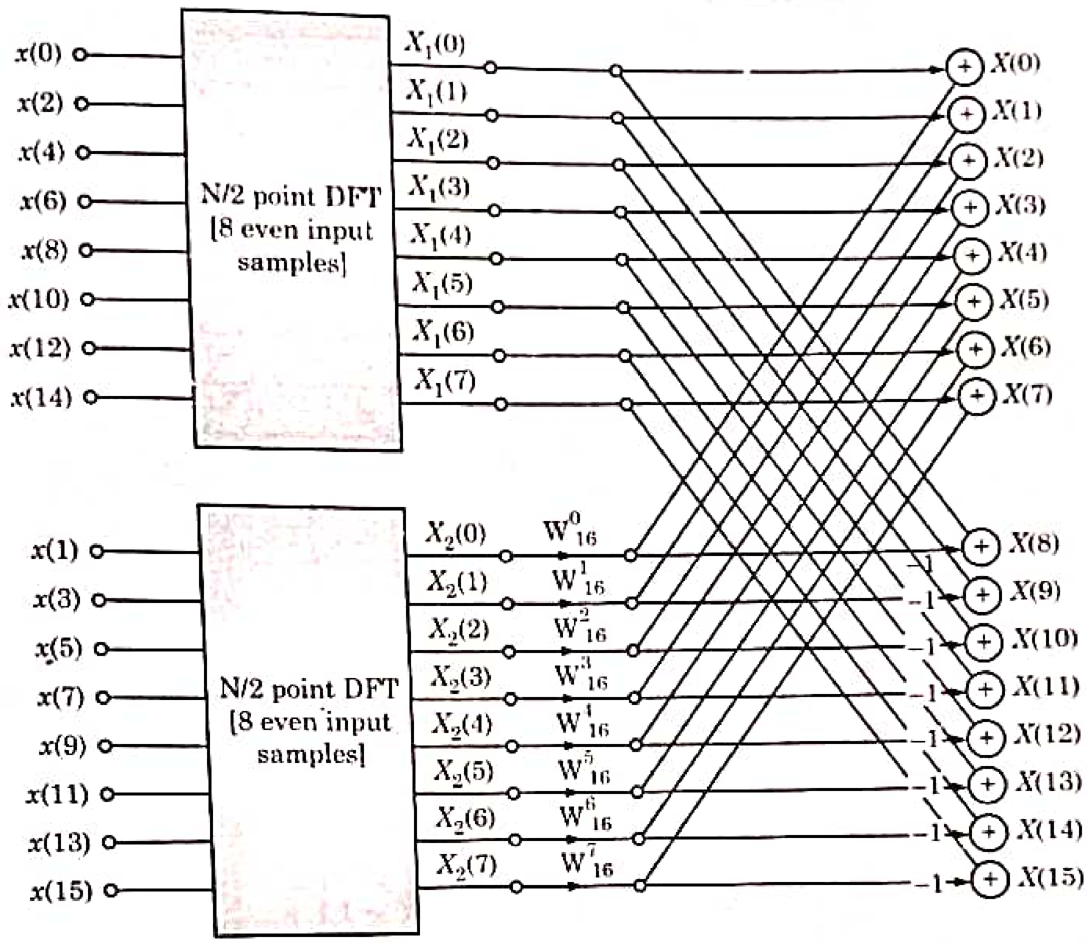


FIGURE 5.15 (*a*).

(4) In the second stage, each 8 point DFT is divided into 2 four point DFTs as shown in figure 5.15 (*b*).

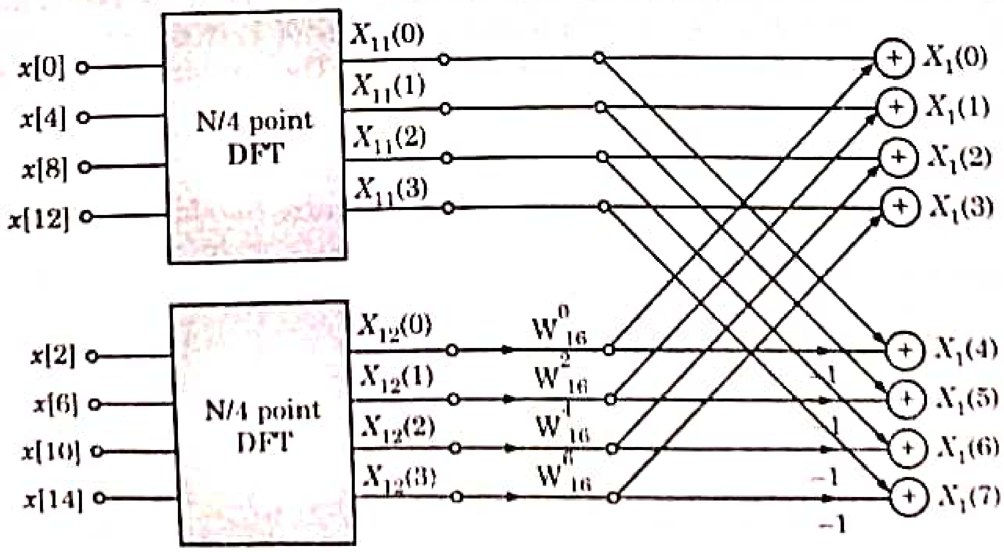


FIGURE 5.15 (*b*) *(Contd...)*

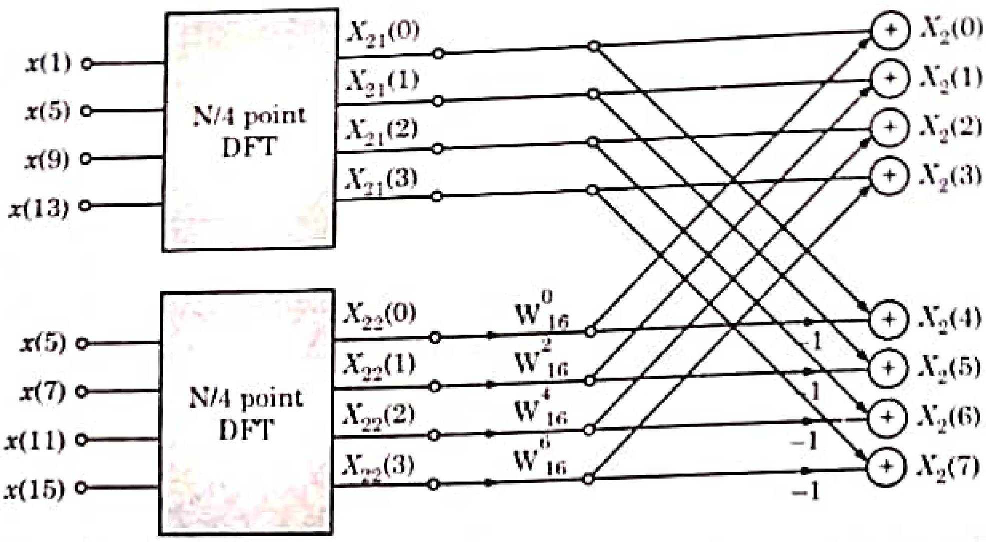


FIGURE 5.15 *(b) (Contd...)*

(5) The total flow graph is shown in figure 5.15 (c).

Comparison of Computational Complexity with Direct Computation: First, let us calculate the computational complexity for direct DFT calculation.

For Direct Computation

According to the definition of DFT, we have,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad k = 0, 1, 2, \ldots\ldots N-1 \qquad \ldots(5.70)$$

Equation (5.70) indicates that we have to take multiplication of $x(n)$ and twiddle factor. Then we have to add all the terms. Since twiddle factor is complex. Therefore, we need to perform complex multiplications and complex additions.

Complex multiplications

As given by equation (5.70), for one value of k, multiplication should be performed for all values of n. The range of n is from 0 to $N-1$. Hence, for one value of k, N complex multiplications are required. Now, the range for k is also from $k = 0$ to $k = N-1$. The total complex multiplications are,

Complex multiplications = $N \times N = N^2$ ...(5.71)

Complex additions

According to equation (5.70), for each value of k, we require to add the product terms of $x(n)$ W_N^{kn}. For example, let us say $N = 4$.

For $k = 0$, we have

$$X(0) = \sum_{n=0}^{3} x(n)\, W_4^{0\times n} = \sum_{n=0}^{3} x(n)\, W_4^0$$

or

$$X(0) = x(0)\, W_4^0 + x(1)\, W_4^0 + x(2)\, W_4^0 + x(3)\, W_4^0 \qquad \ldots(5.72)$$

In equation (5.72), four complex multiplications are required and three complex additions are required. Here, we have considered $N = 4$. Thus, for each value of k, N complex multiplications are required and $N-1$ complex additions are required.

Therefore,

Complex Additions = $N(N-1) = N^2 - N$

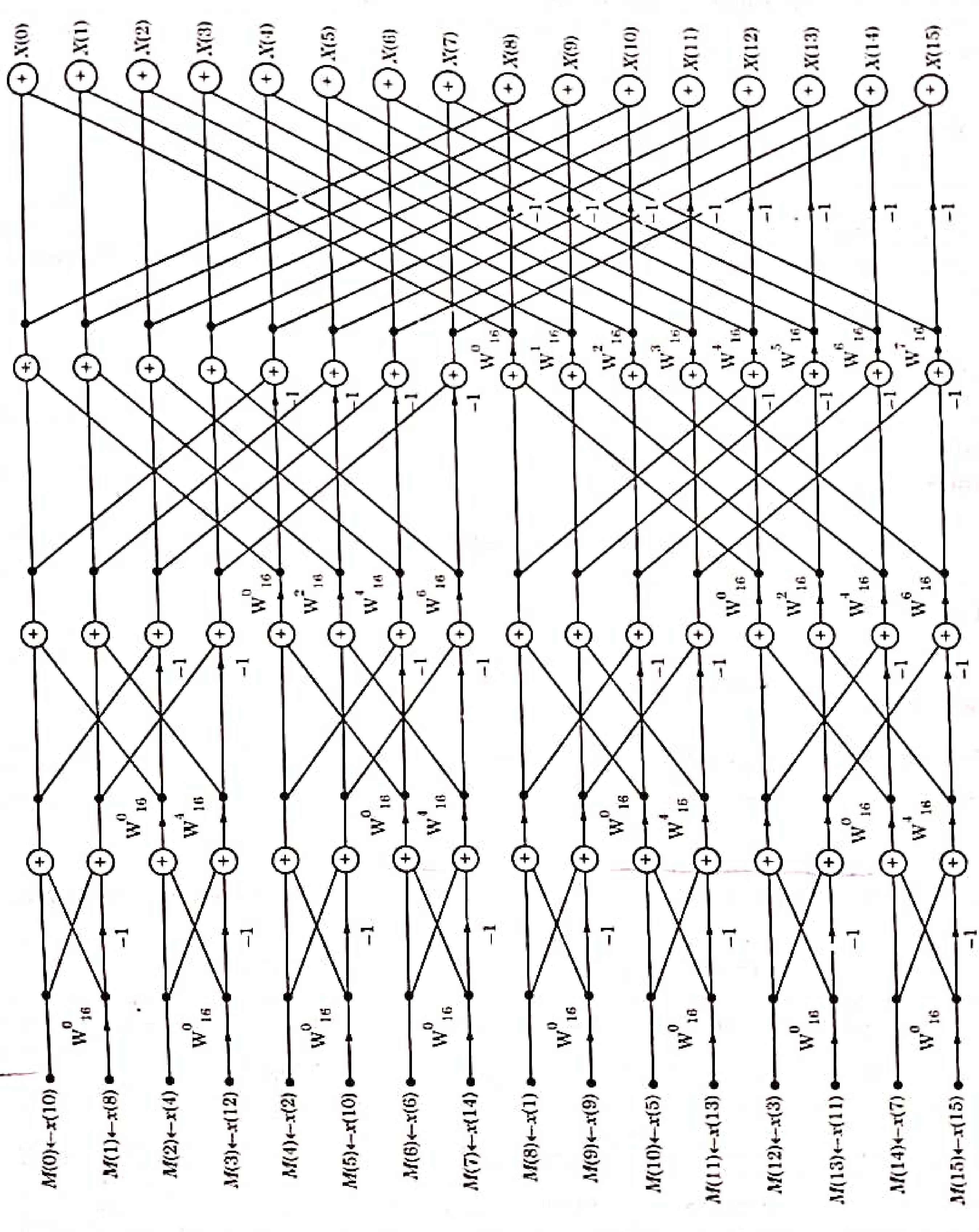


FIGURE 5.15 (c).

Computational Complexity using FFT Algorithm

Firstly, let us calculate the computation complexity required for one butterfly. Let us consider the general structure of butterfly as shown in figure 5.16.

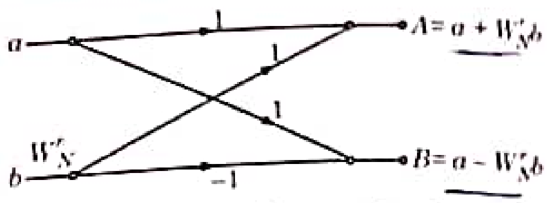


FIGURE 5.16 *General structure of butterfly.*

Here, a and b are inputs and A and B are outputs of butterfly. The outputs are given by,

$$A = a + W_N^r\, b \quad \text{...(5.73)}$$

and $$B = a - W_N^r\, b \quad \text{...(5.74)}$$

(*i*) To calculate any output (A or B), we require to multiply input b by twiddle factor W_n^r. Therefore, **one complex multiplication** is required for one butterfly.

(*ii*) To collect output A, one complex addition is required, while to calculate output B, complex substraction is required as given by equation (5.74). But, the computational complexity of addition and subtraction is same. Therefore, we can say that for one butterfly **two complex additions** are required.

(*iii*) As shown in figure 5.12, for 8 point DFT, 4 butterflies are there at each stage. So, **for N point DFT, at each stage, $\frac{N}{2}$ butterflies are required.**

(*iv*) As shown in figure 5.12, three stages are required to compute 8-point DFT. In general, **for N point DFT, $\log_2 N$ stages are required.**

Complex multiplications

At each stage, there are $\frac{N}{2}$ butterflies. Total number of stages are $\log_3 N$. Also, for each butterfly, one complex multiplication is required.

Therefore,

$$\text{Total complex multiplications} = \frac{N}{2}\log_2 N \quad \text{...(5.75)}$$

Complex additions

Total number of stages are $\log_2 N$. At each stage, $\frac{N}{2}$ butterflies are required. Also, for each butterfly, 2 complex additions are required

Therefore, total complex additions $= 2 \times \frac{N}{2}\log_2 N = N\log_2 N$

Table 5.1 shows comparison of direct DFT computation and computation using FFT algorithms.

TABLE 5.1

S.No.	Number of points N (N^2)	Direct Computation Complex Multiplications ($N^2 - N$)	Complex Additions $\left(\frac{N}{2}\log_2 N\right)$	Using FFT Complex Multiplications ($N\log_2 N$)	Complex Additions
1.	4	16	12	2	4
2.	8	64	56	12	24
3.	16	256	240	32	64
4.	32	1024	992	40	80
5.	64	4096	4032	96	192
6.	128	16,384	16,256	224	448

Table 5.1 shows that by the use of FFT algorithms, the number of complex multiplications are complex additions and reduced. Therefore, there is tremendous improvement in the speed.

5.7 IN-PLACE COMPUTATION TO REDUCE MEMORY SIZE *(Important)*

First, let us discuss the memory requirement of each butterfly. As shown in figure 5.13 (*a*), a butterfly calculates the values of A and B for the inputs a and b. Further, a and b are complex inputs. So, two memory locations are required to store any one of the inputs, a or b. One memory location is required to store real part and other memory location is required to store imaginary part. Now, to store both inputs a and b, $2 + 2 = 4$ memory locations are required.

Now, the outputs are computed as under :

$$A = a + W_N^r\, b \qquad ....(5.76)$$

and

$$B = a - W_N^r\, b \qquad ....5.77)$$

Thus, outputs A and B are calculated by using the values of a and b stored in the memory. Now, A and B are also complex numbers, so $2 + 2 = 4$ memory locations are required to store both the outputs A and B.

Once the computation of A and B is done then, values of 'a' and 'b' are not required. So instead of storing A and B at other memory locations; these values are stored at the same place where a and b were stored. This means that A and B are stored in place of a and b. This is called as **in-place computation**. In-place computation reduces the memory size.

Memory Requirement

We have discussed that four memory locations are needed for every butterfly to store input and output values. Now, there are $\frac{N}{2}$ butterflies per stage. Hence, for each stage, we have

Memory locations required to store inputs and outputs $= 4 \times \frac{N}{2} = 2N$

Also, one value of twiddle factor is required to compute A and B. To store one value of twiddle factor, one memory location is required for each butterfly. Now, there are $\frac{N}{2}$ butterflies at each stage. Thus, for each stage, we have

Memory locations required to store twiddle factor $= \frac{N}{2}$

Therefore, combined memory required per stage is $2N + \frac{N}{2}$. These many number of memory locations are required to store input values, output values and the twiddle factor per stage.

Now, the actual computation of N-point FFT is done stage wise. This means that computation of one stage is done at a time. Hence, these memory locations can be used for other stages also. This will again reduce the memory size.

Therefore, total memory locations $= 2N + \frac{N}{2}$

5.8 RADIX-2 DECIMATION IN FREQUENCY (DIF) FFT ALGORITHM

(U.P. Tech, Sem. Exam., 2006-06)(10 marks)

Decimation in frequency stands for splitting the sequences in terms of frequency. This means that we have to split output sequences into smaller subsequences. This decimation is done as under :

First Stage of Decimation

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(5.78)}$$

Let us divide the summation into two parts as under :

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kN} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{kn} \qquad \text{...(5.79)}$$

Now, let us consider the second summation *i.e.*, $\sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{kn}$

For, $n = n + \frac{N}{2}$, the limits will change as under :

When $\quad n = \frac{N}{2}$, we have

$$\frac{N}{2} = n + \frac{N}{2}$$

Hence, $\quad n = 0$

and when $\quad n = N - 1$, we have

$$N - 1 = n + \frac{N}{2} \quad \text{or} \quad n = N - 1 - \frac{N}{2} = \frac{N}{2} - 1.$$

Therefore, we have

$$\sum_{n=\frac{N}{2}}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)}$$

Substituting this value in equation (5.79), we get

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) . W_N^{k\left(n+\frac{N}{2}\right)}$$

or

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) . W_N^{kn} . W_N^{kn/2}$$

or

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{kn} \qquad \text{...(5.80)}$$

Now, we have $\quad W_N = e^{-\frac{j2\pi}{N}}$

Hence, $\quad W_N^{\frac{kN}{2}} = e^{-\frac{j2\pi}{N} \times \frac{kN}{2}} = e^{-j\pi k} = \left(e^{-j\pi}\right)^k$

or $\quad W_N^{\frac{kN}{2}} = (\cos \pi - j \sin \pi)^k = (-1 - j0)^k$

or
$$W_N^{\frac{kN}{2}} = (-1)^k$$

Substituting this value in equation (5.80), we obtain

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{kn}$$

Taking the summation common, we shall have

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^k\, x\left(n + \frac{N}{2}\right)\right] W_N^{kn} \qquad \text{...(5.81)}$$

Here, we have to split the sequence in terms of frequency. So, we shall split X (k) in terms of even numbered and odd numbered DFT coefficients. Let X (2r) represents even numbered DFT and X(2r + 1) represents odd numbered DFT

Thus, substituting $k = 2r$ in equation (5.81), we shall get even numbered sequence as under:

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{2r}\, x\left(n + \frac{N}{2}\right)\right] W_N^{2rn} \qquad \text{...(5.82)}$$

By substituting $k = 2r + 1$, in equation (5.81), we shall get odd numbered sequence as under:

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{2r+1}\, x\left(n + \frac{N}{2}\right)\right] W_N^{(2r+1)n} \qquad \text{...(5.83)}$$

Here, r is an integer similar to k and it varies from 0 to $\frac{N}{2} - 1$.

Therefore, $(-1)^{2r} = 1$ ...(5.84)

and $(-1)^{2r+1} = (-1)^{2r}(-1)^1 = -1$ ...(5.85)

Substituting these values in equations (5.82) and (5.83), we shall have

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_N^{2rn} \qquad \text{...(5.86)}$$

and
$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^{(2r+1)n} \qquad \text{...(5.87)}$$

Now, let us consider the term W_N^{2rn}

$$W_N^{2rn} = \left(W_N^2\right)^{rn}$$

But, we have $W_N^2 = W_{N/2}$

Therefore, $W_N^{2rn} = (W_{N/2})^{rn} = W_{N/2}^{rn}$ ...(5.88)

Now, we can write

$$W_N^{(2r+1)n} = W_N^{2rn} . W_N^n = W_{N/2}^{rn} . W_N^n \qquad \text{...(5.89)}$$

Substituting these values in equations (5.86) and (5.87), we shall have

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{rn} \qquad \text{...(5.90)}$$

and $$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_{N/2}^{rn} \cdot W_N^n \quad ...(5.91)$$

Now, let $$g(n) = x(n) + x\left(n+\frac{N}{2}\right) \quad ...(5.92)$$

for, $n = 0, 1, 2 \ldots \frac{N}{2}-1$

and $$h(n) = \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^n \quad ...(5.93)$$

Substituting these values in equations (5.90) and (5.91), we obtain

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} g(n)\, W_{N/2}^{rn} \quad ...(5.94)$$

and $$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\, W_{N/2}^{rn} \quad ...(5.95)$$

It may be noted that at this stage, we have decimated the sequence of N point DFT into two $\frac{N}{2}$ point DFTs given by equations (5.94) and (5.95).

Let us consider an example of 8-point DFT *i.e.*, $N = 8$. So, combining equations (5.94) and (5.95) (that means $\frac{N}{2} = 4$), we can obtain N (8 point) point DFT. This is first stage of decimation. Note that equation (5.94) indicates 4 $\left(\frac{N}{2}\right)$ point DFT of $g(n)$ and equation (5.95) indicates 4 $\left(\frac{N}{2}\right)$ point DFT of $h(n)$. For 8-point DFT, equation (5.92) becomes

$$g(n) = x(n) + x(n+4) \quad ...(5.96)$$

Here, we are computing 4 point DFT. So, range of n is n = 0 to n = 3. Substituting these values in equation (5.96), we obtain

For $n = 0$, we have $g(0) = x(0) + x(4)$

For $n = 1$, we have $g(1) = x(1) + x(5)$

For $n = 2$, we have $g(2) = x(2) + x(6)$ ...(5.97)

For $n = 3$, we have $g(3) = x(3) + x(7)$

Similarly, for 8 point DFT, equation (5.93) becomes

$$h(n) = [x(n) - x(n+4)]W_8^n \quad ...(5.98)$$

For $n = 0$, we have $h(0) = [x(0) - x(4)]W_8^0$

For $n = 1$, we have $h(1) = [x(1) - x(5)]W_8^1$

For $n = 2$, we have $h(2) = [x(2) - x(6)]W_8^2$ ...(5.99)

For $n = 3$, we have $h(3) = [x(3) - x(7)]W_8^3$

Using equations (5.97) and (5.99), and equations (5.94) and (5.95), we can draw the flow graph of first stage of decimation as shown in figure 5.17.

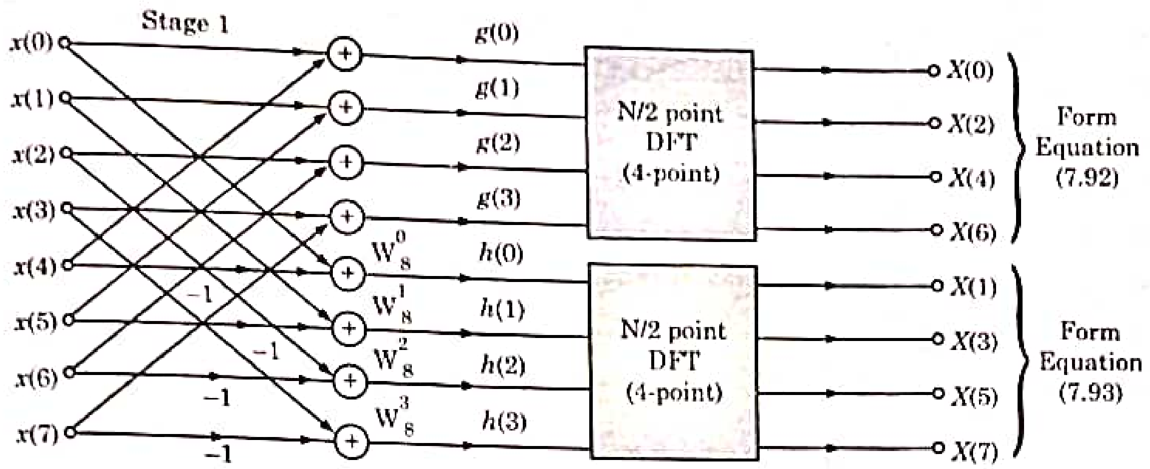


FIGURE 5.17 *First stage of decimation.*

Second Stage of Decimation

In the first stage of decimation, we have used 4-point DFT. We can further decimate the sequence by using 2-point DFT. The second stage of decimation is shown in figure 5.18.

This is similar to DIT-FFT.

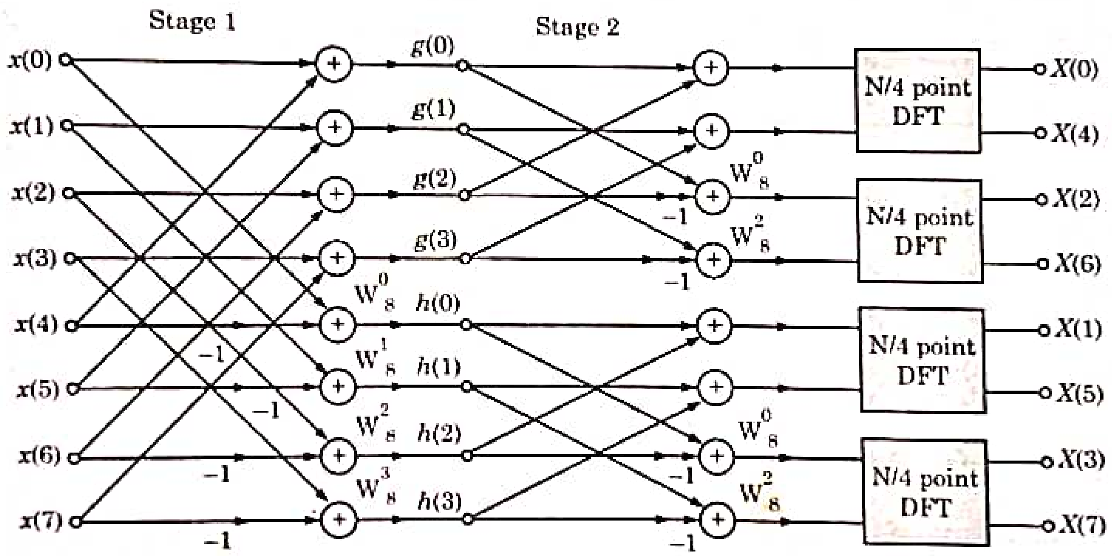


FIGURE 5.18 *Second stage of decimation*

Third Stage of Decimation

In the second stage of decimation, we have used 2-point DFT. So, further decimation is not possible. Now, we shall use a butterfly structure to obtain 2-point DFT. This butterfly is same, as shown in figure 5.16. Thus, the total flow graph for 8 point DIF-FFT is shown in figure. 5.19.

This flow graph is similar to the flow graph of DIF-FFT but it is in the opposite direction. It may be noted that here input $x(n)$ is in sequence but output is shuffled. Similar to DIT-FFT, there are $\log_2 N = \log_2 8 = 3$ stages.

The computational complexity and the memory requirement is same as that of DIT-FFT.

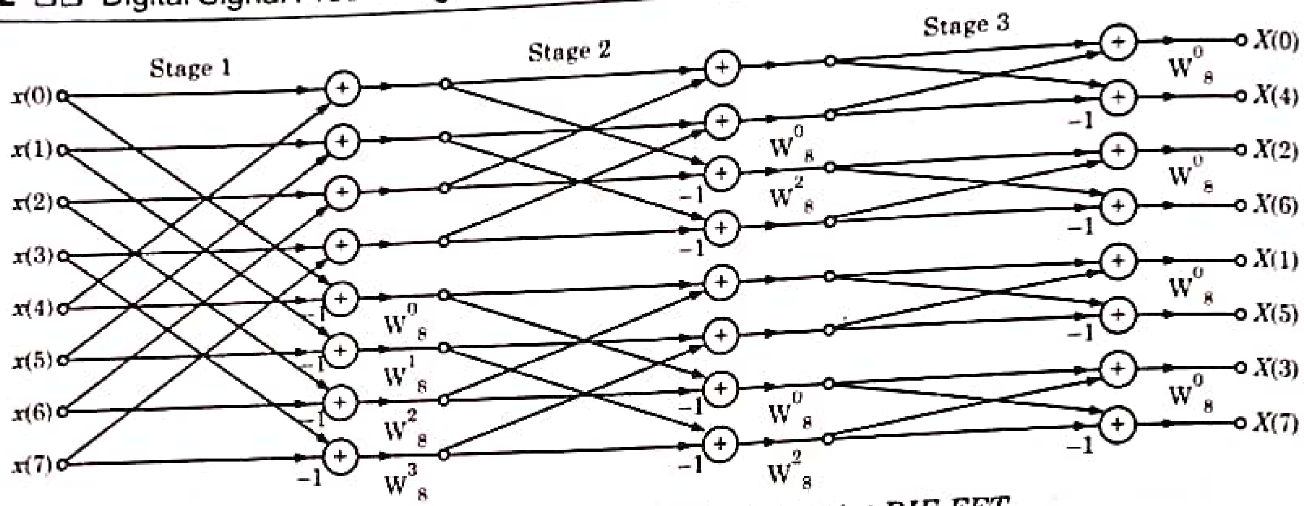


FIGURE 5.19 *Total flow graph for 8-point DIF-FFT*

EXAMPLE 5.5 Obtain DFT of the following sequence :

$$x(n) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right)$$

Using decimation in frequency FFT algorithm.

Solution: The total flow graph has been shown in figure 5.20.

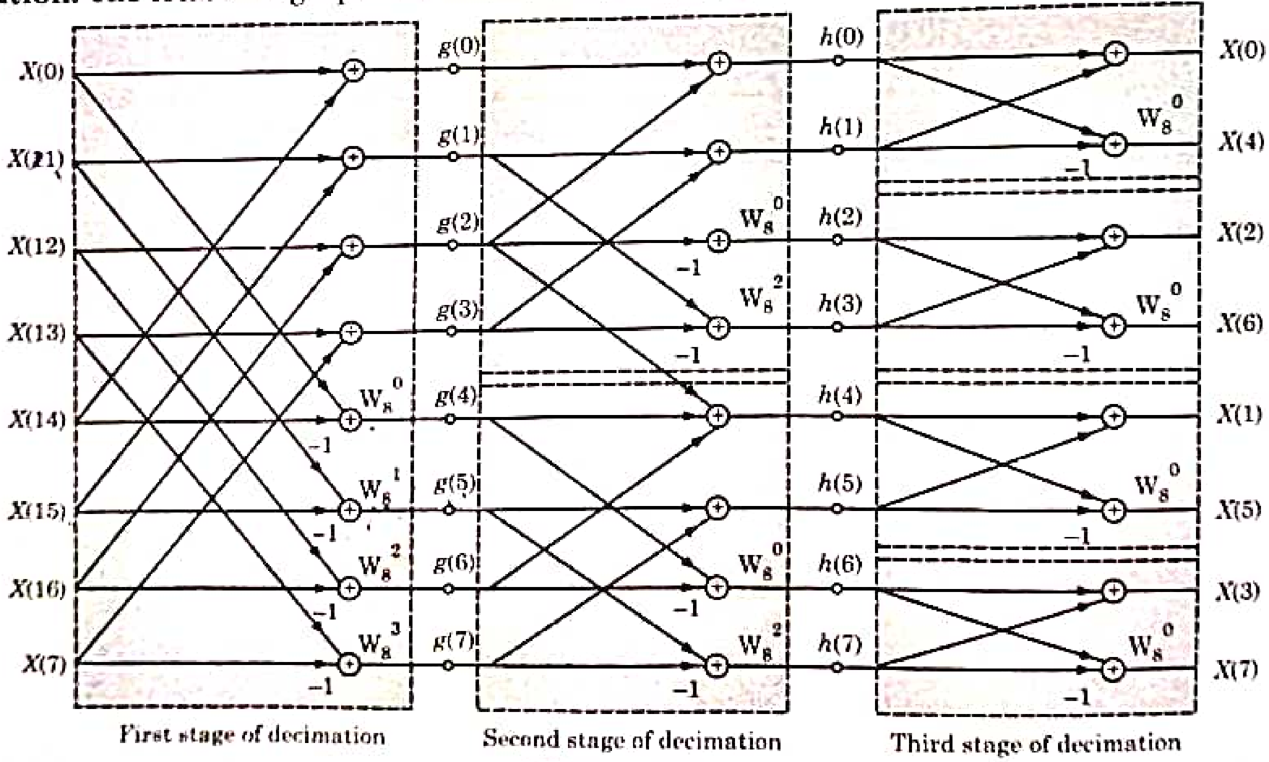


FIGURE 5.20

Here, $g(n)$ is output of first stage and $h(n)$ is output of second stage

The values of twiddle factor are as under :

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-\frac{j\pi}{4}} = 0.707 - j0.707$$

$$W_8^2 = e^{-\frac{j\pi}{2}} = -j$$

$$W_8^3 = -0.707 - j0.705.$$

Output of stage 1

$$g(0) = x(0) + x(4) = \frac{1}{2} + 0 = 0.5$$

$$g(1) = x(1) + x(5) = \frac{1}{2} + 0 = 0.5$$

$$g(2) = x(2) + x(6) = \frac{1}{2} + 0 = 0.5$$

$$g(3) = x(3) + x(7) = \frac{1}{2} + 0 = 0.5$$

$$g(4) = [x(0) - x(4)]\ W_8^0 = \left[\frac{1}{2} - 0\right] 1 = 0.5$$

$$g(5) = [x(1) - x(5)]\ W_8^1 = \left[\frac{1}{2} - 0\right] (0.707 - j0.707)$$

or $$g(5) = 0.3535 - j0.3535$$

$$g(6) = [x(2) - x(6)]\ W_8^2 = \left[\frac{1}{2} - 0\right] (-j) = -j0.5$$

$$g(7) = [x(3) - x(7)]\ W_8^3 = \left[\frac{1}{2} - 0\right] (0.707 = j0.707)$$

Output of stage 2

$$h(0) = g(0) + g(2) = 0.5 + 0.5 = 1$$
$$h(1) = [g(1) + g(3)] = (0.5 + 0.5) = 1$$
$$h(2) = [g(0) - g(2)]\ W_8^0 = (0.5 - 0.5)(+1) = 0$$
$$h(3) = [g(1) - g(3)]\ W_8^2 = (0.5 - 0.5)(-j) = 0$$
$$h(4) = g(4) + g(6) = 0.5 - j\,0.5$$
$$h(5) = g(5) + g(7) = 0.3535 - j0.3535 - j0.3535 - j\,0.3535$$

or $$h(5) = -j0.707$$

$$h(6) = [g(4) - g(6)]\ W_8^0 = [0.5 + j0.5]\ 1 = 0.5 + j\,0.5$$

$$h(7) = [g(5) - g(7)]\ W_8^2$$
$$= [0.3535 - j0.3535 + 0.3535 + j0.3535]\ (-j)$$

or $$h(7) = -j0.707$$

Final Output

$$X(0) = h(0) + h(1) = 1 + 1 = 2$$
$$X(1) = h(4) + h(5) = 0.5 - j0.5 - j.0707 = 0.5 - j1.207$$
$$X(2) = h(2) + h(3) = 0 + 0 = 0$$
$$X(3) = [h(6) + h(7)]\ W_8^0 = [0.5 + j0.5 - j0.707].1 = 0.5 - j0.207$$
$$X(4) = [h(0) - h(1)]\ W_8^0 = [1 - 1].1 = 0$$
$$X(5) = [h(4) - h(5)]\ W_8^0 = [(0.5 - j\,0.5) + j0.707].1$$

or $$X(5) = 0.5 + j0.207$$

$$X(6) = [h(2) - h(3)]\ W_8^0 = 0$$

$$X(7) = [h(6) - h(7)]\, W_8^0 = [0.5 + j0.5 + j0.707] = 0.5 + j\,1.21$$

Therefore, we get

$$X(k) = [X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)]$$

or $\quad X(k) = \{2, 0.5 - j1.207, 0, 0.5 - j0.207, 0, 0.5 + j0.207, 0, 0.5 + j1.21\}$ **Ans.**

5.9 COMPUTATION OF INVERSE DFT (IDFT) USING FFT ALGORITHMS

(Expected

We have discussed Radix-2 DIT and DIF FFT algorithms to compute the DFT, $X(k)$. The sam algorithm can be used to obtain input sequence $x(n)$ from its DFT. This means that we can comput IDFT. We know that the IDFT is expressed

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}, \quad n = 0, 1, \ldots N-1 \qquad \ldots(5.10($$

Just for comparison, let us reproduce the expression for DFT *i..e*,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{+kn}, \quad k = 0, 1, \ldots N-1 \qquad \ldots(5.10$$

This, IDFT differs from DFT by,

(*i*) Multiplication by $\frac{1}{N}$ factor

(*ii*) Negative sign of imaginary part of (W_N).

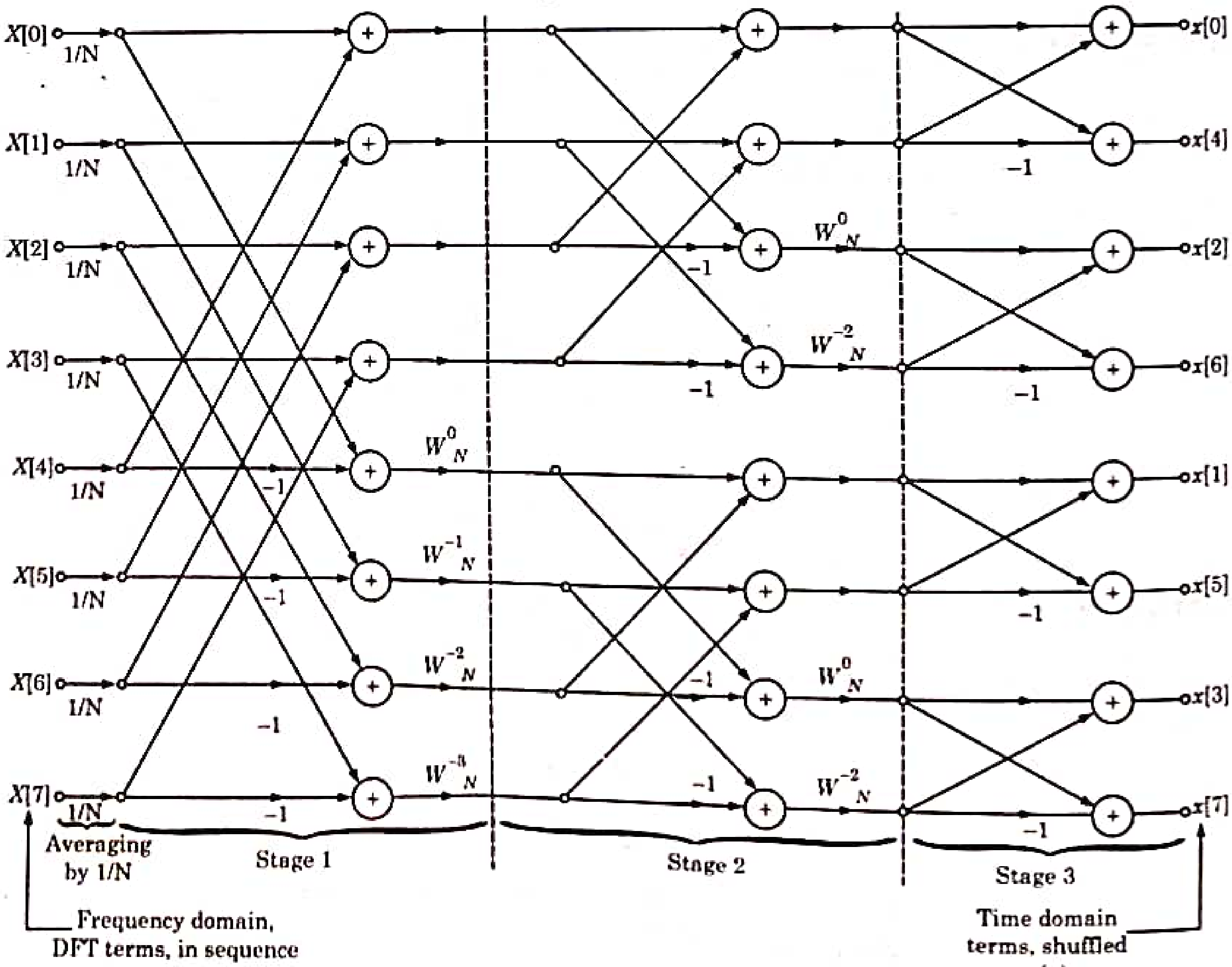


FIGURE 5.21 *Computation of IDFT using FFT.*

Thus, we can use the same algorithm to compute IDFT. But, we have to change the sign of twiddle factor and for DIF FFT algorithm, we have to multiply input sequence X (k) by $\frac{1}{N}$. The total flow graph has been shown in figure 5.21.

EXAMPLE 5.6 Using FFT and IFFT, determine the output of system if input $x(n)$ and impulse response $h(n)$ are given as under :

$$x(n) = \{2, 2, 4\}$$
$$h(n) = \{1, 1\}$$

Solution : If the given sequences are very long, then we have to divide such sequences into smaller segments. In this case, it is not essential to divide the sequences.

Given that $x(n) = \{2, 2, 4\}$

and $h(n) = \{1, 1\}$

Here, L = Number of samples in $x(n) = 3$

M = Number of samples in $h(n) = 2$

Therefore, $L + M - 1 = 3 + 2 - 1 = 4$

This means that we have to make length of $x(n)$ and $h(n)$ equal to 4.

Therefore, $x(n) = \{2, 2, 4, 0\}$

and $h(n) = \{1, 1, 0, 0\}$

First, let us obtain DFT of $x(n)$ using DIT FFT algorithms. We can also use DIF FFT algorithm. The calculations have been shown in figure 5.22.

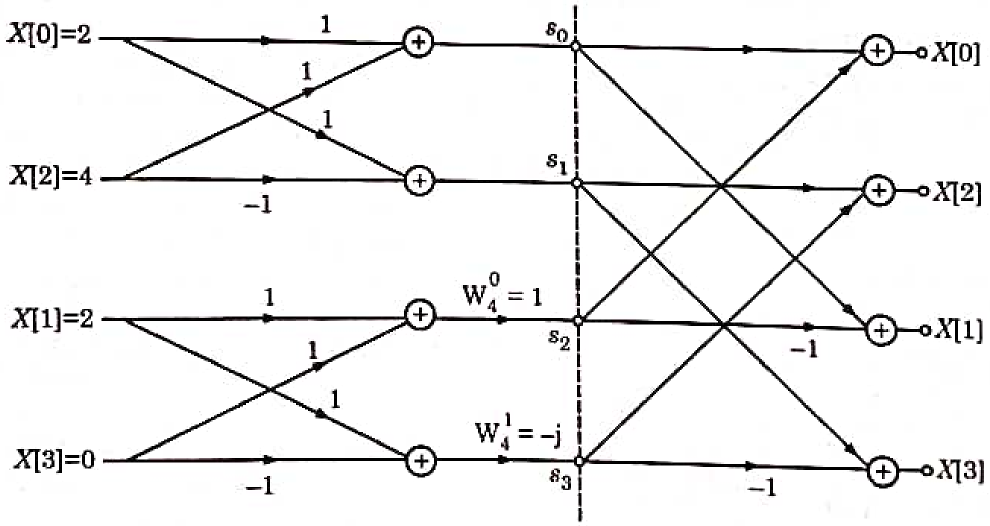


FIGURE 5.22

Here

$$s_0 = x(0) + x(2) = 2 + 4 = 6$$
$$s_1 = x(0) - x(2) = 2 - 4 = -2$$
$$s_2 = [x(1) + x(3)]\ W_4^0 = 2 + 0 = 2$$
$$s_3 = [x(1) - x(3)]\ W_4^1 = (2 - 0)(-j) = -j2$$

The final outupt will be

$$X(0) = s_0 + s_2 = 6 + 2 = 8$$
$$X(1) = s_1 + s_3 = -2 - j2$$
$$X(2) = s_0 - s_2 = 6 - 2 = 4$$
$$X(3) = s_1 - s_3 = -2 + j2$$

Therefore, we have

$$X(k) = \{8, -2-j2, 4, -2+j2\}$$

Now, let us obtain DFT of $h(n)$ using DIT FFT as shown in figure 5.23.

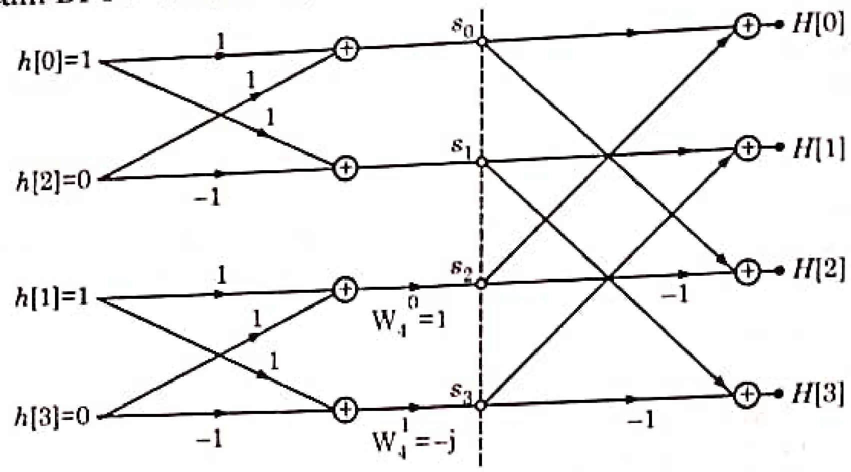


FIGURE 5.23

$$s_0 = h(0) + h(2) = 1 + 0 = 1$$
$$s_1 = h(0) - h(2) = 1 - 0 = 1$$
$$s_2 = [h(1) + h(3)]\ W_4^0 = 1 + 0 = 1$$
$$s_3 = [h(1) - h(3)]\ W_4^1 = (1-0) - j = -j$$

The final output will be

$$H(0) = s_0 + s_2 = 1 + 1 = 2$$
$$H(1) = s_1 + s_3 = 1 - j$$
$$H(2) = s_0 - s_2 = 1 - 1 = 0$$
$$H(3) = s_1 - s_3 = 1 + j$$

Therefore, we have

$$H(k) = \{2, 1-j, 0, 1+j\}$$

Now, we shall multiply $H(k)$ and $X(k)$

Let $\quad Y(k) = X(k) \,.\, H(k)$

Therefore, $\quad Y(k) = \{8, -2-j2, 4, -2+j2\} \,.\, \{2, 1-j, 0, 1+j\}$

or $\quad Y(k) = \{16, -4, 0, -4\}$

Now, let us perform IFFT to obtain sequence $y(n)$. For this, we have to multiply each input by $\frac{1}{N}$ that means $\frac{1}{4}$ and we have to change the sign of imaginary part of twiddle factor. This computation has been shown in figure 5.24.

We have

$$s_0 = \frac{1}{4}Y(0) + \frac{1}{4}Y(2) = \frac{1}{4}(16) + \frac{1}{4}(0) = 4$$

$$s_1 = \frac{1}{4}Y(0) - \frac{1}{4}Y(2) = \frac{1}{4}(16) - \frac{1}{4}(0) = 4$$

$$s_2 = \left[\frac{1}{4}Y(1) + \frac{1}{4}Y(3)\right] . 1 = \frac{1}{4}(-4) + \frac{1}{4}(-4) = -2$$

$$s_3 = \left[\frac{1}{4}Y(1) - \frac{1}{4}Y(3)\right](-j) = \left[\frac{1}{4}(-4) - \frac{1}{4}(-4)\right](-j) = 0$$

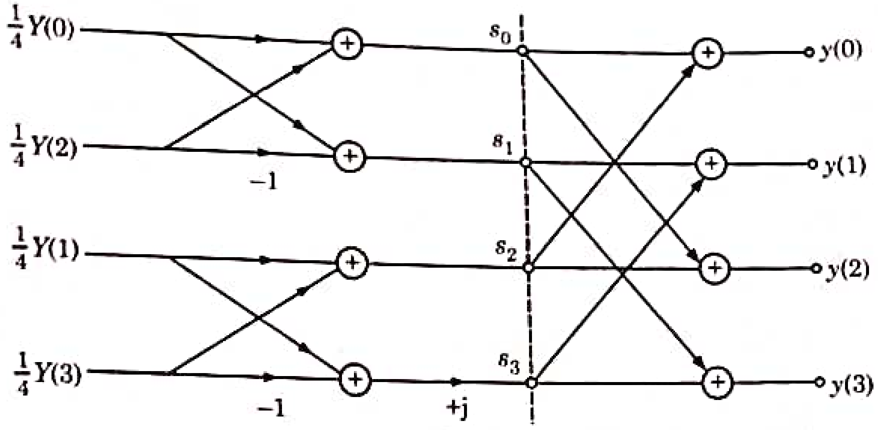


FIGURE 5.24

The final output will be

$$y(0) = s_0 + s_2 = 4 - 2 = 2$$
$$y(1) = s_1 + s_3 = 4 + 0 = 4$$
$$y(2) = s_0 - s_2 = 4 + 2 = 6$$
$$y(3) = s_1 - s_3 = 4 - 0 = 4$$

Therefore, we have

$$y(n) = \{2, 4, 6, 4\}$$ **Ans.**

5.10 APPLICATIONS OF FFT ALGORITHMS

The FFT algorithms described in previous sections find application in a variety of areas, including linear filtering, correlation, and spectrum analysis. Basically, the FFT algorithm is used as an efficient means to compute the DFT and the IDFT.

In this articls, let us consider the use of the FFT algorithm in linear filtering and in the computation of the cross-correlation of two sequences. In addition we iillustrate how to enhance the efficiency of the FFT algorithm by forming complex-valued sequences from real-valued sequences prior to the computation of the DFT.

5.11 EFFICIENT COMPUTATION OF THE DFT OF TWO REAL SEQUENCES

(U.P. Tech, Sem. Exam., 2005-06)(05 marks)

The FFT algorithm is designed to perform complex multiplications and additions, even though the input data may be real valued. The basic reason for this situation is that the phase factors are complex and hence, after the first stage of the algorithm, all variables are basically complex-valued.

In view of the fact that the algorithm can handle complex-valued input sequences, we can exploit this capability in the computation of the DFT of two real-valued sequences.

Let us consider that $x_1(n)$ and $x_2(n)$ are two real-valued sequences of length N, and let $x(n)$ be a complex-valued sequence defined as

$$x(n) = x_1(n) + jx_2(n) \quad 0 \le n \le N-1 \qquad \dots (5.102)$$

The DFT operation is linear and hence the DFT of $x(n)$ can be expressed as

$$X(k) = X_1(k) + jX_2(k) \qquad \dots (5.103)$$

In equation (5.102), the sequences $x_1(n)$ and $x_2(n)$ can be expressed in terms of $x(n)$ as under :

$$x_1(n) = \frac{x(n) + x^*(n)}{2} \qquad \dots (5.104)$$

$$x_2(n) = \frac{x(n) - x^*(n)}{2j} \qquad \dots (5.105)$$