From the above ...

Table 4.1. *Response terms and error terms of first order system for standard test inputs*

| S.No. | *Type of input* | $r(t)$ | $R(s)$ | $c(t)$ | $e(t)$ | $e_{ss}$ |
|---|---|---|---|---|---|---|
| 1. | Unit step input | $u(t)$ | $\frac{1}{s}$ | $1 - e^{-t/T}$ | $e^{-t/T}$ | 0 |
| 2. | Unit ramp input | $t$ | $\frac{1}{s^2}$ | $t - T(1 - e^{-t/T})$ | $T(1 - e^{-t/T})$ | T |
| 3. | Unit impulse input | $\delta(t)$ | 1 | $\frac{1}{T} e^{-t/T}$ | $\delta(t) - c(t)$ | $\lim_{t \to \infty} e(t)$ |

**Table 4.2** *Time response of second order system for different values of $\xi$*

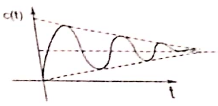
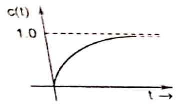
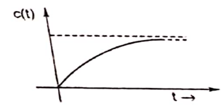
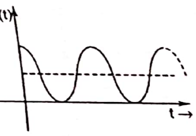
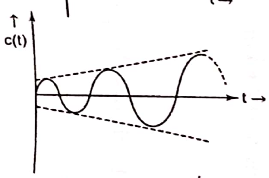
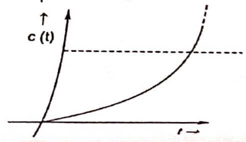
| S. No. | Value of $\xi$ | Nature of System Response | Response $c\,(t)$ | $c\,(t)$ versus $t$ |
|---|---|---|---|---|
| 1. | $0 < \xi < 1$ | Underdamped | $c\,(t) = 1 - \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\,(\omega_d t + \phi)$ where $\omega_d = \omega_n \sqrt{1-\xi^2}$ and $\phi = \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}$ |  |
| 2. | $\xi = 1$ | Critically damped | $c\,(t) = 1 - (1 + \omega_n t)\; e^{-\omega_n t}$ |  |
| 3. | $\xi > 1$ | Overdamped | $c\,(t) = 1 + \dfrac{\left(\xi - \sqrt{\xi^2-1}\right)}{2\sqrt{\xi^2-1}}\, e^{-\left(\xi\omega_n + \omega_n\sqrt{1-\xi^2}\right)t} - \dfrac{\xi + \sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}\, e^{-\left(\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)t}$ |  |
| 4. | $\xi = 0$ | Undamped | $c\,(t) = 1 - \cos\,\omega_n\,t$ |  |
| 5. | $0 > \xi > -1$ | Negative damping | $c\,(t) = A + Be^{at}\cos bt + Ce^{at}\sin bt$ |  |
| 6. | $\xi = -1$ | Negative damping | $c\,(t) = 1 + (\omega_n t - 1)\; e^{\omega_n t}$ |  |

**Table 4.3** *Time response specifications of second order system for unit step input*

| S. No. | "Specifications" | Formula | Definition |
|---|---|---|---|
| 1. | Delay time ($t_d$) | $t_d = \frac{1+0.7\xi}{\omega_n}$ | Time required to reach 50% of the final steady value in first attempt. |
| 2. | Rise Time ($t_r$) | $t_r = \frac{\pi - \phi}{\omega_d}$ <br> $= \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$ | Time required to reach 100% of the final steady value in first attempt. |
| 3. | Peak Time ($t_p$) | $t_p = \frac{\pi}{\omega_d}$ <br> $t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$ | Time required to reach first peak of the response. |
| 4. | Peak overshoot (% $M_p$) | $\% M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \times 100\%$ | It is the normalized difference between first peak value and steady value <br> $\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$ |
| 5. | Settling Time ($t_s$) | $t_s = 4T = \frac{4}{\xi\omega_n}$ <br> for 2% tolerance band, <br> and $t_s = 3T = \frac{3}{\xi\omega_n}$ <br> for 5% tolerance band, | It is the time required to reach the response in a specified band. |
| 6. | Steady state error ($e_{ss}$) | $e_{ss} = 0$ | $e_{ss} = \lim_{t \to \infty} [r(t) - c(t)]$ |

**Table 5.2.** *Steady state error for different types of input for Type-0, Type-1 and Type-2 systems.*

| *Input* | 'Type-0' system | | | 'Type-1' system | | 'Type-2' system | |
|---|---|---|---|---|---|---|---|
| | $\mathcal{L}$ *(input)* | *Static error coefficient* | *steady state error* $e_{ss}$ | *Static error coefficient* | *Steady state error* $e_{ss}$ | *Static error coefficient* | $e_{ss}$ |
| **Step input** | $\frac{A}{s}$ | $K_p = K$ | $\frac{A}{1+K}$ | $K_p = \infty$ | $0$ | $K_p = \infty$ | $0$ |
| **Ramp input** | $\frac{A}{s^2}$ | $K_v = 0$ | $\infty$ | $K_v = K$ | $\frac{1}{K}$ | $K_v = \infty$ | $0$ |
| **Parabolic input** | $\frac{A}{s^3}$ | $K_a = 0$ | $\infty$ | $K_a = 0$ | $\infty$ | $K_v = K$ | $\frac{1}{K}$ |

The following conclusions can be made from the above discussions.

... error of the system. ... value of any system

## Position, velocity and acceleration error constant

| S. No. | Constant | Equation | Steady State Error |
|---|---|---|---|
| 1. | Position error constant ($K_p$) | $K_p = \lim_{s \to 0} G(s) H(s)$ | $e_{ss} = \frac{A}{1+K_p}$ |
| 2. | Velocity error coefficient $K_v$ | $K_v = \lim_{s \to 0} s\,G(s) H(s)$ | $e_{ss} = \frac{A}{K_v}$ |
| 3. | Acceleration error coefficient $K_a$ | $K_a = \lim_{s \to 0} s^2 G(s) H(s)$ | $e_{ss} = \frac{A}{K_a}$ |

## Static error coefficients and steady state error for standard inputs

| Type | Step input | | Ramp input | | Parabolic input | |
|---|---|---|---|---|---|---|
| | $K_p$ | $e_{ss}$ | $K_v$ | $e_{ss}$ | $K_a$ | $e_{ss}$ |
| Type-0 | K | $\frac{A}{1+K}$ | 0 | $\infty$ | 0 | $\infty$ |
| Type-1 | $\infty$ | 0 | K | $\frac{A}{K}$ | 0 | $\infty$ |
| Type-2 | $\infty$ | 0 | $\infty$ | 0 | K | $\frac{A}{K}$ |

**NOTE** Open loop transfer function G (s) H (s) is in time constant form.

...on, the effect of disturbance ... (6.31)

## 6.9 INTRODUCTION TO BASIC CONTROL ACTION OF CONTROLLERS

An automatic controller is used to maintain its output within desirable limits. It determines the deviation of the output from the reference input and generates a control signal to reduce the deviation to zero or a small value. Industrial controllers are classified on the basis of control action as follows :

(a) ON-OFF or Two-Position Controllers
(b) Proportional Controllers (P)
(c) Derivative Controllers (D) or Proportional Plus Derivative Controllers (PD)
(d) Integral Controllers (I) or Proportional Plus Integral Controllers (PI)
(e) Proportional Plus Integral Plus Derivative Controllers (PID)

### 6.9.1 ON-OFF or Two Position Controllers

Let us consider $n(t)$ be the output of the controller and input to the controller be $e(t)$ (*i.e.*, error signal).

A two position controller will produce an output given by

$$\begin{aligned} N(t) &= N_1; e(t) > 0 \\ &= N_2; e(t) < 0 \end{aligned}$$

where $N_1$ and $N_2$ are constants.

Two position controllers are usually electric devices such as solenoid operated valves, relays etc., Figure 6.10 shows a two position controller.

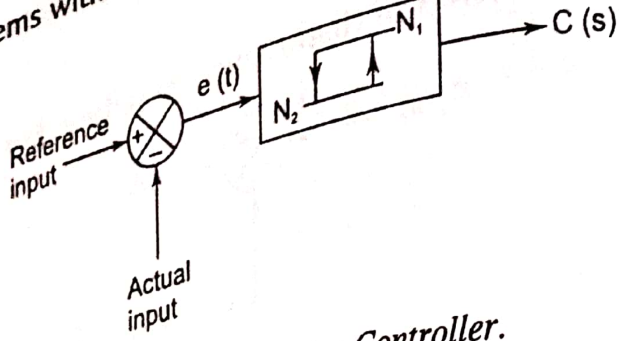



Fig. 6.10. *Two Position Controller.*

From Figure 6.10, following conclusions can be made.

(*i*) If the reference input is less than actual input, the error is negative and the output of the controller is $N_2$.

(*ii*) If the reference input is more than actual input, the error is feedback and the output of the controller is $N_1$.

**Advantage:**

These are simple, economical controllers

**Disadvantage:**

These are not suitable for complex system.

### 6.9.2 Proportional Controllers (P)

In proportional controllers the actuating signal for the control action in a control system is proportional to the error signal. The error signal is the difference between the reference input signal and the feedback signal obtained from the output.

Figure 6.11 shows proportional control action.

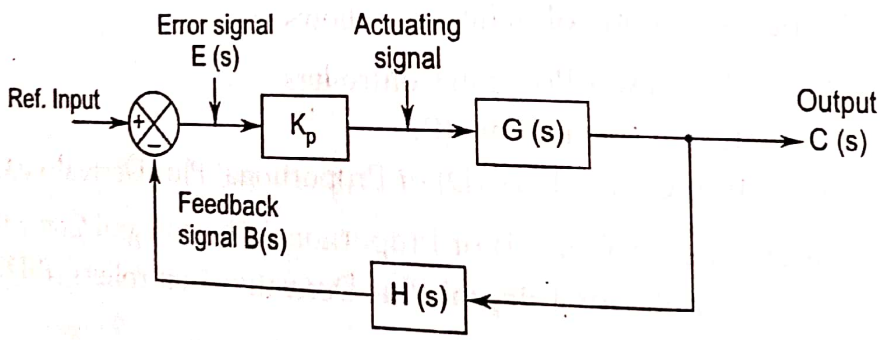



**Fig. 6.11.** *Proportional Control Action.*

As shown in Figure 6.11, the actuating signal is proportional to the error signal [E (s)], therefore the system is called proportional controllers. For quick response the system should be underdamped.

The sluggish overdamped response of a control system can be made faster by increasing forward path gain of the system. The increase in forward path gain reduces the steady state error, but at the same time maximum overshoot is increases. For satisfactory performance of a control system a correct adjustment has to be made between the steady state error and maximum overshoot.

In proportional controllers the output of the controller [*i.e.*, $n(t)$] and input to the controller [*i.e.*, $e(t)$] are related by

$$n(t) = K_P e(t) \qquad \text{... (6.32)}$$

Taking Laplace transform of the equation (6.32)

$$N(s) = K_P E(s)$$

$$\therefore \quad \frac{N(s)}{E(s)} = K_P$$

### 9.3 Derivate Controllers or Proportional Plus Derivative Controllers (PD)

In the derivative controllers or proportional plus derivative controllers, the actuating signal consists of proportional error signal plus derivative of the error signal. Therefore, the actuating signal for derivative control action is given by

$$e_a(t) = K_P e(t) + T_d \cdot \frac{de(t)}{dt} \qquad \text{... (6.33)}$$

where $T_d$ is a constant.

The Laplace transform of equation (6.33) gives,

$$E_a(s) = K_P E(s) + T_d \cdot s \cdot E(s) = (K_P + T_d s) E(s) \qquad \text{... (6.34)}$$

Figure 6.12 shows the block diagram of a second order system with unity feedback, using derivative control.

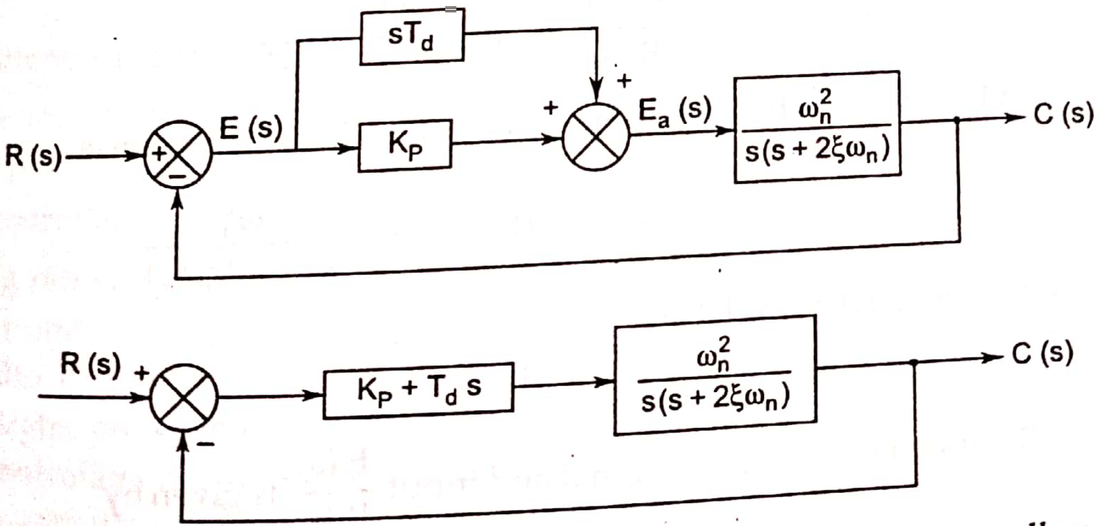



**Fig. 6.12.** *Derivative Controllers or proportional plus derivative controllers.*

From Figure 6.12, the overall T.F. of a closed-loop second order system using derivative control is obtained as follows :

$$\frac{C(s)}{R(s)} = \frac{(K_P + T_d s)\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1 + \left[K_P + (T_d s)\right]\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$\Rightarrow \quad \frac{C(s)}{R(s)} = \frac{(K_P + T_d s)(\omega_n^2)}{s^2 + (2\xi\omega_n + \omega_n^2 T_d)s + K_P\,\omega_n^2} \quad \text{...(6.35)}$$

The characteristic equation is given by

$$s^2 + (2\xi\omega_n + \omega_n^2 T_d)\,s + K_P\,\omega_n^{\,2} = 0 \quad \text{...(6.36)}$$

From the characteristic equation (6.36), the damping ratio using derivative control is given by

$$\xi' = \frac{2\xi\omega_n + \omega_n^2 T_d}{2\omega_n}$$

or

$$\xi' = \xi + \frac{\omega_n T_d}{2} \quad \text{...(6.37)}$$

Therefore, using derivative control the effective damping is increased (equation 6.37) and therefore, the maximum overshoot is reduced.

The overall T.F. given by equation (6.35) can be written as

$$\frac{C(s)}{R(s)} = \omega_n^2 \,.\, T_d \,\frac{\left(s + \dfrac{K_P}{T_d}\right)}{s^2 + 2\xi'\omega_n s + K_P\,\omega_n^2} \quad \text{...(6.38)}$$

The overall T.F. of a second order control system without using derivative control is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{...(6.39)}$$

In Figure 6.16, the forward path T.F. of the Block diagram is

$$G(s) = (K_P + T_d s)\,.\,\frac{\omega_n^2}{s(s + 2\xi\omega_n)} \quad \text{...(6.40)}$$

and the feedback path T.F. is

$$H(s) = 1 \quad \text{...(6.41)}$$

Relation between error signal and input $\dfrac{E(s)}{R(s)}$ is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)\,H(s)} \quad \text{...(6.42)}$$

Putting the values of G (s) and H (s) from equations (6.40) and (6.41) into equation (6.42) for derivative control action, we get

$$\frac{E(s)}{R(s)} = \frac{1}{1 + \dfrac{(K_P + T_d s)\,\omega_n^2}{s(s + 2\xi\omega_n)} \cdot 1}$$

$$= \frac{s\left(s+2\xi\omega_n\right)}{s^2+\left(2\xi\omega_n+\omega_n^2 T_d\right)s+K_P\,\omega_n^2} \qquad \text{... (6.43)}$$

For a unit ramp input $r(t) = t$

$$R(s) = \frac{1}{s^2}$$

$\therefore$

$$E(s) = \frac{1}{s^2}\cdot\frac{s\left(s+2\xi\omega_n\right)}{\left[s^2+\left(2\xi\omega_n+\omega_n^2 T_d\right)s+K_P\,\omega_n^2\right]} \qquad \text{... (6.44)}$$

$\therefore$

The steady state error for unit ramp input is given by

$$e_{ss} = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s\,.\frac{1}{s^2}\,\frac{s\left(s+2\xi\omega_n\right)}{\left[s^2+\left(2\xi\omega_n+\omega_n^2 T_d\right)s+K_P\,\omega_n^2\right]}$$

$\therefore$

$$e_{ss} = \frac{2\xi}{\omega_n} \qquad \text{... (6.45)}$$

It is concluded that the steady state error is not affected by derivative control.

Comparing equations (6.38) and (6.39), it can be concluded that using derivative control the natural frequency ($\omega_n$) is unchanged but a zero at $s = -\frac{1}{T_d}$ is added which results in different expression for time response, where the rise time $t_r$ is reduced.

The determination of the time response of a second order system using derivative control is explained in example 6.3.

In PD controllers, the following effects have been observed :

(i) Damping ratio improves and peak overshoot reduces
(ii) Rise time and settling time are reduced
(iii) Bandwidth increases
(iv) Gain margin, phase margin and resonant peak improves.

As PD controllers are sensitive to rate of change of error, these immediately correct any error by anticipating an error on the slope.

### .9.4 Integral Controller or Proportional Plus Integral (PI) Controllers

For Integral Controller, the actuating signal consists of proportional error signal plus integral of the error signal. Therefore, the actuating signal for integral controller is given by

$$e_a(t) = K_P\,e(t) + K_I\int e(t)\,dt \qquad \text{... (6.46)}$$

The Laplace transform of the actuating signal in equation (6.46) is

$$E_a(s) = K_P E(s) + K_I \frac{E(s)}{s} = \left(K_P + \frac{K_I}{s}\right) E(s)$$

Figure 6.13 shows the block diagram representation of a second order control system with unity feedback using integral control action.

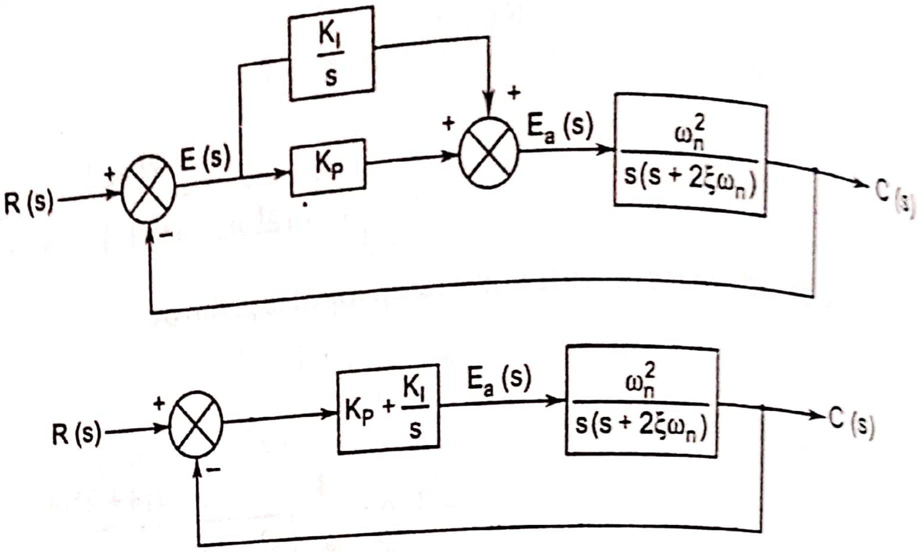



**Fig. 6.13.** *Integral Control Action.*

From the block diagram (Figure 6.13) the transfer function of a closed-loop second order control system using integral control is obtained as follows :

$$\frac{C(s)}{R(s)} = \frac{\left(K_P + \frac{K_I}{s}\right)\left[\frac{\omega_n^2}{s(s+2\xi\omega_n)}\right]}{1+\left[K_P + \frac{K_I}{s}\right]\left[\frac{\omega_n^2}{s(s+2\xi\omega_n)}\right].1}$$

$$= \frac{(sK_P + K_I)\,\omega_n^2}{s^3 + 2\xi\omega_n s^2 + K_P\,\omega_n^2 s + K_I\,\omega_n^2} \quad \text{... (6.48)}$$

The characteristic equation for the overall T.F. (equation 6.48) is

$$s^3 + 2\xi\omega_n s^2 + K_P\,\omega_n^2 s + K_I\,\omega_n^2 = 0 \quad \text{... (6.49)}$$

The characteristic equation (6.49) is of third order. For determining the condition for stability of the system Routh-Hurwitz criterion (discussed later) is used.

If $2\xi w_n > K_I$, then all the three roots have negative real parts indicating a stable system.

From Figure 6.13, the forward path transfer function is

$$G(s) = \frac{(sK_P + K_I)\,\omega_n^2}{s^2\,(s+2\xi\omega_n)} \quad \text{... (6.50)}$$

and the feedback path T.F.

$$H(s) = 1 \quad \text{... (6.51)}$$

The relation between error signal E (s) and input R (s) is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} \quad \text{...(6.52)}$$

Using equations (6.50) and (6.51), equation (6.52) becomes

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{(sK_P + K_I)\,\omega_n^2}{s^2(s+2\xi\omega_n)}.1}$$

or

$$E(s) = \frac{s^2(s+2\xi\omega_n)}{s^3 + 2\xi\omega_n s^2 + K_P\,\omega_n^2 s + K_i\,\omega_n^2}.R(s) \quad \text{...(6.53)}$$

If input is unit ramp function

$$R(s) = \frac{1}{s^2}$$

$$\therefore \quad e_{ss} = \lim_{s\to 0} sE(s) = 0 \quad \text{...(6.54)}$$

Without using integral control action the steady state error (discussed in previous chapter) for a unit ramp input is

$$e_{ss} = \frac{2\xi}{\omega_n} \quad \text{...(6.55)}$$

If the input is unit parabolic function $\left(i.e., R(s) = \frac{1}{s^3}\right)$ the steady state error without integral control action is

$$e_{ss} = \infty \quad \text{...(6.56)}$$

The steady state error using integral control is obtained as follows :
From equation (6.53),

$$E(s) = \frac{s^2(s+2\xi\omega_n)}{s^3 + 2\xi\omega_n s^2 + K_P\,\omega_n^2 s + K_I\,\omega_n^2}.\frac{1}{s^3} \quad \left[\because R(s) = \frac{1}{s^3}\right]$$

Steady state error $e_{ss} = \lim_{s\to 0} sE(s)$

$$e_{ss} = \lim_{s\to 0} s.\frac{1}{s^3}.\frac{s^2(s+2\xi\omega_n)}{s^3 + 2\xi\omega_n s^2 + K_P\,\omega_n^2 s + K_I\,\omega_n^2}$$

$$\therefore \quad e_{ss} = \frac{2\xi}{K_I\,\omega_n} \quad \text{...(6.57)}$$

The main features of a PI controllers are as follows :

1. Due to reduction of steady state error, the behaviour of the system is accurate.
2. Improvement of damping
3. Reduction of peak overshoot
4. Bandwidth increases
5. Noise is filtered out
6. $K_I$ must be designed properly.

The steady state error ($e_{ss}$) for a second order control system with and without using integral action is concluded in Table 6.1 given below :

**Table 6.1** *Steady State Error*

| *Input* | *Without Integral Action* | *With Integral Action* |
|---|---|---|
| Unit Ramp | $\frac{2\xi}{\omega_n}$ | 0 |
| Unit Parabolic | $\infty$ | $\frac{2\xi}{K_I \omega_n}$ |

### 6.9.5 Proportional Plus Integral Plus Derivative Control (PID Control)

For PID control, the actuating signal consists of proportional error signal plus integral and derivative of the error signal. Therefore, the actuating signal for PID control can be written as follows :

$$e_a(t) = e(t) + T_d \frac{de(t)}{dt} + K_I \int e(t)\, dt \qquad \text{... (6.58)}$$

Taking, the Laplace transform of equation (6.58)

$$E_a(s) = E(s) + sT_d E(s) + \frac{K_I}{s} E(s)$$

$$\therefore \quad E_a(s) = E(s)\left[1 + sT_d + \frac{K_I}{s}\right] \qquad \text{... (6.59)}$$

Figure 6.14 shows the block diagram representation of a second order control system having PID control.

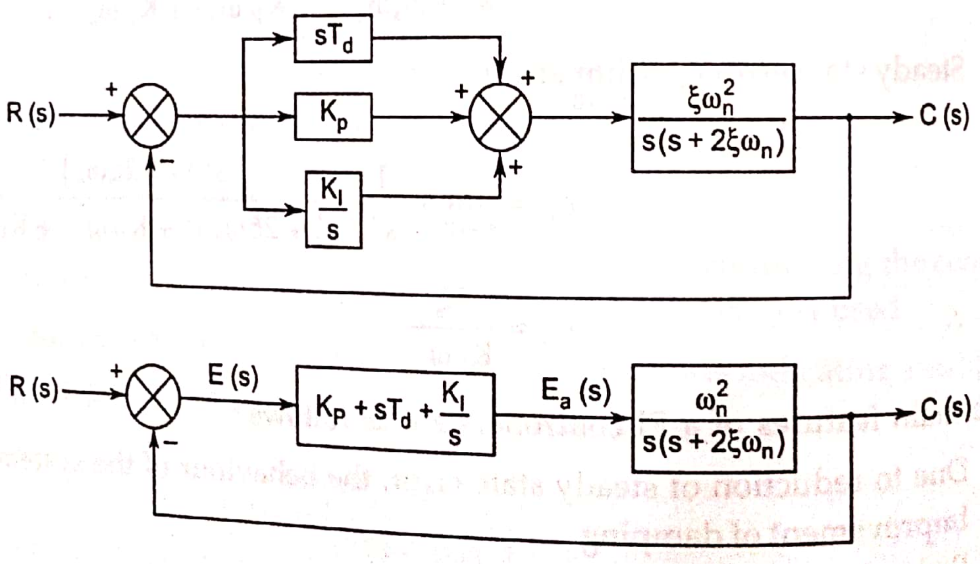



**Fig. 6.14.** *PID controller.*

Since a PD controller improves the transient part and PI controller improves the steady-state part. Hence combination PD and PI improves the overall system.