(ii) High Pass Filter (HPF)

It passes the frequency above some designated frequency called as cut-off frequency. If input signal frequency is less than the cut-off frequency; then this signal is not allowed to pass through it. An ideal HPF characteristic is shown in figure 8.3.

(iii) Band Pass Filter (BPF)

It allows the frequencies between two designated cut-off frequencies (say f_{c1} and f_{c2}). An ideal BPF characteristic is shown in figure 8.4.

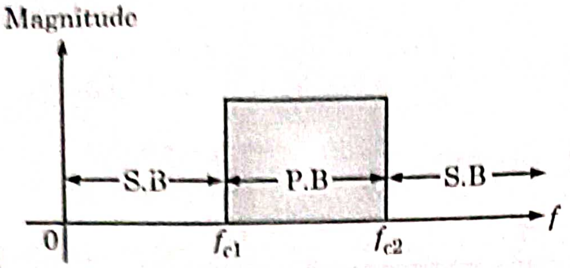

FIGURE 8.3 *Ideal high pass filter (HPF) characteristic.*

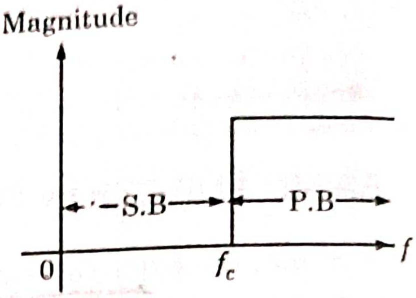

FIGURE 8.4 *Ideal BPF characteristic.*

(iv) Band Reject Filter (BRF)

It attenuates all frequencies between two designated cut-off frequencies. At the sametime, it passes all other frequencies. An ideal band reject filter (BRF) characteristic is shown in figure 8.5.

(v) All Pass Filter :

It passes all the frequencies. However, by using this filter the phase of input signal can be modified.

Figure 8.6 shows all pass filter characteristics.

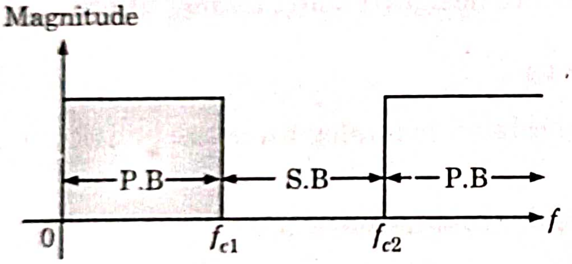

FIGURE 8.5 *Ideal band reject filter (BRF) characteristic.*

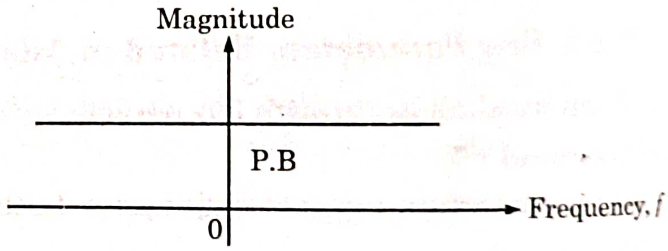

FIGURE 8.6 *All pass filter characteristic.*

8.4 FILTERING CONCEPT

(Important)

Analog filters are designed using analog components like resistors (R), inductors (L) and capacitors (C). On the other hand, digital filters are implemented using difference equation.

The digital filters described by differential equations can be implemented using software like C or assembly language. Since, we can easily change the algorithm so, we can easily change the filter characteristics according to our requirement.

Basically, there are two types of filters as under:

(i) FIR (Finite impulse response) filter.

(ii) IIR (Infinite impulse response) filter.

Let us compare analog and digital filters by studying advantages and disadvantages of digital filters.

8.4.1 Advantages of Digital Filters

1. Many input signals can be filtered by one digital filter without replacing the hardware.

2. Digital filters have characteristic like linear phase response. Such characteristic is not possible to obtain in case of analog filters.
3. The performance of digital filters does not vary with environmental parameters. However, the environmental parameters like temperature, humidity etc., change the values of components in case of analog filters. Therefore, it is required to calibrate analog filters periodically.
4. In case of digital filters, since the filtering is done with the help of digital computer, both filtered and unfiltered data can be saved for further use.
5. Unlike analog filters, the digital filters are portable.
7. The digital filters are highly flexible.
8. Using VLSI technology, the hardware of digital filters can be reduced. Similarly the power consumption can be reduced.
9. Digital filters can be used at very low frequencies, for example, in Biomedical applications.
10. In case of analog filters, maintenance is frequently required. However, for digital filters, it is not required.

8.4.2 Disadvantages of Digital Filters

(i) Speed limitation :

In case of digital filters, ADC and DAC are used. So, the speed of digital filter depends on the conversion time of ADC and the settling time of DAC. Similarly, the speed of operation of digital filter depends on the speed of digital processor. Thus, the bandwidth of input signal processed is limited by ADC and DAC. In real time applications, the bandwidth of digital filter is much lower than analog filters.

(ii) Finite wordlength effect :

The accuracy of digital filter depends on the wordlength used to encode them in binary from. Wordlength should be long enough to obtain the required accuracy.

The digital filters are also affected by the ADC noise, resulting from the quantization of continuous signals. Similarly, the accuracy of digital filters is also affected by the roundoff noise occurred during computation.

(iii) Long design and development time :

An initial design and development time for digital hardware is more than analog filters.

8.5 IDEAL FILTERS AND APPROXIMATIONS

As a matter of fact, ideal frequency selective filters are practically not realizable because of the following two important characteristics :

(i) Ideal filters have constant gain in the passband and zero gain in the stop- band.

(ii) Ideal filter has linear phase response.

Also, in order to design a digital filter, an important condition is that the response of filter must be causal.

Let us consider that the impulse response of an LTI system is denoted by $h(n)$. Now, system is causal when $h(n)$ has some value for positive values of n and $h(n)$ is zero for negative values of n. Therefore, the condition of causality can be expressed as under:

$$h(n) = 0 \text{ for } n < 0 \quad \text{...(8.4)}$$

As an example, let us consider the magnitude response of ideal low pass filter (LPF) as shown in figure 8.7.

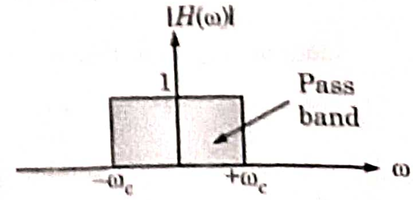

FIGURE 8.7 *Magnitude response of ideal LPF.*

In figure (8.7), ω_c = Cut-off frequency

and $|H(\omega)|$ = Magnitude of filter

According to figure (8.7), the magnitude is unity in the frequency range $-\omega_c$ to $+\omega_c$.

Thus, we have

$$H(\omega) = \begin{cases} 1 & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{elsewhere} \end{cases} \quad \text{...(8.}$$

We can obtain the value of $h(n)$ simply by taking inverse Fourier transform of $H(\omega)$. Hence, we have

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega \quad \text{...(8.}$$

Making use of equation (8.5), we get

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.e^{j\omega n}\, d\omega \quad \text{...(8.}$$

Now, let us consider two conditions as under :

Condition (*i*) : When $n = 0$:

Substituting $n = 0$ in equation (8.7), we get

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.e^0\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega \qquad (\because e^0 = 1)$$

or
$$h(n) = \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c}$$

or
$$h(n) = \frac{\omega_c}{\pi} \qquad \text{for } n = 0 \quad \text{...(8.}$$

Condition (*ii*) : When $n \neq 0$:

Taking integration of equation (8.7), we get

$$h(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

or
$$h(n) = \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] \quad \text{...(8.9)}$$

But according to Euler's identity, we know that

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Thus equation (8.9) becomes,

$$h(n) = \frac{\sin\omega_c n}{\pi n} \qquad \text{...for } n \neq 0 \quad \text{...(8.10)}$$

Combining two conditions, we can write

$$h(n) = \begin{cases} \dfrac{\sin\omega_c n}{\pi n} & \text{for } n \neq 0 \\ \dfrac{\omega_c}{\pi} & \text{for } n = 0 \end{cases} \quad \text{...(8.11)}$$

In figure 8.8, we have sketched $h(n)$ for different values of n by taking $\omega_c = \frac{\pi}{4}$.

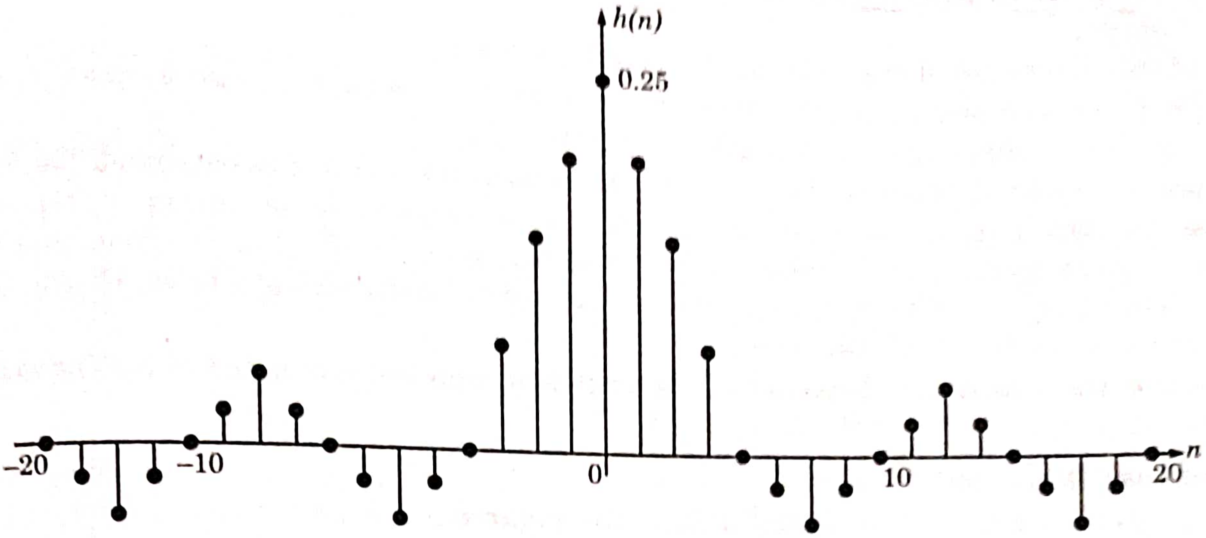

FIGURE 8.8 *Response h(n) of ideal low pass filter (LPF).*

> **Note :** *From figure 8.8, we can easily conclude that h(n) is present for negative values of n. This means that it is noncausal. However, we know that if the frequency response of filter is causal then only it can be realized. Hence, ideal filters are practically not realizable.*
> *Therefore, in practical cases, it is not possible to realize an ideal filter. An important reason for this is that ideal filters are non-causal.*

8.5.1 Magnitude Characteristics of Physically Realizable Filter

(U.P. Tech. Tutorial Question Bank)

The frequency response characteristic of a filter is denoted by $H(\omega)$. An ideal filter cannot be realized because the requirements of ideal filter cannot be fulfilled. But if these requirements are relaxed then it is possible to realize causal filters having characteristics which are approximately similar to ideal filters. The magnitude characteristics of a physically realizable filter is shown in figure 8.9.

The different notations used in figure 8.9 are as under :

δ_1 = Passband ripple
δ_2 = Stopband ripple
ω_p = Passband edge frequency
ω_s = Stopband edge frequency

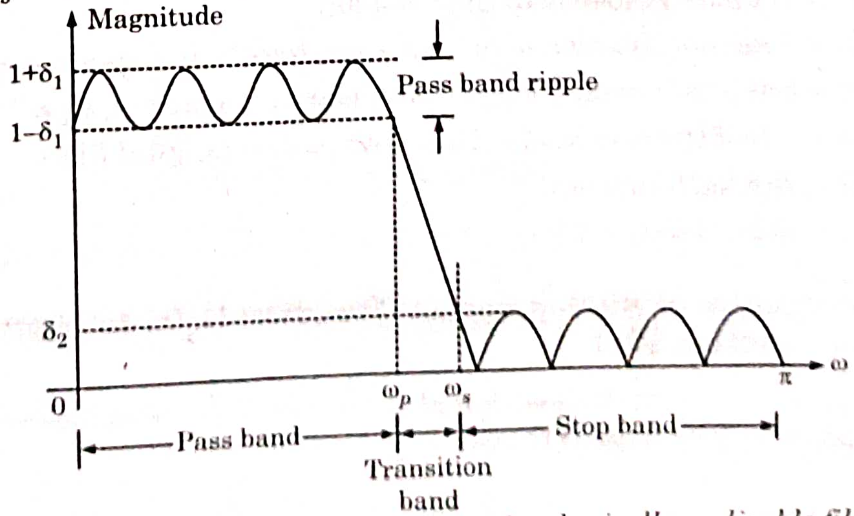

FIGURE 8.9 *Magnitude characteristics of a physically realizable filter.*

The different conditions that can be relaxed to design such filters are as under:

(i) It is not necessary to insist that the magnitude $|H(\omega)|$ is constant in the entire range of passband.

(ii) A small amount of ripple in the passband is allowed.

(iii) It is not necessary that the filter response in the stopband should be perfectly zero. A small amount of ripple in the stopband is tolerable.

As shown in figure 8.9, the frequency response from zero to ω_p is called as passband. The frequency ω_p denotes the edge of passband whereas the frequency ω_s denotes the beginning of stopband. The transition between passband to stopband is called as transition band. Thus the frequency range of transition band is $\omega_p - \omega_s$. The width of passband is called as bandwidth of a filter. Hence, as shown in figure 8.9, the bandwidth of filter is ω_p.

A ripple in the passband is denoted by δ_1 and ripple in stop and is denoted by δ_2. The magnitude in the passband varies between the limits $1 \pm \delta_1$.

Specifications Required

In any filter design, following specifications are required :

(i) The maximum tolerable passband ripple (δ_1)

(ii) The maximum tolerable stopband ripple (δ_2)

(iii) Passband edge frequency (ω_p)

(iv) Stopband edge frequency (ω_s)

8.6 IIR FILTER DESIGN FROM CONTINUOUS-TIME FILTERS

As discussed earlier, in order to design the digital IIR filter, first analog IIR filter is designed. Then analog filter is converted into the digital filter because of following tow reasons :

(i) The procedure to design analog filter is readily available and it is highly advanced.

(ii) When we design digital filter using analog filter then the implementation becomes simple.

8.7 IMPULSE INVARIANT METHOD

In this method, the design starts from the specifications of analog filter. Here, we have to replace analog filter by digital filter. This is achieved if impulse response of digital filter resembles the sampled version of impulse response of analog filter. If impulse response of both, analog and digital filter matches then, both filters perform in a similar manner.

In this method, we shall use the following different notations :

$h_a(t)$ = Impulse response in time domain

$H_a(s)$ = Transfer function of analog filter, here 's' is Laplace operator

$h_a(nT)$ = Sampled version of $h_a(t)$, obtained by replacing t by nT

$H(z)$ = z-transform of $h(nT)$. This is response of digital filter.

Ω = Analog frequency

ω = Digital frequency

8.7.1 Transformation of Analog System Function $H_a(s)$ to Digital System Function $H(z)$

Now, let the system transfer function of analog filter be $H_a(s)$. We can express $H_a(s)$ in terms of partial fraction expansion. This means that

$$H_a(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3} \text{---------}$$

Therefore,
$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s-p_i} \qquad \text{...(8.12)}$$

Here, $A_i = A_1, A_2 \ldots A_N$ are the coefficients of partial fraction expansion.
and $p_i = p_1, p_2 \ldots p_N$ are the poles.

Here, 's' is the Laplace operator. Hence, we can obtain impulse response of analog filter, $h_a(t)$ from $H_a(s)$ by taking inverse Laplace transform of $H_a(s)$. Therefore, using standard relation of inverse Laplace transform, we obtain

$$h_a(t) = \sum_{i=1}^{N} A_i\, e^{p_i t} \qquad \text{...(8.13)}$$

Now, unit impulse response for discrete structure is obtained by sampling $h_a(t)$. This means that $h(n)$ can be obtained from $h_a(t)$ by replacing t by nT in equation (8.13).

Thus,
$$h(n) = \sum_{i=1}^{N} A_i\, e^{p_i nT} \qquad \text{...(8.14)}$$

Here, T is the sampling time.

The system transfer function of digital filter is denoted by $H(z)$. It is obtained by taking Z-transform of $h(n)$. According to the definition of Z-transform for causal system, we have

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \qquad \text{...(8.15)}$$

Substituting equation (8.14) in equation (8.15), we get

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{N} A_i\, e^{p_i nT}\right] . z^{-n} = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} e^{p_i nT} . z^{-n}$$

or
$$H(z) = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} (e^{p_i T} . z^{-1})^n \qquad \text{...(8.16)}$$

Let us use the following standard summation formula:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

With the help of above standard expression, equation (8.16) becomes,

$$H(z) = \sum_{i=1}^{N} A_i . \frac{1}{1-e^{p_i T} z^{-1}} \qquad \text{...(8.17)}$$

This is the required transfer function of digital filter.

8.7.2 Mapping of Poles

Thus, comparing equations (8.12) and (8.17), we can say that the transfer function of digital filter is obtained from the transfer function of analog filter by doing the following transformation:

$$\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T} . z^{-1}} \qquad \text{...(8.18)}$$

Equation (8.18) shows how the poles from analog domain are transferred into the digital domain. This transformation of poles is called as **mapping of poles**.

8.7.3 Relationship of s-plane to z-plane

We know that the poles of analog filters are located at $s = p_i$. Now, from equation (8.18), we can say that the poles of digital filter, $H(z)$ are located at,

$$z = e^{p_i T} \quad ...(8.19)$$

This equation indicates that the poles of analog filter at $s = p_i$ are transformed into the poles of digital filter at $z = e^{p_i} T$. Thus, the relationship between Laplace (s domain) and z-domain is given by,

$$z = e^{s} T \quad ...(8.20)$$

Here, $s = p_i$ and T is the sampling time.

Now, s is the Laplace operator and it is expressed as,

$$s = \sigma + j\Omega \quad ...(8.21)$$

Here, σ = attenuation factor

and Ω = analog frequency

We know that z can be expressed in polar form as under:

$$z = re^{j\omega} \quad ...(8.22)$$

Here, r is magnitude and 'ω' is the digital frequency.

Substituting equations (8.21) and (8.22) in equation (8.20), we obtain

$$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} . e^{j\Omega T} \quad ...(8.23)$$

Separating real and imaginary parts of equation (8.23), we obtain

$$r = e^{\sigma T}$$

and $$e^{j\omega} = e^{j\Omega T} \quad ...(8.24)$$

Thus, we have $$\omega = \Omega T \quad ...(8.25)$$

Now, we will find the relationship between s plane and z plane. Basically, plot in 's'-domain means that σ is plotted on X-axis and $j\Omega$ is plotted on Y-axis. Also, z-domain representation means that real z is plotted on X-axis and imaginary Z is plotted on Y-axis.

Let us consider equation (8.24), *i.e.*,

$$r = e^{\sigma T}$$

we shall discuss the following conditions :

(*i*) If $\sigma < 0$, then r is equal to reciprocal of e raise to some constant. Hence, range of r will be 0 to 1 *i.e.*,

$$\sigma < 0 \Rightarrow 0 < r < 1 \quad ...(8.22)$$

Now, $\sigma < 0$ means the negative values of σ. That is L.H.S. of s plane. We know that r is the radius of circle in z plane.

Therefore, $0 < r < 1$ indicates interior part of unit circle. Thus we can conclude that, **L.H.S. of s plane is mapped inside the unit circle.**

(*ii*) If $\sigma = 0$ then $r = e^0 = 1$ *i.e.*,

$$\sigma = 0 \Rightarrow r = 1$$

Now, $\sigma = 0$ indicates $j\Omega$ axis and $r = 1$ indicates unit circle. Thus, **$j\Omega$ axis in s plane is mapped on the unit circle.**

(*iii*) If $\sigma > 0$ then, r is equal to e raise to some constant. That means $r > 1$ *i.e.*,

$$\sigma > 0 \Rightarrow r > 1$$

Now, $\sigma > 0$ indicates R.H.S. of s plane and $r > 1$ indicates exterior part of unit circle. Thus,

H.S. of s-plane is mapped outside the unit circle.

Combining all the above conditions, this mapping is shown in figure 8.10.

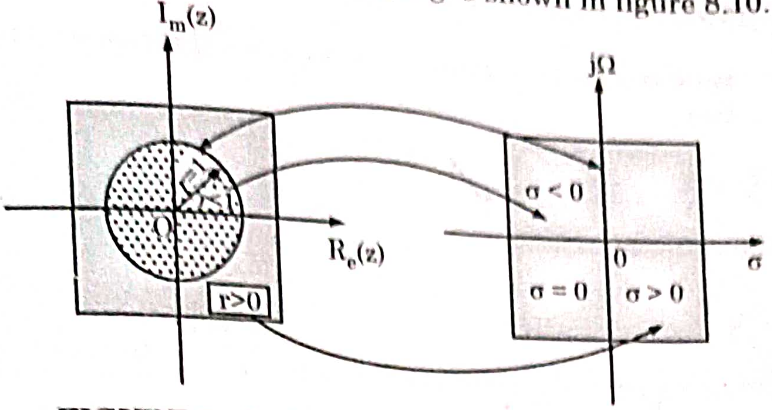

FIGURE 8.10 *Illustration of the mapping* $z = e^{sT}$

8.7.4 Drawbacks of Impulse Invariance Method

(i) We know that Ω is analog frequency and its range is form $\frac{\pi}{T}$ to $-\frac{\pi}{T}$. While the digital frequency ω varies from $-\pi$ to π. This means that from $\frac{\pi}{T}$ to $-\frac{\pi}{T}$, ω maps from $-\pi$ to π.

Let i be any integer. Then, we can write the general rang of Ω as $(i-1)\frac{\pi}{T}$ to $(i+1)\frac{\pi}{T}$. However, for this range also, ω maps from $-\pi$ to π. Hence, mapping from analog frequency Ω to digital frequency ω is termed as **many to one**. This mapping is not one to one.

(ii) Analog filters are not band limited so there will be aliasing due to the sampling process. Because of this aliasing, the frequency response of resulting digital filter will not be identical to the original frequency response of analog filter.

(iii) The change in the value of sampling time (T) has no effect on the amount of aliasing.

8.7.5 Standard Expressions

Some standard formulae for transformation in impulse invariance method are as under :

(1) $$\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T}.z^{-1}}$$

(2) $$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}[\cos bT]z^{-1}}{1-2e^{-aT}[\cos bT]z^{-1}+e^{-2aT}.z^{-2}}$$

(3) $$\frac{b^2}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}[\sin bT]z^{-1}}{1-2e^{-aT}[\cos bT]z^{-1}+e^{-2aT}.z^{-2}}$$

8.7.6 Design Steps for Impulse Invariance Method

(i) In numerical problems, analog frequency transfer function $H_a(s)$ is usually be given. If it is not given, then, we obtain expression of $H_a(s)$ from the given specifications.

(ii) If required, we expand $H_a(s)$ by using partial fraction expansion (PFE).

(iii) Then, we obtain z-transform of each PFE term using impulse invariance transformation equation.

(iv) We obtain $H(z)$, this is required digital IIR filter.

8.7.7 Solved Examples

Now, let us consider few solved examples to illustrate the concept of impulse invariance method.

EXAMPLE 8.1 Determine $H(z)$ using impulse invariance method at 5 Hz sampling frequency from $H_a(s)$ as given below :

$$H_a(s) = \frac{2}{(s+1)(s+2)}$$

Solution. Given analog transfer function is,

$$H_a(s) = \frac{2}{(s+1)(s+2)} \quad ...(i)$$

Let us expand $H_a(s)$ using partial fraction expansion, *i.e.*,

$$H_a(s) = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} \quad ...(ii)$$

Thus, we find that poles are at $p_1 = -1$ and $p_2 = -2$.

Now, values of A_1 and A_2 can be calculated as under:

$$A_1 = (s - p_1)\, H(s)|_{s=p1} = (s+1) \cdot \left.\frac{2}{(s+1)(s+2)}\right|_{s=-1}$$

Therefore, $$A_1 = \frac{2}{-1+2} = 2$$

Also, $$A_2 = (s - P_2)\, H(s)|_{s=p2} = (s+2) \cdot \left.\frac{2}{(s+1)(s+2)}\right|_{s=-2}$$

Therefore, $$A_2 = \frac{2}{-2+1} = -2$$

Substituting values of A_1 and A_2 in equation (*ii*), we obtain

$$H_a(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)} \quad ...(iii)$$

Now, let us obtain the z-transform using impulse invariance transformation equation. It is given by

$$\frac{1}{s-p_i} \to \frac{1}{1-e^{p_i T} \cdot z^{-1}} \quad ...(iv)$$

Here, T = Sampling time. Now, given sampling frequency is $f_s = 5$ Hz.

Also $$T = \frac{1}{f_s} = \frac{1}{5} = 0.2 \text{ sec.}$$

We have poles at $p_1 = -1$ and $p_2 = -2$

Hence, using equation (*iv*), we obtain

$$\frac{1}{s+1} \to \frac{1}{1-e^{-1(0.2)} \cdot z^{-1}} = \frac{1}{1-e^{-0.2} \cdot z^{-1}} \quad ...(v)$$

and $$\frac{1}{s+2} \to \frac{1}{1-e^{-2(0.2)} \cdot z^{-1}} = \frac{1}{1-e^{-0.4} \cdot z^{-1}} \quad ...(vi)$$

The transfer function of digital filter is given by

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1-e^{p_i T} \cdot z^{-1}}$$

In this case, we obtain

$$H(z) = \frac{A_1}{1-e^{p_1 \cdot T} \cdot z^{-1}} + \frac{A_2}{1-e^{p_2 \cdot T} \cdot z^{-1}} \quad ...(vii)$$

Using equations (*v*) and (*vi*), we obtain

$$H(z) = \frac{2}{1-e^{-0.2} \cdot z^{-1}} - \frac{2}{1-e^{-0.4} \cdot z^{-1}} = \frac{2}{1-0.818\, z^{-1}} - \frac{2}{1-0.67\, z^{-1}}$$

To convert each term into positive powers of z, let us multiply each term by z to get

$$H(z) = \frac{2z}{z-0.818} - \frac{2z}{z-0.67} = \frac{2z(z-0.67)-2z(z-0.818)}{(z-0.818)(z-0.67)}$$

or

$$H(z) = \frac{2z^2-1.34\,z-2z^2+1.636\,z}{z^2-0.67z-0.818\,z+0.54} = \frac{0.29\,z}{z^2-1.488\,z+0.54}$$

This is the required transfer function for digital IIR filter. **Ans.**

EXAMPLE 8.2 Determine $H(z)$ using impulse invariance method for the following system function:

$$H_a(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Solution : The given transfer function is,

$$H_a(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} \quad ...(i)$$

In the partial fraction expansion form, $H_a(s)$ can be written as under:

$$H_a(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{(s+0.5)} + \frac{Bs+C}{(s^2+0.5\,s+2)} \quad ...(ii)$$

Let us obtain the values of A, B and C.

$$\frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A(s^2+0.5\,s+2)+(Bs+C)(s+0.5)}{(s+0.5)(s^2+0.5\,s+2)}$$

or
$$A(s^2 + 0.5s + 2) + (Bs + C)(s + 0.5) = 1$$

or
$$As^2 + A\,0.5s + 2A + Bs^2 + B\,0.5s + C.s + 0.5\,C = 1$$

$$s^2(A + B) + s(0.5\,A + 0.5\,B + C) + (2A + 0.5\,C) = 1 \quad ...(iii)$$

Now, s^2 term is absent in RHS, therefore, we have

$$A + B = 0 \quad ...(iv)$$

Similarly, s term is absent in RHS, thus, we have

$$0.5\,A + 0.5\,B + C = 0 \quad ...(v)$$

and
$$2A + 0.5\,C = 1 \quad ...(vi)$$

Now, let us shall solve equations (*iv*), (*v*) and (*vi*) to obtain the values of A, B, C. From equation (*iv*), we have

$$B = -A$$

Substituting this value in equation (*v*), we obtain

$$0.5A - 0.5A + C = 0$$

or
$$C = 0$$

EXAMPLE 8.7 **If $H_a(s) = \dfrac{1}{(s+1)(s+2)}$, find the corresponding $H(z)$ using impluse invariance method for sampling frequency of 5 samples / sec.**

Solution : Let us first expand $H_a(s)$ in partial fractions as under:

$$H_a(s) = \frac{c_1}{s+1} + \frac{c_2}{s+2}$$

$$c_1 = (s+1)\,H_a(s)\big|_{s=-1} = \left.\frac{1}{s+2}\right|_{s=-1} = \frac{1}{-1+2} = 1$$

$$c_2 = (s+2)H_a(s)\big|_{s=-2} = \left.\frac{1}{s+1}\right|_{s=-2} = \frac{1}{-2+1} = -1$$

Therefore, $H_a(s)$ becomes

$$H_a(s) = \frac{1}{s+1} - \frac{1}{s+2} \qquad \text{...(i)}$$

It is given that sampling frequency $f_s = 5\ Hz$

∴ Sampling period

$$T = \frac{1}{f_s} = \frac{1}{5} = 0.2$$

We know that the impluse invariance transformation is given as

$$\frac{1}{s-p_k} \rightarrow \frac{1}{1-e^{p_kT}z^{-1}}$$

Now, let us apply this transformation to individual terms of $H_a(s)$ of equation (*i*), *i.e.*,

$$\frac{1}{s+1} \rightarrow \frac{1}{1-e^{-1\times 0.2}z^{-1}}$$

Here $\qquad p_1 = -1$

and $\qquad \dfrac{1}{s+2} \rightarrow \dfrac{1}{1-e^{-2\times 0.2}z^{-1}}$

Here $\qquad p_2 = -2$

Hence, $H(z)$ of digital filter becomes

$$H(z) = \frac{1}{1-e^{-1\times 0.2}z^{-1}} - \frac{1}{1-e^{-2\times 0.2}z^{-1}}$$

Thus, $\qquad H(z) = \dfrac{1}{1-e^{-0.2}z^{-1}} - \dfrac{1}{1-e^{-0.4}z^{-1}} = \dfrac{1}{1-0.818z^{-1}} - \dfrac{1}{1-0.67\,z^{-1}}$

On simplifying this equation, we get

$$H(z) = \frac{0.148z}{z^2 - 1.48z + 0.548} \quad \textbf{Ans.}$$

8.8 DESIGN OF IIR FILTER BY APPROXIMATION OF DERIVATIVES

(U.P. Tech. Tutorial Question Bank)

As discussed in article 8.1, the analog filter having the rational system function $H(s)$ can also be described by the linear constant coefficient differential equation, *i.e.*,

$$\sum_{k=0}^{N} a_k \cdot \frac{d^k\ y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k\ x(t)}{dt^k} \qquad \text{...(8.27)}$$

Now, in this method, an analog filter is converted into a digital filter by approximating the differential equation in equation (8.27) by an equivalent difference equation.

The backward difference formula is substituted for the derivative $\frac{dy(t)}{dt}$ at time $t = nT$. This means that

$$\left.\frac{dy(t)}{dt}\right|_{\text{at } t=nT} = \frac{y(nT)-y(nT-T)}{T}$$

or

$$\left.\frac{dy(t)}{dt}\right|_{\text{at } t=nT} = \frac{y(n)-y(n-1)}{T} \qquad ...(8.28)$$

here T is called the sampling interval

and $y(n) \equiv y(nT)$.

The system function of an analog differentiator with an output $\frac{dy}{dt}$ is $H(s) = s$, and the digital system which produces the output

$[y(n) - y(n-1)]/T$ has the system function $H(z) = (1 - z^{-1})/T$.

Now, these two may be compared to obtain the frequency-domain equivalent for the relationship expressed in equation (8.28), as under :

$$s = \frac{1-z^{-1}}{T} \qquad ...(8.29)$$

Thus, this is the analog domain to digital domain transformation.

Also, the second derivative $\frac{d^2y}{dt^2}$ can be replaced by the second backward difference *i.e.*,

$$\left.\frac{d^2y(t)}{dt^2}\right|_{\text{at } t=nT} = \frac{d}{dt}\left[\frac{dy(t)}{dt}\right]_{\text{at } t=nT}$$

$$\left.\frac{d^2y(t)}{dt^2}\right|_{\text{at } t=nT} = \frac{[y(nT)-y(nT-T)]/T-[y(nT-T)-y(nT-2T)]T|}{T}$$

$$= \frac{y(n)-2y(n-1)]+y(n-2)}{T^2} \qquad ...(8.30)$$

In the frequency-domain, the equivalent expression for equation (8.30) can be written as

$$s^2 = \frac{1-2z^{-1}+z^{-2}}{T^2} = \left(\frac{1-z^{-1}}{T}\right)^2 \qquad ...(8.31)$$

The i^{th} derivative of function $y(t)$ results in the equivalent frequency-domain relationship as

$$s^i = \left(\frac{1-z^{-1}}{T}\right)^i \qquad ...(8.32)$$

As a result of this, the system function of digital filter can be obtained by the method of approrimation of the derivatives as

$$H(z) = H_a(s)\big|_{at\, s=(1-z^{-1})/T} \qquad ...(8.33)$$

where $H_a(s)$ is the system function of the analog filter characterized by the differential equation. The outcomes of the mapping of the z-plane from the s-plane can be discussed as under :

We can write the equation (8.27) as

$$z = \frac{1}{1 - sT} \quad \text{... (8.34)}$$

Substituting $s = j\Omega$ in the last equation, we have

$$z = \frac{1}{1 - j\Omega T}$$

Simplifying, we have

$$z = \frac{1}{1+\Omega^2 T^2} + j\frac{\Omega T}{1+\Omega^2 T^2} \quad \text{...(8.35)}$$

Now, varying Ω from $-\infty$ to ∞, the corresponding locus of points in the z-plane is a circle with radius $\frac{1}{2}$ and with centre at $z = \frac{1}{2}$ as shown in figure 8.11.

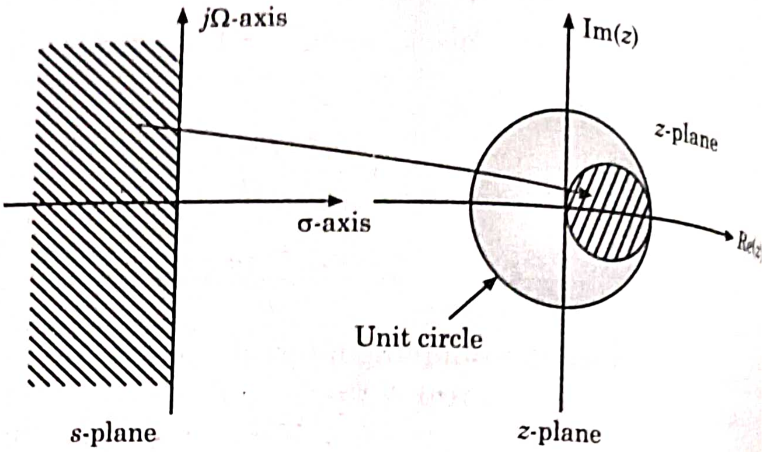

FIGURE 8.11 *Illustration of the mapping of equation (8.6) into the z-plane.*

Now, it may be observed that the mapping of equation (8.6), takes the left-half plane of s domain into the corresponding points inside the circle of radius 0.5 and centre at $z = 0.5$. Also, the right-half of the s-plane is mapped outside the unit circle. Because of this, this mapping results in a stable analog filter transformed into a stable digital filter. However, since the locations of poles in the z-domain are confined to smaller frequencies, this design method can be used only for transforming analog low-pass filters and bandpass filters which are having smaller resonant frequencies. This means that neither a high-pass filter nor a band reject filter can be realised using this technique.

The forward difference can be substituted for the derivative instead of the backward difference. This provides,

$$\frac{dy(t)}{dt} = \frac{y(nT+T) - y(nT)}{T} = \frac{y(n+1) - y(n)}{T} \quad \text{...(8.36)}$$

The transformation formula would be

$$s = \frac{z-1}{T} \quad \text{...(8.37)}$$

or

$$z = 1 + sT \quad \text{...(8.38)}$$

The mapping of equation (8.38) has been shown in figure 8.12. This results in a worse situation than the backward difference substitution for the derivative. When $s = j\,\Omega$, the mapping of these points in the s-domain results in a straight line in the z-domain with coordinates $(z_{real}, z_{imag}) = (1, \Omega T)$. As a result of this, stable analog filters do not always map into stable digital filters.

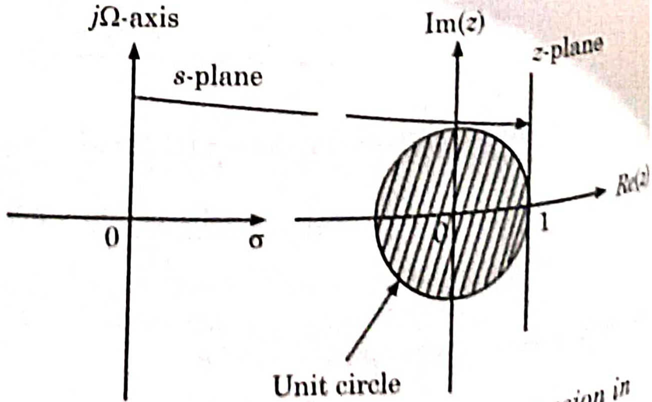

FIGURE 8.12 *Mapping of the expression in equation (10.38) into the z-plane.*

The limitations of the mapping methods discussed above can be overcome by using a more complex substitution for the derivates. An Nth order difference is proposed for the derivative, as shown.

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{1}{T}\sum_{k=1}^{N} a_k \frac{y(nT+kT)-y(nT-kT)}{T} \quad ...(8.39)$$

Here $\{a_k\}$ are a set of parameters selected so as to epitomise the approximation. The transformation from the s-plane to the z-plane will be then,

$$s = \frac{1}{T}\sum_{k=1}^{N} a_k (z^k - z^{-k}) \quad ...(8.40)$$

Thus, if we choose proper values for $\{a_k\}$, then the $j\Omega$ axis can be mapped into the unit circle and the left-half s-plane can be mapped into points inside the unit circle in the z-plane.

EXAMPLE 8.8 Make use of the backward difference for the derivative to convert the analog low-pass filter (LPF) having system function given as under :

$$H_a(s) = \frac{1}{s+2}.$$

Solution : We know that the mapping formula for the backward difference for the derivative is expressed as

$$s = \frac{1-z^{-1}}{T}$$

The system response of the digital filter will be

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{\left(\frac{1-z^{-1}}{T}\right)+2}$$

Simplifying, we get

$$H(z) = \frac{T}{1-z^{-1}+2T}$$

If $T = 1$s, then we have

$$H(z) = \frac{1}{3-z^{-1}} \quad \textbf{Ans.}$$

EXAMPLE 8.9 Make use of the backward difference for the derivative and convert the analog filter having system function :

$$H(s) = \frac{1}{s^2+16}.$$

Solution : We know that,

$$s = \frac{1-z^{-1}}{T}$$

The system response of the digital filter will be

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{\left[\frac{1-z^{-1}}{T}\right]^2+16}$$

Simplifying, we get

$$H(z) = \frac{T^2}{1-2z^{-1}+z^{-2}+16T^2}$$

If $T = 1$ s, then we have

$$H(z) = \frac{1}{z^{-2}-2z^{-1}+17} \quad \textbf{Ans.}$$

EXAMPLE 8.10 An analog filter has the following system function. Convert this filter into a digital filter using backward difference for the derivative given as under:

$$H(s) = \frac{1}{(s+0.1)^2+9}.$$

Solution: We know that the system response of the digital filter is expressed as

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{\left(\frac{1-z^{-1}}{T}+0.1\right)^2+9}$$

Simplifying, we get

$$H(z) = \frac{T^2}{z^{-2}-2(1+0.1T)z^{-1}+(1+0.2T+9.01T^2)}$$

Manipulating, we obtain

$$H(z) = \frac{\dfrac{T^2}{(1+0.2T+9.01T^2)}}{1-2\dfrac{(1+0.1T)}{(1+0.2T+9.01T^2)}z^{-1}+\dfrac{z^{-2}}{(1+0.2T+9.01T^2)}}$$

If $T = 1$s, then we have

$$H(z) = \frac{0.0979}{1-0.2155\,z^{-1}+0.09792\,z^{-2}} \quad \textbf{Ans.}$$

EXAMPLE 8.11 Obtain the system function of the digital filter by approximation of derivation if the system function of the analog filter is as follows :

$$H_a(s) = \frac{1}{(s+0.1)^2+9} \quad \text{...(i)}$$

Solution : We know that the derivative approximation is obtained by substituting

$$s = \frac{1-z^{-1}}{T}$$

$\frac{1-z^{-1}}{T}$

Hence, equation (*i*) becomes

$$H(z) = H_a(s)\Big|_s = \frac{1-z^{-1}}{T} = \frac{1}{\left[\frac{1-z^{-1}}{T}+0.1\right]^2+9}$$

Manipulating, we get

$$H(z) = \frac{\dfrac{T^2}{(1+0.2T+9.01T^2)}}{1-\dfrac{2(1+0.1T)}{1+0.2T+9.01T^2}z^{-1}+\dfrac{1}{1+0.2T+9.01T^2}z^{-2}}$$

This is the required system function. **Ans.**

8.9 DESIGN OF IIR FILTER BY THE BILINEAR TRANSFORMATIONMETHOD

(U.P. Tech. Sem. Exam., 2004-05)(10 marks)

In previous sections, we have studied the IIR filter designing using approximation of derivatives method and the impulse invariant method. However, the IIR filter design using :

(i) the impulse invariant method,

(ii) approximation of derivatives method andare appropriate for the design of low pass filters and bandpass filters whose resonant frequencies are small.

However, these techniques are not suitable for high-pass or band-reject filters. This limitation has been removed in the mapping technique which is popularly known as the **bilinear transformation.** This transformation is a one-to-one mapping from the s-domain to the z-domain. That means that the bilinear transformation is a conformal mapping which transforms the $j\Omega$-axis into the unit circle in the z-plane only once, and hence avoiding aliasing in frequency components. Further, the transformation of a stable analog filter results in a stable digital filter since all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain. We can obtain the bilinear transformation by using the trapezoidal formula for numerical integration.

Let us consider that the system function of the analog filter is expressed as under :

$$H_a(s) = \frac{b}{s+a} \qquad ...(8.41)$$

We can obtain the differential equation which describes the analog filter from last equation i.e., equation (8.41) as belows :

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

or

$$sY(s) + aY(s) = bX(s) \qquad ...(8.42)$$

Taking inverse Laplace transform of both sides, we get

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \qquad ...(8.43)$$

Integrating the above equation between the limits $(nT - T)$ and nT, *i.e.*,

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}dt + a\int_{nT-T}^{nT} y(t)dt = b\int_{nT-T}^{nT} x(t)dt \qquad ...(8.44)$$

The trapezoidal rule for numeric integration is expressed as

$$\int_{nT-T}^{nT} a(t)dt = \frac{T}{2}[a(nT) + a(nT-T)] \qquad ..(8.45)$$

Using equation (8.45) in equation (8.44), we have

$$y(nT) - y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) = \frac{bT}{2}x(nT) + \frac{bT}{2}x(nT-T)$$

Taking z-transform, the system function of the digital filter will be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a} \qquad ...(8.46)$$

Now, comparing equations (8.41) and (8.46), we get

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \qquad ...(8.47)$$

The general characteristic of the mapping $z = e^{sT}$ may be obtained by putting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in equation (8.47).

Thus,
$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

or
$$s = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) \quad ...(8.48)$$

Hence, we have
$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right] \quad ...(8.49)$$

and
$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right] \quad ...(8.50)$$

From equation (8.49), it may be noted that if $r < 1$, then $\sigma < 0$, and if $r > 1$, then $\sigma > 0$. Hence, the left-half of the s-plane maps onto to points inside the unit circle in the z-plane and the transformation results in a stable digital system.

Let us consider equation (8.50), for unity magnitude ($r = 1$), σ is zero. In this case, we have
$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right) = \frac{2}{T}\left(\frac{2\sin\omega/2\cos\omega/2}{\cos^2\omega/2+\sin^2\omega/2+\cos^2\omega/2-\sin^2\omega/2}\right)$$
$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad ...(8.51)$$

or equivalently, we have
$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \quad ...(8.52)$$

Note : *Equation (8.51) provides the relationship between the frequencies in the two domains and this has been shown in figure 8.13. It can be noted that the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. But, as seen in figure 8.13, the mapping is non-linear and the lower frequencies in analog domain are expanded in the-digital domain, whereas the higher frequencies are compressed. This is due to the non-linearity of the acr tangent function and usually known as* ***frequency warping.***

No Alising in Bilinear Transformation

The main difference between impulse invariance and bilinear transformation is that there is no aliasing effect in bilinear transformation. Infact, this is the major advantage of bilinear transformation. Observe that the complete $j\Omega$ axis is mapped on the unit circle only once. But in impulse invariance the segments $\frac{(2k-1)\pi}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$ of $j\Omega$ axis are all mapped on unit circle repeatedly. Thus the transformation is many to one. Hence problem of aliasing takes place in impulse invariance method. The problem with bilinear transformation is that the frequency relationship is nonlinear.

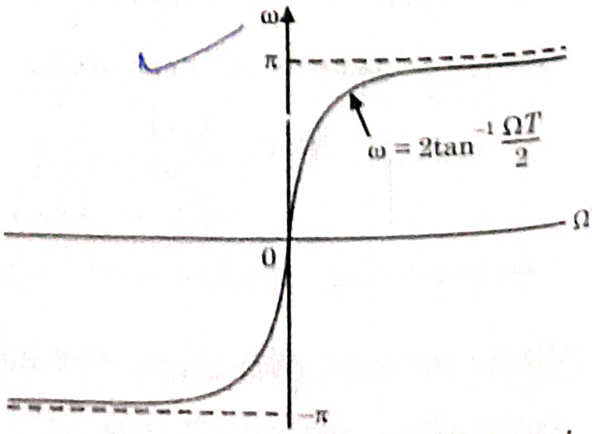

FIGURE 8.13 *Mapping between 'Ω' and ω in bilinear transformation.*

8.9.1 Advangates of Bilinear Transformation Method

(i) There is one to one transformation from the s-domain to the z-domain.

(ii) The mapping is one to one.

(iii) There is no aliasing effect.

(iv) Stable analog filter is transformed into the stable digital filter.

8.9.2 Disadvantages of Bilinear Transformation Method

(i) The mapping is non-linear and because of this, the frequency warping effect takes place.

8.9.3 Comparison between Impulse Invariance Method and Bilinear Transformation Method

S. No.	Impulse invariance method	Bilinear transformation method
1.	Poles are transferred by using the expression: $\frac{1}{s-p_i} \to \frac{1}{1-e^{p_i T} \cdot z^{-1}}$	Poles are transferred by using the expression; $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$
2.	Mapping is many to one.	Mapping is one to one.
3.	Aliasing effect is present.	Aliasing effect is not present.
4.	It is not suitable to design high-pass filter and band reject filter.	High pass filter and bandreject filter can be designed.
5.	Only poles of the system can be mapped.	Poles as well as zeros can be mapped.
6.	No frequency warping effect.	Frequency warping effect is present.

EXAMPLE 8.12 The system function of the analog filter is given as, $s = \sigma + j\Omega$

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

Obtain the system function of the digital filter using bilinear transformation which is resonant at $\omega_r = \frac{\pi}{2}$.

Solution : From the denominator of $H_a(s)$, we can write the poles of analog filter as under:

$$(s+0.1)^2 + 16 = (s+0.1-j4)\,(s+0.1+j4)$$

or

$$s = -0.1 + j4 \text{ and } s = -0.1 - j4$$

There are the two complex conjugate poles. We know that $s = \sigma + j\Omega$. Hence, the values of 'σ' and 'Ω' for these two poles will be

$$\sigma = -0.1 \text{ and } \Omega = \pm 4$$

A function is said to be resonant at its poles. Therefore, $H_a(s)$ will be resonant at,

$$s = -0.1 \pm j4$$

In other words, we can state that $H_a(s)$ will be resonant at $\Omega = 4$. It is needed that the digital filter must be resonant at $\omega_r = \frac{\pi}{2}$. This means that the bilinear transformation must map $\Omega = 4$ into $\omega_r = \frac{\pi}{2}$. We know that the relationship between 'Ω' and 'ω' is given by

$\frac{2}{T}\left(\frac{z-1}{z+1}\right)$

$H_z(s) \cdot H_a(s)\Big|_{s=}$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or

$$T = \frac{2}{\Omega}\tan\frac{\omega}{2}$$

Substituting for $\Omega = 4$ and $\omega_r = \omega = \frac{\pi}{2}$, we get

$$T = \frac{2}{4}\tan\frac{\pi}{4} = \frac{1}{2}.$$

This means that if we select $T = \frac{1}{2}$, then the resonant frequency $\Omega = 4$ of analog filter will map into $\omega_r = \frac{\pi}{2}$ of digital filter in bilinear transformation. The bilinear transformation is given by

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

At $\quad T = \frac{1}{2}$, we have, $s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

Substituting for this value of s in $H_a(s)$, we get $H(z)$ *i.e.*,

$$H(z) = H_a(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right]^2+16}$$

On simplifying the above expression, we get

$$H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.0006z^{-1}+0.975z^{-2}} = \frac{0.128z^2+0.006z-0.122}{z^2+0.0006z+0.975}$$

The roots of denominator of $H(z)$ are poles of $H(z)$. They are located at $z = -0.0003 + j0.9874208$ and $z = -0.0003 - j\,0.9874208$ converting these poles to their polar values,

$$z = 0.9874208\,e^{\pm j\,1.5711001}$$

We know that $z = re^{j\omega}$, hence, $r = 0.9874208$ and $\omega = \pm\,1.5711001 \cong \pm\frac{\pi}{2}$. Hence, the two complex conjugate poles are located at $\omega = \pm\frac{\pi}{2}$. Therefore, $H(z)$ will be resonant at $\omega = \frac{\pi}{2}$.

EXAMPLE 8.13 It is required to design a digital filter with a 7 dB bandwidth of 0.25π from an analog filter having the following system response :

$$H_a(s) = \frac{\Omega_c}{s+\Omega_c}$$

Using bilinear transformation, obtain $H(z)$.

Solution : We have $\quad \Omega_c = \frac{2}{7}\tan\frac{\omega_r}{2} = \frac{2}{7}\tan 0.125\pi = 0.828/T$

The system response of the digital filter is given as

$$H(z) = H(s)\Big|_{s=\frac{2(z-1)}{T(z+1)}} = \frac{\Omega_c}{\frac{2}{T}\frac{(z-1)}{(z+1)} + \Omega_c}$$

or

$$H(z) = \frac{\frac{0.828}{T}}{\frac{2}{T}\frac{(z-1)}{(z+1)} + \frac{0.828}{T}} = \frac{0.828(z+1)}{2(z-1)+0.828(z+1)}$$

Simplifying furhter, we get

$$H(z) = \frac{1+z^{-1}}{3.414 - 1.414z^{-1}} \quad \textbf{Ans.}$$

EXAMPLE 8.14 **Make use of bilinear transformation to obtain $H(z)$ if it is given that**

$$H_a(s) = \frac{1}{(s+1)^2} \quad \text{and } T = 0.1\text{s}.$$

Solution : For the bilinear transformation, we have

$$H(z) = H_a(s)\Big|_{s=\frac{2(z-1)}{T(z+1)}} = \frac{1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)} + 1\right]^2}$$

Putting $T = 0.1$ s in last equation, we have

$$H(z) = \frac{1}{\left[20\frac{(z-1)}{(z+1)} + 1\right]^2} = \frac{(z+1)^2}{(21z-19)^2}$$

Simplifying, we obtain

$$H(z) = \frac{0.0476(1+z^{-1})^2}{(1-0.9048z^{-1})^2} \quad \textbf{Ans.}$$

EXAMPLE 8.15 **An analog filter has the following transfer function $H_a(s) = \frac{1}{s+1}$. Using bilinear transformation technique, determine the transfer function of digital filter $H(z)$ and also write the difference equation of digital filter.**

Solution: The given transfer function is,

$$H_a(s) = \frac{1}{s+1} \quad ...(i)$$

In bilinear transformation, $H(z)$ is obtained by substituting,

$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

Here, T is the sumpling time; which is not given. Hence, let us assume $T = 1$ sec.

Therefore,

$$s = 2\left[\frac{z-1}{z+1}\right] \quad ...(ii)$$

Substituting this value in equation (*i*), we obtain

$$H(z) = \frac{1}{1+2\left(\frac{z-1}{z+1}\right)} = \frac{1}{\frac{(z+1)+2(z-1)}{z+1}} = \frac{z+1}{z+1+2z-2} = \frac{z+1}{3z-1} \quad ...(iii)$$

EXAMPLE 8.19 **The analog transfer function of low pass filter (LPF) is, $H_a(s) = \dfrac{1}{s+2}$ and its bandwidth is 1 rad/sec.**

Design the digital filter using BLT method whose cut-off frequency is 20 π and sampling time is 0.0167 sec by considering the warping effect. *(U.P. Tech. Tutorial Question Bank)*

Solution: Given analog transfer function is

$$H(s) = \frac{1}{s+2} \qquad s \to \frac{s}{\Omega_p^*} \qquad ...(i)$$

For the prewarping procedure, we have

$$\Omega_p^* = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right) = \frac{2}{0.0167}\tan\left(\frac{20\pi\times 0.0167}{2}\right) \text{Here } \omega = 20\pi \text{ (given)}$$

or $$\Omega_p^* = 69.31$$

Now, the value of $H^*_a(s)$ is obtained by putting $s = \dfrac{s}{\Omega_p^*}$ in equation (*i*) *i.e.*,

$$H^*_a(s) = \frac{1}{\dfrac{s}{\Omega_p^*}+2} = \frac{1}{\dfrac{s}{69.31}+2} = \frac{69.31}{s+138.62} \qquad ...(ii)$$

Now, $H(z)$ is obtained by substituting $s = \dfrac{2}{T}\left(\dfrac{z-1}{z+1}\right)$ in equation (*ii*) *i.e.*,

$$H(z) = \frac{69.31}{\dfrac{2}{0.0167}\left(\dfrac{z-1}{z+1}\right)+138.62} = \frac{69.31}{119.76\left(\dfrac{z-1}{z+1}\right)+138.62}$$

or $$H(z) = \frac{69.31(z+1)}{119.76z-119.76+138.62z+138.62}$$

or $$H(z) = \frac{69.31(z+1)}{258.38z+18.86} = \frac{z+1}{3.73z+0.27}$$

This is the required transfer function for digital filter. **Ans.**

8.10 BASIC ANALOG FILTER APPROXIMATIONS

As discussed earlier that the digital IIR filters are designed from the analog filters. Many times, it is necessary to approximate the characteristics of analog filter. This approximation is required because the practical characteristic of a filter is not identical to the ideal characteristics. There are three different types of approximation techniques as under :

(*i*) Butterworth filter approximation.
(*ii*) Chebyshev filter approximation.
(*iii*) Elliptic filter approximation.

8.11 BUTTERWORTH FILTER APPROXIMATION

A typical characteristic of a butterworth low-pass filter (LPF) is as shown in figure 8.14.

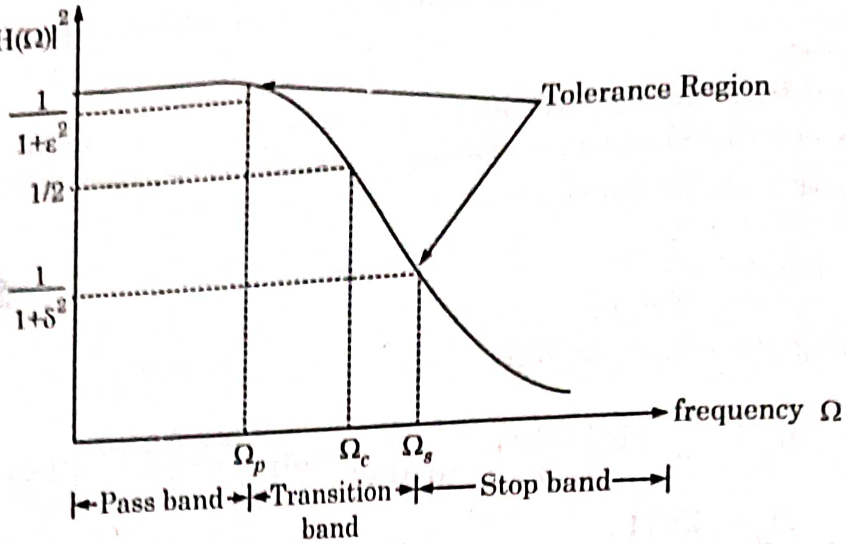

FIGURE 8.14 *Typical characteristics of analog low-pass filter (LPF).*

This type of response is called as Butterworth response because its main characteristic is that the passband is maximally flat. This means that there are no variations (ripples) in the passband.

Now, the magnitude squared response of low pass Butterworth filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad ...(8.53)$$

This equation may also be expressed as,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \quad ...(8.54)$$

Here, $|H(\Omega)|$ = Magnitude of analog low pass filter (LPF).

Ω_c = Cut-off frequency (– 3 dB frequency).

Ω_p = Passband edge frequency.

$1+\epsilon^2$ = Passband edge value.

$1+\delta^2$ = Stopband edge value.

ϵ = Parameter related to ripples in passband.

δ = Parameter related to ripples in stopband.

N = Order of the filter.

We know that, in case of low pass filter (LPF), the frequencies will pass upto the value of cut-off frequency (Ω_c). This is called as passband. After that the frequencies are attenuated. This is called as stop band. Ideal characteristic is shown by dotted line in figure 8.14. Ideally, at the value of cut-off frequency (Ω_c) the frequencies should be stopped. However, in practical cases this is not happening.

Now the order of filter is denoted by N. Roughly we can say order of filter means, the number of stages used in the design of analog filter. As the order of filter N increases, the response of filter is more close to the ideal response as shown in figure 8.15.

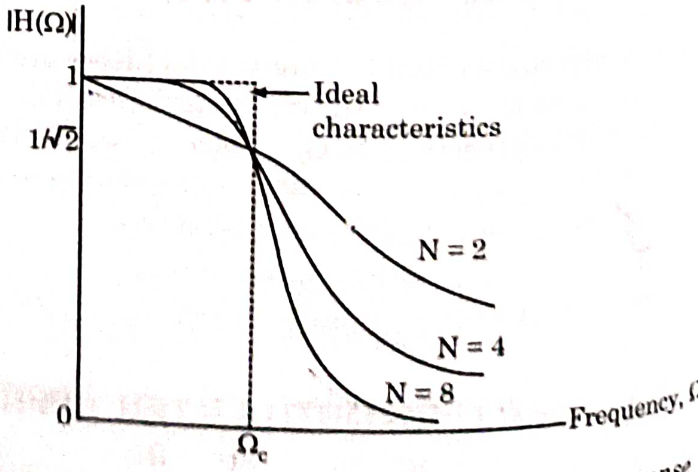

FIGURE 8.15 *Effect of N on frequency response characteristics.*

8.11.1 Salient Features of Low Pass Butterworth Filter

(i) The magnitude response is nearly constant (equal to 1) at lower frequencies. This means that the passband is maximally flat.

(ii) There are no ripples in the pass band and stopband.

(iii) The maximum gain occurs at $\Omega = 0$ and it is $|H(0)| = 1$.

(iv) The magnitude response is monotonically decreasing.

8.11.2 Designing Expressions and Designing Steps

Let A_p = Attenuation in passband.

A_s = Attenuation in stopband.

Ω_p = Pass band edge frequency.

Ω_c = Cut-off frequency

Ω_s = Stopband edge frequency.

In numerical problems, the specifications of required digital filter is usually given and it is asked to design a particular discrete time Butterworth filter. Then the following steps must be used :

1. From the given specifications of digital filter, we obtain equivalent analog filter as under :

(a) For *impulse invariance method*, we have

$$\Omega = \frac{\omega}{T}$$

(b) For *bilinear transformation method*, we have

$$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$

Here, Ω = Frequency of analog filter

ω = Frequency of digital filter

T = Sampling time

2. We evaluate the order N of filter using the following exprression:

$$N = \frac{1}{2} \times \frac{\log\left[\dfrac{\left(\dfrac{1}{A_s^2} - 1\right)}{\left(\dfrac{1}{A_p^2} - 1\right)}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\log\left[\left(1/\delta_s^2\right) - 1\right]}{2\log(\Omega_s/\Omega_c)}$$

Here, δ_s = Attenuation in stop band.

If the specifications are given in decibels (dB) then, we make use of the following expression:

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1A_s(dB)} - 1}{10^{0.1A_p(dB)} - 1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

3. After that we determine cut-off frequency (Ω_c).

The cut-off frequency (Ω_c) of analog filter is calculated as under :

(*a*) For *impulse invariance method*, we have

$$\Omega_c = \frac{\omega_c}{T}$$

(*b*) For *bilinear transformation method*, we have

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2}$$

In case, ω_c is not given then we the following expression:

(*i*) $$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{1/2N}}$$ and if specifications are in dB then

(*ii*) $$\Omega_c = \frac{\Omega_p}{\left[10^{0.1A_p}-1\right]^{1/2N}}$$

4. Next, we determine the poles using the following expression :

$$p_i = \pm\Omega_c\, e^{j(N+2i+1)\pi/2N},\quad i = 0, 1, 2, \ldots N-1.$$

If the poles are complex conjugate then organize the poles (p_i) as complex conjugate pairs that means,

s_1 and s_2^*, s_2 and s_2^* etc.

5. Next, we calculate the system transfer function of analog filter using following expression:

$$H_a(s) = \frac{\Omega_c^N}{(s-p_1)(s-p_2)\ldots}$$

and if poles are complex conjugate then, we have

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)}$$

6. Lastly, we design the digital filter using impulse invariance method or bilinear transformation method.

EXAMPLE 8.20 A digital filter has following frequency specification:

Passband frequency = ω_p = 0.2 π

Stopband frequency = ω_s = 0.3 π

What are the corresponding specifications for passband and stopband frequencies in analog domain if,

(*i*) Impulse invariance technique is used for designing.

(*ii*) Bilinear transformation is used for designing.

Solution: Here, let us assume sampling time $T = 1$ sec

(*i*) For impulse invariance method, we have

$$\Omega_p = \frac{\omega_p}{T} = 0.2\,\pi = 0.63 \text{ rad/sec.}$$

and $$\Omega_s = \frac{\omega_s}{T} = \frac{0.3\pi}{T} = 0.3\pi = 0.94 \text{ rad/sec.}$$

(*ii*) For bilinear transformation, we have

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$

$$= 2\tan\left(\frac{0.2\pi}{2}\right) = 2\tan\left(\frac{0.2\times180}{2}\right) = 0.65 \text{ red/sec.}$$

and
$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$

or
$$\omega_s = 2\tan\left(\frac{0.3\pi}{2}\right) = 2\tan\left(\frac{0.3\times180}{2}\right) = 1.019 \text{ rad/sec.} \qquad \textbf{Ans.}$$

EXAMPLE 8.21 Design a second order discrete-time Butterworth filter with cut-off frequency of 1 kHz and sampling frequency of 10^4 samples/sec by bilinear transformation.

Solution: In this problem, the specifications of digital filter are not given directly. So, first we have to obtain required design specifications for digital filter. Then, for Butterworth approximation, we have to convert the specifications into specifications of equivalent analog filter. Finally, using bilinear transformation we have to obtain $H(z)$.

Given that

Order of filter, $N = 2$

Cut-off frequency of analog filter, $F_c = 1$ kHz = 1000 Hz.

Sampling frequency, $F_s = 10^4$ samples/sec = 10,000 Hz.

First, let us determine the required design specifications of *digital filter*.

We have the equation to convert continuous frequency (F) into discrete frequency (f). It is,

$$\boxed{f = \frac{F}{F_s}}$$

Thus,
$$f_c = \frac{F_c}{F_s} = \frac{1000}{10,000} = 0.1 \text{ cycles/sample}$$

Now, the angular frequency (frequency of digital filter) is

$$\omega_c = 2\pi f_c = 2\pi \times 0.1 = 0.2\pi \text{ radians/sample}$$

Now, let us determine the specifications of analog filter for *Butterworth approximation*.

For bilinear transformation, we have

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Thus,
$$\boxed{\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2}}$$

Here T = sampling time = $\dfrac{1}{F_s} = \dfrac{1}{10,000}$

or
$$\Omega_c = (2\times 10,000)\tan\left(\frac{0.2\pi}{2}\right) = 6498.39 \text{ radians/sec.}$$

The value of N is given so it is not necessary to calculate it. Thus, the specifications of analog filter are as under:

(i) Cut-off frequency = Ω_c = 6498.39 radian/sec.

(ii) Order of filter = N = 2.

Let us Calculate the poles using,

$$p_i = \Omega_c\, e^{j(N+2i+1)\pi/2N} \quad \ldots\; i = 0, 1, \ldots N-1$$

Here $N = 2$, Thus, $i = 0$ to $N - 1$ means $i = 0$ and $i = 1$.

(i) For $i = 0$, we have

$$p_0 = \pm\,\Omega_c\, e^{j(N+1)\pi/2N} = \pm\, 6498.39\, e^{j3\pi/4} = \pm\, 6498.39\left[\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= \pm\, 6498.39\,[-0.707 + j\,0.707]$$

or $\quad p_0 = 6498.39\,[-0.707 + j\,0.707]$ and $-6498.39\,[-0.707 + j\,0.707]$

or $\quad p_0 = -4595.05 + j\,4595.05$ and $4595.05 - j\,4595.05$

(ii) For $i = 1$, we have

$$p_1 = \pm\Omega_c\, e^{j(N+2+1)\pi/2N}$$

$$= \pm 6498.39\, e^{j(5\pi/4)}$$

or $$p_1 = \pm 6498.39\left[\cos\left(\frac{5\pi}{4}\right) + j\sin\left(\frac{5\pi}{4}\right)\right]$$

$$= \pm 6498.39\,[-0.707 - j\,0.707]$$

or $\quad p_1 = 6498.39\,[-0.707 - j\,0.707]$

and $-6498.39\,[-0.707 - j\,0.707]$

or $\quad p_1 = -4595.05 - j\,4595.05$

and $4595.05 + j\,4595.05$

Now, let us plot these poles as shown in figure 8.16.

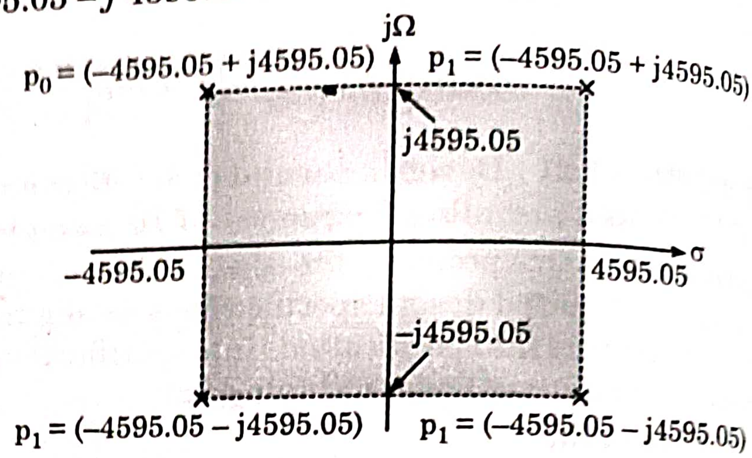

FIGURE 8.16 *Plot of poles, p_i*

> **Note :** *We know that analog filter is stable if the poles lie on the L.H.S. of s-plane. So for the filter to be stable, we have to select the poles which are on the L.H.S. of s-plane. Observe figure 8.16. We shall denote these poles by s_1 and s_1^*; because these poles are complex conjugate of each other.*

Thus, $\quad s_1 = -4595.05 + j\,4595.05$

and $\quad s_1^* = -4595.05 - j\,4595.05$

Now, the transfer function of analog filter is obtained by using the following expression:

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)} = \frac{(6498.39)^2}{(s+4595.05-j\,4595.05)(s+4595.05+j\,4595.05)}$$

Simplifying, we get

$$H_a(s) = \frac{(6498.39)^2}{s^2 + 9190.1s + 42.22\times10^6}$$

Lastly, let us design digital filter using bilinear transformation.

In case of bilinear transformation, $H(z)$ is obtained by putting $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$ in equation of $H_a(s)$

Here, $\quad T = \frac{1}{10{,}000}$.

Thus, we have to put,

$$s = 2\times10^{+4}\left[\frac{z-1}{z+1}\right] \text{ in the equation of } H_a(s)$$

Therefore,

$$H(z) = \frac{(6498.39)^2}{\left[2\times10^4\left(\frac{z-1}{z+1}\right)\right]^2 + 183.802\times10^6\left(\frac{z-1}{z+1}\right) + 42.22\times10^6}$$

This is the required transfer function for digital filter. **Ans.**

Thus specifications of analog filter are :

$$A_p = 1 \text{ dB}, \qquad \Omega_p = 0.65$$

and

$$A_s = 15 \text{ dB}, \qquad \Omega_s = 1.02$$

(*ii*) Next, let us determine the order of filter N.

Since, specifications are given in dB, therefore, we have to use the following expression:

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1 A_s\, dB}-1}{10^{0.1 A_p\, dB}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{1}{2}\frac{\log\left[\dfrac{10^{(0.1\times15)}-1}{10^{(0.1\times1)}-1}\right]}{\log\left(\dfrac{1.02}{0.65}\right)}$$

or

$$N = \frac{1}{2} \times \frac{2.073}{0.1957} = 5.296$$

Thus, the order of filter is $N = 6$. **Ans.**

8.12 CHEBYSHEV FILTER APPROXIMATION

Using Chebyshev filter design, there are two subgroups as under:

(*i*) Type-1 Chebyshev filter

(*ii*) Type-2 Chebyshev filter

8.12.1 Type-1 Chebyshev Filter

These filters are all pole filters. In the passband, these filters show equiripple behaviour and they have monotonic characteristics in the stopband.

The filter characteristics for odd and even values of N are shown in figure 8.20.

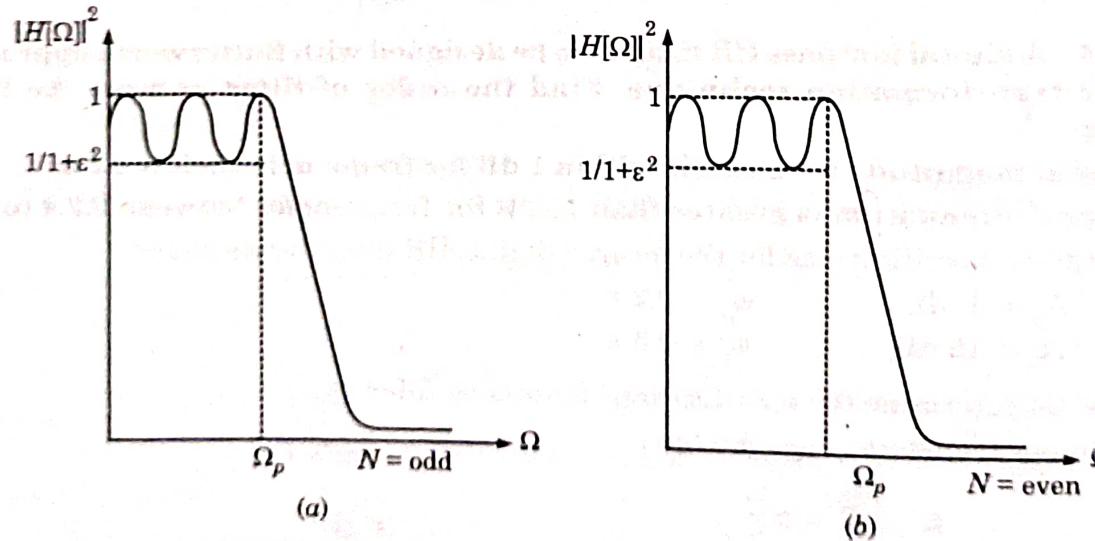

FIGURE 8.20 *Type-1 Chebyshev filter characteristics.*

The magnitude squared frequency response is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega/\Omega_p)} \qquad ...(8.55)$$

Here, ϵ = Ripple parameter in the passband

Ω_p = Passband frequency

$C_N(x)$ = Chebyshev polynomial of order N.

The chebyshev polynomials are determined by using the following expression:

$$C_{N+1}(x) = 2xC_N(x) - C_{N-1}(x) \qquad \text{...(8.56)}$$

with

$$C_0(x) = 1 \text{ and } C_1(x) = x$$

The different polynomials are given in Table 8.1.

Table 8.1. Different Polynomials

S. No.	N	$C_N(x)$
1	0	1
2	1	x
3	2	$2x^2 - 1$
4	3	$4x^3 - 3x$
5	4	$8x^4 - 8x^2 + 1$
6	5	$16x^5 - 20x^3 + 5x$

The major difference between Butterworth and Chebyshev filter is that the poles of Butterworth filter lie on the circle, whereas the poles of Chebsyshev filter lie on ellipse.

8.12.2 Type-II Chebyshev Filter

This filter consists of zeros as well as poles. The characteristic of such filters for even and odd values of N is shown in figure 8.21.

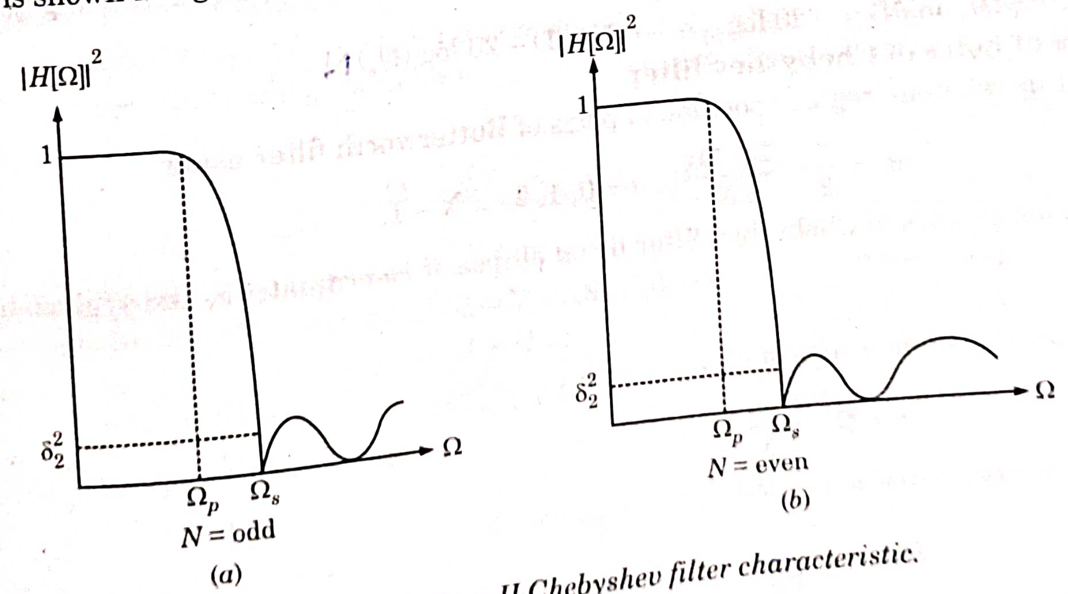

FIGURE 8.21 *Type-II Chebyshev filter characteristic.*

The magnitude squared response is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left[C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)\Big/C_N^2\left(\frac{\Omega_s}{\Omega}\right)\right]}$$

Here $\quad C_N(x) = N^{\text{th}}$ order polynomial

Ω_s = Stopband frequency

Ω_p = Passband frequency

8.12.3 Chebyshev Low Pass Filter Design

(i) Frequency response

The magnitude squared frequency response of chebyshev filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

Here, $C_N\left(\frac{\Omega}{\Omega_p}\right)$ = Chebyshev polynomial of order M.

(ii) Parameter ϵ

It represents ripple parameter in the passband. It is given by

$$\epsilon = [10^{0.1 A_p \, dB} - 1]^{1/2}$$

If A_p is not in dB then ϵ is calculated using the following expression:

$$\epsilon = \left[\frac{1}{A_p^2} - 1\right]^{1/2}$$

(iii) At cut-off frequency $\Omega_c = 1$, the magnitude is given by,

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\epsilon^2}}$$

(iv) Order of a filter

When magnitude is expressed in dB, then order (N) of a filter is obtained by using the following expression:

$$|H(j\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 6(N-1) - 20\log(\Omega_s) N$$

(v) Position of poles of Chebyshev filter

First, let us calculate regular position of poles of Butterworth filter using

$$\theta_i = \frac{\pi}{2} + \frac{(2i+1)\pi}{2N}, \quad i = 0, 1, 2, \ldots N-1$$

The position of poles of Chebyshev filter lie on ellipse at co-ordinates x_i and y_i given by,

$$x_i = r\cos\theta_i, \qquad i = 0, 1, 2, \ldots N-1$$

$$y_i = R\sin\theta_i, \qquad i = 0, 1, 2, \ldots N-1$$

Here r represents minor axis of ellipse and is given by,

$$r = \Omega_p \frac{\beta^2 - 1}{2\beta}$$

and R represents major axis of ellipse and is given by,

$$R = \Omega_p \frac{\beta^2 + 1}{2\beta}$$

Here, the parameter β is given by,

$$\beta = \left[\frac{\sqrt{1+\epsilon^2} + 1}{\epsilon}\right]^{\frac{1}{N}}$$

Then, the pole positions are denoted by s_p, and

$$s_p = r\cos\theta_i + jR\sin\theta_i$$

(ii) **System transfer function**

The system transfer function of analog filter is given by,

$$H_a(s) = \frac{i}{(s-s_0)(s-s_1)(s-s_2)\ldots}$$

After simplification this equation can be written as,

$$H_a(s) = \frac{i}{s^N + b_{N-1}s^{N-1} + \ldots + b_0}$$

Here, b_0 = Constant term in the denominator

Now, the value of i is calculated as under:

$$i = \begin{cases} b_0 & \text{for } N \text{ odd} \\ \dfrac{b_0}{\sqrt{1+\epsilon^2}} & \text{for } N \text{ even} \end{cases}$$ imp

EXAMPLE. 8.25 Design a low-pass 1 rad/sec bandwidth Chebyshev filter with following characteristics.

(i) Acceptable passband ripple of 2 dB.

(ii) Cut-off radian frequency of 1 rad/sec.

(iii) Stopband attenuation of 20 dB or greater beyond 1.3 rad/sec.

Solution: Given data

Given cut-off frequency is 1 rad/sec. This means that it is normalized low pass chebyshev filter.

$$\text{Passband ripple} = A_p = 2 \text{ dB}$$

$$\text{Stopband attenuation} = A_s = 20 \text{ dB}$$

$$\text{Stopband frequency} = \Omega_s = 1.3 \text{ rad/sec.}$$

First, let us determine the parameter ϵ.

We have,

$$\epsilon = \left[10^{0.1A_p dB} - 1\right]^{\frac{1}{2}} = \left[10^{0.1(2)} - 1\right]^{\frac{1}{2}} = 0.765$$

Next, we determine the order N of filter.

Given stopband attenuation = 20 dB for Ω_s = 1.3 rad/sec.

Since, it is altenuation, therefore, we can write

$$\boxed{|H(j\Omega)| = -20 \text{ dB for } \Omega = \Omega_s = 1.3 \text{ rad/sec.}}$$

The order N of filter is calculated by using the following expression:

$$|H(j\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 6(N-1) - 20N\log_{10}\Omega_s$$

or $$-20 = -20\log_{10}(0.765) - 6(N-1) - 20N\log_{10}(1.3)$$

or $$6(N-1) + 20N\log_{10}1.3 = 20 - 20\log_{10}(0.765)$$

$$6N - 6 + 2.28N = 22.327$$

or $$N = 3.421$$

Since, N is order of filter, we can write

$$N \cong 4$$

Now, we shall determine the poles.

The pole positions are given by,

$$\boxed{s_p = r\cos\theta_i + iR\sin\theta_i} \quad \ldots(i)$$

First, we will calculate parameter β, *i.e.*

$$\beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{\frac{1}{N}} = \left[\frac{\sqrt{1+(0.765)^2}+1}{0.765}\right]^{\frac{1}{4}} = 1.31 \quad ...(ii)$$

Now, let us calculate values or r and R.
This is normalized filter. So, $\Omega = 1$ rad/sec.

We have minor axis of ellipse $= r = \Omega_p \cdot \frac{\beta^2 - 1}{2\beta}$

or $$r = 1 \cdot \frac{(1.31)^2 - 1}{2(1.31)} = 0.273 \quad ...(iii)$$

The major axis of an ellipse is given by,

$$R = \Omega_p \frac{\beta^2 + 1}{2\beta} = 1. \frac{(1.31)^2 + 1}{2(1.31)} = 1.04 \quad ...(iv)$$

Now, we shall calculate values of θ_i as under :

$$\theta_i = \frac{\pi}{2} + \frac{(2i+1)\pi}{2N}, \quad i = 0, 1, 2, 3.$$

For $i = 0$: $\theta_0 = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}$

For $i = 1$: $\theta_1 = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}$

For $i = 2$: $\theta_2 = \frac{\pi}{2} + \frac{5\pi}{8} = \frac{9\pi}{8}$

For $i = 3$: $\theta_3 = \frac{\pi}{2} + \frac{7\pi}{8} = \frac{11\pi}{8}$

Using equation (*i*), we can calculate pole positions as under:

For $i = 0$: $s_0 = r\cos\theta_0 + jR\sin\theta_0$

or $$s_0 = 0.273\cos\left(\frac{5\pi}{8}\right) + j(1.04)\sin\left(\frac{5\pi}{8}\right) = -0.1 + j0.96$$

For $i = 1$: $s_1 = r\cos\theta_1 + jR\sin\theta_1$

or $$s_1 = 0.273\cos\left(\frac{7\pi}{8}\right) + j(1.04)\sin\left(\frac{7\pi}{8}\right) = -0.25 + j0.4$$

For $i = 2$: $s_2 = r\cos\theta_2 + jR\sin\theta_2$

or $$s_2 = 0.273\cos\left(\frac{9\pi}{8}\right) + j(1.04)\sin\left(\frac{9\pi}{8}\right) = -0.25 - j04$$

For $i = 3$: $s_3 = r\cos\theta_3 + jR\sin\theta_3$

or $$s_3 = 0.273\cos\left(\frac{11\pi}{8}\right) + j(1.04)\sin\left(\frac{11\pi}{8}\right) = -0.1 - j0.96$$

Note : *Observe that the poles are complex conjugate of each other.*

Lastly, we determine the system function $H_a(s)$.

The system function $H_a(s)$ is given by,

$$H_a(s) = \frac{i}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)}$$

or
$$H_a(s) = \frac{i}{(s+0.1-j0.96)(s+0.25-j0.4)(s+0.25+j0.4)(s+0.1+j0.96)}$$

or
$$H_a(s) = \frac{i}{(s^2+0.2s+0.9316)(s^2+0.5s+0.2225)} = \frac{i}{s^4+0.7s^3+1.25s^2+0.51s+0.207}$$

b_0

Here $N = 4$ which is even. Thus, we have

$$i = \frac{b_0}{\sqrt{1+\epsilon^2}}$$

$$b_0 = \text{constant term in the denominator} = 0.207.$$

or
$$i = \frac{0.207}{\sqrt{1+(0.765)^2}} = 0.16$$

Therefore, we have

$$H_a(s) = \frac{0.16}{s^4+0.7s^3+1.25s^2+0.51s+0.207}$$

This is required transfer function. **Ans.**

EXAMPLE. 8.26 Design a Chebyshev analog filter with maximum passband attenuation of 2.5 dB at Ω_p = 20 rad/sec and stopband attenuation of 30 dB at Ω_s = 50 rad/sec.

Solution: Given data :

$$\text{Passband attenuation} = A_p = 2.5 \text{ dB}$$
$$\text{Passband frequency} = \Omega_p = 20 \text{ rad/sec}$$
$$\text{Stopband attenuation} = A_s = 30 \text{ dB}$$
$$\text{Stopband frequency} = \Omega_s = 50 \text{ rad/sec}$$

First, let us determine the parameter ϵ.

We have,

$$\epsilon = \left[10^{0.1 A_p dB} - 1\right]^{\frac{1}{2}} = \left[10^{0.1(2.5)} - 1\right]^{\frac{1}{2}} = 0.882.$$

Next, we determine the order N of the filter.

Given stopband attenuation = 30 dB for Ω_s = 50 rad/sec

Thus, $\quad |H(j\Omega)| = -30$ dB for $\Omega = \Omega_s = 50$ rad/sec

The order N of filter is calculated using the following expression:

$$|H(j\Omega)| \text{ in dB} = -20\log_{10}\epsilon - 6(N-1) - 20N\log_{10}\Omega_s$$

or
$$-30 = -20\log_{10}(0.882) - 6(N-1) - 20N\log_{10}(50)$$

or
$$-30 = 1.09 - 6(N-1) - 33.98N$$

or
$$-30 = 1.09 - 6N + 6 - 33.98N$$

or
$$39.98N = 1.09 + 30 + 6$$

or
$$N = 0.95$$

Thus, order of filter $= N \equiv 1$.

Here δ is related to δ_1 and δ_2 as follows :

$$\delta_2 = \sqrt{1+\delta^2} \quad \text{...(8.58)}$$

where δ_2 = stopband ripple

and $\delta_2 = 10 \log_{10}(1+\epsilon^2)$

where δ_1 = stepband ripple

The function $K(x)$ is called as elliptic integral of first kind.

8.13.1 The Function K(x) is Called as Elliptic Integral of First Kind

(i) Elliptic filter is more efficient because it provides the smallest order filter for given set of specifications.

(ii) It has smallest transition bandwidth.

8.13.2 Disadvantage of Elliptic Filters

The phase response is more non-linear in passband.

8.14 FREQUENCY TRANSFORMATIONS

Until now we have studied the design of low pass filter only. Now if it is asked to design other filter like high pass, bandpass or bandreject filter then we have to use the frequency transformation.

If the cut-off frequency of LPF is equal to 1 that means if $\Omega_c = 1$ then, it is called as normalized filter. To design the other types of filters; first the system function of normalized LPF is obtained. Then using frequency transformation we can get the system function of the required filter. The following formulae are used for the frequency transformation.

Let, we have a normalized LPF having cut-off frequency Ω_c.

8.14.1 Low Pass to Low Pass

Suppose it is asked to design another LPF with new passband edge frequency Ω_{LP}. Then use the transformation,

$$s \rightarrow \frac{\Omega_c}{\Omega_{LP}} s$$

That means replace 's' by $\frac{\Omega_c}{\Omega_{LP}} s$ in the given equation of $H(s)$.

8.14.2 Low Pass to High Pass

Suppose we have to design HPF with cutoff frequency Ω_{HP} then use,

$$s \rightarrow \frac{\Omega_c \Omega_{HP}}{s}$$

8.14.3 Low Pass to Bandpass

Suppose we have to design bandpass filter with higher cutoff frequency Ω_u and lower cut-off frequency Ω_l then use the transformation.

$$s \rightarrow \Omega_c \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

8.14.4 Lowpass to Bandstop or Notch or Bandreject

Suppose we have to design notch filter with higher cut-off frequency Ω_c and lower cutoff frequency then use the transformation,

$$s \rightarrow \Omega_c \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

The following Table 8.2 gives the summary of frequency transformation :

Table 8.2

S. No.	Conversion type	Transformation
1.	Low pass	$s \rightarrow \frac{\Omega_c}{\Omega_{LP}} s$
2.	High pass	$s \rightarrow \frac{\Omega_c \Omega_{HP}}{s}$
3.	Bandpass	$s \rightarrow \Omega_c \times \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$
4.	Bandpass	$s \rightarrow \times \frac{s(\Omega_u . \Omega_l)}{s^2 + \Omega_u . \Omega_l}$

Note : *The formulae are applicable for analog frequency transformation. Similarly, we can transform the LPF in digital domain by using the formulae for digital frequency transformation.*

EXAMPLE. 8.28 Design second order bandpass digital Butterworth filter with passband of 300 Hz to 500 Hz and sampling frequency of 1500 Hz using bilinear transformation method; given prototype LPF with transfer function $H_a(s) = \frac{\Omega_c}{s + \Omega_c}$.

Solution: First, let us determine the specifications for digital filter.

First, we will calculate the required specification of digital filter. That means specifications in terms of ω.

The given specification are

Higher cut-off frequency = F_u = 500 Hz

Lower cut-off frequency = F_l = 300 Hz

We want the specifications in terms of ω_u and ω_l. For that, first we will calculate values of f_u and f_l by using sampling frequency F_s = 1500 Hz. We have,

$$f_u = \frac{F_u}{F_s} + \frac{500}{1500} = 0.33$$

Now, $\omega_u = 2\pi f_u$

Therefore, $\omega_u = 2\pi \times 0.33 = 2.073$ radians/sample

Similarly, $f_l = \frac{F_l}{F_s} = \frac{300}{1500} = 0.2$

Hence, $\omega_l = 2\pi f_l = 2\pi \times 0.2 = 1.257$ radians/sample

Thus specifications of digital filter are :

(i) Higher cut-off frequency = ω_u = 2.073 radians/sample

(ii) Lower cut-off frequency = ω_l = 1.257 radians/sample

(iii) Order of filter = N = 2.

Next, let us determine the specifications for analog filter.

To use the Butterworth approximation; we will calculate the specifications of equivalent analog filter using BLT method. The frequency relationship for impulse invariance method is given by,

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Here, T is the sampling time.

$$\therefore \qquad \Omega_u = \frac{2}{T}\tan\frac{\omega_u}{2} = (2 \times 1500)\tan\frac{2.073}{2} = 5070.11$$

and

$$\Omega_l = \frac{2}{T}\tan\frac{\omega_l}{2}$$

Hence

$$\Omega_l = (2 \times 1500)\tan\frac{1.257}{2} = 2180.46$$

Now, we determine the system transfer function for analog BPF.

The given transfer function of LPF is,

$$H_a(s) = \frac{\Omega_c}{s+\Omega_c} \qquad ...(i)$$

for the normalized filter $\Omega_c = 1$.

Hence,

$$H_a(s) = \frac{1}{s+1} \qquad ...(ii)$$

Now, $H(s)$ for BPF is obtained by using the transformation equation,

$$s \to \Omega_c \frac{s^2+\Omega_l\Omega_u}{s(\Omega_c-\Omega_l)}$$

Substituting $\Omega_c = 1$ and substituting the values of Ω_c and Ω_l, we get

$$s \to 1 \times \frac{s^2+(2180.46 \times 5070.11)}{s(5070.11-2180.46)}$$

Hence

$$s \to \frac{s^2+11.055\times10^6}{2889.65\,s}$$

Substituting this value of s in equation (ii), we obtain

$$H_a(s) = \frac{1}{\dfrac{s^2+11.055\times10^6}{2889.65\,s}} = \frac{288.65\,s}{s^2+2889.65\,s+11.055\times10^6}$$

This is the transfer function for analog BPF.

Lastly, we obtain $H(z)$ for digital filter.

$H(z)$ can be obtained by putting $s = \dfrac{2}{T_s}\left[\dfrac{z-1}{z+1}\right]$ in the equation of $H_a(s)$.

$$H(z) = \frac{2889.65\times2\times1500\left(\dfrac{z-1}{z+1}\right)}{(2\times1500)^2\left(\dfrac{z-1}{z+1}\right)^2+2889.65\times2\times1500\left(\dfrac{z-1}{z+1}\right)+11.055\times10^6}$$

or

$$H(z) = \frac{8.67\times10^6\left(\dfrac{z-1}{z+1}\right)}{9\times10^6\left(\dfrac{z-1}{z+1}\right)^2+8.67\times10^6\left(\dfrac{z-1}{z+1}\right)+11.055\times10^6}$$

or
$$H(z) = \frac{8.67\left(\frac{z-1}{z+1}\right)}{9\left(\frac{z-1}{z+1}\right)^2 + 8.67\left(\frac{z-1}{z+1}\right) + 11.055} = \frac{8.67\left(\frac{z-1}{z+1}\right) \times (z+1)^2}{9(z-1)^2 + 8.67(z-1)(z+1) + 11.055(z+1)^2}$$

or
$$H(z) = \frac{8.67(z-1)(z+1)}{9(z-1)^2 + 8.67(z^2-1) + 11.055(z+1)^2}$$

EXAMPLE 8.29 Design the digital high pass filter for cut-off frequency of 30 Hz and sampling frequency of 150 Hz using BLT.

Solution: First, let us obtain the specifications for digital filter.

We have
$$f_{HP} = \frac{F_{HP}}{F_s}$$

where F_{HP} = Given cut-off frequency = 30 Hz

F_s = Sampling frequency = 150 Hz

Hence,
$$f_{HP} = \frac{30}{150} = 0.2 \text{ cycles/samples.}$$

Now
$$\omega_{HP} = 2\pi f_{HP} = 2\pi \times 0.2 = 0.4\pi$$

The order of filter is not given; so we can assume order of $N = 1$.

Next, let us obtain the specifications for analog filter.

(i) Determination of Ω_{HP}

For the bilinear transformation we have,
$$\Omega = \frac{2}{T_s}\tan\frac{\omega}{2}$$

Therefore,
$$\Omega_{HP} = \frac{2}{\frac{1}{150}}\tan\frac{\omega_{HP}}{2} = 300\tan\frac{0.4\pi}{2} = 217.96 \text{ radians/sec.}$$

and we have assumed $N = 1$.

(ii) Determination of pole positions

We have,
$$p_i = \pm\Omega_c\, e^{j(N+2i+1)\pi/2N}, \quad i = 0, 1 \ldots N-1$$

We have to use normalized low pass filter to design the required highpass filter. For normalized low pass filter.
$$\Omega_c = 1$$

Hence,
$$p_i = \pm e^{j(1+2i+1)\pi/2N}, \quad i = 0 \text{ to } N-1$$

The range of i is 0 to $N-1$ and $N = 1$; thus we get $i = 0$.

Therefore,
$$p_0 = \pm e^{j\frac{(2)\pi}{2N}} = \pm e^{\frac{j2\pi}{2}} = \pm e^{j\pi}$$

or
$$p_0 = \cos\pi + j\sin\pi = -1$$

Therefore,
$$s_0 = -1 \qquad \ldots(i)$$

Thus, pole is at -1. Therefore, we can write the system transfer function for analog filter.
$$H_a(s) = \frac{1}{s-s_1} = \frac{1}{s+1}$$

Now, using frequency transformation to obtain transfer function of analog high pass filter. We have,