CSC 250: Foundations of Computer Science I

Fall 2023 - Lecture 12

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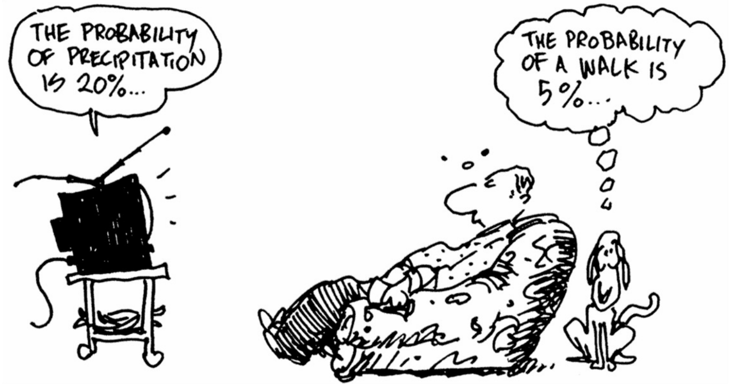
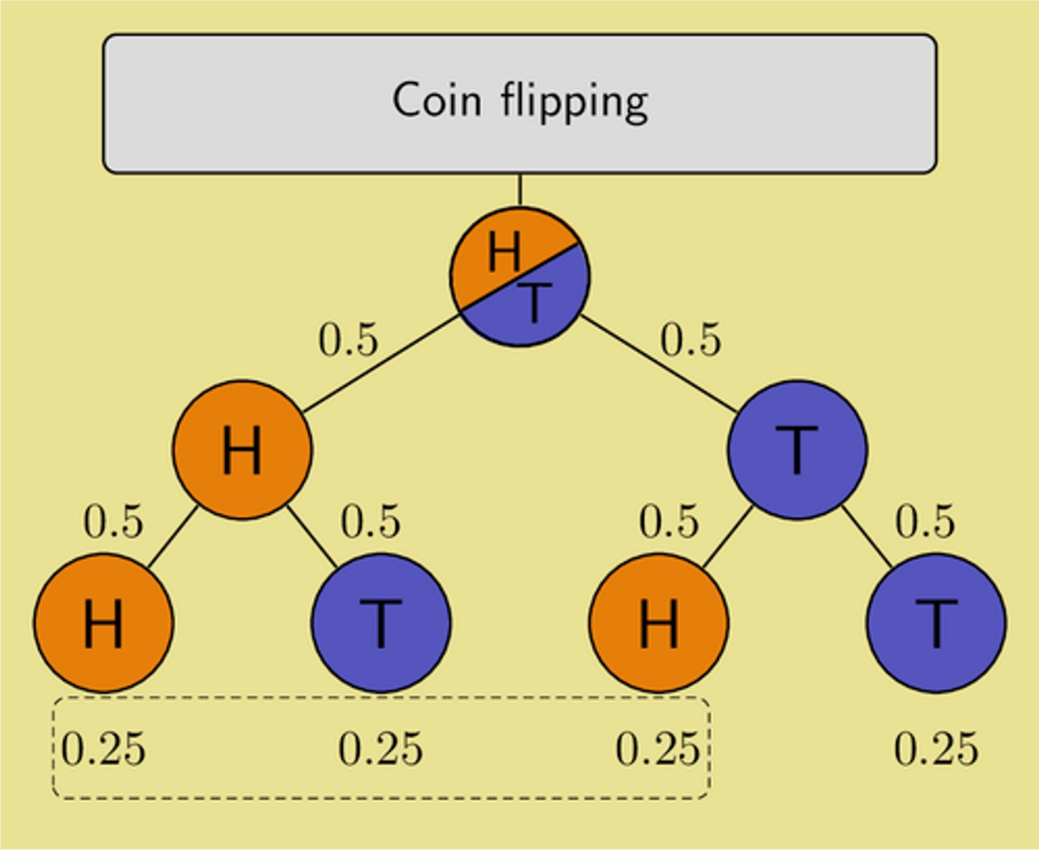
Sample Spaces, Events, Probabilities

Rolling an ordinary six-sided die is a familiar example of a random experiment, an action for which all possible outcomes can be listed, but for which the actual outcome on any given trial of the experiment cannot be predicted with certainty.

In such a situation we wish to assign to each outcome, such as rolling a two, a number, called the probability of the outcome, that indicates how likely it is that the outcome will occur.

Similarly, we would like to assign a probability to any event, or collection of outcomes, such as rolling an even number, which indicates how likely it is that the event will occur if the experiment is performed.





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Sample Spaces, Events, Probabilities

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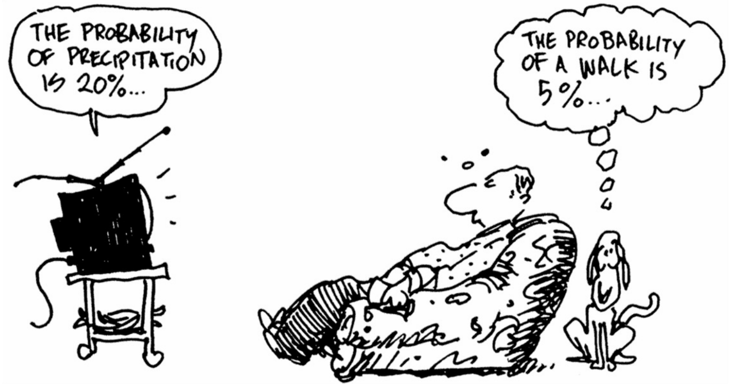
event will occu

A ***random experiment*** is a mechanism that produces a definite outcome that cannot be predicted with certainty. The ***sample space*** associated with a random experiment is the set of all possible outcomes. An ***event*** is a subset of the sample space.



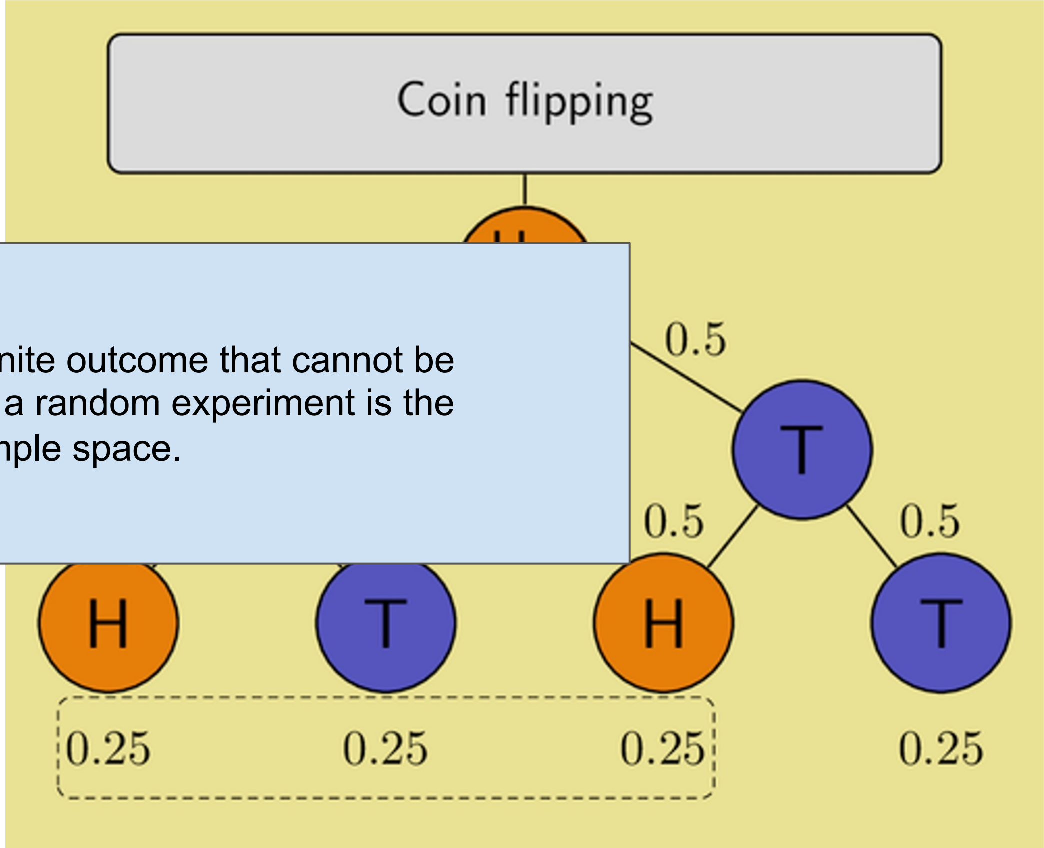


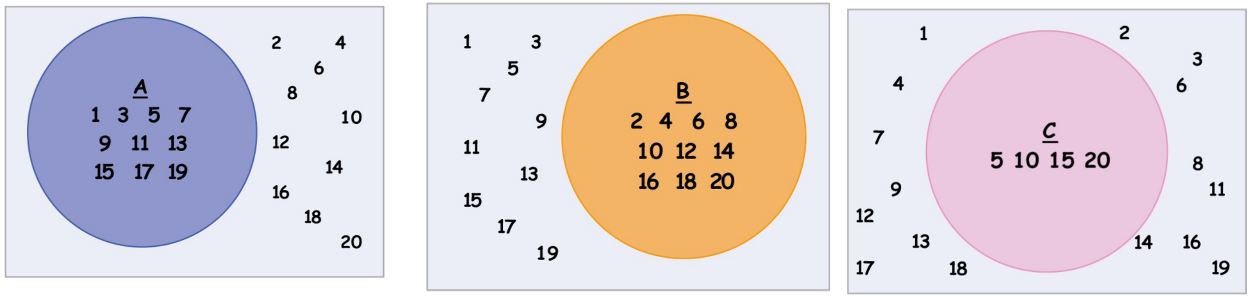
Image credit - https://texample.net/

Probability Theory

Probability is a measure of the likelihood of an event to occur. Many events cannot be predicted with total certainty. We can predict only the chance of an event to occur i.e., how likely they are going to happen, using it.

Probability can range from 0 to 1, where 0 means the event to be an impossible one and 1 indicates a certain event.

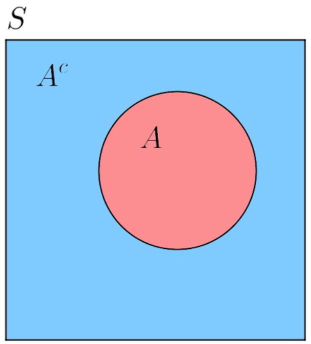
The probability of all the events in a sample space adds up to 1.



$$P(A) = \frac{\#\text{ outcomes in } A}{\#\text{ outcomes in Sample Space}} \qquad P(B) = \frac{\#\text{ outcomes in } B}{\#\text{ outcomes in Sample Space}} \qquad P(C) = \frac{\#\text{ outcomes in } C}{\#\text{ outcomes in Sample Space}}$$

$$P(A) = \frac{5}{20} \qquad P(B) = \frac{5}{20} \qquad P(C) = \frac{4}{20}$$

Image credit - https://calcworkshop.com/



Sample Space S, event A, and complement A^c

Image credit - https://brilliant.org/

$$P(A) + P(\bar{A}) = 1$$

$$P(\bar{A}) = 1 - P(A)$$

$$P(A) = 1 - P(\bar{A})$$

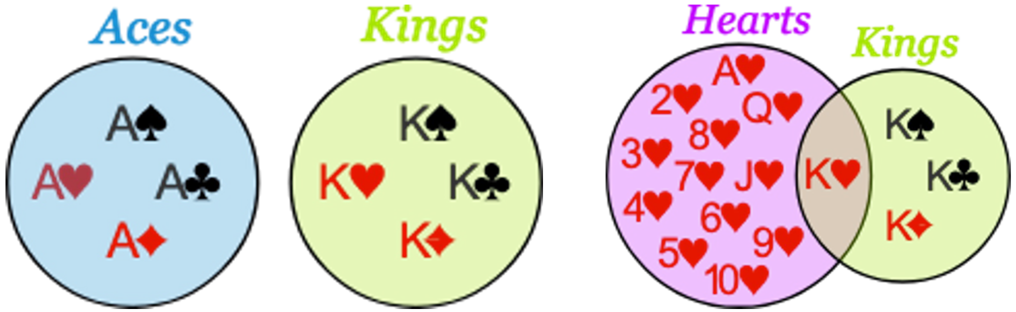
Complementary Events

The complement of A consists of all the outcomes in which the event A does not occur.

Mutually Exclusive/Disjoint Events

In probability theory, two events are said to be mutually exclusive if they cannot occur at the same time or simultaneously. In other words, mutually exclusive events are called disjoint events. If two events are considered disjoint events, then the probability of both events occurring at the same time will be zero.

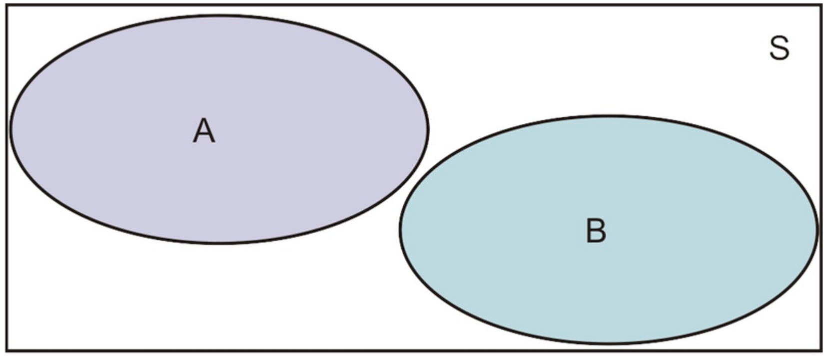
The days of the week - you cannot have a scenario where it is both Monday and Friday!
The outcomes of a dice roll
Selecting a 'diamond' and a 'black' card from a deck



Aces and Kings are Mutually Exclusive (can't be both)

Hearts and Kings are not Mutually Exclusive (can be both)

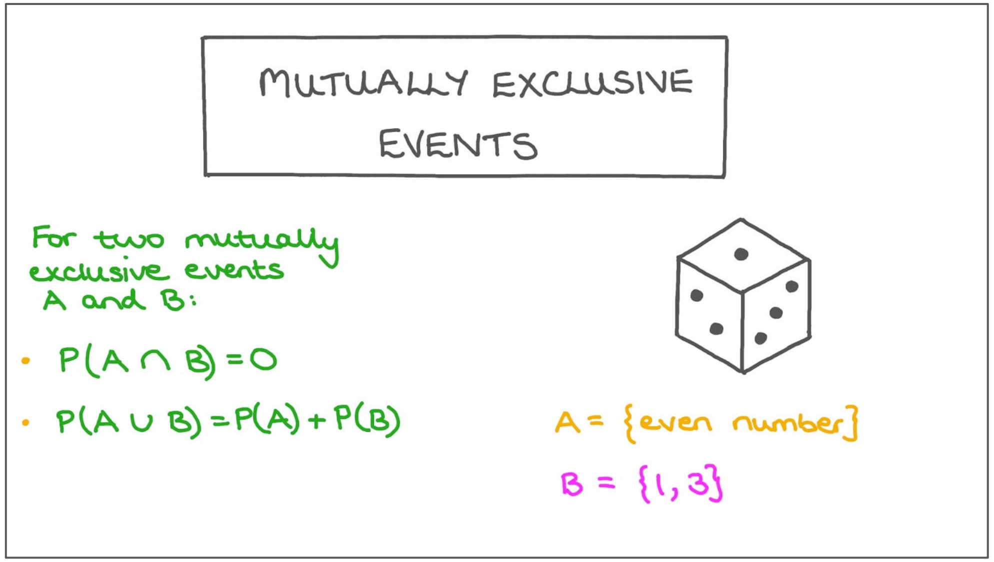
Image credit - https://www.mathsisfun.com/



P(A or B) = P(A) + P(B)

P(A and B) = 0

Image credit - https://www.ck12.org/



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Find your way here

Mutually Exclusive/Disjoint Events

In probability theory, two events are said to be mutually exclusive if they cannot occur

Mutually Exclusive Probability

In this section, we will understand the above formula in detail. To prove mutual exclusivity of two or more events, we need to go through the following probability rules:

1. **Multiplication Rule**: The most straightforward method is to determine the joint probability or integration of two or more events. If the answer is zero, then it means that these events cannot take place simultaneously. Thus, P(A∩B) = 0 shows that these events are disjoint from each other.
2. **Addition Rule**: Alternatively, the individual probabilities of the events can be added up. This results in the probable occurrence of one possibility out of many. It is denoted as P(A∪B) = P(A)+P(B).

Mutually Exclusive Events	Mutually exclusive? Yes	No
Angela goes for her train to work: Event A: she catches the train; Event B: she misses the train	√	
Rory throws a dice: Event A: he gets an odd number; Event B: he gets less than 4		√
Rory throws a dice: Event A: he gets more than 3; Event B: he gets less than 3	√	
Sue takes a card at random from a pack of 52: Event A: she gets a spade; Event B: she gets a club	√	
Sue takes a card at random from a pack of 52: Event A: she gets a spade; Event B: she gets a queen		√

P(A or B) = P(A) + P(B)

P(A and B) = 0

Image credit - https://www.ck12.org/

B = {1, 3}

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Mutually Exclusive/Disjoint Events

Examples

Mutually Exclusive Example

What is the probability of a dice showing a 2 or 5?

$$P(2)=\frac{1}{6} \qquad P(5)=\frac{1}{6}$$

$$\begin{aligned} P(2 \text{ or } 5) &= P(2)+P(5) \\ &= \frac{1}{6}+\frac{1}{6} \\ &= \frac{2}{6}=\frac{1}{3} \end{aligned}$$

The probability of a dice showing 2 or 5 is $\frac{1}{3}$

Example:

The probabilities of three teams A, B and C winning a badminton competition are 1/3, 1/5 and 1/9 respectively.

Calculate the probability that

a) either A or B will win

b) either A or B or C will win

c) none of these teams will win

d) neither A nor B will win

Solution:

a) P(A or B will win) $=\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$

b) P(A or B or C will win) $=\frac{1}{3}+\frac{1}{5}+\frac{1}{9}=\frac{29}{45}$

c) P(none will win) = 1 – P(A or B or C will win) $=1-\frac{29}{45}=\frac{16}{45}$

d) P(neither A nor B will win) = 1 – P(either A or B will win) $=1-\frac{8}{15}=\frac{7}{15}$

Image credit - https://www.onlinemathlearning.com/

Mutually Exclusive/Disjoint Events
Examples

3. **JOBS** Adelaide is the employee of the month at her job. Her reward is to select at random from 4 gift cards, 6 coffee mugs, 7 DVDs, 10 CDs, and 3 gift baskets. What is the probability that an employee receives a gift card, coffee mug, or CD?

SOLUTION:

Let event G represent receiving a gift card. Let event C represent receiving a coffee mug. Let event D represent receiving a CD.
There are a total of 4 + 6 + 7 + 10 + 3 or 30 items.

$$P(G \text{ or } C \text{ or } D) = P(G) + P(C) + P(D)$$
$$= \frac{4}{30} + \frac{6}{30} + \frac{10}{30}$$
$$= \frac{20}{30}$$
$$= \frac{2}{3}$$
$$\approx 67\%$$

Image credit - www.chino.k12.ca.us

2. A box containing 4 bulbs, the probability of having one defected bulb is 0.5 and the probability to have zero defected bulb is 0.4. Calculate the probability of one defected bulb and zero defected bulb.

Solution: Probability of single bulb being defected is P(X) = 0.5
Probability of zero bulbs being defected is P(Y) = 0.4

As there can be either zero defected bulb or 1 defected bulb because these two events cannot occur simultaneously. Hence, they are considered as mutually exclusive.

P(X or Y) = 0.5 + 0.4 = 0.9

Image credit - https://www.vedantu.com/

Example 1

It is known that the probability of obtaining zero defectives in a sample of 40 items is 0.34 whilst the probability of obtaining 1 defective item in the sample is 0.46. What is the probability of

(a) obtaining not more than 1 defective item in a sample?

(b) obtaining more than 1 defective items in a sample?

Answer

"Obtaining not more than one" means we choose either 0 or 1 defective.

Let event E_1 be "obtaining zero defectives" and E_2 be "obtaining 1 defective item".

(a) Events E_1 and E_2 are mutually exclusive, so

$$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) = 0.34 + 0.46 = 0.8$$

(b) $P(\text{more than } 1) = 1 - 0.8 = 0.2$

Image credit - https://www.intmath.com/

Mutually Exclusive/Disjoint Events

Examples

A couple has two children. What is the probability that at least one child is a boy?

Solution

Our sample space consists of the different possible combinations that the couple can have. Let B denote a boy and G denote a girl.

Our sample space is therefore S = {GG, GB, BB, BG}. Since none of these options can occur simultaneously, they are all mutually exclusive. We can therefore apply the 'sum' rule.

$$P(''atleastonechildisaboy'') = P(GBorBBorBG) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

You roll a regular 6-sided dice. What is the probability of rolling an even number?

Solution

The sample space is the possible outcomes from rolling the dice: 1, 2, 3, 4, 5, 6. The even numbers on the dice are 2, 4, and 6. Since these results are **mutually exclusive**, we can apply the sum rule to find the probability of rolling either 2, 4 or 6.

$$P(''rollinganevennumber'') = P(''rollinga2, 4, or6'') = P(''rolling2'') + P(''rolling4'') + P(''rolling6'') = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

Image credit - https://www.hellovaia.com/

Example 1:

A pair of dice is rolled. The events of rolling a 5 and rolling a double have NO outcomes in common so the two events are mutually exclusive.
A pair of dice is rolled. The events of rolling a 4 and rolling a double have the outcome $(2, 2)$ in common so the two events are not mutually exclusive.

Example 2:

From a group of 6 freshmen and 5 sophomores, 3 students are to be selected at random to form a committee. What is the probability that at least 2 freshmen are selected?

The committee will have at least 2 freshmen if either 2 freshmen and 1 sophomore are selected (event A)or 3 freshmen are selected (event B). Since the two events are mutually exclusive

$$P(A \cup B) = P(A) + P(B)$$

$$P(A) = \frac{{}_6C_2 \cdot {}_5C_1}{{}_{11}C_3} = \frac{15 \cdot 5}{165} = \frac{75}{165} = \frac{5}{11}$$

$$P(B) = \frac{{}_6C_3}{{}_{11}C_3} = \frac{20}{165} = \frac{4}{33}$$

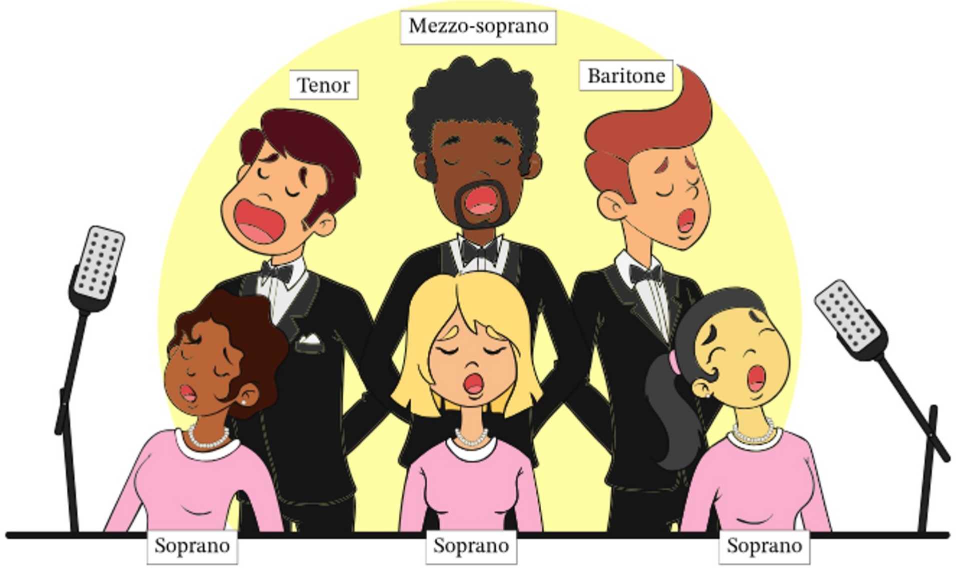
$$P(A \cup B) = \frac{5}{11} + \frac{4}{33} = \frac{15}{33} + \frac{4}{33} = \frac{19}{33}$$

Image credit - https://www.varsitytutors.com/

Mutually Exclusive/Disjoint Events
Examples

Example 4: Using the Addition Rule to Determine the Probability of Union of Mutually Exclusive Events

A small choir has a tenor singer, 3 soprano singers, a baritone singer, and a mezzo-soprano singer. If one of their names was randomly chosen, determine the probability that it was the name of the tenor singer or soprano singer.



If we assume that the singers stick to their parts so that, for example, a soprano singer does not sing tenor or baritone parts and vice versa, then the events of choosing a soprano, tenor, baritone, or mezzo-soprano singer are mutually exclusive, since no two events can occur at the same time.

This being the case, to find the probability that a randomly chosen singer is either a tenor or a soprano singer, we can use the probability rule $P(A \cup B) = P(A) + P(B)$, since, by mutual exclusivity of A and B, $P(A \cap B) = 0$.

As there are 6 singers in total and only one is a tenor singer, the probability that a randomly chosen singer is a tenor singer is

$$P(\text{tenor}) = \frac{\text{number of tenors}}{\text{size of choir}} = \frac{1}{6}.$$

Similarly, there are 3 soprano singers; hence,

$$P(\text{soprano}) = \frac{\text{number of sopranos}}{\text{size of choir}} = \frac{3}{6} = \frac{1}{2}.$$

Applying the rule that says that for mutually exclusive events, $P(A \cup B) = P(A) + P(B)$, we have

$$P(\text{tenor} \cup \text{soprano}) = P(\text{tenor}) + P(\text{soprano}) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}.$$

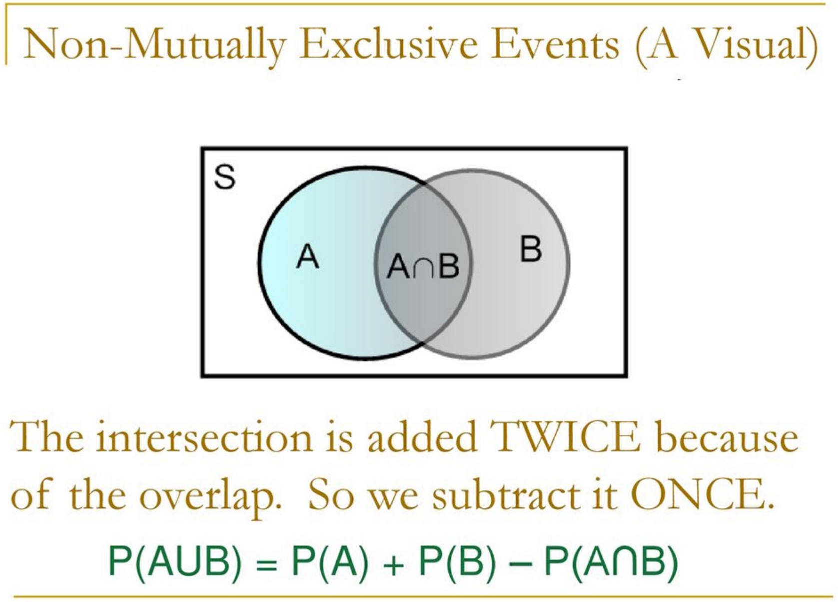
Image credit - https://www.nagwa.com/

Non-Mutually Exclusive/Union Events

Non-mutually exclusive events are events that can happen at the same time. Examples include: driving and listening to the radio, even numbers and prime numbers on a die, losing a game and scoring, or running and sweating. Non-mutually exclusive events can make calculating probability more complex.

For example, in the case of rolling a die the event of getting an 'odd-face' and the event of getting 'less than 4' are not mutually exclusive and they are also known as compatible event.

Image credit - https://www.mathsisfun.com/



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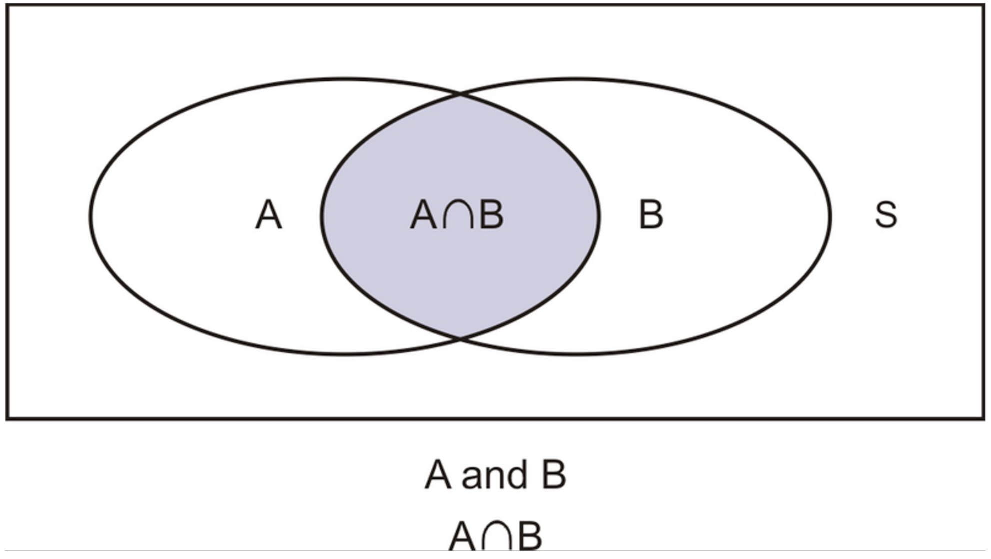


Image credit - https://www.ck12.org/

Non-Mutually Exclusive/Union Events

Examples

4. **CLUBS** According to the table, what is the probability that a student in a club is a junior or on the debate team?

Club	Soph.	Junior	Senior
Key	12	14	8
Debate	2	6	3
Math	7	4	5
French	11	15	13

SOLUTION:

Since some juniors are on the debate team, these events are not mutually exclusive. Use the rule for two events that are not mutually exclusive. The total number of students is 100.

$$\begin{aligned} P(\text{j or d}) &= P(\text{j}) + P(\text{d}) - P(\text{j and d}) \\ &= \frac{39}{100} + \frac{11}{100} - \frac{6}{100} \\ &= \frac{44}{100} \\ &= \frac{11}{25} \\ &= 44\% \end{aligned}$$

Image credit - www.chino.k12.ca.us

Example 2

The probability that a student passes Mathematics is $\frac{2}{3}$ and the probability that he passes English is $\frac{4}{9}$. If the probability that he will pass at least one subject is $\frac{4}{5}$, what is the probability that he will pass both subjects?

(We assume it is based on probability only.)

Answer

It is possible for a student to either:

- Pass math only
- Pass English only
- Pass both math and English

So we conclude that these are not mutually exclusive events. We have:

$$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$$

Substituting:

$$\frac{4}{5} = \frac{2}{3} + \frac{4}{9} - P(E_1 \text{and} E_2)$$

So

$$P(E_1 \text{and} E_2) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$$

Image credit - https://www.intmath.com/

Non-Mutually Exclusive/Union Events

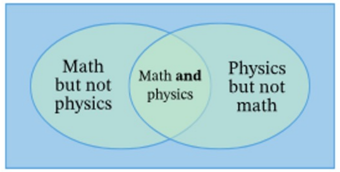
Examples

Example 6: Finding the Probability of a Difference of Two Events given the Probability of Each Event as well as Their Intersection

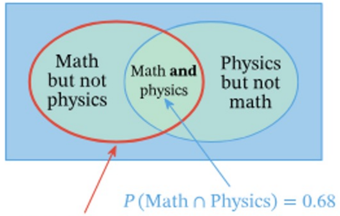
The probability that a student passes their physics exam is 0.71. The probability that they pass their mathematics exam is 0.81. The probability that they pass both exams is 0.68. What is the probability that the student only passes their mathematics exam?

Answer

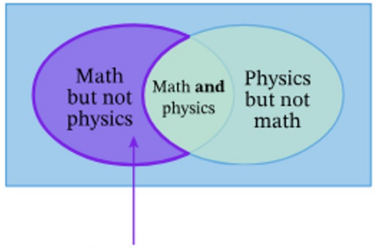
To find the probability that the student passes mathematics but not physics, let's illustrate the events in a Venn diagram. We note that since there is an overlap, the events are not mutually exclusive and can occur together. This gives us the following:



Now, if we highlight the probabilities that we know concerning the event "passes math" on the diagram, that is, $P(\text{Math}) = 0.81$ and $P(\text{Math} \cap \text{Physics}) = 0.68$, we have



The event "passes math" is everything inside the red oval, which has a probability of 0.81. The overlap in the center of the diagram covers "passes both math and physics" and has a probability of 0.68. But we want to find the probability of passing mathematics but not physics, which covers the dark purple section in the diagram below.



Since the probability of passing math is made up of the probability of passing math but not physics and the probability of passing both, we have

$$P(\text{Math}) = P(\text{Math but not physics}) + P(\text{Math} \cap \text{Physics})$$

$$0.81 = P(\text{Math but not physics}) + 0.68.$$

Rearranging this gives us

$$P(\text{Math but not physics}) = 0.81 - 0.68$$

$$= 0.13.$$

Hence, the probability that a student passes mathematics but not physics is 0.13.

Image credit - https://www.nagwa.com/

Non-Mutually Exclusive/Union Events

Examples

7. rolling a pair of dice and getting doubles or a sum of 8

SOLUTION:

If you have the outcome (4, 4), it is both doubles and the sum is 8. Because these two events can happen at the same time, these are not mutually exclusive. Use the rule for two events that are not mutually exclusive.

The total number of possible outcomes when rolling a pair of dice is 36.

$$\begin{aligned} P(\text{d or } 8) &= P(\text{d}) + P(8) - P(\text{d and } 8) \\ &= \frac{6}{36} + \frac{5}{36} - \frac{1}{36} \\ &= \frac{10}{36} \\ &\approx 27.8\% \end{aligned}$$

ANSWER:

not mutually exclusive; $\frac{10}{36}$ or 27.8%

9. selecting a number at random from integers 1 to 20 and getting an even number or a number divisible by 3

SOLUTION:

18 is between 1 and 20, and is both even and divisible by 3. Because these two events can happen at the same time, these are not mutually exclusive. Use the rule for two events that are not mutually exclusive.

Let e represent an even number and d represent divisible by 3.

$$\begin{aligned} P(\text{e or d}) &= P(\text{e}) + P(\text{d}) - P(\text{e and d}) \\ &= \frac{10}{20} + \frac{6}{20} - \frac{3}{20} \\ &= \frac{13}{20} \\ &= 65\% \end{aligned}$$

ANSWER:

not mutually exclusive; $\frac{13}{20}$ or 65%

11. drawing an ace or a heart from a standard deck of 52 cards

SOLUTION:

Because these two events can happen at the same time, these are not mutually exclusive. Use the rule for two events that are not mutually exclusive.

$$\begin{aligned} P(\text{a or h}) &= P(\text{a}) + P(\text{h}) - P(\text{a and h}) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \\ &\approx 30.8\% \end{aligned}$$

ANSWER:

not mutually exclusive; $\frac{4}{13}$ or 30.8%

Image credit - https://school.ckseattle.org

Independent Events

In probability, two events are **independent** if the incidence of one event does not affect the probability of the other event. If the incidence of one event does affect the probability of the

EXAMPLE

There is a red 6-sided fair die and a blue 6-sided fair die. Both dice are rolled at the same time. Let A be the event that the red die's result is even. Let B be the event that the blue die's result is odd. Are the events independent?

Consider whether rolling an even number on the red die will affect whether or not the blue dies rolls odd. The outcome of the red die has no impact on the outcome of the blue die. Likewise, the outcome of the blue die does not affect the outcome of the red die.

$P(A) = \frac{1}{2}$ regardless of whether B happens or not.

$P(B) = \frac{1}{2}$ regardless of whether A happens or not.

Therefore, the events are independent.

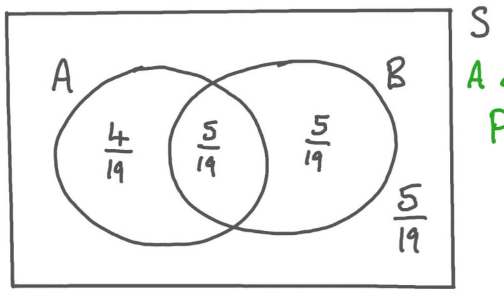
Image credit - https://brilliant.org/

Probability of Independent Events

When two events, A and B, are independent the probability of both occurring is:

P(A and B) = P(A) x P(B)

In a Sample Space S, the probabilities are shown for the Combinations of events occuring. Are A and B independent events?



A and B independent if:
$P(A \cap B) = P(A)P(B)$

Image credit - https://www.nagwa.com/

Independent Events

Independent events are events which are not affected by the occurrence of other events. For example, if we roll a die twice, the outcome of the first roll and second roll have no effect on each other - they are independent.

If two events are independent then

P(A and B) = P(A) x P(B).

For example,
When we roll a dice twice the probability of getting a 6 is $\frac{1}{6}$.

So the probability of getting a 6 and a 6 is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

THIRD SPACE LEARNING

Independent Events
Examples

Exercise 3

If the independent probabilities that three people A, B and C will be alive in 30 years time are $0.4, 0.3, 0.2$ respectively, calculate the probability that in 30 years' time,

(a) all will be alive

(b) none will be alive

(c) only one will be alive

(d) at least one will be alive

Answer

(a) $P = P(A) \times P(B) \times P(C) = 0.4 \times 0.3 \times 0.2 = 0.024$

(b) We use the notation $P(\overline{A})$ to mean "the probability that A will **not** occur". So:

$P = P(\overline{A}) \times P(\overline{B}) \times P(\overline{C})$

$= 0.6 \times 0.7 \times 0.8$

$= 0.336$

(c) $P = P(A \text{ only alive}) + P(B \text{ only alive}) + P(C \text{ only alive})$

$= [P(A) \times P(\overline{B}) \times P(\overline{C})] + [P(\overline{A}) \times P(B) \times P(\overline{C})] + [P(\overline{A}) \times P(\overline{B}) \times P(C)]$

$= 0.4 \times 0.7 \times 0.8 + 0.6 \times 0.3 \times 0.8 + 0.6 \times 0.7 \times 0.2$

$= 0.452$

(d) $P = 1 - \{P(\overline{A}) \times P(\overline{B}) \times P(\overline{C})\} = 1 - 0.336 = 0.664$

Image credit - https://www.intmath.com/

When combined events A and B are ***independent***:

$$\boxed{\mathbf{P(A \text{ and } B) = P(A) \times P(B)}}$$

For example, if a coin is tossed and a card is taken at random from a pack of 52

$$P(\text{Head}) = \frac{1}{2} \qquad P(\text{King}) = \frac{4}{52} = \frac{1}{13}$$

$$P(\text{Head } \mathbf{and} \text{ King}) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$$

For independence, if 2 cards are taken from the pack the first must be replaced before the second is taken. In this case

$$P(2 \text{ Kings}) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

Image credit - www.nuffieldfoundation.org

Independent Events
Examples

Example:
If a dice is thrown twice, find the probability of getting two 5's.

Solution:

$$P(\text{getting a 5 on the first throw}) = \frac{1}{6}$$

$$P(\text{getting a 5 on the second throw}) = \frac{1}{6} \quad P(\text{two 5's}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Example:
Two sets of cards with a letter on each card as follows are placed into separate bags.

Bag 1:	I	L	J	A	U	
Bag 2:	L	R	H	E	C	A

Sara randomly picked one card from each bag. Find the probability that:
a) She picked the letters 'J' and 'R'.
b) Both letters are 'L'.
c) Both letters are vowels.

Solution:

a) Probability that she picked J and R = $\frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$

b) Probability that both letters are L = $\frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$

c) Probability that both letters are vowels = $\frac{3}{5} \times \frac{2}{6} = \frac{3}{15} = \frac{1}{5}$

Image credit - https://www.onlinemathlearning.com/

Step 1: Find the probability of each individual event.

P(ace) $= \frac{4}{52} = \frac{1}{13}$ There are 4 aces in a deck of cards / There are 52 outcomes in a deck of cards

P(spade) $= \frac{13}{52} = \frac{1}{4}$ There are 13 spades in a deck of cards / There are 52 outcomes in a deck of cards

P(4) $= \frac{4}{52} = \frac{1}{13}$ There are 4 fours in a deck of cards / There are 52 outcomes in a deck of cards

Step 2: Multiply the probabilities of each individual event.

P(ace and spade and 4) = $\frac{1}{13} \cdot \frac{1}{4} \cdot \frac{1}{13} = \frac{1}{676}$

The probability of drawing an ace, replacing it, and then drawing a spade, replacing it, and then drawing a 4 is 1/676.

Image credit -https://www.algebra-class.com/

Definition: Two events, A and B, are **independent** if the fact that A occurs does not affect the probability of B occurring.

Some other examples of independent events are:

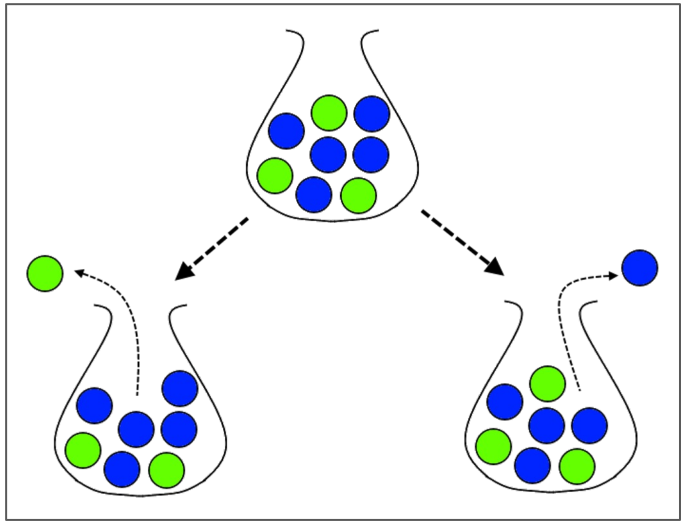
- Landing on heads after tossing a coin **AND** rolling a 5 on a single 6-sided die.
- Choosing a marble from a jar **AND** landing on heads after tossing a coin.
- Choosing a 3 from a deck of cards, replacing it, **AND** then choosing an ace as the second card.
- Rolling a 4 on a single 6-sided die, **AND** then rolling a 1 on a second roll of the die.

To find the probability of two independent events that occur in sequence, find the probability of each event occurring separately, and then multiply the probabilities. This multiplication rule is defined symbolically below. Note that multiplication is represented by AND.

Dependent Events

Dependent events: Two events are dependent when the outcome of the first event influences the outcome of the second event.

Suppose a bag has 3 red and 6 green balls. Two balls are drawn from the bag, one after the other. Let A be the event of drawing a red ball in the first draw and B be the event of drawing a green ball in the second draw. If the ball drawn in the first draw is not replaced back in the bag, then A and B are dependent events because P(B) is decreased or increased according to the first draw results as a red or green ball.



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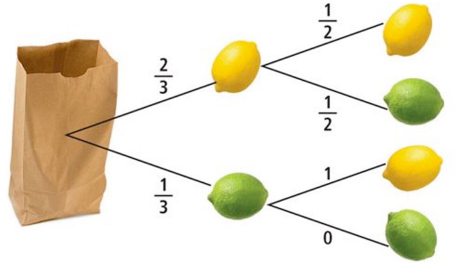
Independent and Dependent Events

Events are **dependent events** if the occurrence of one event affects the probability of the other.

For example, suppose that there are 2 lemons and 1 lime in a bag. If you pull out two pieces of fruit, the probabilities change depending on the outcome of the first.



The probability of a specific event can be found by multiplying the probabilities on the branches that make up the event.
For example, the probability of drawing two lemons is

Independent vs. Dependent Events

Using the bag of marbles on the left, what is the probability of pulling a black marble two times in a row? $P(black, black)$

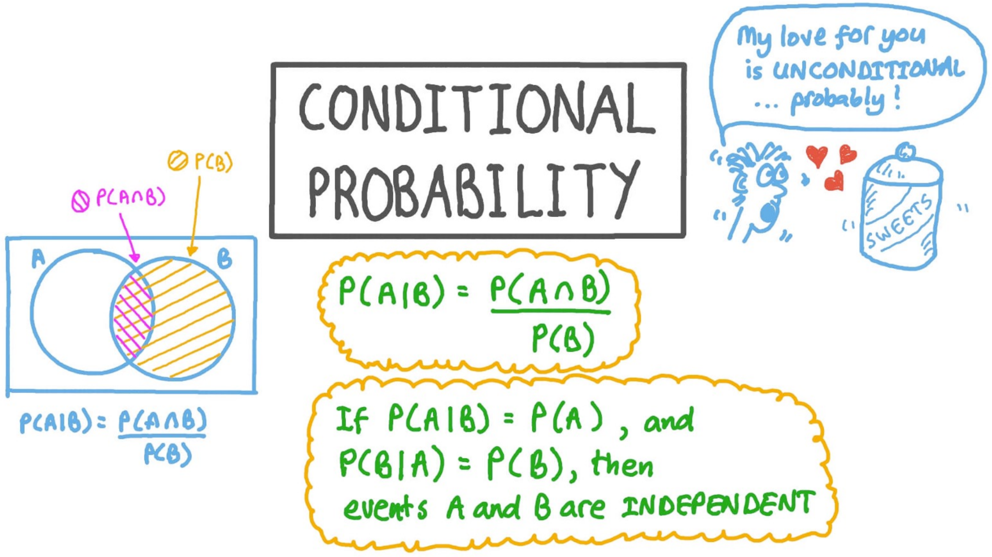
When you put 1st marble back in (*Independent Events*)	When you KEEP 1st marble (*Dependent Events*)
$\frac{2}{10} * \frac{2}{10}$	$\frac{2}{10} * \frac{1}{9}$
$\frac{1}{5} * \frac{1}{5} = \boxed{\frac{1}{25}}$	$\frac{1}{5} * \frac{1}{9}$

Dependent Events - Conditional Probability

Conditional probability is defined as the likelihood of an event or outcome occurring, based on the occurrence of a previous event or outcome.

The conditional probability of an event B is the probability that the event will occur given the knowledge that an event A has already occurred. This probability is written P(B|A), notation for the probability of B given A.

In the case where events A and B are independent (where event A has no effect on the probability of event B), the conditional probability of event B given event A is simply the probability of event B, that is P(B).



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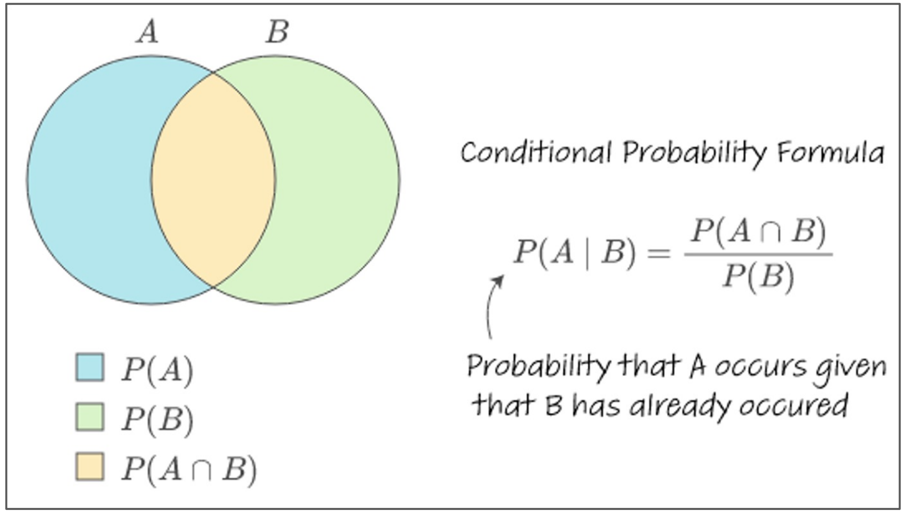
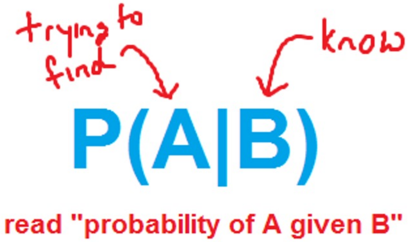
	Yes	No	Total
Males	15	5	20
Females	17	3	20
Total	32	8	40

$$P(yes \mid male) = \frac{P(yes\ and\ male)}{P(male)} = \frac{15}{20}$$





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Conditional Probability
Examples

Example

> A survey asked full time and part time students how often they had visited the college's tutoring center in the last month. The results are shown below.

	Number of times students visited tutoring			
	One or fewer times	Two to three times	Four or more times	Total
Full time student	12	25	8	45
Part time student	2	5	6	13
Total	14	30	14	58

> (a) What is the probability the student visited the tutoring center four or more times, given that the student is full time?

Conditional probability is all about focusing on the information you know. When calculating this probability, we are given that the student is full time. Therefore, we should only look at full time students to find the probability.

	Number of times students visited tutoring			
	One or fewer times	Two to three times	Four or more times	Total
Full time student	12	25	8	45
Part time student	2	5	6	13
Total	14	30	14	58

P(four or more times| full time student) $= \frac{8}{45} \approx \boxed{0.18}$

find given

Conditional Probability
Examples

> (b) Suppose that a student is part time. What is the probability that the student visited the tutoring center one or fewer times?

This one is a bit more tricky due to the wording. Think of it in the following way:

- **Find:** probability student visited the tutoring center one or fewer times
- **Assume or given:** student is part time ("suppose that a student is part time")

Since we are assuming (or supposing) the student is part time, we will only look at part time students for this calculation.

	Number of times students visited tutoring			
	One or fewer times	Two to three times	Four or more times	Total
Full time student	12	25	8	45
Part time student	2	5	6	13
Total	14	30	14	58

P(one or fewer times | part time) = $\frac{2}{13} \approx \boxed{0.15}$

find

given

> (c) If the student visited the tutoring center four or more times, what is the probability he or she is part time?

As above, we need to make sure we know what is given, and what we are finding.

- **Find:** probability he or she is part time
- **Assume or given:** student visited the tutoring center four or more times ("if the student visited the tutoring center four or more times…")

For this question, we are only looking at students who visited the tutoring center four or more times.

	Number of times students visited tutoring			
	One or fewer times	Two to three times	Four or more times	Total
Full time student	12	25	8	45
Part time student	2	5	6	13
Total	14	30	14	58

P(part time | visited four or more times) = $\frac{6}{14} \approx \boxed{0.43}$

find

given

As you can see, when using a table, you just need to pay attention to which group from the table you should focus on.

Find your way here

Conditional Probability

Examples

Problem: A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?

Analysis: This problem describes a conditional probability since it asks us to find the probability that the second test was passed given that the first test was passed. In the last lesson, the notation for conditional probability was used in the statement of Multiplication Rule 2.

Multiplication Rule 2: When two events, A and B, are dependent, the probability of both occurring is:

$P(A \text{ and } B) = P(A) \cdot P(B|A)$

Problem: A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?

Solution:

$$P(\text{Second}|\text{First}) = \frac{P(\text{First and Second})}{P(\text{First})} = \frac{0.25}{0.42} = 0.60 = 60\%$$

Let's look at some other problems in which we are asked to find a conditional probability.

Example 1: A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

Solution:

$$P(\text{White}|\text{Black}) = \frac{P(\text{Black and White})}{P(\text{Black})} = \frac{0.34}{0.47} = 0.72 = 72\%$$

Example 2: The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?

Solution:

$$P(\text{Absent}|\text{Friday}) = \frac{P(\text{Friday and Absent})}{P(\text{Friday})} = \frac{0.03}{0.2} = 0.15 = 15\%$$

Example 3: At Kennedy Middle School, the probability that a student takes Technology and Spanish is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes Spanish given that the student is taking Technology?

Solution:

$$P(\text{Spanish}|\text{Technology}) = \frac{P(\text{Technology and Spanish})}{P(\text{Technology})} = \frac{0.087}{0.68} = 0.13 = 13\%$$

Image credit -https://www.mathgoodies.com/

Conditional Probability

Examples

In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{4}$. If it's rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{8}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is 0.25. You pick a random day.

a. What is the probability that it's not raining and there is heavy traffic and I am not late?
b. What is the probability that I am late?
c. Given that I arrived late at work, what is the probability that it rained that day?

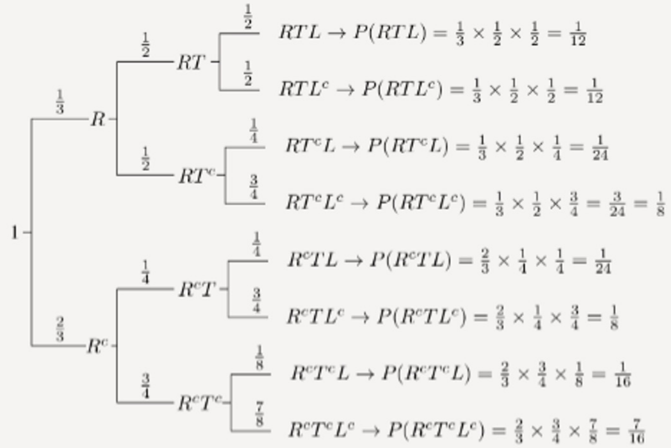


Fig.1.27 - Tree diagram for Problem 5.

a. The probability that it's not raining and there is heavy traffic and I am not late can be found using the tree diagram which is in fact applying the chain rule:

$$
\begin{aligned}
P(R^c \cap T \cap L^c) &= P(R^c)P(T|R^c)P(L^c|R^c \cap T) \\
&= \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{4} \\
&= \frac{1}{8}.
\end{aligned}
$$

Image credit - https://www.probabilitycourse.com/

b. The probability that I am late can be found from the tree. All we need to do is sum the probabilities of the outcomes that correspond to me being late. In fact, we are using the law of total probability here.

$$
\begin{aligned}
P(L) &= P(R,T,L) + P(R,T^c,L) + P(R^c,T,L) + P(R^c,T^c,L) \\
&= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} \\
&= \frac{11}{48}.
\end{aligned}
$$

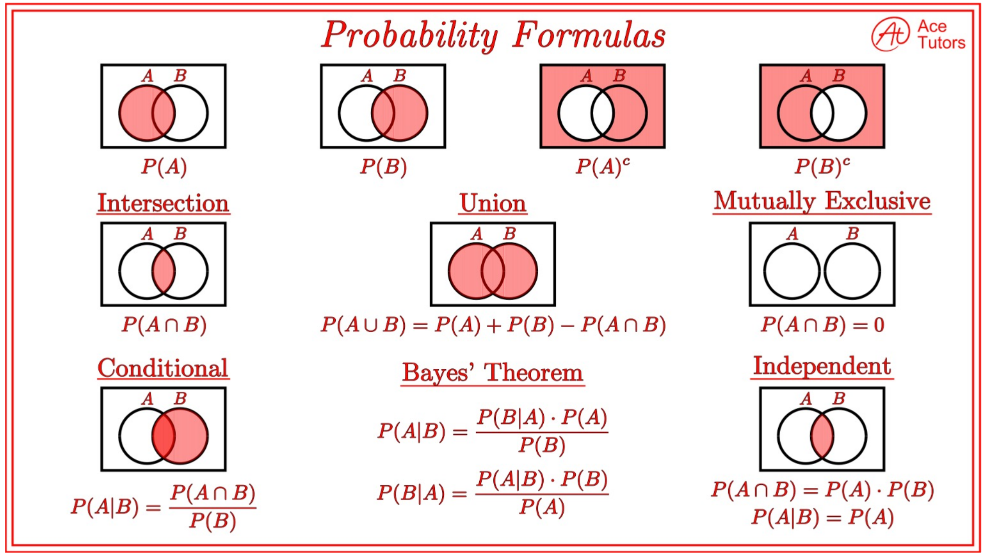
c. We can find $P(R|L)$ using $P(R|L) = \frac{P(R \cap L)}{P(L)}$. We have already found $P(L) = \frac{11}{48}$, and we can find $P(R \cap L)$ similarly by adding the probabilities of the outcomes that belong to $R \cap L$. In particular,

$$
\begin{aligned}
P(R \cap L) &= P(R,T,L) + P(R,T^c,L) \\
&= \frac{1}{12} + \frac{1}{24} \\
&= \frac{1}{8}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
P(R|L) &= \frac{P(R \cap L)}{P(L)} \\
&= \frac{1}{8} \cdot \frac{48}{11} \\
&= \frac{6}{11}.
\end{aligned}
$$

Chapter Summary - Formula Sheet



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for Mutually Exclusive Events

$$P(A \cup B) = P(A) + P(B)$$

for non-Mutual Events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

for Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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Probability Rules Cheat Sheet

complement rule

$$P(A) = 1 - P(A')$$

multiplication rules (joint probability)

dependent $P(A \cap B) = P(A) * P(B|A)$

independent $P(A \cap B) = P(A) * P(B)$

mutually exclusive $P(A \cap B) = 0$

addition rules (union of events)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

mutually exclusive $P(A \cup B) = P(A) + P(B)$

conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' Theorem

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$

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