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CSC 250: Foundations of Computer Science I

Fall 2023 - Lecture 9

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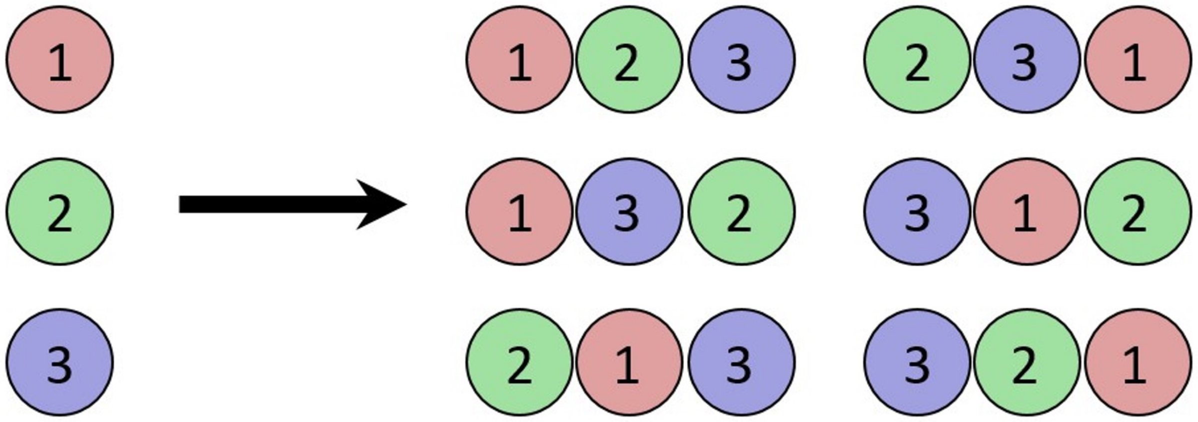
Counting Technique
Systematic Listing

There are only four cyclists: A, B, C and D. There are four cyclists to choose from for the first place, three cyclists to choose from for the second place and two cyclists to choose from for the third place.

The actual list of all possible arrangements is ABC, ABD, ACB, ACD, ADB, ADC, BAC, BAD, BCA, BCD, BDA, BDC, CAB, CAD, CBA, CBD, CDA, CDB, DAB, DAC, DBA, DBC, DCA, DCB.

Image credit - https://www.slideshare.net/rejzmaalam/four-counting-techniquespptx



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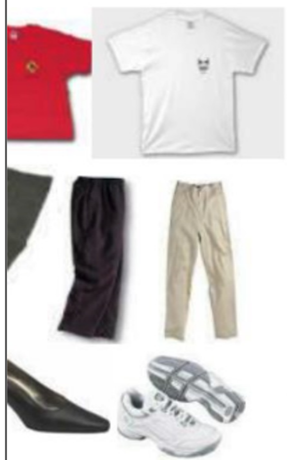
Jo packed 2 shirts (red, white) and 3 slacks (gray, black, khaki) for her trip to NYC

How many different outfits are possible?

- Red shirt with gray slacks {R,G}
- Red shirt with black slacks {R,B}
- Red shirt with khaki slacks {R,K}
- White shirt with gray slacks {W,G}
- White shirt with black slacks {W,B}
- White shirt with khaki slacks {W,K}

Let's say Jo packed 2 shirts (red, white), 3 slacks (gray, black, khaki), and 2 shoes (dress, tennis)



How many different outfits are possible?

Dress	{R,G,D}	{R,B,D}	{R,K,D}
Shoes	{W,G,D}	{W,B,D}	{W,K,D}
Tennis	{R,G,T}	{R,B,T}	{R,K,T}
Shoes	{W,G,T}	{W,B,T}	{W,K,T}

12 possible outfits

Image credit - https://www.slideserve.com/kosey/counting-methods

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Counting Technique
Tabular Method

What are the possible outcomes of rolling two fair die?

		Green Die					
		1	2	3	4	5	6
Red Die	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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Let's take three independent events A, B, and C. Suppose A can have two possible outcomes, B can have three possible outcomes, and C can have four possible outcomes. We can create a table like this.

The table shows all the possible outcomes for events A, B, and C, with each row representing a unique combination of outcomes. In this case, there are
2 x 3 x 4 = 24 possible outcomes.

A	B	C
1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
1	2	2
1	2	3
1	2	4
1	3	1
1	3	2
1	3	3
1	3	4
2	1	1
2	1	2
2	1	3
2	1	4
2	2	1
2	2	2
2	2	3
2	2	4
2	3	1
2	3	2
2	3	3
2	3	4

Image credit - ChatGPT

Determine the number of two-digit numbers that can be written using the digits from the set $\{2, 4, 6\}$.

The task consists of two parts:

1. Choose a first digit
2. Choose a second digit

The results for a two-part task can be pictured in a **product table**.

		Second Digit		
		2	4	6
First Digit	2	22	24	26
	4	42	44	46
	6	62	64	66

9 possible numbers

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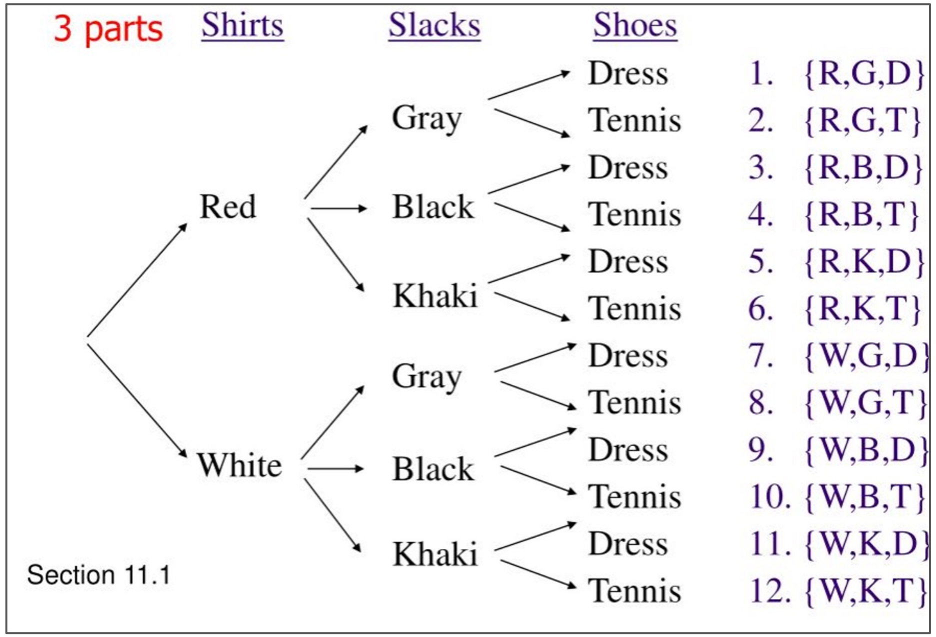
Counting Technique

Tree Diagrams



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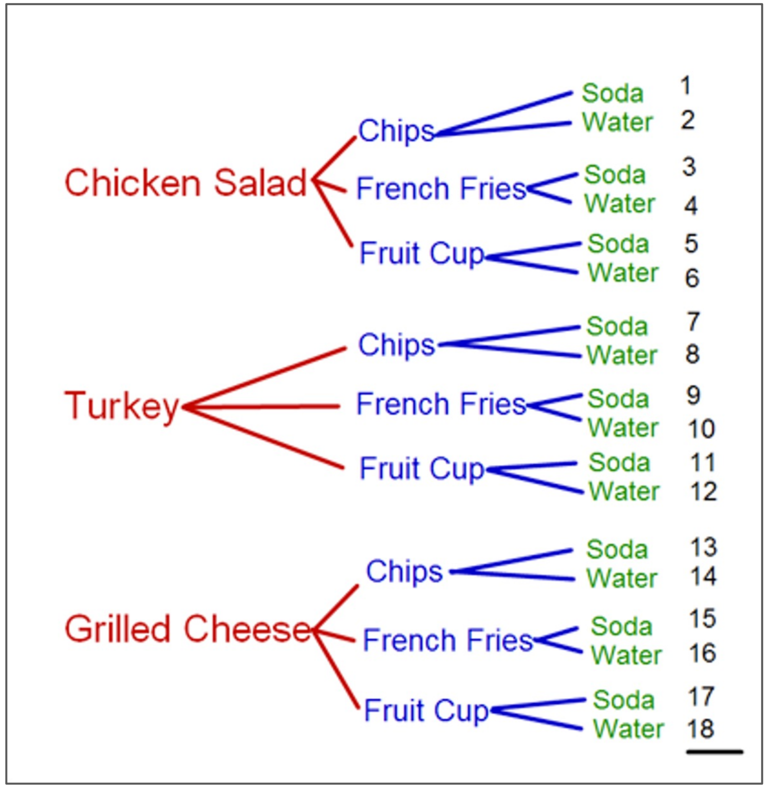

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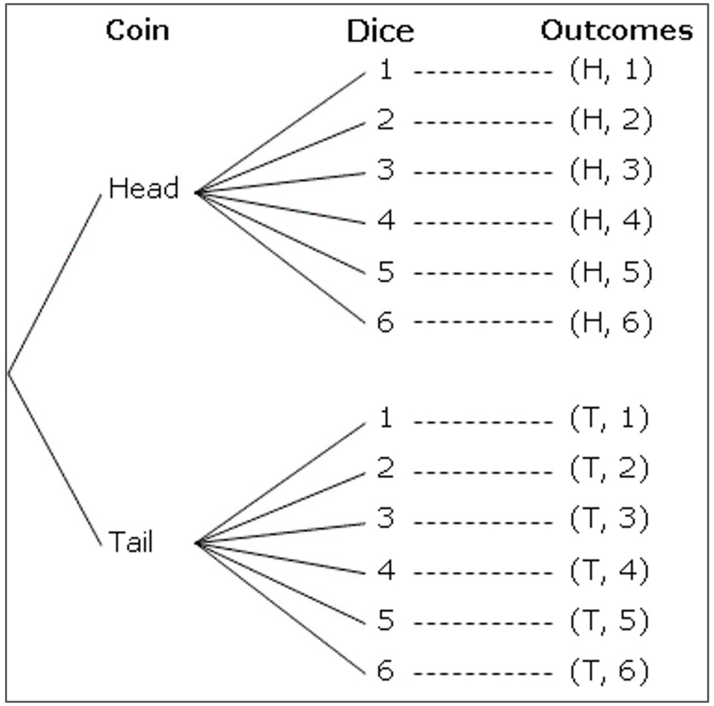

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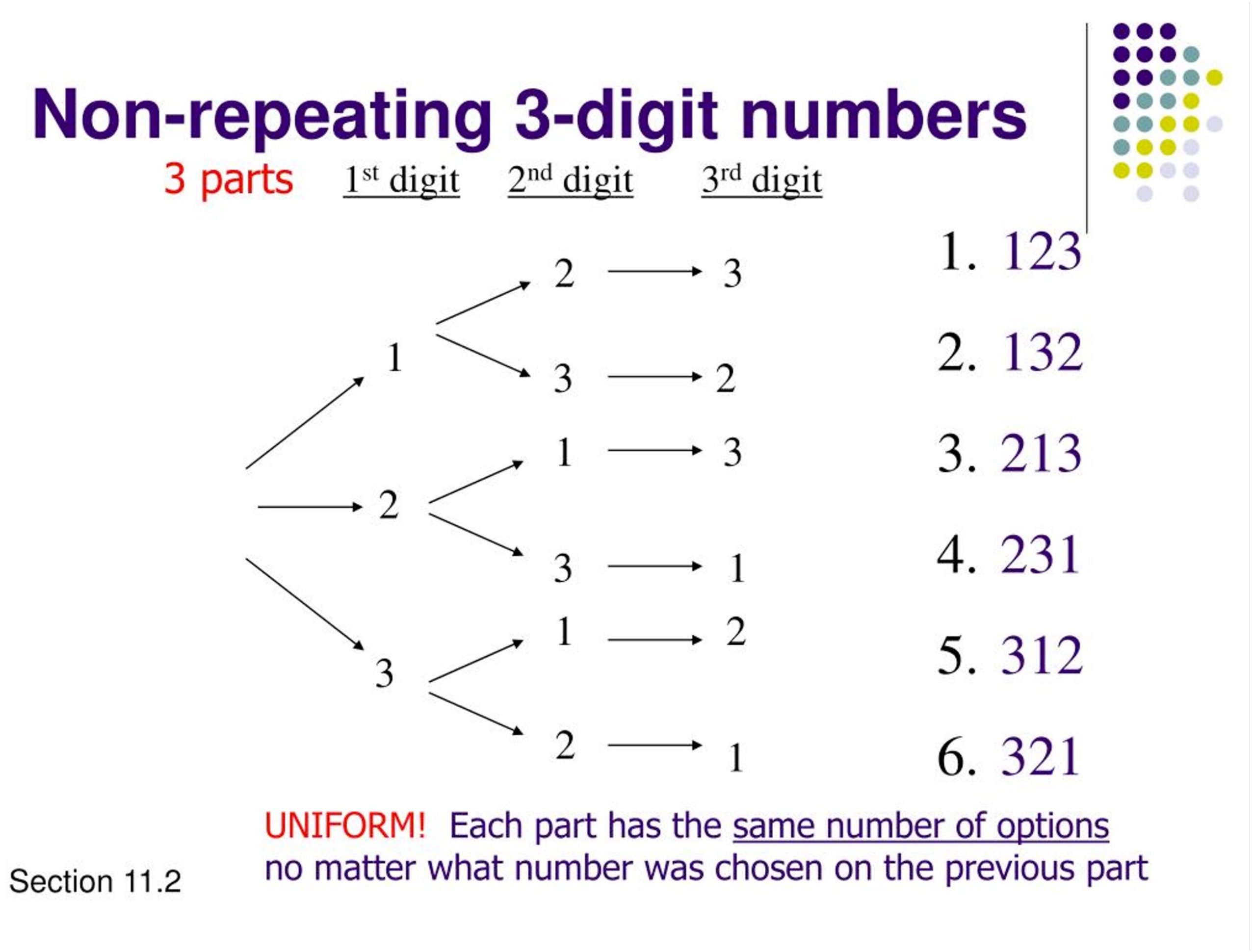
Counting Technique

Tree Diagrams - Non repeating options



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Fundamental Counting Principle

States that if there are **m** ways to do a task, and **n** ways to do another, then there are **m x n** ways of doing both.

Example 1:

Suppose you have 3 shirts (call them A , B , and C), and 4 pairs of pants (call them w , x , y , and z). Then you have

$3 \times 4 = 12$

possible outfits:

$$\begin{aligned} &Aw, Ax, Ay, Az \\ &Bw, Bx, By, Bz \\ &Cw, Cx, Cy, Cz \end{aligned}$$

Example 2:

Suppose you roll a 6 -sided die and draw a card from a deck of 52 cards. There are 6 possible outcomes on the die, and 52 possible outcomes from the deck of cards. So, there are a total of

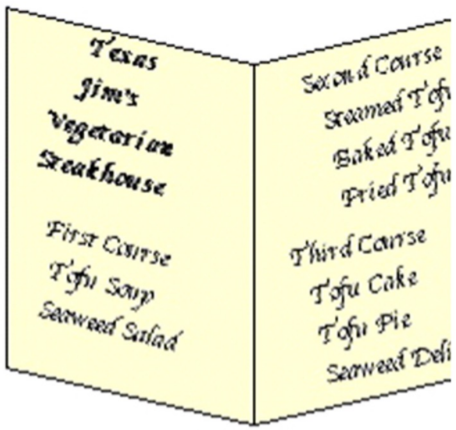
$6 \times 52 = 312$

possible outcomes of the experiment.

The counting principle can be extended to situations where you have more than 2 choices. For instance, if there are p ways to do one thing, q ways to a second thing, and r ways to do a third thing, then there are $p \times q \times r$ ways to do all three things.

Image credit - https://www.varsitytutors.com/

Fundamental Counting Principle



Plato is going to choose a three-course meal at his favorite restaurant. He must choose one item from each of the following three categories.
First course: Tofu Soup (TS); Seaweed Salad (SS)
Second course: Steamed Tofu (ST); Baked Tofu (BT); Fried Tofu (FT);
Third course: Tofu Cake (TC); Tofu Pie (TP); Seaweed Delight (SD)
How many different three-course meals are possible?

Solve this problem by listing every possible 3-course meal.

SOLUTION
We will list every possible 3-course meal:

1. TS-ST-TC	2. TS-ST-TP	3. TS-ST-SD	4. TS-BT-TC
5. TS-BT-TP	6. TS-BT-SD	7. TS-FT-TC	8. TS-FT-TP
9. TS-FT-SD	10. SS-ST-TC	11. SS-ST-TP	12. SS-ST-SD
13. SS-BT-TC	14. SS-BT-TP	15. SS-BT-SD	16. SS-FT-TC
17. SS-FT-TP	18. SS-FT-SD		

We see that there are 18 different three-course meals.

However, there is an easier way to get at this answer. Notice that in order to choose a three-course meal, we need to make three decisions:
1. Choose item for first course: There are 2 options.
2. Choose item for second course: There are 3 options
3. Choose option for third course: There are 3 options.
Now observe that (2)(3)(3) = 18.

At an Ice Cream shop they have 5 different flavors of ice cream and you can pick one of 4 toppings.

How many choices do you have?

5 choices of flavors,
4 choices of toppings

5x4 = 20

How many ways can you flip 4 coins?

The 1st coin can be flipped 2 ways.
The 2nd coin can be flipped 2 ways.
The 3rd coin can be flipped 2 ways.
The 4th coin can be flipped 2 ways.

$$\underbrace{2 \times 2 \times 2 \times 2}_{\text{four tasks—each done in two ways}} = 16 \text{ ways.}$$

Someone wants to know how many different outfits they can make with 3 coats, 5 pants, 7 shirts, and 4 ties.

Task	Number of Ways to Perform Task
Select coat	3
Select pants	5
Select shirt	7
Select tie	4

$3 \times 5 \times 7 \times 4 = 420$ outfits

number of coats . . . times number of pants . . . times number of shirts . . . times number of ties

Fundamental Counting Principle

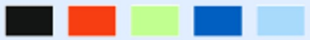
Independent Events

Example: You are buying a new car.

There are **2** body styles:

sedan or hatchback

There are **5** colors available:

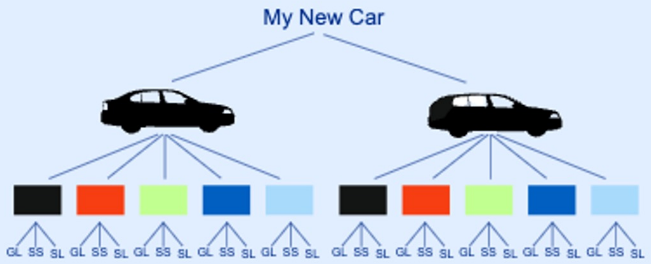


There are **3** models:

- GL (standard model),
- SS (sports model with bigger engine)
- SL (luxury model with leather seats)

How many total choices?

You can see in this "tree" diagram:



You can count the choices, or just do the simple calculation:

Total Choices = $2 \times 5 \times 3 = 30$

Independent or Dependent?

But it only works when all choices are **independent** of each other.

If one choice affects another choice (i.e. **depends** on another choice), then a simple multiplication is not right.

Example: You are buying a new car ... but ...

the salesman says **"You can't choose black for the hatchback"** ... well then things change!

You now have only 27 choices.

Because your choices are **not independent** of each other.

But you can still make your life easier with this calculation:

Choices = $5\times3 + 4\times3 = 15 + 12 = 27$

Image credit - https://www.mathsisfun.com/

Fundamental Counting Principle

Dependent Events

A bag contains **three red balls** and **two blue balls**. If two balls are drawn from the bag without replacement, what is the total number of possible outcomes?

The number of outcomes for the first draw is 5 (since there are 5 balls in total).

The number of outcomes for the second draw depends on the outcome of the first draw. If the first ball drawn is red, there are 4 balls left in the bag, of which 2 are red and 2 are blue.

If the first ball drawn is blue, there are 4 balls left in the bag, of which 3 are red and 1 is blue. Therefore, the number of outcomes for the second draw is 4 if the first ball drawn is red, and 4 if the first ball drawn is blue.

The total number of possible outcomes is the product of the number of outcomes for each event, which is
5 x 4 = 20.

Therefore, there are 20 possible outcomes when two balls are drawn from the bag without replacement.

Dependent events: the outcome of one event does affect the outcome of another event.

The fundamental counting principle applies to dependent events as well as independent events.

Ex. Taking a yellow skittle from a jar and then taking a second skittle without replacing the yellow one; is a dependent event.

Often time factorials can be used to represent a dependent event. (recall; $4! = 4\cdot 3\cdot 2\cdot 1 = 24$)

Image credit - https://slideplayer.com/slide/14123764/

Fundamental Counting Principle
Examples

To use an ATM at a certain bank, you must enter a 5-digit code, using the digits 0−9. How many 5-digit codes are there if you are allowed to repeat a number.

Key facts:

5-digit code, using the digits 0-9, allowed to repeat a number.

Find out how many 5-digit codes are there.

There 5 options: ___ ___ ___ ___ ___

There are 10 possibilities
(0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) for each of the
5 digits $10*10*10*10*10=100{,}000$

EXAMPLE 1

John wants to give a novel and a cookbook to his friend. He has four different novels and three different cookbooks to choose from. How many different choices can he make?

Solution:

John has two tasks to do: pick a novel and pick a cookbook. As he has four different novels to choose from, he has four ways to accomplish that task; as he has three cookbooks to choose from, he has three ways to do that. Thus, he has $4 \cdot 3 = 12$ ways to pick a novel and then a cookbook.

EXAMPLE 2

The Thursday special at Lisa's Pizza is a large pizza with any combination of toppings (but no more than one of each). The available toppings are pepperoni, mushrooms, onions, sausage, peppers, olives, and pineapple. How many different Thursday specials can be ordered?

Solution:

Think of ordering a Thursday special as a sequence of seven different tasks: choose whether or not to have pepperoni, then choose whether or not to have mushrooms, and so on. There are two choices for each task, so there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$ different ways to order.

Image credit - https://sites.radford.edu/

Difficulty: Easy

Fundamental Counting Principle
Examples

EXAMPLE

Lily is trying to decide what to wear. She has shirts in the following colors: red, purple, and blue, and she has pants in the following colors: black and white. How many different outfits can Lily choose from (assuming she selects one shirt and one pair of pants)?

We know from the definition of the rule of product that if there are n options for doing one thing (like choosing a shirt), and m options for doing another thing (like choosing a pair of pants), then there are $n \times m$ total combinations we can choose from. In this case, there are 3 options for choosing a shirt, and there are 2 options for choosing pants. Thus, there are $3 \times 2 = 6$ total options.

Here is a table where each row represents a possible outfit.

Shirt	Pants
Red	Black
Blue	Black
Purple	Black
Red	White
Blue	White
Purple	White

As expected, there are 6 possible combinations. □

EXAMPLE

There are 8 daily newspapers and 5 weekly magazines published in Chicago. If Colin wants to subscribe to exactly one daily newspaper and one weekly magazine, how many different choices does he have?

Colin has $8 \times 5 = 40$ choices. □

Image credit - https://brilliant.org/

Q. How many meal combos are possible?
A. There are 4 stages:

1. Choose a sandwich.
2. Choose a side.
3. Choose a dessert.
4. Choose a drink.

There are 4 different types of sandwich, 3 different types of side, 2 different types of desserts and five different types of drink.

The number of meal combos possible is 4 * 3 * 2 * 5 = 120.

Image credit - https://www.statisticshowto.com/

Fundamental Counting Principle
Examples

Difficulty: Intermediate

EXAMPLE

Calvin wants to go to Milwaukee. He can choose from 3 bus services or 2 train services to head from home to downtown Chicago. From there, he can choose from 2 bus services or 3 train services to head to Milwaukee. How many ways are there for him to get to Milwaukee?

Since Calvin can either take a bus or a train downtown , he has $3+2=5$ ways to head downtown (Rule of sum). After which, he can either take a bus or a train to Milwaukee, and hence he has another $2+3=5$ ways to head to Milwaukee (Rule of sum). Thus in total, he has $5\times 5=25$ ways to head from home to Milwaukee (Rule of product). $\square$

EXAMPLE

Six friends Andy, Bandy, Candy, Dandy, Endy, and Fandy want to sit in a row at the cinema. If there are only six seats available, how many ways can we seat these friends?

For the first seat, we have a choice of any of the 6 friends. After seating the first person, for the second seat, we have a choice of any of the remaining 5 friends. After seating the second person, for the third seat, we have a choice of any of the remaining 4 friends. After seating the third person, for the fourth seat, we have a choice of any of the remaining 3 friends. After seating the fourth person, for the fifth seat, we have a choice of any of the remaining 2 friends. After seating the fifth person, for the sixth seat, we have a choice of only 1 of the remaining friends. Hence, by the rule of product, there are $6\times 5\times 4\times 3\times 2\times 1=720$ ways to seat these 6 people. More generally, this problem is known as a permutation. There are $n!=n\times(n-1)\times(n-2)\times\cdots\times 1$ ways to seat n people in a row. $\square$

Image credit - https://brilliant.org/

Addition vs Multiplication Principle

Addition Principle (Rule of Sum)

The Sum Rule states that if a task can be performed in either two ways, where the two methods cannot be performed simultaneously, then completing the job can be done by the sum of the ways to perform the task.

Sum Rule:

A task can be performed either in n_1 ways **OR** in n_2 ways, where the two tasks cannot be performed simultaneously, then there are $n_1 + n_2$ ways to perform the task

For instance, suppose a bakery has a selection of 20 different cupcakes, 10 different donuts, and 15 different muffins. If you are to select a tasty treat, how many different choices of sweets can you choose from?

Here's how this works.

Because we have to choose from either a **cupcake** or **donut** or **muffin** (notice the "OR"), we have **20** + **10** + **15** = 45 treats to choose from.

Image credit - https://calcworkshop.com/

Multiplication Principle (Rule of Product)

The Product Rule states that if a task can be performed in a sequence of tasks, one after the other, then completing the job can be done by the product of the ways to perform the task

Product Rule:

A task can be performed either in n_1 ways **AND** in n_2 ways, after the first task is complete, then there are $n_1 \cdot n_2$ ways to perform the task.

Continuing our story from above, suppose a bakery has a selection of 20 different cupcakes, 10 different donuts, and 15 different muffins — how many different orders are there?

What makes this question different from the first problem is that we are **not** asking how many total choices there are. We are asking how many different ways we can select a treat.

It's possible that you only want one treat, but you can quite easily want more than one.

So how many different orders can you create, if you're allowed to choose as few or as many as you like?

This is the job for the product rule!

Because we can choose treats from a selection of **cupcakes** and **donuts** and **muffins** (notice the "AND"), we **20** x **10** x **15** = 3,000 ordering options.

Difficulty: Hard

Addition vs Multiplication Principle

Rule of Product and Rule of Sum Combined

Suppose four cards are chosen at random from a standard 52-card deck, with replacement. And we wish to determine the number of four-card sequences where all four cards are from the same suit.

There are four suits in a deck (diamonds, hearts, clubs, spades), which means there are 13 cards in each suit. And if we want to determine the number of four-card sequences that are all the same suit, then this would mean that we would be looking for either all hearts or all diamonds or all clubs or all spades.

There are **114,244** possibilities!

$$\underbrace{HHHH}_{\text{all Hearts}} \text{ or } \underbrace{DDDD}_{\text{all Diamonds}} \text{ or } \underbrace{CCCC}_{\text{all Clubs}} \text{ or } \underbrace{SSSS}_{\text{all Spades}}$$

$$\underbrace{13 \cdot 13 \cdot 13 \cdot 13}_{\text{all Hearts}} + \underbrace{13 \cdot 13 \cdot 13 \cdot 13}_{\text{all Diamonds}} + \underbrace{13 \cdot 13 \cdot 13 \cdot 13}_{\text{all Clubs}} + \underbrace{13 \cdot 13 \cdot 13 \cdot 13}_{\text{all Spades}}$$

Image credit - https://calcworkshop.com/

Example set of 52 playing cards; 13 of each suit: clubs, diamonds, hearts, and spades

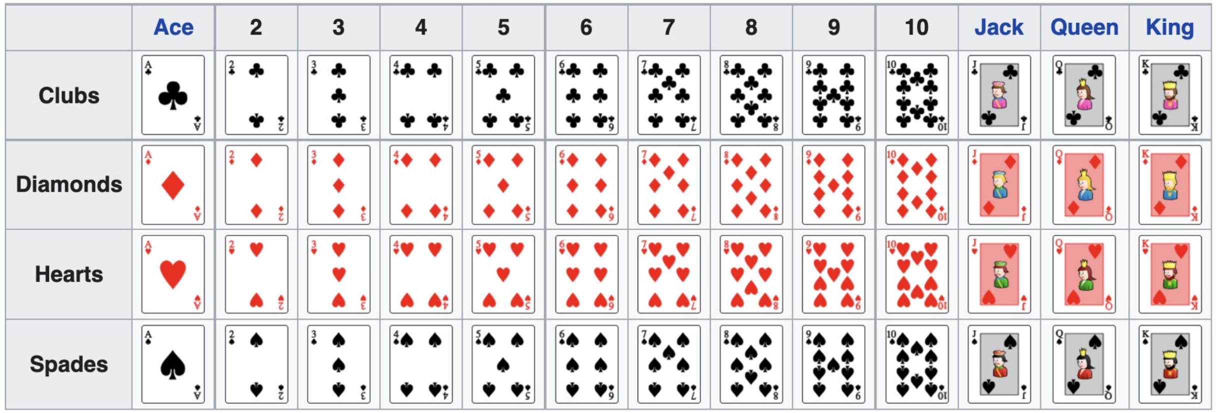


Image credit - Colin M.L. Burnett via Wikipedia

Difficulty: Hard

Addition vs Multiplication Principle

Rule of Product and Rule of Sum Combined

Passwords consist of character strings of 6 to 8 characters.
Each character is an upper case letter or a digit.
Each password must contain at least one digit.
How many passwords are possible?

Total number is
passwords with 6 char. + # passwords with 7 char. + # pws 8 char. (=P6+P7+P8).

P6: # possibilities without constraint : 36^6
exclusions is # passwords without any digits is 26^6
And so, $P6 = 36^6-26^6$
Similarly, $P7 = 36^7-26^7$ and $P8 = 36^8-26^8$
Giving a final answer of $P = P6+P7+P8 = 36^6-26^6 + 36^7-26^7 + 36^8-26^8$

Image credit - UCI ICS/Math 6A, Summer 2007

How Safe Is Your Password?

Time it would take a computer to crack a password with the following parameters

Number of characters	Lowercase letters only	At least one uppercase letter	At least one uppercase letter +number	At least one uppercase letter +number+symbol
1	Instantly	Instantly	-	-
2	Instantly	Instantly	Instantly	-
3	Instantly	Instantly	Instantly	Instantly
4	Instantly	Instantly	Instantly	Instantly
5	Instantly	Instantly	Instantly	Instantly
6	Instantly	Instantly	Instantly	Instantly
7	Instantly	Instantly	1 min	6 min
8	Instantly	22 min	1 hrs	8 hrs
9	2 min	19 hrs	3 days	3 wks
10	1 hrs	1 mths	7 mths	5 yrs
11	1 day	5 yrs	41 yrs	400 yrs
12	3 wks	300 yrs	2,000 yrs	34,000 yrs

Source: Security.org

statista

Addition vs Multiplication Principle

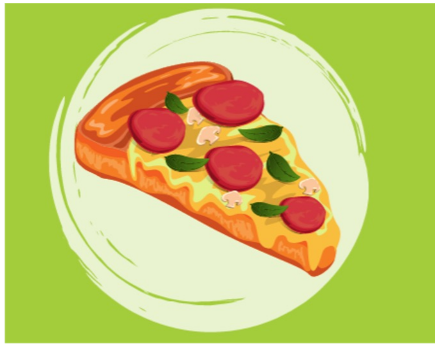
Rule of Product and Rule of Sum Combined

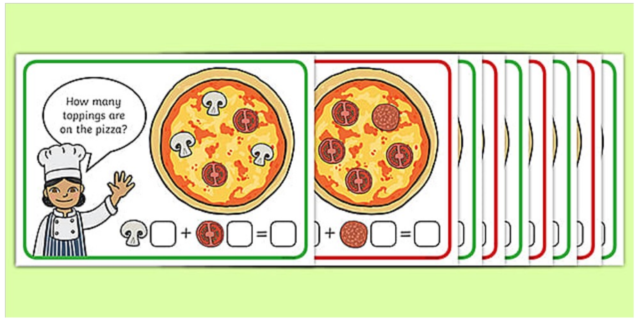
Difficulty: Hard

Suppose you are ordering a pizza, and you have the following options for toppings:

1. You can choose up to 2 types of meat toppings from a list of 4 (e.g., pepperoni, sausage, bacon, ham).
2. You can choose up to 3 types of vegetable toppings from a list of 5 (e.g., mushrooms, onions, bell peppers, olives, tomatoes).

Now, you want to calculate the total number of different pizza combinations you can create based on these topping choices.





You can choose up to 2 types of meat toppings from a list of 4. Using the Rule of Sum, you can calculate the number of ways to choose 0, 1, or 2 meat toppings:

- 0 meat toppings: 1 choice (no meat)
- 1 meat topping: Choose 1 from 4 (4 choices)
- 2 meat toppings: Choose 2 from 4 (combinations: 4 choose 2)

Total choices for meat toppings = 1 + 4 + (4 choose 2) = 1 + 4 + 6 = 11 choices for meat toppings.

You can choose up to 3 types of vegetable toppings from a list of 5. Using the Rule of Sum, you can calculate the number of ways to choose 0, 1, 2, 3 vegetable toppings:

- 0 vegetable toppings: 1 choice (no vegetables)
- 1 vegetable topping: Choose 1 from 5 (5 choices)
- 2 vegetable toppings: Choose 2 from 5 (combinations: 5 choose 2)
- 3 vegetable toppings: Choose 3 from 5 (combinations: 5 choose 3)

Total choices for vegetable toppings = 1 + 5 + (5 choose 2) + (5 choose 3) = 1 + 5 + 10 + 10 = 26 choices for vegetable toppings.

To calculate the total number of different pizza combinations, you apply the Rule of Product by multiplying the choices for meat and vegetable toppings:

Total Pizza Combinations = (Choices for Meat Toppings) × (Choices for Vegetable Toppings) = 11 choices × 26 choices = 286 different pizza combinations.

So, you have 286 different pizza combinations you can create based on your topping choices.

Difficulty: Hard

Addition vs Multiplication Principle
Rule of Product and Rule of Sum Combined

How many integer numbers less than 500 ends with 0?

- How many one-digit number end with 0? Well, there's only one number (i.e., "0") that fits that description, so there's only one possible way to get this value.

- How many two-digit numbers end with 0? Well, for a number to have two digits, the first digit can't be 0, so that means we are limited to choosing digits ranging from 1-9, and the second digit has to be zero, so there's only 1 possible value for this.

- How many three-digit numbers that are less than 500 end with zero? First digit can't be 0 and the number has to be less than 500, so the digit must be a value of 4 or less. Now the second digit can be any value, so that means it can range from 0-9, and our third digit has to be zero, so there's only 1 possible value.

How many ways to get a one-digit number that ends in zero?	$\underset{\textit{digit value}}{\underline{0}}$	1 way
How many ways can we get a two-digit number that ends in zero?	$\left(\underset{\textit{digit value}}{\underline{\#1-9}}\right)\left(\underset{\textit{digit value}}{\underline{0}}\right)$	$(9\ \textit{ways})(1\ \textit{way})$
How many ways can we get a three-digit number that ends in zero and is less than 500?	$\left(\underset{\textit{digit value}}{\underline{\#1-4}}\right)\left(\underset{\textit{digit value}}{\underline{\#0-9}}\right)\left(\underset{\textit{digit value}}{\underline{0}}\right)$	$(4\ \textit{ways})(10\ \textit{ways})(1\ \textit{way})$

$$1+(9)(1)+(4)(10)(1)=50$$

Image credit - https://calcworkshop.com/

There are 50 different numbers that are less than 500 and end with 0!

Pigeonhole Principle

Consider a flock of pigeons nestled in a set of **n** pigeonholes.

If there are **n** pigeons, then it is possible for all of the pigeons to rest happily in separate pigeonholes.

However, if at least one more pigeon arrives, making a total of more than n pigeons, then at least one of the pigeonholes, inevitably, will end up with more than one pigeon.

Peter Dirichlet, German Mathematician

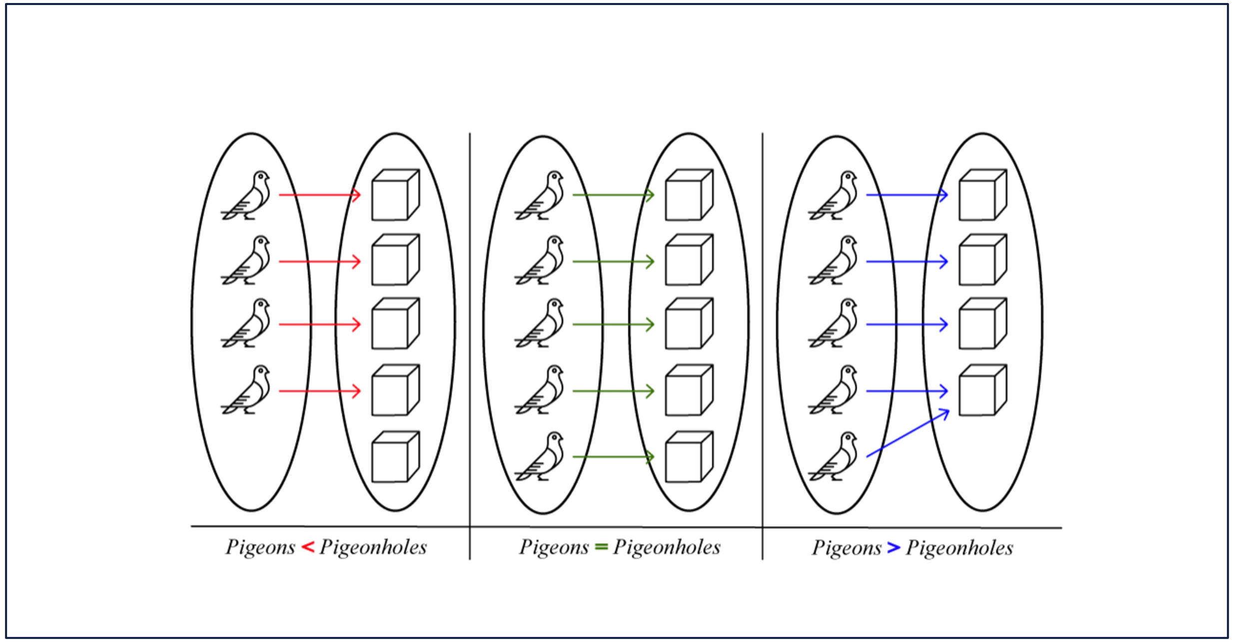


Image credit - https://calcworkshop.com/

If n objects (pigeons) are placed into m boxes (pigeonholes), there are at least one box containing at least ceiling (n/m) objects.

Let $n = 679$ (pigeons) and $m = 35$ (pigeonholes)

Then $\left\lceil \frac{679}{35} \right\rceil = \lceil 19.4 \rceil = 20$

Image credit - https://calcworkshop.com/

Pigeonhole Principle
Examples

Ten people are swimming in the lake. Prove that at least two of them were born on the same day of the week.

The people are the pigeons and the days of the week are the pigeonholes. There are only 7 days in a week and 10 people, therefore at least two of them were born on the same day of the week.

Ceiling (10/7) = 2

Seventeen children are in an elevator. Prove that at least three of them were born on the same day of the week.

If no more than two children were born on each day of the week, then there could be at most 14 children. Since there are 17 children, there must be a day of the week on which at least 3 of them were born.

Ceiling (17/7) = 3

Three people are running for student government. There are 202 people who vote. What is the minimum number of votes needed for someone to win the election?

Solution: By pigeonhole, there exists a person who has gotten at least $\lceil 202/3 \rceil = 68$ votes. So, someone could win with a $67-67-68$ split.

There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, what is the minimum number of different rooms that will be needed?

Solution: There exists a time period will have at least $\lceil 677/38 \rceil = 18$ classes during it. So 18 different rooms will be needed.

Show that in a 8×8 grid, it is impossible to place 9 rooks so that they all don't threaten each other.

Solution: By Pigeonhole, there exists one row with at least two rooks, so they must threaten each other.