Probability Basics

Question 1: Suppose a lift has 3 occupants A, B and C and there are three possible floors (1, 2 and 3) on which they can get out. Assuming that each person acts independently of the others and that each person has an equally likely chance of getting off at each floor, calculate the probability that exactly one person will get out on each floor.

Answer: As there are 3 floors and each person has an equally likely chance of getting off at each floor,

$$total\ number\ of\ possible\ outcomes:\ 3 \times 3 \times 3 = 27$$

Now, considering exactly one person will get off on each floor,

$$favorable\ number\ of\ outcomes\ will\ be:\ 3 \times 2 \times 1 = 6$$

Therefore,

$$required\ probability\ =\ \frac{6}{27}$$

Question 2: n points are taken at random and independently of one another inside a sphere of radius R. What is the probability that the distance from the center of the sphere to the nearest point will not be less than r?

Answer: n points must lie on or outside a sphere of radius r, having the same center as the original sphere of radius R.

For any of the n points,

$$P(lie\ inside\ the\ smaller\ sphere) = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} = \frac{r^3}{R^3}$$

$$P(lie\ on\ or\ outside\ the\ smaller\ sphere) = 1 - \frac{r^3}{R^3}$$

As the points are taken independently, the probability that all n points will lie outside is given by

$$= \left(1 - \frac{r^3}{R^3}\right)^n$$

Question 3: What is the probability of the occurrence of a number that is even or less than 5 when a fair die is rolled?

Answer: Let the event of the occurrence of a number that is even be 'A' and the event of the occurrence of a number that is less than 5 be 'B'. We need to find P(A or B).

$$P(A) = \frac{3}{6}\ \text{(even numbers are 2,4 and 6)}$$

$$P(B) = \frac{4}{6} \text{ (numbers less than 5 are 1,2,3 and 4)}$$

$$P(A \; and \; B) = \frac{2}{6} \text{ (numbers that are both even and less than 5 are 2 and 4)}$$

Now,

$$\begin{aligned} P(A \; or \; B) &= P(A) + P(B) - P(A \; and \; B) \\ &= \frac{3}{6} + \frac{4}{6} - \frac{2}{6} \\ &= \frac{5}{6} \end{aligned}$$

Question 4: A coin is thrown 3 times .What is the probability that at least one head is obtained?

Answer: Sample space = [HHH, HHT, HTH, THH, TTH, THT, HTT, TTT]
Total number of ways = 2 × 2 × 2 = 8.
Favorable Cases = 7

$$P(\text{of getting at least one head}) = \frac{7}{8}$$

$$OR$$

$$P(\text{of getting at least one head}) = 1 - P(\text{of getting no head}) = 1 - (\frac{1}{8}) = \frac{7}{8}$$

Question 5: A problem is given to three persons P, Q, R whose respective chances of solving it are $\frac{2}{7}, \frac{4}{7}, \frac{4}{9}$ respectively. What is the probability that the problem is solved?

Answer: Problem will be solved if at least one of them solves it .

$$\text{P(none of them solves the problem)} = (1 - \frac{2}{7})(1 - \frac{4}{7})(1 - \frac{4}{9})$$

$$= (\frac{5}{7})(\frac{3}{7})(\frac{5}{9}) = \frac{75}{441}$$

Hence,

$$\text{P(at least one of them solves the problem)} = 1 - (\frac{75}{441}) = \frac{366}{441} = \frac{122}{147}$$

Question 6: According to hospital records, 75% of patients suffering from a disease die from that disease. Find out the probability that 4 out of the 6 randomly selected patients survive.

Answer: This has to be a binomial distribution as there are only 2 outcomes - death or survive. Here n = 6, and x = 4.

Probability of survive, p=0.25
Probability of death, q = 0.75
Then, probability that 4 out of the 6 randomly selected patients survive will be given as

$$P(X) = {}^{n}C_r \times p^n \times q^{(n-x)} = {}^{6}C_4 \times (0.25)^4 \times (0.75)^2 = 0.03295$$

Conditional Probability

Question 7: Consider we have purchased a certain product. The manual states that the lifetime of the product, T, defined as the amount of time (in years) the product works properly until it breaks down, satisfies the below condition.

$$P(T \geq t) = e^{\frac{-t}{5}}, \text{for all t} \geq 0$$

For example, the probability that the product lasts more than (or equal to) 2 years is

$$P(T \geq 2) = e^{\frac{-2}{5}} = 0.6703$$

After the purchase we have used it for two years without any problems. What is the probability that it breaks down in the third year?

Answer: Let 'A' be the event that a purchased product breaks down in the third year. Also, let 'B' be the event that a purchased product does not break down in the first two years. We are interested in P(A|B).
We have

$$P(B) = P(T \geq 2) = e^{-\frac{2}{5}}$$

We also have

$$\begin{aligned} P(A) &= P(2 \geq T \geq 3) \\ &= P(T \geq 2) - P(T \geq 3) \\ &= e^{-\frac{2}{5}} - e^{-\frac{3}{5}} \end{aligned}$$

Finally, since A ⊂ B, we have A ∩ B = A. Therefore,

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)}{P(B)} \\ &= \frac{e^{-\frac{2}{5}} - e^{-\frac{3}{5}}}{e^{-\frac{2}{5}}} \\ &= 0.1813 \end{aligned}$$

Question 8: We toss a fair coin three times.

a) What is the probability of getting three heads?

b) What is the probability that we can observe exactly one head?

c) Given that we have observed at least one head, what is the probability that we can observe at least two heads?

Answer: We assume that the coin tosses are independent. For a coin tossed three times sample space will be
$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$$a)P(HHH) = P(H).P(H).P(H) = (\frac{1}{2})^3 = \frac{1}{8}$$

$$\begin{aligned} &b)\text{To find the probability of exactly one head,}\\ P(\text{One head}) &= P(HTT \cup THT \cup TTH)\\ &= P(HTT) + P(THT) + P(TTH)\\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\\ &= \frac{3}{8} \end{aligned}$$

c) Let 'A1' be the event that we observe at least one head, and 'A2' be the event that we observe at least two heads.
Then

$$A1 = S - \{TTT\},\ and\ P(A1) = \frac{7}{8};$$

$$A2 = \{HHT, HTH, THH, HHH\},\ and\ P(A2) = \frac{4}{8}.$$

Thus,we can write,

$$\begin{aligned} P(A2|A1) &= \frac{P(A2 \cap A1)}{P(A1)}\\ &= \frac{P(A2)}{P(A1)}\\ &= \frac{4}{8} * \frac{8}{7}\\ &= \frac{4}{7}. \end{aligned}$$

Question 9: A box contains three coins: two regular coins and one fake two-headed coin (P(H)=1),

a) You pick a coin at random and toss it. What is the probability that it lands heads up?

b) You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

Answer: This is another typical problem for which the law of total probability is useful. Let C1 be the event that you choose a regular coin, and let C2 be the event that you choose the two-headed coin. Note that C1 and C2 form a partition of the sample space.
We already know that

$$P(H|C1) = 0.5, \; and \; P(H|C2) = 1$$

a) Thus, we can use the law of total probability to write

$$P(H) = P(H|C1)P(C1) + P(H|C2)P(C2)$$
$$= \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{3}$$
$$= \frac{2}{3}$$

b) Now, for the second part of the problem, we are interested in P(C2|H). Using Bayes rule,

$$P(C2|H) = \frac{P(H|C2)P(C2)}{P(H)}$$
$$= \frac{1 \times \frac{1}{3}}{\frac{2}{3}}$$
$$= \frac{1}{2}.$$

Question 10: At a park there are 8 girls playing, 3 are named Casey. There are 11 boys playing at that park, 1 of them is named Cane. Let G represent the girls playing at the park. Let B represent the boys playing at the park. Let C represent the kids named Casey playing at the park. Then calculate each of the following:

1. P(C|G)

2. P(B|C)

Answer: Total number of girls, n(G)=8
Number of girls with name Casey, n(C)=3
Total number of boys, n(B)=11
Number of boys with name Cane =1
G: Girls playing at the park
B: Boys playing at the park
C: Kids named Casey

1) Computing P(C|G):

$$\begin{aligned} P(C|G) &= \frac{P(G \cap C)}{P(G)} \\ &= \frac{n(G \cap C)}{n(G)} \\ &= \frac{3}{8} \\ &= 0.375 \end{aligned}$$

2) Because C and B are mutually exclusive events i.e. both do not exist together, therefore,

$$\begin{aligned} P(B|C) &= \frac{P(C \cap B)}{P(C)} \\ &= \frac{0}{P(C)} \\ &= 0 \end{aligned}$$

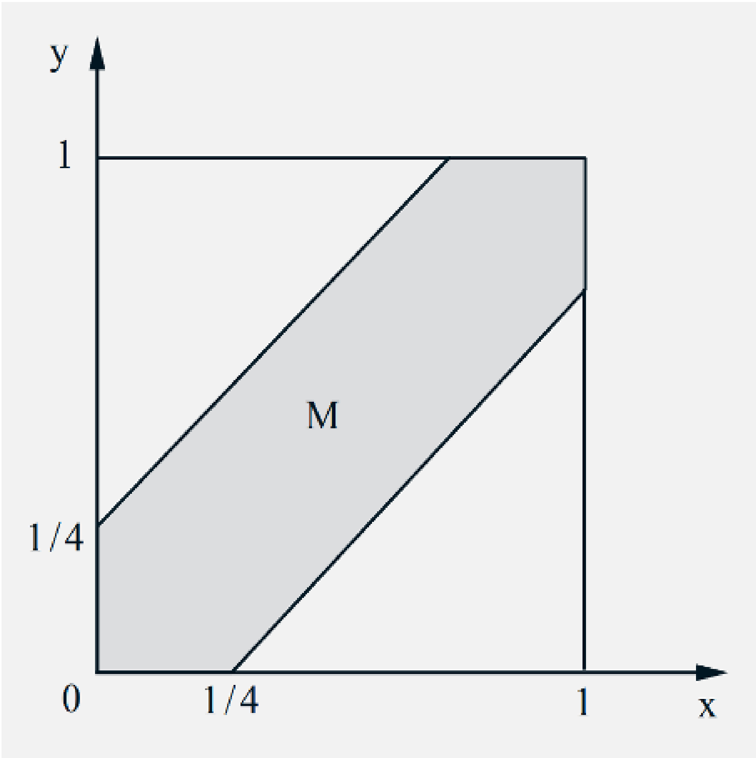
Question 11: Ram and Laxman decide to meet at their hut in the forest between 3 pm and 4 pm (yes, they had time measuring devices). But, they said that either person would only wait for a maximum of 15 minutes, or until 4 pm - whichever is earlier. Assume that they can come at any time between 3 and 4 pm with equal probability (up to infinitesimal second counting - HINT: Do not think of discretizing time). What is the probability that they meet?

Options:
a) 7/16
b) 9/16
c) 1
d) Can't be found

Draw the sample space which is a unit square of side 1. Let the horizontal axis represent the time Ram arrives (Random Variable X), and let the vertical axis represent the time Laxman arrives (Random Variable Y). We need the probability that |X-Y|<1/4.

Answer:
Use a unit square as the sample space. Let X and Y be the two independent, randomly chosen times by person X and Y respectively from the one hour period.
These assumptions imply that every point in the unit square is equally likely, where the first coordinate represents the time when person X shows

up and the second coordinate represents when person Y shows up.
As 15 minutes is quarter of the time between 3 and 4 pm, the required probability in this situation is the area of the shaded region - the set of all points satisfying $|X - Y| \leq 1/4$



Your probability fraction is the area between the lines upon the total area of the square.

Total area = 60 * 60 = 3600 (since there are 60 minutes)
Area between the lines = Total Area - 2 x Area of each triangle
Here total area = 1x1 = 1
Area of each triangle = (Height of triangle x Base of triangle) / 2
The required probability is :-

$$= 1 - 2 \times (\frac{1}{2} \times \ base\ of\ triangle\ \times\ height\ of\ triangle)$$

$$= 1 - 2 \times (\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4})$$

$$= 1 - \frac{9}{16}$$

$$= \frac{7}{16}$$

Bayes Probability

Question 12: Three factories produce light bulbs to supply the market. Factory A produces 20%, 50% of the bulbs are produced in factories B and 30% in factory C. Also, 2% of the bulbs produced in factory A, 1% of the bulbs produced in factory B and 3% of the bulbs produced in factory C are defective.
A bulb is selected at random in the market and found to be defective. What is the probability that this bulb was produced by factory B?
Answer: Let P(A)=20%, P(B)=50%, and P(C)=30% represent the probabilities that a bulb selected at random is from factory A, B and C respectively.
Let P(D) be the probability that a defective bulb is selected.
Let P(D|A)=2%, P(D|B)=1% and P(D|C)=3% represent the conditional probabilities that a bulb is defective given that it is selected from factory A, B and C respectively.
Now, using Bayes' theorem, we can calculate the conditional probability that the bulb was produced by factory B given that it is defective, P(B|D)

$$\begin{aligned} P(B|D) &= \frac{P(D|B)P(B)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\ &= \frac{0.01 \times 0.5}{0.02 \times 0.2 + 0.01 \times 0.5 + 0.03 \times 0.3} \\ &= 0.2777 \end{aligned}$$

Question 13: A factory has two Machines - I and II. Machine-I produces 60% of items and Machine-II produces 40% of the items of the total output. Further 2% of the items produced by Machine-I are defective whereas 4% items produced by Machine-II are defective. If an item is drawn at random what is the probability that it is defective?
Answer: Probability of items produced by Machine-I,

$$P(M1) = \frac{60}{100}$$

Probability of defective items produced by Machine-I,

$$P(D|M1) = \frac{2}{100}$$

Probability of items produced by Machine-II,

$$P(M2) = \frac{40}{100}$$

Probability of defective items produced by Machine-II,

$$P(D|M2) = \frac{4}{100}$$

Now, we need to find the probability that a randomly selected item is defective.
Randomly selected item will be defective either by Machine-I or Machine-II.

$$\begin{aligned} P(D) &= P(M1).P(D|M1) + P(M2).P(D|M2) \\ &= (\frac{60}{100})(\frac{2}{100}) + (\frac{40}{100})(\frac{4}{100}) \\ &= \frac{120}{10000} + \frac{160}{10000} \\ &= \frac{280}{10000} \\ &= 0.0280 \end{aligned}$$

Question 14: There are two identical urns containing respectively 6 black and 4 red balls, 2 black and 2 red balls. An urn is chosen at random and a ball is drawn from it.
(i) find the probability that the ball is black.
(ii) if the ball is black, what is the probability that it is from the first urn?
Answer: Total number of balls in Urn 1

$$\begin{aligned} &= 6\ black\ +\ 4\ red \\ &= 10\ balls \end{aligned}$$

Probability of getting black ball from Urn 1,

$$P(B|U1) = \frac{6}{10}$$

Total number of balls in Urn 2

$$\begin{aligned} &= 2\ black\ +\ 2\ red \\ &= 4\ balls \end{aligned}$$

Probability of getting black ball from Urn 2,

$$P(B|U2) = \frac{2}{4}$$

$$P(U1) = \frac{1}{2},\ and\ P(U2) = \frac{1}{2}$$

(i) Probability that the ball is black,

$$\begin{aligned} P(B) &= P(U1).P(B|U1) + P(U2).P(B|U2) \\ &= \frac{1}{2} \times \frac{6}{10} + \frac{1}{2} \times \frac{2}{4} \\ &= \frac{3}{10} + \frac{1}{4} \\ &= \frac{11}{20} \end{aligned}$$

(ii) We need to find the probability that the selected black ball is from the first urn.

$$\begin{aligned} P(U1|B) &= \frac{P(B|U1).P(U1)}{P(B|U1).P(U1) + P(B|U2).P(U2)} \\ &= \frac{\frac{6}{10} \times \frac{1}{2}}{\frac{6}{10} \times \frac{1}{2} + \frac{2}{4} \times \frac{1}{2}} \\ &= \frac{(\frac{6}{20})}{(\frac{11}{20})} \\ &= \frac{6}{11} \end{aligned}$$

Question 15: A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% are sick with 1 measles.
Let F stand for an event of a child being sick with flu and M stand for an event of a child being sick with measles.
Assume for simplicity that $\mathbf{F \cup M = \Omega}$, i.e., that there no other maladies in that neighborhood.
A well-known symptom of measles is a rash (the event of having which we denote R). Assume that the probability of having a rash if one has measles is $\mathbf{P(R|M) = 0.95}$.
However, occasionally children with flu also develop rash, and the probability of having a rash if one has flu is $\mathbf{P(R|F) = 0.08}$.
Upon examining the child, the doctor finds a rash. What is the probability that the child has measles?
Answer:

Using Bayes' theorem,

$$\begin{aligned} P(M|R) &= \frac{P(R|M).P(M)}{P(R|M).P(M) + P(R|F).P(F)} \\ &= \frac{0.95 \times 0.10}{(0.95 \times 0.10 + 0.08 \times 0.90)} \\ &= 0.57 \end{aligned}$$

