Multi Variable Limits

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Question 1

Compute the following limit: $\lim_{(x,y)\to(0,0)} \frac{e^y+xy}{1+xy}$

Solution

The function $f(x,y) = \frac{e^y+xy}{1+xy}$ is continious at $(0,0)$ and thus $\lim_{(x,y)\to(0,0)} \frac{e^y+xy}{1+xy} = \frac{e^0+0}{1+0} = \boxed{1}$

Question 2

Compute the following limit: $\lim_{(x,y)\to(0,0)} \frac{x^2-y^6}{xy^3}$

Solution

We can show this limit does not exist by using the *Heine criterion*. To do so we observe the following two sequences: $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$, $(\frac{1}{n}, \frac{1}{n^{\frac{1}{6}}}) \to (0,0)$, Now it holds that:

$\lim_{n\to\infty} \frac{(\frac{1}{n})^2-(\frac{1}{n})^6}{\frac{1}{n}\cdot(\frac{1}{n})^3} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-\frac{1}{n^6}}{\frac{1}{n^4}} = \lim_{n\to\infty} n^4(\frac{1}{n^2}+\frac{1}{n^6}) = \lim_{n\to\infty} n^2 - \frac{1}{n^2} = \infty$

$\lim_{n\to\infty} \frac{(\frac{1}{n})^2-(\frac{1}{n^{\frac{1}{6}}})^6}{\frac{1}{n}\cdot(\frac{1}{n^{\frac{1}{6}}})^3} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-\frac{1}{n}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} n^{\frac{3}{2}}(\frac{1}{n^2}-\frac{1}{n}) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} - \sqrt{n} = -\infty$

And because the limits of the sequences are different we have that the limit does not exist.

Question 3

Compute the following limit: $\lim_{(x,y)\to(2,1)} \frac{x^2-2xy}{x^2-4y^2}$

Solution

In this case simple algebraic manipulation does the trick, notice it holds that:

$\lim_{(x,y)\to(2,1)} \frac{x^2-2xy}{x^2-4y^2} = \lim_{(x,y)\to(2,1)} \frac{x(x-2y)}{(x+2y)(x-2y)} \overset{(*)}{=} \lim_{(x,y)\to(2,1)} \frac{x}{x+2y} = \frac{2}{2+2\cdot 1} = \boxed{\frac{1}{2}}$

$(*) \exists 0 < \delta : (x,y) \in B_\delta(2,1) \setminus \{(2,1)\} \implies \frac{x^2-2xy}{x^2-4y^2} = \frac{x}{x+2y}$

Question 4

Compute the following limit: $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Solution

Lemma. *Let* $f:\mathbb{R}^2\to\mathbb{R}$*. Then* $\lim_{(x,y)\to(x_0,y_0)} |f(x,y)| = 0 \implies \lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0$

Indeed this follows immediately from $|f(x,y)-0| = ||f(x,y)|-0|$ and the definition of a limit.

Now we have $0 \leq |\frac{3x^2y}{x^2+y^2}| = |\frac{x^2}{x^2+y^2}||3y| \leq |3y|$ And thus by the squeeze theorem we have $\lim_{(x,y)\to(0,0)} |\frac{3x^2y}{x^2+y^2}| = 0$ and so using the lemma we obtain $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \boxed{0}$

Question 5

Compute the following limit: $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Solution

Lemma. *Let* $f,g:\mathbb{R}^2\to\mathbb{R}, h:\mathbb{R}\to\mathbb{R}$*. If* $f = h\circ g$ *for some* $B_\delta(x_0,y_0)\setminus\{(x_0,y_0)\}$ *and* $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = t_0, \lim_{t\to t_0} h(t) = L$ *then* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

Indeed, the proof is exactly the same as the proof for a limit of a composition of two single variable functions.

Now we can define $h(x) = \frac{\sin x}{x}$ and $h(x,y) = x^2+y^2$, it is known that $\lim_{x\to 0} h(x) = 1$ also $h(x,y)$ is continious so we have $\lim_{(x,y)\to(0,0)} x^2+y^2 = 0$ so by the lemma $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \boxed{1}$