Calculus 1 - 2022aa

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Question 1

(a) $\forall \epsilon > 0 \; \exists \delta > 0 : 0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$

(b) Using the *Weierstrass theorem* we get that there exists a minimum and maximum we can denote by $m, M \in \mathbb{R}$ respectively that f attains on $[a, b]$. i will prove that $\{f(x) : x \in [a, b]\} = [m, M]$. One direction is trivial: $\{f(x) : x \in [a, b]\} \subseteq [m, M]$ (directly from the definition of m, M), the other direction follows from the *intermediate value theorem* which states that for every $m \leq y \leq M$ there exists $x_y \in [a, b]$ such that $f(x_y) = y$ which precisely means that $[m, M] \subseteq \{f(x) : x \in [a, b]\}$.

Question 2

(a) We will calculate the Taylor polynomial of degree 3 of $f(x) = \sqrt{1 + 2x}$ around $x = 0$ and use the *Lagrange error bound* to prove the needed inequality. First let us calculate the derivatives: $f(x) = (1 + 2x)^{1/2}, f'(x) = (1 + 2x)^{-1/2}, f''(x) = -(1 + 2x)^{-3/2}, f'''(x) = 3(1 + 2x)^{-5/2}$ and thus it holds that $f(0) = 1, f'(0) = 1, f''(0) = -1, f'''(0) = 3 \implies P_3(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + e_3(x)$. Now it is only left to show that $x > -\frac{1}{2} \implies e_3(x) \leq 0$. We know from the *Lagrange error bound* that $e_3(x) = \frac{f^{(4)}(c)}{24}x^4$ for some c between 0 and x. Because $f^{(4)}(c) = -15(1 + 2c)^{-7/2}$ it holds that: $x > -\frac{1}{2} \implies c > -\frac{1}{2} \implies (1+2c)^{-7/2} \geq 0 \implies f^{(4)}(c) \leq 0 \implies e_3(x) \leq 0$ as needed.

(b) Note that $f : [1, \infty) \to [1, \infty)$ and thus $f(x) \geq 1$ so $\frac{f(x)}{f(2x)} \geq 0$ (note that $f(2x) \neq 0$). In addition f is monotonic increasing because $f'(x) \geq f(x) \geq 1 \geq 0$. Now, using the *mean value theorem* on $[x, 2x]$ (for some $x \geq 1$, note that f is differentiable in $[1, \infty)$ so we can use it) we get that $f(2x) - f(x) \overset{2x \geq c \geq x}{=} f'(c)x \overset{f' \geq f}{\geq} f(c)x \overset{c \geq x \implies f(c) \geq f(x)}{\geq} f(x)x \implies f(2x) \geq f(x)(x + 1)$ so $\frac{f(x)}{f(2x)} \leq \frac{f(x)}{f(x)(x+1)} \overset{f \geq 0}{=} \frac{1}{x+1}$. Now we have $0 \leftarrow 0 \leq \frac{f(x)}{f(2x)} \leq \frac{1}{x+1} \to 0$ so by the *sandwich theorem* we get $\lim_{x \to \infty} \frac{f(x)}{f(2x)} = 0$.

Question 3

(a) From the fact that $a_{n+1} \geq 2a_n$ for some $n > N$ for all $N \in \mathbb{N}$ we can build a subsequence such that $a_{n_k+1} \geq 2a_{n_k}$ whoever a_n is bounded and thus a_{n_k} is also bounded so by *Bolzano Weierstrass* there exists some sub-subsequence $a_{n_{k_l}}$ which converges. Let us denote the limit by K. By definition $K \geq \liminf_{n\to\infty} a_n$ and also $a_{n_{k_l}+1} \geq 2a_{n_{k_l}}$ now $a_{n_{k_l}+1}$ does not necessarily converge so we take one last convergent subsequence of $a_{n_{k_l}+1}$ (it is bounded as a subsequence of the bounded sequence a_n so one exists) and we get a subsequence $a_{n_{k_{l_m}}+1}$ which converges (denote the limit by D). Now because limits preserve weak inequalities we get $D \geq 2K \geq 2\liminf_{n\to\infty} a_n$ as needed.

(b) Assume by contradiction that the limit of f at 0 does not exist, this means (from the negation of the *Heine criterion*) that there exists two sequences $(a_n), (b_n) \underset{n\to\infty}{\to} 0$ for which it holds that $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$. Now note that we can build two subsequences a_{n_k}, b_{n_k} such that $|a_{n_k}|, |b_{n_k}| \leq 1/k^2$ (by induction on k for example, always take far enough elements such that the index is greater than the previous element and the element smaller than $1/k^2$). This means that $\lim_{n\to\infty} f(a_{n_k}) \neq \lim_{n\to\infty} f(b_{n_k})$ (subsequence converge to the same limit as the convergent sequence) and also from the comparison test and absolute convergence implies convergence we get that $\sum a_{n_k}, \sum b_{n_k}$ converge. Now define a new sequence $c_k = \begin{cases} a_{n_{\lceil \frac{k}{2} \rceil}} & k \text{ odd} \\ b_{n_{\frac{k}{2}}} & k \text{ even} \end{cases}$ $(a_{n_1}, b_{n_1}, a_{n_2}, b_{n_2} ...)$ and note that $\sum c_k$ converges (The partial sum can be expressed with the sum of the partial sums of $\sum a_{n_k}, \sum b_{n_k}$ which converge) and thus $\lim_{k\to\infty} f(c_k) \in \mathbb{R}$ which is a contradiction to the *Heine criterion* because it has two subsequences with different limits ($\lim_{k\to\infty} f(a_{n_k}) \neq \lim_{k\to\infty} f(b_{n_k})$).

Question 4

(a) We will prove that $f'(x)$ is bounded on $[1, \infty)$ and then $f(x)$ will be lipschitz continuous and also uniformly continuous on $[1, \infty)$, and because f is continuous on $[0, 1]$ we know it is uniformly continuous there from the *Heine Cantor theorem* and we will get that f is uniformly continuous on $[0, \infty)$ as needed. Indeed $f''(x) \leq 0$ so from a theorem in class $f'(x)$ is monotonic decreasing and thus $0 \leq f'(x) \leq f'(1)$ for all $x \in [1, \infty)$ which means that $f'(x)$ is bounded as needed.

(b) From the definition of the limit we have $x \in [M, \infty) \implies 4 \leq f(x) + f'(x) \leq 6$ for some $M \in \mathbb{R}$. Now i will prove that $f(x)$ must be bounded in $[M, \infty)$ and thus $f'(x)$ is also bounded in $[M, \infty)$ which means $f(x)$ is uniformly continuous there (from the same argument as 4.a) and it is also uniformly continuous in $[0, M]$ from the *Heine Cantor theorem* and thus uniformly continuous on $[0, \infty)$. Define the set $A = \{x \geq M : f(x) \geq 7\}$ and assume by contradiction that it is not empty, then it is bounded from below and thus it has an infimum we can denote by x_0. Notice that because f is continuous on $[0, \infty)$ then for every $x' \in A$ there exists some $\delta > 0$ such that $x \in [x' - \delta, x' + \delta] \implies f(x) \geq 6.5$ and thus it must hold that $x \in [x' - \delta, x' + \delta] \implies f'(x) \leq 0$ which means that f is monotonic decreasing in $[x' - \delta, x' + \delta]$ so $f(x' - \delta) \geq f(x') \geq 7$. Whoever if we apply this to x_0 we get that $x_0 - \delta \in A$ in contradiction to the fact that it is the infimum. So the set A must be empty and thus $f(x)$ is bounded from above by 7 on $[M, \infty)$ as needed.