Calculus 1 - 2022ba

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Question 1

Let us define the following set: $A := \{x \in [a,b] : f(x) \leq 0\}$ It is non-empty because $f(a) < 0$ and also bounded from above because $x \in A \Rightarrow x \leq b$. We can thus conclude from the *axiom of completeness* that it has a supremum we can denote by c and i will prove that $f(c) = 0$. First note that we showed in class that there must exist a sequence of elements from A which converges to c, let us denote it as follows: $(x_n) \underset{n\to\infty}{\rightarrow} c$ Now because of the definition of A it holds that $f(x_n) \leq 0$ for all n whoever limits preserve weak inequalities and thus $\underset{n\to\infty}{lim} f(x_n) \leq 0$ whoever f is continuous and thus $f(c) = f(\underset{n\to\infty}{lim} x_n) = \underset{n\to\infty}{lim} f(x_n) \leq 0$ and thus $f(c) \leq 0$. Now assume by contradiction that $f(c) < 0$. because f is continuous there is a Neighbourhood of c we can denote by $B_c := \{x : |x-c| < \delta\}$ such that $x \in B_c \Rightarrow f(x) < 0$ (the delta is from the definition of the limit $\underset{x\to c}{lim} f(x) = f(c)$ and a sufficiently small epsilon for example $\epsilon = \frac{|f(c)|}{17}$). This is a contradiction to the fact that c is the supremum of A because $f(c + \frac{\delta}{2}) < 0$. This proves that $f(c) = 0$.

Question 2

(a) We saw in class that a sequence converges if and only if it is a *Cauchy Sequence*. Thus it is sufficient to show that the sequence is *Cauchy*. We will do this by first proving that the following inequality holds for all n using induction: $|a_{n+1} - a_n| \leq \lambda^{n-1}|a_2 - a_1|$
for $n = 1$ we get $|a_2 - a_1| \leq |a_2 - a_1|$ which trivially holds. Now by induction we get:
$|a_{n+1} - a_n| \leq \lambda|a_n - a_{n-1}| \leq \lambda\lambda^{n-2}|a_2 - a_1| = \lambda^{n-1}|a_2 - a_1|$ which proves the result. Now note:
$|a_n - a_m| \overset{\text{triangle inequality}}{\leq} \sum_{k=n}^{m-1} |a_{k+1} - a_k| \overset{\text{Our result}}{\leq} \sum_{k=n}^{m-1} \lambda^{k-1}|a_2 - a_1| \leq |a_2 - a_1| \sum_{k=n}^{\infty} \lambda^{k-1}$
whoever $|a_2 - a_1| \sum_{k=n}^{\infty} \lambda^{k-1} \overset{\text{converging geometric series}}{=} |a_2 - a_1| \frac{\lambda^{n-1}}{1-\lambda} \to 0$. This proves that for some ϵ there is a sufficiently large N for which $N < n < m \Rightarrow |a_n - a_m| < \epsilon$ which proves the result.

(b) (i) By the *Inequality of arithmetic and geometric means* we get $\sqrt{a_n a_{n+1}} \leq \frac{a_n + a_{n+1}}{2}$ so it is sufficient to show that $\sum_{n=1}^{\infty} \frac{a_n + a_{n+1}}{2}$ converges ($0 < a_n$ so we can use comparison test). Whoever this is trivial because we know that $\sum_{n=1}^{\infty} a_n < \infty$.
(ii) First notice that (a_n) must be monotonic decreasing, if we assume by contradiction that it is monotonic increasing we get $\sqrt{a_n a_{n+1}}$ is monotonic increasing and thus $0 < \sqrt{a_1 a_2} \leq \sqrt{a_n a_{n+1}}$ which means that $\sqrt{a_n a_{n+1}} \nrightarrow 0$ which is a contradiction because $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} < \infty \implies \sqrt{a_n a_{n+1}} \to 0$. Now we can finish the proof using the comparison test and the following observation: $a_{n+1} = \sqrt{a_{n+1}^2} \leq \sqrt{a_n a_{n+1}}$ (Also the theorem about changing a finite number of elements).

Question 3

(a) First note that $a_{n+1} = \ln(e^{a_n} - a_n) \iff e^{a_{n+1}} = e^{a_n} - a_n \iff a_n = e^{a_n} - e^{a_{n+1}}$. Thus:
$\sum_{n=1}^{\infty} a_n = \underset{n\to\infty}{lim} \sum_{k=1}^{n} a_k = \underset{n\to\infty}{lim} \sum_{k=1}^{n}(e^{a_k} - e^{a_{k+1}}) = \underset{n\to\infty}{lim}(e^{a_1} - e^{a_{n+1}}) = \underset{n\to\infty}{lim}(e - e^{a_{n+1}})$.
Now we only need to prove that $(a_n) \downarrow 0$ and we are done because then $\underset{n\to\infty}{lim}(e - e^{a_{n+1}}) = e - 1$.
Indeed i will prove by induction that a_n is monotonic decreasing and bounded from below and then from a theorem in class (which is easy to prove) will will get that $a_n \to \inf\{a_n : n \in \mathbb{N}\}$.
Using induction: $0 \leq a_1 = 1 \wedge 0 \leq a_n \implies 0 \leq \ln(e^{a_n} - a_n) = a_{n+1}$. The last inequality holds because $0 \leq \ln(e^x - x) \iff 1 \leq e^x - x \iff 0 \leq e^x - x - 1$ and indeed that is the case for $x \in [0, \infty)$ because: $f(x) = e^x - x - 1 \implies f'(x) = e^x - 1 \geq 0$ for $0 \leq x$ so f is monotonic increasing and thus $0 = f(0) \leq f(x) = e^x - x - 1$ for $x \in [0, \infty)$. Thus $0 \leq a_n$, now i will prove by induction that it is monotonic decreasing:
$a_1 = 1 = \ln(e) \geq \ln(e - 1) = a_2 \wedge a_{n+2} = \ln(e^{a_{n+1}} - a_{n+1}) \leq \ln(e^{a_n} - a_n) = a_{n+1}$ such that the inequality follows from the induction hypothesis that $a_{n+1} \leq a_n$ and the fact that (easy to show using derivatives) $f(x) = \ln(e^x - x)$ is monotonic increasing for $x \in [0, \infty)$. This proves that (a_n) converges. To show the limit is 0 notice that from the uniqueness of the limit (denoted by L):
$a_{n+1} = \ln(e^{a_n} - a_n) \implies L = \ln(e^L - L) \implies L = 0$

(b) Because $e^x \neq 0$ for all $x \in \mathbb{R}$ we get that $f(x)$ and $g(x) = 9e^{x/3}f(x)$ have the same roots. let $x_1 < x_2 \cdots < x_7$ be some roots of $g(x)$. From *Rolle's theorem* it follows that there are $y_1 \in [x_1, x_2], \ldots y_6 \in [x_6, x_7]$ such that $g'(y_i) = 0$. If we apply *Rolle's theorem* again on $g'(x)$ (remember $f(x)$ is twice differentiable and thus so is $g(x)$) we get $z_1 \in [y_1, y_2] \ldots z_5 \in [y_5, y_6]$ such that $g''(z_i) = 0$. After computation we get that $g''(x) = e^{x/3}(f(x) + 6f'(x) + 9f''(x))$ and again because $e^x \neq 0$ for all $x \in \mathbb{R}$ we get that $z_1, \ldots z_5$ are roots of $f(x) + 6f'(x) + 9f''(x)$ as needed.

Question 4

(a) On one direction: $f(c) = g(c) \implies h(c) = f(c) = g(c)$ now because $f(x)$ and $g(x)$ are continuous at $x = c$ we can take $\delta = \min\{\delta_f, \delta_g\}$ that fit some ϵ and get that $|x - c| < \delta \implies |h(x) - h(c)| < \epsilon$
On the other direction: both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense sets in $\mathbb{R}$ and thus there are sequences $x_n \subset \mathbb{Q}, y_n \subset \mathbb{R} \setminus \mathbb{Q}$ such that $x_n, y_n \to c$. Now because $h(x)$ is continuous at $x = c$ it follows from *heine* limit definition that $\underset{n\to\infty}{lim} h(x_n) = \underset{n\to\infty}{lim} h(y_n)$ and Thus $\underset{n\to\infty}{lim} f(x_n) = \underset{n\to\infty}{lim} g(y_n)$ whoever $f(x)$ and $g(x)$ are continuous at $x = c$ so $f(c) = f(\underset{n\to\infty}{lim} x_n) = \underset{n\to\infty}{lim} f(x_n) = \underset{n\to\infty}{lim} g(y_n) = g(\underset{n\to\infty}{lim} y_n) = g(c)$.

(b) Let $x < y \in (0, 1)$. Note that by *Lagrange's theorem* it holds that $g(y) - g(x) = g'(c)(y - x)$ whoever $g'(c) = \frac{1}{c}(f'(c) - \frac{f(c)}{c})$. If we apply *Lagrange's theorem* on $[0, x]$ we get that $f'(c)(x - 0) = f(x) - f(0)$ for some $c \in (0, x)$ whoever this means that $f'(x) - \frac{f(x)}{x} = f'(x) - f'(c) \geq 0$ because $f'(x)$ is monotonic increasing and thus $g'(c) = \frac{1}{c}(f'(c) - \frac{f(c)}{c}) \geq 0$ and from a theorem in class we get that $g(x) = \frac{f(x)}{x}$ is monotonic increasing in $(0, 1)$.