Calculus 2 - 2022ba

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Question 1

(a) Proof: using *l'hopital's rule* (We can because $x \to \infty$) and the *fundamental theorem of calculus* which states that $F' = f$ on $[0, \infty)$ (because f is continuous on $[0, \infty)$) we can calculate the limit:
$\lim_{x\to\infty} \frac{F(x)}{x} = \lim_{x\to\infty} \frac{F'(x)}{x'} = \lim_{x\to\infty} \frac{f(x)}{1} = \lim_{x\to\infty} f(x) = 1$.

(b) Disprove: define $f(x) = 1 + cos(x)$, now $f(x) \underset{x\to\infty}{\nrightarrow} 1$ and $\lim_{x\to\infty} \frac{F(x)}{x} = \lim_{x\to\infty} \frac{x+sin(x)}{x} = 1$

Question 2

(a) Let I be a compact interval, now because f is continuous we know from the *Cantor-Heine theorem* that f is uniformly continuous on I. Now let $\epsilon > 0$, from uniform continuity there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Now let us take some $n > \lceil \frac{1}{\delta} \rceil$ and notice that now $x \in [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}] \implies |f(x) - f(x_0)| < \epsilon$ so by the definition of the supremum and infimum we get $f(x) - \epsilon \leq \inf\{f(y) : y \in [x - \frac{1}{n}, x + \frac{1}{n}]\} \leq \sup\{f(y) : y \in [x - \frac{1}{n}, x + \frac{1}{n}]\} = f_n(x) \leq f(x) + \epsilon$ which means that $|f_n(x) - f(x)| < \epsilon$ for all $x \in I$. This proves that $f_n \xrightarrow{u} f$ on I as needed.

(b) Let (a, b) be an open interval and $x_0 \in (a, b)$ be some point and $\epsilon > 0$ be some epsilon. Because $f_n \xrightarrow{u} f$ there exists some $N \in \mathbb{N}$ such that $n > N \implies |f_n(x) - f(x)| < \epsilon$ for all $x \in (a, b)$. This means that $\sup\{f(y) : y \in [x - \frac{1}{n}, x + \frac{1}{n}]\} - \epsilon < f(x) < \sup\{f(y) : y \in [x - \frac{1}{n}, x + \frac{1}{n}]\} + \epsilon$.
Now choose $\delta = \lfloor \frac{1}{n} \rfloor$ then $|x - x_0| < \delta \implies x_0 \in (x - \frac{1}{n}, x + \frac{1}{n})$ which means that $f(x_0) - \epsilon < \sup\{f(y) : y \in [x - \frac{1}{n}, x + \frac{1}{n}]\} - \epsilon < f(x)$. In the same way we can show that $f(x) - \epsilon < f(x_0)$ (switch x with x_0 in the inequality which follows from the uniform convergence and continue in the same fashion) for the same delta. This proves that $f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$ for $|x - x_0| < \delta$ which means f is continuous at x_0 by definition as needed.

Question 3

(a) We have that $S_N f \xrightarrow{u} f$ and because uniform convergence implies pointwise convergence we also have $S_N f(x_0) \underset{N\to\infty}{\to} f(x_0)$ which implies $S_{N^2} f(x_0) \underset{N\to\infty}{\to} f(x_0)$ (subsequence has the same limit) whoever we also have $S_{N^2} f(x_0) \underset{N\to\infty}{\to} c$ so by the *uniqueness of the limit* we get $c = f(x_0)$.

(b) First we will prove that $\min\limits_{a,b,c\in\mathbb{C}} ||f-(a+be^{it}+ce^{-it})||_2 = ||f-F_1||_2$. Let P_1 be a trigonometric polynomial of degree 1 and let F_1 be the Fourier approximation of degree 1 of f. Now:
$||f-P_1||_2^2 = ||(f-F_1)+(F_1-P_1)||_2^2 \overset{*}{=} ||f-F_1||_2^2 + ||F_1-P_1||_2^2 \geq ||f-F_1||_2^2$ and $*$ follows from the fact that $f-F_1 \perp F_1 - P_1$ and the *Pythagoras theorem.* Now we will find the Fourier coefficients of F_1:

$$\begin{aligned}
&\langle f, e_0\rangle = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} t dt = 0\\
&\langle f, e_1\rangle = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} te^{it}dt = \tfrac{1}{2\pi}(t\tfrac{e^{it}}{i}\ |_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\tfrac{e^{it}}{i}) = \tfrac{1}{2\pi}(\tfrac{-2\pi}{i}-0) = i\\
&\langle f, e_{-1}\rangle = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} te^{-it}dt = \begin{bmatrix} u=-t \\ du=-dt \end{bmatrix} = \tfrac{1}{2\pi}\int_{\pi}^{-\pi} ue^{iu} = -\langle f, e_1\rangle = -i
\end{aligned}$$

And thus Using *Parseval's identity* and the *Pythagoras theorem* we get that:

$$||f-F_1||_2 = \sqrt{||f||_2^2 - ||F_1||_2^2} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} t^2dt - (|i|^2 + |-i|^2 + |0|^2)} = \sqrt{\frac{\pi^2}{3} - 2}$$

Question 4

(a) The proof that (X, d_1+d_2) is a metric space is trivial and follows immediately from the properties of d_1 and d_2 as metrics. I will now prove that (X, d_1+d_2) cannot be a compact space. Assume by contradiction that it is and let $x_n \subset X$ be a sequence from X. Now take some converging subsequence with respect to d_1+d_2 (exists because we assumed by contradiction that (X, d_1+d_2) is compact) denoted by x_{n_k}. Now take another converging subsequence with respect to d_1 (exists because of compactness) denoted by $x_{n_{k_l}}$. This means that there exists $x_0 \in X$ for which it holds that $(d_1+d_2)(x_{n_{k_l}}, x_0) \to 0$ and $d_1(x_{n_{k_l}}, x_0) \to 0$. So now by arithmetic of limits we get:
$d_2(x_{n_{k_l}}, x_0) = (d_1+d_2)(x_{n_{k_l}}, x_0) - d_1(x_{n_{k_l}}, x_0) \to 0 - 0 = 0$ which means (X, d_2) is compact in contradiction (we found a converging subsequence for a given sequence).

(b) On one direction assume that $x^{(n)} \to x$, this means that $||x^{(n)} - x||_2 \to 0$ so by the reverse triangle inequality $0 \leq |||x^{(n)}||_2 - ||x||_2| \leq ||x^{(n)} - x||_2 \to 0$ so $||x^{(n)}||_2 \to ||x||_2$. In addition $||x^{(n)} - x||_2 \to 0 \implies 0 \leq |x_j^{(n)} - x_j| = \sqrt{(x_j^n - x_j)^2} \leq \sqrt{\sum_{i=0}^{\infty}(x_i^{(n)} - x_i)^2} \to 0$ so $x_j^{(n)} \to x_j$ for all $j \in \mathbb{N}$. On the other direction assume $x_j^{(n)} \to x_j$ for all $j \in \mathbb{N}$ and $||x^{(n)}||_2 \to ||x||_2$. First i will prove that $\langle x^{(n)}, x\rangle \to \langle x, x\rangle$. This will help me prove the wanted result. Indeed:
$|\sum_{i=0}^{\infty} x_i^{(n)}x_i - \sum_{i=0}^{\infty} x_i^2| = |(\sum_{i=0}^{M} x_i^{(n)}x_i - \sum_{i=0}^{M} x_i^2) + (\sum_{i=M+1}^{\infty} x_i^{(n)}x_i - \sum_{i=M+1}^{\infty} x_i^2)| \leq$
$|(\sum_{i=0}^{M} x_i^{(n)}x_i - \sum_{i=0}^{M} x_i^2)| + |(\sum_{i=M+1}^{\infty} x_i^{(n)}x_i - \sum_{i=M+1}^{\infty} x_i^2)|$ now because every series converges we can choose an arbitrary $M \in \mathbb{N}$ such that the second term is $< \epsilon/2$ (the tail sum converges to zero) and also because the first term converges to 0 (remember that $x_i^{(n)} \to x_i$) there is a $N \in \mathbb{N}$ such that $n > N$ implies the second term is also $< \epsilon/2$ and thus by definition $\langle x^{(n)}, x\rangle \to \langle x, x\rangle$.
Now we get $||x^{(n)} - x||_2 = \sqrt{\langle x^{(n)} - x, x^{(n)} - x\rangle} = \sqrt{||x^{(n)}||^2 - 2\langle x^{(n)}, x\rangle + ||x||^2} \to 0$ (remember $||x^{(n)}||_2 \to ||x||_2$) so by definition we get $x^{(n)} \to x$ as needed.