Calculus 1 - 2022bb

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Question 1

The existence of $f^{-1} : J \to I$ follows from the fact that $f : I \to f(I) = J$ is surjective (by definition) and injective (because it is strictly monotone increasing).
Now assume that I is an interval, i will show that J must also be an interval. Let $y_1, y_2 \in J$ be elements in the Image and assume without loss of generality that $y_1 < y_2$, this means there are $x_1, x_2 \in I$ such that $f(x_1) = y_1, f(x_2) = y_2$. Whoever from the intermediate value theorem we know that for each $y_1 < c < y_2$ there is a $x_c \in I$ such that $f(x_c) = c$. This means that $c \in J$ and thus J is an interval. Now i will prove that f^{-1} is continuous by definition. Indeed:
$x_0 \in I \Rightarrow f^{-1}(f(x_0)) = x_0$ So i need to prove that for each $0 < \epsilon$ there is a $0 < \delta$ such that $|x - f(x_0)| < \delta \Rightarrow |f^{-1}(x) - x_0| < \epsilon$. Assume by contradiction that this is not the case, meaning there is a $0 < \epsilon_0$ such that for each $0 < \frac{1}{n}$ there is a x_n such that $|x_n - f(x_0)| < \frac{1}{n}$ whoever $|f^{-1}(x_n) - x_0| \geq \epsilon_0$. This means by definition that $|x_n - f(x_0)| \to 0$ and thus $x_n \to f(x_0)$. Whoever there is a sequence $(y_n) \in I$ such that $f(y_n) = x_n$ so we proved that $f(y_n) \to f(x_0)$. This means that $f(x_0) = f(\lim y_n)$ (because f is continuous and the limit is unique) whoever this means that $x_0 = \lim y_n$ (because f is injective) and this is a contradiction to the fact that $|y_n - x_0| = |f^{-1}(x_n) - x_0| \geq \epsilon_0$ for all n. From this we deduce that f^{-1} is continuous.

Question 2

(a) Let us denote $\limsup a_n = \lambda$, we also know that $-\infty < \lambda < \infty$ because (a_n) is bounded. Now i will prove by definition that $S_n \to \lambda$. Take some $0 < \epsilon$. Using the properties of limit supremum we know that there exists some N such that $n \geq N \implies a_n < \lambda + \epsilon$ so by definition of the supremum we get $S_N < \lambda + \epsilon$ and because the series is monotone decreasing (because we take the sup of a set that is a subset of the previous) we get $n \geq N \implies S_n < \lambda + \epsilon$. Additionally from the properties of the supremum we know that there are infinitly many elements such that $\lambda - \epsilon < a_n$, so we get that $\lambda - \epsilon < S_n$ for all n. Thus $n \geq N \implies |S_n - \lambda| < \epsilon$ so by definition $S_n \to \lambda$.

(b) Let us define the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \begin{cases} 0 & n = 0 \\ (\frac{n}{n+1})^n & 0 < n \end{cases}$ and calculate the radius of convergence using the *Cauchy Hadamard* formula. It is easy to see that $\limsup |a_n|^{1/n} = 1$ and Thus the radius of convergence is $\frac{1}{1} = 1$. Now we only need to check $x = -1, 1$ whoever it is easy to see that in both points the power series diverges because the limit of the sequence won't be zero. Indeed: $(\frac{n}{n+1})^n \to \frac{1}{e}$ so at $x = 1$ the limit is not zero and $(-1)^{2n}(\frac{2n}{2n+1})^{2n} \to \frac{1}{e}$ so we have a subsequential limit which is not zero and thus at $x = -1$ the limit cannot be zero too. We can now conclude that: $-\infty < \sum_{n=1}^{\infty} (\frac{an}{n+1})^n < \infty \iff |a| < 1$.

Question 3

(a) Because f is uniformly continuous there is a $0 < \delta$ such that $|x-y| < \delta \implies |f(x)-f(y)| < 1$. Now note that (remember that $f(0) = 0$):
$|f(x)| = |f(x)-f(0)| \overset{\text{triangle inequality}}{\leq} \sum_{n=1}^{\lfloor \frac{2x}{\delta} \rfloor} |f((n-1)\frac{\delta}{2}) - f(n\frac{\delta}{2})| + |f(\lfloor \frac{2x}{\delta} \rfloor \frac{\delta}{2}) - f(x)| \leq \lfloor \frac{2x}{\delta} \rfloor + 1 \leq \frac{2x}{\delta} + 1$ and thus we can choose $a = \frac{2}{\delta}, b = 1$ and we proved $|f(x)| \leq ax + b$.

(b) We will prove $f(x) = (3x^2 + cos(2x) + 5x^4 - 7)D(x)$ has a point of continuity in the interval $(0, 2)$. Indeed notice that $g(x) = 3x^2 + cos(2x) + 5x^4 - 7$ is continuous because it is a composition of continuous functions, and $g(0) = -6 < 0, g(2) = cos(4) + 85 > 0$ so by the intermediate value theorem there exists some $0 < x_0 < 2$ such that $g(x_0) = 0$ whoever $g(x)$ is continuous so $\lim_{x \to x_0} g(x) = g(x_0) = 0$. Now because $D(x)$ is a bounded function we know from a theorem in class that $\lim_{x \to x_0} g(x)D(x) = 0 = g(x_0)D(x_0)$ which means that $f(x) = g(x)D(x)$ is continuous at x_0.

Question 4

(a) First i will prove that f' is uniformly continuous on $[M, \infty)$ for some $M \in \mathbb{R}$ (we only need uniform continuity past a certain point). from the fact that $f''(x) \underset{x \to \infty}{\to} L$ ($L \in \mathbb{R}$) we get that there exists some interval $[M, \infty)$ on which f'' is bounded (can be easily shown from the definition of the limit). This means that f' is uniform continuous on $[M, \infty)$ (bounded derivative implies the function is Lipschitz which implies it is uniformly continuous). Now assume by contradiction that $f'(x) \underset{x \to \infty}{\nrightarrow} 0$ this means that there exists some sequence $(x_n) \to \infty$ such that $|f(x_n)| \geq \epsilon_0$ (Assume that $x_n \geq M$ otherwise just take the subsequence for which that is the case) whoever because we proved f' is uniformly continuous on $[M, \infty)$ there exists some $\delta > 0$ for which it holds that $x \in [x_n - \delta, x_n + \delta] \implies |f'(x)| \geq \frac{\epsilon_0}{2}$ whoever if we use the *mean value theorem* on $[x_n - \delta, x_n + \delta]$ this means that $|f(x_n - \delta) - f(x_n + \delta)| = f'(c_n)|(x_n - \delta) - (x_n + \delta)| \geq \frac{\epsilon_0}{2}|2\delta| = \epsilon_0 \delta$ whoever this is a contradiction because from the fact that $f(x) \underset{x \to \infty}{\to} L$ ($L \in \mathbb{R}$) we get that (using the *Cauchy criterion*) on some interval $[K, \infty)$ it must hold that $x, y \in [K, \infty) \implies |f(x) - f(y)| \leq (\epsilon_0 \delta)/2$ and if we choose some $x_n > K + \delta$ we get the contradiction that $(\epsilon_0 \delta)/2 \geq |f(x_n - \delta) - f(x_n + \delta)| \geq \epsilon_0 \delta$ and thus $f'(x) \underset{x \to \infty}{\to} 0$.

(b) Notice that: $\limsup_{n \to \infty} (\frac{n^{n^2+1}}{(n+1)^{n^2}})^{1/n} = \lim_{n \to \infty} \frac{n^{n+1/n}}{(n+1)^n} = \lim_{n \to \infty} \sqrt[n]{n}(\frac{n}{n+1})^n = 1 \cdot \frac{1}{e} = \frac{1}{e}$ (the last two are very known limits) and thus from the *Cauchy Hadamard* formula $R = \frac{1}{\frac{1}{e}} = e$. Now we can check the edges, because $(\frac{n}{n+1})^n \downarrow \frac{1}{e}$ and x^n is monotone increasing we get that: $(\frac{n^{n^2+1}}{(n+1)^{n^2}})e^n = (\frac{n}{n+1})^{n^2} ne^n \geq (\frac{1}{e})^n ne^n = n \to \infty$ so by the squeeze theorem $(\frac{n^{n^2+1}}{(n+1)^{n^2}})e^n \to \infty$ and thus the series diverges at $x = e$ (because the sequence does not converge to zero). Similarly the sequence wont converge to zero at $x = -e$ (and thus the series will diverge there as well) because it has a subsequential limit of ∞ at the even indexes (because the even indexes at $x = -e$ are a sub sequence of the sequence at $x = e$ which tends to ∞). To sum up the series will converge in the interval $(-e, e)$.