Multi Variable Limits

Amit Bajar

Question 1

Compute the following limit: $\lim_{(x,y)\to(0,0)}\frac{e^y+xy}{1+xy}$

Solution

The function $f(x,y) = \frac{e^y + xy}{1+xy}$ is continuous at (0,0) and thus $\lim_{(x,y)\to(0,0)} \frac{e^y + xy}{1+xy} = \frac{e^0 + 0}{1+0} = \boxed{1}$

Question 2

Compute the following limit: $\lim_{(x,y)\to(0,0)}\frac{x^2-y^6}{xy^3}$

Solution

We can show this limit does not exist by using the Heine criterion. To do so we observe the following two sequences: $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$, $(\frac{1}{n}, \frac{1}{n^{\frac{1}{6}}}) \to (0,0)$, Now it holds that:

$$\lim_{n \to \infty} \frac{(\frac{1}{n})^2 - (\frac{1}{n})^6}{\frac{1}{n} \cdot (\frac{1}{n})^3} = \lim_{n \to \infty} \frac{\frac{1}{n^2} - \frac{1}{n^6}}{\frac{1}{n^4}} = \lim_{n \to \infty} n^4 (\frac{1}{n^2} + \frac{1}{n^6}) = \lim_{n \to \infty} n^2 - \frac{1}{n^2} = \infty$$

$$\lim_{n \to \infty} \frac{(\frac{1}{n})^2 - (\frac{1}{n})^6}{\frac{1}{n^2} \cdot (\frac{1}{n})^3} = \lim_{n \to \infty} \frac{\frac{1}{n^2} - \frac{1}{n}}{\frac{1}{n^2}} = \lim_{n \to \infty} n^{\frac{3}{2}} (\frac{1}{n^2} - \frac{1}{n}) = \lim_{n \to \infty} \frac{1}{\sqrt{n}} - \sqrt{n} = -\infty$$

$$\lim_{n \to \infty} \frac{\frac{(\frac{1}{n}) - (\frac{1}{n})}{n \frac{1}{n}}}{\frac{1}{n} \cdot (\frac{1}{n})^{\frac{1}{n}}} = \lim_{n \to \infty} \frac{\frac{1}{n^2} - \frac{1}{n}}{\frac{1}{n^2}} = \lim_{n \to \infty} n^{\frac{3}{2}} (\frac{1}{n^2} - \frac{1}{n}) = \lim_{n \to \infty} \frac{1}{\sqrt{n}} - \sqrt{n} = -\infty$$

And because the limits of the sequences are different we have that the limit does not exist.

Question 3

Compute the following limit: $\lim_{(x,y)\to(2,1)}\frac{x^2-2xy}{x^2-4y^2}$

Solution

In this case simple algebraic manipulation does the trick, notice it holds that:

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2} = \lim_{(x,y)\to(2,1)} \frac{x(x - 2y)}{(x + 2y)(x - 2y)} \stackrel{(*)}{=} \lim_{(x,y)\to(2,1)} \frac{x}{x + 2y} = \frac{2}{2 + 2 \cdot 1} = \boxed{\frac{1}{2}}$$

$$(*)\exists 0 < \delta : (x,y) \in B_{\delta}(2,1) \setminus \{(2,1)\} \implies \frac{x^2 - 2xy}{x^2 - 4y^2} = \frac{x}{x + 2y}$$

Question 4

Compute the following limit: $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Solution

Lemma. Let $f: \mathbb{R}^2 \to \mathbb{R}$. Then $\lim_{(x,y)\to(x_0,y_0)} |f(x,y)| = 0 \implies \lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0$ Indeed this follows immediately from |f(x,y)-0| = ||f(x,y)|-0| and the definition of a limit.

Now we have $0 \le |\frac{3x^2y}{x^2+y^2}| = |\frac{x^2}{x^2+y^2}||3y| \le |3y|$ And thus by the squeeze theorem we have $\lim_{(x,y)\to(0,0)} |\frac{3x^2y}{x^2+y^2}| = 0$ and so using the lemma we obtain $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \boxed{0}$

Question 5

Compute the following limit: $\lim_{(x,y)\to(0,0)}\frac{\sin(x^2+y^2)}{x^2+y^2}$

Solution

Lemma. Let $f, g : \mathbb{R}^2 \to \mathbb{R}, h : \mathbb{R} \to \mathbb{R}$. If $f = h \circ g$ for some $B_{\delta}(x_0, y_0) \setminus \{(x_0, y_0)\}$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = t_0, \lim_{t\to t_0} h(t) = L$ then $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

Indeed, the proof is exactly the same as the proof for a limit of a composition of two single variable functions.

Now we can define $h(x)=\frac{\sin x}{x}$ and $h(x,y)=x^2+y^2$, it is known that $\lim_{x\to 0}h(x)=1$ also h(x,y) is continuous so we have $\lim_{(x,y)\to(0,0)}x^2+y^2=0$ so by the lemma $\lim_{(x,y)\to(0,0)}\frac{\sin(x^2+y^2)}{x^2+y^2}=\boxed{1}$