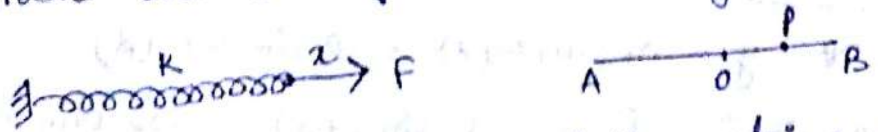
SHM Motion : Translation, rotation, vibration/oscillation

periodic motion $f(t) = f(t+T)$ e.g. $\sin\frac{2\pi t}{T}$, $\cos\frac{2\pi t}{T}$

if periodic over same path → oscillatory motion

$\boxed{\text{elasticity \& inertia}}$



SHM is a special type of periodic motion where restoring force on particle is proportional to displacement & directed to the mean position.

oscillation between point A & B, mean position O. at time t, particle is at P & displacement is x. F - restoring force

"small oscillation approximation"

$F \propto -x$ or $F = -kx$ or $ma = -kx$

$\therefore \quad a = -\frac{k}{m}x = -\omega^2 x$

characteristics (1) linear motion → to-n-fro in straight line.

(2) $F \propto -x$

linear harmonic motion ↔ angular harmonic motion.

(pendulum) (torsional pendulum)

$F \propto -x$ $\tau \propto -\theta$

complete oscillation : one point to same point. (time period)

amplitude : maximum displacement on both sides

frequency : no. of oscillations in 1 second.

phase : displacement, velocity, accelaration & direction of motion. After 1 oscillation, phase is same.

t=0, initial phase.

Relation between SHM & uniform circular motion

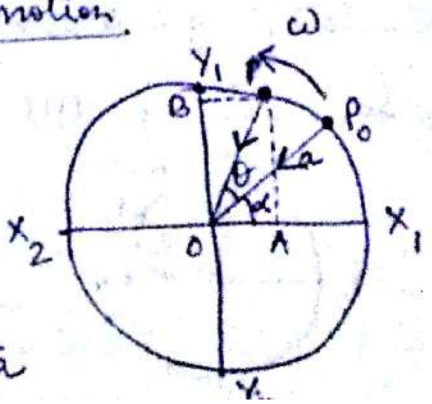
$\theta = \omega t$

$s = a\theta$

$OA = x, \; OB = y$

$= OP\cos(\theta+\alpha) = a\cos(\theta+\alpha)$

$= a\cos(\omega t+\alpha)$



speed $v = \omega a$, centripetal acc $f_r = \frac{v^2}{a} = \omega^2 a$

Accelaration of A is component of f_r along X_1OX_2

$$f_A = -f_r \cos(\omega t+\alpha) = -\omega^2 a \cos(\omega t+\alpha) = -\omega^2 x$$

$$\therefore \underline{f_A \propto -x.}$$

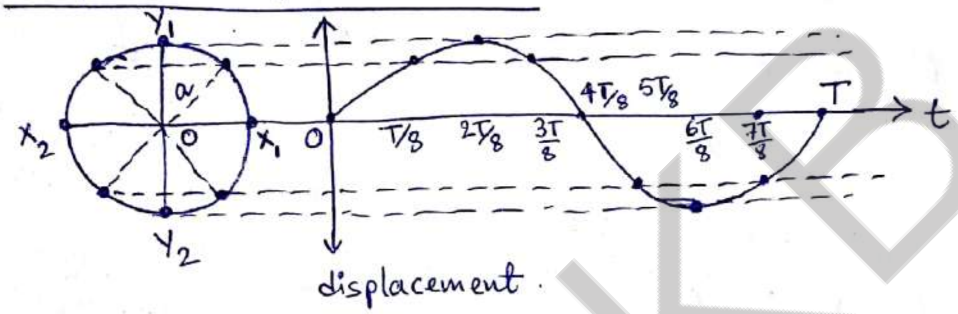
Similarly, $OB = y = OP\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

Accelaration of B is $f_B = -f_r \sin(\theta+\alpha) = -\omega^2 a\sin(\omega t+\alpha) = -\omega^2 y$

$$\therefore \underline{f_B \propto -y.}$$

$\therefore$ SHM is defined as the projection of uniform circular motion along diameter of circle.

<u>Graphical representation</u>

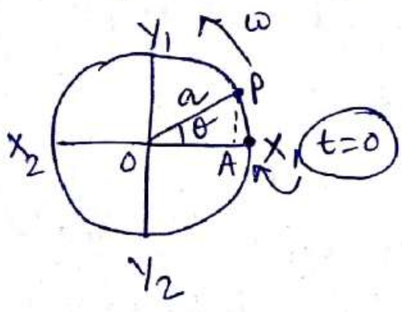


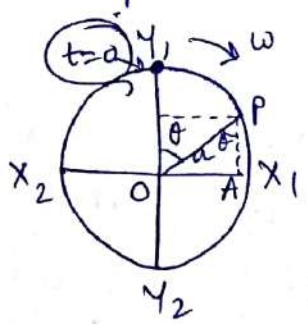
Time period = T.

$$y = a\sin\frac{2\pi}{T}t$$

(SHM along y-axis)

<u>Displacement</u> In SHM, displacement at time t is the distance of the particle from the mean position.





$OA = OP\cos\theta$

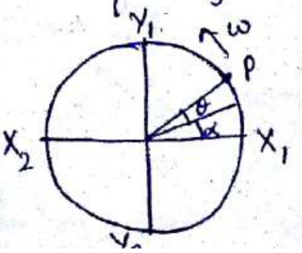
$x = a\cos\omega t$

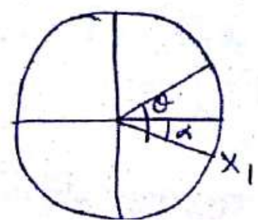
$OA = OP\cos(\pi/2-\theta)$

$x = a\sin\theta = a\sin\omega t$

Similarly, $y = a\cos\omega t$ & $y = a\sin\omega t$.

So, eq^n. of SHM can be derived from any instant t.



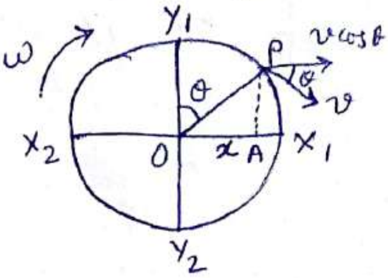


$x = a\cos(\theta+\alpha) = a\cos(\omega t+\alpha)$

Similarly, $x = a\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

if initial position is X_1 (2^{nd} pic) then $x = a\cos(\omega t-\alpha)$

or $x = a\sin(\omega t-\alpha)$



Velocity & accelaration

velocity of SHM is component of the particle's velocity along x-axis at time t.

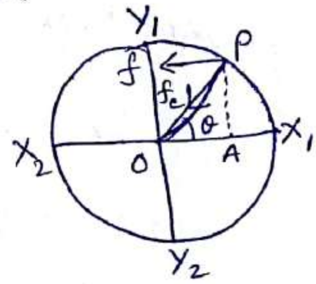
$V = a\omega$, V parallel to OA $= v\cos\theta$

$= a\omega\cos\theta = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$x = a\sin\theta$

$\therefore \boxed{v = \omega\sqrt{a^2-x^2}}$

v_{max} is at $x=0$, $v_{max} = a\omega$. & $x=a$, $v_{min}=0$.



Same with accelaration ⇒ SHM is the projection along X-axis is component of accelaration along X-axis. $f_c = -\omega^2 a$ & component around X_1X_2 is $-\omega^2 a\cos\theta = -\omega^2 a\cos\omega t = -\omega^2 x$.

$\therefore f = -\omega^2 x$.

$f_{max} = -\omega^2 a$ when $x=\pm a$, $f_{max} = \pm\omega^2 a$.

$f_{min} = 0$ when $x=0$.

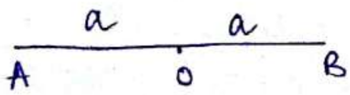
Calculus: $x = a\sin\omega t$, $v = \dot{x} = a\omega\cos\omega t = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$= \omega\sqrt{a^2-x^2}$.

$f = \ddot{x} = -a\omega^2\sin\omega t = -\omega^2 x$.

$\omega^2 = f/x$ (neglect)

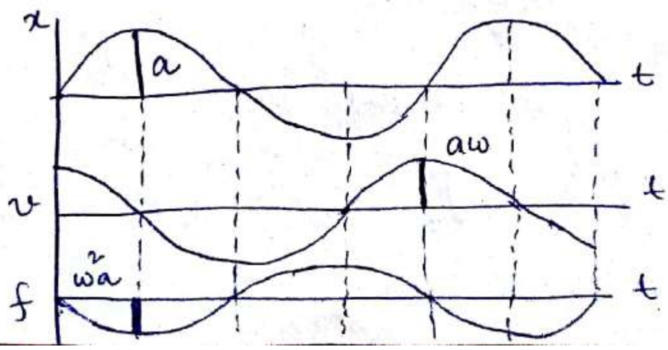
Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{f}}$



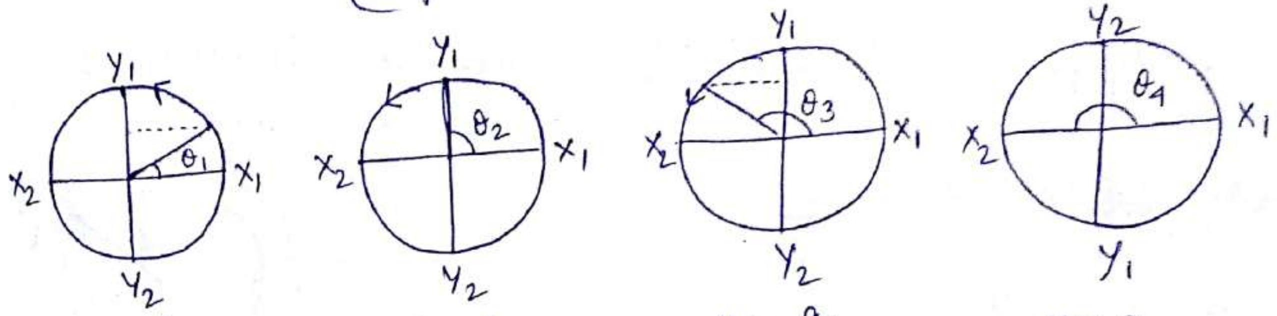
$x = a\sin\omega t = a\sin\frac{2\pi}{T}t$

$v = a\omega\cos\omega t = a\omega\cos\frac{2\pi}{T}t$

$f = -a\omega^2\sin\omega t = -a\omega^2\sin\frac{2\pi}{T}t$



Phase you see, a & ω (angular velocity) are constant.
(amplitude) $\theta = \omega t$ is changing = phase.



$y = a/2$, $\theta_1 = 30^\circ$, v upwards

$y_1 = a$, $\theta_2 = 90^\circ$, $V = 0$

$y = a/2$, $\theta_3 = 150^\circ$, V = downwards

$y = 0$, $\theta_4 = 180^\circ$, v = downwards.

phase difference 2 particles. $\phi = \theta_1 - \theta_2 = 0$ (in phase)
$= 180^\circ$ (out of phase)

Differential form & solution Homogeneous, 2nd order, ODE with constant coefficients

$F = -kx$ or $m\ddot{x} = -kx$ or $\ddot{x} + \omega^2 x = 0$, $\omega = \sqrt{\frac{k}{m}}$.

Solution: Multiply by $2\dot{x}$, $2\dot{x}\ddot{x} + 2\omega^2 x\dot{x} = 0$

Integrating $\dot{x}^2 = -\omega^2 x^2 + C$

when displacement is maximum, $x = a$, $\dot{x} = 0$ $\Rightarrow$ $C = \omega^2 a^2$

$\therefore$ $v = \dot{x} = \pm\omega\sqrt{a^2 - x^2}$

or $\pm\frac{dx}{\sqrt{a^2-x^2}} = \omega dt$, Integrating $\sin^{-1}\frac{x}{a} = \omega t + \phi$

or $x = a\sin(\omega t + \phi)$

See, $x = a\cos(\omega t + \phi)$ also satisfy $\ddot{x} + \omega^2 x = 0$.

$x = a\sin(\omega t + \phi) = a\sin\omega t\cos\phi + a\cos\omega t\sin\phi$
$= A\sin\omega t + B\cos\omega t$.

In operator form, $\frac{d^2x}{dt^2} = D^2 x$, $\frac{dx}{dt} = Dx$

$D^2 x + \omega^2 x = 0$ or $D^2 = -\omega^2$ or $D = \pm i\omega$

$\therefore$ General solution $x = Ae^{i\omega t} + Be^{-i\omega t}$

For real value of x, $A = B^*$ $\quad A = a+ib, \quad B = a-ib$

you can also have $x = ae^{i(\omega t+\phi)}$

sinusoidal or cosinusoidal.

CW 1. Oscillatory motion of a particle is represented by $x = ae^{i\omega t}$. Establish the motion is SHM. Similarly if $x = a\cos\omega t + b\sin\omega t$ then SHM.

$$x = ae^{i\omega t}, \quad \dot{x} = ai\omega e^{i\omega t}, \quad \ddot{x} = -a\omega^2 e^{i\omega t}$$
$$= -\omega^2 x \quad \text{(SHM)}$$

$$x = a\cos\omega t + b\sin\omega t, \quad \dot{x} = -a\omega\sin\omega t + b\omega\cos\omega t$$
$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \quad \text{(SHM)}.$$

2. Which periodic motion is not oscillatory?

→ earth around sun or moon around earth.

3. Dimension of force constant of vibrating spring.

$$F = -Kx \qquad [K] = \frac{[\text{Force}]}{[\text{displacement}]} = \frac{[\text{Newton}]}{[\text{Metre}]}$$
$$= \frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$$

also called "stiffness".

HW 1. In SHM, displacement is $x = a\sin(\omega t+\phi)$. at $t=0$, $x = x_0$ with velocity v_0, show that $a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$ & $\tan\phi = \frac{\omega x_0}{v_0}$.

2. Particle is vibrated at frequency 5 Hz in SHM. Show that when displacement exceeds 10^{-2} metre, the particle loses contact with the vibrator. Given $g = 9.8$ m/s^2

3. In SHM, a particle has speed 80 cm/s & 60 cm/s with displacement 3 cm & 4 cm. Calculate amplitude of vibration

Energy of a particle in SHM

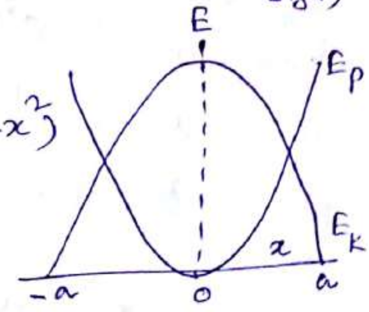
Work is done on particle to displace → restoring force. So P.E. in spring stored & motion is K.E. Total energy constant

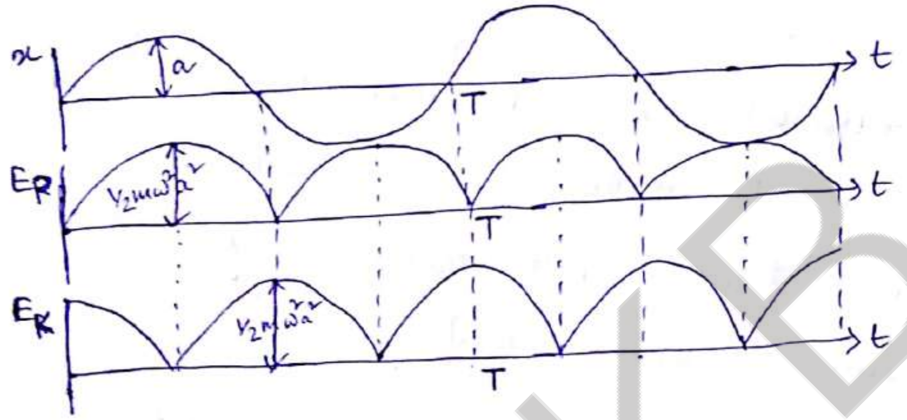
P.E. $F = mf = -m\omega^2 x \quad \therefore dW = F\,dx = m\omega^2 x\,dx$ (against so no -ive sign)

$$\therefore E_p = \int_0^x m\omega^2 x\,dx = \frac{1}{2} m\omega^2 x^2.$$

K.E. $v = \omega\sqrt{a^2-x^2}, \quad E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2-x^2)$

$$E_{Tot} = E_K + E_p = \frac{1}{2}m\omega^2 a^2 = \text{constant}.$$





Examples of SHM

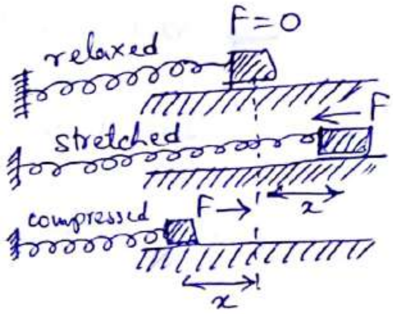
Horizontal oscillations

$F = -kx = m\ddot{x}$

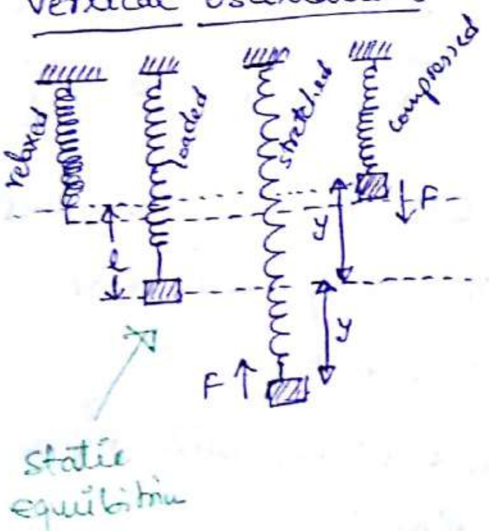
$\ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{K}{m}}$

$x = A\cos(\omega t + \phi), \quad T = 2\pi\sqrt{\frac{m}{K}}$

initial cond^n. material.



Vertical oscillations



Tension on spring $F_0 = Kl$

Force on mass $= mg$.

Static eq. $mg = Kl$.

stretched Tension on spring $= K(l+y)$

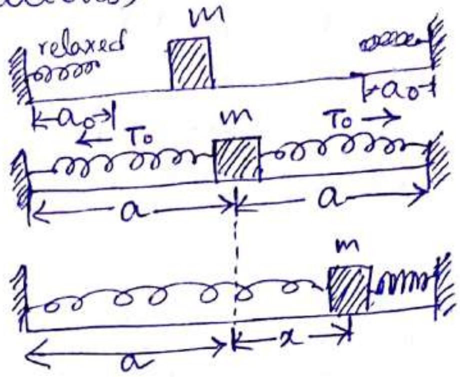
$$mg - F = K(l+y) = Kl + Ky = \cancel{mg} + Ky$$

$$F = -Ky.$$

compressed $mg + F = K(l-y) = \cancel{mg} - Ky$

$$F = -Ky.$$

Two spring system (Longitudinal oscillations)



horizontal frictionless surface, rigid wall, massless spring, relaxed length a_0.

After connection, static equilibrium

$$T_0 = K(a - a_0)$$

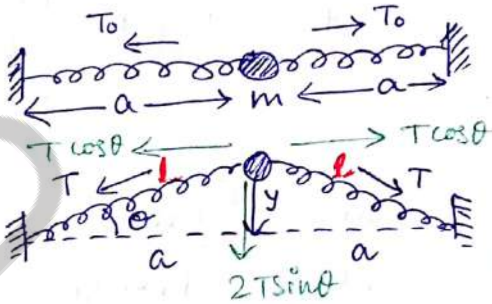
x = displacement to right. restoring force by left spring $-K(a + x - a_0)$

force on right spring $K(a - x - a_0)$

$$\therefore \; F_x = K(a-x-a_0) - K(a+x-a_0) = -2Kx$$

$$m\ddot{x} = -2Kx \quad \text{or} \quad \ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{long} = 2\pi\sqrt{\frac{m}{2K}}}$$

Two spring system (Transverse oscillations)



$$T_0 = K(a - a_0)$$

$$T = K(l - a_0)$$

$$F_y = -2T\sin\theta = -2T\frac{y}{l}$$

$$l = \sqrt{y^2 + a^2}$$

$$\text{or} \quad m\ddot{y} + \frac{2T}{l}y = 0 \quad \text{or} \quad \ddot{y} + \omega^2 y = 0$$

$$\omega^2 = \frac{2T}{ml} = \frac{2K(l-a_0)}{ml}, \quad \text{but } l = f(y).$$

So $\ddot{y} + \frac{2K}{m}\left(1 - \frac{a_0}{f(y)}\right)y = 0$ is not a SHM.

ⓐ Slinky approximation $a \gg a_0$ or $\frac{a_0}{a} \ll 1$.

$$\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{l}\right) = \frac{2K}{m}\left(1 - \frac{a_0}{a}\frac{a}{l}\right) \quad \text{as } l > a.$$

$$= \frac{2K}{m}. \quad \text{Then SHM.} \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K}}}$$

Spring

"large" harmonic oscillations

ⓑ small oscillation approximation $a \not\gg a_0$ but $y \ll a$ or l.

$$\therefore \; l = \sqrt{y^2 + a^2} = a\sqrt{\frac{y^2}{a^2} + 1} \sim a$$

Then also $\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{a}\right)$ or $\boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K\left(1 - \frac{a_0}{a}\right)}}}$

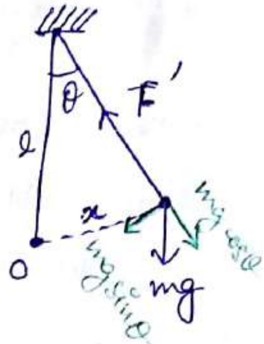
Rubber

SHM.

$$\therefore \; T_{long} = \sqrt{1 - \frac{a_0}{a}}\; T_{trans}.$$

So longitudinal is faster than transverse.

Simple pendulum



$F' = mg\cos\theta$
(tension in string)

$\left[\lim_{\theta\to 0}\right]$

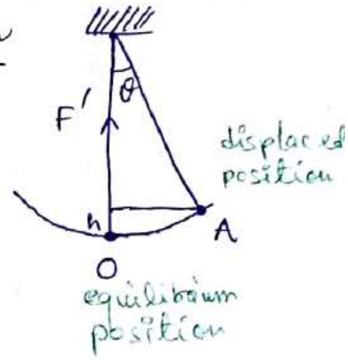
$F = -mg\sin\theta$
(restoring force) $= -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \simeq -mg\theta$

$x = l\theta$

or, $m\ddot{x} = -mg\frac{x}{l}$ or $\ddot{x} + \frac{g}{l}x = 0.$

$\therefore\ \omega = \sqrt{g/l},\ T = 2\pi\sqrt{\frac{l}{g}}.$ (mass independent)

string tension when pendulum at mean position



$F' = mg + \frac{mv^2}{l}$ (centrifugal force)

at A, Energy = KE + PE = $0 + mgh = mgh$

at O, Energy = KE + PE = $\frac{1}{2}mv^2 + 0 = \frac{1}{2}mv^2$

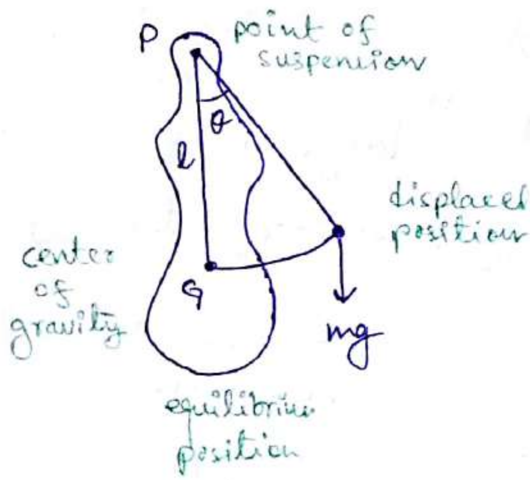
Conservation of energy $\Rightarrow \frac{1}{2}mv^2 = mgh$ or $v^2 = 2gh.$

or $v^2 = 2g(l - l\cos\theta) = 2gl(1-\cos\theta) = 2gl \times 2\sin^2\frac{\theta}{2}$

$\simeq 4gl\left(\frac{\theta}{2}\right)^2 = gl\theta^2.$

$\therefore\ F' = mg + \frac{m}{l}gl\theta^2 = mg(1+\theta^2).$

Compound Pendulum



arbitrary shaped rigid body oscillating about a horizontal axis passing through it.

restoring force $\Leftrightarrow$ reactive couple or torque

moment of restoring force
$= -mgl\sin\theta$

angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$, moment of inertia = I.

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$\text{or } \frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\sin\theta \simeq -\frac{mgl}{I}\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

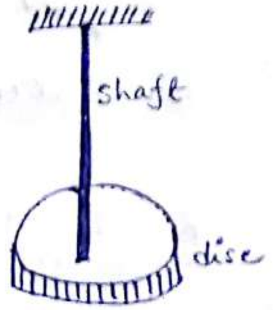
$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

If we consider moment of inertia about a parallel axis through G, K = radius of gyration then using parallel axis theorem,

$$I = mK^2 + ml^2 \qquad \therefore T = 2\pi\sqrt{\frac{K^2/l + l}{g}} = 2\pi\sqrt{\frac{l'}{g}}.$$

equivalent length of simple pendulum $= \frac{K^2}{l} + l$.

Torsional Pendulum



twist of shaft → torsional oscillations

torsional couple $= -\tau\theta$

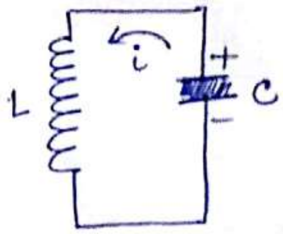
couple due to accelaration $= I\frac{d^2\theta}{dt^2}$

$$I\frac{d^2\theta}{dt^2} = -\tau\theta \quad , \quad T = 2\pi\sqrt{\frac{I}{\tau}}$$

From classical mechanics course, $\tau = \frac{\pi\eta d^4}{32\,l} = \frac{\pi\eta r^4}{2l}$

d = shaft diameter, η = modulus of rigidity,
$= 2r$.

Electrical oscillator



Capacitor is charged ⇒ electrostatic energy in dielectric media. It discharges through the inductor electrostatic energy ⇔ magnetic energy. (no dissipation of heat)

voltage across inductor $= -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$ $\quad q$ = charge

voltage across capacitor $= \frac{q}{C}$.

No e.m.f. circuit, $\frac{q}{C} = -L\frac{d^2q}{dt^2}$ or $\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$

$\omega^2 = \frac{1}{LC}$, $q = q_0\sin(\omega t + \phi)$. charge on capacitor varies harmonically.

$$i = \frac{dq}{dt} = \omega q_0 \cos(\omega t + \phi)$$

$$V = \frac{q}{C} = \frac{q_0}{C} \sin(\omega t + \phi)$$

Total energy = magnetic energy + electric energy

$$= \int iV\,dt + \frac{1}{2}CV^2 = \int i\,L\frac{di}{dt}\,dt + \frac{1}{2}CV^2$$

$$= \int Li\,di + \frac{1}{2}CV^2 = \frac{1}{2}Li^2 + \frac{1}{2}CV^2 = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}CV^2$$

In mechanical oscillation, Total energy $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

equivalence

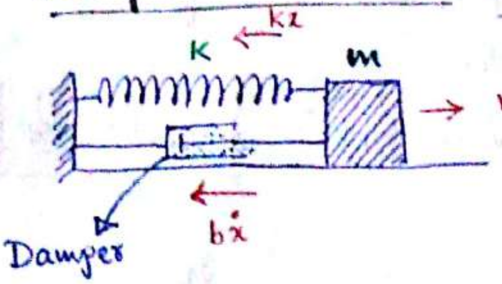
$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

In electrical oscillation, Total energy $= \frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2$

Free Damped harmonic motion

Damping of a real system is a complex phenomena involving several kind of damping force. Damping force of a body in a fluid is a function of velocity. This is called "**viscous damping**." When an oscillating body is contact with a surface, the frictional force is called "**Coulomb friction**". Also in solids, energy is partly lost due to internal friction & imperfect elasticity of the material. Experiments suggest that such resistive force is independent of frequency & proportional to amplitude. This is called "**structural damping**." The viscouse damping force may be represented as $F = -Av + \cancel{Bv^2} - \cancel{Cv^3} + \cdots \rightarrow 0$ and such approximation is "linear damping".

Damped oscillation of a system with 1 degree of freedom



inertial force $m\ddot{x}$ is balanced by elastic restoring force kx & viscous damping force $b\dot{x}$

$$\therefore \; m\ddot{x} = -b\dot{x} - kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \underbrace{\frac{b}{m}}_{=\gamma}\frac{dx}{dt} + \underbrace{\frac{K}{m}}_{=\omega_0^2}x = 0.$$

This is a linear homogeneous 2nd order ODE.

Let the trial solution $x = Ae^{\alpha t}$, substituting we get

$$(\alpha^2 + \gamma\alpha + \omega_0^2)Ae^{\alpha t} = 0 \Rightarrow \alpha^2 + \gamma\alpha + \omega_0^2 = 0.$$

$$\therefore \; \alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2}$$

$$\therefore \text{ Solution } \; x = A_1 \exp\left[-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t + A_2 \exp\left[-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t$$

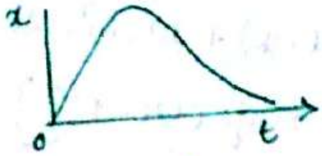
$$= e^{-\gamma t/2}\left[A_1 \exp\left(\sqrt{\gamma^2/4 - \omega_0^2}\, t\right) + A_2 \exp\left(-\sqrt{\frac{\gamma^2}{4} - \omega_0^2}\, t\right)\right]$$

We can have three possibilities:

(a) Heavy damping $\gamma^2/4 > \omega_0^2$ $q = \sqrt{\gamma^2/4 - \omega_0^2} > 0$.

$x = e^{-\gamma t/2}(A_1 e^{qt} + A_2 e^{-qt})$. This means that x cannot be negative and at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & e^{qt} contributes like exponenti then at $t \to \infty$, it'll damp to x (initial). If we had started at $x = 0$, after a time interval it decays back to zero $\Rightarrow$ "Dead beat Galvanometer."

no oscillation.

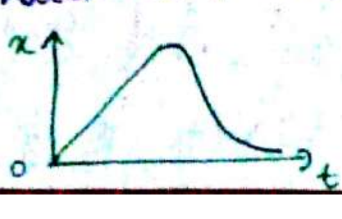


(b) Critical damping $\gamma^2/4 = \omega_0^2$: $x = (A_1 + A_2)e^{-\gamma t/2}$. The damping is slower but it has a discrepancy that at $x=0$ at $t=0$, $v=0$ which is not true. Changing the trial solution, we can derive

$x \sim t e^{-\gamma t/2}$ mean at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & $x \propto t$

& later $t \to \infty$, $e^{-\gamma t/2}$ dominates. x is never negative $\Rightarrow$ no oscillati

"pointer_type galvanometer"



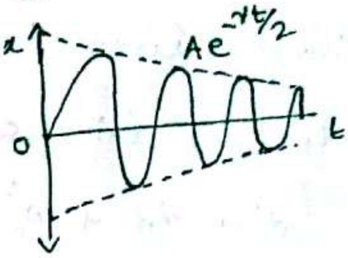
(c) Weak damping $\frac{\gamma^2}{4} < \omega_0^2$ $\quad q = \sqrt{\gamma^2/4 - \omega_0^2}$ = imaginary.

This gives oscillatory damped harmonic motion

$$x = e^{-\gamma t/2}\left[A_1 e^{i\sqrt{\omega_0^2 - \gamma^2/4}\,t} + A_2 e^{-i\sqrt{\omega_0^2-\gamma^2/4}\,t}\right] \qquad \boxed{\omega = \sqrt{\omega_0^2 - \gamma^2/4}}$$

$$= e^{-\gamma t/2}\left(A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right)$$

$$= e^{-\gamma t/2}\left[\underbrace{(A_1 + A_2)}_{A\cos\delta}\cos\omega t + \underbrace{i(A_1 - A_2)}_{A\sin\delta}\sin\omega t\right] = A e^{-\gamma t/2}\cos(\omega t - \delta)$$



Amplitude decreases in due time

Angular frequency is less than undamped motion.

$\tau = 2/\gamma$ = mean life time of oscillation.

Energy of a weakly damped oscillator

Using $x = Ae^{-\gamma t/2}\cos(\omega t - \delta)$ we develop expression for average energy. $\dot{x} = -\frac{\gamma}{2}Ae^{-\gamma t/2}\cos(\omega t - \delta) - Ae^{-\gamma t/2}\omega\sin(\omega t - \delta)$

$\therefore$ Kinetic energy (instantaneous) of the vibrating body

$$\frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \gamma\omega\cos(\omega t - \delta)\sin(\omega t - \delta)\right]$$

$$\text{Potential energy} = \int_0^x F\,dx = \int_0^x Kx\,dx = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2e^{-\gamma t}\cos^2(\omega t - \delta)$$
$$= \frac{1}{2}m\omega_0^2A^2e^{-\gamma t}\cos^2(\omega t + \delta)$$

$\therefore$ Total energy = KE + PE =

$$\frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \omega_0^2\cos^2(\omega t - \delta) + \frac{\gamma\omega}{2}\sin\{2(\omega t - \delta)\}\right]$$

For small damping, $\gamma << 2\omega_0$, then $e^{-\gamma t}$ does not change appreciably during one time period $T = \frac{2\pi}{\omega}$, then time averaged energy of the oscillator is

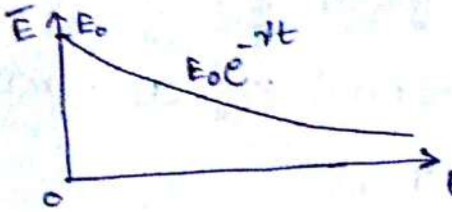
$$\langle E\rangle = \frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\langle\cos^2(\omega t - \delta)\rangle + \omega^2\langle\sin^2(\omega t - \delta)\rangle + \omega_0^2\langle\cos^2(\omega t - \delta)\rangle + \frac{\gamma\omega}{2}\underbrace{\langle\sin\{2(\omega t - \delta)\}\rangle}_{0}\right]$$

$$\text{Now } \langle\cos^2(\omega t - \delta)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\omega t - \delta)\,d(\omega t) = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}2\cos^2x\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(1 + \cos 2x)\,dx = \frac{1}{2} = \langle\sin^2(\omega t - \delta)\rangle$$

$$\therefore \langle E \rangle = \frac{1}{2} m A^2 e^{-\gamma t}\left[\cancel{\frac{\gamma^2}{8}} + \left(\omega_0^2 - \cancel{\frac{\gamma^2}{4}}\right)\frac{1}{2} + \frac{\omega_0^2}{2}\right] = \frac{1}{2} m\omega_0^2 A^2 e^{-\gamma t}$$

$\boxed{\langle E \rangle = E_0 e^{-\gamma t}}$ where $E_0 = \frac{1}{2} m\omega_0^2 A^2$ is energy of undamped oscillator



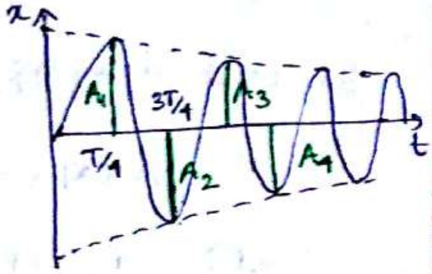
The average power dissipation in one time period

$\langle P(t) \rangle = \frac{d}{dt}\langle E(t) \rangle = \gamma \langle E(t) \rangle$. due to friction

Estimation of Damping

There are various ways of estimation of the damping of an oscillator. Let us choose initial condition at $t=0$, $x=0$, $\frac{dx}{dt} = v_0$ and $\delta = \pi/2$, $x = Ae^{-\gamma t/2}\cos(\omega t - \pi/2) = Ae^{-\gamma t/2}\sin\omega t$

Logarithmic Decrement



$$x = Ae^{-\gamma t/2}\sin\omega t = Ae^{-\gamma t/2}\sin\frac{2\pi t}{T}$$

at $t = \frac{T}{4}$, $x_1^{max} = Ae^{-\gamma T/8}\sin\frac{2\pi}{T}\frac{T}{4} = Ae^{-\gamma t/8}$

at $t = \frac{3T}{4}$, $x_2^{max} = Ae^{-3\gamma T/8}$

at $t = \frac{5T}{4}$, $x_3^{max} = Ae^{-5\gamma T/8}$ etc.

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \frac{x_3^{max}}{x_4^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\gamma T/4} = d \text{ (constant)}$$

"d" is called decrement of the motion. $\lambda = \ln d$ is the logarithmic decrement of the motion $= \ln e^{\gamma T/4} = \frac{\gamma T}{4}$

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\lambda}$$

Multiplying, $\frac{x_1^{max}}{x_n^{max}} = e^{(n-1)\lambda}$ or $\lambda = \frac{1}{n-1}\ln\left(\frac{x_1^{max}}{x_n^{max}}\right)$

$$\lambda = \frac{2.303}{n-1}\log_{10}\left(\frac{x_1^{max}}{x_n^{max}}\right)$$

This method is used to determine the corrected last throw of a Ballistic galvanometer due to damping. Relation between undamped throw θ_0 & first throw θ_1 is

$\theta_1 = \theta_0 e^{-\gamma T/8}$ $\therefore \theta_0 = \theta_1 e^{\gamma T/8} = \theta_1 e^{\lambda/2} \simeq \theta_1\left(1 + \frac{\lambda}{2}\right)$ for $\lambda \ll 1$

So knowing λ, we can correct θ_1 for damping.

Quality Factor (Q-Value)

Another method to express damping in an oscillatory system is to measure the rate of decay of energy. Quality factor $Q = \frac{\omega}{\gamma}$ $= \frac{\omega_0}{\gamma}\sqrt{1-\frac{\gamma^2}{4\omega_0^2}}$. while $\langle E\rangle = E_0 e^{-\gamma t}$, power $\langle P(t)\rangle = \frac{d}{dt}\langle E\rangle = \gamma\langle E\rangle$

So the average energy dissipated in time period T is

$$\gamma T\langle E\rangle = \frac{2\pi\gamma}{\omega}\langle E\rangle = \frac{2\pi}{Q}\langle E\rangle = \frac{2\pi}{Q} \times \text{average energy stored.}$$

$$\therefore \boxed{Q = 2\pi \times \frac{\text{Average energy stored in one time period}}{\text{Average energy lost in one time period}}}$$

In weak damping limit $\frac{\gamma^2}{4\omega_0^2} << 1$, $Q = \frac{\omega_0}{\gamma}$. As $\gamma \to 0$, $Q \to \infty$

$\therefore$ $x = A\exp\left(-\frac{\omega_0 t}{2Q}\right)\cos(\omega_0 t - \delta)$ in limit $\gamma^2/4\omega_0^2 << 1$

$\langle E\rangle = E_0\exp\left(-\frac{\omega_0 t}{Q}\right)$ and see that $\tau_1 = \frac{Q}{\omega_0}$, $\langle E\rangle = E_0 e^{-1}$

and no. of complete oscillation if is n, then $n = \frac{\omega_0}{2\pi}\tau_1 = \frac{Q}{2\pi}$

So $\langle E\rangle$ reduces to e^{-1} of $\langle E\rangle$ in $Q/2\pi$ cycles of oscillation.

Note that $\boxed{\lambda = \frac{\gamma T}{4},\ \tau = \frac{2}{\gamma}\ \&\ Q = \frac{\omega_0}{\gamma},\ \tau_1 = \frac{Q}{\omega_0} = \frac{1}{\gamma}}$

"Moving coil Galvanometer" is the example of damped harmonic motion. Similarly, current or charge oscillation in LCR circuit, mechanical vibration of a string or tuning fork etc.

Forced Vibration

Vibrating system with damping + periodic force = forced vibration. Vibration dies out, system tunes to the frequency of natural vibration. For example, a bridge vibrates in the influence of marching soldiers. Contributions are restoring force kx, damping force $b\dot{x}$, inertial force $m\ddot{x}$ & external periodic force $F(t) = F_0\cos\omega t$.

$\therefore$ Equation of motion of the body is

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$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F(t)$$

or $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f_0\cos\omega t$, $\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{K}{m}$, $f_0 = \frac{F_0}{m}$.

linear homogeneous 2^{nd} order ODE. Solution of this we can separate out as $\frac{d^2x_1}{dt^2} + \gamma\frac{dx_1}{dt} + \omega_0^2 x_1 = f_0\cos\omega t$ & $\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \omega_0^2 x_2 = 0$ so that $x_1 + x_2$ is a solution. Now we know $x_2 = Ae^{-\gamma t/2}\cos(\omega^* t - \delta)$ with $\omega^* = \sqrt{\omega_0^2 - \gamma^2/4}$ & will die out in time (transient state). For x_1, we can write $x_1 = B\cos(\omega t - \delta)$ where B & δ are to be determined

$$X = Re(Be^{i(\omega t-\delta)}).$$ In this notation,

$$\frac{d^2X}{dt^2} + \gamma\frac{dX}{dt} + \omega_0^2 X = f_0 e^{i\omega t} = f_0 e^{i(\omega t-\delta)} e^{i\delta}$$

or $$\left[B\left[(\omega_0^2 - \omega^2) + i\omega\gamma\right] - f_0 e^{i\delta}\right] e^{i(\omega t-\delta)} = 0, \forall t.$$

$$B(\omega_0^2 - \omega^2 + i\omega\gamma) - f_0 e^{i\delta} = 0 \quad \text{or} \quad Be^{-i\delta} = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

or $$B\cos\delta - iB\sin\delta = \frac{f_0\left[\omega_0^2 - \omega^2 - i\omega\gamma\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\therefore B\cos\delta = \frac{f_0(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}, \quad B\sin\delta = \frac{f_0\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \quad \therefore B = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)$$

$$\therefore \boxed{x_1 = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)\right)}$$

<u>Steady state solution</u>

Its dependent on $F_0, m, \omega, \omega_0, \gamma$ & there is a phase difference δ between force & displacement. When $D = (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2$ is minimum B is maximum amplitude. If this frequency is ω_r then $\left.\frac{dD}{d\omega}\right|_{\omega=\omega_r} = 0$ and $\left.\frac{d^2D}{d\omega^2}\right|_{\omega=\omega_r} > 0$. $\therefore -2(\omega_0^2 - \omega_r^2)2\omega_r + 2\omega_r\gamma^2 = 0$

or $\omega_r = \sqrt{\omega_0^2 - \gamma^2/2}$ and convince yourself $\frac{d^2D}{d\omega^2} > 0$ if $\frac{\gamma^2}{2} < \omega_0^2$.

Thus amplitude of forced oscillation is maximum if frequency of the driving force is nearly equal to frequency of natural oscillation

At $\omega = \omega_r$, $B_{max} = \dfrac{f_0}{\gamma(\omega_0^2 - \gamma^2/4)^{1/2}}$ and $\gamma << \omega_0$, $B_{max} \approx \dfrac{f_0}{\gamma\omega_0} = \dfrac{F_0}{m\gamma\omega_0} = \dfrac{F_0}{b\omega_0}$

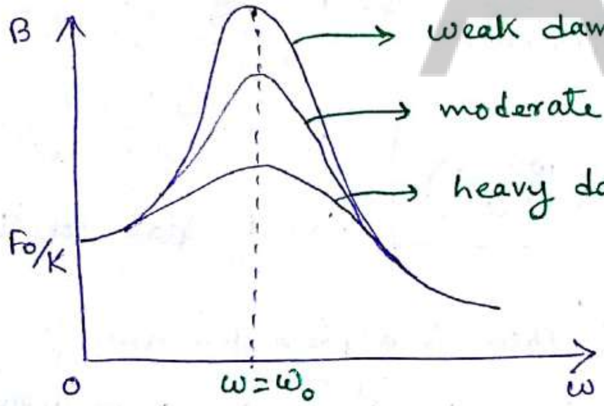
Thus in this limit $\omega_r \simeq \omega_0$ and the amplitude is controlled by "b" and the forced oscillator is "**resistance controlled.**"

Recall $B = \dfrac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$, In limit $\omega << \omega_0$, $B \approx \dfrac{F_0/m}{\omega_0^2\sqrt{1 + \frac{\omega^2}{\omega_0^2}\frac{\gamma^2}{\omega_0^2}}}$

$\simeq \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{K}$

This displacement a constant force F_0 would produce. when $\omega \to 0$, $F(t) \to F_0$ or we get back $m\dfrac{d^2x}{dt^2} = -m\omega^2 x$ very small role than Kx term. ∴ Response of the oscillator is controlled by the stiffness constant K & the oscillator is "**stiffness controlled.**"

Similarly for $\omega >> \omega_0$, $B \simeq \dfrac{F_0/m}{\omega^2\sqrt{1 + \frac{\gamma^2}{\omega_0^2}\frac{\omega_0^2}{\omega^2}}}$ which for weak damping $\gamma << \omega_0$ is $B \simeq \dfrac{F_0}{m\omega^2}$ and $m\omega^2 x$ is dominating. and the oscillator is "**mass or inertia controlled.**"



amplitude resonance at $\omega = \omega_0$ when $\gamma^2/2 < \omega_0^2$.

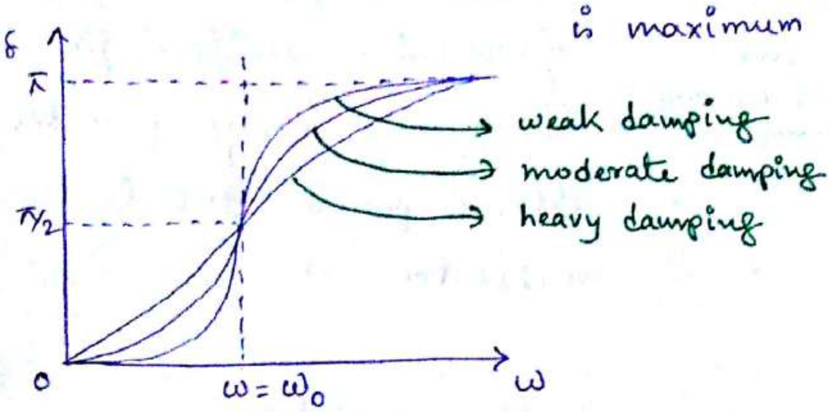
Also when $\omega << \omega_0$,

$\tan\delta = \dfrac{\omega\gamma}{\omega_0^2 - \omega^2} \simeq \dfrac{\omega}{\omega_0}\dfrac{\gamma}{\omega_0}$

as $\omega \to 0$, $\delta \to 0$. Thus for low frequency of driving force, displacement is nearly in phase with driving force. If $\omega >> \omega_0$, $\tan\delta \simeq -\dfrac{\gamma}{\omega} \simeq \dfrac{\gamma}{\omega_0}\dfrac{\omega_0}{\omega}$ which for weak damping $\gamma << \omega_0$ has small negative value or $\delta \simeq \pi$.

∴ If frequency of driving force >> natural frequency of free oscillations, then displacement will be out of phase with driving force. Also ~~when~~ ~~[illegible]~~ accelaration will be in phase with driving force.

But at resonance, $\omega \simeq \omega_0$ & $\tan\delta = \infty$ $\therefore$ $\delta = \pi/2$ or displacement is maximum when driving force is zero.



Displacement x_1 lags the force $F(t)$ by δ.

<u>Velocity Resonance</u> $\quad x_1 = B\cos(\omega t - \delta)$ $\therefore$ $\dot{x}_1 = -\omega B\sin(\omega t - \delta)$

or $v = v_0 \cos(\omega t - \phi)$ where $v_0 = \omega B = \dfrac{F_0/m}{\sqrt{\dfrac{(\omega_0^2 - \omega^2)^2}{\omega^2} + \nu^2}}$

$= v_0 \cos(\omega t - \delta + \pi/2)$

and $\phi = \delta - \pi/2$. $[-\sin(\omega t - \delta) = \cos(\omega t - \delta + \pi/2)]$

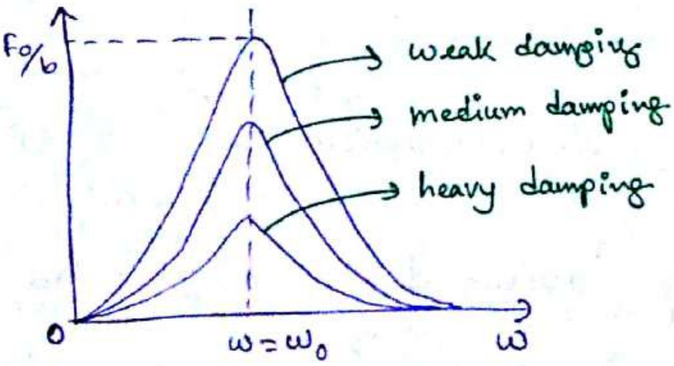
$\therefore$ Velocity leads the displacement in phase by $\pi/2$. V_0 is maximum when denominator is minimum. $\dfrac{d}{d\omega}\left[\dfrac{(\omega_0^2-\omega^2)^2}{\omega^2} + \nu^2\right]\Bigg|_{\omega=\omega_r} = 0$

or $\omega_r = \omega_0$. So at $\omega = \omega_0$, V_0 is maximum, velocity resonance

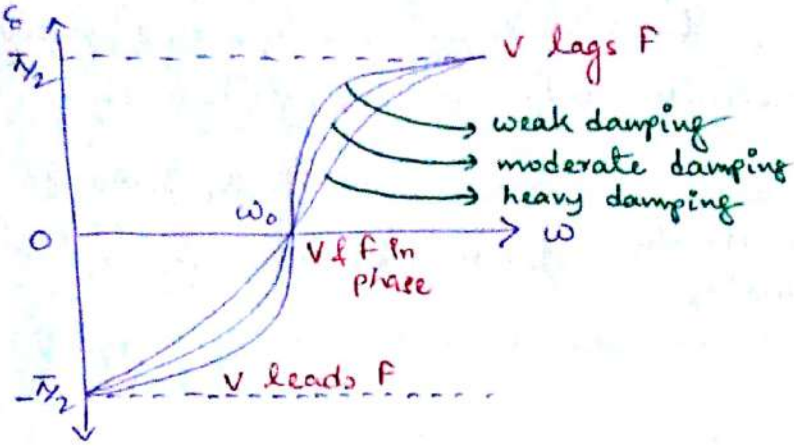
$V_0^{max} = \dfrac{F_0/m}{\nu} = \dfrac{F_0}{b}$, so as "b" increases, V_0^{max} decreases.

For $\omega >> \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega}$ and if ν is not large then $V_0 \to 0$ for $\omega \to \infty$

for $\omega << \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{m\omega^2}\dfrac{\omega^2}{\omega_0^2} \to 0$ for $\omega \to \infty$.



Phase of velocity relative to the force is $\phi = \delta - \pi/2$. For $\omega << \omega_0$, $\delta \simeq 0$, so $\phi = -\pi/2$. As ϕ is angle by which velocity lags behind the force, so here velocity leads the force by an angle $\pi/2$. For $\omega >> \omega_0$, $\delta \simeq \pi$, $\phi = \pi - \pi/2 = \pi/2$ so for very high frequencies, velocity lags the force by $\pi/2$. At resonance $\omega = \omega_0$, $\delta = \pi/2$ and $\phi = 0$ & velocity is in phase with force.



This is therefore the most favourable condition for transfer of energy from the external periodic force to the oscillator.

Power transfer from driving force to the oscillator

Energy of a damped oscillator decreases exponentially as $E(t) = E_0 e^{-\gamma t}$. In order to maintain steady state oscillation, driving force transfers energy to oscillator. Now

$$x = B\cos(\omega t - \delta) = B\cos\delta\cos\omega t + B\sin\delta\sin\omega t$$
$$= B_{el}\cos\omega t + B_{ab}\sin\omega t$$

where B_{el} = elastic amplitude $B\cos\delta = \dfrac{f_0(\omega_0^2 - \omega^2)}{(\omega_0^2 + \omega^2)^2 + \gamma^2\omega^2}$ [in phase with force]

B_{ab} = absorptive amplitude $B\sin\delta = \dfrac{f_0\,\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$ [out of phase $\pi/2$ with force]

$v = \dot{x} = \omega(-B_{el}\sin\omega t + B_{ab}\cos\omega t)$ & thus the power by driving force $F_0\cos\omega t$ / second is the work done by the force/second

$$P(t) = F_0\cos\omega t\; v = F_0\omega\cos\omega t(-B_{el}\sin\omega t + B_{ab}\cos\omega t).$$

∴ Time averaged input power over one complete cycle is

$$P_{input} = \langle P(t)\rangle = \frac{1}{T}\int_0^T P(t)\,dt = -\frac{F_0\,\omega B_{el}}{T}\int_0^T \underbrace{\sin(\omega t)\cos(\omega t)\,dt}_{0} + \frac{F_0\,\omega B_{ab}}{T}\int_0^T \underbrace{\cos^2(\omega t)\,dt}_{\to 1/2} = \frac{1}{2}F_0\,\omega B_{ab}.$$

This input power supplied by driving force is not stored in oscillator but dissipated as work done in moving the system against friction. Instantaneous power dissipated through friction is

$$P(t) = bv\cdot v = b\left(\frac{dx}{dt}\right)^2 = b\omega^2(B_{ab}^2\cos^2\omega t + B_{el}^2\sin^2\omega t - 2B_{ab}B_{el}\cos\omega t\sin\omega t)$$

$\therefore$ Time averaged power $\langle P(t) \rangle = P_{dissipation} = \frac{b\omega^2}{2}(B_{el}^2 + B_{ab}^2)$.

$$= \frac{b\omega^2 f_0^2}{2[(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2]} = \frac{1}{2} F_0 \omega B_{ab}$$

$\therefore P_{input} = P_{dissipate}$ (steady state).

Energy of the forced oscillator. Instantaneous KE is

$$\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(B_{ab}^2\cos^2\omega t + B_{el}^2\sin^2\omega t - 2B_{ab}B_{el}\cos\omega t\sin\omega t)$$

Instantaneous PE $\frac{1}{2}Kx^2 = \frac{1}{2}m\omega_0^2(B_{ab}^2\sin^2\omega t + B_{el}^2\cos^2\omega t + 2B_{ab}B_{el}\cos\omega t\sin\omega t)$

$\therefore$ Time averaged total energy is $E = \langle E(t) \rangle = \frac{1}{4}m(\omega^2+\omega_0^2)(B_{ab}^2 + B_{el}^2)$

$E_{resonance} = \frac{1}{2}m\omega_0^2(B_{ab}^2 + B_{el}^2)$ at $\omega \simeq \omega_0$

$\langle KE \rangle = \frac{1}{4}m\omega^2(B_{ab}^2 + B_{el}^2)$, $\langle PE \rangle = \frac{1}{4}m\omega_0^2(B_{ab}^2 + B_{el}^2)$

Maximum input power & Bandwidth

Time averaged input power $P_{input} = \frac{1}{2}F_0\omega B_{ab}$

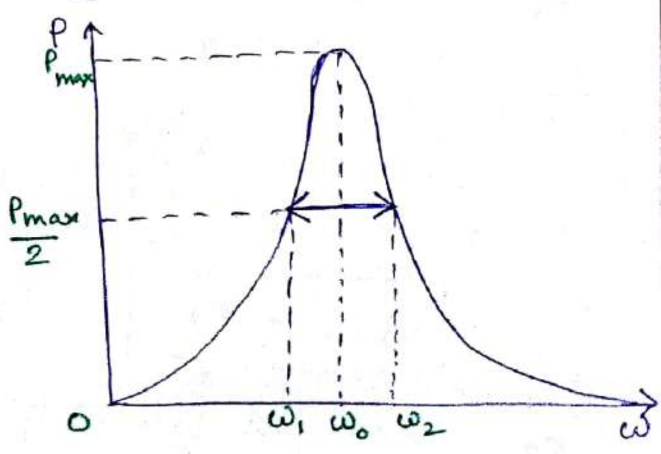
$$= \frac{F_0^2\gamma}{2m}\left[\frac{\omega^2}{(\omega_0^2-\omega)^2 + \gamma^2\omega^2}\right]$$

This will be maximum for $\frac{dP}{d\omega} = 0$

& that yields $\omega = \omega_0$. Thus at resonance frequency P_{input} is maximum.

$P_{input}^{max} = \frac{F_0^2}{2m\gamma}$ $\therefore$ $P = P_{input}^{max}\frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

Frequency ω_1 & ω_2 at which the power drops down to ½ of maximum is the half power freq.



$$\frac{1}{2} = \frac{P_{input}}{P_{input}^{max}} = \frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

or $\omega^2 = \omega_0^2 \pm \gamma\omega$

$$\begin{cases} \omega_1 = -\frac{\gamma}{2} + (\omega_0^2 + \frac{\gamma^2}{4})^{1/2} \\ \omega_2 = \frac{\gamma}{2} + (\omega_0^2 + \frac{\gamma^2}{4})^{1/2} \end{cases}$$, band width $\Delta\omega = \omega_1 - \omega_2 = \gamma$.

<u>Quality Factor</u> Q is a parameter that gives the sharpness of resonance & defined as $Q = \frac{\text{resonant frequency}}{\text{band width}} = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\gamma}$

$$= 2\pi \frac{\text{Avg. energy stored in one cycle}}{\text{Avg. energy lost in one cycle}}$$

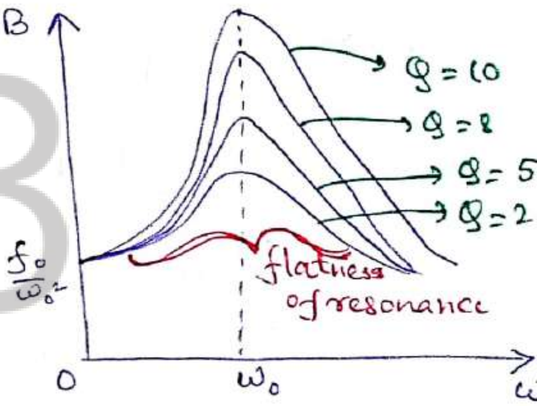
$$\therefore Q = 2\pi \frac{\langle E(t)\rangle}{P_{dissipate}\, T} = \overbrace{\left(\frac{2\pi}{T}\right)}^{=\omega} \frac{1}{\cancel{4}_{\,2}} m(\omega^2+\omega_0^2)\cancel{(B_{ab}^2+B_{el}^2)} \frac{\cancel{2}}{b\omega^{\cancel{2}}\cancel{(B_{ab}^2+B_{el}^2)}}$$

$$= \frac{\omega^2+\omega_0^2}{2\gamma\omega} \quad \text{and for } \omega \approx \omega_0, \quad Q^{\text{resonance}} = \frac{\omega_0}{\gamma}$$

Thus for low damping, $\gamma \ll \omega_0$ and Q is high. that makes the resonance very ~~high~~. sharp. Thus Q measures the sharpness of resonance

Using $Q = \frac{\omega_0}{\gamma}$, the amplitude is

$$B = \frac{f_0 Q}{\omega\omega_0\sqrt{1+Q^2\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2}}$$



Q large, B large. Q can be regarded as amplification factor. at low driving force $\omega \to 0$, $B_0 = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \simeq \frac{f_0}{\omega_0^2}$ and we know

$$B_{max} = \frac{f_0}{\gamma\sqrt{\omega_0^2-\gamma^2/4}}. \quad \text{So} \quad \frac{B_{max}}{B_0} = \frac{\omega_0^2}{\gamma\sqrt{\omega_0^2-\gamma^2/4}} = \frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}$$

(for low damping) Q is very large $= Q\left(1-\frac{1}{4Q^2}\right)^{-1/2} \simeq Q\left(1+\frac{1}{8Q^2}\right)$

$$= Q.$$

$\therefore \boxed{B_{max} = QB_0}$ The resonant amplitude is Q times the amplitude at low frequencies of the driving force.