(c) We know, velocity of radio signal = c

and it took 1125 sec to reach human observer on Earth, so distance of spaceship from Earth is 1125 c metre.

$\therefore$ According to Earth's observer, time taken for spaceship to return $t = \dfrac{1125c}{0.5c} = \underline{2250 \text{ sec}}$.

Now as velocity of spaceship $= 0.5c$, so $\gamma = \dfrac{1}{\sqrt{1-0.5^2}} = 1.155$.

So for the crew of spaceship this time measured will be the proper time t_0 (the time dilated measurements are done by Earth's observer). $t_0 = \dfrac{t}{\gamma} = \dfrac{2250}{1.155} = \underline{1948.6 \text{ sec}}$.

Assignment - 2 (Relativistic Dynamics)

(1)(a) Let E be the energy of the rocket's emitted radiation, v = velocity of rocket propulsion, m_i & m_f are initial & final rest mass of the rocket.

From the conservation of relativistic momentum, we have

$\gamma m_f v = \dfrac{E}{c}$; $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$ and from conservation of relativistic energy, we have $m_i c^2 = \gamma m_f c^2 + E$.

Eliminating E from both of the above equations, we have

$$m_i c^2 = \gamma m_f c^2 + \gamma m_f v c$$

$$\text{or } \frac{m_i}{m_f} = \frac{\gamma(c^2+vc)}{c^2} = \gamma\left(1+\frac{v}{c}\right) = \frac{1+v/c}{\sqrt{1-v^2/c^2}} = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}$$

$$\text{or } \frac{m_i}{m_f} = \sqrt{\frac{1+\beta}{1-\beta}}$$ (Hence proved).

(b) Given, the density of stationary body is ρ_0 having volume $dV_0 = dx_0 dy_0 dz_0$. In the moving frame, let ρ be the density and volume $dV = dx\,dy\,dz$.

Now. ~~[illegible]~~ density $\rho = \frac{m}{V}$ and using Einstein's formula, we have $\rho = \frac{\gamma m_0}{V_0/\gamma} = \rho_0 \gamma^2 = \frac{\rho_0}{1 - v^2/c^2}$.

We want to find $\rho = \rho_0 + \frac{25}{100}\rho_0 = \frac{5}{4}\rho_0$

$\therefore \frac{5}{4}\cancel{\rho_0} = \frac{\cancel{\rho_0}}{1 - v^2/c^2}$ or $1 - v^2/c^2 = \frac{4}{5}$ or $\frac{v^2}{c^2} = 1 - \frac{4}{5} = \frac{1}{5}$

or $v = \frac{c}{\sqrt{5}}$. So the velocity of the reference frame is $\frac{c}{\sqrt{5}}$ unit

(c) Given that for two lumps of clay, rest mass m_0 and speed $v = \frac{3}{5}c$ after collision stick together. Let the mass of the composite clay is M.

According to the relativistic energy conservation, we have

$Mc^2 = mc^2 + mc^2$ where $m = \gamma m_0$ is the mass of two moving clays before collision. $= \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{\sqrt{1 - 9/25}} = \frac{5}{4}m_0$

$\therefore M\cancel{c^2} = 2mc^2 = 2 \times \frac{5}{4} m_0 \cancel{c^2}$ $\quad \therefore M = \frac{5}{2}m_0 = 2.5m_0$

So mass of the composite lump is $2.5m_0$.

(2) (a) In the accelerator, rest mass energy of the particle = $m_0c^2 = 1$ GeV and energy of the accelerated particle = $mc^2 = 5$ GeV.

From Einstein's relativistic mass relation, we have $m = \gamma m_0$

So that $\gamma = \frac{m}{m_0} = \frac{mc^2}{m_0c^2} = \frac{5}{1} = 5$.

or $\frac{1}{\sqrt{1 - v^2/c^2}} = 5$ or $1 - v^2/c^2 = 1/25$ or $v^2/c^2 = \frac{24}{25}$

or $v = \sqrt{\frac{24}{25}}\, c = 0.98c$. This is the velocity of the particle in rest frame of the accelerator.

(b) Rest mass of the electron $m_0 c^2 = 0.51$ MeV.

Kinetic energy of the electron $T = (m - m_0)c^2 = 0.25$ MeV.

$\therefore (\gamma m_0 - m_0)c^2 = 0.25$ or $(\gamma - 1)\, 0.51 = 0.25$.

or $\gamma = 1 + \frac{0.25}{0.51} = \frac{0.76}{0.51}$ or $1 - v^2/c^2 = \left(\frac{0.51}{0.76}\right)^2$

or $v^2 = c^2\left(1 - \frac{0.51^2}{0.76^2}\right)$ or $\boxed{v = 0.7414\, c}$. This is the velocity of the electron.

(c) $\pi \rightarrow \mu + \nu$. Rest mass of pion $= m_\pi$

Mass of muon $= m_\mu$ and $m_\nu = 0$

From the conservation of relativistic momentum, we have

$0 = P_\mu + P_\nu$ or $P_\nu = -P_\mu$. and

from the conservation of relativistic energy we have

$m_\pi c^2 = E_\mu + E_\nu$. Now, energy of the neutrino is

$E_\nu = P_\nu c$. So we have,

$(\because m_\nu = 0$, so $E_\nu = \sqrt{m_\nu^2 c^4 + P_\nu^2 c^2} = P_\nu c)$

$$m_\pi c^2 = \sqrt{m_\mu^2 c^4 + P_\mu^2 c^2} + P_\nu c = \sqrt{m_\mu^2 c^4 + P_\mu^2 c^2} - P_\mu c.$$

$$\text{or } (m_\pi c^2 + P_\mu c)^2 = m_\mu^2 c^4 + \cancel{P_\mu^2 c^2} = m_\pi^2 c^4 + \cancel{P_\mu^2 c^2} + 2 m_\pi P_\mu c^3$$

$$\text{or } P_\mu = \frac{(m_\mu^2 - m_\pi^2) c^4}{2 m_\pi c^3} = \frac{m_\mu^2 - m_\pi^2}{2 m_\pi} c.$$

$\therefore$ Energy of the muon $E_\mu = \sqrt{P_\mu^2 c^2 + m_\mu^2 c^4}$

$$\text{or } E_\mu^2 = \frac{(m_\mu^2 - m_\pi^2)^2}{4 m_\pi^2} c^4 + m_\mu^2 c^4 = \frac{(m_\mu^2 - m_\pi^2)^2 + 4 m_\mu^2 m_\pi^2}{4 m_\pi^2} c^4$$

$$= \frac{(m_\mu^2 + m_\pi^2)^2}{4 m_\pi^2} c^4.$$

$$\therefore \boxed{E_\mu = \frac{m_\mu^2 + m_\pi^2}{2 m_\pi} c^2.}$$

(d) $K^+ \rightarrow e^+ + \pi^0 + \nu_e$

Given mass of kaon $m_{K^+} = 494\ MeV/c^2$

mass of pion $m_{\pi^0} = 135\ MeV/c^2$

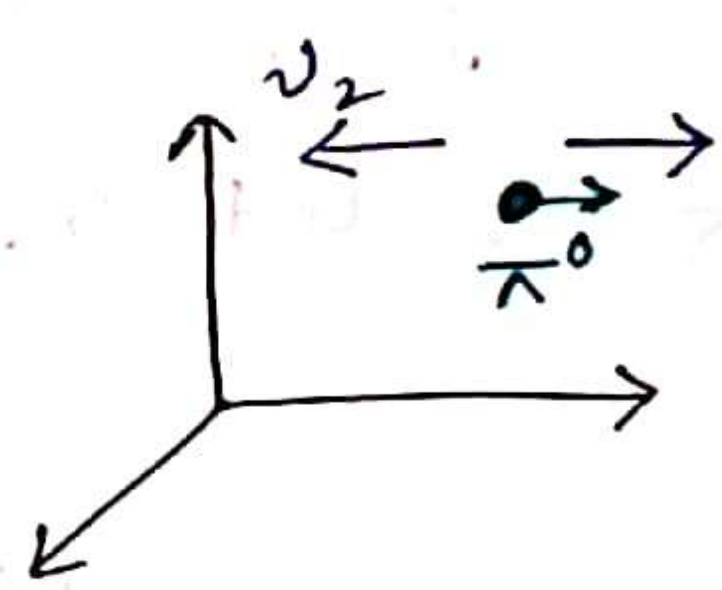
mass of electron $m_{e^+} = 0.5\ MeV/c^2$.

From the relativistic formula for conservation of energy, we have

$$E_{K^+} = E_{e^+} + E_{\pi^0} + E_{\nu_e}$$

$$\therefore\ E_{e^+} = E_{K^+} - E_{\pi^0} - E_{\nu_e} = m_{K^+}c^2 - m_{\pi^0}c^2 - m_{\nu_e}c^2$$

$$= 494 - 135 - 0.5 = \underline{358.5\ MeV}.$$ This is the maximum energy emitted by the positron.



(3) (a) $\pi^0 \rightarrow \nu_1 + \nu_2$

Given rest mass of $\pi^0 = m$ and

relativistic momentum $P_{\pi^0} = \frac{3}{4} mc$.

So the total relativistic energy of the pion is

$$E_{\pi^0} = \sqrt{P_{\pi^0}^2 c^2 + m_{\pi^0}^2 c^4} = \sqrt{\frac{9}{16} m^2c^4 + m^2c^4} = \sqrt{\frac{25}{16} m^2c^4} = \frac{5}{4} mc^2.$$

$\therefore$ From the relativistic energy conservation relation, we have

$$E_{\pi^0} = E_{\nu_1} + E_{\nu_2} = \frac{5}{4} mc^2. \quad (1)$$ and from relativistic momentum conservation, we have $P_{\pi^0} = P_{\nu_1} - P_{\nu_2}$. While $m_{\nu_1} = m_{\nu_2} = 0$

(rest mass of photon $= 0$), $E_{\nu_1} = \sqrt{P_{\nu_1}^2 c^2}$ $\quad \therefore \quad P_{\nu_1} = \frac{E_{\nu_1}}{c}$

$E_{\nu_2} = \sqrt{P_{\nu_2}^2 c^2}$ $\quad P_{\nu_2} = \frac{E_{\nu_2}}{c}$

$$\therefore\ \frac{3}{4} mc = \frac{E_{\nu_1}}{c} - \frac{E_{\nu_2}}{c} \quad \text{or} \quad E_{\nu_1} - E_{\nu_2} = \frac{3}{4} mc^2 \quad (2).$$

Adding (1) & (2) we have, $2E_{\nu_1} = 2mc^2$ or $\boxed{E_{\nu_1} = mc^2}$

Substituting back this in either (1) or (2), we have

$\boxed{E_{\nu_2} = \frac{1}{4} mc^2}$ These are the required relativistic energy of the two photons ν_1 and ν_2.

(b) Rest mass energy of π^+- meson $m_\pi c^2 = 135$ MeV.

At a height $h = 120$ km, above sea level, total energy of the π^+ meson $E_\pi = 1.35 \times 10^5$ MeV. $= \gamma m_\pi c^2$ from Einstein's mass formula.

$$\therefore \gamma = \frac{E_\pi}{m_\pi c^2} = \frac{1.35\times10^5}{135} \quad \text{or} \quad 1 - v^2/c^2 = \left(\frac{\overset{10^2}{\cancel{135}}}{\cancel{1.35}} \times 10^{-5}\right)^2 = 10^{-6}.$$

$$\therefore v = \sqrt{1-10^{-6}}\, c = 0.99c.$$

While after creation of π^+ in rest frame it disintegrates in $\Delta t_0 = 2\times10^{-8}$ sec, so due to time dilation, in the lab frame, time of disintegration $\Delta t = \gamma \Delta t_0 = \frac{1.35}{135}\times10^5 \times 2\times10^{-8}$

$= 2\times10^3 \times 10^{-8} = 2\times10^{-5}$ sec.

$\therefore$ Total distance travelled by π^+ meson before disintegration in laboratory frame $= v\Delta t = 0.99c \times 2\times10^{-5}$ m

$= 5.999\times10^3$ m $= 5.999$ km, so it will disintegrate at height $= h - v\Delta t = 120 - 5.999$ Km $= \underline{\underline{114.001 \text{ Km}}}$.

(c) We know $m = \gamma m_0$ & if $m_0 = 0$ then $m\sqrt{1-v^2/c^2} = 0$. while $m \neq 0$, only way to achieve this is $v = c$. Hence the particle has to move at the speed of light.

Alternatively, using relativistic energy relation

$E^2 = p^2c^2 + m_0^2c^4$ for rest mass zero particle $m_0 = 0$

yields $E^2 = p^2c^2 = m^2v^2c^2$. Again from Einstein's mass-energy relation, $E^2 = \cancel{m^2} c^{\cancel{4}\,2} = \cancel{m^2} v^2 \cancel{c^2}$.

$\therefore v^2 = c^2$ or $\underline{\underline{v = c}}$ (Proved)

(4) (a) Rest mass energy of a particle $= m_0c^2$

Total relativistic energy of the particle $E = \sqrt{p^2c^2 + m_0^2c^4}$.

Given, total energy is equal to twice of relativistic energy.

$\therefore \quad E = 2m_oc^2 = \sqrt{p^2c^2 + m_o^2c^4}$

$\Rightarrow 4m_o^2c^4 = p^2c^2 + m_o^2c^4 \quad \Rightarrow \quad p^2c^2 = 3m_o^2c^4 \quad \Rightarrow \quad p = \sqrt{3}\,m_oc.$

So, magnitude of relativistic momentum will be $\underline{\sqrt{3}\,m_oc \text{ unit}}$.

(b) Mass of the body at rest before breaking $= m$. Its two parts m_1 & m_2 move with velocity v_1 and v_2.

Total energy of the body before break $E_m = mc^2$ and for m_1, it is $E_{m_1} = m_1'c^2 = \gamma_1 m_1 c^2$ & $E_{m_2} = m_2'c^2 = \gamma_2 m_2 c^2$

where $m_1' = \gamma_1 m_1 = \frac{m_1}{\sqrt{1 - v_1^2/c^2}}$ is the moving mass m_1 in the comoving frame with velocity v_1. Similarly $m_2' = \gamma_2 m_2 = \frac{m_2}{\sqrt{1-v_2^2/c^2}}$.

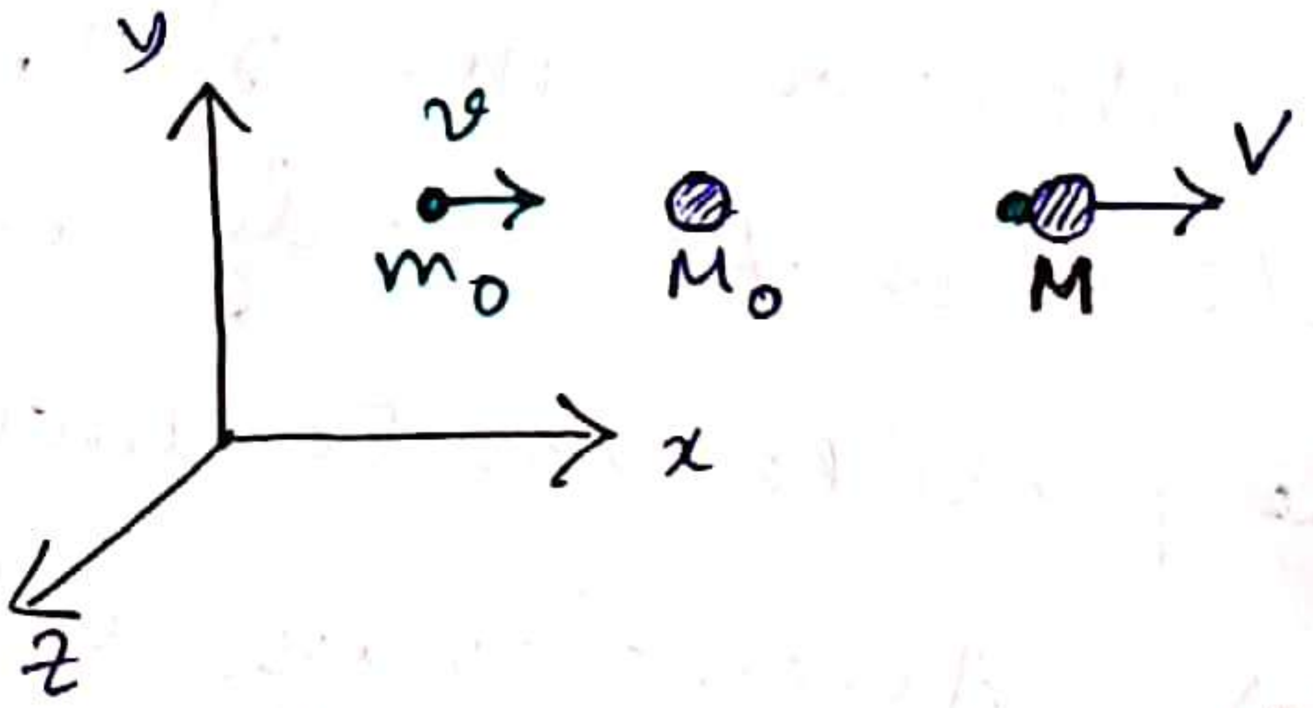
Now, from the relativistic total energy conservation formula,

$$E_m = E_{m_1} + E_{m_2}$$

$$\Rightarrow mc^2 = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2 \quad \Rightarrow \quad m = \gamma_1 m_1 + \gamma_2 m_2$$

as $v_1, v_2 < c$, so $\gamma_1, \gamma_2 > 1$, so $\boxed{m > m_1 + m_2}$. (Proved)

(5) (a) Rest mass of the particle is m_o which is moving with velocity v, while M_o rest mass particle is actually static before collision.



Suppose the sticky composite particle is of rest mass M which moves with velocity V.

From relativistic total energy conservation formula, we have

$$\gamma m_oc^2 + M_oc^2 = \gamma' M c^2 \; ; \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}, \quad \gamma' = \frac{1}{\sqrt{1-V^2/c^2}}$$

Again, from relativistic momentum conservation, we have

$$\gamma m_o v = \gamma' M V.$$ Eliminating γ' from two equations

we have

$$\frac{\gamma' M V}{\gamma' M \cancel{c^2}} = \frac{\gamma m_0 v}{\gamma m_0 \cancel{c^2} + M_0 \cancel{c^2}} \quad \text{or} \quad \boxed{V = \frac{\gamma m_0 v}{\gamma m_0 + M_0}} \quad \text{(Proved).}$$

This is the speed of the composite particle.

(b) Given $v = 0.8c$, so $\gamma = \frac{1}{\sqrt{1-0.8^2}} = 1.67$.

$M_0 = m_0$ as two particles are of identical rest mass.

So from above formula, $V = \frac{\gamma m_0 v}{\gamma m_0 + m_0} = \frac{1.67 \times 0.8c}{1.67 + 1} = \underline{\underline{0.5c}}$

$$\therefore \gamma' = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - 0.5^2}} = 1.1547.$$

from momentum conservation, $\gamma m_0 v = \gamma' M V$

$$\therefore M = \frac{\gamma m_0 v}{\gamma' V} = \frac{1.6667 \times 0.8\cancel{c} \times m_0}{1.1547 \times 0.5\cancel{c}} = \underline{\underline{2.31 m_0}}.$$

So the composite particle of rest mass $2.31 m_0$ will move with velocity $c/2$.