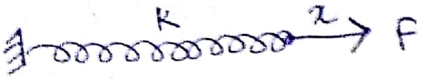
SHM   Motion : Translation, rotation, vibration/oscillation

periodic motion $f(t) = f(t+T)$ e.g. $\sin \frac{2\pi t}{T}$, $\cos \frac{2\pi t}{T}$

if periodic over same path → oscillatory motion

elasticity & inertia



SHM is a special type of periodic motion where restoring force on particle is proportional to displacement & directed to the mean position.

oscillation between point A & B, mean position O. at time t, particle is at P & displacement is x. F = restoring force

$F \propto -x$ or $F = -kx$. or $ma = -kx$ small oscillation approximation

$\therefore \; a = -\frac{k}{m}x = -\omega^2 x$

Characteristics (1) Linear motion → to-n-fro in straight line

(2) $F \propto -x$.

Linear harmonic motion ⟷ angular harmonic motion
(pendulum)   (torsional pendulum)
$F \propto -x$ $\tau \propto -\theta$

complete oscillation : one point to same point. (time period)

amplitude : maximum displacement on both sides.

frequency : no. of oscillations in 1 second.

phase : displacement, velocity, accelaration & direction of motion. After 1 oscillation, phase is same.

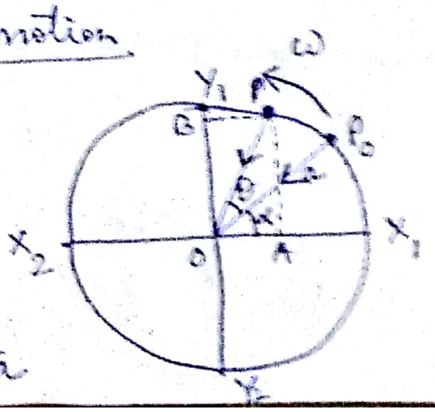
t=0, initial phase.

Relation between SHM & uniform circular motion

$OA = x$, $OB = y$ $\theta = \omega t$
$s = a\theta$

$= OP \cos(\theta + \alpha) = a\cos(\theta+\alpha)$

$= a\cos(\omega t + \alpha)$



Speed $v = \omega a$, centripetal acc $f_r = \frac{v^2}{a} = \omega^2 a$

Accelaration of A is component of f_r along X_1OX_2

$$f_A = -f_r \cos(\omega t+\alpha) = -\omega^2 a \cos(\omega t+\alpha) = -\omega^2 x$$

$$\therefore \; f_A \propto -x.$$

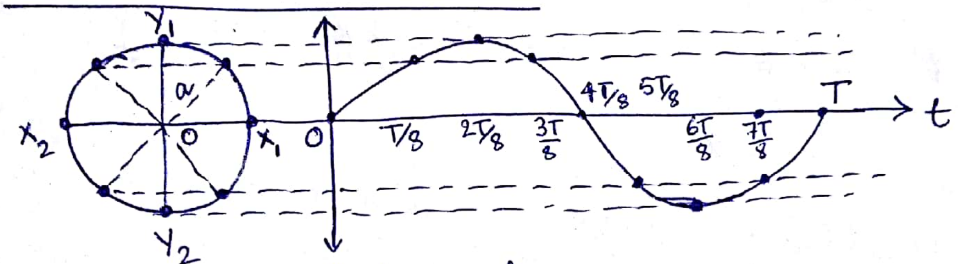
Similarly, $OB = y = OP\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$

Accelaration of B is $f_B = -f_r \sin(\theta+\alpha) = -\omega^2 a \sin(\omega t+\alpha) = -\omega^2 y$

$$\therefore \; f_B \propto -y.$$

$\therefore$ SHM is defined as the projection of uniform circular motion along diameter of circle.

Graphical representation

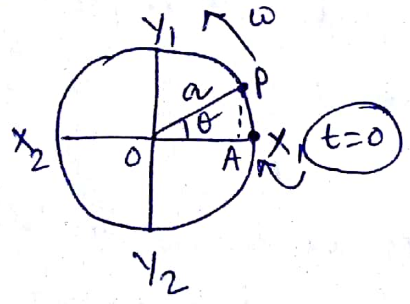


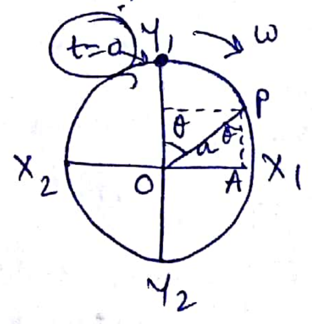
Time period $= T$.

$$y = a\sin\frac{2\pi}{T}t$$

(SHM along y-axis)

Displacement In SHM, displacement at time t is the distance of the particle from the mean position.





$OA = OP\cos\theta$

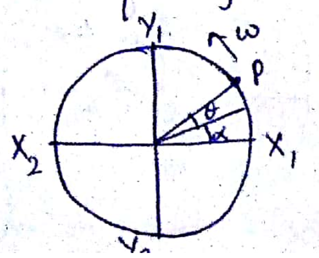
$x = a\cos\omega t$

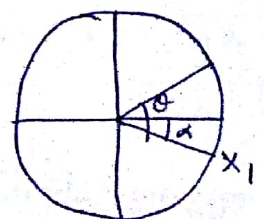
$OA = OP\cos(\pi/2 - \theta)$

$x = a\sin\theta = a\sin\omega t$

Similarly, $y = a\cos\omega t$ & $y = a\sin\omega t$.

So, eqn. of SHM can be derived from any instant t.



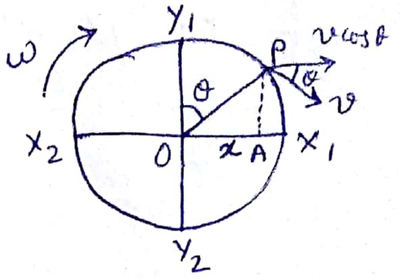


$x = a\cos(\theta+\alpha) = a\cos(\omega t+\alpha)$

Similarly, $x = a\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

if initial position is X_1 (2^{nd} pic) then $x = a\cos(\omega t-\alpha)$

or $x = a\sin(\omega t-\alpha)$



Velocity & accelaration

velocity of SHM is component of the particle's velocity along x-axis at time t.

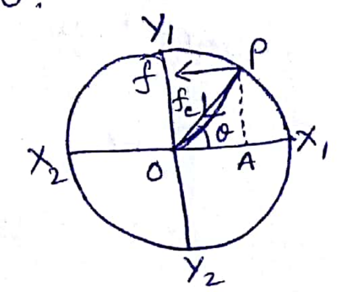
$V = a\omega$, V parallel to OA $= V\cos\theta$

$= a\omega\cos\theta = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$x = a\sin\theta$

$\therefore \boxed{v = \omega\sqrt{a^2-x^2}}$

v_{max} is at $x=0$, $v_{max} = a\omega$. & $x=a$, $v_{min}=0$.



Same with accelaration ⇒ SHM is the projection along X-axis is component of accelaration along X-axis. $f_c = -\omega^2 a$ & component around X_1X_2 is $-\omega^2 a\cos\theta = -\omega^2 a\cos\omega t = -\omega^2 x$.

$\therefore f = -\omega^2 x$.

$f_{max} = -\omega^2 a$ when $x = \pm a$, $f_{max} = \pm\omega^2 a$.

$f_{min} = 0$ when $x = 0$.

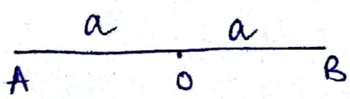
Calculus: $x = a\sin\omega t$, $v = \dot{x} = a\omega\cos\omega t = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$= \omega\sqrt{a^2-x^2}$.

$f = \ddot{x} = -a\omega^2\sin\omega t = -\omega^2 x$.

$\omega^2 = f/x$ (neglect)

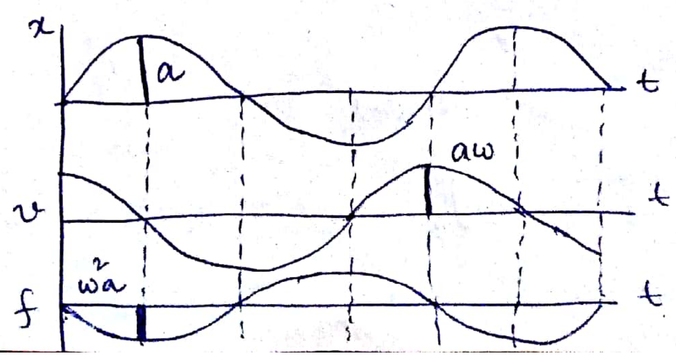
Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{f}}$



$x = a\sin\omega t = a\sin\frac{2\pi}{T}t$

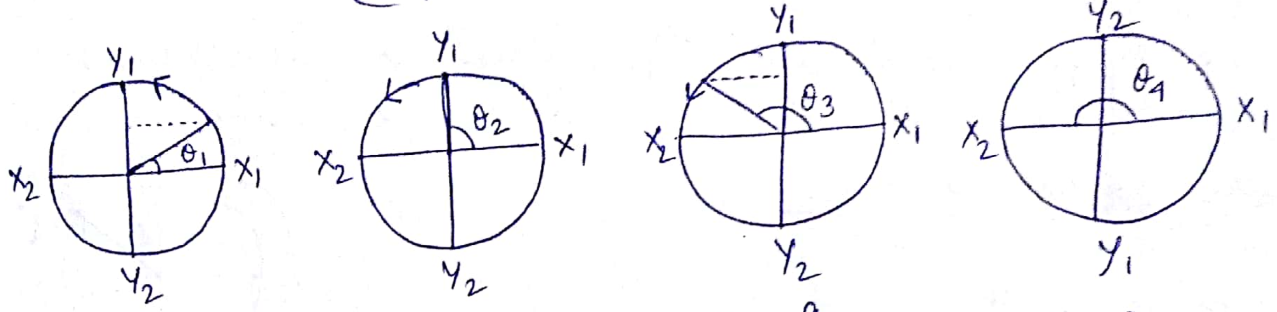
$v = a\omega\cos\omega t = a\omega\cos\frac{2\pi}{T}t$

$f = -a\omega^2\sin\omega t = -a\omega^2\sin\frac{2\pi}{T}t$



Scanned by CamScanner

Phase you see, a & ω (angular velocity) are constant.
(amplitude) $\theta = \omega t$ is changing = phase.



$y = a/2$	$y_1 = a$	$y = a/2$	$y = 0$
$\theta_1 = 30^\circ$	$\theta_2 = 90^\circ$	$\theta_3 = 150^\circ$	$\theta_4 = 180^\circ$
v upwards	$V = 0$	V= downwards	v= downwards.

phase difference 2 particles. $\phi = \theta_1 - \theta_2 = 0$ (in phase)
$= 180^\circ$ (out of phase)

Differential form & solution Homogeneous, 2^{nd} order, ODE with constant coefficient

$F = -kx$ or $m\ddot{x} = -kx$ or $\ddot{x} + \omega^2 x = 0$, $\omega = \sqrt{\frac{k}{m}}$.

Solution : Multiply by $2\dot{x}$, $2\dot{x}\ddot{x} + 2\omega^2 x\dot{x} = 0$

Integrating $\dot{x}^2 = -\omega^2 x^2 + c$

when displacement is maximum, $x = a$, $\dot{x} = 0$ $\Rightarrow$ $c = \omega^2 a^2$

$\therefore$ $v = \dot{x} = \pm \omega\sqrt{a^2 - x^2}$

or $\pm \frac{dx}{\sqrt{a^2 - x^2}} = \omega dt$, Integrating $\sin^{-1}\frac{x}{a} = \omega t + \phi$

or $x = a\sin(\omega t + \phi)$

See, $x = a\cos(\omega t + \phi)$ also satisfy $\ddot{x} + \omega^2 x = 0$.

$x = a\sin(\omega t + \phi) = a\sin\omega t\cos\phi + a\cos\omega t\sin\phi$
$= A\sin\omega t + B\cos\omega t$.

In operator form, $\frac{d^2x}{dt^2} = D^2 x$, $\frac{dx}{dt} = Dx$

$D^2 x + \omega^2 x = 0$ or $D^2 = -\omega^2$ or $D = \pm i\omega$

$\therefore$ General solution $x = Ae^{i\omega t} + Be^{-i\omega t}$

For real value of x, $A = B^*$ $\quad A = a+ib, \quad B = a-ib$

you can also have $x = a e^{i(\omega t+\phi)}$

sinusoidal or cosinusoidal.

CW 1. Oscillatory motion of a particle is represented by $x = a e^{i\omega t}$. Establish the motion is SHM. Similarly if $x = a\cos\omega t + b\sin\omega t$ then SHM.

$$x = a e^{i\omega t}, \quad \dot{x} = a i\omega e^{i\omega t}, \quad \ddot{x} = -a\omega^2 e^{i\omega t} = -\omega^2 x \quad \text{(SHM)}$$

$$x = a\cos\omega t + b\sin\omega t, \quad \dot{x} = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \quad \text{(SHM)}.$$

2. Which periodic motion is not oscillatory?

→ earth around sun or moon around earth.

3. Dimension of force constant of vibrating spring.

$$F = -Kx \qquad [K] = \frac{[\text{Force}]}{[\text{displacement}]} = \frac{[\text{Newton}]}{[\text{Metre}]}$$

also called "stiffness".

$$= \frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$$

HW 1. In SHM, displacement is $x = a\sin(\omega t+\phi)$. at $t=0$, $x = x_0$ with velocity v_0, show that $a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$ & $\tan\phi = \frac{\omega x_0}{v_0}$.

2. Particle is vibrated at frequency 5 Hz in SHM. Show that when displacement exceeds 10^{-2} metre, the particle loses contact with the vibrator. Given $g = 9.8\ m/s^2$

3. In SHM, a particle has speed 80 cm/s & 60 cm/s with displacement 3 cm & 4 cm. Calculate amplitude of vibration