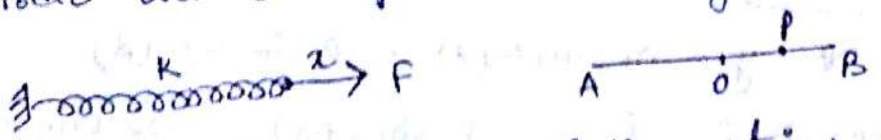
SHM

Motion : Translation, rotation, vibration/oscillation

periodic motion $f(t) = f(t+T)$ e.g. $\sin \frac{2\pi t}{T}$, $\cos \frac{2\pi t}{T}$

if periodic over same path $\rightarrow$ oscillatory motion

$\boxed{\text{elasticity \& inertia}}$



SHM is a special type of periodic motion where restoring force on particle is proportional to displacement & directed to the mean position.

oscillation between point A & B, mean position O. at time t, particle is at P & displacement is x. F = restoring force

"small oscillation approximation"

$F \propto -x$ or $F = -kx$ or $ma = -kx$

$\therefore \quad a = -\frac{k}{m}x = -\omega^2 x$

Characteristics (1) Linear motion $\rightarrow$ to-n-fro in straight line.

(2) $F \propto -x$

Linear harmonic motion $\leftrightarrow$ angular harmonic motion.

(pendulum) (torsional pendulum)

$F \propto -x$ $\quad \tau \propto -\theta$

complete oscillation : one point to same point. (time period)

amplitude : maximum displacement on both sides

frequency : no. of oscillations in 1 second.

phase : displacement, velocity, accelaration & direction of motion. After 1 oscillation, phase is same.

$t = 0$, initial phase.

Relation between SHM & uniform circular motion.

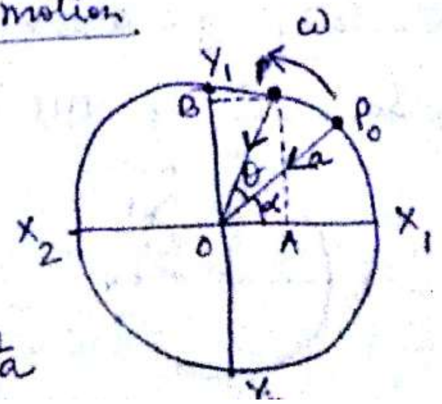
$\theta = \omega t$

$s = a\theta$

$OA = x$, $OB = y$.

$= OP \cos(\theta + \alpha) = a\cos(\theta+\alpha)$

$= a\cos(\omega t + \alpha)$



Speed $v = \omega a$, Centripetal acc $f_r = \frac{v^2}{a} = \omega^2 a$

Accelaration of A is component of f_r along X_1OX_2

$$f_A = -f_r \cos(\omega t+\alpha) = -\omega^2 a \cos(\omega t+\alpha) = -\omega^2 x$$

$\therefore$ $f_A \propto -x$.

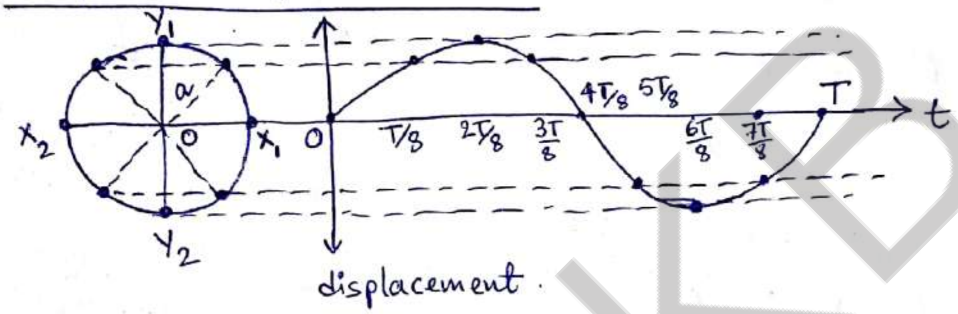
Similarly, $OB = y = OP\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

Accelaration of B is $f_B = -f_r \sin(\theta+\alpha) = -\omega^2 a \sin(\omega t+\alpha) = -\omega^2 y$

$\therefore$ $f_B \propto -y$.

$\therefore$ SHM is defined as the projection of uniform circular motion along diameter of circle.

Graphical representation

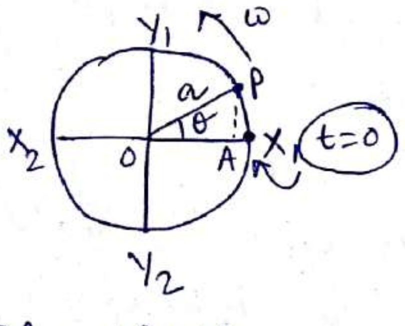


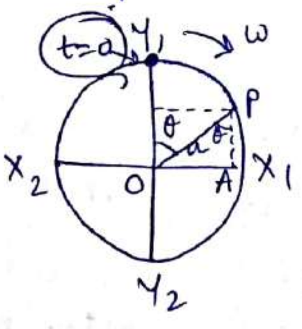
Time period $= T$.

$$y = a\sin\frac{2\pi}{T}t$$

(SHM along y-axis)

Displacement In SHM, displacement at time t is the distance of the particle from the mean position.





$OA = OP\cos\theta$

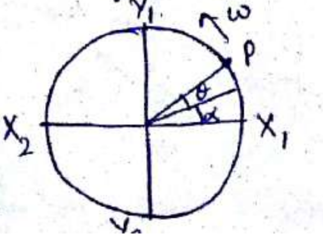
$x = a\cos\omega t$

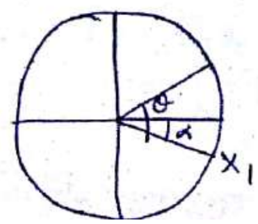
$OA = OP\cos(\pi/2 - \theta)$

$x = a\sin\theta = a\sin\omega t$

Similarly, $y = a\cos\omega t$ & $y = a\sin\omega t$.

So, eq^n. of SHM can be derived from any instant t.





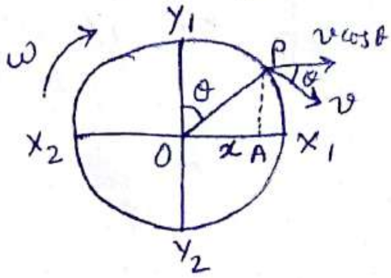
$x = a\cos(\theta+\alpha) = a\cos(\omega t+\alpha)$

Similarly, $x = a\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

if initial position is X_1 (2^{nd} pic) then $x = a\cos(\omega t - \alpha)$

or $x = a\sin(\omega t - \alpha)$

Velocity & accelaration



Velocity of SHM is component of the particle's velocity along x-axis at time t.

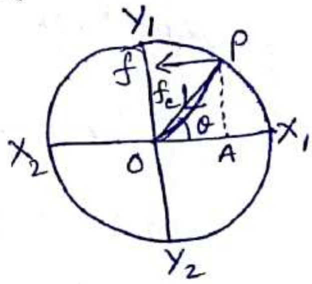
$V = a\omega$, V parallel to OA $= V\cos\theta$

$= a\omega\cos\theta = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$x = a\sin\theta$

$\therefore \boxed{v = \omega\sqrt{a^2-x^2}}$

v_{max} is at $x=0$, $v_{max} = a\omega$. & $x=a$, $v_{min}=0$.



Same with accelaration ⇒ SHM is the projection along X-axis is component of accelaration along X-axis. $f_c = -\omega^2 a$ & component aroun X_1X_2 is $-\omega^2 a\cos\theta = -\omega^2 a\cos\omega t = -\omega^2 x$.

$\therefore f = -\omega^2 x$.

$f_{max} = -\omega^2 a$ when $x=\pm a$, $f_{max} = \pm\omega^2 a$.

$f_{min} = 0$ when $x=0$.

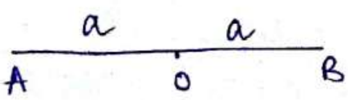
Calculus: $x = a\sin\omega t$, $V = \dot{x} = a\omega\cos\omega t = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$= \omega\sqrt{a^2-x^2}$.

$f = \ddot{x} = -a\omega^2\sin\omega t = -\omega^2 x$.

$\omega^2 = f/x$ (neglect)

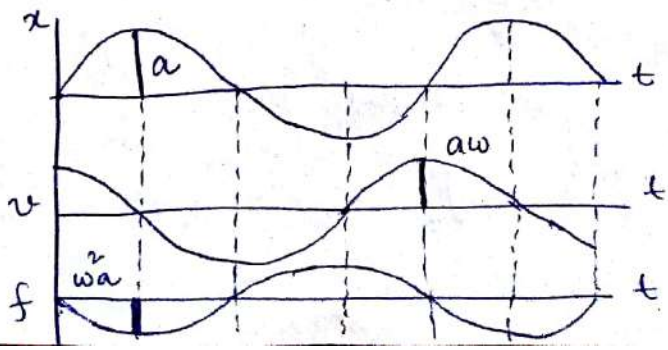
Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{f}}$



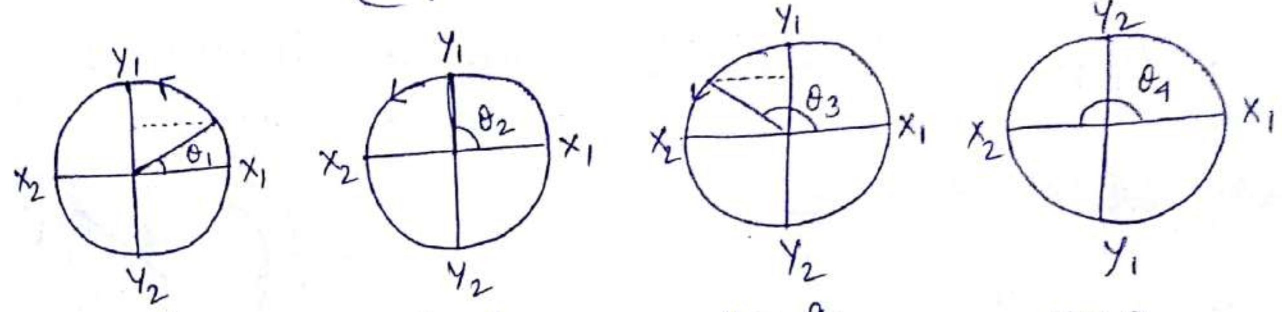
$x = a\sin\omega t = a\sin\frac{2\pi}{T}t$

$v = a\omega\cos\omega t = a\omega\cos\frac{2\pi}{T}t$

$f = -a\omega^2\sin\omega t = -a\omega^2\sin\frac{2\pi}{T}t$



Phase you see, a & ω (angular velocity) are constant.
(amplitude) $\theta = \omega t$ is changing = phase.



$y = a/2$
$\theta_1 = 30^\circ$
v upwards

$y_1 = a$
$\theta_2 = 90^\circ$
$V = 0$

$y = a/2$
$\theta_3 = 150^\circ$
V = downwards

$y = 0$
$\theta_4 = 180^\circ$
v = downwards.

phase difference 2 particles. $\phi = \theta_1 - \theta_2 = 0$ (in phase)
$= 180^\circ$ (out of phase)

Differential form & solution
Homogeneous, 2nd order, ODE with constant coefficients

$F = -kx$ or $m\ddot{x} = -kx$ or $\ddot{x} + \omega^2 x = 0$, $\omega = \sqrt{\frac{k}{m}}$.

Solution: Multiply by $2\dot{x}$, $2\dot{x}\ddot{x} + 2\omega^2 x\dot{x} = 0$

Integrating $\dot{x}^2 = -\omega^2 x^2 + c$

when displacement is maximum, $x = a$, $\dot{x} = 0$ $\Rightarrow$ $c = \omega^2 a^2$

$\therefore$ $v = \dot{x} = \pm\omega\sqrt{a^2 - x^2}$

or $\pm\frac{dx}{\sqrt{a^2 - x^2}} = \omega dt$, Integrating $\sin^{-1}\frac{x}{a} = \omega t + \phi$

or $x = a\sin(\omega t + \phi)$

See, $x = a\cos(\omega t + \phi)$ also satisfy $\ddot{x} + \omega^2 x = 0$.

$x = a\sin(\omega t + \phi) = a\sin\omega t\cos\phi + a\cos\omega t\sin\phi$
$= A\sin\omega t + B\cos\omega t$.

In operator form, $\frac{d^2x}{dt^2} = D^2x$, $\frac{dx}{dt} = Dx$

$D^2x + \omega^2 x = 0$ or $D^2 = -\omega^2$ or $D = \pm i\omega$

$\therefore$ General solution $x = Ae^{i\omega t} + Be^{-i\omega t}$

For real value of x, $A = B^*$ $\quad A = a+ib, \quad B = a-ib$

you can also have $x = ae^{i(\omega t+\phi)}$

sinusoidal or cosinusoidal.

CW 1. Oscillatory motion of a particle is represented by $x = ae^{i\omega t}$. Establish the motion is SHM. Similarly if $x = a\cos\omega t + b\sin\omega t$ then SHM.

$$x = ae^{i\omega t}, \quad \dot{x} = ai\omega e^{i\omega t}, \quad \ddot{x} = -a\omega^2 e^{i\omega t} = -\omega^2 x \quad (SHM)$$

$$x = a\cos\omega t + b\sin\omega t, \quad \dot{x} = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \quad (SHM).$$

2. Which periodic motion is not oscillatory?

→ earth around sun or moon around earth.

3. Dimension of force constant of vibrating spring.

$$f = -Kx \qquad [K] = \frac{[Force]}{[displacement]} = \frac{[Newton]}{[Metre]}$$

also called "stiffness".

$$= \frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$$

HW 1. In SHM, displacement is $x = a\sin(\omega t+\phi)$. at $t=0$, $x = x_0$ with velocity v_0, show that $a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$ & $\tan\phi = \frac{\omega x_0}{v_0}$.

2. Particle is vibrated at frequency 5 HZ in SHM. Show that when displacement exceeds 10^{-2} metre, the particle loses contact with the vibrator. Given $g = 9.8\ m/s^2$

3. In SHM, a particle has speed 80 cm/s & 60 cm/s with displacement 3 cm & 4 cm. Calculate amplitude of vibration

Energy of a particle in SHM

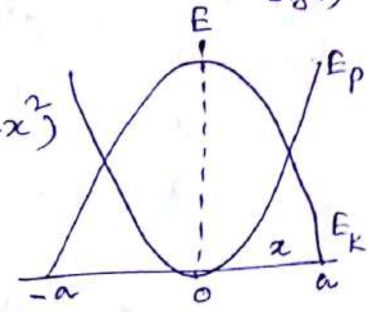
Work is done on particle to displace → restoring force. So P.E. in spring stored & motion is K.E. Total energy constant

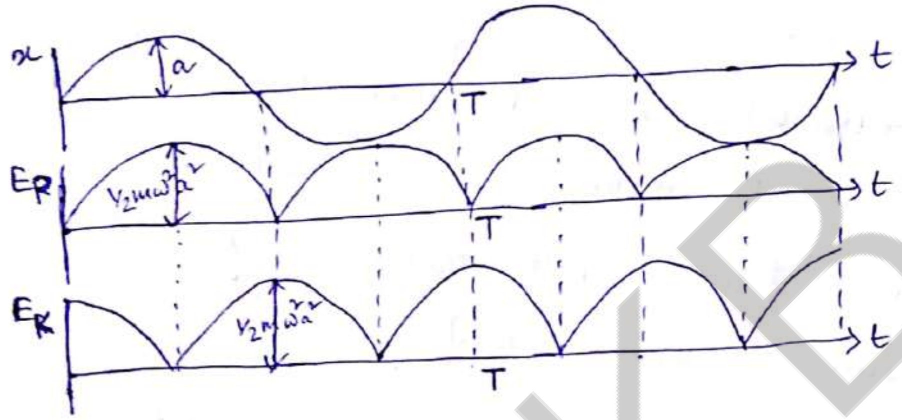
P.E. $F = mf = -m\omega^2 x$ $\therefore$ $dW = F\,dx = m\omega^2 x\,dx$ (against so no −ive sign)

$\therefore E_p = \int_0^x m\omega^2 x\,dx = \frac{1}{2} m\omega^2 x^2$.

K.E. $v = \omega\sqrt{a^2 - x^2}$, $E_K = \frac{1}{2} m v^2 = \frac{1}{2} m\omega^2 (a^2 - x^2)$

$E_{Tot} = E_K + E_p = \frac{1}{2} m\omega^2 a^2 = \text{constant}.$





Examples of SHM

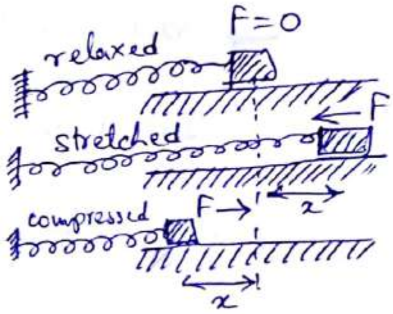
Horizontal oscillations

$F = -Kx = m\ddot{x}$

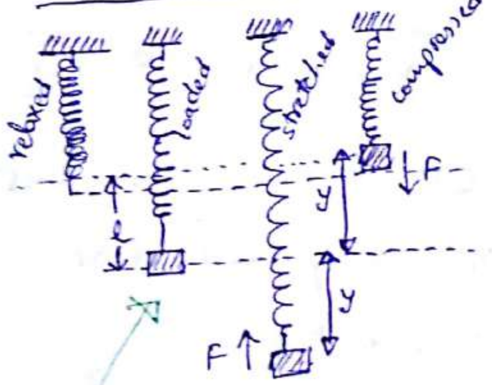
$\ddot{x} + \omega^2 x = 0$ $\omega = \sqrt{\frac{K}{m}}$

$x = A\cos(\omega t + \phi)$, $T = 2\pi\sqrt{\frac{m}{K}}$ → material.

initial cond^n.



Vertical oscillations



static equilibrium

Tension on spring $F_0 = Kl$

Force on mass $= mg$.

Static eq. $mg = Kl$.

stretched tension on spring $= K(l+y)$

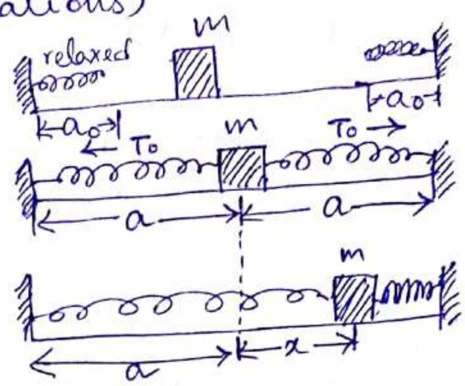
$mg - F = K(l+y) = Kl + Ky$
$= \cancel{mg} + Ky$

$F = -Ky$.

compressed $mg + F = K(l-y) = \cancel{mg} - Ky$

$F = -Ky$.

Two spring system (Longitudinal oscillations)



horizontal frictionless surface,
rigid wall, massless spring,
relaxed length a_0.
After connection, static equilibrium

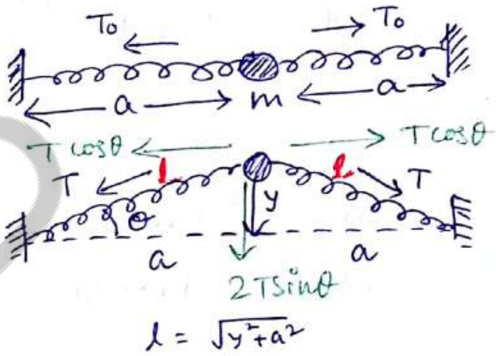
$$T_0 = K(a-a_0)$$

x = displacement to right. restoring force by left spring $-K(a+x-a_0)$
force on right spring $K(a-x-a_0)$

$$\therefore\ F_x = K(a-x-a_0) - K(a+x-a_0) = -2Kx$$

$$m\ddot{x} = -2Kx \quad \text{or} \quad \ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{long} = 2\pi\sqrt{\frac{m}{2K}}}$$

Two spring system (Transverse oscillations)



$$T_0 = K(a-a_0)$$

$$T = K(l-a_0)$$

$$F_y = -2T\sin\theta = -2T\frac{y}{l}$$

$l = \sqrt{y^2+a^2}$

$$\text{or} \quad m\ddot{y} + \frac{2T}{l}y = 0 \quad \text{or} \quad \ddot{y} + \omega^2 y = 0$$

$$\omega^2 = \frac{2T}{ml} = \frac{2K(l-a_0)}{ml}, \quad \text{but } l = f(y).$$

So $\ddot{y} + \frac{2K}{m}\left(1 - \frac{a_0}{f(y)}\right)y = 0$ is not a <u>SHM</u>.

ⓐ <u>Slinky approximation</u> $a >> a_0$ or $\frac{a_0}{a} << 1$.

$$\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{l}\right) = \frac{2K}{m}\left(1 - \frac{a_0}{a}\frac{a}{l}\right) \quad \text{as } l > a.$$

$$= \frac{2K}{m}. \quad \text{Then SHM.} \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K}}}$$

Spring

"large" harmonic oscillations

ⓑ <u>small oscillation approximation</u> $a \not\gg a_0$ but $y << a$ or l.

$$\therefore\ l = \sqrt{y^2+a^2} = a\sqrt{\frac{y^2}{a^2}+1} \sim a$$

Then also $\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{a}\right)$ or $\boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K\left(1-\frac{a_0}{a}\right)}}}$

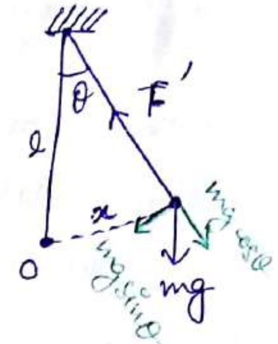
Rubber

<u>SHM</u>.

$$\therefore\ T_{long} = \sqrt{1 - \frac{a_0}{a}}\ T_{trans}.$$

So longitudinal is faster than transverse.

Simple pendulum



$F' = mg\cos\theta$
(tension in string)

$\left[\lim_{\theta\to 0}\right]$

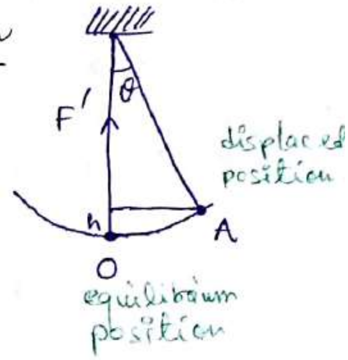
$F = -mg\sin\theta$
(restoring force) $= -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \simeq -mg\theta$

$x = l\theta$

or, $m\ddot{x} = -mg\frac{x}{l}$ or $\ddot{x} + \frac{g}{l}x = 0$.

$\therefore \omega = \sqrt{g/l}$, $T = 2\pi\sqrt{\frac{l}{g}}$. (mass independent)

String tension when pendulum at mean position



$F' = mg + \frac{mv^2}{l}$
(centrifugal force)

at A, Energy $= KE + PE = 0 + mgh = mgh$

at O, Energy $= KE + PE = \frac{1}{2}mv^2 + 0 = \frac{1}{2}mv^2$

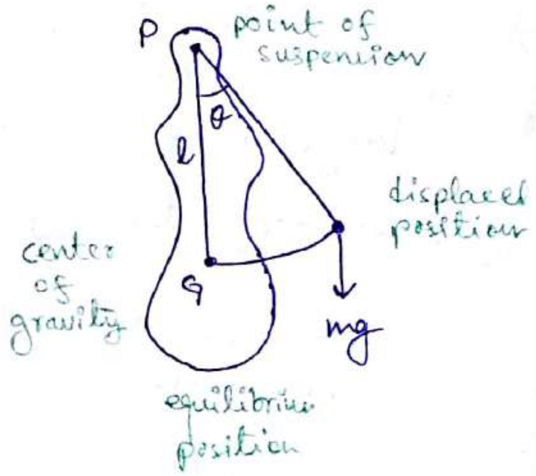
Conservation of energy $\Rightarrow \frac{1}{2}mv^2 = mgh$ or $v^2 = 2gh$.

or $v^2 = 2g(l - l\cos\theta) = 2gl(1-\cos\theta) = 2gl \times 2\sin^2\frac{\theta}{2}$

$\simeq 4gl\left(\frac{\theta}{2}\right)^2 = gl\theta^2$.

$\therefore F' = mg + \frac{m}{\cancel{l}}g\cancel{l}\theta^2 = mg(1+\theta^2)$.

Compound Pendulum



arbitrary shaped rigid body oscillating about a horizontal axis passing through it.

restoring force $\Leftrightarrow$ reactive couple or torque

moment of restoring force
$= -mgl\sin\theta$

angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$, moment of inertia $= I$.

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

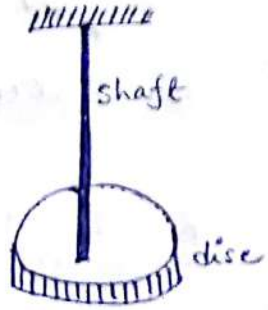
or $$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\sin\theta \simeq -\frac{mgl}{I}\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

If we consider moment of inertia about a parallel axis through G,
K = radius of gyration then using parallel axis theorem,

$$I = mK^2 + ml^2 \quad \therefore T = 2\pi\sqrt{\frac{K^2/l + l}{g}} = 2\pi\sqrt{\frac{l'}{g}}.$$

equivalent length of simple pendulum $= \frac{K^2}{l} + l$.



Torsional Pendulum

twist of shaft $\rightarrow$ torsional oscillations

torsional couple $= -\tau\theta$

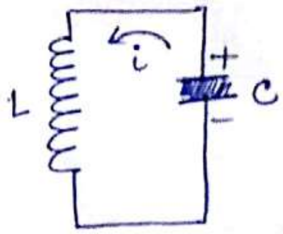
couple due to accelaration $= I\frac{d^2\theta}{dt^2}$

$$I\frac{d^2\theta}{dt^2} = -\tau\theta \quad , \quad T = 2\pi\sqrt{\frac{I}{\tau}}$$

From classical mechanics course, $\tau = \frac{\pi\eta d^4}{32 l} = \frac{\pi\eta r^4}{2l}$

d = shaft diameter, η = modulus of rigidity,
$= 2r$.

Electrical oscillator



Capacitor is charged $\Rightarrow$ electrostatic energy in dielectric media. It discharges through the inductor electrostatic energy $\Leftrightarrow$ magnetic energy. (no dissipation of heat)

voltage across inductor $= -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$ q = charge

voltage across capacitor $= \frac{q}{C}$.

No e.m.f. circuit, $\frac{q}{C} = -L\frac{d^2q}{dt^2}$ or $\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$

$\omega^2 = \frac{1}{LC}$, $q = q_0\sin(\omega t + \phi)$. charge on capacitor varies harmonically.

$$i = \frac{dq}{dt} = \omega q_0 \cos(\omega t + \phi)$$

$$V = \frac{q}{C} = \frac{q_0}{C} \sin(\omega t + \phi)$$

Total energy = magnetic energy + electric energy

$$= \int iV\,dt + \frac{1}{2}CV^2 = \int i L \frac{di}{dt} dt + \frac{1}{2}CV^2$$

$$= \int Li\,di + \frac{1}{2}CV^2 = \frac{1}{2}Li^2 + \frac{1}{2}CV^2 = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}CV^2$$

In mechanical oscillation, Total energy $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

equivalence

$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

In electrical oscillation, Total energy $= \frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2$

Resultant / Superposition of Harmonic oscillations

The resultant of two or more harmonic displacements is the algebraic sum of individual displacements. For linear homogeneous differential equations, sum of any two solutions is also a solution.

Realize that if $\frac{d^2x}{dt^2} = -\omega^2 x + \alpha x^2 + \beta x^3 + \cdots$ then if

$$\frac{d^2x_1}{dt^2} = -\omega^2 x_1 + \alpha x_1^2 + \beta x_1^3 + \cdots \quad \& \quad \frac{d^2x_2}{dt^2} = -\omega^2 x_2 + \alpha x_2^2 + \beta x_2^3 + \cdots$$

then $x_1 + x_2$ isn't a solution because if $x_1 + x_2 = x_3$. then

$$\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} = -\omega^2(x_1+x_2) + \alpha(x_1^2 + x_2^2) + \beta(x_1^3 + x_2^3) + \cdots$$

$$\therefore \frac{d^2x_3}{dt^2} = -\omega^2 x_3 + \alpha(x_3^2 - 2x_1x_2) + \beta(x_3^3 - 3x_1^2x_2 - 3x_1x_2^2) + \cdots$$

Composition of two colinear SHM of same frequency but different amplitude & phase:

Frequency $\omega = 2\pi\nu$, amplitude a & b, phase difference ϕ

$$x_1 = a\sin\omega t, \quad x_2 = b\sin(\omega t + \phi)$$

Time period for both motion is same & so phase difference is also same.

resultant displacement $x = x_1 + x_2 = a\sin\omega t + b\sin(\omega t + \phi)$

$= (a + b\cos\phi)\sin\omega t + b\sin\phi\cos\omega t = A\cos\theta\sin\omega t + A\sin\theta\cos\omega t$

$x = A\sin(\omega t + \theta) \Rightarrow$ S.H.M.

Amplitude of resultant wave $A^2 = (a + b\cos\phi)^2 + b^2\sin^2\phi$

or $A = (a^2 + b^2 + 2ab\cos\phi)^{1/2}$

phase of resultant wave $\tan\theta = \dfrac{b\sin\phi}{a + b\cos\phi}$

$\therefore \quad x = \sqrt{a^2 + b^2 + 2ab\cos\phi}\,\sin\left(\omega t + \tan^{-1}\left\{\dfrac{b\sin\phi}{a + b\cos\phi}\right\}\right)$

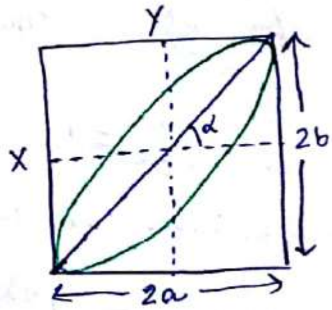
if $\phi = 0$ then $\theta = 0$, $A = a + b$, $x = (a+b)\sin\omega t$

if $\phi = \pi$ then $\theta = 0$ (opposite phase), $A = a - b$, $x = (a-b)\sin\omega t$.

if $a = b$, $x = 0 \Rightarrow$ no resultant motion.

Composition of two SHM at right angle with same frequency but different in phase & amplitude

Again, say two SHM acting in X & Y axis, amplitude a & b, phase difference ϕ.



$x = a\sin\omega t$, $y = b\sin(\omega t + \phi)$

$\therefore \cos\omega t = \sqrt{1 - x^2/a^2}$

and $\sin\omega t\cos\phi + \cos\omega t\sin\phi = y/b$.

or $\dfrac{x}{a}\cos\phi + \sqrt{1 - \dfrac{x^2}{a^2}}\sin\phi = \dfrac{y}{b}$

or $\left(\dfrac{y}{b} - \dfrac{x}{a}\cos\phi\right)^2 = \left(1 - \dfrac{x^2}{a^2}\right)\sin^2\phi$

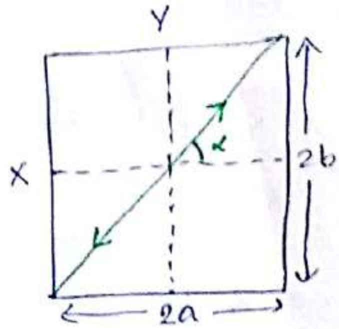
or $\boxed{\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2} - \dfrac{2xy}{ab}\cos\phi = \sin^2\phi}$ —— (1)

This is equation of ellipse confined to rectangle of side 2a & 2b with direction of major axis $\tan\alpha = \dfrac{2ab}{a^2 - b^2}\cos\phi$

(a) $\phi = 0$ $\quad \sin\phi = 0,\ \cos\phi = 1,\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{2xy}{ab} = 0$

or $\left(\dfrac{y}{b} - \dfrac{x}{a}\right)^2 = 0$ or $y = \dfrac{b}{a}x$

Straight line passing through origin & inclined to x-axis at angle $\alpha = \tan^{-1}\dfrac{b}{a}$ & with resultant amplitude $= \sqrt{a^2+b^2}$



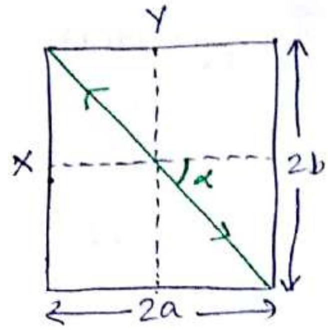
(b) $\phi = \pi$ Two motions are in opposite phase

Then the combined equation is

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} + \frac{2xy}{ab} = 0 \quad \text{or} \quad \left(\frac{y}{b} + \frac{x}{a}\right)^2 = 0$$

$\therefore\ y = -\dfrac{b}{a}x.$

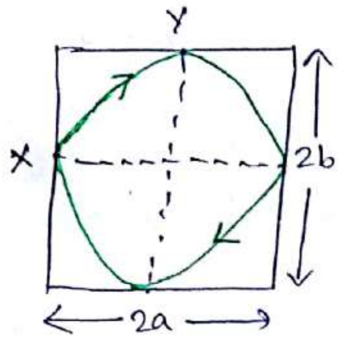
straight line passing through origin & inclined to x-axis at angle $\tan\alpha = -\dfrac{b}{a}$. If $a = b$, $\alpha = 135^\circ$



(c) $\phi = \pi/2$ Then the combined equation is

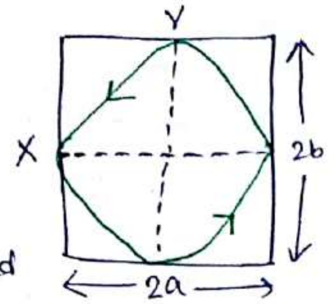
$\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2} = 1$, elliptical motion with major axis 2a, minor axis 2b.

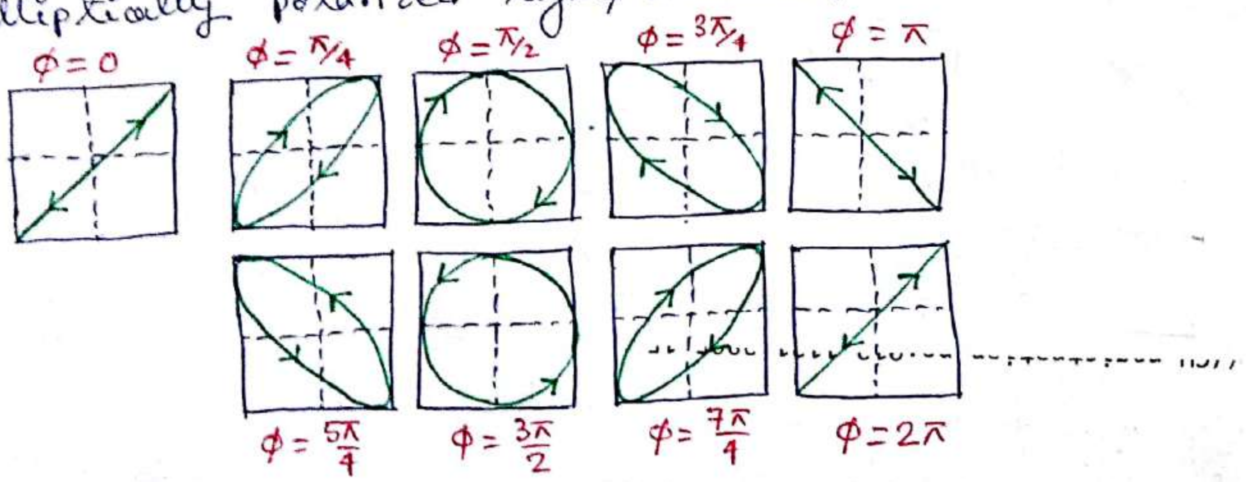
If $a = b$, then circular motion with $x^2 + y^2 = a^2$



(d) $\phi = \dfrac{3\pi}{2}$ Then the combined equation is

$\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2} = 1$, elliptic motion but counter-clockwise. In ray optics, this is called left-handed elliptically polarized light/vibration.





Composition of two SHM at right angle with different frequency, different phase, different amplitude:

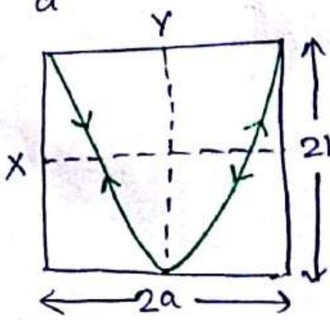
Complicated motion $\longrightarrow$ Lissajous figures. Suppose frequencies are in 1:2 ratio $x = a\cos\omega t$, $y = b\cos(2\omega t + \phi)$

$$\therefore \frac{y}{b} = \cos(2\omega t)\cos\phi - \sin(2\omega t)\sin\phi$$
$$= (2\cos^2\omega t - 1)\cos\phi - 2\sin\omega t\cos\omega t\sin\phi .$$
$$= \left(2\frac{x^2}{a^2} - 1\right)\cos\phi - 2\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\,\sin\phi .$$
$$\text{or } \left(\frac{y}{b} + \cos\phi\right) - \frac{2x^2}{a^2}\cos\phi = -\frac{2x}{a}\sqrt{1-\frac{x^2}{a^2}}\,\sin\phi .$$
$$\text{or } \left(\frac{y}{b} + \cos\phi\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 - \frac{y}{b}\cos\phi\right) = 0 \Rightarrow 4^{th} \text{ degree equation}$$

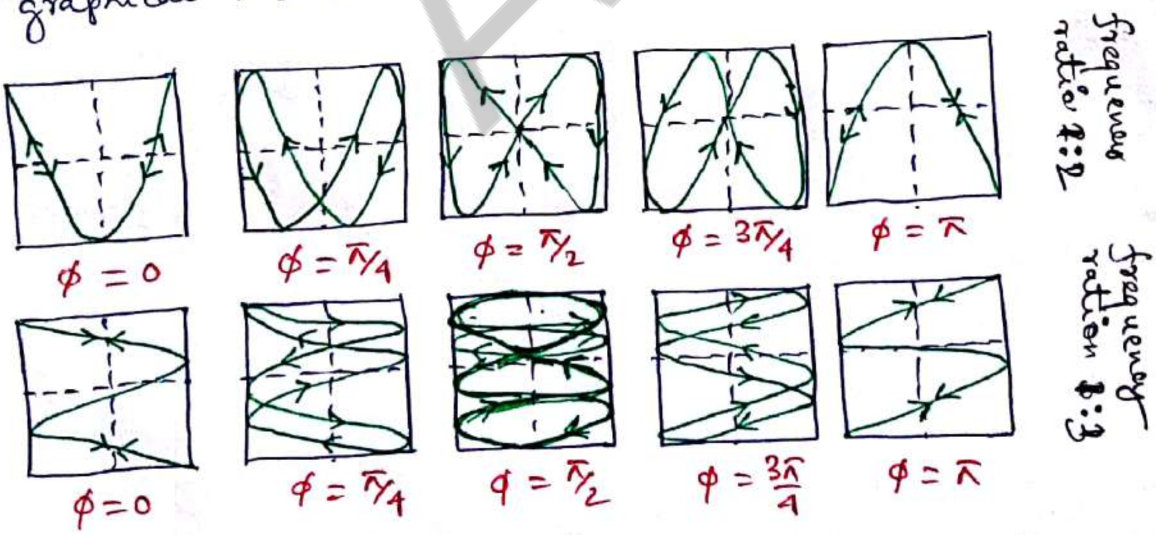
$\underline{\phi = 0}$ $\quad \left(\frac{y}{b}+1\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 - \frac{y}{b}\right) = 0$ or $\left(\frac{y}{b} - \frac{2x^2}{a^2} + 1\right)^2 = 0$

Two coincident parabola with vertex at $(0,-b)$ with equation $\frac{y}{b} - \frac{2x^2}{a^2} + 1 = 0$

or $x^2 = \frac{a^2}{2b}(y+b)$.



$\phi \neq 0$ very complex to resolve analytically & graphical method is the most convenient method.



So frequencies need to be in a commensurate ratio to give a periodic motion. Notice the interesting features that (1) resultant curve is always inside rectangle & the motion is periodic, (2) Number of tangential point in $x:y$ is the frequency ratio inverse.

HW 1. A particle is simultaneously subjected to two SHM in same direction, each of frequency 5 Hz. If amplitudes are 0.005 m & 0.002 m & phase difference is 45°, find the amplitude of the resultant ~~direction~~ displacement & its phase relative to the first component. Write down the expression for the resultant displacement as a function of time.

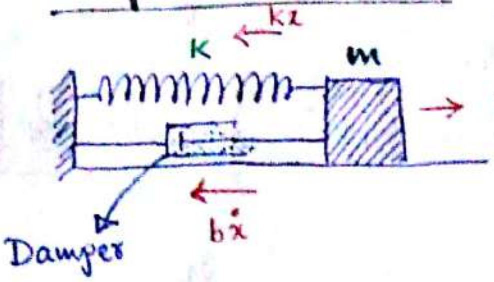
2. Two vibrations along the same line are described by

$x_1 = 0.03 \cos 10\pi t$, $x_2 = 0.03 \cos 12\pi t$, x_1, x_2 in metres & t in seconds. Obtain the equation describing the resultant motion and the beat period (beat period is the time interval between two consecutive maximum amplitude).

Free Damped harmonic motion

Damping of a real system is a complex phenomena involving several kind of damping force. Damping force of a body in a fluid is a function of velocity. This is called "**viscous damping**." When an oscillating body is contact with a surface, the frictional force is called "**Coulomb friction**". Also in solids, energy is partly lost due to internal friction & imperfect elasticity of the material. Experiments suggest that such resistive force is independent of frequency & proportional to amplitude. This is called "**structural damping**." The viscouse damping force may be represented as $F = -Av + \cancel{Bv^2} - \cancel{Cv^3} + \cdots$ and such approximation is "linear damping".

Damped oscillation of a system with 1 degree of freedom



inertial force $m\ddot{x}$ is balanced by elastic restoring force kx & viscous damping force $b\dot{x}$

$$\therefore\ m\ddot{x} = -b\dot{x} - kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \underbrace{\frac{b}{m}}_{\gamma}\frac{dx}{dt} + \underbrace{\frac{k}{m}}_{\omega_0^2}x = 0.$$

This is a linear homogeneous 2nd order ODE.

Let the trial solution $x = Ae^{\alpha t}$, substituting we get

$$(\alpha^2 + \gamma\alpha + \omega_0^2)Ae^{\alpha t} = 0 \Rightarrow \alpha^2 + \gamma\alpha + \omega_0^2 = 0.$$

$$\therefore\ \alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2}$$

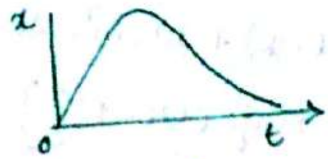
$$\therefore \text{ Solution } x = A_1 \exp\left[-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t + A_2 \exp\left[-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t$$

$$= e^{-\gamma t/2}\left[A_1 \exp\left(\sqrt{\gamma^2/4 - \omega_0^2}\, t\right) + A_2 \exp\left(-\sqrt{\frac{\gamma^2}{4} - \omega_0^2}\, t\right)\right]$$

We can have three possibilities:

(a) Heavy damping $\gamma^2/4 > \omega_0^2$ $\quad q = \sqrt{\gamma^2/4 - \omega_0^2} > 0$.

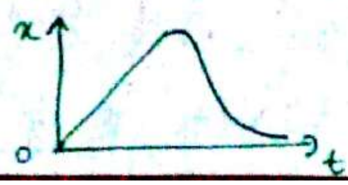
$x = e^{-\gamma t/2}(A_1 e^{qt} + A_2 e^{-qt})$. This means that x cannot be negative and at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & e^{qt} contributes like exponenti[al] then at $t \to \infty$, it'll damp to x (initial). If we had started at $x = 0$, after a time interval it decays back to zero $\Rightarrow$ "Dead beat Galvanometer." no oscillation



(b) Critical damping $\gamma^2/4 = \omega_0^2$: $x = (A_1 + A_2)e^{-\gamma t/2}$. The damping is slower but it has a discrepancy that at $x=0$ at $t=0$, $v=0$ which is not true. Changing the trial solution, we can derive

$x \sim t e^{-\gamma t/2}$ mean at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & $x \propto t$ & later $t \to \infty$, $e^{-\gamma t/2}$ dominates. x is never negative $\Rightarrow$ no oscillati[on]

"pointer_type galvanometer"



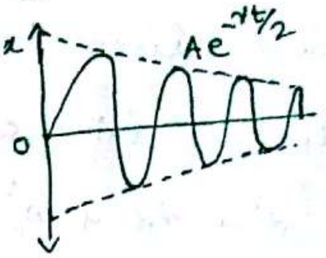
(c) Weak damping $\gamma^2/4 < \omega_0^2$ $\quad q = \sqrt{\gamma^2/4 - \omega_0^2}$ = imaginary.

This gives oscillatory damped harmonic motion $\quad \boxed{\omega = \sqrt{\omega_0^2 - \gamma^2/4}}$

$$x = e^{-\gamma t/2}\left[A_1 e^{i\sqrt{\omega_0^2 - \gamma^2/4}\,t} + A_2 e^{-i\sqrt{\omega_0^2-\gamma^2/4}\,t}\right]$$

$$= e^{-\gamma t/2}\left(A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right)$$

$$= e^{-\gamma t/2}\left[\underbrace{(A_1 + A_2)}_{A\cos\delta}\cos\omega t + \underbrace{i(A_1 - A_2)}_{A\sin\delta}\sin\omega t\right] = A e^{-\gamma t/2}\cos(\omega t - \delta)$$



Amplitude decreases in due time

Angular frequency is less than undamped motion.

$\tau = 2/\gamma$ = mean life time of oscillation.

Energy of a weakly damped oscillator

Using $x = Ae^{-\gamma t/2}\cos(\omega t - \delta)$ we develop expression for average energy. $\quad \dot{x} = -\frac{\gamma}{2}Ae^{-\gamma t/2}\cos(\omega t - \delta) - Ae^{-\gamma t/2}\omega\sin(\omega t - \delta)$

∴ Kinetic energy (instantaneous) of the vibrating body

$$\frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \gamma\omega\cos(\omega t - \delta)\sin(\omega t - \delta)\right]$$

Potential energy $= \int_0^x F\,dx = \int_0^x Kx\,dx = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2e^{-\gamma t}\cos^2(\omega t - \delta)$

$$= \frac{1}{2}m\omega_0^2A^2e^{-\gamma t}\cos^2(\omega t + \delta)$$

∴ Total energy = KE + PE =

$$\frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \omega_0^2\cos^2(\omega t - \delta) + \frac{\gamma\omega}{2}\sin\{2(\omega t - \delta)\}\right]$$

For small damping, $\gamma << 2\omega_0$, then $e^{-\gamma t}$ does not change appreciably during one time period $T = \frac{2\pi}{\omega}$, then time averaged energy of the oscillator is

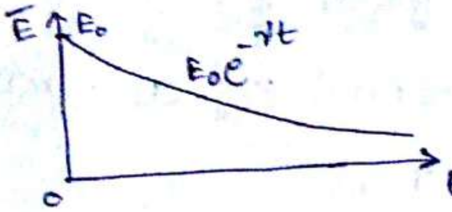
$$\langle E\rangle = \frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\langle\cos^2(\omega t - \delta)\rangle + \omega^2\langle\sin^2(\omega t - \delta)\rangle + \omega_0^2\langle\cos^2(\omega t - \delta)\rangle + \frac{\gamma\omega}{2}\cancelto{0}{\langle\sin\{2(\omega t - \delta)\}\rangle}\right]$$

Now $\langle\cos^2(\omega t - \delta)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\omega t - \delta)\,d(\omega t) = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}2\cos^2 x\,dx$

$$= \frac{1}{4\pi}\int_0^{2\pi}(1 + \cos 2x)\,dx = \frac{1}{2} = \langle\sin^2(\omega t - \delta)\rangle$$

$$\therefore \langle E\rangle = \frac{1}{2}mA^2e^{-\gamma t}\left[\cancel{\frac{\gamma^2}{8}} + \left(\omega_0^2 - \cancel{\frac{\gamma^2}{4}}\right)\frac{1}{2} + \frac{\omega_0^2}{2}\right] = \frac{1}{2}m\omega_0^2A^2e^{-\gamma t}$$

$\boxed{\langle E\rangle = E_0\, e^{-\gamma t}}$ where $E_0 = \frac{1}{2}m\omega_0^2A^2$ is energy of undamped oscillator



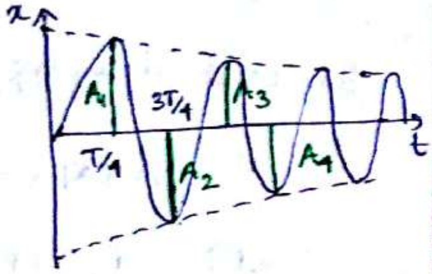
The average power dissipation in one time period

$$\langle P(t)\rangle = \frac{d}{dt}\langle E(t)\rangle = \gamma\langle E(t)\rangle.$$ due to friction

Estimation of Damping

There are various ways of estimation of the damping of an oscillator. Let us choose initial condition at $t=0$, $x=0$, $\frac{dx}{dt} = v_0$ and $\delta = \pi/2$, $x = Ae^{-\gamma t/2}\cos(\omega t - \pi/2) = Ae^{-\gamma t/2}\sin\omega t$

Logarithmic Decrement



$$x = Ae^{-\gamma t/2}\sin\omega t = Ae^{-\gamma t/2}\sin\frac{2\pi t}{T}$$

at $t = \frac{T}{4}$, $x_1^{max} = Ae^{-\gamma T/8}\sin\frac{2\pi}{T}\frac{T}{4} = Ae^{-\gamma t/8}$

at $t = \frac{3T}{4}$, $x_2^{max} = Ae^{-3\gamma T/8}$

at $t = \frac{5T}{4}$, $x_3^{max} = Ae^{-5\gamma T/8}$ etc.

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \frac{x_3^{max}}{x_4^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\gamma T/4} = d \text{ (constant)}$$

"d" is called decrement of the motion. $\lambda = \ln d$ is the logarithmic decrement of the motion $= \ln e^{\gamma T/4} = \frac{\gamma T}{4}$

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\lambda}$$

Multiplying, $\frac{x_1^{max}}{x_n^{max}} = e^{(n-1)\lambda}$ or $\lambda = \frac{1}{n-1}\ln\left(\frac{x_1^{max}}{x_n^{max}}\right)$

$$\lambda = \frac{2.303}{n-1}\log_{10}\left(\frac{x_1^{max}}{x_n^{max}}\right)$$

This method is used to determine the corrected last throw of a Ballistic galvanometer due to damping. Relation between undamped throw θ_0 & first throw θ_1 is

$\theta_1 = \theta_0 e^{-\gamma T/8}$ $\therefore \theta_0 = \theta_1 e^{\gamma T/8} = \theta_1 e^{\lambda/2} \simeq \theta_1\left(1+\frac{\lambda}{2}\right)$ for $\lambda \ll 1$

So knowing λ, we can correct θ_1 for damping.

Quality Factor (Q-Value)

Another method to express damping in an oscillatory system is to measure the rate of decay of energy. Quality factor $Q = \frac{\omega}{\nu}$ $= \frac{\omega_0}{\nu}\sqrt{1-\nu^2/4\omega_0^2}$. While $\langle E\rangle = E_0 e^{-\nu t}$, power $\langle P(t)\rangle = \frac{d}{dt}\langle E\rangle = \nu\langle E\rangle$

So the average energy dissipated in time period T is

$$\nu T\langle E\rangle = \frac{2\pi\nu}{\omega}\langle E\rangle = \frac{2\pi}{Q}\langle E\rangle = \frac{2\pi}{Q} \times \text{average energy stored.}$$

$$\therefore \boxed{Q = 2\pi \times \frac{\text{Average energy stored in one time period}}{\text{Average energy lost in one time period}}}$$

In weak damping limit $\frac{\nu^2}{4\omega_0^2} << 1$, $Q = \frac{\omega_0}{\nu}$. As $\nu \to 0$, $Q \to \infty$

$\therefore$ $x = A\exp\left(-\frac{\omega_0 t}{2Q}\right)\cos(\omega_0 t - \delta)$ in limit $\nu^2/4\omega_0^2 << 1$

$\langle E\rangle = E_0 \exp\left(-\frac{\omega_0 t}{Q}\right)$ and see that $\tau_1 = \frac{Q}{\omega_0}$, $\langle E\rangle = E_0 e^{-1}$

and no. of complete oscillation if is n, then $n = \frac{\omega_0}{2\pi}\tau_1 = \frac{Q}{2\pi}$

So $\langle E\rangle$ reduces to e^{-1} of $\langle E\rangle$ in $Q/2\pi$ cycles of oscillation.

Note that $\boxed{\lambda = \frac{\nu T}{4},\ \tau = \frac{2}{\nu}\ \&\ Q = \frac{\omega_0}{\nu},\ \tau_1 = \frac{Q}{\omega_0} = \frac{1}{\nu}}$

"Moving coil Galvanometer" is the example of damped harmonic motion. Similarly, current or charge oscillation in LCR circuit, mechanical vibration of a string or tuning fork etc.

Forced Vibration

Vibrating system with damping + periodic force = forced vibration natural vibration dies out, system tunes to the frequency of force. For example, a bridge vibrates in the influence of marching soldiers. Contributions are restoring force kx, damping force $b\dot{x}$, inertial force $m\ddot{x}$ & external periodic force $F(t) = F_0\cos\omega t$.

$\therefore$ Equation of motion of the body is