Using 1st law $ds = \dfrac{dU + PdV}{T}$ and $U = U(V,T)$

$dU = \left(\dfrac{\partial U}{\partial V}\right)_T dV + \left(\dfrac{\partial U}{\partial T}\right)_V dT$ we get $ds = \left\{\dfrac{P}{T} + \dfrac{1}{T}\left(\dfrac{\partial U}{\partial V}\right)_T\right\} dV + \dfrac{1}{T}\left(\dfrac{\partial U}{\partial T}\right)_V dT$

Now $U = U(V,T) = BT^n \ln(V/V_0) + f(T)$

$$ds = \left(\frac{AT^2}{V} + \frac{V_0 BT^{n-1}}{V}\right) dV + \left[\frac{f'(T)}{T} + nBT^{n-2} \ln\frac{V}{V_0}\right] dT$$

for ds to be exact differential, we have

Entropy

$$\frac{\partial}{\partial T}\left(\frac{V_0 BT^{n-1} + AT^2}{V}\right) = \frac{\partial}{\partial V}\left(\frac{f'(T)}{T} + nBT^{n-2} \ln\frac{V}{V_0}\right)$$

$$\frac{V_0 B(n-1)T^{n-2}}{\cancel{V}} + \frac{2AT}{\cancel{V}} = nBT^{n-2}\frac{V_0}{\cancel{V}}$$

$$\cancel{V_0 nBT^{n-2}} - BT^{n-2} + 2AT = \cancel{V_0 nBT^{n-2}} \quad \therefore \quad 2AT = BT^{n-2}$$

$\therefore$ $B = 2A$, $n = 3$.

2. $$\Delta S_{universe} = \int_{T_1}^{T_c} \frac{C_p dT}{T} + \int_{T_2}^{T_c} \frac{C_p dT}{T} = C_p \ln\frac{T_c^2}{T_1 T_2} \geq 0$$

$\therefore$ $T_c^2 \geq T_1 T_2$ or $T_c \geq \sqrt{T_1 T_2}$.

Maximum work can be obtained using reversible engine $\Delta S = 0$.

$$W_{max} = C_p(T_1 + T_2 - 2T_c^{min}) = C_p(T_1 + T_2 - 2\sqrt{T_1 T_2})$$

$$= C_p(\sqrt{T_1} - \sqrt{T_2})^2$$