Special Theory of Relativity

Books :

(1.) Introduction to Special Relativity → Resnick (Willey) ⇒ Good for first time readers to build concepts of SR (Einsteinian) without 4-vector (Poincare).

(2.) The Special Theory of Relativity → Bohm (Routledge)/ It's about time understanding Einstein's Relativity (Princeton University Press) → Mermin / Introduction to Special Relativity → Smith (Dover) / Relativity - The Special & General Theory → Einstein ⇒ Good for concept building.

(3.) Special Relativity → French. (CRC) ⇒ Berkeley Physics style book ; very good for minute details.

(4.) The Special Theory of Relativity → Banerji and Banerjee (PHI) ⇒ Student friendly nice book that consider both Einsteinian & Poincarian relativity without going too deep.

(5.) Classical Electrodynamics → Jackson (John Willey) ⇒ Chapter 11 ⇒ Tough but very good book to read about invariances and 4-vector formalism.

(6.) Classical Theory of Fields → Landau & Lifshitz (HB) ⇒ A book that any serious physics student can never afford to miss.

(7.) Modern Physics → Beiser / Mani-Mehta ⇒ Any standard modern physics book briefly touches Eeinsteinian Relativity which is usually easy to read for first time readers.

(8.) Tensor Analysis → B. Spain / Spiegel Schaum Series on Vector analysis ⇒ Read about tensors.

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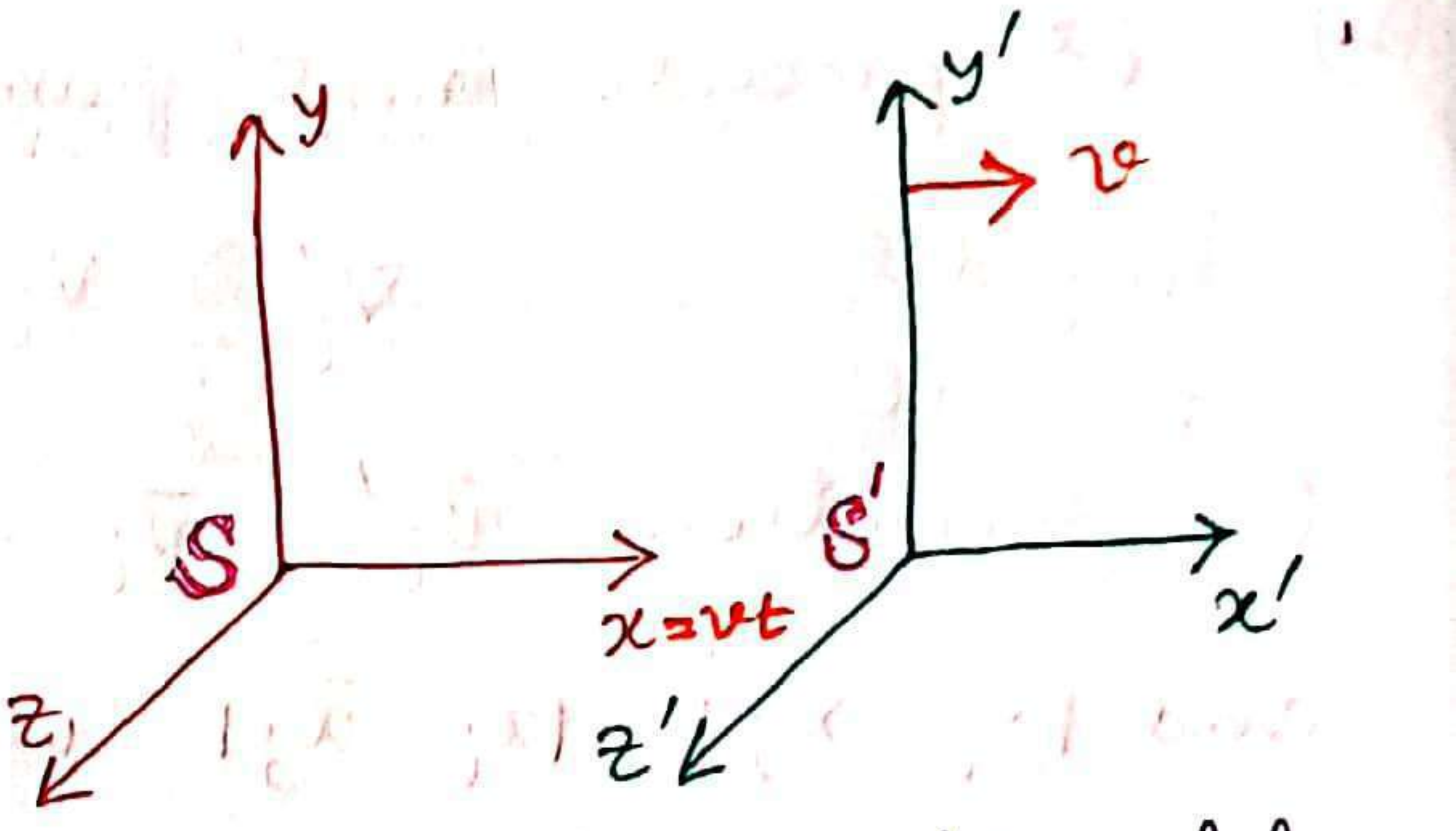
Background: Special theory of relativity, as originally proposed by Einstein in 1905, resulted due to two basic inconsistencies that were posed from theory and experiment. The incompatibility of Newtonian mechanics (Galilean relativity) with Maxwell's equation of Electrodynamics, and the hard-to-throw "luminiferous aether" hypothesis that originated due to study of Optics were void after the null result obtained from Michelson-Morley Nobelprize winning experiment, lead Einstein to develop this theory that works for inertial frame of reference only. The noninertial effects were bundled into the General theory of Relativity. Special theory is accurate in producing results at relativistic speed, i.e close to the velocity of light. The theory is "special" because it's the special case of the "general" theory where the curvature of spacetime, designated by the energy-momentum tensor to cause gravity, is negligible - so it's flat!

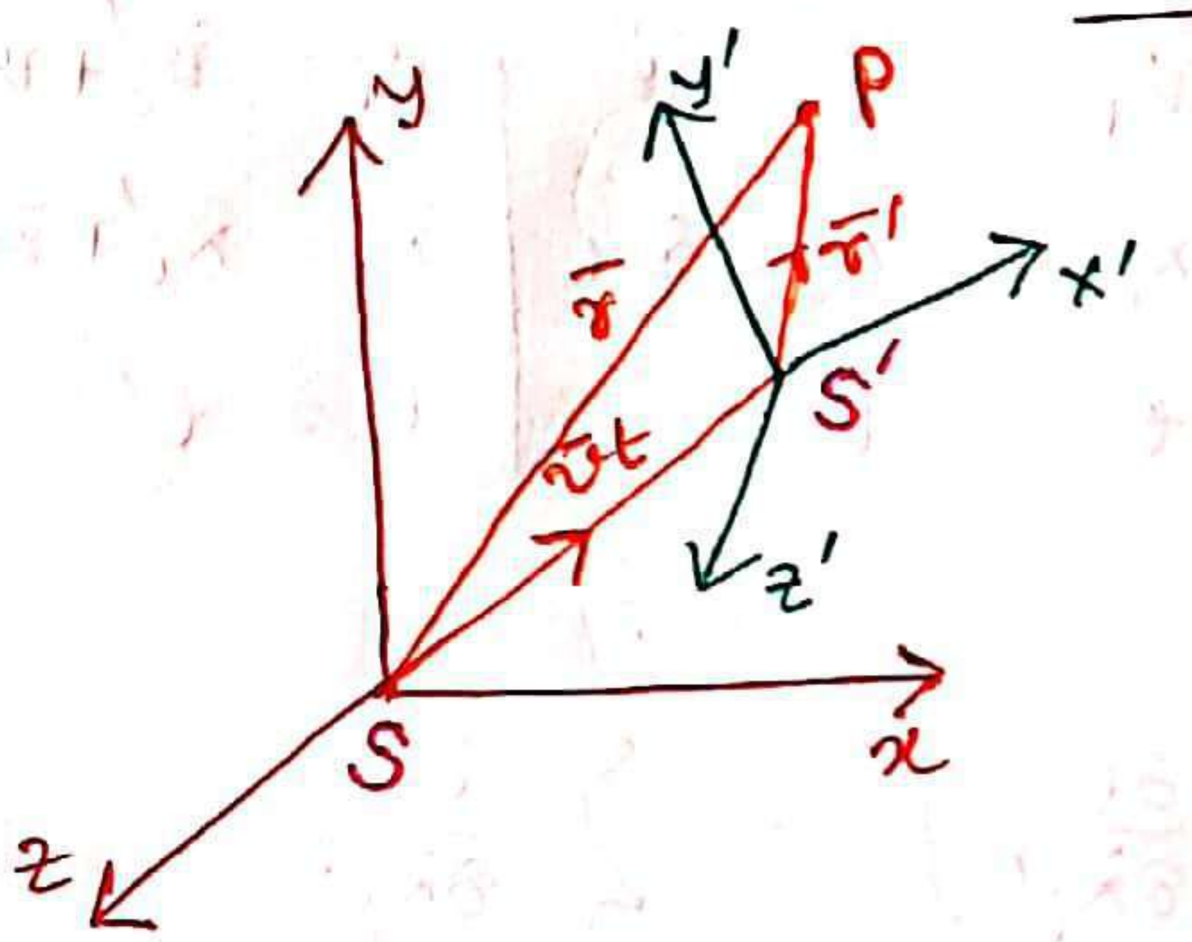
In Newtonian mechanics, speed of light has no special significance, so that according to $E = \frac{1}{2}mv^2$ in a particle accelerator, if energy of electron is increased 4 times, then velocity must be doubled, but experimentally it was found that a change from $0.9988c$ to $0.9999c$ happens for 10 MeV to 40 MeV increase of energy. So the connection between classical mechanics and electromagnetism was not understood. Here we look at the mathematical problem closely next.

Newtonian & Galilean Relativity



At non-relativistic speed, laws of physics are invariant in all inertial (Galilean) frames of reference. This is the principle of Galilean relativity which was established by taking Galilean transformation between two inertial frames S and S'. $x' = x - vt$, $y = y'$, $z = z'$ so that $u_{x'} = u_x - v$, $u_{y'} = u_y$, $u_{z'} = u_z$ & $a_{x'} = a_x$, $a_{y'} = a_y$, $a_{z'} = a_z$.

This means that "true force" measured in different inertial frames are equal and there is no way to distinguish among the infinitely many frames in which frame the true force is measured. This also applies for frames that are not parallel or relative velocities are also not parallel.



$$\bar{r}' = \bar{r} - \bar{v}t, \quad t' = t \quad \text{so that } \bar{u}' = \bar{u} - \bar{v} \text{ and } \bar{a}' = \bar{a}.$$

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In Galilean relativity, the length of a rod is invariant for both observers in S and S'.

If (x_1, y_1, z_1) and (x_2, y_2, z_2) are coordinates of two end points of the rod in S frame and (x_1', y_1', z_1') and (x_2', y_2', z_2') are that in S' frame which are in relative motion with each other having velocity v along x-direction, then

$$x_2' - x_1' = x_2 - vt - (x_1 - vt) = x_2 - x_1, \quad y_2' - y_1' = y_2 - y_1$$
$$z_2' - z_1' = z_2 - z_1$$

According to Pythagoras theorem, $l = l'$ where $l = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

In general case, $(\bar{r}_2' - \bar{r}_1')^2 = (\bar{r}_2 - \bar{r}_1)^2$, where $\bar{x}' = \bar{x} - \bar{v}t$.

Consider a group of Newtonian particles interacting via 2-body central field potential $V_{ij}(|\bar{x}_i - \bar{x}_j|)$. Equation of motion of the i^{th} particle in S' frame is

$$\boxed{m_i \frac{d\bar{v}_i'}{dt'} = -\bar{\nabla}_i' \sum_j V_{ij}(|\bar{x}_i' - \bar{x}_j'|)}$$

with the Galilean transformation $\bar{v}_i' = \bar{v}_i - \bar{v}$, $\bar{\nabla}_i' = \bar{\nabla}_i$, $\frac{d\bar{v}_i'}{dt'} = \frac{d\bar{v}_i}{dt}$ as $\bar{v} \neq \bar{v}(t)$ and $|\bar{x}_i' - \bar{x}_j'| = |\bar{x}_i - \bar{x}_j|$, we get back Newton's law in S-frame.

$$\boxed{m_i \frac{d\bar{v}_i}{dt} = -\bar{\nabla}_i \sum_j V_{ij}(|\bar{x}_i - \bar{x}_j|)}$$

(Galilean invariance)

If however the same transformation is applied to wave equation then a field $\phi(\bar{x}', t')$ satisfying wave equation in S' frame

$$\boxed{\Big(\underbrace{\sum_i \frac{\partial^2}{\partial \bar{x}_i'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}}_{\Box'^2}\Big) \phi(\bar{x}', t') = 0} \quad \text{— (1)}$$

$$\left\{ \begin{array}{l} \bar{v}_i' = \bar{v}_i - \bar{v} \\ \bar{x}_i' = \bar{x}_i - \bar{v}t \\ t' = t \end{array} \right. \quad \left| \quad \begin{array}{l} \bar{v}_i = \bar{v}_i' + \bar{v} \\ \bar{x}_i = \bar{x}_i' + \bar{v}t \\ t = t' \end{array} \right.$$

So that $\sum_i \frac{\partial^2}{\partial \bar{x}_i'^2} = \sum_i \frac{\partial}{\partial \bar{x}_i'}\frac{\partial}{\partial \bar{x}_i'}$

$$= \sum_i \Big(\frac{\partial}{\partial \bar{x}_i} \underbrace{\frac{\partial \bar{x}_i}{\partial \bar{x}_i'}}_{=1} + \frac{\partial}{\partial t} \underbrace{\frac{\partial t}{\partial \bar{x}_i'}}_{=0} \Big)\Big(\frac{\partial}{\partial \bar{x}_i} \underbrace{\frac{\partial \bar{x}_i}{\partial \bar{x}_i'}}_{=1} + \frac{\partial}{\partial t} \underbrace{\frac{\partial t}{\partial \bar{x}_i'}}_{=0} \Big) = \sum_i \frac{\partial^2}{\partial \bar{x}_i^2}$$

but $$\frac{\partial^2}{\partial t'^2} = \frac{\partial}{\partial t'}\frac{\partial}{\partial t'} = \Big(\frac{\partial}{\partial t} \underbrace{\frac{\partial t}{\partial t'}}_{=1} + \frac{\partial}{\partial \bar{x}_i} \underbrace{\frac{\partial \bar{x}_i}{\partial t'}}_{=\bar{v}} \Big)\Big(\frac{\partial}{\partial t} \underbrace{\frac{\partial t}{\partial t'}}_{=1} + \frac{\partial}{\partial \bar{x}_i} \underbrace{\frac{\partial \bar{x}_i}{\partial t'}}_{=\bar{v}} \Big)$$

$$= \Big(\frac{\partial}{\partial t} + \bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\Big)\Big(\frac{\partial}{\partial t} + \bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\Big) = \frac{\partial^2}{\partial t^2} + 2\bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\frac{\partial}{\partial t} + \Big(\bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\Big)\Big(\bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\Big)$$

So (1) becomes

$$\boxed{\Big(\sum_i \frac{\partial^2}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\Big(\bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\Big)\frac{\partial}{\partial t} - \frac{1}{c^2}\Big(\bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\Big)\Big(\bar{v}\cdot\frac{\partial}{\partial \bar{x}_i}\Big)\Big)\phi(\bar{x}, t) = 0}$$

So wave equation is not invariant under Galilean transformation. This is reasonable for sound consisting of compression & rarefaction that changes with the choice of reference frame, but not EM wave.

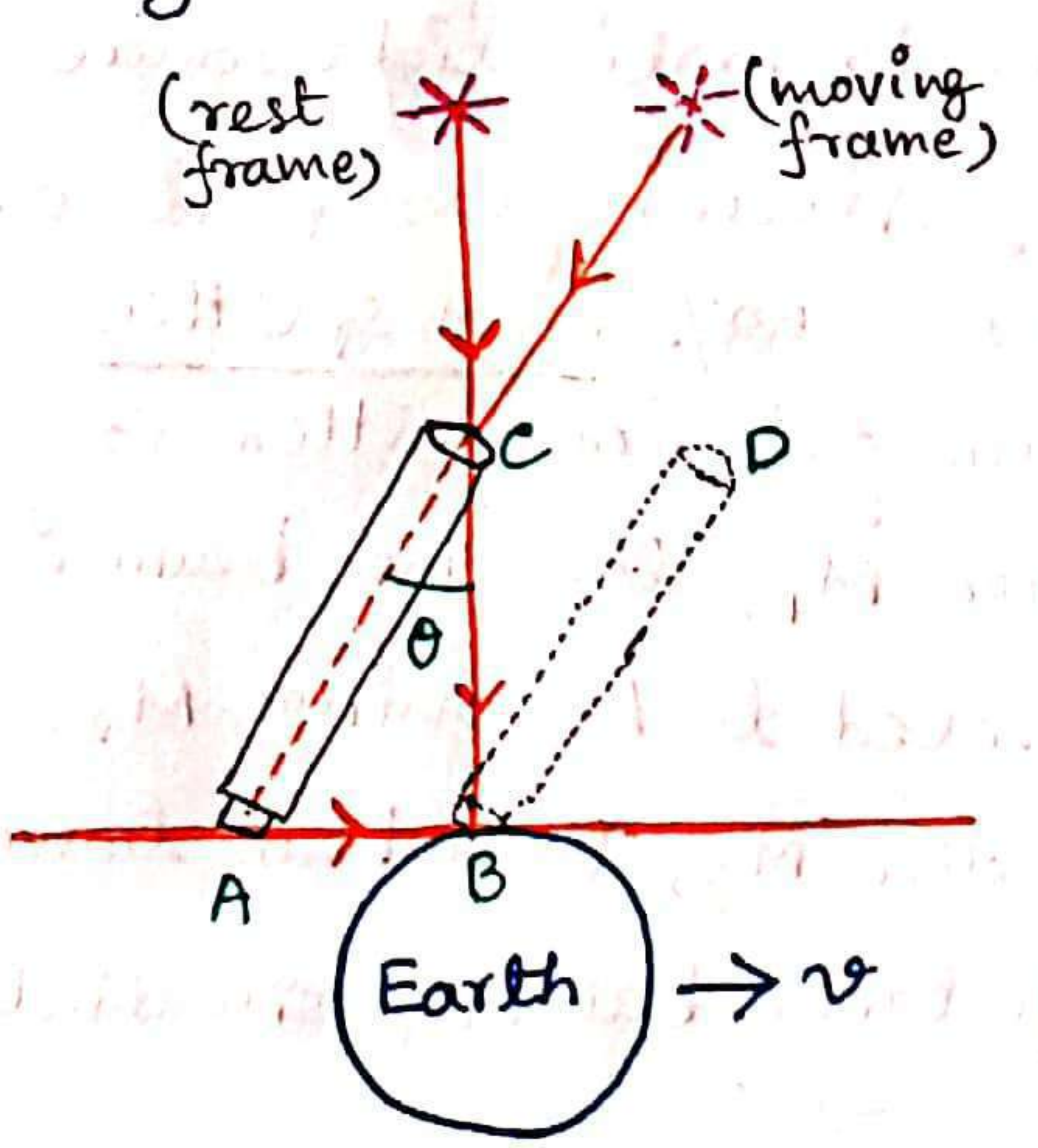
Schrödinger equation however is invariant under Galilean transformation, $-\frac{\hbar^2}{2m}\bar{\nabla}'^2\psi' + V'\psi' = i\hbar\frac{\partial \psi'}{\partial t'}$ becomes $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$ with $V' = V$ and $\psi = \psi' e^{\left[i\frac{m}{\hbar}\bar{v}\cdot\bar{x} - i\frac{mv^2}{2\hbar}t\right]}$

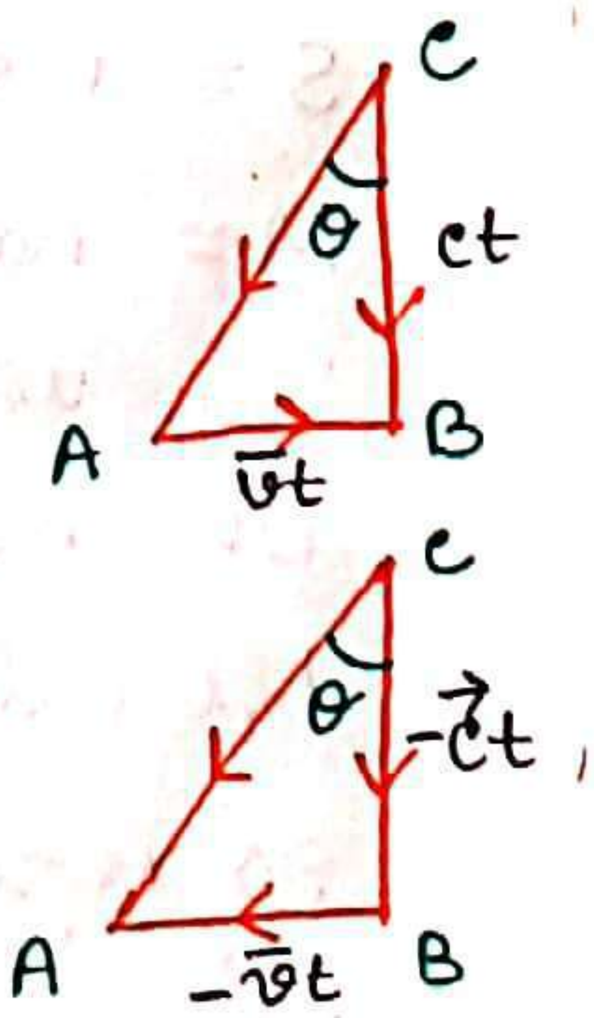
The luminiferous Ether & Search for Special Privileged frame

"Luminiferous aether", meaning light-bearing ether, was came into theory since Newton to accomodate the idea of light propagation through an invisible medium ether, similar to sound propagation. Einstein remarked in 1895 that the velocity of a wave is proportional to square root of elastic forces & inversely proportional to the mass of ether moved (dragged) by these forces. $v = \sqrt{E/\rho}$. 19th century believed that this velocity is the absolute velocity of Earth (with respect to ether frame) and tried to find this special frame using a series of optics experiments. The contradictory and negative result with prediction framed the theoretical ground of special relativity.

(a) Aberration of light (Bradley, 1727)

"Aberration" means propagation in moving bodies. If Earth is considered an ~~non~~ inertial frame, then on a windless rainy day a man standstill on ground will see raindrops atop (zenith) coming vertically downwards, which isn't the case if the man starts to move. Similarly if light is seen coming from a star to a man with telescope (astronomer) from zenith, the apparent direction of light from star will not be vertical because of Earth velocity v. The angle θ between actual & apparent direction is called aberration.





By the time light enters through C to come out from B of the telescope, due to Earth's velocity A has moved to B.

So $CB = ct$ and $AB = vt$

$$\therefore \tan\theta = 1/\frac{CB}{BA} = \frac{v}{c} \text{ or } \theta = \tan^{-1}\left(\frac{v}{c}\right)$$

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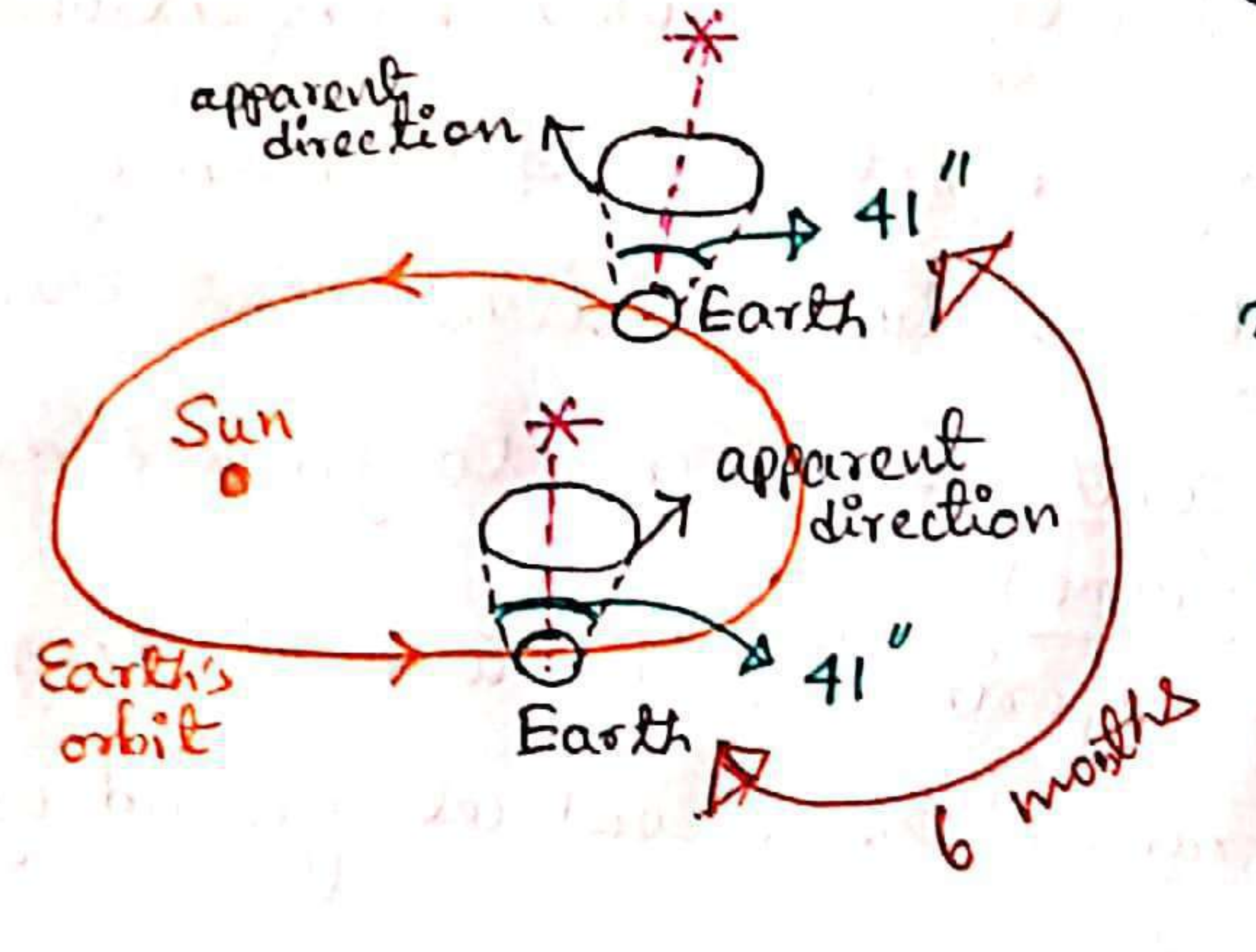
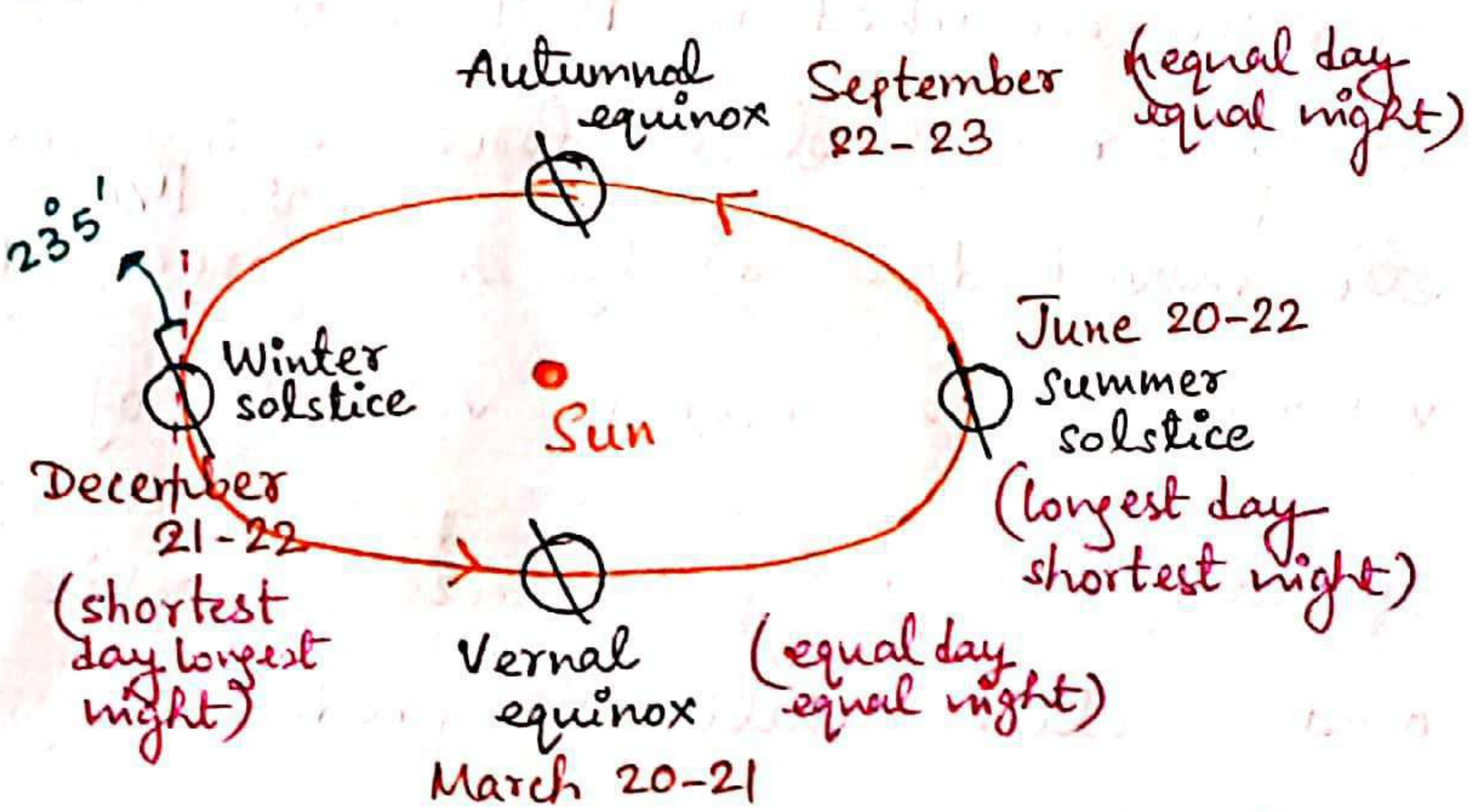
We know that Earth has daily and annual rotation, so for an equitorial observer, Earth's daily velocity is approximately $\frac{1}{2}$ km/s while that of annual is 30 km/s. Taking this into account

$$\theta = \tan^{-1}\frac{v}{c} = \tan^{-1}\frac{3\times10^4}{3\times10^8} = 20.5 \text{ arc seconds.}$$





So if an astronomer observes a star overhead for a year, it will create an ellipse of angular diameter of 41" with the zenith which Bradley tested with γ-Draconis star to experimentally confirm Earth's absolute velocity to be 30 km/s, and ether is not dragged around with Earth.

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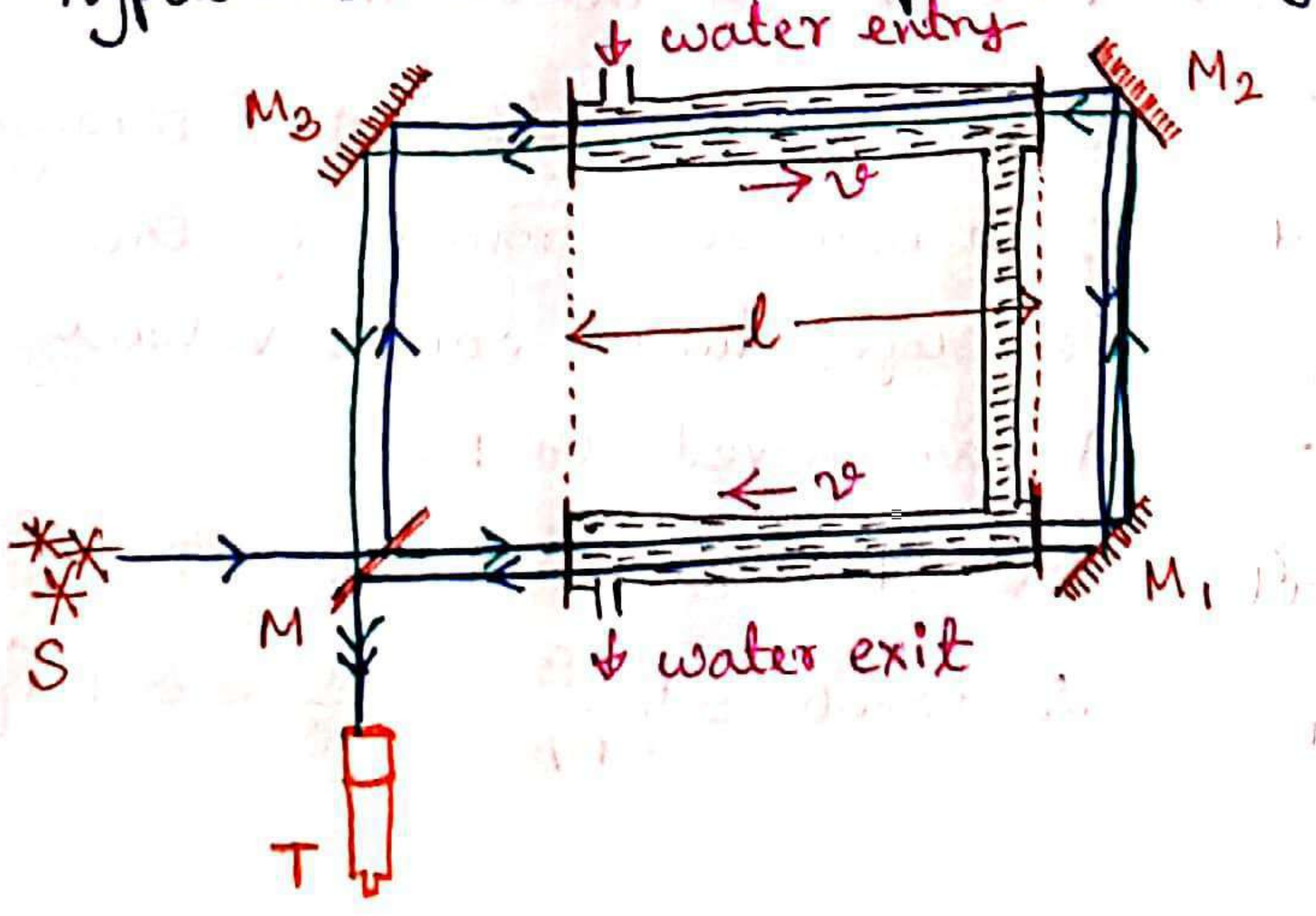
(b) Fizeau's Experiment (1851) after Fresnel's hypothesis (1817)

In 1817, Fresnel theoretically predicted that light will be partially dragged along flowing water to contradict the ether-drag hypothesis that was experimentally verified by Fizeau.



S = monochromatic light source

M = half-silvered glass plate as used as 50% beam splitter.

One beam gets transmitted to hit mirror M_1, the other beam is 90° reflected to hit mirror M_3. Using another M_2, both beam traverse equal distance but in opposite direction $(= 2l)$

Water flows through U-shaped tube. When both beams arrive at M_1 part of $MM_1M_2M_3$ beam is reflected (and $MM_3M_2M_1$ beam is transmitted) towards source S while the other parts interfere at telescope T. A particular fringe represents a particular "optical path" (distance in vacuum equivalent to actual path) difference between the two interfering beams. Distance l through water of refractive index n represents an optical path $= nl$.

If water flow had no dragging effect on ether then fringes before & after the start of water flow should not change, but a shift was experimentally seen. If water drags the ether with velocity v (with drag coefficient f) then periods of travel of two beams are $\frac{2l}{c/n - fv}$ and $\frac{2l}{c/n + fv}$, so that time period difference is

$$\Delta t = \frac{2l}{c/n - fv} - \frac{2l}{c/n + fv} = \frac{4flv}{c^2/n^2 - f^2v^2} = \frac{4flvn^2}{c^2\left(1 - \frac{f^2v^2n^2}{c^2}\right)}$$

$$\simeq \frac{4flvn^2}{c^2} \quad (\text{if } f^2v^2n^2/c^2 << 1)$$

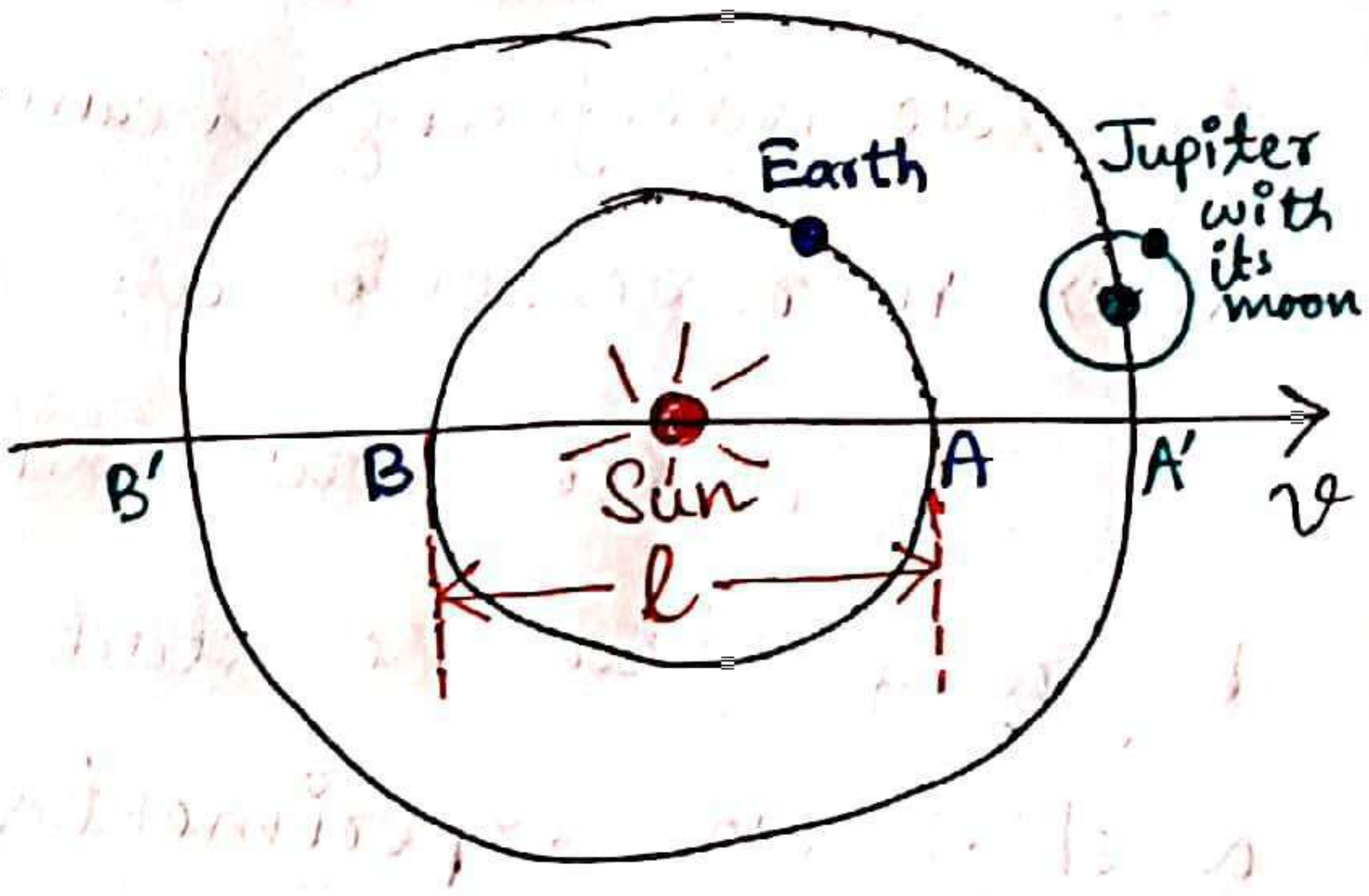
So optical path difference will be $c\Delta t = \frac{4flvn^2}{c}$ and shift of fringe is $\nu\Delta t = \frac{c\Delta t}{\lambda} = \frac{4flv}{c\lambda}n^2$. In Fizeau's experiment, $l = 1.5\,m$, $v = 7\,m/s$, $\lambda = 5.3 \times 10^{-7}\,m$, $n_{water} = 1.33$, $\nu\Delta t = 0.23$ fringe

$\therefore f \simeq 0.48$.

Fresnel in 1818 tried to justify this value theoretically by introducing "Fresnel's drag coefficient" $\boxed{f = 1 - \frac{1}{n^2}} = 0.43$. This was taken as a confirmation of partial ether drag hypothesis, that was established by more precision experiments by Michelson & Morley in 1886 and Zeeman in 1922.

Michelson - Morley Experiment (Nobel prize 1907)

Prelude In 1668, Rømer measured the speed of light by observing the eclipses of Jupiter's moon, by noticing a discrepancy of time between eclipses that increase when Earth was moving away from Jupiter and decrease when Earth was approaching. Maxwell in 1879 tried to measure the velocity of Solar system through ether using this idea. While Jupiter has a period of 12 years so as Earth in 6 months come from A to B, Jupiter doesn't move very far in its orbit. But in 6 years if Jupiter have moved from A′ to B′ then by looking at eclipse, solar system's velocity v through ether can be determined, as time difference



$$\Delta t = \frac{l}{c-v} - \frac{l}{c+v} = \frac{2lv}{c^2-v^2} = \frac{2lv}{c^2(1-\cancel{v^2/c^2})} \simeq \frac{2lv}{c^2} = \frac{2v}{c}t_0$$

where $t_0 \simeq 16$ minute, but then to detect $\Delta t = 1$ sec, $v = 150$ km/s which was too high (usually 20 km/s). So Maxwell concluded that effect proportional to first power of $\frac{v}{c}$ isn't good enough as to measure a second order effect, the light beam has to return to its starting point. So if one way path length is l & it moves along the direction v then total time $= \frac{l}{c+v} + \frac{l}{c-v} = \frac{2lc}{c^2-v^2}$ $\simeq \frac{2l}{c}(1+v^2/c^2)$, so that change of time is $\frac{2l}{c}v^2/c^2$. Taking $v = 30$ km/s as orbital speed of Earth, $v^2/c^2 \sim (10^{-4})^2 \simeq 10^{-8}$ which Maxwell thought was an impossible task to achieve, that Michelson at an age of 25 started to think of devicing an experiment. Michelson was the first American to receive a Nobel prize for this experiment, whose outcome we will discuss next.

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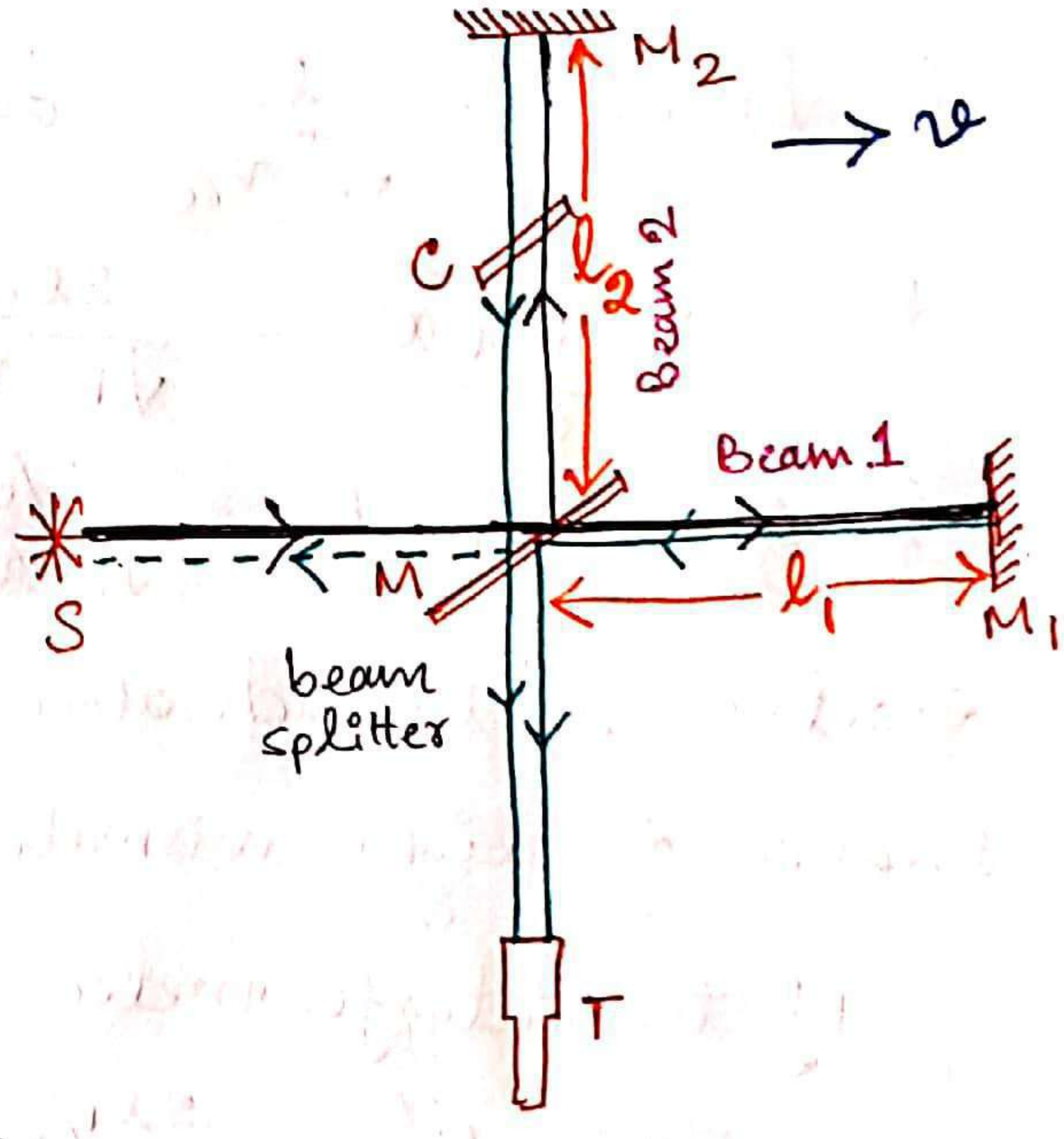
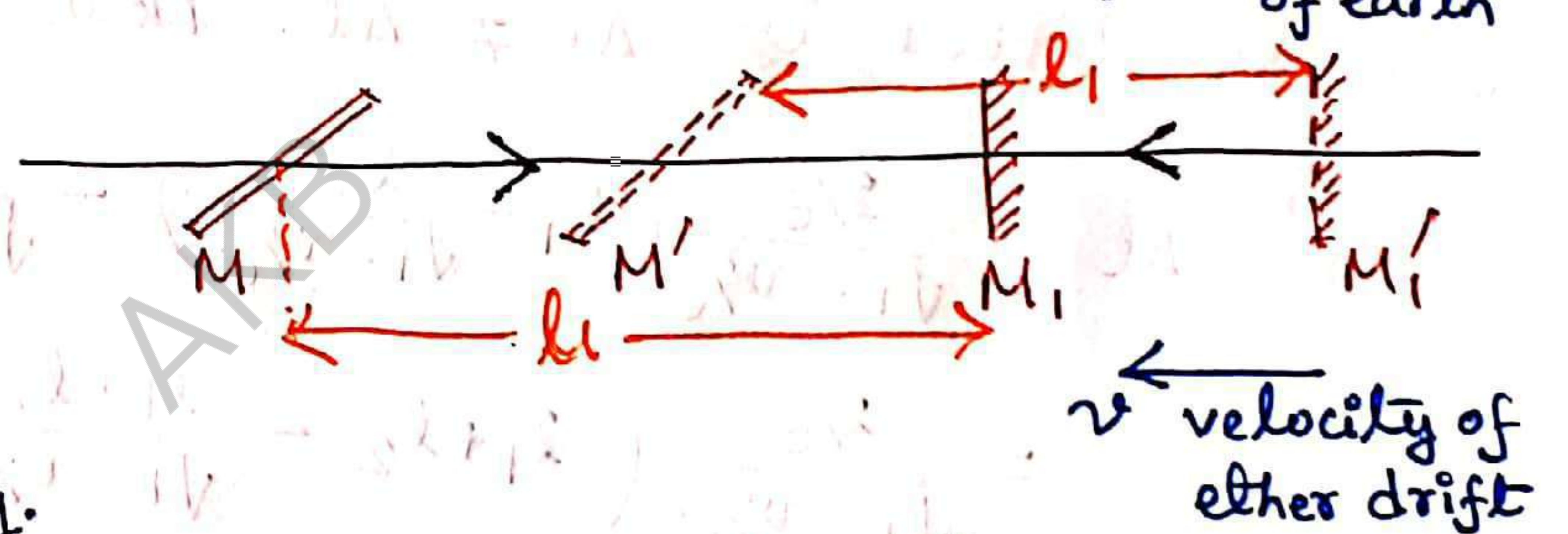
Michelson's Interferometer



In 1881, Michelson deviced an interferometer which was subsequently improved by Morley in 1887 that can measure the fringe shift in 2nd order. Light from monochromatic source S falls at 45° to a half-silvered glass plate that splits the beam & directs them towards mirror M_1 and M_2. Like Fizeau's interferometer, these beams interfere at telescope T to result in interference fringes. A compensating plate C is kept to equalize the traversed path by beam 1 and beam 2.

Beam 1 :

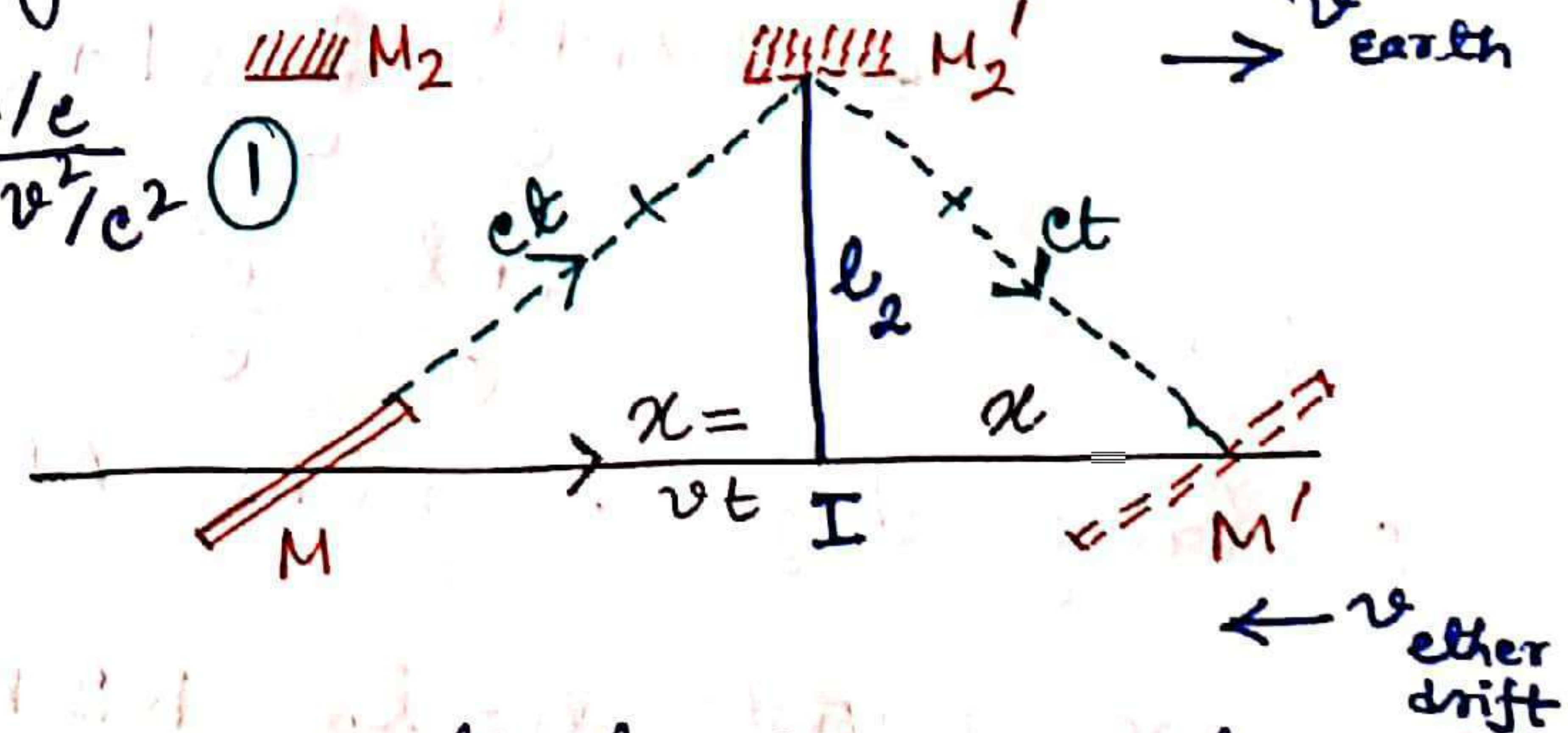


This beam travels along arm MM_1 along the direction of Earth's velocity.

Similar to a tidal speed in "up-and-down stream" experienced by a swimmer, time required for light to traverse back is

$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = \frac{2l_1/c}{1-v^2/c^2} \quad (1)$$

Beam 2 :



This beam moves "cross-stream" while in time beam 2 goes from MM_2, M_2 has reached M_2' due to Earth's velocity & by the time it retraces back, silvered glass plate M reaches M'. Drawing a perpendicular on MM' makes the median $M_2'I$ of the isosceles triangle $MM_2'M'$. If $MI = IM' = x$ then $MM_2' = M_2'M' = \sqrt{l_2^2 + x^2}$.

Now $\frac{x}{\sqrt{l_2^2+x^2}} = \frac{v}{c}$ or $1 - \frac{x^2}{l_2^2+x^2} = 1 - \frac{v^2}{c^2}$

$\therefore \sqrt{l_2^2+x^2} = \dfrac{l_2}{\sqrt{1-v^2/c^2}}$ and so the time for to-n-fro motion

$$t_2 = \frac{2\sqrt{l_2^2+x^2}}{c} = \frac{2l_2/c}{\sqrt{1-v^2/c^2}} \quad -② $$

So $\Delta t = t_1 - t_2 = \dfrac{2/c}{\sqrt{1-v^2/c^2}}\left(\dfrac{l_1}{\sqrt{1-v^2/c^2}} - l_2\right)$. If mirrors are kept exactly $90°$ to each other then fringes are concentric circles but because of slight mismatch, parallel line fringes are obtained.

If the interferometer is rotated by $90°$ to interchange the arms then $t_1' = \dfrac{2l_1/c}{\sqrt{1-v^2/c^2}}$ and $t_2' = \dfrac{2l_2/c}{\sqrt{1-v^2/c^2}}$, so that

$\Delta t' = t_1' - t_2' = \dfrac{2/c}{\sqrt{1-v^2/c^2}}\left(l_1 - \dfrac{l_2}{\sqrt{1-v^2/c^2}}\right)$. The interference fringes will be shifted if $\Delta t \neq \Delta t'$. Now,

$$\Delta t' - \Delta t = \frac{2/c}{\sqrt{1-v^2/c^2}}\left(l_1 - \frac{l_2}{\sqrt{1-v^2/c^2}}\right) - \frac{2/c}{\sqrt{1-v^2/c^2}}\left(\frac{l_1}{\sqrt{1-v^2/c^2}} - l_2\right)$$

$$= \frac{2/c}{\sqrt{1-v^2/c^2}}\left(l_1+l_2 - \frac{l_1+l_2}{\sqrt{1-v^2/c^2}}\right) = -\frac{2(l_1+l_2)}{c\sqrt{1-v^2/c^2}}\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$

$$\simeq -\frac{\cancel{2}(l_1+l_2)}{c}\left(1+\frac{v^2}{2c^2}\right)\left[\cancel{1}+\frac{v^2}{\cancel{2}c^2} - \cancel{1}\right] \quad \text{Binomial expansion}$$

$$= -\frac{l_1+l_2}{c}\frac{v^2}{c^2}\left(1+\cancel{\frac{v^2}{2c^2}}\right) \quad \text{of } O(v^4/c^4)$$

$$\simeq -\frac{l_1+l_2}{c}\frac{v^2}{c^2}$$

$\therefore$ fringe shift $= \nu(\Delta t' - \Delta t) = -\left(\dfrac{c}{\lambda}\right)\left(\dfrac{l_1+l_2}{c}\right)\left(\dfrac{v^2}{c^2}\right) = -\dfrac{l_1+l_2}{\lambda}\dfrac{v^2}{c^2}$

Using $v = 30\,km/s$, $l_1 = l_2 = 1.2$ metre, $\lambda = 6000\,Å$, Michelson obtained a fringe shift of 0.04. However in experiment, he found no shift. The precision was increased substantially without success & later Morley used $l_1+l_2 = 22$ metre, $\lambda = 5500\,Å$ to find fringe shift $= 0.4$ without any experimental success. It was concluded that absolute velocity of Earth must be less than $10\,km/s$.

Contradictory inferences on ether-drag hypothesis

Experiment	Absolute velocity of Earth through Ether	Inference on ether-drag by Earth	particle model	Wave/ether model
1. Stellar aberration (Bradley)	30 km/s	No drag	✓	OK if Earth moves w.r.t. ether
2. Fizeau & Airy	< 30 Km/s	Partial drag	requires partial drag of light by medium	requires partial drag of light by medium
3. Michelson-Morley	0	Total drag	✓	Implies Earth doesn't move w.r.t. ether.
4. Trouton-Noble	0	Total drag	✓	"

In 1892 Fitzgerald and Lorentz independently came up with the
length contraction hypothesis which was partially flawed due to
the lack of first principle connection and trial-error estimate
of contracted length. Einstein solved the problem in his Special
Relativity theory. To explain the null result, he argued that the
reason why it's not possible to detect relative motion through ether
is as velocity of light is an invariant & doesn't depend on the motion
of source, observer or medium. This mathematically establishes Lorentz-
Fitzgerald hypothesis that object moving relative to ether shortens
in the direction parallel to relative velocity in ratio $1 : \sqrt{1 - v^2/c^2}$.
Check that this gives $\Delta t = 0$, a perfect explanation of the null
result when $l_1 = l_2$ but not true ($\Delta t \neq 0$) if $l_1 \neq l_2$ as they thought
this contraction happens due to interaction between ether & matter,
which failed severely to explain Kennedy-Thorndike's experimental
setup of Michelson's interferometer.

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Fundamental Postulates of SR

In 1905, Einstein discarded the notion of time invariance as posed by Newton & Galileo in the principle of Galilean relativity and said that 'absolute motion' is meaningless concept.

(i) All laws of physics (mechanics & electrodynamics) remain invariant in all inertial (Galilean) frames. No experiment will be able to differentiate between two frames & so its not possible to find relative velocities w.r.t. ether.

(ii) The speed of light in vacuum is constant, independent not only of the direction of propagation but also of the relative velocity between the source of light & observer.

Lorentz in 1904 mathematically found the set of transformation that leaves Maxwell's equation invariant (which we know name as "Lorentz transformation"), but didn't proceed further as the transformation breaks the invariance of laws of Mechanics. Scientists before Lorentz also saw that Galilean invariance is true only for Mechanics & not for electrodynamics, hence to preserve the notion of Galilean invariance, the ether hypothesis came to remove the contradictions. Einstein modified ~~both~~ the transformation to accomodate electrodynamics.

Simultaneity & Time order

Einstein conjectured that measurement of both space & time depend on the observer's frame of reference as follows:

If two observer, one on Earth & another moving towards source S with velocity v measures the speed of light using their clock then

$$c = \frac{x}{t} = \frac{x'}{t'}$$ because in rocket frame too, velocity of light is not $c-v$.

Since $x' < x$, then $t' < t$.

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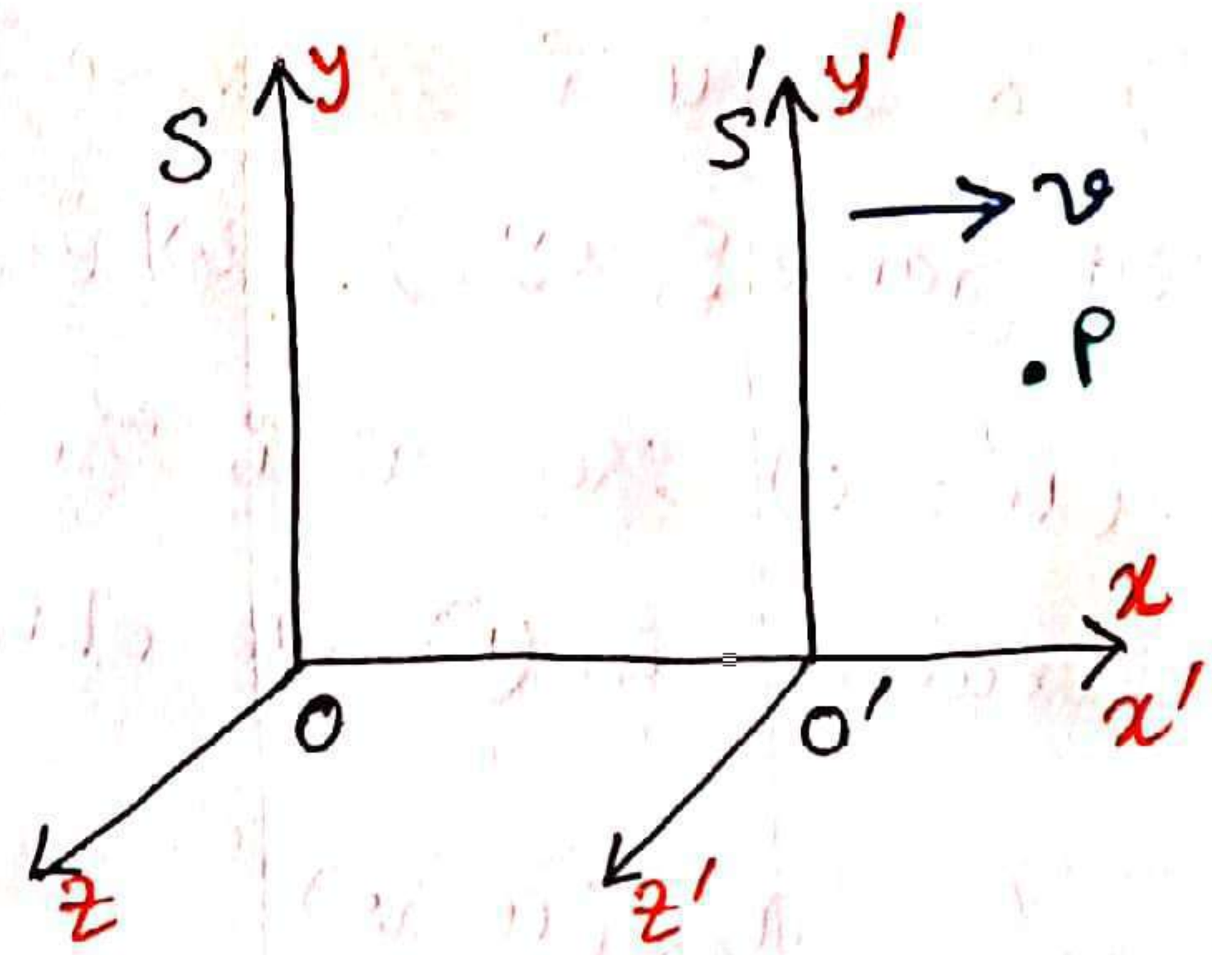
Just using two chronometers to synchronize at two different places it can be realized (read Resnick sec. 2.1) that no information can be transmitted at a velocity greater than light.

Lorentz transformation



Suppose S and S' are two inertial frame of reference, in which S is at rest and S' is moving with respect to S with a velocity v along the x-axis.

As there is no motion along y and z direction, so the y and z components will remain unchanged, i.e. $y' = y$ and $z' = z$. But x must transform linearly into x' and t', so as t. Linearity is needed for homogeneity of space and time. When the transformation is linear, length of a measuring rod will not depend on what region of space it is located. If l_1 is the length of the measuring rod and l_2 is the reading in the meter scale in unprimed frame, then if $l_1 = l_2$ then also in the primed frame $l_1' = l_2'$, so that rod and scale both contract / elongate equally.

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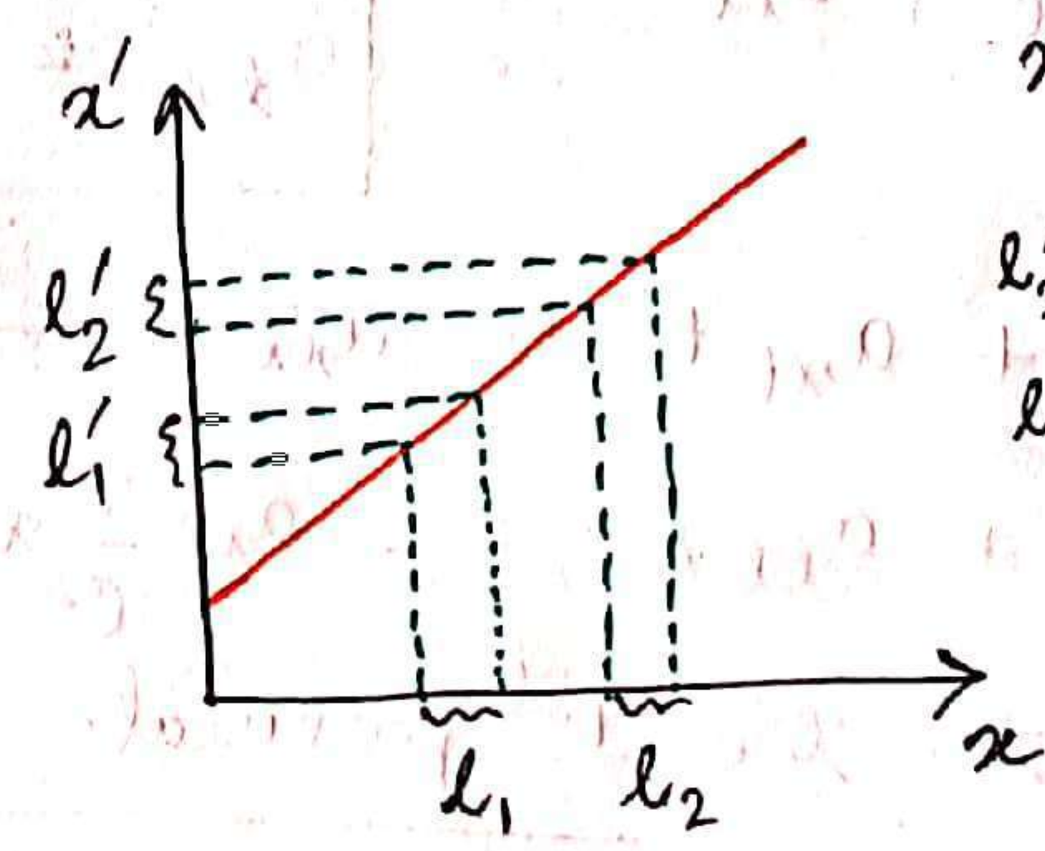
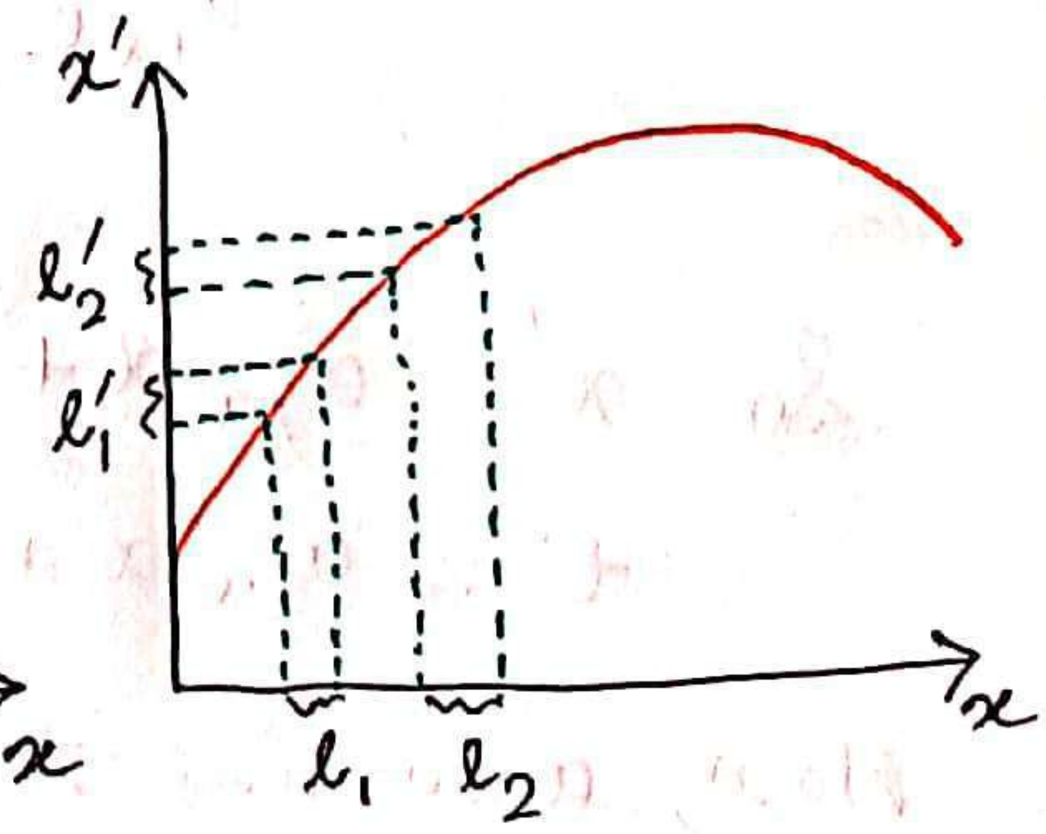




If the transformation is nonlinear then if $l_1 = l_2$, $l_1' \neq l_2'$, i.e. the length of the rod would depend upon the region of space where it's located. This is against the principle of homogeneity of space. A similar argument is also applicable for time.

$\therefore$ We seek the transformation,
$$\boxed{\begin{aligned} x' &= a_{xx}\, x + a_{xt}\, t \\ t' &= a_{tx}\, x + a_{tt}\, t \end{aligned}}$$
where $a_{xx}, a_{xt}, a_{tx}, a_{tt}$ are constants to be determined. The interval therefore can be defined as

$$\Delta x' = a_{xx}\,\Delta x + a_{xt}\,\Delta t\,, \quad \Delta t' = a_{tx}\,\Delta x + a_{tt}\,\Delta t.$$

Dividing both side, we obtain

$$u' = \frac{\Delta x'}{\Delta t'} = \frac{a_{xx}\Delta x + a_{xt}\Delta t}{a_{tx}\Delta x + a_{tt}\Delta t} = \frac{a_{xx}\frac{\Delta x}{\Delta t} + a_{xt}}{a_{tx}\frac{\Delta x}{\Delta t} + a_{tt}} = \frac{a_{xx}u + a_{xt}}{a_{tx}u + a_{tt}} \quad \text{— (1)}$$

where u' is the velocity of the particle P in the primed frame $S'(x'y'z')$ and u is the velocity of the same particle P in the unprimed frame $S(xyz)$. When the particle is at rest in the primed frame ($u' = 0$), then v is the velocity in the unprimed frame S.

Substituting in (1), we obtain $0 = \dfrac{a_{xx}v + a_{xt}}{a_{tx}v + a_{tt}}$ or $\boxed{a_{xt} = -a_{xx}v}$

$\therefore u' = \dfrac{a_{xx}(u-v)}{a_{tx}u + a_{tt}}$. On the contrary, if the particle is at rest in the unprimed frame S, then $u = 0$ and $u' = -v$ as S' is moving at a velocity v in the +ive x-axis.

So $-v = \dfrac{a_{xx}(0-v)}{a_{tt}}$ or $-va_{tt} = -va_{xx}$ or $\boxed{a_{xx} = a_{tt}}$

$\therefore u' = \dfrac{a_{xx}(u-v)}{a_{tx}u + a_{xx}}$. According to Einstein's 2nd postulate, the velocity of light is constant in both S and S' frame, so $u = u' = c$

$\therefore c = \dfrac{a_{xx}(c-v)}{a_{tx}c + a_{xx}}$ or $a_{tx}c^2 + \cancel{a_{xx}c} = \cancel{a_{xx}c} - va_{xx}$

or $\boxed{a_{tx} = -a_{xx}\, v/c^2}$

So
$$\left.\begin{aligned} x' &= a_{xx}x + a_{xt}t = a_{xx}x - a_{xx}vt = a_{xx}(x - vt) \\ t' &= a_{tx}x + a_{tt}t = -a_{xx}\frac{v}{c^2}x + a_{xx}t = a_{xx}(t - vx/c^2) \end{aligned}\right\} \text{— (2)}$$

Now, according to the 1st postulate of Einstein, i.e. the equivalence of all inertial frames, the above relations must hold if S' is at rest and S moves with a velocity $-v$, so

$$x = a_{xx}(x' + vt') \text{ and } t = a_{xx}(t' + vx'/c^2) \quad \text{— (3)}$$

Substituting the values of x & t from (3) into (2), we get

$$\cancel{x'} = a_{xx}^2\left[x' + \cancel{vt'} - v\left(\cancel{t'} + \frac{vx'}{c^2}\right)\right] = a_{xx}^2\cancel{x'}(1 - v^2/c^2)$$

or $\boxed{a_{xx} = \dfrac{1}{\sqrt{1 - v^2/c^2}}} = \gamma$ (say)

So, $\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}}$ $\Longleftrightarrow$ $\boxed{\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned}}$

Lorentz transformation (L.T.) Reciprocal Lorentz transformation. (R.L.T.)

Lorentz transformation in Hyperbolic form

The L.T. relations can be expressed in terms of hyperbolic trigonometric functions by measuring space and time in the same units (by replacing t to ct), so that the symmetry between the equations are obvious.

R.L.T. $x = \gamma(x' + vt') = \gamma\left(\underline{x'} + \frac{v}{c}\,\underline{ct'}\right)$

$y = y'$, $z = z'$

$ct = \gamma\left(\underline{ct'} + \frac{v}{c}\,\underline{x'}\right)$. We define "rapidity" via $\frac{v}{c} = \tanh\beta$

So that $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \dfrac{1}{\sqrt{1 - \tanh^2\beta}} = \dfrac{1}{\sqrt{1 - \frac{\sinh^2\beta}{\cosh^2\beta}}} = \dfrac{\cosh\beta}{\sqrt{\cosh^2\beta - \sinh^2\beta}} = \cosh\beta$

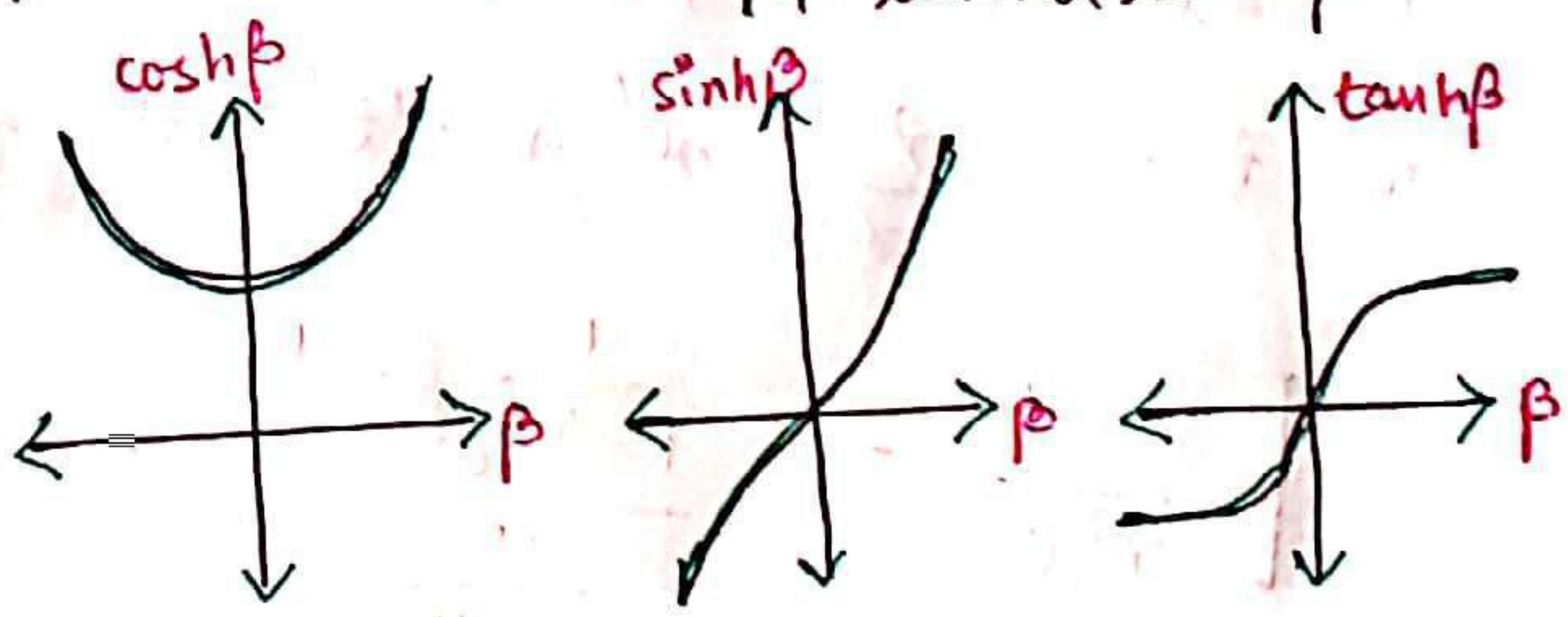
[Using trigonometric identities, $\cosh\beta = \dfrac{e^{\beta} + e^{-\beta}}{2}$, $\sinh\beta = \dfrac{e^{\beta} - e^{-\beta}}{2}$,

$\cosh^2\beta - \sinh^2\beta = 1$, $\sinh(\alpha + \beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$

$\cosh(\alpha + \beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$, $\tanh(\alpha + \beta) = \dfrac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta}$

$\frac{d}{d\beta}\sinh\beta = \cosh\beta$ $\quad \tanh\beta = \dfrac{\sinh\beta}{\cosh\beta}$

$\frac{d}{d\beta}\cosh\beta = \sinh\beta$]



$\frac{v}{c}\gamma = \tanh\beta\cosh\beta = \sinh\beta$.

Inserting these identities into the R.L.T. takes the form in 1+1 dimension

$$x = x'\gamma + ct'\gamma\frac{v}{c} = x'\cosh\beta + ct'\sinh\beta$$

$$ct = ct'\gamma + x'\gamma\frac{v}{c} = x'\sinh\beta + ct'\cosh\beta.$$

$$\therefore \boxed{\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}\begin{pmatrix} x' \\ ct' \end{pmatrix}}.$$

This means that with analogy to the Euclidean space (xyz), L.T. resembles to hyperbolic rotation in the Minkowski space (x, ct). We will see that due to this hyperbolic form, distance $x^2 - c^2t^2$ remains invariant, i.e. $x^2 - c^2t^2 = x'^2 - c^2t'^2$. (Lorentz Scalar)

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Properties of Lorentz Transformation

① Galelian transformations as special case of L.T.

When speed is not at relativistic speed, ie $v/c << 1$ then $v/c^2 \to 0$, $v^2/c^2 \to 0$, so that $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = 1$ and we obtain from L.T.

$x' = x - vt$, $y' = y$, $z' = z$, $t' = t$. So L.T. equations reduce to G.T. at nonrelativistic speeds. Recall that the Schrödinger equation, that designate of a nonrelativistic electron, is invariant under G.T. but Maxwell's equations are invariant under L.T.

② L.T. is Symmetric

We have defined R.L.T. ~~as~~ when the relative velocity of S as observed from S' to be $-v$. We can derive them from L.T.

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{v}{c^2}x\right) \quad — ①$$

or $$\frac{x' + vt'}{\gamma} = x - \cancel{vt} + \cancel{vt} - \frac{v^2}{c^2}x = x\left(1 - \frac{v^2}{c^2}\right) = \frac{x}{\gamma^2}$$

∴ $\boxed{x = \gamma(x' + vt')}$. Substituting back to ①, we obtain

$$\frac{x'}{\gamma} = x - vt = \gamma(x' + vt') - vt$$

or $$vt = \left(\gamma - \frac{1}{\gamma}\right)x' + \gamma v t' = \frac{\gamma^2 - 1}{\gamma}x' + \gamma v t'$$

or $$t = \frac{\gamma^2 - 1}{\gamma v}x' + \gamma t' = \gamma\left(t' + \frac{\gamma^2 - 1}{\gamma^2 v}x'\right) = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

$$\left[\frac{\gamma^2 - 1}{\gamma^2 v} = \frac{\frac{1}{1 - v^2/c^2} - 1}{\frac{v}{1 - v^2/c^2}} = \frac{v^2/c^2}{v} = \frac{v}{c^2}\right]$$

∴ $\boxed{t = \gamma\left(t' + \frac{v}{c^2}x'\right)}$

R.L.T.

③ Existence of Invariant Interval & volume element

An "event" is described by three coordinates of the particle & the time as measured in an inertial frame of reference. In a fictitional 4-dimensional space made of 3 space-coordinate & 1 time coordinate, events are represented by points (called "world points") & successive events are connected with line (called "world line"). If we consider two events P_1 and P_2 occuring at (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) in frame S and (x_1', y_1', z_1', t_1') & (x_2', y_2', z_2', t_2')

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in frame S' then distance covered by light signal $c(t_1-t_2)$, is equal to $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$. Because c is same in both S and S', so $c(t_1'-t_2')$ is equal to $\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2+(z_2'-z_1')^2}$. So we define "interval"

$$S_{12} = \left[c^2(t_2-t_1)^2-(x_2-x_1)^2-(y_2-y_1)^2-(z_2-z_1)^2\right]^{1/2} = \sqrt{c^2dt^2-dx^2-dy^2-dz^2}$$

and if it's zero in S then is zero also in S'.

L.T. equations $x' = \gamma(x-vt)$, $y'=y$, $z'=z$, $t' = \gamma(t-\frac{v}{c^2}x)$

So $x'^2 - c^2t'^2 = \gamma^2(x-vt)^2 - c^2\gamma^2(t-\frac{v}{c^2}x)^2$

$$= \gamma^2\left[x^2+v^2t^2-\cancel{2vxt}-c^2(t^2-\cancel{\tfrac{2v}{c^2}xt}+\tfrac{v^2}{c^4}x^2)\right]$$

$$= \gamma^2\left[x^2(1-v^2/c^2)-c^2t^2(1-v^2/c^2)\right] = \frac{1-v^2/c^2}{1-v^2/c^2}(x^2-c^2t^2)$$

$$= x^2-c^2t^2. \text{ (Lorentz scalar)}$$

It follows apparently that $dS_{12}^2 = c^2dt^2-dx^2 = dS_{12}'^2 = c^2dt'^2-dx'^2$

Before we establish a relation between dx' & dx, dt' & dt (length contraction & time dilation), we see the invariance of $\Box^2 = \partial_x^2+\partial_y^2+\partial_z^2-\frac{1}{c^2}\partial_t^2$ under L.T.

L.T. $x' = \gamma(x-vt)$, $y'=y$, $z'=z$, $t' = \gamma(t-\frac{v}{c^2}x)$

$$\text{So } \partial_x = \frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \gamma\frac{\partial}{\partial x'} - \frac{v\gamma}{c^2}\frac{\partial}{\partial t'} = \gamma\partial_{x'} - \frac{v\gamma}{c^2}\partial_{t'}$$

$$\partial_y = \frac{\partial}{\partial y} = \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = \frac{\partial}{\partial y'} = \partial_{y'}, \quad \partial_z = \partial_{z'}$$

$$\partial_t = \frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \gamma\frac{\partial}{\partial t'} - \gamma v\frac{\partial}{\partial x'} = \gamma\partial_{t'} - \gamma v\partial_{x'}$$

$$\therefore \partial_x^2 = \left(\gamma\partial_{x'} - \frac{v\gamma}{c^2}\partial_{t'}\right)\left(\gamma\partial_{x'} - \frac{v\gamma}{c^2}\partial_{t'}\right)$$

$$= \gamma^2\partial_{x'}^2 + \frac{v^2}{c^4}\gamma^2\partial_{t'}^2 - \cancel{\frac{2v\gamma}{c^2}\partial_{x'}\partial_{t'}}$$

$$\partial_t^2 = (\gamma\partial_{t'}-v\gamma\partial_{x'})(\gamma\partial_{t'}-v\gamma\partial_{x'}) = \gamma^2\partial_{t'}^2 + v^2\gamma^2\partial_{x'}^2 - 2v\gamma^2\partial_{x'}\partial_{t'}$$

$$\therefore \Box^2 = \partial_x^2+\partial_y^2+\partial_z^2-\frac{1}{c^2}\partial_t^2 = \underbrace{\gamma^2\left(1-\frac{v^2}{c^2}\right)}_{=1}\partial_{x'}^2 + \partial_{y'}^2+\partial_{z'}^2 - \frac{1}{c^2}\underbrace{\gamma^2\left(1-\frac{v^2}{c^2}\right)}_{=1}\partial_{t'}^2$$

$$= \partial_{x'}^2+\partial_{y'}^2+\partial_{z'}^2-\frac{1}{c^2}\partial_{t'}^2 = \Box'^2 \quad \text{(Lorentz scalar)}$$

We will define fourdimensional volume element next & show that its also a Lorentz scalar or invariant under L.T.

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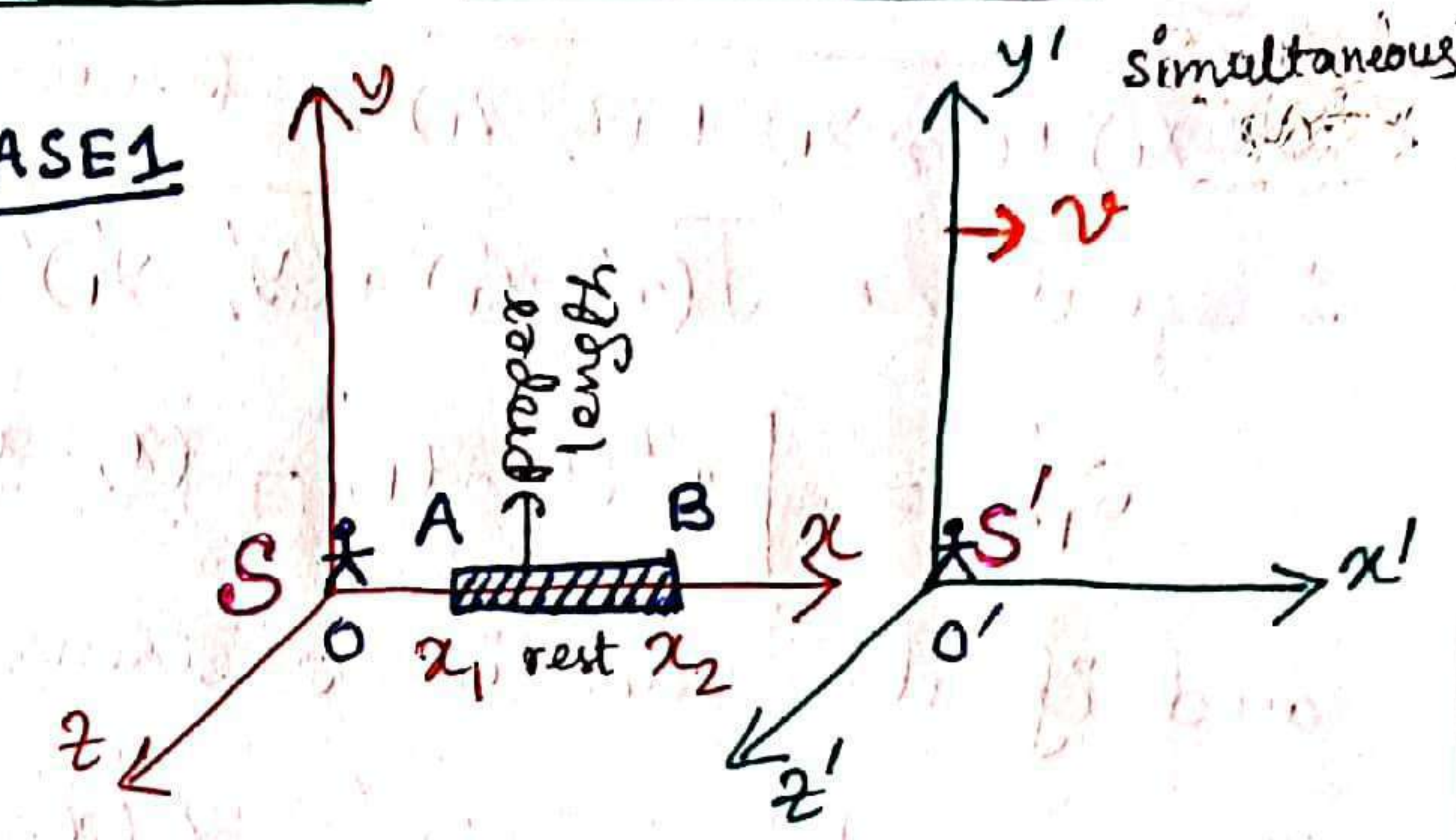
④ Lorentz - Fitzgerald length contraction & Proper length

CASE 1



Let us consider a rod of length l at rest in a reference frame S parallel & on the x-axis. If the coordinates of two ends A & B are x_1 & x_2, then length of the rod measured by an observer is $l = x_2 - x_1$. If an observer on S' frame (which is moving with velocity v w.r.t. S frame) measures the same length by calculating the end coordinates A & B at the same instant of time t' according to his clock,

From R.L.T. we have $x_1 = \gamma(x_1' + vt')$ and $x_2 = \gamma(x_2' + vt')$

So $l = x_2 - x_1 = \gamma(x_2' - x_1')$

$\therefore \boxed{l' = x_2' - x_1' = l/\gamma}$ $\therefore \underline{l' < l}$

In other words, observer in S' finds the rod AB of S-frame contracted to $l/\gamma = l\sqrt{1 - v^2/c^2}$. Recall that $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1$

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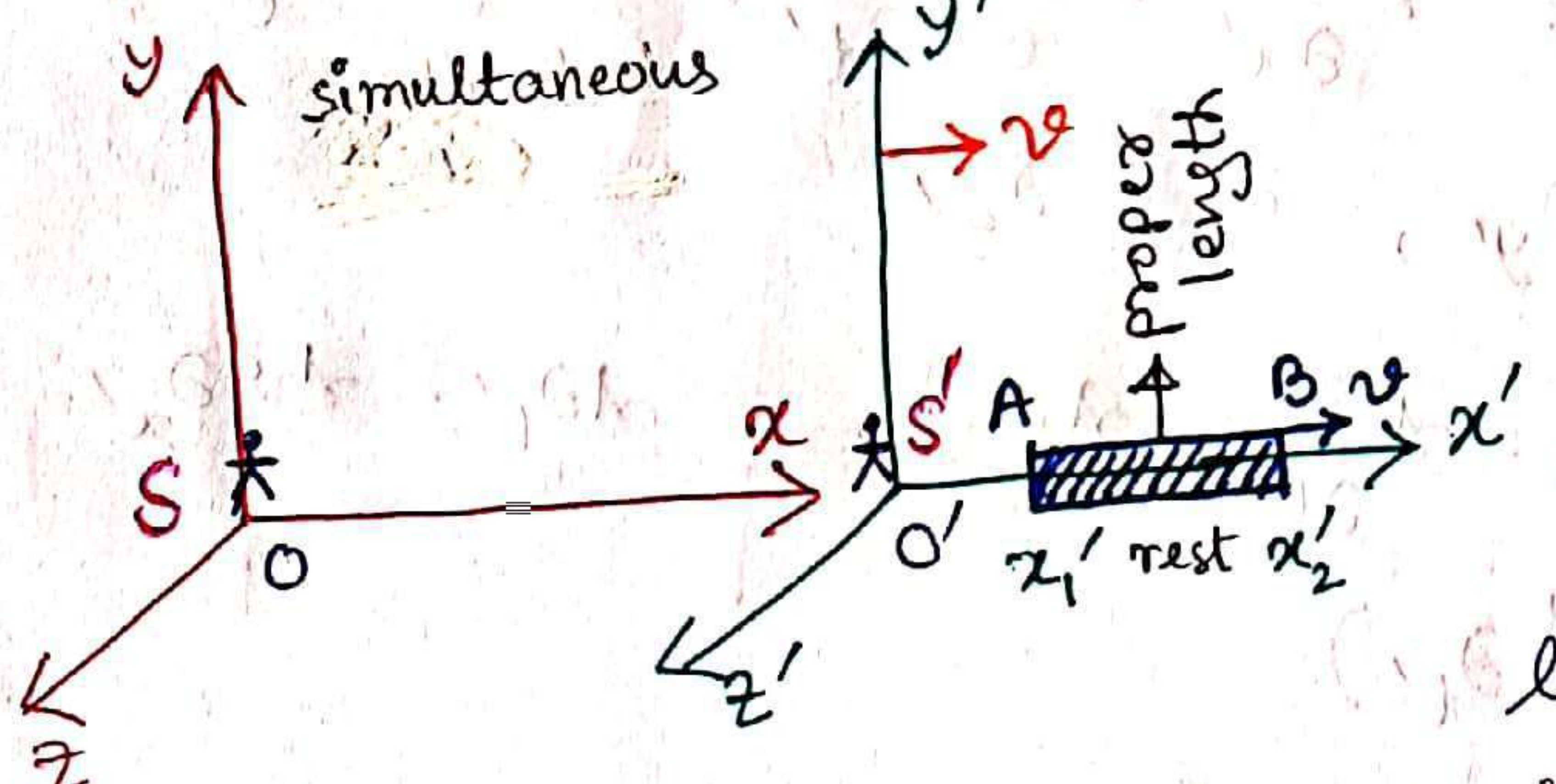
CASE 2

Now suppose that if we place the rod in S' frame at rest with an observer measuring its two ends A and B at an instant of time t' with a clock, then $l' = x_2' - x_1'$.



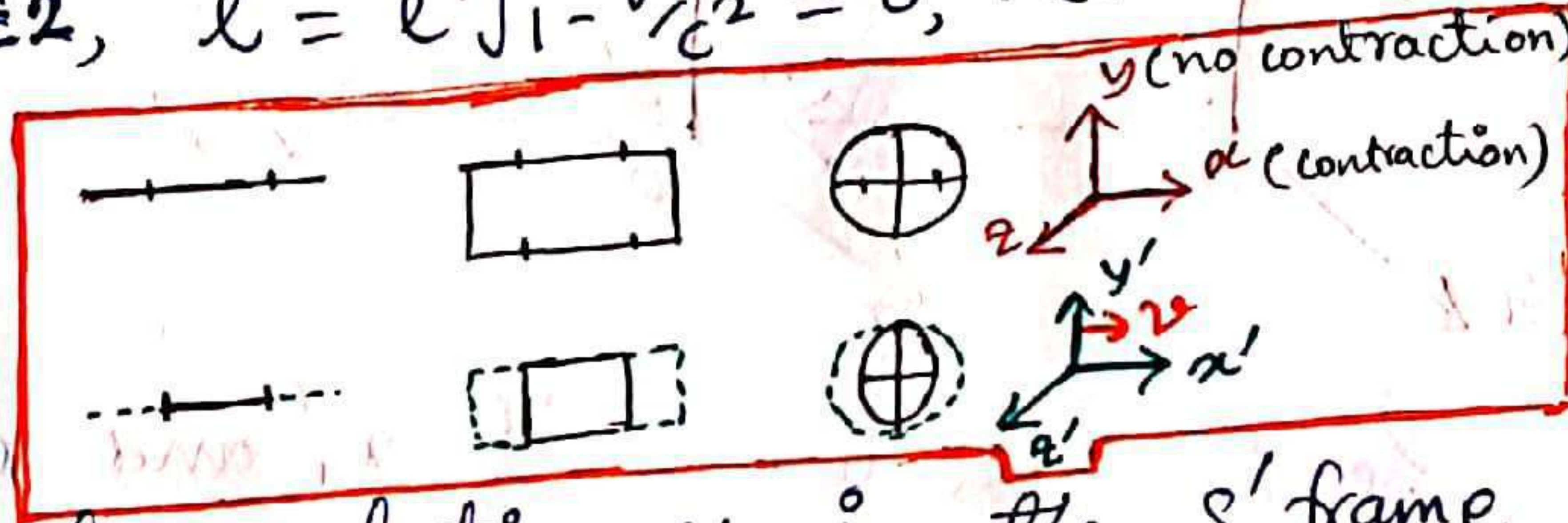
Now if observer in S measures the length of the rod with his scale at an instant of time t with his clock, then according to L.T.

$x_1' = \gamma(x - vt)$ and $x_2' = \gamma(x_2 - vt)$, so that

$l' = x_2' - x_1' = \gamma(x_2 - x_1) = l\gamma$. So, $\boxed{l = l'/\gamma}$ $\therefore \underline{l < l'}$

So the length of the rod is <u>longest</u> in the reference frame in which it is at <u>rest</u>, while the same length appears to be contracted by $\sqrt{1 - v^2/c^2}$ in the direction of relative motion in all other reference frame. However it is not contracted in the perpendicular direction y and z.

The longest length is called the "proper length" and replacing v to $-v$ doesnot affect the Transformation, so it's perfectly "reciprocal". In the original derivation of Lorentz-Fitzgerald, this reciprocity was absent which came into existence due to Einstein's 1^{st} postulate. If the contraction occured in one frame only, then we could determine its absolute velocity, violating the relativity principle. If $v=c$ we see that, for CASE1, $l' = l\sqrt{1-v^2/c^2} = 0$ i.e. the observer in S' frame will find no rod & for CASE2, $l = l'\sqrt{1-v^2/c^2} = 0$, i.e. the observer in S will not find a rod.



⑤ Time dilation & Proper time

Let us consider CASE 2, where a clock stationary in the S' frame measures time t' at a space coordinate (x', y', z'). If an observer at S frame measures the time t with his clock, then from R.L.T. we have $t = \gamma(t' + \frac{vx'}{c^2})$. ∴ A time interval measured by both clocks will be related as $\Delta t = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma\Delta t'$

As $\gamma > 1$, so $\Delta t > \Delta t'$. So the clock in the S-frame appears to run slow compared to the rest (comoving) clock in S' frame.

Considering CASE1, we will now obtain $t' = \gamma(t - \frac{vx}{c^2})$ & ∴ $\Delta t' = \gamma\Delta t$ or $\Delta t' > \Delta t$, otherwise the clock in S' frame appears to run slow relative to the clock in S-frame. So time dilation is also "reversible." "Proper time" is the time elapsed in a reference frame at rest with respect to the events. So in opposite to the argument for "proper length", propertime of a moving object is always less than the corresponding interval in rest frame. So moving clocks go more slowly than those at rest.

$$\boxed{\begin{array}{ll} \text{case I} : & l' = l_0/\gamma\,; \quad \Delta t' = \gamma\Delta t_0 \\ \text{case II} : & l = l_0'/\gamma\,; \quad \Delta t = \gamma\Delta t_0' \end{array}}$$

l_0 = proper length (longest)
t_0 = proper time (shortest)
l_0' = proper length (longest)
t_0' = proper time (shortest)

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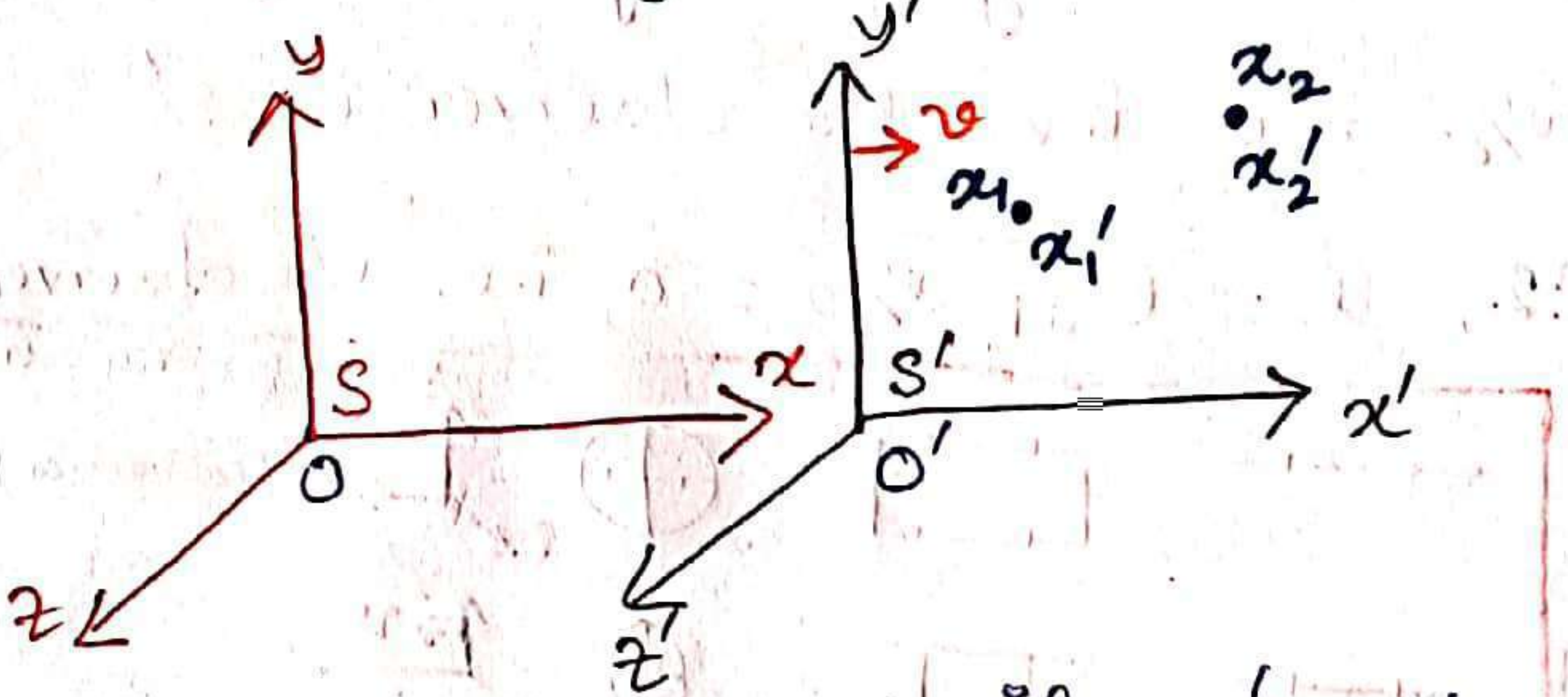
Another thing to note from $t = \gamma(t' + \frac{v}{c^2}x')$ is that a phase difference of $\frac{v}{c^2}x'$ exists between S' clock relative to S clock, so farther the S' clock position is from the origin O', the slower it appears to the observer in S frame.

⑥ Relativity of simultaneity ; sequence of Events



Let us consider two events that take place at x_1 & x_2 position coordinate in a reference frame S, which are simultaneous, meaning $\Delta t = t_2 - t_1 = 0$

So the question is if x_1' and x_2' are the position coordinate of the same two events, then whether they are simultaneous ($\Delta t' \overset{?}{=} 0$) or not. If t_2' & t_1' are the time recorded by the S' clock then using L.T. we have $\Delta t' = t_2' - t_1' = \gamma(\cancel{t_2} - \frac{v}{c^2}x_2) - \gamma(\cancel{t_1} - \frac{v}{c^2}x_1) = \frac{\gamma v}{c^2}(x_1 - x_2)$

$[\text{as } (t_2 - t_1) = 0]$

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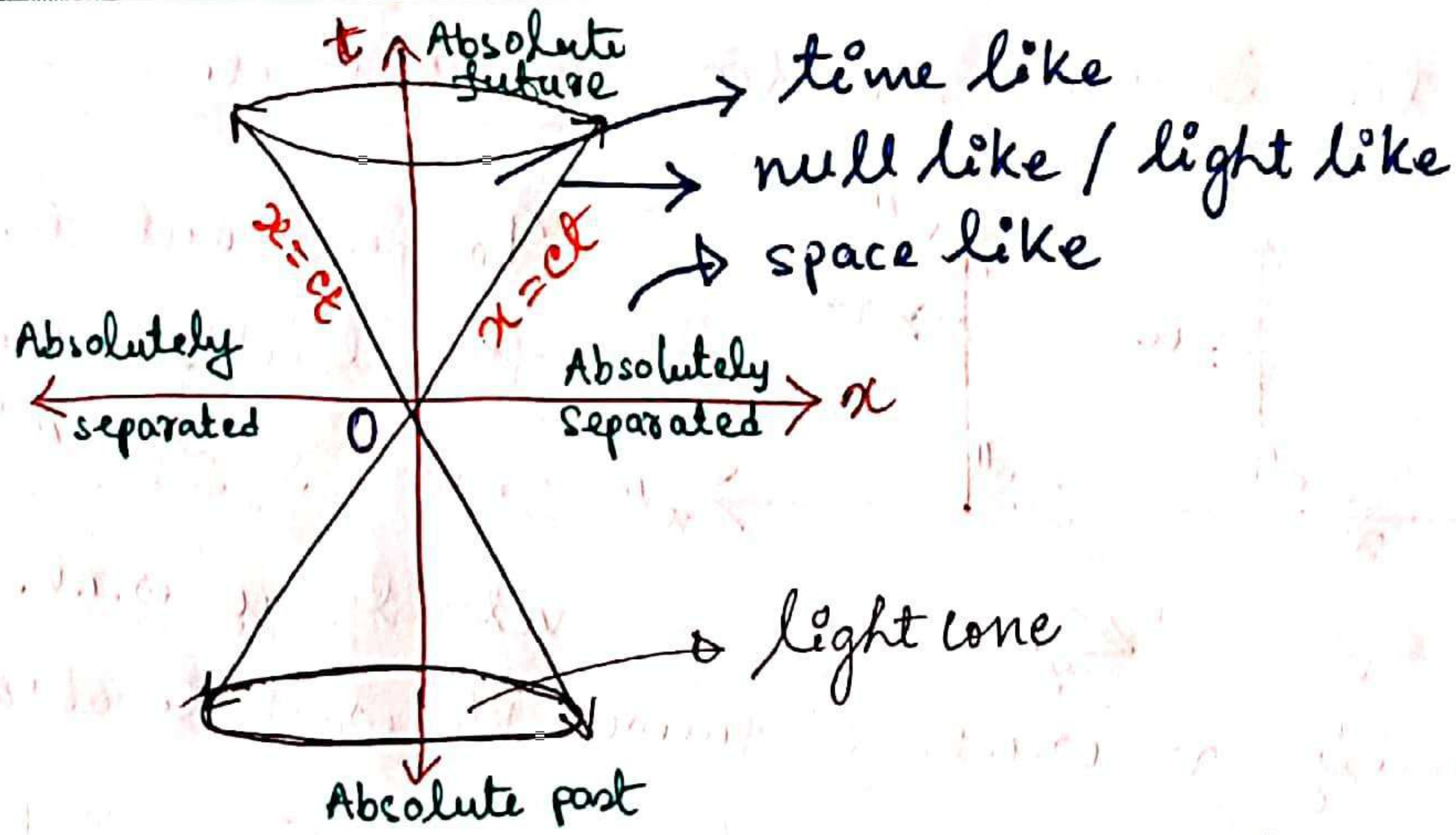
As, $x_1 \neq x_2$ in S-frame, we see that $\Delta t' \neq 0$, so the events are not simultaneous. If & only if $x_1 = x_2$, then two events must occur at same place can only make it simultaneous in a comoving S' frame.

Also, $\Delta t' = \gamma(\Delta t - \frac{v}{c^2}(x_2 - x_1))$ for non-simultaneous event ($\Delta t \neq 0$) and if so happens that in a sequence, $t_2' > t_1'$ (or $\Delta t' > 0$) then we have, $t_2 - t_1 > \frac{v}{c^2}(x_2 - x_1)$ or $\boxed{c(t_2 - t_1) > \frac{v}{c}(x_2 - x_1)}$

As $t_2 - t_1$ is the time required to connect two points x_2 & x_1 using a light signal, it means that distance which the points move between two events cannot be greater than $c\Delta t$. (timelike). Because of the invariance of intervals $c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 - \Delta x'^2$, we see that if $\Delta x' = 0$ (both occur at same place in s'), then $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 > 0$ (timelike). However if $\Delta t' = 0$ (both occur at same time in S'), then $\Delta s^2 = -\Delta x'^2 < 0$, so that $\Delta x' = x_2' - x_1' = i\Delta s$ (spacelike). If $c^2\Delta t'^2 = \Delta x'^2$ then $\Delta s^2 = 0$ (lightlike).



So due to length contraction & time dilation, $dx' = dx/\gamma$ and $dt' = \gamma dt$ where dx & dt are proper length & time measured in S frame. So 4-dimensional volume element in S' frame is

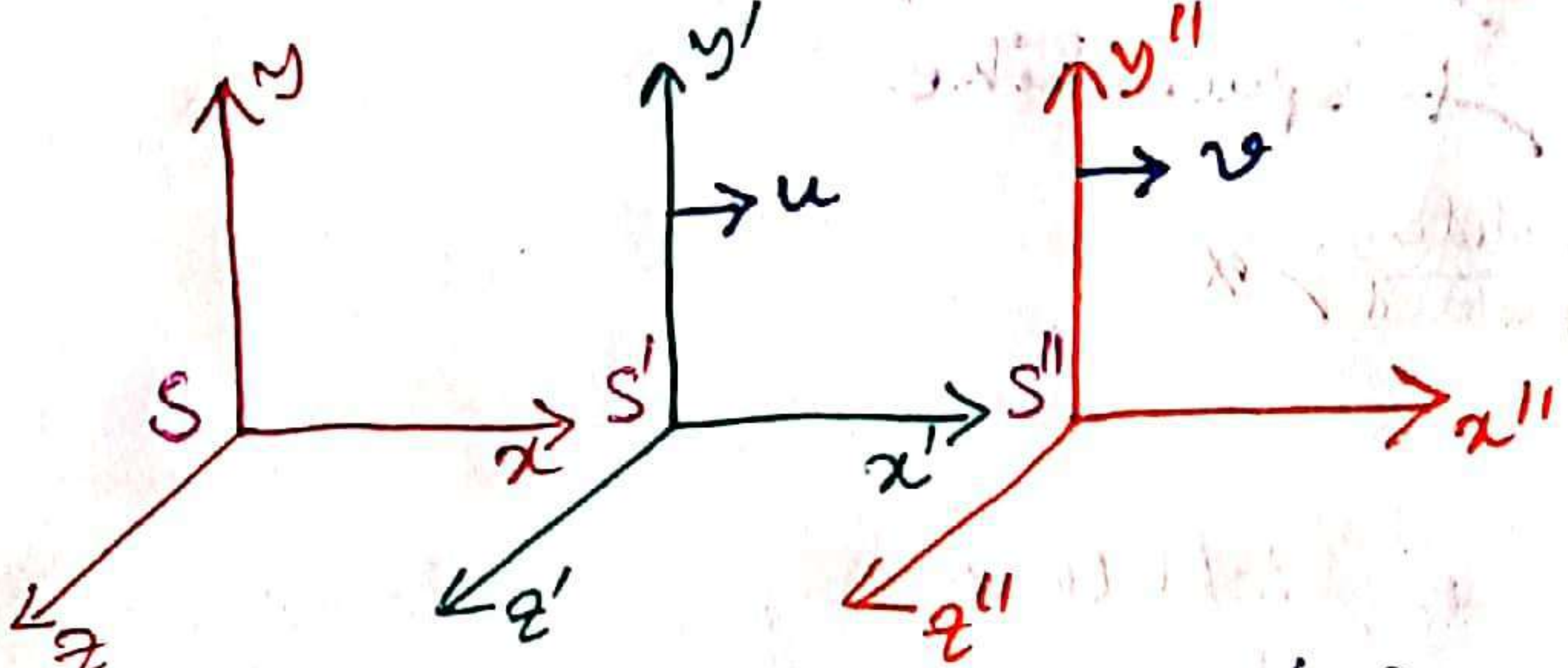
$$dV' = dx'dy'dz'dt' = \frac{dx}{\gamma} dy\, dz\, \gamma dt = dV \quad \text{(Lorentz scalar)}$$

Twin Paradox: The time dilation effect is reciprocal in nature, so S-frame observer feels that S' clock is running slow whereas S' frame observer feels that S clock is running slow. Does either of these two clocks run slow? Suppose A & B are two twins & B is travelling on a spaceship at $v = \frac{\sqrt{3}}{2}c$. For 15 years he travels & comes back to Earth (years measured by B in S' frame). So in accord to time dilation, A finds $T = \frac{T'}{\sqrt{1-v^2/c^2}} = \frac{30}{\sqrt{1-3/4}} = 60$ years is the age of B

But as time dilation is reciprocal, then B finds that Earth has moved away from him at $v = -\frac{\sqrt{3}}{2}c$ and so $T' = \frac{T}{\sqrt{1-v^2/c^2}} = 60$ years.

So although one can expect B to be 30 years old when A is 60, due to reversibility one can also see that B can be 60 years old while A is only 30. This is called "Twin Paradox". The paradox cannot be resolved in special relativity as it doesn't accomodate non-inertial frames. The start, turning around & stop of a rocket require acceleration / deceleration & can be taken into account in GR. So B is indeed younger than A after returning (as seems by A) but B's 30 years is not 30 years of A.

Relativistic law of Addition of Velocities



To find out the transformation for velocities, we consider inertial frames S, S' & S'', in which S' moves with velocity u w.r.t. S frame & S'' moves with velocity v w.r.t. S' frame. We want to obtain velocity w of S'' w.r.t. S. Using L.T. we have

$x' = \gamma(x - ut)$, $y' = y$, $z' = z$ and $t' = \gamma\left(t - \frac{ux}{c^2}\right)$ & $\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$

$x'' = \gamma'(x' - vt')$, $y'' = y'$, $z'' = z'$ and $t'' = \gamma'\left(t' - \frac{vx'}{c^2}\right)$; $\gamma' = \frac{1}{\sqrt{1 - v^2/c^2}}$

Replacing x', y', z', t', we have $y'' = y$, $z'' = z$.

$$x'' = \gamma'\left[\gamma(x - ut) - v\gamma\left(t - \frac{ux}{c^2}\right)\right] = \gamma\gamma'\left[x\left(1 + \frac{uv}{c^2}\right) - (u+v)t\right]$$

$$\text{or } x'' = \underbrace{\gamma\gamma'\left(1 + \frac{uv}{c^2}\right)}\left[x - \underbrace{\frac{u+v}{1 + \frac{uv}{c^2}}}_{\omega} t\right]$$

If $\omega = \frac{u+v}{1 + \frac{uv}{c^2}}$, then $1 - \frac{\omega^2}{c^2} = 1 - \frac{(u+v)^2}{c^2\left(1 + \frac{uv}{c^2}\right)^2} = \frac{c^2 + \frac{u^2v^2}{c^2} + \cancel{2uv} - u^2 - v^2 - \cancel{2uv}}{c^2\left(1 + \frac{uv}{c^2}\right)^2}$

$$\text{or, } 1 - \frac{\omega^2}{c^2} = \frac{1 - \frac{u^2}{c^2} - \frac{v^2}{c^2}\left(1 - \frac{u^2}{c^2}\right)}{\left(1 + \frac{uv}{c^2}\right)^2} = \frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{uv}{c^2}\right)^2} = \frac{1}{\gamma^2\gamma'^2\left(1 + \frac{uv}{c^2}\right)^2}$$

$\therefore \gamma'' = \gamma\gamma'\left(1 + \frac{uv}{c^2}\right) = \frac{1}{\sqrt{1 - \omega^2/c^2}}$, so that $\boxed{x'' = \gamma''(x - \omega t)}$

$$\text{Similarly } t'' = \gamma'\left[\gamma\left(t - \frac{ux}{c^2}\right) - \frac{v}{c^2}\gamma(x - ut)\right]$$

$$\text{or } t'' = \gamma\gamma'\left[\left(1 + \frac{uv}{c^2}\right)t - \frac{1}{c^2}(u+v)x\right] = \gamma\gamma'\left(1 + \frac{uv}{c^2}\right)\left[t - \frac{u+v}{c^2\left(1 + \frac{uv}{c^2}\right)}x\right]$$

$$\text{or } \boxed{t'' = \gamma''\left(t - \frac{\omega}{c^2}x\right)}$$

So two LT (consecutive) are equivalent to one LT (true only if u & v are parallel) and commutative (S' velocity v & S'' velocity u). This is the Einstein's law of addition of velocities. To note,

$\omega = \frac{u+v}{1 + \frac{uv}{c^2}} = c\left[1 - \frac{(1 - u/c)(1 - v/c)}{1 + uv/c^2}\right]$*. As $u, v < c$, we have $\omega < c$

So maximum velocity cannot be greater than c & can be c if $u = v = c$.

* (Show)

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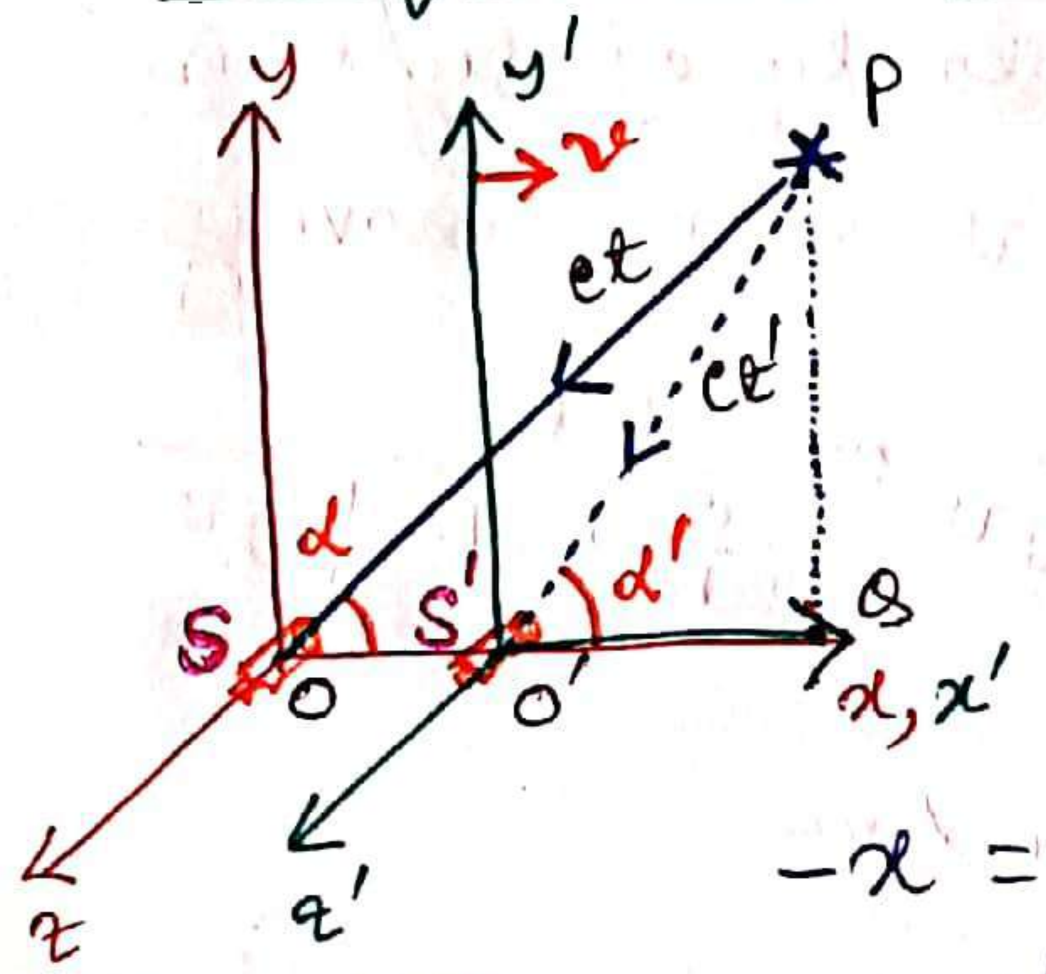
Consequence I: Aberration of Light (Bradley)



We again address Bradley's experiment on light aberation as seen by two inertial frames S and S' moving with the velocity v (w.r.t. S) of a star at P. From diagram, we have

$$-x = -ct\cos\alpha,\ -y = -ct\sin\alpha,\ -x' = -ct'\cos\alpha',\ -y' = -ct'\sin\alpha'$$

Omitting the negative signs & using R.L.T. we have

$$x = ct\cos\alpha \quad \text{or, } \cancel{\gamma}(x' + vt') = c\cancel{\gamma}\left(t' + \frac{v}{c^2}x'\right)\cos\alpha.$$

$$\text{or, } x'\left(1 - \frac{v}{c}\cos\alpha\right) = c\left(\cos\alpha - \frac{v}{c}\right)t' \quad \text{or, } x' = \frac{c\left(\cos\alpha - v/c\right)}{1 - v/c\cos\alpha}t' \quad \text{— (1)}$$

$$= ct'\cos\alpha'.$$

$$\therefore \cos\alpha' = \frac{\cos\alpha - \frac{v}{c}}{1 - \frac{v}{c}\cos\alpha}. \quad \text{— (A)}$$

Similarly $y = ct\sin\alpha$ or, $y' = c\gamma\left(t' + \frac{v}{c^2}x'\right)\sin\alpha.$ = (using (1))

$$c\gamma\left[t' + \frac{v}{c^{\cancel{2}}}\frac{\cancel{c}(\cos\alpha - v/c)}{1 - v/c\cos\alpha}t'\right]\sin\alpha = ct'\sin\alpha\left[\frac{1 - \cancel{v/c\cos\alpha} + \cancel{\frac{v}{c}\cos\alpha} - v^2/c^2}{1 - v/c\cos\alpha}\gamma\right]$$

$$= ct'\sin\alpha\left[\frac{\gamma}{\gamma^2(1 - v/c\cos\alpha)}\right] = ct'\frac{\sin\alpha}{\gamma(1 - v/c\cos\alpha)} = ct'\sin\alpha'$$

$$\therefore \sin\alpha' = \frac{\sin\alpha}{\gamma(1 - v/c\cos\alpha)} \quad \text{— (B)}$$

Dividing (A) and (B) we have $\boxed{\cot\alpha' = \frac{\gamma\left[\cos\alpha - \frac{v}{c}\right]}{\sin\alpha} = \gamma\left(\cot\alpha - \frac{v}{c}\operatorname{cosec}\alpha\right)}$

$$= \left(\cot\alpha - \frac{v}{c}\operatorname{cosec}\alpha\right)\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \simeq \left(\cot\alpha - \frac{v}{c}\operatorname{cosec}\alpha\right)\left(1 + \frac{v^2}{2c^2}\right) \quad (\because v < c) \text{ non relativistic}$$

$$\simeq \cot\alpha - \frac{v}{c}\operatorname{cosec}\alpha + \frac{v^2}{2c^2}\cot\alpha + \cancel{O\left(\frac{v^3}{c^3}\right)} \to 0 \quad \left(\text{only upto } O(v^2/c^2)\right)$$

In S frame, star P was at zenith (top), so that $\alpha = \frac{\pi}{2}$ and angle of aberration $\theta = -\alpha + \alpha' = -\frac{\pi}{2} + \alpha'$, so that

$$\cot\alpha' = \cot\left(\pi/2 + \theta\right) = \frac{\cot\pi/2\cot\theta + 1}{\cot\theta + \cot\pi/2} = -\frac{1}{\cot\theta} = -\tan\theta$$

$\left(\cot\frac{\pi}{2} = \frac{\cos\pi/2}{\sin\pi/2} = 0,\ \operatorname{cosec}\frac{\pi}{2} = \frac{1}{\sin\pi/2} = 1\right)$

But $\cot\alpha' = \cot\alpha - \frac{v}{c}\operatorname{cosec}\alpha + \frac{v^2}{2c^2}\cot\alpha = \cot\frac{\pi}{2} - \frac{v}{c}\operatorname{cosec}\frac{\pi}{2} + \frac{v^2}{2c^2}\cot\pi/2 = -\frac{v}{c}.$

Equating, $\tan\theta = \frac{v}{c}$ or $\boxed{\theta = \tan^{-1}\frac{v}{c}}$. So we get Newtonian mechanics correctly as observed in Bradley's experiment.

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Consequence 2 : Fizeau's Experiment

If n was the refractive index of water then velocity of light in water is c/n and due to cross-stream velocity v of the moving water in the tube, periods of travel was $\dfrac{2l}{c/n \pm fv} \simeq \dfrac{2l}{\frac{c}{n} \pm (1-\frac{1}{n^2})v}$ according to fresnel.

From Einstein's velocity addition theorem, we have

$$w = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{c/n + v}{1 + \frac{c}{n}\frac{v}{c^2}} = \left(\frac{c}{n}+v\right)\left(1+\frac{v}{nc}\right)^{-1} \simeq \left(\frac{c}{n}+v\right)\left(1-\frac{v}{nc}\right) \quad (\because v<<c)$$

$$= \frac{c}{n} + v - \frac{v}{n^2} - \cancel{\frac{v^2}{nc}}^{\,0} \quad O(v^2) \text{ neglected} \quad = \frac{c}{n} + v\left(1-\frac{1}{n^2}\right) = \frac{c}{n} + fv$$

This restores back fresnel's result of drag coefficient $\boxed{f = 1-\frac{1}{n^2}}$

To note here that if $n<1$ (e.g. radio waves in an ionized medium) $w>c$ or phase velocity of a monochromatic wave can increase above light's speed, but not the group velocity of the wave packet, because to convey information, superposition of many harmonic wave is required whose different frequency response propagate with c or less than c.

$$v_{group} = v_{phase}\left(1 - \frac{w}{v_{phase}}\frac{dv_{phase}}{dw}\right)^{-1}$$

$w =$ angular velocity

$v_{group} = \dfrac{dw}{dk}$,

$v_{phase} = \dfrac{w}{k}$.

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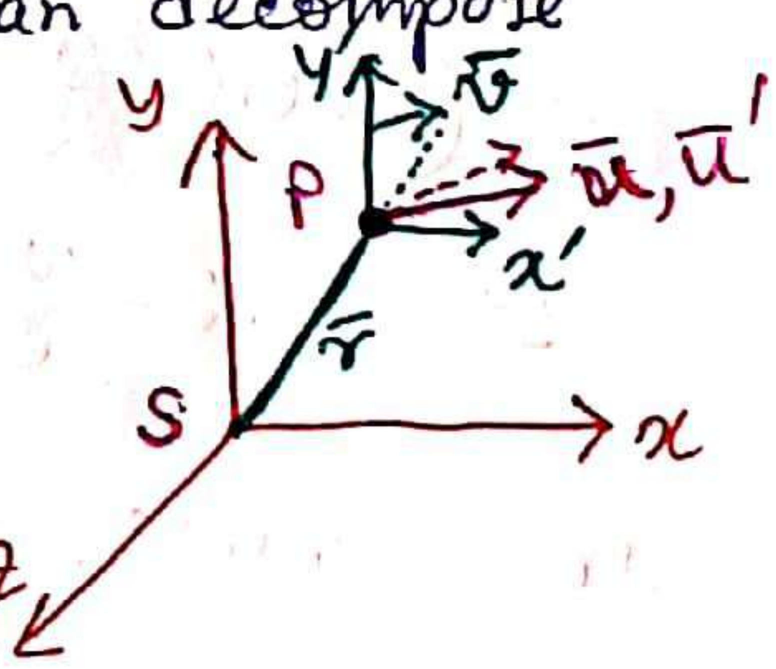
Generalized 3-dimensional L.T.

When v is not parallel to x-coordinate then we can decompose



$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$ where $\parallel$ means r's component parallel to $\vec{v}$ and $\perp$ means perpendicular to $\vec{v}$.

The choice is such that $\vec{r}_{\parallel}$ will transform as x-coordinate in LT & $\vec{r}_{\perp}$ will remain invariant like y-z coordinate.

So $\vec{r}_{\parallel} = (\vec{r}\cdot\hat{v})\hat{v}$ and $\vec{r}_{\parallel}' = \gamma(\vec{r}_{\parallel} - \vec{v}t)$, $\vec{r}_{\perp}' = \vec{r}_{\perp}$

$\vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel} = \vec{r} - (\vec{r}\cdot\hat{v})\hat{v}$

So in S' frame, $\vec{r}' = \vec{r}_{\parallel}' + \vec{r}_{\perp}' = \gamma(\vec{r}_{\parallel} - \vec{v}t) + \vec{r}_{\perp}$

$= \gamma[(\vec{r}\cdot\hat{v})\hat{v} - \vec{v}t] + \vec{r} - (\vec{r}\cdot\hat{v})\hat{v}$

$$= \vec{r} - \gamma\vec{v}t + (\gamma-1)(\vec{r}\cdot\hat{v})\hat{v} = \vec{r} - \vec{v}t + \vec{v}t - \gamma\vec{v}t + (\gamma-1)(\vec{r}\cdot\hat{v})\hat{v}$$

$$= \vec{r} - \vec{v}t + (\gamma-1)\left[(\vec{r}\cdot\hat{v})\hat{v} - \vec{v}t\right] = \vec{r} - \vec{v}t + (\gamma-1)\frac{\vec{v}}{v^2}(\vec{r}\cdot\vec{v} - v^2 t)$$

$$\therefore \boxed{\begin{aligned} \vec{r}\,' &= \vec{r} - \vec{v}t + (\gamma-1)\frac{\vec{v}}{v^2}(\vec{r}\cdot\vec{v} - v^2 t) \\ t' &= \gamma\left(t - \frac{\vec{r}\cdot\vec{v}}{c^2}\right) \end{aligned}} \quad \text{L.T.} \quad \text{— (1)}$$

$$\boxed{\begin{aligned} \vec{r} &= \vec{r}\,' + \vec{v}t' + (\gamma-1)\frac{\vec{v}}{v^2}(\vec{r}\,'\cdot\vec{v} + v^2 t') \\ t &= \gamma\left(t' + \frac{\vec{r}\,'\cdot\vec{v}}{c^2}\right) \end{aligned}} \quad \text{R.L.T.} \quad \text{— (2)}$$

In component form,

$$x' = x\left[1 + (\gamma-1)\frac{v_x^2}{v^2}\right] + y(\gamma-1)\frac{v_x v_y}{v^2} + z(\gamma-1)\frac{v_x v_z}{v^2} - \gamma v_x t$$

$$y' = x(\gamma-1)\frac{v_x v_y}{v^2} + y\left[1 + (\gamma-1)\frac{v_y^2}{v^2}\right] + z(\gamma-1)\frac{v_y v_z}{v^2} - \gamma v_y t$$

$$z' = x(\gamma-1)\frac{v_x v_z}{v^2} + y(\gamma-1)\frac{v_y v_z}{v^2} + z\left[1 + (\gamma-1)\frac{v_z^2}{v^2}\right] - \gamma v_z t$$

$$t' = \gamma\left[t - (x v_x + y v_y + z v_z)/c^2\right]$$

check!

or

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} 1 + (\gamma-1)\frac{v_x^2}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_z}{v^2} & -\gamma v_x \\ (\gamma-1)\frac{v_x v_y}{v^2} & 1 + (\gamma-1)\frac{v_y^2}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} & -\gamma v_y \\ (\gamma-1)\frac{v_x v_z}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} & 1 + (\gamma-1)\frac{v_z^2}{v^2} & -\gamma v_z \\ -\frac{\gamma v_x}{c^2} & -\frac{\gamma v_y}{c^2} & -\frac{\gamma v_z}{c^2} & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

Λ matrix

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L.T. is also called Lorentz "boost" if pure rotation is extracted out.

If we take L.T. along x-axis only then $v_x = v$, $v_y = v_z = 0$, $\beta_x = \frac{v_x}{c}$ and $\gamma_x = \frac{1}{\sqrt{1-\beta_x^2}}$. and for y-axis only case $v_y = v$, $v_x = v_z = 0$, $\beta_y = \frac{v_y}{c}$ and $\gamma_y = \frac{1}{\sqrt{1-\beta_y^2}}$.

$$\therefore L_x = \begin{pmatrix} \gamma_x & 0 & 0 & -\gamma_x v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_x\beta_x/c & 0 & 0 & \gamma_x \end{pmatrix}, \quad L_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_y & 0 & -\gamma_y v \\ 0 & 0 & 1 & 0 \\ 0 & -\gamma_y\beta_y/c & 0 & \gamma_y \end{pmatrix}.$$

check!

$\boxed{L_x L_y \neq L_y L_x}$. So for different direction L.T. do not commute. but for equal direction, they commute. Notice that to obtain (1) or (2), we haven't rotated x' axis w.r.t. x but only relative velocity in arbitrary direction, so to obtain 3D L.T. has to incorporate rotation.

From ① and ②, we have $d\bar{r}' = d\bar{r} - \bar{v}dt + (\gamma-1)\frac{\bar{v}}{v^2}(d\bar{r}\cdot\bar{v} - v^2 dt)$

$$dt' = \gamma\left(dt - \frac{d\bar{r}\cdot\bar{v}}{c^2}\right).$$

So, $$\vec{u}' = \frac{d\bar{r}'}{dt'} = \frac{d\bar{r}'/dt}{dt'/dt} = \frac{\bar{u} - \bar{v} + (\gamma-1)\frac{\bar{v}}{v^2}(\bar{u}\cdot\bar{v} - v^2)}{\gamma\left(1 - \frac{\bar{u}\cdot\bar{v}}{c^2}\right)} \quad \text{— ④}$$

Similarly $$\bar{u} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}/dt'}{dt/dt'} = \frac{\bar{u}' + \bar{v} + (\gamma-1)\frac{\bar{v}}{v^2}(\bar{u}\cdot\bar{v} - v^2)}{\gamma\left(1 - \frac{\bar{u}\cdot\bar{v}}{c^2}\right)} \quad \text{— ③}$$

Using vector identity, (Show) $$\bar{u}'^2 = \frac{(\bar{u}-\bar{v})^2 - \frac{1}{c^2}(\bar{u}\times\bar{v})^2}{\left(1 - \frac{\bar{u}\cdot\bar{v}}{c^2}\right)^2}$$ and

$$1 - \frac{\bar{u}'^2}{c^2} = \frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{\bar{u}\cdot\bar{v}}{c^2}\right)^2}. \quad \text{(Show) — ⑥}$$

As $u, v < c$, $1 - \frac{\bar{u}'^2}{c^2} > 0$ ∴ $\bar{u}' < c$. So Einstein's 2nd postulate is also established for velocity transformation between S & S'.

Case 1 If velocity of the particle $\vec{u}$ is parallel to relative velocity of S' w.r.t. S $\vec{v}$, means $\vec{u}\times\vec{v} = 0$ then from ③,

$$\boxed{\bar{u}' = \frac{\bar{u} - \bar{v}}{1 - \frac{\bar{u}\cdot\bar{v}}{c^2}}} \quad \text{— ⑤}$$

Case 2 If velocity of the particle $\vec{u}$ is perpendicular to relative velocity $\bar{v}$, means $\bar{u}\cdot\bar{v} = 0$ then from ④,

$$\bar{u}' = \frac{\bar{u} - \cancel{\bar{v}} + (\cancel{\gamma-1})(-v)}{\gamma} = \frac{\bar{u}}{\gamma} - \bar{v}$$

Differentiating ④, $$\bar{u}'' = \frac{d^2\bar{r}'}{dt'^2} = \ddot{\bar{r}}'$$

(Show) $$= \frac{\ddot{\bar{r}}}{\gamma^2\left(1 - \frac{\dot{\bar{r}}\cdot\bar{v}}{c^2}\right)^2} + \frac{(\ddot{\bar{r}}\cdot v)\left[\gamma\frac{\dot{\bar{r}}}{c^2} - (\gamma-1)\frac{\bar{v}}{v^2}\right]}{\gamma^3\left(1 - \frac{\dot{\bar{r}}\cdot\bar{v}}{c^2}\right)^3}$$

∴ Even if $\dot{\bar{r}}\cdot\bar{v} = \ddot{\bar{r}}\cdot\bar{v} = 0$,

$\ddot{\bar{r}}' \neq \ddot{\bar{r}}$ and so Newton's law is not in general valid with LT.

Only if $v \ll c$ with the above condition, then only $\ddot{\bar{r}}' = \ddot{\bar{r}}$

Case 3 As a special case 1, other than $\vec{u} \times \vec{v} = 0$ and $\vec{v}$ is parallel to x' axis only, so that $\bar{u} \cdot \bar{v} = u_x v$. Here $\bar{u} = (u_x = \frac{dx}{dt}, u_y = \frac{dy}{dt}, u_z = \frac{dz}{dt})$ and $\bar{u}' = (u_{x'} = \frac{dx'}{dt'}, u_y' = \frac{dy'}{dt'}, u_z' = \frac{dz'}{dt'})$ are the components of velocity in S & S' frame, v = relative velocity of S' frame w.r.t. S frame.

from L.T. $dx' = \gamma(dx - v\,dt)$, $dy' = dy$, $dz' = dz$, $dt' = \gamma(dt - \frac{v}{c^2}dx)$

$$\therefore u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - v/c^2\, dx} = \frac{dx/dt - v}{1 - v/c^2\, dx/dt} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - v/c^2\, dx)} = \frac{u_y}{\gamma(1 - u_x v/c^2)}$$

$$u_z' = \frac{dz'}{dt'} = \frac{dz}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{u_z}{\gamma(1 - u_x v/c^2)}$$

Realize that these are nothing but 5 written in component form.

$$u'^2 = u_x'^2 + u_y'^2 + u_z'^2 = \frac{(u_x - v)^2 + \frac{1}{\gamma^2}(u_y^2 + u_z^2)}{(1 - \frac{u_x v}{c^2})^2} = \frac{(u_x - v)^2 + \frac{1}{\gamma^2}(u^2 - u_x^2)}{(1 - \frac{u_x v}{c^2})^2}$$

Realize that this is nothing but 3 written in component form.

The reverse transformations can be easily obtained as

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}, \quad u_y = \frac{u_y'}{\gamma(1 + \frac{u_x' v}{c^2})}, \quad u_z = \frac{u_z'}{\gamma(1 + \frac{u_x' v}{c^2})} \quad \text{and}$$

$$u^2 = \frac{(u_x' + v)^2 + (u'^2 - u_x'^2)/\gamma^2}{(1 - \frac{u_x' v}{c^2})^2}.$$ Einstein's velocity addition theorem is a special case where $u_x = u$, $u_y = u_z = 0$ then $u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$.

Similar to 6, starting from u'^2 expression it is easy to show that

$$1 - \frac{u'^2}{c^2} = \frac{(1 - \frac{u^2}{c^2})(1 - \frac{v^2}{c^2})}{(1 - \frac{u_x v}{c^2})^2}, \quad 1 - \frac{u^2}{c^2} = \frac{(1 - \frac{u'^2}{c^2})(1 - \frac{v^2}{c^2})}{(1 - \frac{u_x' u}{c^2})^2}$$ to claim

Einstein's 2nd postulate. Also, $\ddot{x}' = \frac{\ddot{x}}{\gamma^3 (1 - \frac{u_x v}{c^2})^3}$

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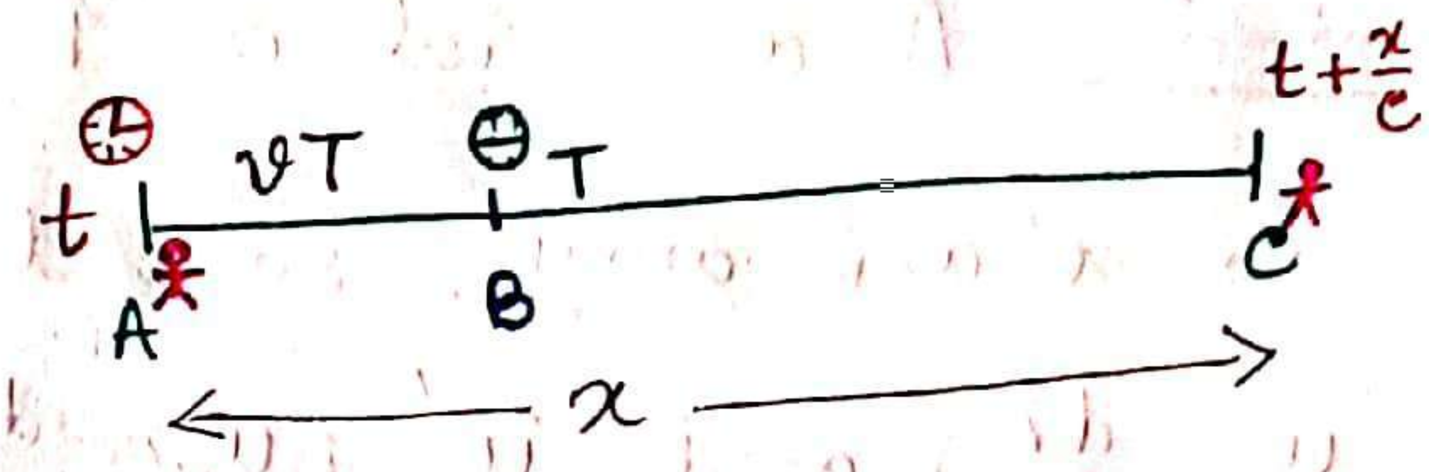
Relativistic Doppler Effect

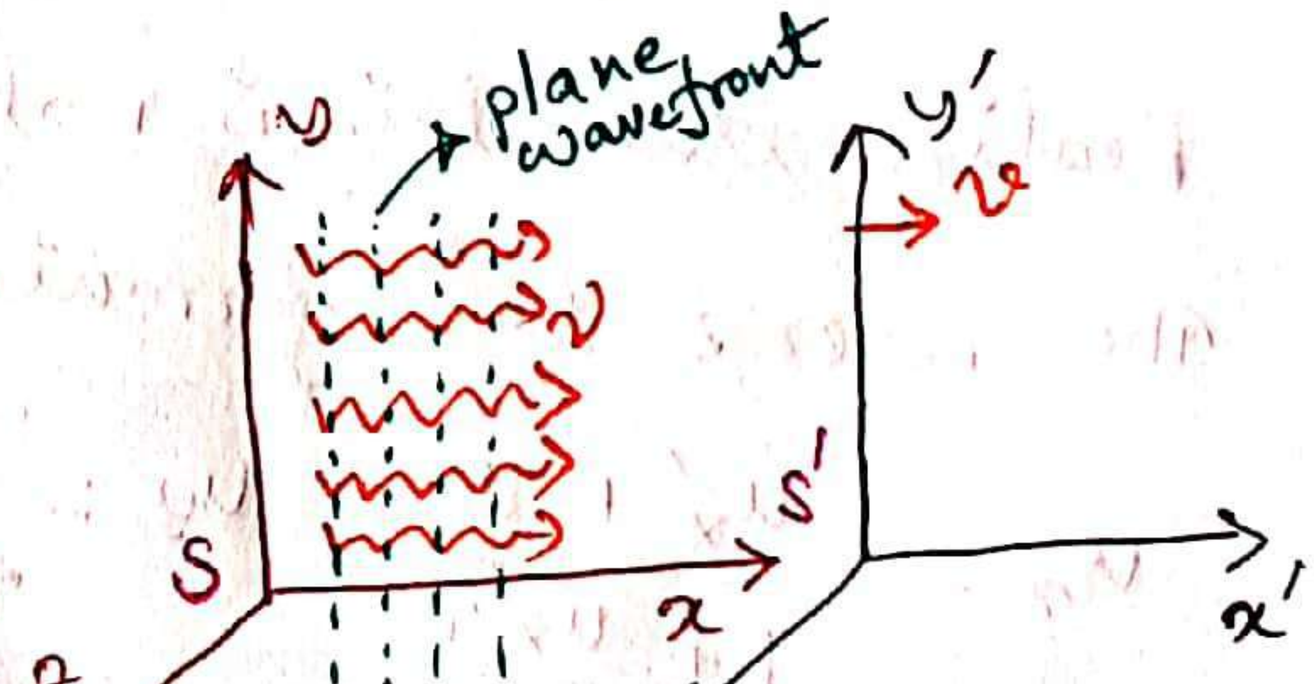


Let us consider that an emitter at A sends a light signal at his clock time t and the light reaches to receiver C at a distance x at his clock time $t+\frac{x}{c}$. If now A moves with uniform velocity v for T time and sends another signal from B after $t+T$ time, then receiver C finds this signal after $t+T+\frac{x-vt}{c}$ time. The time differance measured by C for these two light pulses are

$$\left(\cancel{t}+T+\frac{\cancel{x}-vT}{c}\right)-\left(\cancel{t}+\frac{\cancel{x}}{c}\right)=T\left(1-\frac{v}{c}\right).$$

So the moving emitter's clock time T will be more than that of observer at C as it's reduced by $1-\frac{v}{c}$, or the frequency increases by $\left(1-\frac{v}{c}\right)^{-1}$. So Doppler effect tells that when an emitter approaches an observer, the frequency appears to increase while for a receding emitter the frequency appears to decrease.

Longitudinal Doppler Effect:



Suppose a monochromatic plane-wave of light with frequency ν is travelling in the +ive x-direction, in the S frame represented by $\psi=\psi_0\cos(\omega t-kx)=\psi_0\cos\frac{2\pi}{\lambda}(ct-x)$. In the moving frame S', this wave is represented by $\psi'=\psi_0'\cos(\omega't'-k'x')$. The S' frame wave will still be a plane wave because L.T. is linear and phase $\phi=\omega t-kx$ is a Lorentz scalar, because the number of wavefronts as counted by two observers at S and S' must be equal.

$$\therefore\ \omega t-kx=\omega't'-k'x'$$

Using R.L.T. we have $\omega\gamma\left(t'+\frac{v}{c^2}x'\right)-k\gamma(x'+vt')=\omega't'-k'x'$

or $\gamma\left(\frac{\omega v}{c^2}-k\right)x'-\gamma(kv-\omega)t'=\omega't'-k'x'$

Equating the coefficients of x' and t' on both sides, we have

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$\omega' = \gamma(\omega - kv)$ and $k' = \gamma(k - \frac{\omega v}{c^2})$

or $2\pi\nu' = \gamma(2\pi\nu - \frac{2\pi}{\lambda}v)$ and $\frac{2\pi}{\lambda'} = \gamma(\frac{2\pi}{\lambda} - \frac{2\pi\nu v}{c^2})$

or $\nu' = \gamma(\nu - \frac{v}{\lambda})$ and $\frac{1}{\lambda'} = \gamma(\frac{1}{\lambda} - \frac{\nu v}{c^2})$ $[\because \nu\lambda = \nu'\lambda' = c$, $\frac{1}{\lambda} = \frac{\nu}{c}]$

or $\nu' = \gamma(\nu - \frac{v\nu}{c})$ and $\frac{1}{\lambda'} = \gamma(\frac{1}{\lambda} - \frac{1}{\lambda}\frac{v}{c})$

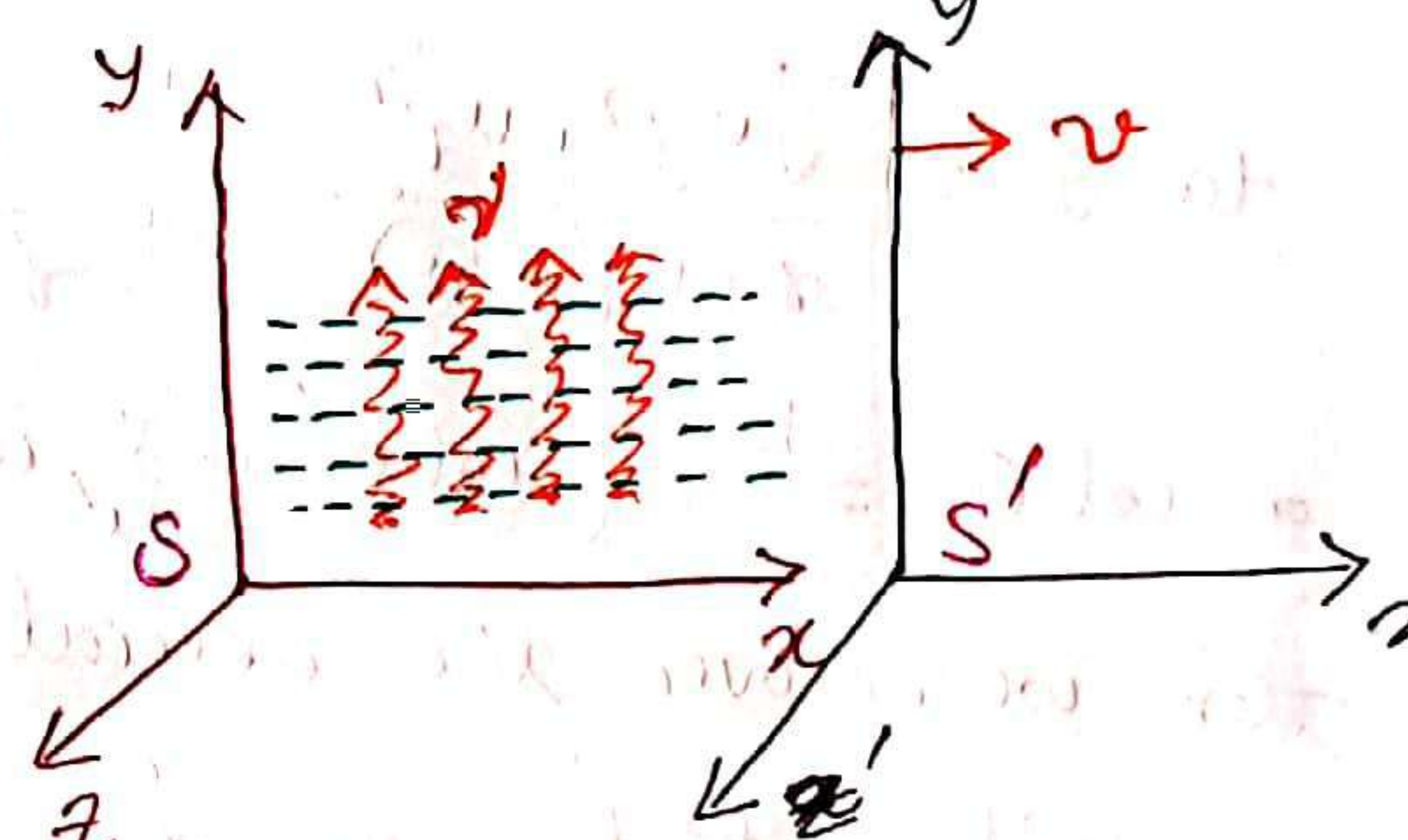
or $\nu' = \frac{1}{\sqrt{1-v^2/c^2}}\nu(1 - \frac{v}{c})$ and $\frac{1}{\lambda'} = \frac{1}{\lambda}\frac{1}{\sqrt{1-v^2/c^2}}(1-\frac{v}{c})$.

or $\nu' = \nu\sqrt{\frac{1-v/c}{1+v/c}}$ and $\frac{1}{\lambda'} = \frac{1}{\lambda}\sqrt{\frac{1-v/c}{1+v/c}}$

or $\boxed{\nu' = \nu\sqrt{\frac{c-v}{c+v}} \text{ and } \lambda' = \lambda\sqrt{\frac{c+v}{c-v}}}$ → Red shift || Blue shift

So $\nu' < \nu$ and $\lambda' > \lambda$. So as S' frame moves away from emitter at S, frequency of light appears to decrease or wavelength to increase. This is known as "**red shift**" as red is the longest λ in visible spectrum. When S' frame is approaching S frame then $v \to -v$ and so $\nu' > \nu$ and $\lambda' < \lambda$. This is known as "**blue shift**". In astronomy red shift implies the stars are moving away from us.

Transverse Doppler Effect:



Let us now consider that the plane wave is propagating along y-direction such that $\psi = \psi_0 \cos(\omega t - ky)$ in S frame & $\psi' = \psi_0\cos(\omega' t' - k' y')$ as observed in S' frame moving with the velocity $\vec{v}$ along +ive x axis.

As the phase is invariant under L.T., we have

$\omega t - ky = \omega' t' - k' y'$. Using R.L.T. we have

or $\omega\gamma(t' + \frac{v}{c^2}x') - ky' = \omega' t' - k' y'$

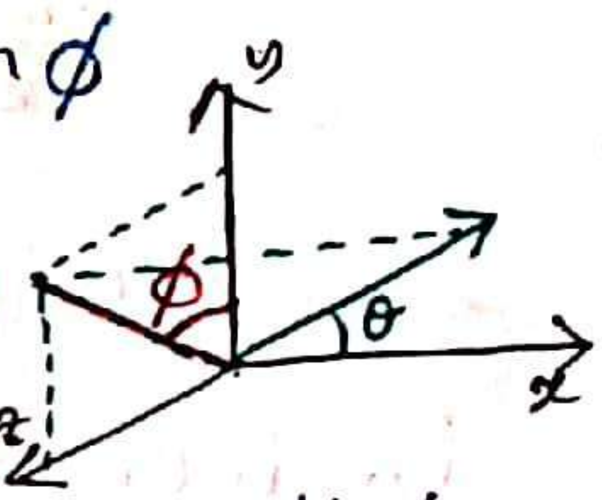
Comparing coefficients of y', t' we have $k = k'$, $\omega' = \omega\gamma$

or $\boxed{\nu' = \gamma\nu}$ So $\nu' < \nu$ and $\lambda' > \lambda \Rightarrow$ Red shift for receding frame & Blue shift for the other. In nonrelativistic case (first power of v/c) there is no transverse Doppler effect, and in relativistic case this arises due to Einstein time dilation. In a general 2D motion of planewave $\psi = \psi_0 \cos 2\pi\left[\frac{x\cos\theta + y\sin\theta}{\lambda} - \nu t\right]$ by the same calculation, $\nu' = \nu\gamma\left(1-\frac{v}{c}\cos\theta\right) = \nu\gamma\left(1-\frac{\vec{v}\cdot\hat{n}}{c}\right)$

$\nu'\cos\theta' = \nu\gamma\left(\cos\theta - \frac{v}{c}\right)$, $\nu'\sin\theta' = \nu\sin\theta$, so that $\cot\theta' = \gamma\left(\cot\theta - \frac{v}{c}\operatorname{cosec}\theta\right)$ which is the same result obtained in explaining Bradley's Aberration of light.

Headlight Effect ; Terrell Effect

By changing from cartesian to spherical polar coordinate, we have $u_x = u\cos\theta$, $u_y = u\sin\theta\cos\phi$, $u_z = u\sin\theta\sin\phi$



$\therefore \tan\theta = \frac{\sqrt{u_y^2 + u_z^2}}{u_x}$, $\tan\phi = \frac{u_z}{u_y}$.

Using velocity transformation relation $u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}$, $u_y = \frac{u_y'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}$

$u_z = \frac{u_z'}{\gamma\left(1+\frac{u_x' v}{c^2}\right)}$ we have

$$\tan\theta = \frac{\sqrt{u_y'^2 + u_z'^2}}{\gamma(u_x' + v)} = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}$$

$[u_x' = u'\cos\theta'$
$u_y' = u\sin\theta'\cos\phi'$
$u_z' = u\sin\theta'\sin\phi']$

or $\cot\theta = \gamma\left(\cot\theta' + \frac{v}{u'}\operatorname{cosec}\theta'\right)$. If $u = u' = c$ then we recover the aberration result $\cot\theta = \gamma\left(\cot\theta' + \frac{v}{c}\operatorname{cosec}\theta'\right)$.

Not only that, $\tan\phi = \frac{u_z}{u_y} = \frac{u_z'}{u_y'} = \tan\phi'$, so $\phi = \phi'$.

So if relative velocity between S and S′ frames is the polar axis then a particle does not suffer azimuthal rotation. The light beam become more & more convergent in the forward direction as $v \to c$. This is called collimation / headlight effect. So a star approaching Earth with relativistic speed will appear to be brighter & blue (Doppler Blue shift) & receding star will be faint & red (Doppler red shift). Terell-Penrose coined the fact that shape of a moving objett remains unchanged to observer's eye.

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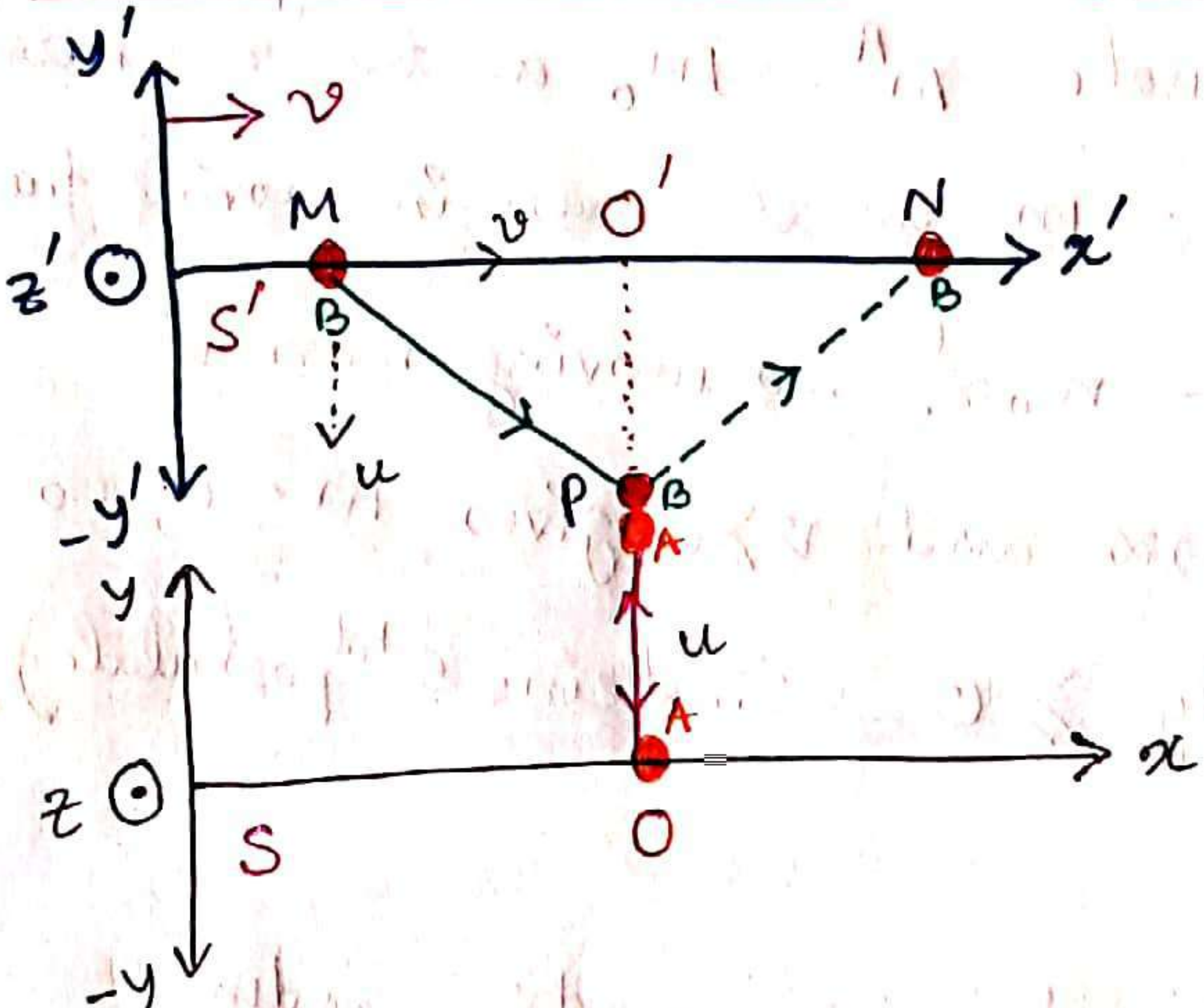
Relativistic variation of mass with velocity



Consider two observers O and O' in inertial frames S and S' at rest w.r.t. each other with their x-coordinate lying at a distance apart. Both observers have an identical hard sphere (A & B) meaning same mass & size at rest. S' now moves to the right along x-axis with a uniform velocity v. Observer O at S-frame throws the hard sphere A at +y direction with speed u as measured in his frame. Simultaneously, O' throws his sphere B from M along $-y'$ direction with velocity u. After coming to P, B suffers elastic collision with A & then reaches N along PN.

If observer at O in S-frame measures velocity of A as u^A and that of S' frame at O' as $u^{B'}$ velocity of B then $u^A_x = 0$, $u^A_y = u$, $u^{B'}_x = 0$, $u^{B'}_y = -u$. If observer O in S frame measures velocity of B as u^B having components u^B_x, u^B_y then applying LT, we have

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$u^B_x = \frac{u'^B_x + v}{1+\frac{v}{c^2}u'^B_x} = v, \quad u^B_y = \frac{u^{B}_{y'}}{\gamma\left(1+\frac{v}{c^2}u'^B_x\right)} = -\frac{u}{\gamma}$$

If m^A is the mass of A measured by observer O ($= m^{B'}$ which is the mass of B measured by observer O') and m^B is that of B measured by observer O at S-frame. Total momentum of the spheres along Y direction (measured by O) is $= m^A u^A_y + m^B u^B_y$ (before collision)

$$= m^A u - m^B \frac{u}{\gamma}.$$ After collision,

Y component of velocity is reversed, so $u^A_y = -u$, $u^B_y = \frac{u}{\gamma}$ and total momentum $= -m^A u + m^B \frac{u}{\gamma}$. From the principle of conservation of linear momentum, we have $m^A u - m^B \frac{u}{\gamma} = -m^A u + m^B \frac{u}{\gamma}$

or $m^A u = m^B \frac{u}{\gamma}$ or $\boxed{m^B = \gamma m^A}$

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So although $m^A = m^{B'}$, $m^A \neq m^B$ due to movement of frame S' w.r.t. S with velocity v. If we denote $m^A = m_0$ as the "rest mass" of A measured in rest-frame S & $m^B = m$ as the mass in moving frame S' as seen by O, then $m = \frac{m_0}{\sqrt{1-v^2/c^2}} = m_0\gamma$. So moving mass is greater than rest. As $v = c$ gives $m \to \infty$ and $v > c$ gives $m < 0$, so a material particle cannot have velocity $\geq c$ (Einstein's 2nd postulate).

Mass - Energy Relation

From Newton's 2nd law we know $F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$

and due to the force F, work done stored in the mass to move dx distance is the kinetic energy

$$\therefore dE_K = F dx = m\frac{dv}{dt}dx + v\frac{dm}{dt}dx = m\,dv\frac{dx}{dt} + v\,dm\frac{dx}{dt}$$

$$= mv\,dv + v^2 dm \quad (\because v = \tfrac{dx}{dt}) \quad \text{— (1)}$$

Now, $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ or, $m^2 = \frac{m_0^2c^2}{c^2-v^2}$ $\therefore$ $m^2c^2 - m^2v^2 = m_0^2c^2$

Differentiating both sides, we have

$$2m\,dm\,c^2 - 2m\,dm\,v^2 - 2m^2v\,dv = 0$$

$$\therefore mv\,dv + v^2 dm = dm\,c^2 \quad \text{— (2)}$$

Substituting (2) in (1), we have $dE_K = dm\,c^2$. Integrating

$$\int dE_K = \int_{m_0}^{m} dm\,c^2 = c^2\underbrace{(m-m_0)}_{\text{mass converted into K.E.}}$$

So increase in mass indicates the measure of KE, $E_K = (m-m_0)c^2$

$$\therefore \underbrace{mc^2}_{\text{Total Energy}} = \underbrace{E_K}_{\text{K.E.}} + \underbrace{m_0c^2}_{\text{Rest mass energy}} = E.$$

So $\boxed{E = mc^2}$ is Einstein's mass-energy relationship.

So mass can be converted into Energy and unlike classical mechanics, where separate conservation of mass and conservation of energy exists, in special relativity, conservation of mass & energy is the law. In nuclear fission & fusion, large amount of mass converted into energy happen.

Note that when $v < c$, $E_K = (m - m_0)c^2 = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2$

$$\simeq m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = \frac{1}{2} m_0 v^2$$

So we recover the classical Newtonian expression for KE at non-relativistic velocities.

Relativistic Momentum & Energy transformation

Let us consider a particle m with velocity $\vec{u}$ & momentum $\vec{p}$ in S-frame, such that $\vec{p} = m\vec{u}$ with $p_x = m u_x = \frac{m_0 u_x}{\sqrt{1-u^2/c^2}}$,

$p_y = m u_y = \frac{m_0 u_y}{\sqrt{1-u^2/c^2}}$, $p_z = m u_z = \frac{m_0 u_z}{\sqrt{1-u^2/c^2}}$ and $E = mc^2 = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}}$

If $\vec{p}'$ is the momentum in S'-frame with $\vec{u}'$ velocity & momentum $\vec{p}'$ then $\vec{p}' = m'\vec{u}'$ so that $p_x' = m' u_x' = \frac{m_0 u_x'}{\sqrt{1-u'^2/c^2}}$, $p_y' = m' u_y' = \frac{m_0 u_y'}{\sqrt{1-u'^2/c^2}}$, $p_z' = m' u_z' = \frac{m_0 u_z'}{\sqrt{1-u'^2/c^2}}$ and $E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}}$. To establish the transformation,

$$p_x' = m_0 \frac{u_x'}{\sqrt{1-u'^2/c^2}} = m_0 \frac{(u_x - v)\cancel{\left(1 - \frac{v}{c^2}u_x\right)}}{\cancel{\left(1-\frac{v}{c^2}u_x\right)}\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}}$$

$$= \frac{m_0 u_x - m_0 v}{\sqrt{1-u^2/c^2}\sqrt{1-v^2/c^2}}$$

$\left[\because u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}\right.$ & $\left.\sqrt{1-u'^2/c^2} = \frac{\sqrt{(1-v^2/c^2)(1-u^2/c^2)}}{1 - v/c^2\, u_x}\right]$

$$= \frac{p_x - E\frac{v}{c^2}}{\sqrt{1-v^2/c^2}}$$

$$\therefore \boxed{p_x' = \frac{p_x - \frac{v}{c^2}E}{\sqrt{1-v^2/c^2}} = \gamma\left(p_x - \frac{v}{c^2}E\right)}$$

$$p_y' = \frac{m_0 u_y'}{\sqrt{1-u'^2/c^2}} = \frac{m_0 u_y \cancel{\sqrt{1-v^2/c^2}}\cancel{\left(1-\frac{v}{c^2}u_x\right)}}{\cancel{\left(1-\frac{v}{c^2}u_x\right)}\sqrt{\cancel{\left(1-\frac{v^2}{c^2}\right)}\left(1-\frac{u^2}{c^2}\right)}} = \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} = p_y.$$ So $\boxed{p_y' = p_y}$

$\boxed{p_z' = p_z}$ L.T. (momentum)

$p_x = \gamma\left(p_x' + \frac{v}{c^2}E\right)$, $p_y = p_y'$, $p_z = p_z'$ R.L.T. (momentum)

Now, $$E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}} = \frac{m_0 c^2\left(1 - \frac{v}{c^2}u_x\right)}{\sqrt{(1-v^2/c^2)(1-u^2/c^2)}} = \frac{\frac{m_0 c^2}{\sqrt{1-u^2/c^2}} - \frac{m_0 u_x \cdot v}{\sqrt{1-u^2/c^2}}}{\sqrt{1-v^2/c^2}}$$

or $$\boxed{E' = \frac{E - v p_x}{\sqrt{1-v^2/c^2}} = \gamma(E - v p_x)}$$ L.T. (Energy) &

$E = \gamma(E' + v p_x')$ R.L.T. (Energy)

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Now $p'^2 - \frac{E'^2}{c^2} = p_x'^2 + p_y'^2 + p_z'^2 - \frac{(E - vp_x)^2}{c^2(1-v^2/c^2)}$

$$= \frac{(p_x - \frac{v}{c^2}E)^2}{1 - v^2/c^2} + p_y^2 + p_z^2 - \frac{(E - v p_x)^2}{c^2(1 - v^2/c^2)}$$

$$= \frac{p_x^2 - \cancel{\frac{2vp_xE}{c^2}} + \frac{v^2E^2}{c^4} - \frac{E^2}{c^2} - \frac{v^2p_x^2}{c^2} + \cancel{\frac{2vp_xE}{c^2}}}{1 - v^2/c^2} + p_y^2 + p_z^2$$

$$= \frac{p_x^2(1 - v^2/c^2) - E^2/c^2(1 - v^2/c^2)}{1 - v^2/c^2} + p_y^2 + p_z^2 = p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$$

$= p^2 - \frac{E^2}{c^2}$. So this quantity is a "Lorentz scalar."

A few more relationship

Momentum of a particle with moving mass m (rest mass m_0) at velocity $\vec{u}$ is $\vec{p} = m\vec{u} = \gamma m_0 \vec{u}$.

$\therefore \vec{p}\cdot\vec{p} = p^2 = \gamma^2 m_0^2 u^2 = \frac{m_0^2 u^2}{1 - u^2/c^2} = \frac{m_0^2 u^2 c^2}{c^2 - u^2}$ — 1

or $p^2c^2 = u^2(p^2 + m_0^2c^2)$ or $\boxed{u = \frac{pc}{\sqrt{p^2 + m_0^2c^2}}}$ relativistic velocity

Again, $E = mc^2 = \frac{m_0c^2}{\sqrt{1 - u^2/c^2}}$ or $E^2 = \frac{m_0^2c^4}{1 - u^2/c^2} = \frac{m_0^2c^6}{-u^2 + c^2}$.

Using 1, we have $E^2 - p^2c^2 = \frac{m_0^2c^6}{-u^2 + c^2} - \frac{m_0^2u^2c^4}{c^2 - u^2} = \frac{m_0^2c^4(c^2 - u^2)}{c^2 - u^2}$

or $E^2 - p^2c^2 = m_0^2c^4$ or $\boxed{E = \sqrt{p^2c^2 + m_0^2c^4}}$

relativistic Energy-momentum

Rest mass for photon $= 0$. So $E = pc$ and $p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}$.

Again $E_K = (m - m_0)c^2 = mc^2 - m_0c^2 = E - m_0c^2$.

$\therefore E = E_K + m_0c^2 = \sqrt{p^2c^2 + m_0^2c^4}$. Squaring both sides we have

$E_K^2 + \cancel{m_0^2c^4} + 2m_0E_Kc^2 = p^2c^2 + \cancel{m_0^2c^4}$ $\Rightarrow$ $\boxed{p = \sqrt{\frac{E_K^2}{c^2} + 2m_0E_K}}$

relativistic momentum - K.E.

Now we find LT for force. Suppose S' moves with velocity v wrt S and $F = (F_x = \frac{dp_x}{dt}, F_y = \frac{dp_y}{dt}, F_z = \frac{dp_z}{dt})$ is the force measured in S and

$F' \left(F_x' = \frac{dP_x'}{dt'}, P_y' = \frac{dP_y'}{dt'}, F_z' = \frac{dP_z'}{dt'} \right)$ is the same force measured in S' frame.

So $F_x' = \frac{dP_x'}{dt'} = \frac{dP_x'}{dt}\frac{dt}{dt'} = \frac{\frac{dP_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}}{\cancel{\sqrt{1-v^2/c^2}}} \cdot \frac{\cancel{\sqrt{1-v^2/c^2}}}{1-\frac{v}{c^2}\frac{dx}{dt}}$ $\left[P_x' = \frac{P_x - v/c^2 E}{\sqrt{1-v^2/c^2}}, \quad t' = \frac{t - v/c^2 x}{\sqrt{1-v^2/c^2}} \right]$

$$= \frac{F_x - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{v}{c^2}u_x} \quad \text{— (2)}$$

Now we know, $E^2 = p^2c^2 + m_0^2c^4 = \vec{p}\cdot\vec{p}\,c^2 + m_0^2c^4$.

Differentiating, $2E\frac{dE}{dt} = \left(\vec{p}\cdot\frac{d\vec{p}}{dt} + \frac{d\vec{p}}{dt}\cdot\vec{p}\right)c^2 = 2c^2\,\vec{p}\cdot\frac{d\vec{p}}{dt}$

or $2\cancel{mc^2}\frac{dE}{dt} = 2mc^2\,\vec{u}\cdot\vec{F}$ $\quad [\vec{p} = m\vec{u}]$

$= \cancel{2mc^2}(u_xF_x + u_yF_y + u_zF_z)$

Substituting in (2) we have, $F_x' = \frac{F_x - \frac{v}{c^2}(u_xF_x + u_yF_y + u_zF_z)}{1 - \frac{v}{c^2}u_x}$

or $\boxed{F_x' = \frac{\left(1-\frac{v}{c^2}u_x\right)F_x - \frac{v}{c^2}(u_yF_y + u_zF_z)}{1-\frac{v}{c^2}u_x} = F_x - \frac{v/c^2\,(u_yF_y + u_zF_z)}{1-\frac{v}{c^2}u_x}}$

Again, $\boxed{F_y' = \frac{dP_y'}{dt'} = \frac{dP_y'}{dt}\frac{dt}{dt'} = \frac{dP_y}{dt}\frac{\sqrt{1-v^2/c^2}}{1-\frac{v}{c^2}\frac{dx}{dt}} = \frac{F_y\sqrt{1-v^2/c^2}}{1-\frac{v}{c^2}u_x}}$ $\quad [P_y' = P_y]$

and $\boxed{F_z' = \frac{F_z\sqrt{1-v^2/c^2}}{1-\frac{v}{c^2}u_x}}$ If velocity of particle $u = u_x = v$, $u_y = 0, u_z = 0$

then $\boxed{F_x' = F_x, \quad F_y' = \gamma F_y \text{ and } F_z' = \gamma F_z}$

Finally we calculate the acceleration of relativistic particle. from

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{u}) = m\frac{d\vec{u}}{dt} + \vec{u}\frac{dm}{dt}.$$

from $m = \frac{E}{c^2}$ we have $\frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d}{dt}(E_K + m_0c^2) = \frac{1}{c^2}\frac{dE_K}{dt}$

$$= \frac{1}{c^2}\frac{\vec{F}\cdot d\vec{x}}{dt} = \frac{1}{c^2}\vec{F}\cdot\frac{d\vec{x}}{dt} = \frac{1}{c^2}\vec{F}\cdot\vec{u}.$$

Substituting back, we have $\vec{F} = m\frac{d\vec{u}}{dt} + \frac{1}{c^2}\vec{u}(\vec{F}\cdot\vec{u}) = m\vec{a} + \frac{\vec{u}(\vec{F}\cdot\vec{u})}{c^2}$

∴ Acceleration $\boxed{\vec{a} = \frac{d\vec{u}}{dt} = \frac{\vec{F}}{m} - \frac{\vec{u}}{mc^2}(\vec{F}\cdot\vec{u})}$ So $\vec{a} \nparallel \vec{F}$ in general.

(Charge in electric field)

When $\vec{a}, \vec{F}$ & $\vec{u}$ are parallel then we have $F_\parallel = \frac{m_0}{(1-u^2/c^2)^{3/2}}a_\parallel$

↳ longitudinal mass

from $F = m\frac{du}{dt} + u\frac{dm}{dt}$ & substituting $m = m_0/\sqrt{1-u^2/c^2}$

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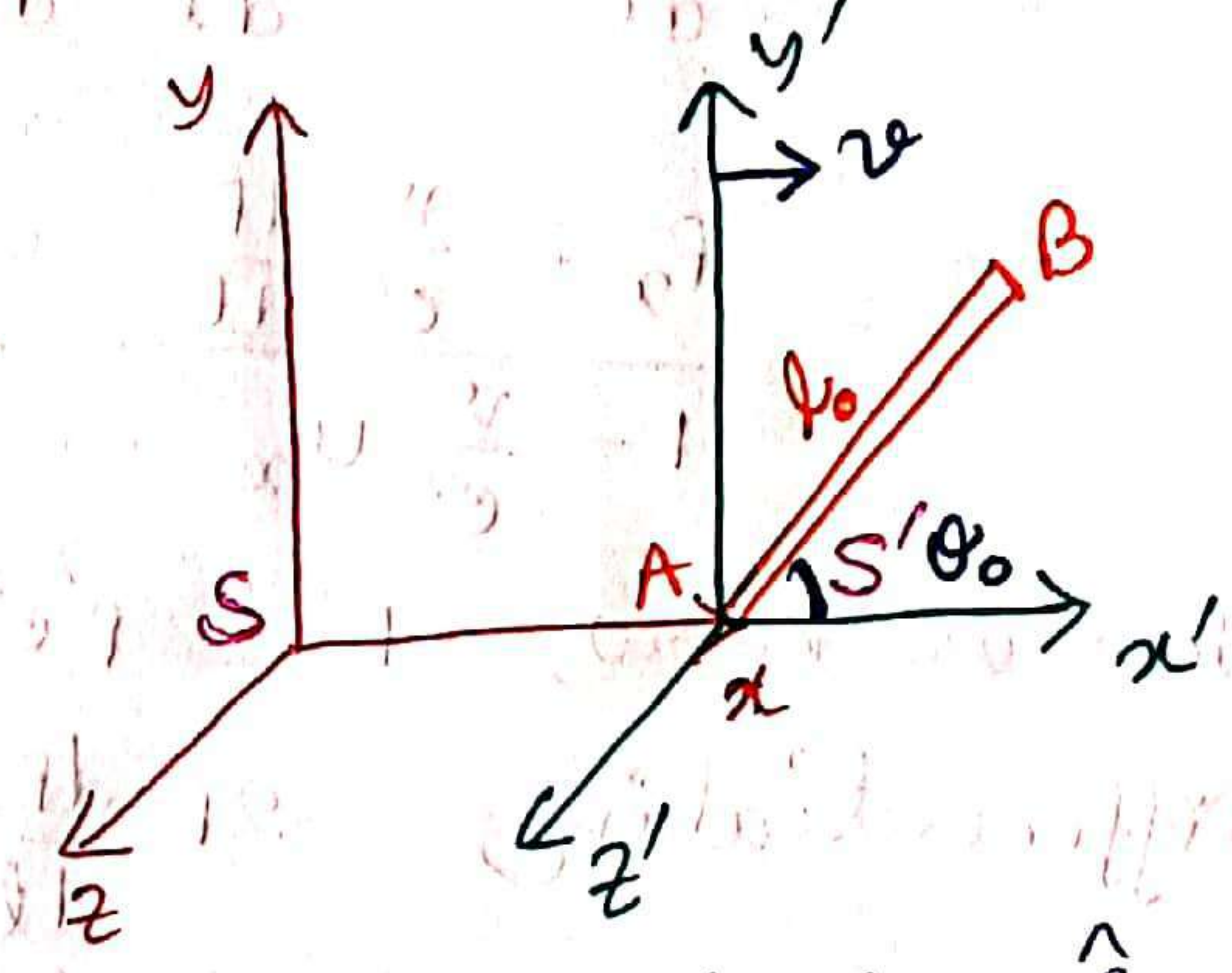
When $\vec{F} \perp \vec{u}$, then $\vec{F}\cdot\vec{u} = 0$ & and $F_\perp = \frac{m_0}{\sqrt{1-u^2/c^2}}\, a_\perp$ ($\vec{a} \perp \vec{u}$)

(charge in magnetic field) $\vec{a}\cdot\vec{u} = 0$ ↳ Transverse mass

Orientation of a Moving Rod



A rod of proper length l_0 lies in the $x'y'$ plane of S' frame and makes an angle θ_0 with x' axis. We want to obtain the length & orientation of the rod in the inertial frame S in which the rod moves to the right with a velocity $v\hat{i}$.

Let the end points of the rod have coordinates (x', y') in the S' frame. If one end of the rod is at the origin of S' frame, then $x_1' = 0$, $x_2' = l_0\cos\theta_0$ & $y_1' = 0$, $y_2' = l_0\sin\theta_0$. To find out the end point coordinates of the rod at time t in S frame, we apply L.T.

$$x_1' = 0 = \gamma(x_1 - vt), \quad y_1' = 0 = y_1$$

$$x_2' = l_0\cos\theta_0 = \gamma(x_2 - vt), \quad y_2' = l_0\sin\theta_0 = y_2.$$

$$\therefore x_2' - x_1' = l_0\cos\theta_0 = \gamma(x_2 - x_1) \quad \text{or} \quad x_2 - x_1 = \frac{l_0\cos\theta_0}{\gamma} \;\&\; y_2 - y_1 = l_0\sin\theta_0$$

$\therefore$ Length of the rod as measured in S-frame is

$$l = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = l_0\sqrt{\left(1-\frac{v^2}{c^2}\right)\cos^2\theta_0 + \sin^2\theta_0} = l_0\sqrt{1-\frac{v^2}{c^2}\cos^2\theta_0}$$

and the angle that the rod makes with x-axis is

$$\theta = \tan^{-1}\left(\frac{y_2-y_1}{x_2-x_1}\right) = \tan^{-1}(\gamma\tan\theta_0).$$

as $\gamma \geq 1$, so $\theta \geq \theta_0$. Thus moving rod appears to not only be contracted but more rotated as well.

$$\boxed{\begin{aligned} l &= l_0\sqrt{1-\frac{v^2}{c^2}\cos^2\theta_0} \\ \theta &= \tan^{-1}(\gamma\tan\theta_0) \end{aligned}}$$

If $\theta_0 = \pi/2$, then $l = l_0$, $\theta = \frac{\pi}{2}$ as expected → no contraction, no rotation in perpendicular direction.

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