Resultant / Superposition of Harmonic oscillations

The resultant of two or more harmonic displacements is the algebraic sum of individual displacements. For linear homogeneous differential equations, sum of any two solutions is also a solution.

Realize that if $\frac{d^2x}{dt^2} = -\omega^2 x + \alpha x^2 + \beta x^3 + \cdots$ then if $\frac{d^2x_1}{dt^2} = -\omega^2 x_1 + \alpha x_1^2 + \beta x_1^3 + \cdots$ & $\frac{d^2x_2}{dt^2} = -\omega^2 x_2 + \alpha x_2^2 + \beta x_2^3 + \cdots$

then $x_1 + x_2$ isn't a solution because if $x_1 + x_2 = x_3$. then

$$\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} = -\omega^2(x_1+x_2) + \alpha(x_1^2+x_2^2) + \beta(x_1^3+x_2^3) + \cdots$$

$$\therefore \frac{d^2x_3}{dt^2} = -\omega^2 x_3 + \alpha(x_3^2 + 2x_1x_2) + \beta(x_3^3 - 3x_1^2x_2 - 3x_1x_2^2) + \cdots$$

Composition of two colinear SHM of same frequency but different amplitude & phase:

Frequency $\omega = 2\pi\nu$, amplitude a & b, phase difference ϕ

$x_1 = a\sin\omega t$, $x_2 = b\sin(\omega t + \phi)$

Time period for both motion is same & so phase difference is also same.

resultant displacement $x = x_1 + x_2 = a\sin\omega t + b\sin(\omega t + \phi)$

$= (a + b\cos\phi)\sin\omega t + b\sin\phi\cos\omega t = A\cos\theta\sin\omega t + A\sin\theta\cos\omega t$

$x = A\sin(\omega t + \theta) \Rightarrow$ S.H.M.

Amplitude of resultant wave $A^2 = (a + b\cos\phi)^2 + b^2\sin^2\phi$

or $A = (a^2 + b^2 + 2ab\cos\phi)^{1/2}$

phase of resultant wave $\tan\theta = \dfrac{b\sin\phi}{a + b\cos\phi}$

$\therefore \quad x = \sqrt{a^2+b^2+2ab\cos\phi}\ \sin\left(\omega t + \tan^{-1}\left\{\dfrac{b\sin\phi}{a+b\cos\phi}\right\}\right)$

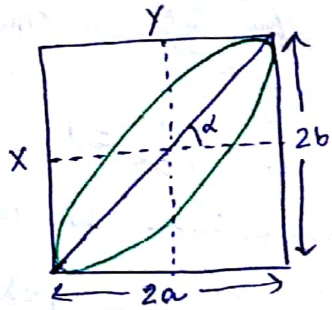
if $\phi = 0$ then $\theta = 0$, $A = a+b$, $x = (a+b)\sin\omega t$

if $\phi = \pi$ then $\theta = 0$ (opposite phase), $A = a-b$, $x = (a-b)\sin\omega t$.

if $a = b$, $x = 0 \Rightarrow$ no resultant motion.

Composition of two SHM at right angle with same frequency but different in phase & amplitude

Again, say two SHM acting in X & Y axis, amplitude a & b, phase difference ϕ.



$x = a\sin\omega t, \quad y = b\sin(\omega t + \phi)$

$\therefore \cos\omega t = \sqrt{1 - x^2/a^2}$

and $\sin\omega t\cos\phi + \cos\omega t\sin\phi = y/b$.

or $\dfrac{x}{a}\cos\phi + \sqrt{1 - \dfrac{x^2}{a^2}}\sin\phi = \dfrac{y}{b}$

or $\left(\dfrac{y}{b} - \dfrac{x}{a}\cos\phi\right)^2 = \left(1 - \dfrac{x^2}{a^2}\right)\sin^2\phi$

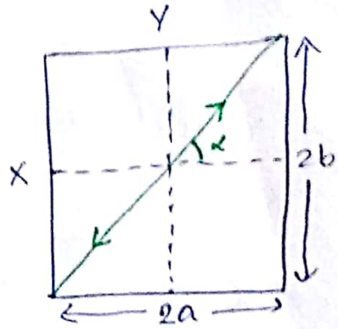
or $\boxed{\dfrac{y^2}{b^2} + \dfrac{x^2}{a^2} - \dfrac{2xy}{ab}\cos\phi = \sin^2\phi}$ ——— (1)

This is equation of ellipse confined to rectangle of side 2a & 2b with direction of major axis $\tan\alpha = \dfrac{2ab}{a^2 - b^2}\cos\phi$.

(a) $\phi = 0$ $\quad \sin\phi = 0, \; \cos\phi = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$

or $\left(\frac{y}{b} - \frac{x}{a}\right)^2 = 0$ or $y = \frac{b}{a}x$

Straight line passing through origin & inclined to x-axis at angle $\alpha = \tan^{-1}\frac{b}{a}$ & with resultant amplitude $= \sqrt{a^2+b^2}$

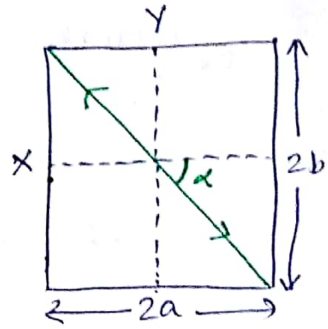


(b) $\phi = \pi$ $\quad$ Two motions are in opposite phase

Then the combined equation is

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} + \frac{2xy}{ab} = 0 \quad \text{or} \quad \left(\frac{y}{b} + \frac{x}{a}\right)^2 = 0$$

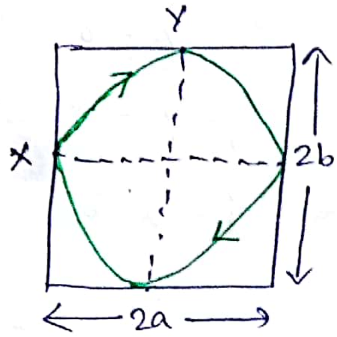
$\therefore \; y = -\frac{b}{a}x$



straight line passing through origin & inclined to x-axis at angle $\tan\alpha = -\frac{b}{a}$. If $a = b$, $\alpha = 135^\circ$

(c) $\phi = \pi/2$ $\quad$ Then the combined equation is

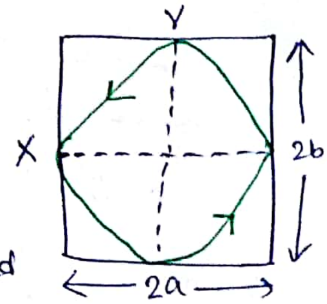
$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$, elliptical motion with major axis $2a$, minor axis $2b$.

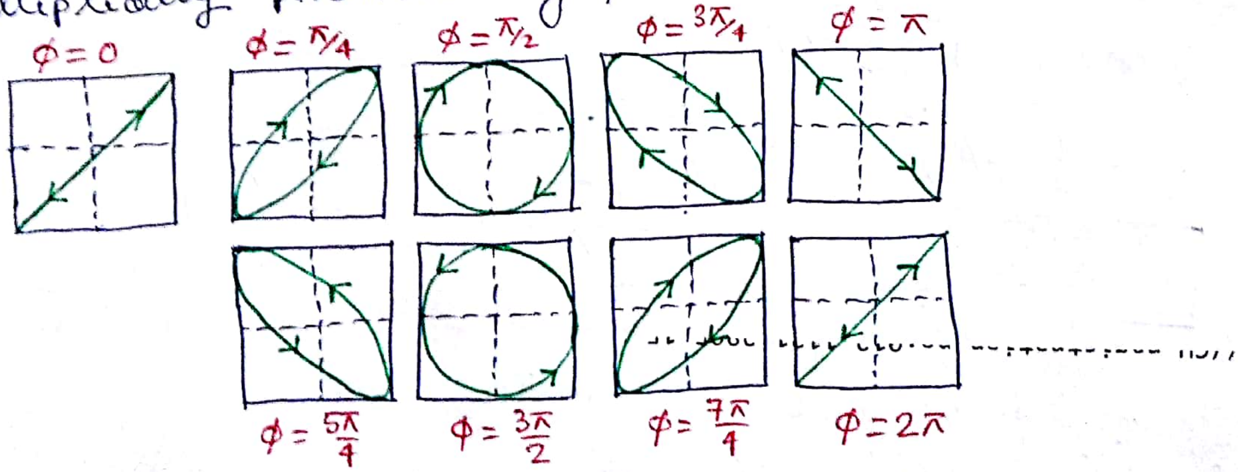


If $a = b$, then circular motion with $x^2 + y^2 = a^2$

(d) $\phi = \frac{3\pi}{2}$ $\quad$ Then the combined equation is

$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$, elliptic motion but counter-clockwise. In ray optics, this is called left-handed elliptically polarized light/vibration.





Composition of two SHM at right angle with different frequency, different phase, different amplitude:

Complicated motion $\rightarrow$ Lissajous figures. Suppose frequencies are in 1 : 2 ratio $x = a\cos\omega t$, $y = b\cos(2\omega t + \phi)$.

$$\therefore \frac{y}{b} = \cos(2\omega t)\cos\phi - \sin(2\omega t)\sin\phi$$

$$= (2\cos^2\omega t - 1)\cos\phi - 2\sin\omega t\cos\omega t\sin\phi.$$

$$= \left(2\frac{x^2}{a^2} - 1\right)\cos\phi - 2\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\sin\phi.$$

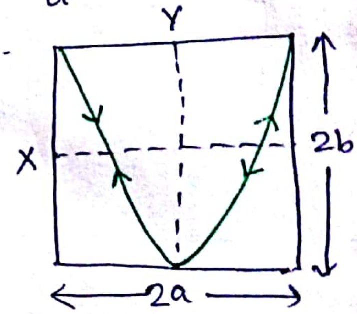
$$\text{or } \left(\frac{y}{b} + \cos\phi\right) - \frac{2x^2}{a^2}\cos\phi = -\frac{2x}{a}\sqrt{1 - \frac{x^2}{a^2}}\sin\phi.$$

$$\text{or } \left(\frac{y}{b} + \cos\phi\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 - \frac{y}{b}\cos\phi\right) = 0 \Rightarrow 4^{th} \text{ degree equation}$$

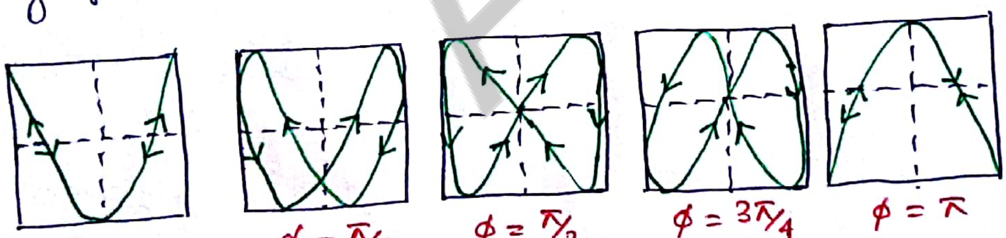
$\underline{\phi = 0}$ $\quad \left(\frac{y}{b} + 1\right)^2 + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1 - \frac{y}{b}\right) = 0$ or $\left(\frac{y}{b} - \frac{2x^2}{a^2} + 1\right)^2 = 0$

Two coincident parabola with vertex at $(0, -b)$ with equation $\frac{y}{b} - \frac{2x^2}{a^2} + 1 = 0$

$$\text{or } x^2 = \frac{a^2}{2b}(y + b).$$

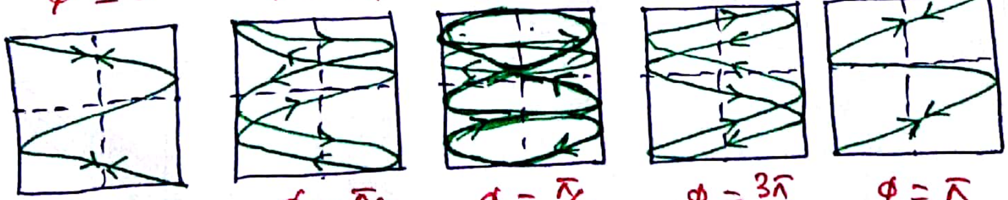


$\phi \neq 0$ very complex to resolve analytically & graphical method is the most convenient method.



$\phi = 0$ $\quad \phi = \pi/4$ $\quad \phi = \pi/2$ $\quad \phi = 3\pi/4$ $\quad \phi = \pi$

frequency ratio 1:2



$\phi = 0$ $\quad \phi = \pi/4$ $\quad \phi = \pi/2$ $\quad \phi = \frac{3\pi}{4}$ $\quad \phi = \pi$

frequency ratio 1:3

So frequencies need to be in a commensurate ratio to give a periodic motion. Notice the interesting features that (1) resultant curve is always inside rectangle & the motion is periodic, (2) Number of tangential point in $x : y$ is the frequency ratio inverse.

HW 1. A particle is simultaneously subjected to two SHM in same direction, each of frequency 5 Hz. If amplitudes are 0.005 m & 0.002 m & phase difference is 45°, find the amplitude of the resultant ~~direction~~ displacement & its phase relative to the first component. Write down the expression for the resultant displacement as a function of time.

2. Two vibrations along the same line are described by

$x_1 = 0.03 \cos 10\pi t$, $x_2 = 0.03 \cos 12\pi t$, x_1, x_2 in metres & t in seconds. Obtain the equation describing the resultant motion and the beat period (beat period is the time interval between two consecutive maximum amplitude).