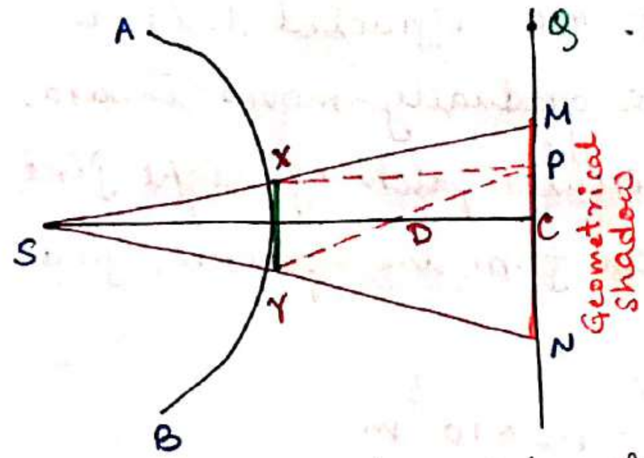
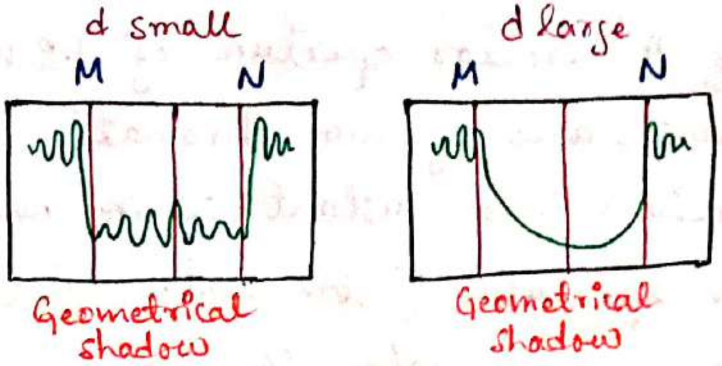
Diffraction at a wire





S = rectangular slit, XY = wire with thickness d. At point Q outside the geometrical shadow intensity distribution is same as straight edge at X and so diffraction bands of unequal width is formed above M & below N. These bands are independent of thickness of wire and effect on other side is negligible (because wire stops the important half-period strips).

Within geometric shadow, interference fringes appear. Effect due to AX of cylindrical wave-front at P in geometrical shadow is entirely due to few half-period strips at X, so a small luminous source can be thought at X. Similarly for BY portion, Y is a luminous source. If $PY - PX = n\lambda$, point P will be bright and $= (2n+1)\frac{\lambda}{2}$, P will be dark.

Equal interference fringe width $\beta = \frac{D}{d}\lambda$.

D = distance of screen from obstacle (wire)

d = thickness of obstacle (wire's diameter).

Point C will be always bright as waves from X & Y always meet in phase. For moderate value of d pattern is shown, while as d is increased, β decreases finally to disappear, so only diffraction band above M & below N is seen.

CW A circular aperture of 1.2 mm diameter is illuminated by plane waves of monochromatic light. The diffracted light is received on a distant screen which is gradually moved towards the aperture. If the centre of the circular patch of light first becomes dark, when the screen is 30 cm from the aperture, find the wavelength of light used.

Aperture diameter $2r = 1.2\text{ mm} = 1.2\times10^{-3}\text{ m}$

So radius $r = 0.6\times10^{-3}\text{ m}$, $a = 30\text{cm} = 0.3\text{m}$.

for the first minimum on the axial point, the aperture must have 2 half-period zones. So $n = \frac{\pi r^2}{\pi a\lambda} = 2$.

$$\therefore \lambda = \frac{r^2}{2a} = \frac{(0.6\times10^{-3})^2}{2\times0.3} = 6\times10^{-7}\text{m} = 6000\text{Å}.$$

CW Light of wavelength 6000 Å passes through a narrow circular aperture of radius 0.9×10^{-3} m. At what distance along the axis will the first maximum intensity be observed?

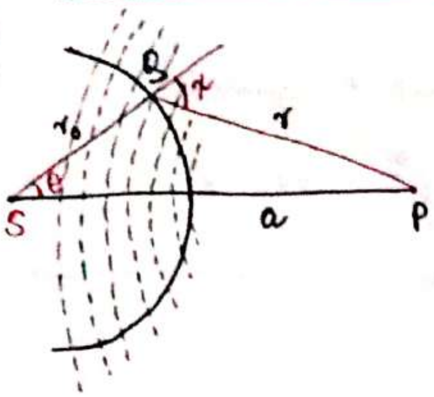
$\lambda = 6000\times10^{-10}$ m, $r = 0.9\times10^{-3}$ m. $a = ?$

for first maximum intensity $a^2+r^2 = (a+\frac{\lambda}{2})^2$

$$\therefore a^2+r^2 = a^2 + \cancel{\frac{\lambda^2}{4}}^{0} + 2a\lambda/2 \quad \therefore r^2 = a\lambda.$$

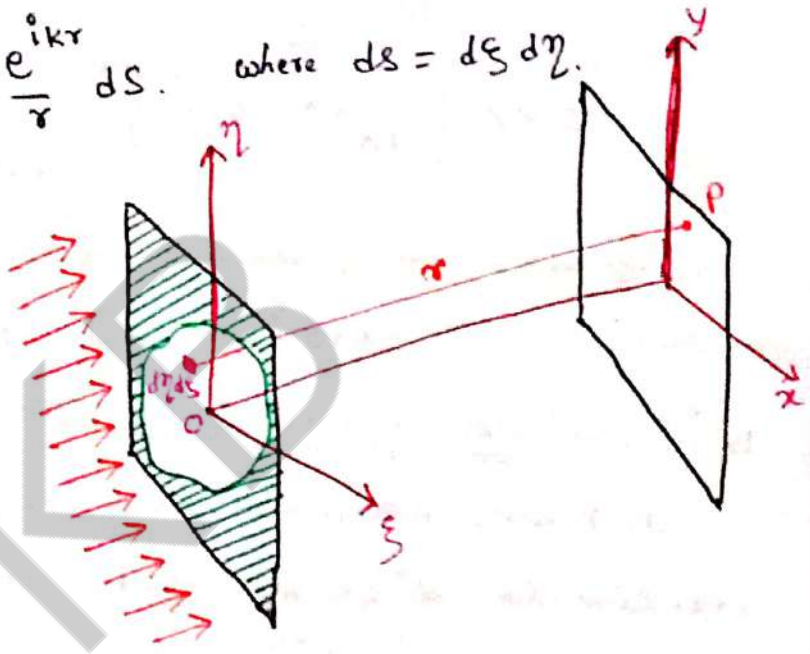
$$\therefore a = \frac{r^2}{\lambda} = \frac{(0.9\times10^{-3})^2}{6000\times10^{-10}} = 1.35\text{m}.$$

Fresnel Diffraction Integral



Omitting the time dependent factor $e^{-i\omega t}$, disturbance recieved at Q from a source S that emit monochromatic spherical wave, is $A\dfrac{e^{ikr_0}}{r_0}$. According to Fresnel-Huygen's theory we consider each element as the source of secondary wavelet, so that contribution at screen at P due to elementary area dS at Q is

$$du(P) = \underbrace{C(\chi)}_{\text{inclination factor}} \frac{Ae^{ikr_0}}{r_0}\frac{e^{ikr}}{r}\,dS. \quad \text{where } dS = d\xi\, d\eta.$$



For a plane wave of amplitude A incident normally on aperture

$$du(P) = \frac{Ae^{ikr}}{r}\,dS.$$

So total contribution at P due to whole wave-front is

$$u(P) = C\iint \frac{Ae^{ikr}}{r}\,d\xi\, d\eta$$

In principle $A = A(\xi,\eta)$ but in absence of aperture $A = \text{constant}$.

and $C = -\dfrac{ik}{2\pi} = \dfrac{1}{i\lambda}$, so that $u(P) = \dfrac{A}{i\lambda}\displaystyle\iint \frac{e^{ikr}}{r}\,d\xi\, d\eta$

Two assuptions that are made is (i) screen does not affect the field at P, so dimension of aperture >> wavelength, (ii) $C(\chi) = C$ so that $C(\chi) = C(1+\cos\chi) \simeq C$.

$$\text{Now } r = \sqrt{(x-\xi)^2+(y-\eta)^2+z^2} = z\sqrt{1+\frac{(x-\xi)^2+(y-\eta)^2}{z^2}} = z\sqrt{1+\alpha}$$

$$\simeq z\left(1+\frac{\alpha}{2}-\frac{\alpha^2}{8}+\cdots\right) = z+\frac{(x-\xi)^2}{2z}+\frac{(y-\eta)^2}{2z}$$

$$\therefore\ u(P) = \frac{1}{i\lambda z}e^{ikz}\iint A(\xi,\eta)\, e^{\frac{ik}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]}\,d\xi\, d\eta$$

where we have replaced r by z in denominator safely.

$\therefore\ u(P) \simeq \frac{1}{i\lambda z} e^{ikz} e^{\frac{ik}{2z}(x^2+y^2)} \iint A(\xi,\eta)\, e^{\frac{ik}{2z}(\xi^2+\eta^2)} e^{-i(u\xi+v\eta)} d\xi d\eta$

where $u = \frac{2\pi x}{\lambda z}$ and $v = \frac{2\pi y}{\lambda z}$ (Spatial frequencies)

This is called "Fresnel diffraction integral" with "Fresnel approximation" $\alpha^2 <<$ neglected, so maximum phase change is less than π, so $\frac{1}{8} kz\alpha^2 << \pi$

or $\boxed{z >> \left\{ \frac{1}{4\lambda}\left[(x-\xi)^2 + (y-\eta)^2\right]^2_{max} \right\}^{1/3}}$ — ①

$z >> \left\{ \frac{1}{4\lambda}(x^2+y^2)^2 \right\}^{1/3}$ for circular aperture with radius a when observed in region of dimension > a

So if $a = 0.1$ cm & observed region is radius 1 cm then $x^2+y^2 = r^2 = 1\,cm^2$ then for $\lambda = 5\times10^{-5}$ cm, $z >> 17$ cm.

In "Fraunhofer approximation", $z >>$ large so that $e^{\frac{ik}{2z}(\xi^2+\eta^2)} \simeq 1$, which means maximum phase change is way less than π. So in addition to $\alpha^2 <<$ neglected (or condition ①),

$z >> \frac{[\xi^2+\eta^2]_{max}}{\lambda}$, so $z >> a^2/\lambda$ (circular aperture)

so that $u(P) \simeq \frac{1}{i\lambda z} e^{ikz} e^{\frac{ik}{2z}(x^2+y^2)} \iint A(\xi,\eta)\, e^{-i(u\xi+v\eta)} d\xi d\eta$

This is called "Fraunhofer diffraction integral"

So Fresnel number $N_F = \frac{a^2}{\lambda z} << 1$ for Fraunhofer approximation.

Fresnel Integral $\Rightarrow$ $C(\tau) = \int_0^\tau \cos\left(\frac{\pi u^2}{2}\right) du$, $S(\tau) = \int_0^\tau \sin\left(\frac{\pi u^2}{2}\right) du$

Property: $C(-\tau) = -C(\tau)$ $\quad C(\infty) = S(\infty) = 0.5$

$S(-\tau) = -S(\tau)$ $\quad C(0) = S(0) = 0$.

"Cornu's spiral"