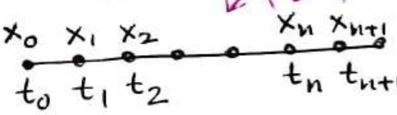
Non linear Dynamics

Dynamics is the study of the time-evolutionary process of a system and the corresponding set of equations is known as the Dynamical System. A system of n 1^{st} order differential equations is called a dynamical system of dimension n. If a process's entire future and entire past are **uniquely** defined, then it is called as deterministic process (e.g. 1 & 2 body Newtonian dynamics), if not uniquely determined then semi-deterministic (e.g. heat propagation in a metal where future is determined by the present but past is not), otherwise a non-deterministic process (e.g. Brownian motion).

An evolutionary process can be categorized into:

(a) A continuous time process which is represented by differential equations (or "flow"). $\boxed{\dot{\underline{x}} = \underline{f}(\underline{x}, t)}$ ⇒ nonlinear, sufficiently smooth function

(b) A discrete time process which is represented by difference equations (or "maps")



$$x_{n+1} = g(x_n) = g(g(x_{n-1})) = g^2(x_{n-1}) = \cdots$$

If $\underline{f}(\underline{x}, t)$ is explicitly time independent ⇒ "Autonomous system" but if $\underline{f}(\underline{x}, t)$ is explicitly time dependent ⇒ "Non-autonomous system"

n-dimensional nonautonomous system

$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, \ldots, x_n, t)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \ldots, x_n, t)$$

⟹ [define $x_{n+1} = t$]

$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n, x_{n+1})$$
$$\dot{x}_2 = f_2(x_1, x_2, \ldots, x_n, x_{n+1})$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \ldots, x_n, x_{n+1})$$

(n+1) dimensional autonomous system

The dynamical variable $\underline{x}, \underline{f}$ can be anything as generalized coordinate, a real coordinate as in Newton's law or wavefunction in Schrödingers equation or Fokker-Planck equation that describes the probability of finding a Brownian particle. $P(\underline{X},t)$. It can be one particular function $\underline{u}(\underline{X},t)$ that satisfy wave equation for a given geometry, may be a set of functions $\underline{E}(X,t)$, $\underline{B}(X,t)$ satisfying Maxwell's equations.

Given initial coordinates (X_0, P_0), one can completely solve Newton's 2^{nd} order differential equation that canbe written as a set of 2 1^{st} order equation $\dot{x} = P/m$, $\dot{P} = f(X)$. Because this is a linear problem one can apply superposition principle to decouple, which is not true if f is non-autonomous. Such set of equations $\dot{x} = P/m$, $\dot{P} = f(\underline{X},t)$ exhibit dynamical chaos. For deterministic yet nonlinear $f(\underline{X},t)$ sensitive dependence on initial conditions is observed in the resulting trajectory (butterfly effect). The reason being even though we can caste a n-dimensional non-autonomous system into a $n+1$ dimensional autonomous system, increasing the dimensionality changes the dynamics completely.

Examples: Autonomous Systems:

(a) Damped linear harmonic oscillator $\Rightarrow \ddot{x} + \alpha\dot{x} + \beta x = 0 \quad (\beta > 0)$

(b) Undamped nonlinear oscillator (pendulum) $\Rightarrow \ddot{x} + \omega^2 \sin x = 0 \quad (\omega = \sqrt{g/L})$

(c) Nonlinearly damped van der Pol oscillator $\Rightarrow \ddot{x} - \mu(1-x^2)\dot{x} + \beta x = 0 \quad (\mu > 0)$

(d) Lotka-Volterra predator-prey model $\Rightarrow$

$$\dot{x} = \alpha x - \beta xy \quad (\alpha, \beta, \gamma, \delta > 0)$$
$$\dot{y} = -\gamma y + \delta xy$$

Non-autonomous systems:

(a) Forced linear harmonic oscillator $\Rightarrow \ddot{x} + \alpha\dot{x} + \beta x = f\cos\omega t \quad (\alpha, \beta > 0)$

(b) Duffing nonlinear oscillator $\Rightarrow \ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta x^3 = f\sin\omega t$

(c) Forced nonlinearly damped van der Pol oscillator $\Rightarrow \ddot{x} - \mu(1-x^2)\dot{x} + \beta x = f\cos\omega t \quad (\mu > 0)$

(d) Forced nonlinearly damped Duffing-van der Pol oscillator $\Rightarrow$

$$\ddot{x} - \mu(1-x^2)\dot{x} + \omega_0^2 x + \beta x^3 = f\cos\omega t \quad (\mu > 0)$$

Conservative & Dissipative Dynamical system:

In the formulation of Energy mechanics by Lagrange & Hamilton in terms of generalized coordinate & generalized momenta, we have noticed for forces to be conservative should be derivable from a scalar potential depending only on coordinates so that $\bar{F}_i = -\nabla_i V(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$ so that $\sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = -\sum_{i=1}^{N} \vec{\nabla}_i V \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha}$

$$= -\frac{\partial V}{\partial q^\alpha}.$$

Using this we had defined a Lagrangian that satisfies the Euler-Lagrange equation (EL) $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0$

One of the direct consequences of the Lagrangian formulation is the relation of conservation laws to symmetries of dynamical system, or the Noether's theorem. Eg. Coulomb potential $V \sim r^{-1}$ has no angular dependence (symmetric under rotation), so that θ becomes a **cyclic** ("ignorable") coordinate ($\partial L/\partial\theta = 0$) which from EL equation means the conjugate (angular) momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}}$ is conserved or constant of the motion. (**COM**).

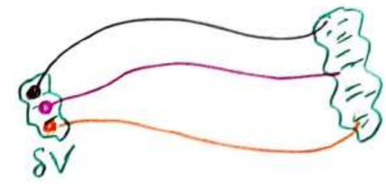
In terms of Hamiltonian function $H = H(q_i, p_i)$ that is independent of time is a conservative system. Using Hamilton's equations $\dot{q}_i = \frac{\partial H}{\partial p_i}(q_i, p_i, t)$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}(q_i, p_i, t)$, the rate of change

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = 0.$$

So H is the COM which is the total energy E. We note that phase velocity $v = (\dot{q}_i, \dot{p}_i) = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i}\right)$ of a conservative system is equal in magnitude and perpendicular to $\bar{\nabla} H(q_i, p_i) = \left(\frac{\partial H}{\partial q_i}, \frac{\partial H}{\partial p_i}\right)$, so the motion is along contours of constant H and the phase diagram consists of several contours, that are invariant sets of the system, along with fixed point of H $v = 0$ which

is given also by $\bar{\nabla} H = 0$. The system remains in equilibrium at these fixed points (CFP).

"**Liouville's theorem**" states that Hamilton's equations preserve the dimension (area, volume) of the phase space. Suppose we have several neighbouring initial conditions for a dynamical system and calculate the rate of change of an initial volume element



$\delta V = \delta x_1 \delta x_2 \cdots \delta x_N$. The dynamical equation takes the form

$\dot{x}_1 = f_1(x_1, x_2, \dots, x_N)$ and for the neighbouring point

$\dot{x}_1 + \delta\dot{x}_1 = f_1(x_1 + \delta x_1, x_2, \dots, x_N)$.

$$\therefore \ \delta\dot{x}_1 = f_1(x_1+\delta x_1, x_2, \dots, x_N) - f_1(x_1, x_2, \dots, x_N) = \frac{\partial f_1}{\partial x_1}\delta x_1 + \cancel{O(\delta x_1^2)}$$

$$\therefore \ \delta\dot{V} = \delta\dot{x}_1 \delta x_2 \cdots \delta x_N + \delta x_1 \delta\dot{x}_2 \cdots \delta x_N + \cdots + \delta x_1 \delta x_2 \cdots \delta\dot{x}_N$$

$$= \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n}\right]\delta x_1 \delta x_2 \cdots \delta x_N = (\vec{\nabla}\cdot\vec{f})\,\delta V$$

While volume element does not change for a conservative system, $\vec{\nabla}\cdot\vec{f} = 0$. For a dissipative system $\vec{\nabla}\cdot\vec{f} < 0$ so that the phase volume gradually shrinks to zero at $t \to \infty$. For an antidissipative system, $\vec{\nabla}\cdot\vec{f} > 0$ and the phase volume gradually expands.

Note that if $\vec{\nabla}\cdot\vec{f}$ = constant (say c) then using the evolution equation $\dot{V} = cV$ yields $V(t) = V(0)\,e^{ct} = V(0)$ only when $c = 0$ or the vector field is solenoidal.

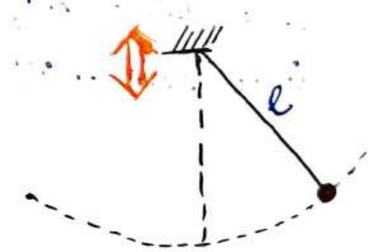
Note that for nonpotential forces are functions of velocities (damping, coriolis force etc) so that they are not derivable from a potential. EL takes the form $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = \sum_{i=1}^{N} \vec{F}_i^D \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha}$ where dissipative force $F_i^D \neq -\bar{\nabla} V(x_i)$.

Examples : (A) **Linear Harmonic oscillator:** Equation of motion $\ddot{x} + \omega^2 x = 0$. If we set $\dot{x} = y$ to rewrite two 1^{st} order equation $\dot{x} = y$, $\dot{y} = -\omega^2 x$ so that we can caste it in dynamical form $\dot{\underline{X}} = f(\underline{X})$ with $f(\underline{X}) = \begin{pmatrix} y \\ -\omega^2 x \end{pmatrix}$ where $\omega = \sqrt{g/l} > 0$. The divergence of the vector field $f(\underline{X})$ is : $\vec{\nabla}\cdot\vec{f} = \partial_x y + \partial_y(-\omega^2 x) = 0$. This means the system is conservative and the area occupied in the 2D phase portrait $(x-y)$ is constant.

(B) **Damped Harmonic oscillator :** The governing equation is $\ddot{x} + \alpha\dot{x} + \beta x = 0$ $(\alpha, \beta > 0)$. Setting $\dot{x} = y$ we can rewrite the system and vector field as $\dot{x} = y$, $\dot{y} = -\alpha y - \beta x$ with $f(\underline{X}) = \begin{pmatrix} y \\ -\alpha y - \beta x \end{pmatrix}$.

Now $\vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-\alpha y - \beta x) = -\alpha < 0$ (while $\alpha > 0$).

This means $\frac{dV}{dt} = -\alpha V$ or $V(t) = V(0)e^{-\alpha t} \to 0$ as $t \to \infty$. So the system is dissipative in nature and the area in the phase plane decays with time. For $\alpha < 0$, this becomes anti-dissipative as $\vec{\nabla}\cdot\vec{f} > 0$ and $\lim_{t\to\infty} V(t) \to \infty$.



(C) **Parametric (Mathieu) oscillator :**

A Mathieu oscillator is a simple pendulum connected by a rigid rod to a pivot having alternating vertical motion without friction. Consider the parametric forcing to be a time dependent gravitational field $g(t) = g_0 + \beta(t)$ so that the linearized equation of motion is $\ddot{\theta} + \frac{g(t)}{l}\theta = 0$. When $g(t)$ is periodic and sinusoidal $g(t) = g_0 + g_1\cos(2\omega t)$ we have $\ddot{\theta} + \omega_0^2[1 + h\cos(2\omega t)]\theta = 0$ where $\omega_0^2 = g_0/l$ and $h = g_1/g_0 > 0$.

Setting $\dot{\theta} = \psi$, $\dot{\psi} = -\omega_0^2[1 + h\cos(2\omega t)]\theta$ with $f(\underline{\theta}) = \begin{pmatrix} \psi \\ -\omega_0^2[1 + h\cos(2\omega t)]\theta \end{pmatrix}$

Now $\vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial\theta}(\psi) + \frac{\partial}{\partial\psi}(-\omega_0^2(1 + h\cos 2\omega t)\theta) = 0$. for $\omega \neq \omega(\dot{\theta})$. In such circumstance forced parametric oscillator acts as a conservative system.

(d) Duffing oscillator: $\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta x^3 = f\sin\omega t$. Here $\dot{x} = y$, $\dot{y} = -\alpha y - \omega_0^2 x - \beta x^3 + f\sin\omega t$ so that $\vec{\nabla}\cdot\vec{f} = -\alpha < 0$

So the system is dissipative.

(e) van der Pol oscillator: $\ddot{x} + \mu(x^2-1)\dot{x} + \beta x = 0$, $(\mu > 0)$. Setting $\dot{x} = y$, $\dot{y} = -\mu(x^2-1)y - \beta x$ we have $\vec{\nabla}\cdot\vec{f} = -\mu(x^2-1)$. So

$$\frac{dV}{dt} = -\int \mu(x^2-1)\,dx\,A = -A\mu\left(\frac{1}{3}x^3 - x\right) = A\mu x(1 - x^2/3).$$

$$\therefore A\frac{dx}{dt} = A\mu x(1 - x^2/3) \quad \therefore \quad x\left[\frac{1 - x/\sqrt{3}}{1 + x/\sqrt{3}}\right]^{1/\sqrt{12}} = e^{\mu t}.$$

This shows that for $x > 1$ and $x < -1$, $\vec{\nabla}\cdot\vec{f} < 0$, for $x < 1$ and $x > -1$, $\vec{\nabla}\cdot\vec{f} > 0$ and for $x = \pm 1$, $\vec{\nabla}\cdot\vec{f} = 0$. The phase area (line) periodically shrinks & expands.

(f) Lolka-Volterra Predator-Prey population model:

$\dot{x} = ax - bxy$, $\quad a, b, c > 0$, $\quad \vec{\nabla}\cdot\vec{f} = \partial_x(ax - bxy) + \partial_y(bxy - cy)$

$\dot{y} = bxy - cy$ $\qquad\qquad = a - by + bx - c$.

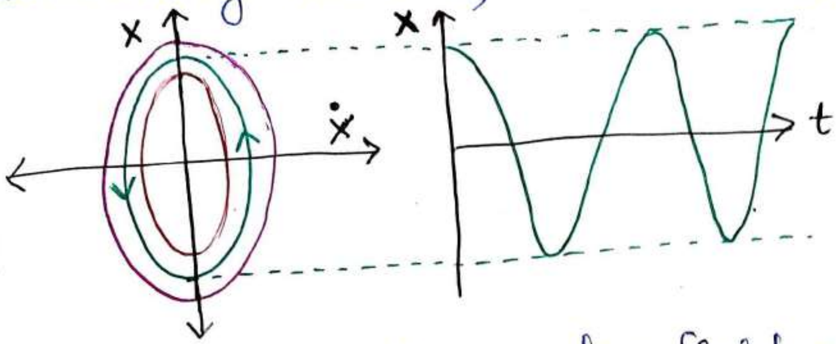
$$\therefore \frac{dV}{dt} = \int \vec{\nabla}\cdot\vec{f}\,dx\,dy\,\ell = \ell\int(a - c - by + bx)\,dx\,dy$$

$= \ell(a-c)xy - b\ell(xy^2 - x^2y)$. So the phase area periodically shrinks & expands.

Now, let us try to understand the behaviour of the simplest dynamical systems in the x-y (or x-$\dot{x}$) plane which is called the phase portrait. The general solution of the EOM for SHM $\ddot{x} + \omega^2 x = 0$ is $x = A\sin\omega t + B\cos\omega t$ where the integration constants A and B are evaluated by initial condition $t = 0$, $x = x_0$, $\dot{x} = \dot{x}_0$. Substituting this to x and $\dot{x}$ $(= A\omega\cos\omega t - B\omega\sin\omega t)$ we have $x = \frac{\dot{x}_0}{\omega}\sin\omega t + x_0\cos\omega t$, $\dot{x} = \dot{x}_0\cos\omega t - x_0\omega\sin\omega t$

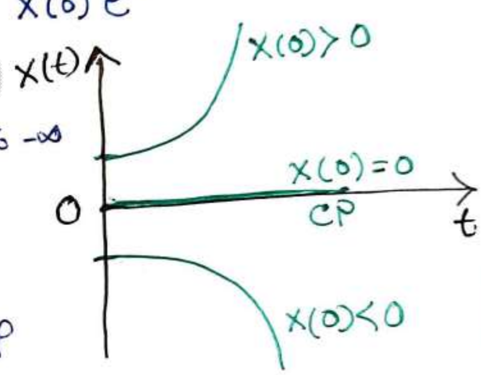
Eliminating time t, we have $x^2 + \frac{\dot{x}^2}{\omega^2} = x_0^2 + \frac{\dot{x}_0^2}{\omega^2}$

PHASE PORTRAIT



The family of ellipses in x-$\dot{x}$ plane designate the phase portrait for different initial conditions, so that $\dot{\underline{x}} = f(\underline{x})$ represents a vector field on the line. Naturally, the equilibrium (no flow) situation $f(\underline{x}) = 0$ and the roots are called "fixed points" or "critical points" (FP/CP). There are two class of FPs, stable FP ("attractor" or "sink") and unstable FP ("repellor" or "source"). In a dynamical system, nature of the flow (phase trajectory) is controlled by the behaviour of the system near its CP.

Flow in 1D: When $\dot{x} = x$, then $x(t) = x(0)e^{t}$

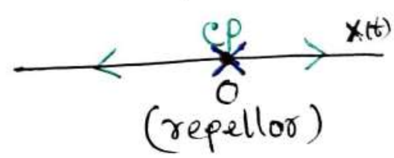


When $x(0) > 0$, trajectory diverges upwards to ∞

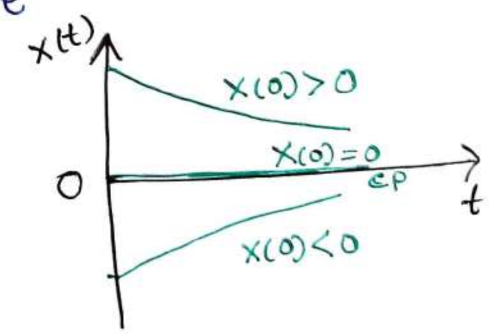
$x(0) < 0$, trajectory diverges downwards to $-\infty$

$x(0) = 0$, $\dot{x} = 0$ is the CP that serves as the separatrix.

In the phase space (line) 0 is an unstable CP because any perturbation to $x(t) > 0$ leads to $+\infty$ and $x(t) < 0$ leads to $-\infty$.



Similarly when $\dot{x} = -x$, then $x(t) = x(0)e^{-t}$

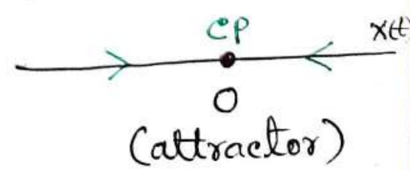


When $x(0) > 0$, trajectory tends to $x(0) = 0$ at $t \to \infty$

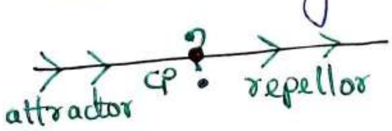
$x(0) < 0$, trajectory tends to $x(0) = 0$ at $t \to \infty$.

$x(0) = 0$, $\dot{x} = 0$ is the CP towards which both the flow happen.

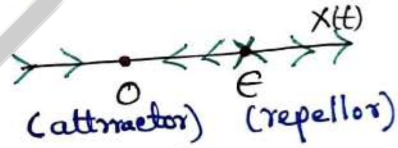
In the phase space (line) 0 is a stable CP because any perturbation in any direction of the axis tends to flow towards $x(0) = 0$.

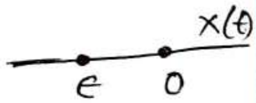
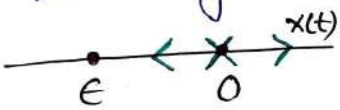
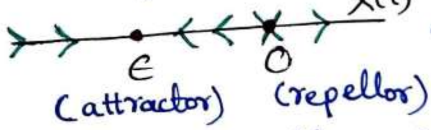


Because this is the only attractor, its also called global attractor.

But for $\dot{x} = x^2$, the phase line  for $x > 0$ and

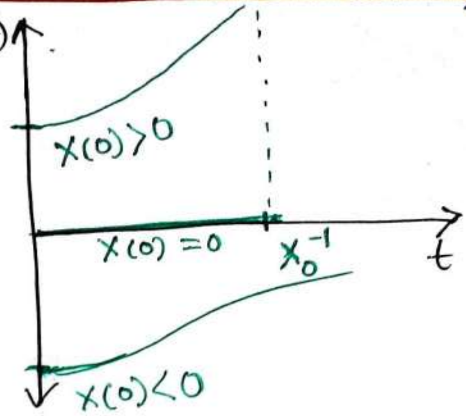
$x < 0$, so that the cp is neither an attractor or repellor but a higher order (degenerate) cp because $f(x)$ is nonlinear.

Bifurcation: To understand the nature of double root $x^2 = 0$ ~~let~~ let us consider $f(x) = x(x-\epsilon)$ so that one shifts one of the roots in the vicinity $\lim_{\epsilon \to 0} f(x) = x^2$. For $\epsilon > 0$, we have the phase portrait  to understand which one is attractor/ repellor, let us linearize the problem in the vicinity of $x = 0$ and $x = \epsilon$ separately. In the neighbourhood of $x = 0$, $x^2 \simeq 0$ so that $\dot{x} = x(x-\epsilon) \simeq -\epsilon x$, which tells that $x = 0$ cp is an attractor. . Now in the neighbourhood of $x = \epsilon$ to linearize, lets change variable $u = x - \epsilon$ so that $\dot{u} = (u+\epsilon)u \simeq \epsilon u$ as $u^2 \simeq 0$. While $\epsilon > 0$, $\dot{u} = \epsilon u$ tells that $x = \epsilon$ is a repellor. The phase line becomes  and in the limit $\epsilon \to 0$, we get back degenerate cp phaseline.

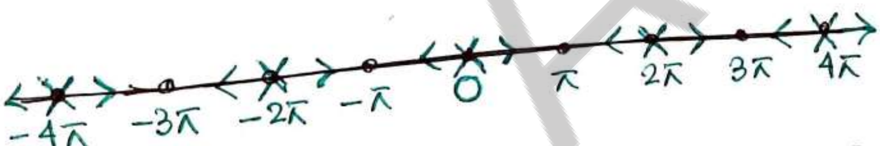
Now for $\epsilon < 0$, we have the phase portrait inverted  so that in the neighbourhood of $x = 0$, $\dot{x} \simeq -\epsilon x$ makes the cp a repellor.  and in the neighbourhood at $x = \epsilon$ $\dot{u} = \epsilon u$ makes the cp an attractor. .

So qualitatively the sign of ϵ decides whether the cp will remain an attractor or repellor, so the dynamical system bifurcates at $\epsilon = 0$, which is called an "exchange of stability bifurcation". Note that for $\epsilon \neq 0$, direct solution of $\dot{x} = x(x-\epsilon)$ will yield $x \sim e^{t} f(x_0, \epsilon)$ or $e^{-t} f(x_0, \epsilon)$ that diverges or converges in time in accordance with the phase line. However for $\epsilon = 0$, $\dot{x} = x^2 \Rightarrow \frac{1}{x(0)} - \frac{1}{x(t)} = t$ or $x(t) = \frac{x(0)}{1 - x(0)t}$.

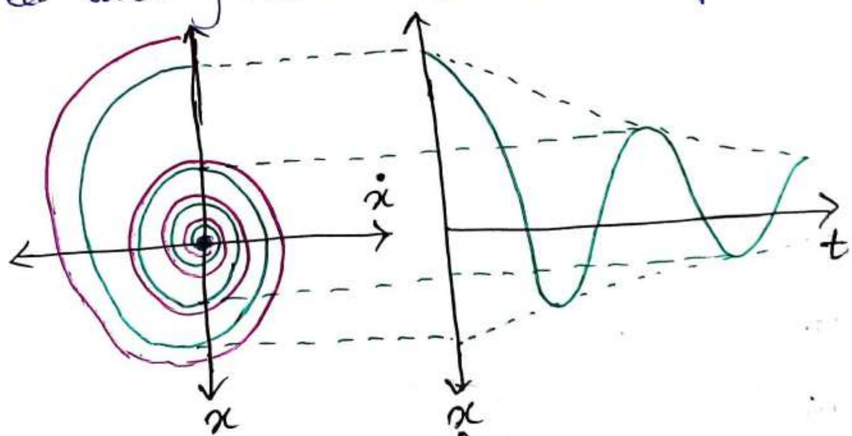
For $x(0)>0$, $f(x)$ diverges at $t = x(0)^{-1}$ so that phase point disappears at a finite time. which happens only in a nonlinear higher order dynamics. For $x(0)<0$, $f(x)$ never diverges and $f(x) \sim -t^{-1}$ for $t \to \infty$.

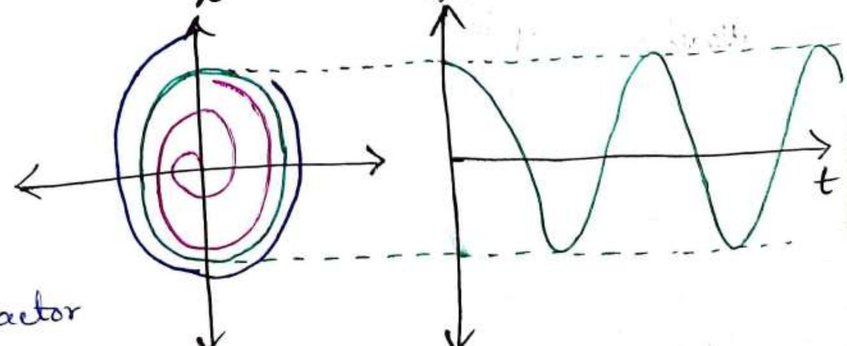


Suppose now $f(x) = \sin x$ with cp's at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$
To know the nature of the cp, we have seen in the earlier nonlinear example that because of the continuity of the phase flow lines, stable & unstable cp should alternate because physically there cannot be two adjacent stable or unstable cp because the system cannot decide where to flow in equilibrium.
Now $\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and so in the vicinity of $x=0$, $f(x) = \sin x \simeq x$. which is a repellor. In the neighbourhood of $x=\pi$, casting $u = x-\pi$, we have $f(u) = u+\pi - \frac{(u+\pi)^3}{3!} + \frac{(u+\pi)^5}{5!} - \cdots$
$\simeq u(1-\pi^2/2)$ which immediately tells $x=\pi$ is an attractor.



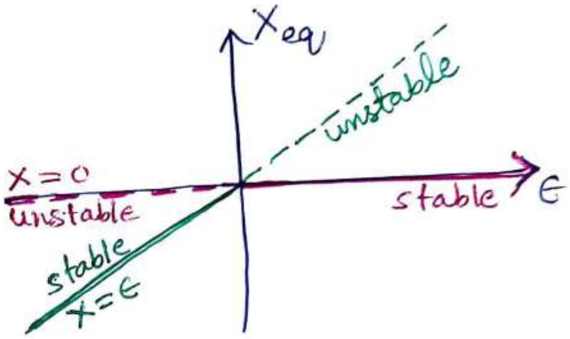
We refocus back to damped and forced oscillator example.
The elliptical trajectories become spirals that end at an "attractor" (point). in the weak damping limit for a damped oscillator. For a harmonically excited viscously damped oscillator, trajectories approach asmptotically to closed curves ("limit cycles") which also is an attractor. In 2D, point attractor





and limit cycles are the only possible attractor. The situation for nonlinear equation is different, e.g. Duffing oscillator exhibit chaotic behaviour.

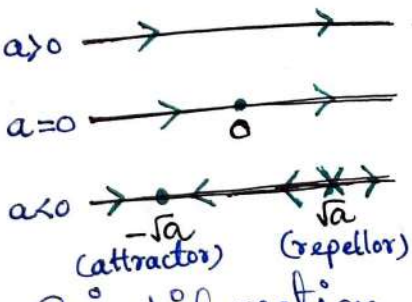
Bifurcation Diagram: Reconsider the exchange of stability bifurcation $f(x) = x(x-\epsilon)$ with $\epsilon = 0$ and $\epsilon \neq 0$. If we draw $x_{eq}-\epsilon$ diagram, then we have two lines $x=0$ & $x=\epsilon$. If we draw unstable with dotted line, then $x=\epsilon$ for $\epsilon>0$ and $x=0$ for $\epsilon<0$ should be dotted line:



Saddle Node Bifurcation

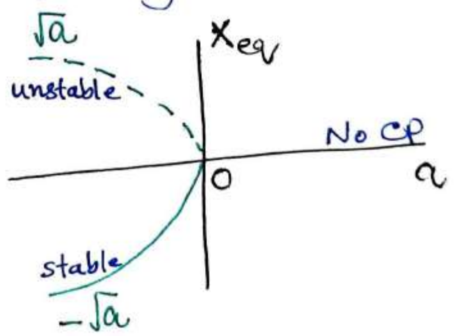
In this class of bifurcation, pairs of critical points are created one of them is stable and the other one unstable, as one varies the bifurcation parameter (which is ϵ in the example of exchange of stability bifurcation).

Consider the 1st order system $\dot{x} = a + x^2$, a is the bifurcation parameter that is $a=0$ and $a\neq 0$ (both signs). For $a>0$, for both signs of x no critical points exist. For $a=0$ the system has a degenerate higher order critical point at $x=0$. For $a<0$ the system has two critical points $x = \pm\sqrt{a}$, $\sqrt{a}$ points act as a repellor and $-\sqrt{a}$ acts as an attractor. The phase line is portrayed below.



phase portrait

bifurcation diagram



This bifurcation is also called as "turning point bifurcation" or a "fold bifurcation" or a "blue sky bifurcation".

Stability can be checked by determining the linear stability.