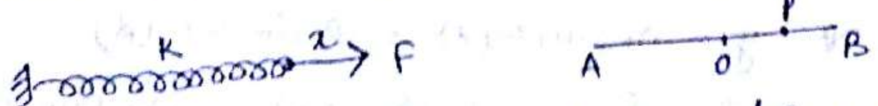
SHM

Motion : Translation, rotation, vibration/oscillation

periodic motion $f(t) = f(t+T)$ e.g. $\sin \frac{2\pi t}{T}$, $\cos \frac{2\pi t}{T}$

if periodic over same path → oscillatory motion

elasticity & inertia



SHM is a special type of periodic motion where restoring force on particle is proportional to displacement & directed to the mean position.

oscillation between point A & B, mean position O. at time t, particle is at P & displacement is x. F- restoring force

$F \propto -x$ or $F = -kx$ or $ma = -kx$ "small oscillation approximation"

$\therefore \quad a = -\frac{k}{m}x = -\omega^2 x$

Characteristics (1) linear motion → to-n-fro in straight line.
(2) $F \propto -x$.

linear harmonic motion ↔ angular harmonic motion.
(pendulum) (torsional pendulum)
$F \propto -x$ $\tau \propto -\theta$

complete oscillation : one point to same point. (time period)

amplitude : maximum displacement on both sides

frequency : no. of oscillations in 1 second.

phase : displacement, velocity, accelaration & direction of motion. After 1 oscillation, phase is same.

$t=0$, initial phase.

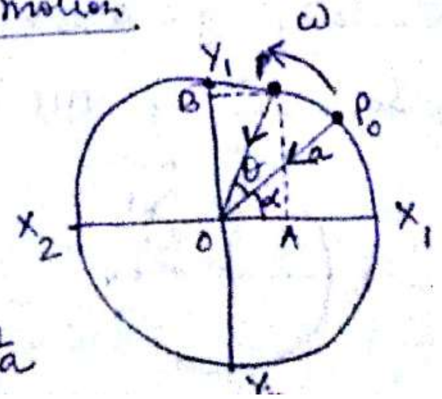
Relation between SHM & uniform circular motion.

$\theta = \omega t$
$s = a\theta$

$OA = x$, $OB = y$.

$= OP\cos(\theta+\alpha) = a\cos(\theta+\alpha)$

$= a\cos(\omega t+\alpha)$



speed $v = \omega a$, centripetal acc $f_r = \frac{v^2}{a} = \omega^2 a$

Accelaration of A is component of f_r along X_1OX_2

$$f_A = -f_r \cos(\omega t+\alpha) = -\omega^2 a \cos(\omega t+\alpha) = -\omega^2 x$$

$$\therefore \underline{f_A \propto -x.}$$

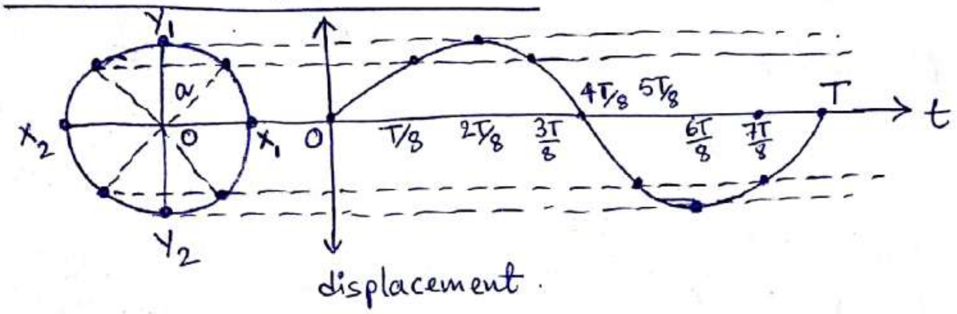
Similarly, $OB = y = OP\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

Accelaration of B is $f_B = -f_r \sin(\theta+\alpha) = -\omega^2 a \sin(\omega t+\alpha) = -\omega^2 y$

$$\therefore \underline{f_B \propto -y.}$$

$\therefore$ SHM is defined as the projection of uniform circular motion along diameter of circle.

<u>Graphical representation</u>

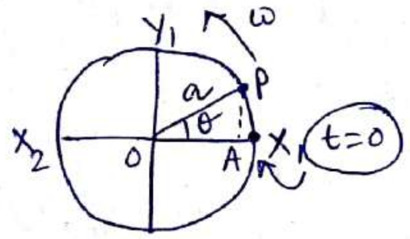


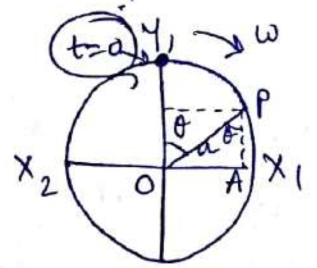
Time period = T.

$$y = a\sin\frac{2\pi}{T}t$$

(SHM along y-axis)

<u>Displacement</u> In SHM, displacement at time t is the distance of the particle from the mean position.





$OA = OP\cos\theta$

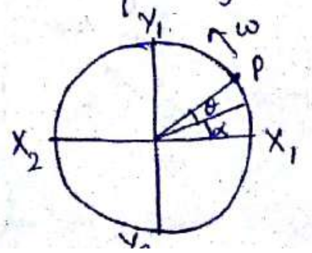
$x = a\cos\omega t$

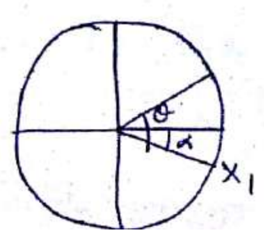
$OA = OP\cos(\pi/2 - \theta)$

$x = a\sin\theta = a\sin\omega t$

Similarly, $y = a\cos\omega t$ & $y = a\sin\omega t$.

So, eq^n of SHM can be derived from any instant t.





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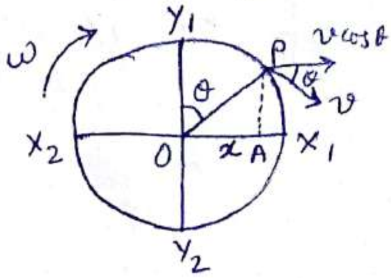
$x = a\cos(\theta + \alpha) = a\cos(\omega t + \alpha)$

Similarly, $x = a\sin(\theta + \alpha) = a\sin(\omega t + \alpha)$.

if initial position is X_1 (2^{nd} pic) then $x = a\cos(\omega t - \alpha)$

or $x = a\sin(\omega t - \alpha)$



Velocity & accelaration

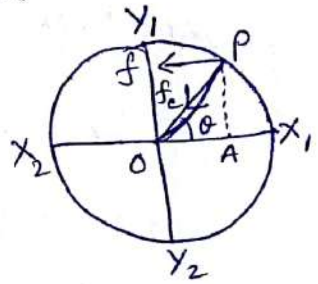
velocity of SHM is component of the particle's velocity along x-axis at time t.

$V = a\omega$, V parallel to OA $= V\cos\theta$ $\qquad x = a\sin\theta$

$= a\omega\cos\theta = a\omega\sqrt{1 - \frac{x^2}{a^2}}$

$\therefore \boxed{v = \omega\sqrt{a^2 - x^2}}$

v_{max} is at $x = 0$, $v_{max} = a\omega$. & $x = a$, $v_{min} = 0$.



Same with accelaration ⇒ SHM is the projection along X-axis is component of accelaration along X-axis. $f_c = -\omega^2 a$ & component arouth X_1X_2 is $-\omega^2 a\cos\theta = -\omega^2 a\cos\omega t = -\omega^2 x$.

$\therefore f = -\omega^2 x$.

$f_{max} = -\omega^2 a$ when $x = \pm a$, $f_{max} = \pm\omega^2 a$.

$f_{min} = 0$ when $x = 0$.

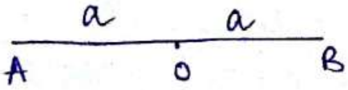
Calculus : $x = a\sin\omega t$, $v = \dot{x} = a\omega\cos\omega t = a\omega\sqrt{1 - \frac{x^2}{a^2}}$

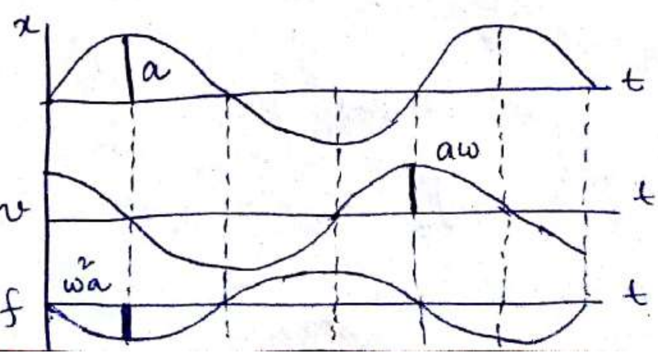
$= \omega\sqrt{a^2 - x^2}$.

$f = \ddot{x} = -a\omega^2\sin\omega t = -\omega^2 x$.

$\omega^2 = f/x$ (neglect)

Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{f}}$



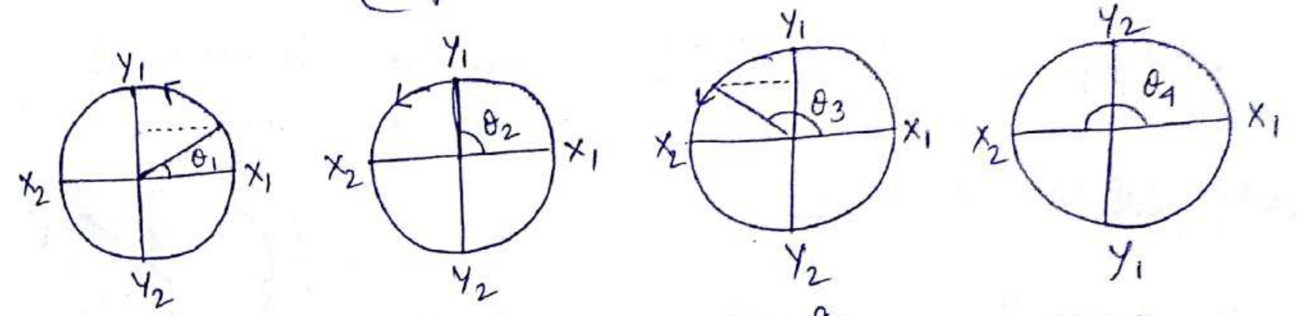


$x = a\sin\omega t = a\sin\frac{2\pi}{T}t$

$v = a\omega\cos\omega t = a\omega\cos\frac{2\pi}{T}t$

$f = -a\omega^2\sin\omega t = -a\omega^2\sin\frac{2\pi}{T}t$

Phase you see, a & ω (angular velocity) are constant.
(amplitude) $\theta = \omega t$ is changing = phase.



$y = a/2$, $\theta_1 = 30^\circ$, v upwards

$y_1 = a$, $\theta_2 = 90^\circ$, $V = 0$

$y = a/2$, $\theta_3 = 150^\circ$, V = downwards

$y = 0$, $\theta_4 = 180^\circ$, v = downwards.

phase difference 2 particles. $\phi = \theta_1 - \theta_2 = 0$ (in phase)
$= 180^\circ$ (out of phase)

Differential form & solution Homogeneous, 2nd order, ODE with constant coefficients

$F = -kx$ or $m\ddot{x} = -kx$ or $\ddot{x} + \omega^2 x = 0$, $\omega = \sqrt{\frac{k}{m}}$.

Solution: Multiply by $2\dot{x}$, $2\dot{x}\ddot{x} + 2\omega^2 x\dot{x} = 0$

Integrating $\dot{x}^2 = -\omega^2 x^2 + C$

when displacement is maximum, $x = a$, $\dot{x} = 0 \Rightarrow C = \omega^2 a^2$

$\therefore \quad v = \dot{x} = \pm\omega\sqrt{a^2 - x^2}$

or $\pm \dfrac{dx}{\sqrt{a^2 - x^2}} = \omega dt$, Integrating $\sin^{-1}\dfrac{x}{a} = \omega t + \phi$

or $x = a\sin(\omega t + \phi)$

See, $x = a\cos(\omega t + \phi)$ also satisfy $\ddot{x} + \omega^2 x = 0$.

$x = a\sin(\omega t + \phi) = a\sin\omega t\cos\phi + a\cos\omega t\sin\phi$
$= A\sin\omega t + B\cos\omega t$.

In operator form, $\dfrac{d^2x}{dt^2} = D^2 x$, $\dfrac{dx}{dt} = Dx$

$D^2 x + \omega^2 x = 0$ or $D^2 = -\omega^2$ or $D = \pm i\omega$

$\therefore$ General solution $x = Ae^{i\omega t} + Be^{-i\omega t}$

For real value of x, $A = B^*$ $\quad A = a+ib, \quad B = a-ib$

you can also have $x = a e^{i(\omega t+\phi)}$

sinusoidal or cosinusoidal.

CW 1. Oscillatory motion of a particle is represented by $x = a e^{i\omega t}$. Establish the motion is SHM. Similarly if $x = a\cos\omega t + b\sin\omega t$ then SHM.

$$x = a e^{i\omega t}, \quad \dot{x} = a i\omega e^{i\omega t}, \quad \ddot{x} = -a\omega^2 e^{i\omega t} = -\omega^2 x \quad (\text{SHM})$$

$$x = a\cos\omega t + b\sin\omega t, \quad \dot{x} = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \quad (\text{SHM}).$$

2. Which periodic motion is not oscillatory?

→ earth around sun or moon around earth.

3. Dimension of force constant of vibrating spring.

$$F = -Kx \qquad [K] = \frac{[\text{Force}]}{[\text{displacement}]} = \frac{[\text{Newton}]}{[\text{Metre}]}$$

also called "stiffness".

$$= \frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$$

HW 1. In SHM, displacement is $x = a\sin(\omega t+\phi)$. at $t=0$, $x = x_0$ with velocity v_0, show that $a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$ & $\tan\phi = \frac{\omega x_0}{v_0}$.

2. Particle is vibrated at frequency 5 Hz in SHM. Show that when displacement exceeds 10^{-2} metre, the particle loses contact with the vibrator. Given $g = 9.8\ m/s^2$

3. In SHM, a particle has speed 80 cm/s & 60 cm/s with displacement 3 cm & 4 cm. Calculate amplitude of vibration

Energy of a particle in SHM

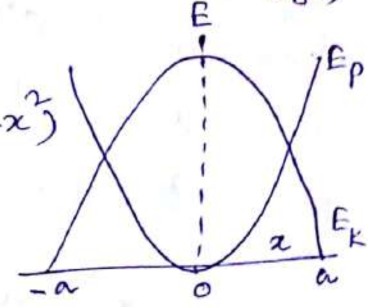
Work is done on particle to displace → restoring force. So P.E. in spring stored & motion is K.E. Total energy constant

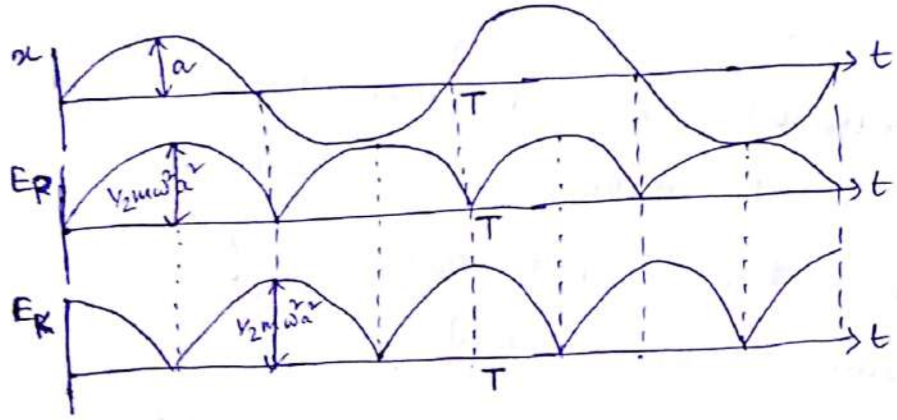
P.E. $F = mf = -m\omega^2 x$ $\therefore$ $dW = F\,dx = m\omega^2 x\,dx$ (against so no −ive sign)

$\therefore$ $E_p = \int_0^x m\omega^2 x\,dx = \frac{1}{2} m\omega^2 x^2.$

K.E. $v = \omega\sqrt{a^2 - x^2}$, $E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2 - x^2)$

$E_{Tot} = E_K + E_p = \frac{1}{2}m\omega^2 a^2 = \text{constant}.$





Examples of SHM

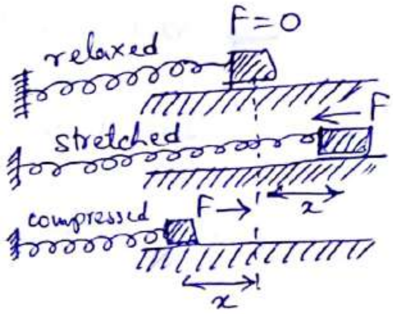
Horizontal oscillations

$F = -kx = m\ddot{x}$

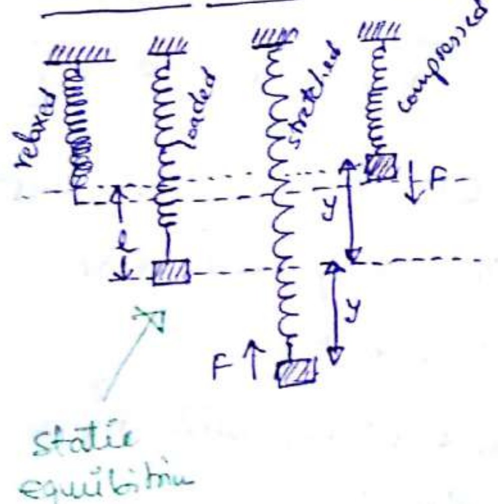
$\ddot{x} + \omega^2 x = 0$ $\quad \omega = \sqrt{\frac{k}{m}}$

$x = A\cos(\omega t + \phi)$, $T = 2\pi\sqrt{\frac{m}{k}}$ → material

initial condⁿ.



Vertical oscillations



Tension on spring $F_0 = kl$

Force on mass $= mg$.

Static eq. $mg = kl$.

stretched $\quad$ Tension on spring $= k(l+y)$

$\cancel{mg} - F = k(l+y) = kl + ky$

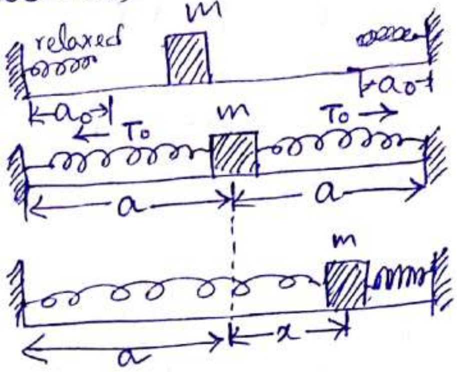
$= \cancel{mg} + ky$

$F = -ky.$

compressed $\quad mg + F = k(l-y) = mg - ky$

$F = -ky.$

Two spring system (Longitudinal oscillations)



horizontal frictionless surface,
rigid wall, massless spring,
relaxed length a_0.
After connection, static equilibrium

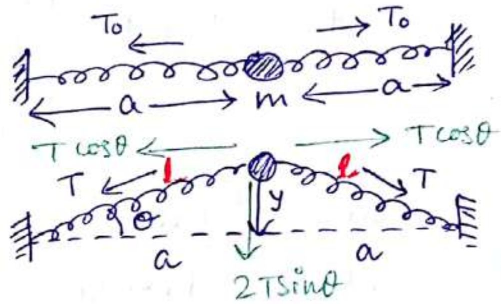
$$T_0 = K(a - a_0)$$

x = displacement to right. restoring force by left spring $-K(a+x-a_0)$
force on right spring $K(a-x-a_0)$

$$\therefore \quad F_x = K(a-x-a_0) - K(a+x-a_0) = -2Kx$$

$$m\ddot{x} = -2Kx \quad \text{or} \quad \ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{long} = 2\pi\sqrt{\frac{m}{2K}}}$$

Two spring system (Transverse oscillations)



$$T_0 = K(a - a_0)$$

$$T = K(l - a_0)$$

$$F_y = -2T\sin\theta = -2T\frac{y}{l}$$

$l = \sqrt{y^2+a^2}$

$$\text{or} \quad m\ddot{y} + \frac{2T}{l}y = 0 \quad \text{or} \quad \ddot{y} + \omega^2 y = 0$$

$$\omega^2 = \frac{2T}{ml} = \frac{2K(l-a_0)}{ml}, \quad \text{but} \quad l = f(y).$$

So $\ddot{y} + \frac{2K}{m}\left(1 - \frac{a_0}{f(y)}\right)y = 0$ is not a SHM.

(a) Slinky approximation $a \gg a_0$ or $\frac{a_0}{a} \ll 1$.

$$\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{l}\right) = \frac{2K}{m}\left(1 - \frac{a_0}{a}\frac{a}{l}\right) \quad \text{as } l > a.$$

$$= \frac{2K}{m}. \quad \text{Then SHM.} \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K}}}$$

Spring

"large" harmonic oscillations

(b) small oscillation approximation $a \not\gg a_0$ but $y \ll a$ or l.

$$\therefore l = \sqrt{y^2+a^2} = a\sqrt{\frac{y^2}{a^2}+1} \sim a.$$

Then also $\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{a}\right)$ or $\boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K\left(1-\frac{a_0}{a}\right)}}}$

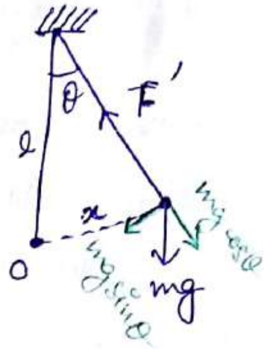
Rubber

SHM.

$$\therefore \quad T_{long} = \sqrt{1 - \frac{a_0}{a}}\; T_{trans}.$$

So longitudinal is faster than transverse.

Simple pendulum



$F' = mg\cos\theta$
(tension in string)

$\left[\lim_{\theta \to 0}\right]$

$F = -mg\sin\theta$

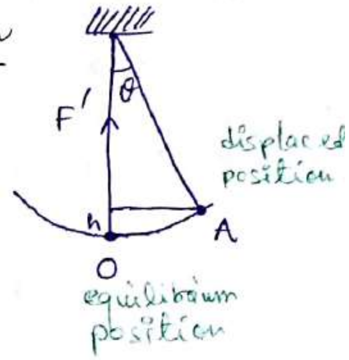
(restoring force) $= -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \simeq -mg\theta$

$x = l\theta$

or, $m\ddot{x} = -mg\frac{x}{l}$ or $\ddot{x} + \frac{g}{l}x = 0.$

$\therefore \omega = \sqrt{g/l}, \quad T = 2\pi\sqrt{\frac{l}{g}}.$ (mass independent)

String tension when pendulum at mean position



$F' = mg + \frac{mv^2}{l}$
(centrifugal force)

at A, Energy $= KE + PE = 0 + mgh = mgh$

at O, Energy $= KE + PE = \frac{1}{2}mv^2 + 0 = \frac{1}{2}mv^2$

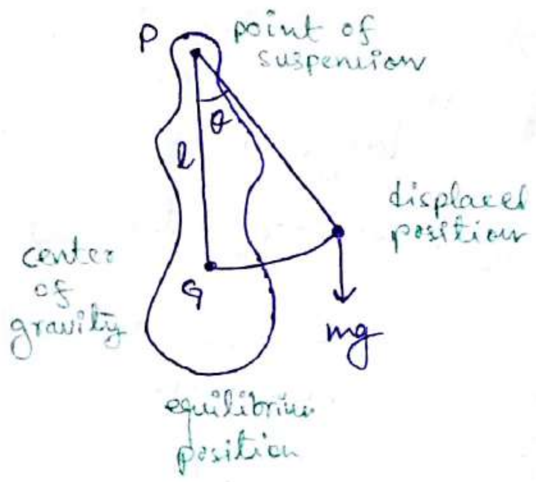
Conservation of energy $\Rightarrow \frac{1}{2}mv^2 = mgh$ or $v^2 = 2gh$.

or $v^2 = 2g(l - l\cos\theta) = 2gl(1-\cos\theta) = 2gl \times 2\sin^2\frac{\theta}{2}$

$\simeq 4gl\left(\frac{\theta}{2}\right)^2 = gl\theta^2.$

$\therefore F' = mg + \frac{m}{\cancel{l}}g\cancel{l}\theta^2 = mg(1+\theta^2).$

Compound Pendulum



arbitrary shaped rigid body oscillating about a horizontal axis passing through it.

restoring force $\Leftrightarrow$ reactive couple or torque

moment of restoring force
$= -mgl\sin\theta$

angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$, moment of inertia $= I$.

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

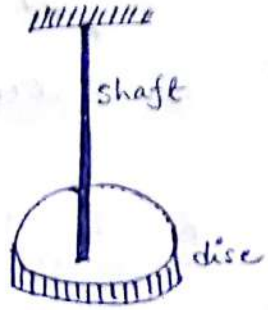
or $$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\sin\theta \simeq -\frac{mgl}{I}\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

If we consider moment of inertia about a parallel axis through G, K = radius of gyration then using parallel axis theorem,

$$I = mK^2 + ml^2 \quad \therefore T = 2\pi\sqrt{\frac{K^2/l + l}{g}} = 2\pi\sqrt{\frac{l'}{g}}.$$

equivalent length of simple pendulum $= \frac{K^2}{l} + l$.



Torsional Pendulum

twist of shaft → torsional oscillations

torsional couple $= -\tau\theta$

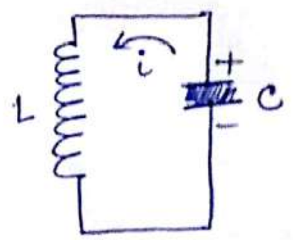
couple due to accelaration $= I\frac{d^2\theta}{dt^2}$

$$I\frac{d^2\theta}{dt^2} = -\tau\theta \quad , \quad T = 2\pi\sqrt{\frac{I}{\tau}}$$

From classical mechanics course, $\tau = \frac{\pi\eta d^4}{32l} = \frac{\pi\eta r^4}{2l}$

$d =$ shaft diameter, $\eta =$ modulus of rigidity,
$= 2r$.

Electrical Oscillator



Capacitor is charged ⇒ electrostatic energy in dielectric media. It discharges through the inductor electrostatic energy ⇔ magnetic energy. (no dissipation of heat)

voltage across inductor $= -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$ $\quad$ $q =$ charge

voltage across capacitor $= \frac{q}{C}$.

No e.m.f. circuit, $\frac{q}{C} = -L\frac{d^2q}{dt^2}$ or $\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$

$\omega^2 = \frac{1}{LC}$, $q = q_0\sin(\omega t + \phi)$. charge on capacitor varies harmonically.

$$i = \frac{dq}{dt} = \omega q_0 \cos(\omega t + \phi)$$

$$V = \frac{q}{C} = \frac{q_0}{C} \sin(\omega t + \phi)$$

Total energy = magnetic energy + electric energy

$$= \int iV\,dt + \frac{1}{2}CV^2 = \int i L \frac{di}{dt} dt + \frac{1}{2}CV^2$$

$$= \int L i\,di + \frac{1}{2}CV^2 = \frac{1}{2}Li^2 + \frac{1}{2}CV^2 = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}CV^2$$

In mechanical oscillation, Total energy $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2$

equivalence

$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

In electrical oscillation, Total energy $= \frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2$