PHYSICAL OPTICS

(Diffraction and Holography)

Books*:

1. Opticks → Ghatak (6th Edition, Tata Mc Graw Hill)
   ⇒ Standard textbook, Good for first time readers.
2. Introduction to Geometrical and Physical Optics → B.K. Mathur (Old Book) ⇒ Good for concept building and theory learning.
3. Fundamental of Optics (Tata McGraw Hill) → Jenkins & White ⇒ Concise book, good for Problem solving.
4. Principles of Optics (Pergamon Press) → Born & Wolf ⇒ Very good book for theory learning.
5. Feynman lectures on Physics Vol-1 → Feynman/ Leighton/ Sands (Narosa) ⇒ Short and concise for concept building.
6. Optics → Hecht (Addison Wesley) ⇒ Good for problem solving and first time readers.
7. Introduction to Holography → Toal (CRC Press) ⇒ New age book for basic holography principles.

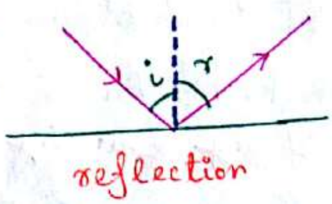
Opticks → Geometrical Optics
Opticks → Physical Optics.

Geometrical Optics deals with refraction and reflection at surfaces, lenses, Matrix method, dispersion through prism, Aberrations and eyepieces and it terms on the particle (corpuscular) theory of light using Fermat's principle. Physical optics on the other hand deals with wave theory of light as Fresnel-Hugen's principle and discusses on Interference and Coherence, Diffraction, Polarisation (crystal optics), Fiber optics and Holography.

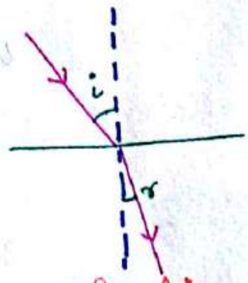
DIFFRACTION

Fermat's principle says that when a ray of light goes from one point to another through a set of media, it always follow a path along which the time taken is minimum.

$\frac{dt}{dx} = 0$ yields the "law of reflection" $i = r$. and the "Snell's law of refraction" $v_1 \sin i = v_2 \sin r$ by conservation of the horizontal component of momentum.



reflection

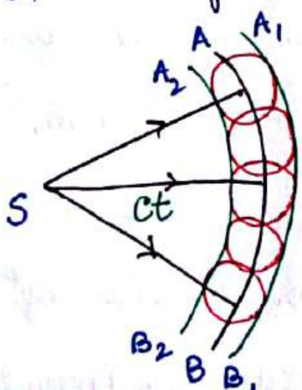


refraction

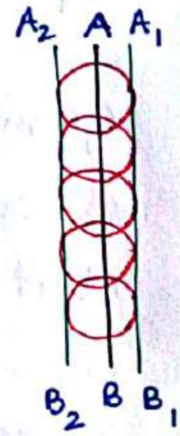
The corpuscular model of light establish the rectilinear (straight line) propagation of light and propagation of light through vacuum.

Wave theory and Huygens-Fresnel principle

A source of light transmit wave that contain energy in all directions. A "wave front" is defined as the locus of all points which are in the same state of vibration (same phase). For example, circular ripples spreading out in a pond if a pebble is dropped, each circumferential point oscillating at same amplitude & same phase. Similarly for a light source, at a nearby location $x = ct$ where AB is a spherical wavefront, while at large distance, AB is a plane wavefront. Surface AB is called "primary wavefront". The direction in which the wave is propagated is known as "ray" which is perpendicular to the wavefront.



spherical wavefront



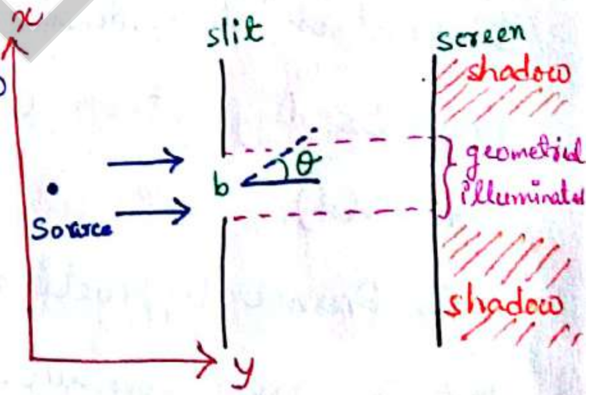
plane wavefront

Hygen-Fresnel principle tells that all points on the primary wavefront are considered to be the centres of disturbance and they transmit secondary waves in all direction with the same velocity as the primary. So A_1B_1 surface that touch the spheres after ct_1 distance is the "Secondary wavefront"

Using Huygen-Fresnel principle, law of reflection $(i = r)$, law of refraction $(v_1 \sin i = v_2 \sin r)$, refraction of spherical wave at concave spherical surface $\left(\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{R}\right)$ and convex sphereical surface $\left(\frac{\mu-1}{R} = \frac{1}{v} - \frac{1}{u}\right)$, lens formula for thin convex / concave lens $\left(\frac{1}{f} = (\mu-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right)$ can be obtained.

Why Diffraction? Wave-particle duality as in deBroglie's matter wave theory $\lambda = \frac{h}{p}$ gives rise to Heisenberg's uncertainty principle $\Delta x \Delta p_x \geq h$.

If we illuminate a single slit (narrow opening) and if light propagation is rectilinear then there is no bending of light in the geometrical shadow.



But if a light quanta (photon) or electron pass through slit, then $\Delta x \sim b$, so $\Delta p_x \sim \frac{h}{b}$. As $p_x = p \sin\theta$

, So $\sin\theta \sim \frac{h}{pb} \sim \frac{\lambda}{b}$.

When $b \gg \lambda$, $\sin\theta \to 0$ or almost no bending in geometrical shadow, while for $b \sim \lambda$ then there will be significant bending.

The bending of light round corners and spreading of light waves into the geometrical shadow of an object is called Diffraction.

Difference between Interference & Diffraction

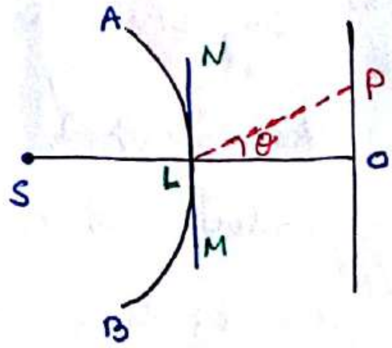
Interference	Diffraction
1. Result due to superposition of light from two different wavefront emanating from the same source.	1. Result due to superposition of light from different parts of the same wavefront.
2. Fringes may/may-not be of same width.	2. Fringes are never of same width
3. All bright bands are of uniform intensity	3. All bright bands are of different intensity
4. Points of minimum intensity are perfectly dark.	4. Points of minimum intensity are not perfectly dark.

Classification of Diffraction

Diffraction phenomena are divided into two distinct classes, as Fresnel's diffraction (near field) and Fraunhofer diffraction (far field).

In Fresnel diffraction, source of light & screen are at finite distance from aperture. No concave/convex lenses are used so that incident wavefront is either spherical/cylindrical but not planar. So phase of secondary wavefront isn't same in the plane of aperture.

Fresnel's assumptions



(a) A wavefront is divided into a large number of small area (Fresnel's zone). Secondary waves originating from various zones will interfere and the resultant effect can be noted at point P on the screen.

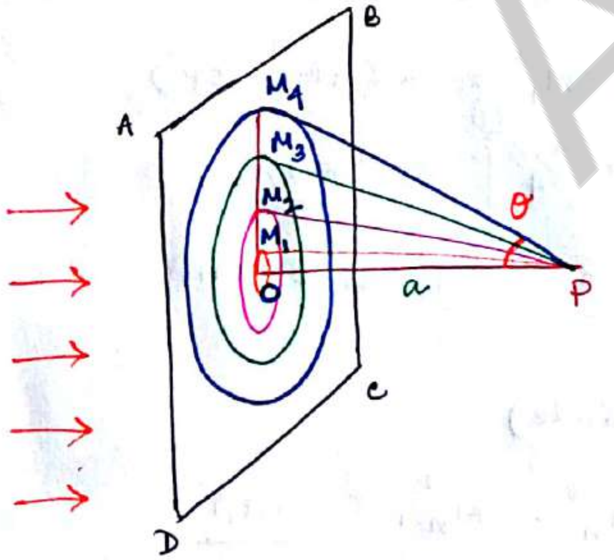
(b) Resultant at P due to a particular zone will depend on the distance of the point from the zone.

(c) Resultant at P will also depend on obliquity factor, which is proportional to $(1+\cos\theta)$. So for a wavefront at L, maximum at O occurs for $\theta = 0$, while in LN or LM direction, intensity is half of O, as $\theta = \pi/2$. Along LS, $\theta = \pi$, so no intensity in reverse direction. (Zone plate)

Fraunhofer diffraction occur when source of light/screen are effectively infinite distances from aperture. Two convex lenses are used & incident wavefront is plain. Secondary wavelet from exposed portion of the wavefront at aperture are in the same phase at all points in plane of the aperture.

(plane transmission grating, concave reflection grating)

Fresnel's half-period zone of a plain wavefront



- First half period zone $a + \lambda/2$
- Second half period zone $a + \lambda$
- Third half period zone $a + 3\lambda/2$
- fourth half period zone $a + 2\lambda$

Let us consider a plane wavefront of a monochromatic light at any particular instant. We want to find out the resultant amplitude at P due to all the wavelets coming from this wavefront.

According to Huygen's principle, every point on the plane wavefront may be regarded as the origin of the secondary wavelets & therefore the resultant effect at P due to the whole wavefront will be equal to the resultant of all these secondary wavelets.

The wavefront is divided into a number of Fresnel's half period zones - from P drop a perpendicular on ABCD at O (pole of the wave). Let $OP = a$ and P as centre & radius $(a+\lambda/2)$, draw a sphere cutting the wavefront in a circle at M_1,

$PM_1 = a + \lambda/2$ so that the secondary wavelets from O & from the points on the circumference of M_1 on reaching P will differ in phase by $\frac{2\pi}{\lambda}(PM_1 - OP) = \frac{2\pi}{\cancel{\lambda}} \times \frac{\cancel{\lambda}}{\cancel{2}} = \pi = \frac{T}{2}$ (half period)

Similarly other sphere of radii $(a+\frac{2\lambda}{2}), (a+\frac{3\lambda}{2}), (a+\frac{4\lambda}{2}),$ ... can be drawn that intersect at $M_2, M_3, M_4, \cdots$ so that the whole wavefront can be divided into several half period zones.

Amplitude due to wavelets produced by each zone is

(i) Directly proportional to the area of the zone which is approximately equal.

(ii) Varies inversely with the distance of zone from P.

(iii) Varies with the obliquity factor $(1+\cos\theta)$.

Area of 1st half period zone $= \pi OM_1^2 = \pi(PM_1^2 - OP^2)$

$= \pi[(a+\lambda/2)^2 - a^2] = \pi[a\lambda + \cancel{\frac{\lambda^2}{4}}^{\,0}] \simeq \pi a\lambda$.

Similarly $OM_n^2 = PM_n^2 - OP^2 = (a + \frac{n\lambda}{2})^2 - a^2 = an\lambda$

(n^{th} circle)

$OM_{n-1}^2 = a(n-1)\lambda$ ($(n-1)^{th}$ circle).

So Area of n^{th} zone $= \pi(OM_n^2 - OM_{n-1}^2) = \pi a\lambda$.

So
$$\boxed{\begin{array}{l} \text{radii of zone} \propto \sqrt{n} \\ \text{area of zone independent of } n \end{array}}$$

Schuster's Method :

For visible light, $\lambda \simeq$ small & so area of zone $= \pi a \lambda$ but if λ is not very small then the area of half period zones of higher order decreases gradually. If the phase of the wavelets coming from O is zero then the phase of wavelets from intermediate points between O and M_1 will vary from 0 to π (because $\frac{2\pi}{\lambda}(PM_1 - OP) = \pi$).

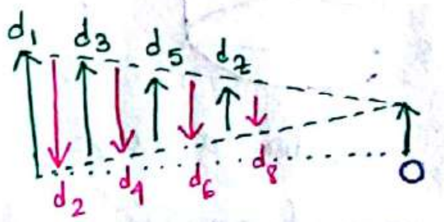
$\therefore$ Average phase of all wavelets from 1^{st} zone $= \frac{0+\pi}{2} = \frac{\pi}{2}$.

Similarly phase difference of wavelets from M_1 & M_2 will be between π and 2π, so that average phase of all wavelets from 2^{nd} zone $= \frac{\pi + 2\pi}{2} = \frac{3\pi}{2}$, from 3^{rd} zone $\frac{5\pi}{2}$, from 4^{th} zone $\frac{7\pi}{2}$ & so on...

Resultant phase-difference between two consecutive zones $= \pi$.

Resultant phase-difference between two alternate zones $= 2\pi$.

So if resultant from 1^{st} half period zone is positive then 2^{nd} half period zone is negative.



Amplitude decreases due to obliquity factor $(1+\cos\theta)$, so resultant amplitude

$$D = d_1 - d_2 + d_3 - d_4 + d_5 - \cdots \pm d_n.$$

(i) If $n =$ odd, to a first approximation $d_2 = \frac{d_1 + d_3}{2}$, $d_4 = \frac{d_3 + d_5}{2}$

so that $D = \frac{d_1}{2} + \left(\frac{d_1 + d_3}{2} - d_2\right) + \left(\frac{d_3 + d_5}{2} - d_4\right) + \cdots + \frac{d_n}{2}$

$$= \frac{d_1}{2} + \frac{d_n}{2}.$$

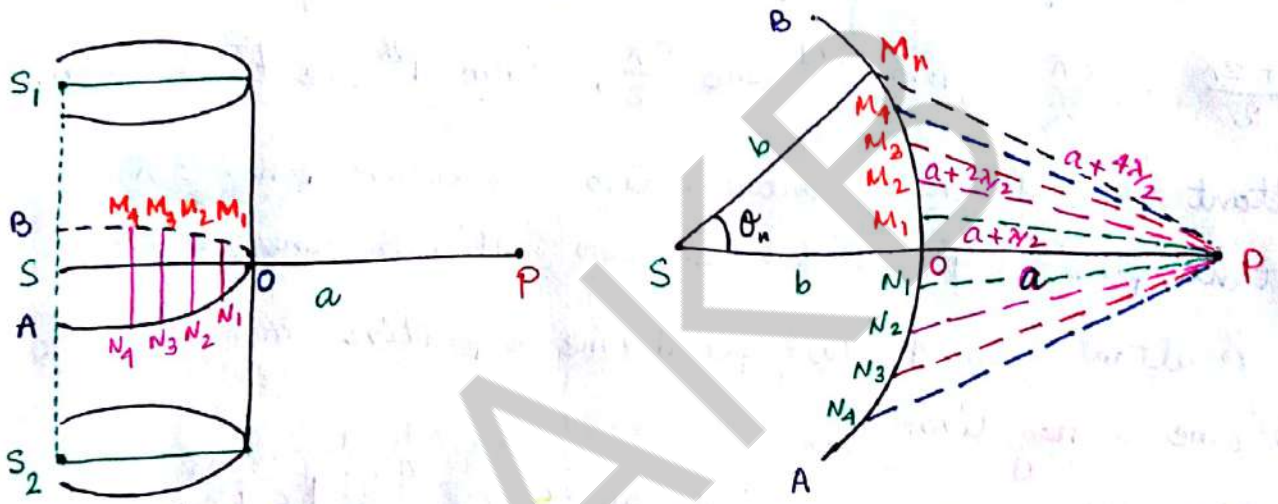
(ii) If $n =$ even, $D = \frac{d_1}{2} + \frac{d_{n-1}}{2} - d_n$

If n is very large, then effect from n^{th} zone is negligible &

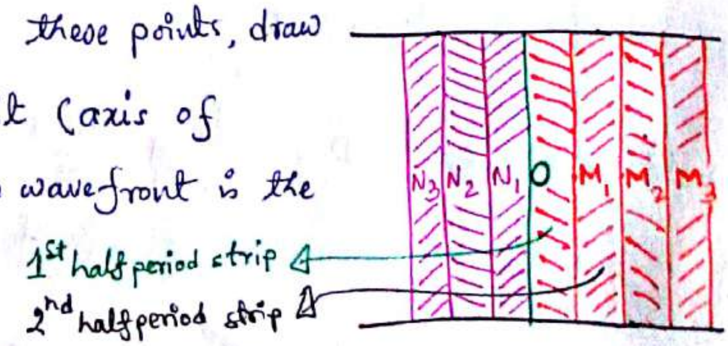
resultant amplitude due to whole wave is $D = \frac{d_1}{2}$

intensity $I = D^2 = \frac{d_1^2}{4}$. If an obstacle is placed at O then the resultant disturbance at P is = half the disturbance due to wavelets from the 1st half-period zone with one-fourth the intensity. If obstackle at O blocks a considerable number of half-period zones, effect is negligible & no light is received at P - or light travels approximately in a straight line.

Fresnel's half-period strip of a cylindrical wavefront



Consider a long and narrow slit S_1S_2, when illuminated by monochromatic light of wavelength λ, produces cylindrical wavefront. To find the resultant amplitude, the wavefront can be divided into half period strips, with O as pole. Consider an equatorial section AOB through O in plane of paper. With P as centre & radius $(a+\frac{\lambda}{2})$, $(a+\frac{2\lambda}{2})$, ... etc, draw arcs that cut AOB at point M_1N_1, M_2N_2, ... etc. Through these points, draw lines parallel to length of slit (axis of wavefront) and the area of the wavefront is the half period strip.



Amplitude of the waves reaching P due to wavelets produced by each half-period strip is

(i) Directly proportional to the area of the strip. (not equal)

(ii) Average distance of strip from P

(iii) Varies with the obliquity factor $(1+\cos\theta)$

As length of strip is same, so areas are proportional to arcs $OM_1, M_1M_2, M_2M_3, \dots$ where $PM_n = a + \frac{n\lambda}{2}$

from triangle PM_nS, we have $PM_n^2 = SM_n^2 + PS^2 - 2SM_n PS\cos\theta_n$

$$\text{or } \left(a+\frac{n\lambda}{2}\right)^2 = b^2 + (a+b)^2 - 2b(a+b)\underbrace{\cos\theta_n}_{(1-\theta_n^2/2!)}$$

$$\text{or } \cancel{a^2} + an\lambda + \cancel{\frac{n^2\lambda^2}{4}}^{\to 0} = \cancel{2b^2} + \cancel{a^2} + \cancel{2ab} - \cancel{2ab} - \cancel{2b^2} + b(a+b)\theta_n^2$$

$$\text{or } an\lambda = b(a+b)\theta_n^2 \quad \text{or} \quad \theta_n = \sqrt{\frac{an\lambda}{b(a+b)}} = K\sqrt{n}$$

Now $OM_n = b\theta_n = bK\sqrt{n}$

So $OM_1 = bK$, $OM_2 = bK\sqrt{2}$, $OM_3 = bK\sqrt{3}$.

So $M_1M_2 = bK(\sqrt{2}-1) = 0.414\ bK$

$M_2M_3 = bK(\sqrt{3}-\sqrt{2}) = 0.318\ bK$

$M_3M_4 = bK(\sqrt{4}-\sqrt{3}) = 0.268\ bK$, $M_4M_5 = 0.236\ bK, \dots$

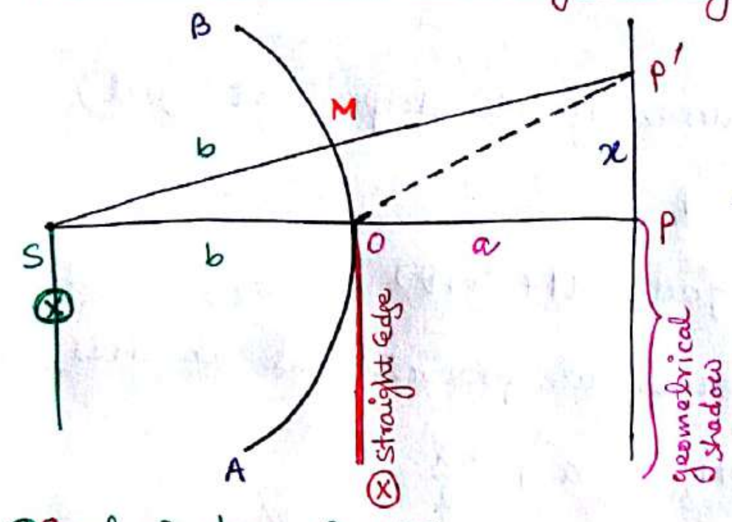
So area of strip initially decreases rapidly & then for increasing order more slowly. and because of opposite sign they cancel out each other. So the resultant at P is only due to first few half period strips.

$$D = d_1 - d_2 + d_3 - d_4 + \dots \approx \frac{d_1}{2} \text{ (left side) (from half wavefront)}$$
$$\approx \frac{d_1}{2} \text{ (from right side half wavefront)}$$

So resultant due to whole wavefront $= \frac{d_1}{2} \pm \frac{d_1}{2} = d_1$ (n odd)

$= 0$ (n even)

Diffraction at a straight Edge

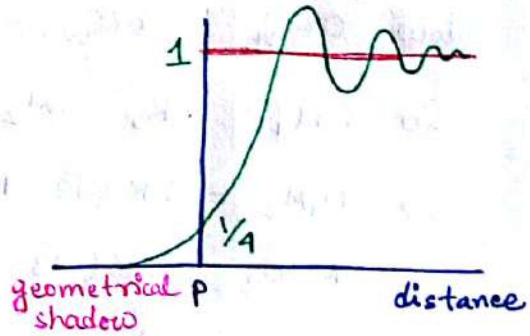


⊗ out of plane of paper

Consider a straight edge at O and an illuminated narrow slit S parallel to each other. Dark & bright bands of unequal width of varying intensity is observed in geometrical shadow. We study intensity at P' with M as pole and construct Fresnel's half-period strip. The effect at P' depends upon the number of half-period strips contained in OM & BM.

Due to straight edge, the effect at P is due to the upper half of the wavefront only, so displacement at P is $\frac{1}{2}$ of the displacement for whole wavefront or $\frac{1}{4}$ of the full wavelet intensity.

# of half-period strips contained in OM depends on the path difference $OP' - MP'$



Now $OP' = \sqrt{a^2+x^2} = a\left(1+\frac{x^2}{a^2}\right)^{1/2}$

$\simeq a\left(1+\frac{x^2}{2a^2}\right) = a + \frac{x^2}{2a}$

$SP' = \sqrt{(a+b)^2+x^2} = a+b+\frac{x^2}{2(a+b)}$

$\therefore MP' = SP' - SM = a + \frac{x^2}{2(a+b)}$

$\therefore$ Path difference $OP' - MP' = a + \frac{x^2}{2a} - a - \frac{x^2}{2(a+b)} = \frac{bx^2}{2a(a+b)}$

for the displacement to be maximum,

$$\frac{bx^2}{2a(a+b)} = (2n+1)\frac{\lambda}{2} \quad \text{or} \quad x = \left[\frac{a(a+b)(2n+1)\lambda}{b}\right]^{1/2}, \; n=0,1,2,\ldots$$

$x \propto \sqrt{2n+1}$ (bright band)

for the displacement to be minimum, $\frac{bx^2}{2a(a+b)} = n\lambda$

or $x = \left[\frac{2a(a+b)n\lambda}{b}\right]^{1/2}$, $n = 1, 2, 3$, $x \propto \sqrt{n}$ (dark band)

Using these, wavelength of light can be found.

CW A narrow slit illuminated by light of $\lambda = 5890\text{Å}$ is located at a distance of 0.1 m from a straight edge. If the measurements are made at a distance of 0.5 m from the edge, calculate the distance between 1st & 2nd dark band.

$a = 0.5\,m$
$b = 0.1\,m$
$\lambda = 5890 \times 10^{-10}\,m$

For nth dark band $x = \sqrt{\frac{2a(a+b)n\lambda}{b}}$

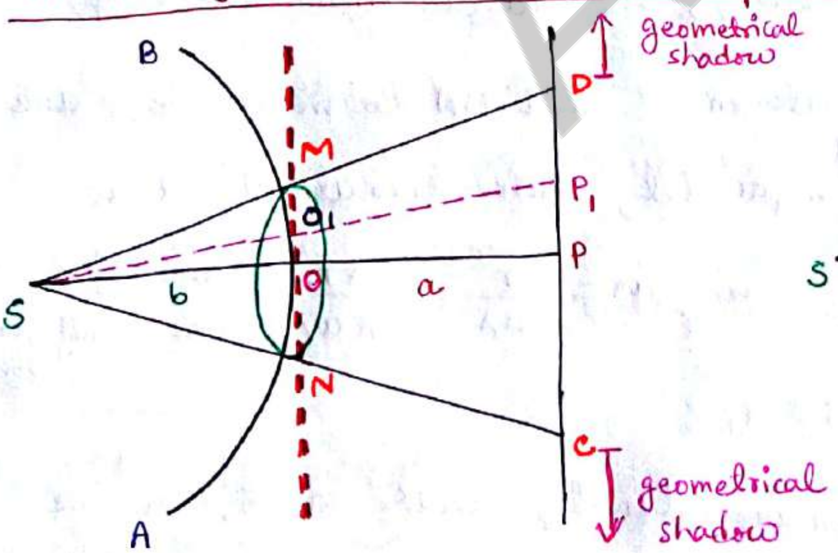
$\therefore x_2 - x_1 = \sqrt{\frac{2a(a+b)\lambda}{b}}\,(\sqrt{2} - 1)$

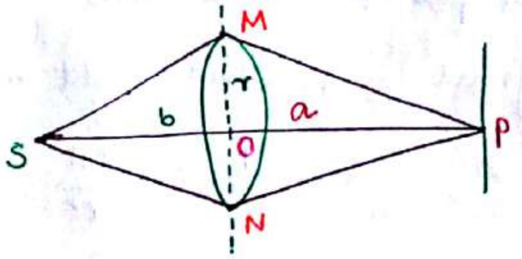
$= 0.7786 \times 10^{-3}\,m.$

Read about diffraction of light by a thin wire. Fringe width

$\beta = \frac{D\lambda}{d}$, D = distance between obstacle & crosswire of Eyepiece, λ = wavelength of light.

Fresnel's diffraction at a circular aperture





From a point source S a wavefront (spherical) touches a circular aperture MN. To calculate the amplitude at screen P, we need to divide the wavefront MON into Fresnel's half-period zones about the pole O.

Intensity at an axial point P :

If only the 1^{st} half period zone is exposed then amplitude at P is twice the amplitude if the whole wavefront is exposed, or intensity will be four times. Let the amplitude is d_1.

If the screen is moved towards the aperture so that 1^{st} & 2^{nd} half-period zones are exposed then amplitude $= d_1 - d_2 \simeq 0$ as $d_1 \simeq d_2$

So dark & bright fringes will form as more half-period zones are exposed.

Path difference for waves reaching P along SMP & SOP is

$$= (SM + MP) - (SO + OP) = \sqrt{b^2 + r^2} + \sqrt{a^2 + r^2} - (b + a)$$

$$\simeq b\left(1 + \frac{r^2}{2b^2}\right) + a\left(1 + \frac{r^2}{2a^2}\right) - (b + a) = \frac{r^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

If the aperture contains n half-period zones then path difference is $\frac{n\lambda}{2}$. So $\frac{r^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{n\lambda}{2}$ or $\frac{1}{a} + \frac{1}{b} = \frac{n\lambda}{r^2}$.

$n =$ even (minimum intensity), $n =$ odd (maximum intensity).

If we put a convex lens between S and MN to make the plane wavefront (incident light is parallel, source lies at ∞) then

$b = \infty$, or $\frac{1}{a} = \frac{n\lambda}{r^2}$ or $n = \frac{r^2}{a\lambda} = \frac{\pi r^2}{\pi a\lambda} = \frac{\text{area of aperture}}{\text{area of half-period zone}}$.

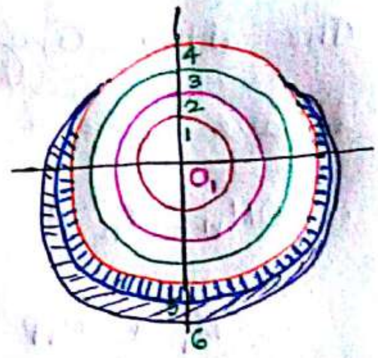
Intensity at a non-axial point P_1 :

Suppose at P we have maximum intensity with $n = 5$. As we move up to P_1, the pole shifts to O_1. Here suppose only 4 zones are completely exposed while $\frac{1}{2}$ of 5^{th} and 6^{th} zone are exposed, so that resultant displacement at P_1

$$= d_1 - d_2 + d_3 - d_4 + d_5/2 - d_6/2$$

$= \frac{d_1}{2} + \left(\frac{d_1+d_3}{2} - d_2\right) + \left(\frac{d_3+d_5}{2} - d_4\right) - \frac{d_6}{2}$

$= \frac{d_1}{2} - \frac{d_6}{2}$. So the intensity will be minimum.



If we move up to P_2 then intensity will be maximum as first 3 zones are completely exposed and $\frac{1}{2}$ of 4^{th}, 5^{th}, 6^{th}, 7^{th} zones are exposed. So there will be concentric alternating bright & dark rings.

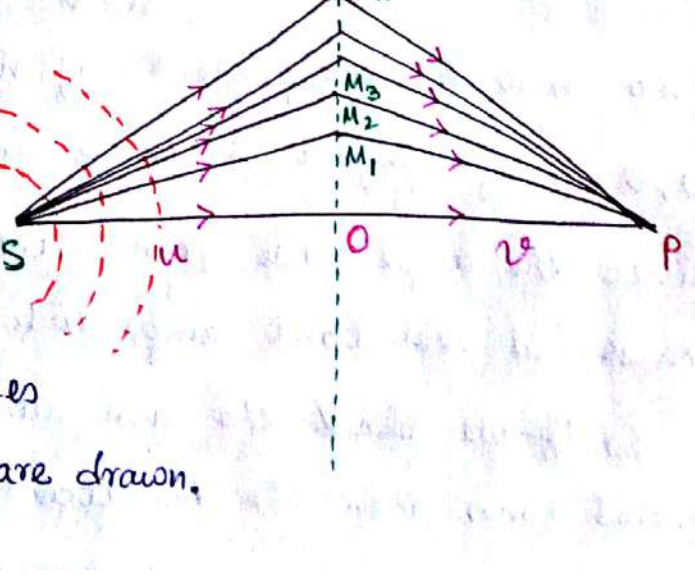
Diffraction at a circular obstackle

Here if the obstacle obstruct only the 1^{st} half-period zone then at P, displacement $= -d_2 + d_3 - d_4 + d_5 \cdots = -\frac{d_2}{2}$. $\therefore I \propto \frac{d_2^2}{4}$.

If size of obstacle is increased or point P is brought near so that 2^{nd}, 3^{rd}, ... etc zones are obstructed then displacement is $\frac{d_3}{2}, -\frac{d_4}{2},$ or $I \propto \frac{d_3^2}{4}, \frac{d_4^2}{4}, \ldots$ So P remain always bright, which is actually the geometrical shadow.

For any other point $P_1, P_2,$ etc within the geometrical shadow, diffracted waves interfere due to phase difference & produce interference band (circular). Outside the geometrical shadow, we get diffraction band of unequal width.

Zone Plate



The idea of Fresnel's half period zone can be used to construct a transparent plate on which circles with radii proportional to $\sqrt{n}$ are drawn.

The alternating annular zones are blocked, so that the plate behaves like a convex lens. So by construction $OM_1 = r_1$, $OM_2 = r_2$, ... $OM_n = r_n$ are the radius of the circles and

$$SM_1 + M_1P = SO + OP + \lambda/2$$
$$SM_2 + M_2P = SO + OP + \lambda$$
$$\vdots$$
$$SM_n + M_nP = SO + OP + n\lambda/2 \quad \text{— (1)}$$

So annular rings are half-period zones, consecutive zone differs by $\lambda/2$.

If $SO = u$, $OP = v$ then $SM_n = \sqrt{SO^2 + OM_n^2} = \sqrt{u^2 + r_n^2} \simeq u\left(1 + \frac{r_n^2}{2u^2}\right)$

$$= u + \frac{r_n^2}{2u}, \quad (u \gg r_n).$$

Similarly $M_nP = \sqrt{v^2 + r_n^2} \simeq v + \frac{r_n^2}{2v}$.

So from equation (1), $\cancel{u} + \frac{r_n^2}{2u} + \cancel{v} + \frac{r_n^2}{2v} = \cancel{u} + \cancel{v} + n\lambda/2$

or $r_n^2\left(\frac{1}{u} + \frac{1}{v}\right) = n\lambda$

Applying the sign convention $u \to -u$, $r_n^2 = \dfrac{n\lambda uv}{u - v}$

So $\boxed{r_n \propto \sqrt{n}}$ for $\lambda, u, v =$ constant.

Area of n^{th} zone $= \pi(r_n^2 - r_{n-1}^2) = \pi\left[\dfrac{n\lambda uv}{u-v} - \dfrac{(n-1)\lambda uv}{u-v}\right]$

$$= \frac{\pi\lambda uv}{u-v} \neq f(n)$$

So area is independent of n & decreases for decreasing u, v or if object or image are brought near to the zoneplate.

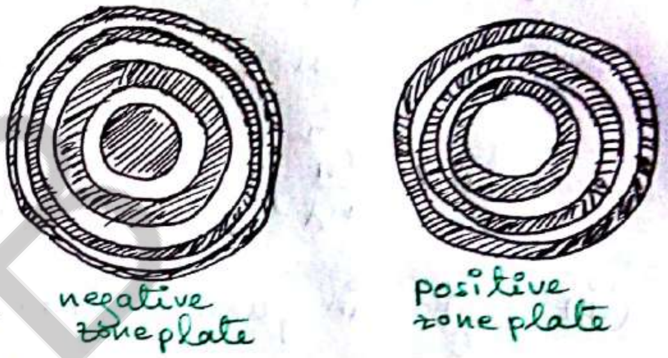
Since the amplitude from alternate zones will have opposite ~~faces~~ phases so resultant amplitude at P will be $d = d_1 - d_2 + d_3 - d_4 + \ldots$

So if we block the even number or the odd number of half period zones then the resultant amplitude at P will be either

$d = d_1 + d_3 + d_5 + \cdots$ or $d = d_2 + d_4 + d_6 + \cdots$

So intensity at P is many times brighter than that due to all exposed zones, so that light from S can be focussed at P. So the result is similar to a convex lens. When $u = \infty$, $v = f$ so that $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ where $f = \frac{r_n^2}{n\lambda}$ is the focal length of the zoneplate. So zoneplate acts like a convex lens.

Construction of zoneplate:



Zoneplate is a system of areas corresponding to half-period zones. Concentric circles with radii $\propto \sqrt{\text{natural number}}$ are drawn of a white paper/ glass. Alternate zones are painted black - if odd zones are transparent & even zones are opaque then its a positive zone plate, otherwise a negative zone plate.

Phase reversal in a zoneplate:

R.W. Wood coated the even no. zones with a thin film of transparent substance made of Gelatin mixed with $K_2Cr_2O_7$, instead of painting them black. As a result, the phase of the waves traversing the even number zones change phase π, producing intensity at P 4-fold as

$$d = d_1 + d_2 + d_3 + d_4 + \cdots$$

Such type of zoneplates are used as objectives of telescope, photograhic camera etc.

Difference between a zoneplate & convex lens

(i) For a particular wavelength, a convex lens have a single focal length, but for a zone plate, there are a number of focal length between the plate and brightes focus (multiple foci) $f = \frac{r_n^2}{n\lambda}$. So for a fixed distance of object, lens produces one image whereas zoneplate produces a number of image. Depending on the position of screen, it may contain 3 or 5 or 7 half-period zone & intensity of image decreases with decreasing focal length

$$f_1 = \frac{r_n^2}{n\lambda}, \quad f_2 = \frac{r_n^2}{3n\lambda}, \quad f_3 = \frac{r_n^2}{5n\lambda}, \ldots$$

(ii) Light in passing through the lens takes equal time to go from S to P through any part of lens whereas in a zoneplate disturbances from any transparent zone reach P one-period later than the disturbances from the next inner zone.

(iii) Focal length of a lens is $\frac{1}{f} = (\mu-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ whereas for zoneplate is $\frac{1}{f} = \frac{n\lambda}{r_n^2}$.

(iv) Focal length of lens is proportional to λ, so is greater for red rays than violet. Focal length is inversely proportional to λ for zoneplate & so is greater for violet than red rays.

Q) What is the radius of 1^{st} zone of a zoneplate of focal length 0.2 m for a light of wavelength 5000 Å?

$\lambda = 5000\,\text{Å} = 5 \times 10^{-7}\,\text{m}$, $f = 0.2\,\text{m}$, $n = 1$, So $f = \frac{r_1^2}{\lambda}$.

or $r_1 = \sqrt{f\lambda} = 3.16 \times 10^{-4}\,\text{m}$.

CW Calculate the radii of the first 3 clear elements of a zone plate which is designed to bring a parallel beam of light of wavelength 6000 Å to the first focus at a distance of 2 metres.

parallel light means $u = \infty$, $v = f$ (first focus)

$v = f = 2$ m., $\lambda = 6000$ Å $= 6 \times 10^{-7}$ m., $r_n = \sqrt{n\lambda f}$.

For 1st clear zone $n = 1$, $r_1 = \sqrt{\lambda f} = 1.095 \times 10^{-3}$ m.

For 2nd clear zone $n = 3$, $r_2 = \sqrt{3\lambda f} = 1.897 \times 10^{-3}$ m.

For 3rd clear zone $n = 5$, $r_3 = \sqrt{5\lambda f} = 2.449 \times 10^{-3}$ m.

CW A plane wavefront ($\lambda = 6000$ Å) advancing towards a point is divided into a number of half-period zones. Amplitude contribution of these half-period zones is 1, 0.98, 0.96, ... 0. Compare the intensities at the point when first 31 & 36 half period zone are only exposed.

$d_1 = 1$, $d_2 = 0.98$, $d_3 = 0.96$, ... $d_n = 0$.

So $\frac{d_1 + d_3}{2} = d_2$. , $d_1 - d_2 = 1 - 0.98 = 0.02$

$d_{31} = 1 - 30 \times 0.02 = 0.40$

$d_{35} = 1 - 34 \times 0.02 = 0.32$

$d_{36} = 1 - 35 \times 0.02 = 0.30$

So resultant amplitude of 31 zones $= \frac{d_1}{2} + \frac{d_{31}}{2} = \frac{1}{2} + \frac{0.4}{2} = 0.7$

resultant amplitude of 36 zones $= \frac{d_1}{2} + \frac{d_{35}}{2} - d_{36}$

$= \frac{1}{2} + \frac{0.32}{2} - 0.3 = 0.36$.

$\therefore \frac{I_{31}}{I_{36}} = \frac{0.7^2}{0.36^2} = 3.78$.

CW A zoneplate is found to give series of images of a point source on the axis. If the strongest and the 2nd strongest images are at distances of 0.3 m and 0.06 m respectively from the zoneplate (both on same side) calculate the distance of the source from the zoneplate, principle focal length and radius of 1st zone for $\lambda = 5\times10^{-7}$ m.

$\lambda = 5\times10^{-7}$ m, $v_1 = 0.3$ m, $v_2 = 0.06$ m.

$f_1 = \frac{r_n^2}{\lambda}$, $f_2 = \frac{r_n^2}{3\lambda}$ $\therefore f_2 = f_1/3$.

If u is the distance of the object from zone plate then

$$\frac{1}{u} = \frac{1}{v_1} - \frac{1}{f_1} \quad \text{and} \quad \frac{1}{u} = \frac{1}{v_2} - \frac{1}{f_2}$$

$$\text{or } \frac{1}{v_1} - \frac{1}{f_1} = \frac{1}{v_2} - \frac{3}{f_1} \quad \text{or } \frac{2}{f_1} = \frac{1}{v_2} - \frac{1}{v_1} = \frac{1}{0.06} - \frac{1}{0.3}$$

or $f_1 = 0.15$ m. → principal focal length

Now $r_1 = \sqrt{f_1 \lambda} = 0.274\times10^{-4}$ m. → radius of 1st zone and from

$$\frac{1}{u} = \frac{1}{v_1} - \frac{1}{f_1} = \frac{1}{0.3} - \frac{1}{0.15}, \quad u = -0.3 \text{ m.}$$

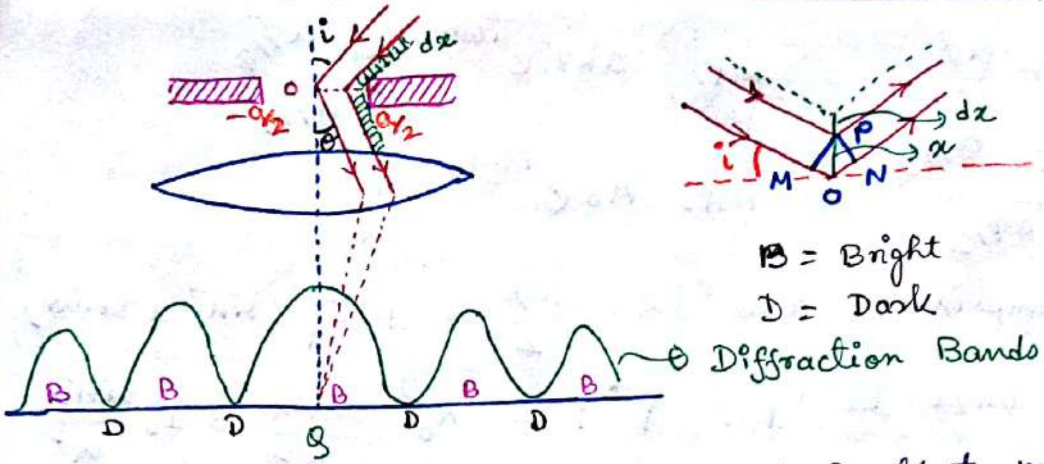
↳ distance from source to zone plate

Fraunhofer Diffraction (far-field)

Single slit diffraction : A single slit is a vacant space which is obstructed by two sharp opaque regions. When a monochromatic light is incident on such a slit, it is found that the intensity on the opposite side has a variation and maximum & minimum brightness are observed, which are called diffraction bands. This phenomena occurs due to diffraction of light.

Theory :

Let a parallel beam of monochromatic light of wavelength λ is incident on a narrow slit of width a in a direction making an angle i with the normal. After diffraction, they are scattered in various direction. We will calculate the intensity at screen due to the rays diffracted at an angle θ with the normal.



$\angle OPM = i$

$\angle OPN = \theta$.

The rays coming from O and P that meet at Q will have path difference $= OM + ON = x(\sin i \pm \sin\theta)$. So the phase difference between this two waves is $\frac{2\pi}{\lambda} x (\sin i \pm \sin\theta)$.

Let the displacement at any point Q due to secondary waves from the origin O (midpoint of the slit) is proportional to

$y =$ R.P. of $re^{i\omega t}$. [For derivation, we consider the nature of wavefront's spatial part outside the calculation]

Then the amplitude of the wave coming from the point P that meets at point Q is $y =$ R.P. $kr\,e^{i[\omega t \pm \frac{2\pi}{\lambda} x(\sin i \pm \sin\theta)]}$

$= $ R.P. $kr\,e^{i(\omega t \pm \delta)}$, $k =$ constant.

If we consider the change of phase over a small distance dx which is negligible compared to x, then the displacement at Q due to waves from a region dx after x will be given by

$$y = \text{R.P. } kr\,e^{i(\omega t \pm \delta)}\,dx.$$

$\therefore$ The total displacement at Q due to the whole slit is

$$Y = \text{R.P. } kr\,e^{i\omega t}\int_{-a/2}^{a/2} e^{\pm i\delta}\,dx$$, let $\delta = \phi x$ where $\phi = \pm\frac{2\pi}{\lambda}(\sin i \pm \sin\theta)$

$$= \text{R.P. } kr\,e^{i\omega t}\int_{-a/2}^{a/2} e^{i\phi x}\,dx = \text{R.P. } kr\,e^{i\omega t}\left[\frac{e^{i\phi x}}{i\phi}\right]_{-a/2}^{a/2}$$

$$= \text{R.P. } \frac{kr\,e^{i\omega t}}{\phi}\, 2\left[\frac{e^{ia\phi/2} - e^{-ia\phi/2}}{2i}\right]$$

$$= R.P.\ \frac{2Kr e^{i\omega t}}{\phi} \sin\frac{a\phi}{2} = R.P.\ aKr e^{i\omega t} \frac{\sin a\phi/2}{a\phi/2}$$

$$= R.P.\ A_0 e^{i\omega t} \frac{\sin a\phi/2}{a\phi/2} = R.P.\ A_0 e^{i\omega t} \frac{\sin\alpha}{\alpha}.$$

where A_0 is total amplitude and $\boxed{\alpha = \frac{a\phi}{2} = \pm \frac{a\pi}{\lambda}(\sin i \pm \sin\theta)}$

$\therefore$ Intensity at Q will be $I = Y^* Y = A_0^2 \frac{\sin^2\alpha}{\alpha^2} = I_0 \frac{\sin^2\alpha}{\alpha^2}$

which will be extremum (minimum or maximum) when

$$\frac{dI}{d\alpha} = 0. \Rightarrow \frac{d}{d\alpha}\left(I_0 \frac{\sin^2\alpha}{\alpha^2}\right) = 0.$$

or $2I_0 \frac{\sin\alpha}{\alpha} \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} = 0$ or $2I_0 \frac{\sin^2\alpha}{\alpha^2}\left(\cot\alpha - \frac{1}{\alpha}\right) = 0$

or $2I\left(\cot\alpha - \frac{1}{\alpha}\right) = 0.$

Also, $\frac{d^2I}{d\alpha^2} = 2\frac{dI}{d\alpha}\left(\cot\alpha - \frac{1}{\alpha}\right) + 2I\left(-\operatorname{cosec}^2\alpha + \frac{1}{\alpha^2}\right)$

$$= 4I\left(\cot\alpha - \frac{1}{\alpha}\right)^2 + 2I\left[-(1+\cot^2\alpha) + \frac{1}{\alpha^2}\right]$$

$$= 4I\left(\cot\alpha - \frac{1}{\alpha}\right)^2 - 2I\left(1+\cot^2\alpha - \frac{1}{\alpha^2}\right)$$

So $\frac{dI}{d\alpha} = 0$ only when (i) $I = 0$ or (ii) $\cot\alpha = \frac{1}{\alpha}$.

when $\boxed{\cot\alpha = \frac{1}{\alpha}}$, $\frac{d^2I}{d\alpha^2} = -2I < 0$ (maximum)

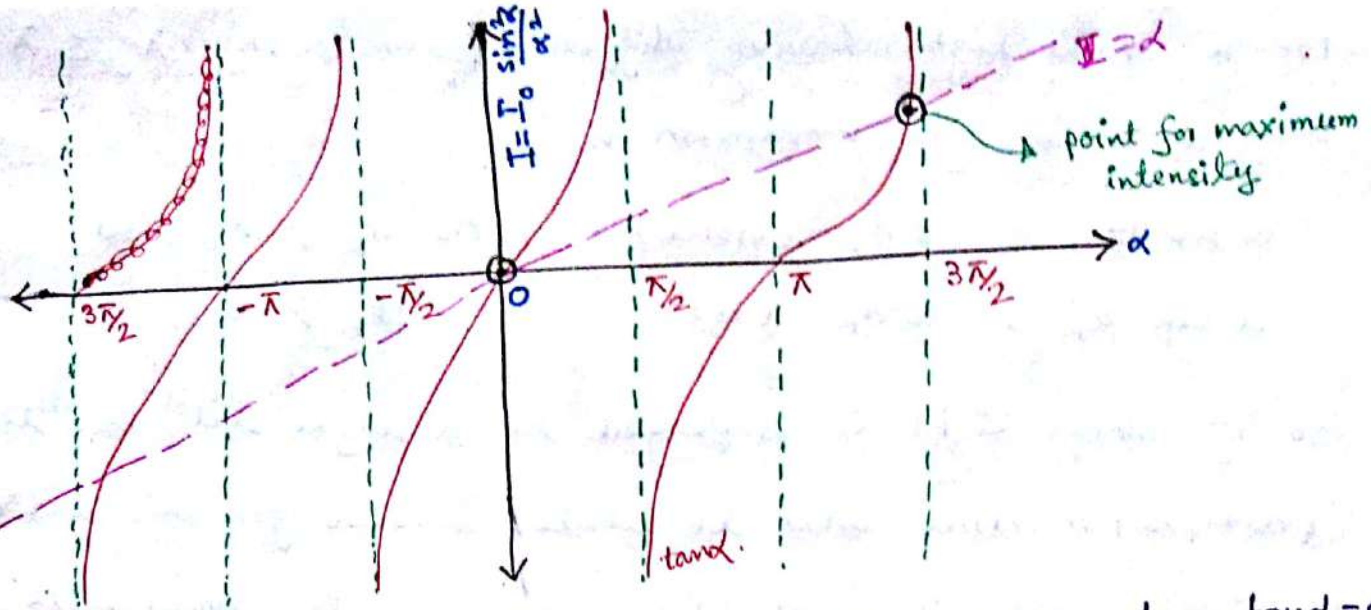
So other condition will give us the condition for minimum, $I = 0$

or $\sin\alpha = 0$. But $\alpha \neq 0$, so $\alpha = s\pi$, $s = \pm1, \pm2, \pm3, \ldots$

or $\alpha = \frac{a\phi}{2} = \frac{a}{2}\frac{2\pi}{\lambda}(\sin i \pm \sin\theta) = s\pi.$

or $\boxed{a(\sin i \pm \sin\theta) = s\lambda}$ (minimum) (Oblique incidence)

For normal incidence $i = 0$, $\therefore a\sin\theta = s\lambda$ (normal incidence)



The values of α that satisfy the relation $\cot\alpha = \frac{1}{\alpha}$ or $\tan\alpha = \alpha$ can be obtained graphically by plotting $I = \alpha$ and $I = \tan\alpha$ on the same graph. The intensity is maximum when $\alpha = 0$ is called the principal maximum. The other value of α which will give maximum will be obtained from intersection of this points which will occur at $\alpha = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, etc.

Principal Maxima : For the principle maxima, $\alpha = 0$ and taking

$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1$, we get $I_{principal} = I_0$

Hence when $\alpha \to 0$, we obtain $\frac{a\phi}{2} \to 0$

or $\varphi = \frac{2\pi}{\lambda}(\sin i \pm \sin\theta) \to 0$

i.e. $i \to 0$ & $\theta \to 0$. Thus, principal maxima is obtained at the middle when angle of incidence & diffraction is zero. As the value of α increases, the value of $\frac{\sin\alpha}{\alpha}$ decreases, so the intensity of secondary maxima reduces.

White light :

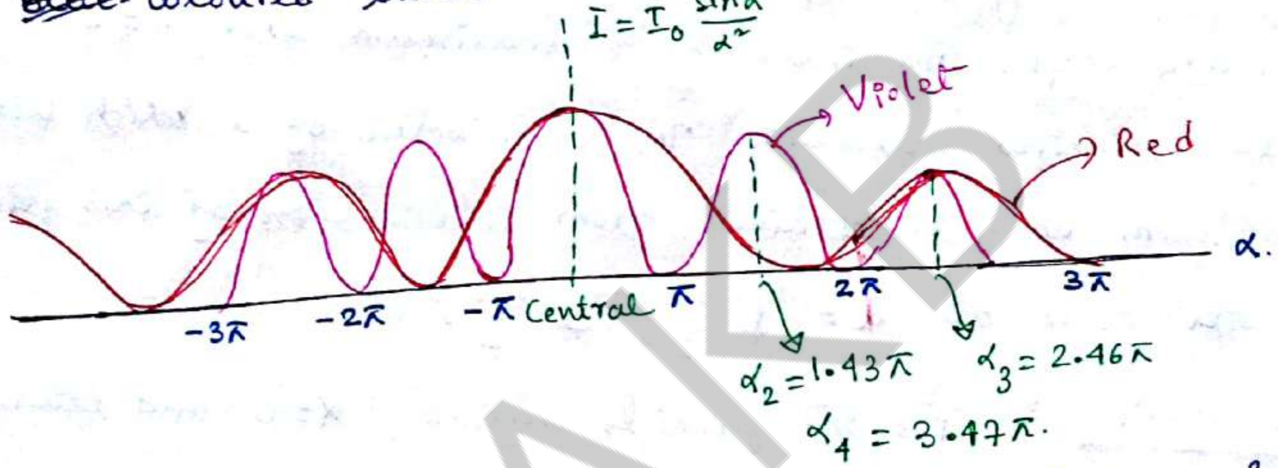
For normal incidence ($i = 0$), the condition of minimum for diffraction in single slit is $a\sin\theta = l\lambda$.

For a white light having different wavelength $\lambda_V \leq \lambda \leq \lambda_R$ then condition for minimum is

$$a \sin\theta_V = s\lambda_V \quad \text{(violet)}$$
$$a \sin\theta_R = s\lambda_R \quad \text{(red)}$$

As $\lambda_R > \lambda_V$, so $\boxed{\theta_R > \theta_V}$

So if white light is employed to a single slit, then the central (principle) maxima will be white, because for this condition $\theta \to 0$. But for other secondary maxima, the band will be coloured with the red-coloured band further apart from the ~~blue~~ violet-coloured band.



So secondary maxima do not fall half-way between two minima.

Secondary Minima: The direction of secondary minima is

$$a\sin\theta = s\lambda \quad \text{(normal incidence)}$$

So $\alpha = \frac{a\pi}{\lambda}\sin\theta = s\pi$, $s = \pm 1, \pm 2, \dots, \pm n$.

If $\alpha = \pm\pi$, $\sin\alpha = 0$, Intensity $I = 0$

So various diffraction minima occur at $\alpha = \pm\pi, \pm 2\pi, \dots, \pm n\pi$. ($n$ = integer)

Again, $\frac{a\pi}{\lambda}\sin\theta = s\pi$ gives $\sin\theta = s\frac{\lambda}{a} = \pm n\frac{\lambda}{a}$

So position of 1st, 2nd, 3rd etc minima are given by $\pm\frac{\lambda}{a}, \pm\frac{2\lambda}{a}, \pm\frac{3\lambda}{a}, \dots$ for various values of $\sin\theta$.

Secondary Maxima The direction of n^{th} secondary maximum is

$$\sin\theta = \pm \frac{(2n+1)\lambda}{2a}.$$

$$\text{So } \alpha = \pm \frac{\pi}{\lambda} a \frac{(2n+1)\lambda}{2a} = \pm (2n+1)\frac{\pi}{2} = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

1^{st} secondary maximum $\alpha = \frac{3\pi}{2}$, so $I = I_0 \frac{\sin^2\alpha}{\alpha^2} = \frac{4}{9\pi^2} I_0 \simeq \frac{I_0}{22}$

2^{nd} secondary maximum $\alpha = \frac{5\pi}{2}$, so $I = I_0 \frac{\sin^2\alpha}{\alpha^2} = \left(\frac{2}{5\pi}\right)^2 I_0 \simeq \frac{I_0}{61}$

CW A parallel beam of light of wavelength 5×10^{-7} m is incident normally on a narrow slit of width 0.2 mm. The Fraunhofer diffraction pattern is observed on a screen which is placed at the focal plane of a convex lens of focal length 50 cm. Calculate the distance between the first two minima and the first two maxima on the screen. Assume that the lens is placed very close to the slit.

$a = 0.2\text{ mm} = 2\times10^{-4}$ m, $\lambda = 5\times10^{-7}$ m, $f = 0.5$ m.

For secondary minima, $a\sin\theta = \pm 8\lambda$.

and $\theta << 1$, $\sin\theta \simeq \theta$. $\therefore$ Angular diffraction for 1^{st} minima

$$\theta_1 = \frac{\lambda}{a} = \frac{5\times10^{-7}}{2\times10^{-4}} = 2.5\times10^{-3} \text{ radian}$$

Angular diffraction for 2^{nd} minima $\theta_2 = \frac{2\lambda}{a} = \frac{10\times10^{-7}}{2\times10^{-4}} = 5\times10^{-3}$ rad

$\therefore$ Separation between 1^{st} & 2^{nd} minimum $x = f(\theta_2 - \theta_1)$

$$= (5-2.5)\times10^{-3}\times0.5 = 0.125\times10^{-3} \text{ m.}$$

1^{st} and 2^{nd} maxima will occur at $\alpha = 1.43\pi$ & 2.46π.

$\therefore$ Angular diffraction for 1^{st} maxima $\theta_1' = 1.43\times2.5\times10^{-3}$ rad

Angular diffraction for 2^{nd} maxima $\theta_2' = 2.46\times2.5\times10^{-3}$ rad

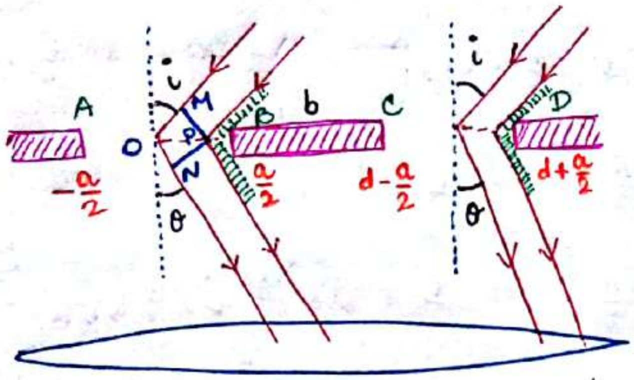
$\therefore$ Separation between 2^{nd} & 1^{st} maxima $x' = f(\theta_2' - \theta_1')$

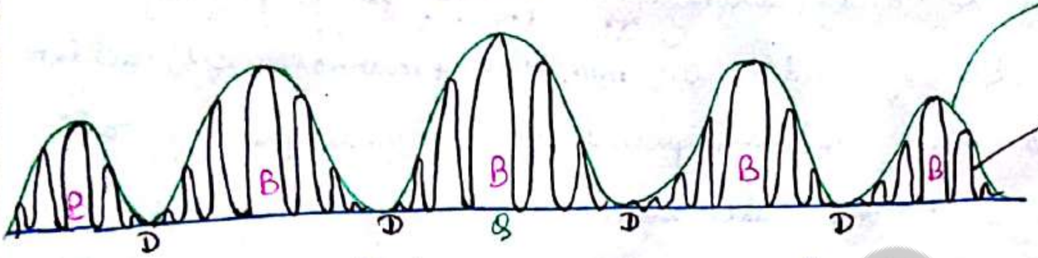
$$= 0.1288\times10^{-2} \text{ m.}$$

Double slit diffraction :

$a+b = d$ $\angle OPM = i$
$\angle OPN = \theta$

B = Bright
D = Dark





When a parallel beam of wavelength λ is made incident normally on a surface containing two narrow and close slits AB and CD, each of width a and kept separated by an opaque space BC of width b, we get a series of dark & bright bands on the screen. The two points on two slits (A, C) or (B, D) which are separated by a distance $a+b=d$ are called "Corresponding points."

If δ is the phase difference between the rays diffracted from origin O (which is the mid point of the first slit) and from P (which is at a distance x from O), then

$$\delta = \frac{2\pi}{\lambda} x(\sin i \pm \sin\theta) = \frac{2\pi x}{\lambda}\phi \quad \text{where } \phi = \sin i \pm \sin\theta.$$

If $y = \text{R.P. of } re^{i\omega t}$ is the displacement of the wave at any point Q due to diffraction from the origin O, then the displacement at Q due to the rays diffracted from P at a distance x from O is

$$y = \text{R.P. of } Kr\,e^{i(\omega t \pm \delta)}, \quad K = \text{constant}.$$

Hence the displacement at Q due to rays diffracted from a small region dx of the first slit is

$$y = R.P.\ Kr\, e^{i(\omega t \pm \delta)} dx = R.P.\ R\, e^{i\frac{2\pi}{\lambda}(ct \pm x\phi)}$$

$$= R.P.\ R\, e^{i\sigma} e^{ilx} dx \qquad [R = Kr]$$

$\left[\omega = \frac{2\pi c}{\lambda} = 2\pi\nu\right]$
$\left[\delta = \frac{2\pi x\phi}{\lambda}\right]$
$\left[\sigma = \frac{2\pi ct}{\lambda}\right]$
$\left[l = \frac{2\pi\phi}{\lambda}\right]$

So resultant displacement at Q due to the two slits together would be

$$Y = R.P.\ R e^{i\sigma}\left[\int_{-a/2}^{a/2} e^{ilx} dx + \int_{d-a/2}^{d+a/2} e^{ilx} dx\right]$$

$$= R.P.\ aR\, e^{i\sigma}\left[\frac{\sin(al/2)}{(al/2)} + e^{ild}\frac{\sin(al/2)}{(al/2)}\right]$$

$$= R.P.\ aR\frac{\sin\alpha}{\alpha}(1+e^{ild})\, e^{i\sigma} \qquad \left[\alpha = \frac{al}{2} = \frac{\pi a\phi}{\lambda}\right] \quad \left[A_0 = aR,\ A = A_0\frac{\sin\alpha}{\alpha}\right]$$

$$= R.P.\ A(1+\cos(ld) + i\sin(ld))(\cos\sigma + i\sin\sigma) \qquad [C = 1+\cos(ld),\ D = \sin(ld)]$$

$$= R.P.\ A(C+iD)(\cos\sigma + i\sin\sigma)$$

$$= A(C\cos\sigma - D\sin\sigma) \qquad [C = E\cos\gamma',\ D = E\sin\gamma'$$

$$= AE\cos(\sigma+\gamma')$$

so that $E = \sqrt{C^2+D^2}$

$= \sqrt{1+\cos(ld)^2 + \sin(ld)^2}$

$= \sqrt{2(1+\cos ld)}$

$= \sqrt{4\cos^2 ld/2}\ ]$

So the amplitude of resultant displacement is $A\sqrt{C^2+D^2}$ or the resultant intensity

$$I = A^2(C^2+D^2)$$

$$= 4A^2\cos^2\left(\frac{ld}{2}\right) = 4A^2\cos^2\beta \qquad \left[\beta = \frac{ld}{2} = \frac{\pi d}{\lambda}(\sin i \pm \sin\theta)\right]$$

$$= 4A_0^2\frac{\sin^2\alpha}{\alpha^2}\cos^2\beta$$

$$= I_1 \times I_2 \qquad \left[I_1 = 4A_0^2\frac{\sin^2\alpha}{\alpha^2},\ I_2 = \cos^2\beta\right]$$

This indicate that two different system of fringes will appear. Intensity I_1 is due to diffraction produced by each individual slit while intensity I_2 is due to the interference of the diffracted rays at an angle θ from the corresponding points of each slit. (interference).

Condition for Minima: The intensity I would be zero when either $I_1 = 0$ (Diffraction minima) or $I_2 = 0$ (Interference minima).

(i) $I_1 = 4A_0^2 \frac{\sin^2\alpha}{\alpha^2} = 0$ gives $\alpha = s\pi$, $s = \pm 1, \pm 2, \ldots, \pm n$

or $\frac{\pi a}{\lambda} \sin\theta = s\pi$ (for normal incidence) [Diffraction minima]

or $\boxed{a\sin\theta = s\lambda}$

and $\boxed{a(\sin i \pm \sin\theta) = s\lambda}$ (for oblique incidence)

(ii) $I_2 = \cos^2\beta = 0$ gives $\beta = (2n+1)\frac{\pi}{2}$

or $\frac{\pi d}{\lambda} \sin\theta = (2n+1)\frac{\pi}{2}$ (for normal incidence) [Interference minima]

or $\boxed{d\sin\theta = (2n+1)\lambda/2}$

and $\boxed{d(\sin i \pm \sin\theta) = (2n+1)\lambda/2}$ (for oblique incidence)

Angular separation between two consecutive minima

$\sin\theta_1 = \pm \frac{3\lambda}{2d}$, $\sin\theta_2 = \pm \frac{5\lambda}{2d}$, so $\boxed{\sin\theta_2 - \sin\theta_1 = \pm \frac{\lambda}{d}}$

Condition for Maxima:

For central maxima $\alpha = 0$ so that $\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1$.

So for diffraction maxima (secondary), $\alpha = \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \pm \ldots$

So $\sin\theta_1 = \pm\frac{3\lambda}{a}, \pm \frac{5\lambda}{a}, \pm \frac{7\lambda}{a}, \ldots$ [Diffraction maxima].

when width a of each slit is very small, then α = small, so $I_1 \to 4A_0^2$ becomes a constant. Under this condition, the maxima of the resultant I will be solely controlled by I_2. Hence for normal incidence, interference maxima is given by $\cos^2\beta = 1$ or

$$\beta = \frac{\pi}{\lambda} d\sin\theta = n\pi \quad \text{or} \quad \boxed{d\sin\theta = n\lambda}, \quad n = 0, \pm1, \pm2, \ldots$$

(normal incidence)

and $\boxed{d(\sin i \pm \sin\theta) = n\lambda}$ (oblique incidence) [Interference maxima]

Angular separation between two consecutive maxima

$$\sin\theta_1 = \pm\frac{\lambda}{d}, \quad \sin\theta_2 = \pm\frac{2\lambda}{d}, \quad \text{so} \quad \boxed{\sin\theta_2 - \sin\theta_1 = \pm\frac{\lambda}{d}}$$

So the angular separation between two consecutive maxima & minima are equal. $= \pm\frac{\lambda}{d}$.

Missing Order in double slit diffraction pattern

For normal incidence when the condition for maxima of interference pattern and minima of diffraction pattern for a given value of θ are simultaneously satisfied, then the interference maxima will be missing.

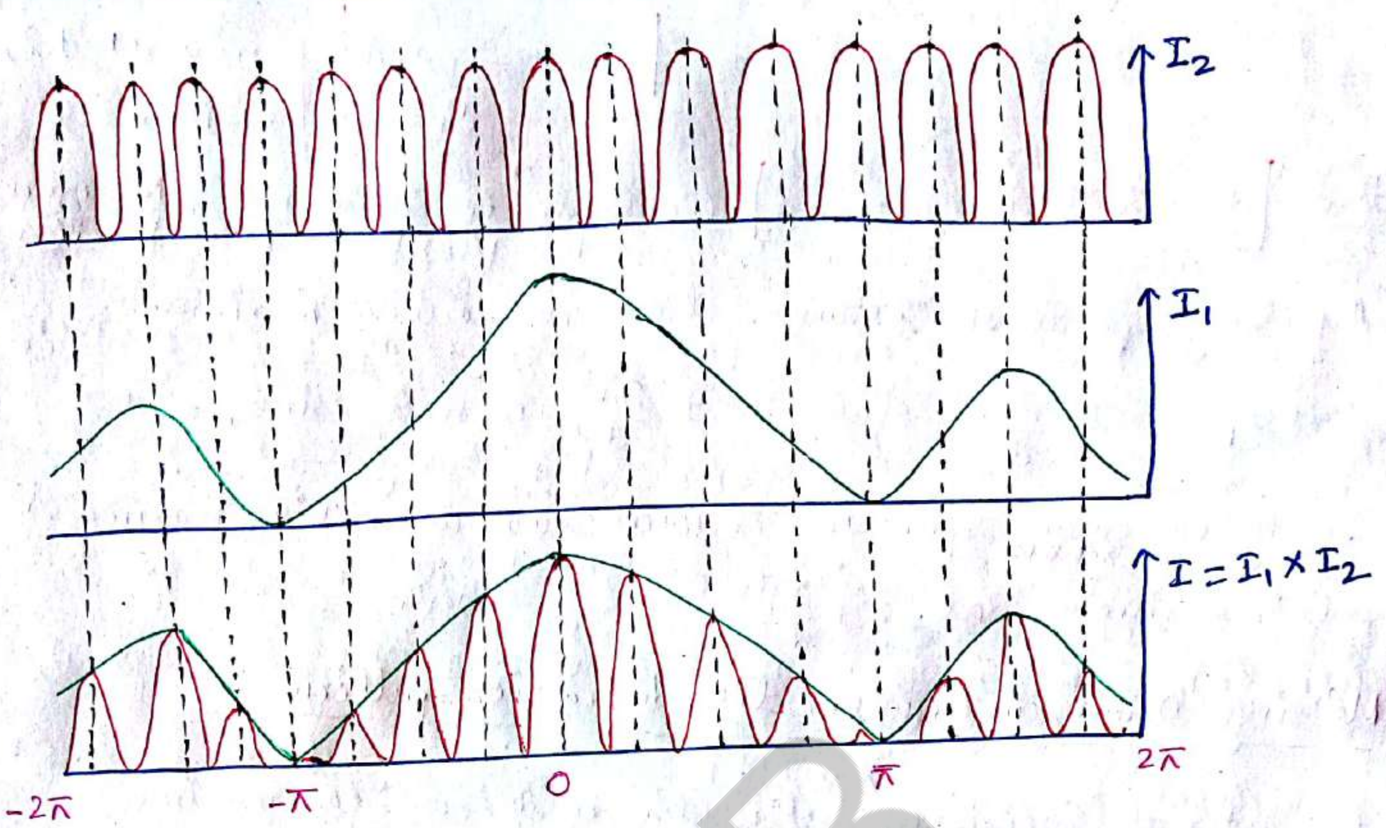
Thus $\alpha = \frac{\pi}{\lambda} a\sin\theta = s\pi$ (diffraction minima)

$\beta = \frac{\pi}{\lambda} d\sin\theta = n\pi$ (interference maxima)

$\therefore \frac{\beta}{\alpha} = \frac{d}{a} = \frac{n}{s}$. When $d = 2a$, $n = 2s$

Hence 2, 4, 6, ... etc orders of interference maxima are absent which corresponds to 1, 2, 3, ... etc orders of diffraction dark bands. Similarly when $d = 3a$, $n = 3s$, so 3, 6, 9, ... etc orders of interference maxima are absent, which corresponds to 1, 2, 3, ... etc orders of diffraction dark bands.

Complete double slit pattern

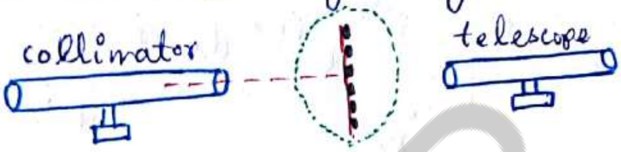


Plane Transmission Diffraction Grating

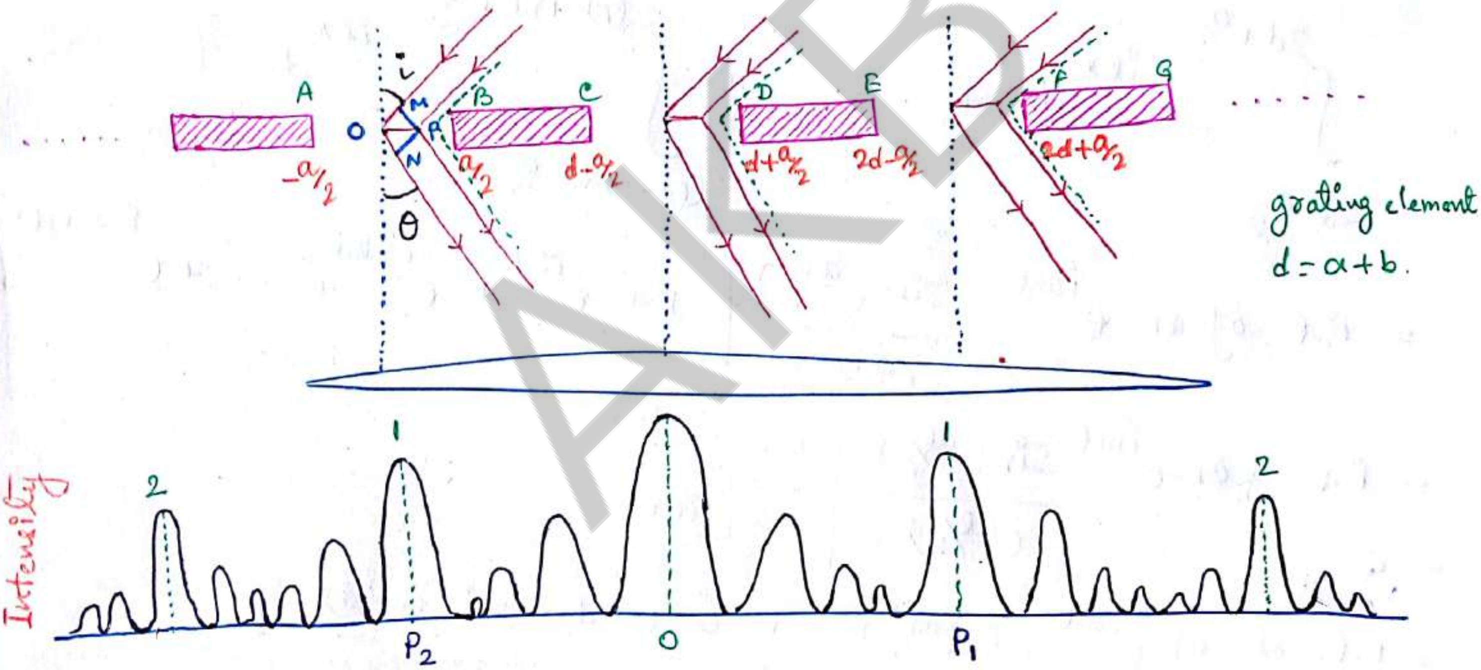
A plane transmission grating consists of a number of parallel and equidistant lines ruled over optically plane & parallel glass plate by means of a fine diamond point worked with a ruling engine. Number of such ruled lines per inch ranges between 2000 - 15000. Each ruled line behaves as an opaque space and the transparent portion between two consecutive ruled lines behaves as a slit. If "a" be the width of the clear space and "b" be the width of the ruled lines (opaque space) then the distance $d = a+b$ is called "Grating constant". The two points on the consecutive clear space whose distance is d are called "corresponding points."

Master / Replica Grating.

Drawing exactly parallel & equidistant lines on a glass plate by a diamond point is an extremely difficult task & so the grating becomes very costly. To cut the cost, a cast of these ruled surface is made with some transparent material & such cast is called "replica grating." Cellulose Acetate in properly diluted is put on the surface of master grating and then dried to a thin film. This film is mounted on a transparent glass plate to form the replica grating.



Theory



A parallel beam of monochromatic light of wavelength λ is incident at an angle i with the normal to the grating surface, in which there are N number of parallel equidistant slits of width a and opaque space of width b, and are diffracted at an angle θ.

Let $y = \text{R.P. } r e^{i\omega t}$ is the displacement of the wave reaching a point on screen which is diffracted from the middle point O of the first slit. Then the displacement of the wave from a point P at a distance x from O will be

$$y = \text{R.P. of } Kr\, e^{i(\omega t \pm \delta)} \quad , K = \text{constant}$$

$$= \text{R.P. of } R\, e^{i(\omega t + lx)}$$

$\left[R = Kr \quad \delta = lx \quad l = \frac{2\pi}{\lambda}(\sin i \pm \sin\theta) \right]$

and the displacement from a further small distance dx after x, will be $y = \text{R.P. of } R\, e^{i(\omega t + lx)} dx$.

$\therefore$ Amplitude due to all the point in slit will be

$$Y = \text{R.P. of } R\, e^{i\omega t} \left[\int_{-a/2}^{a/2} e^{ilx} dx + \int_{d-a/2}^{d+a/2} e^{ilx} dx + \int_{2d-a/2}^{2d+a/2} e^{ilx} dx + \dots + \int_{(N-1)d-a/2}^{(N-1)d+a/2} e^{ilx} dx \right]$$

$$= \text{R.P. of } aR\, e^{i\omega t} \frac{\sin(al/2)}{(al/2)} \left[1 + e^{ild} + e^{i2ld} + \dots + e^{i(N-1)ld} \right]$$

$$= \text{R.P. of } aR\, e^{i\omega t} \frac{\sin(al/2)}{(al/2)} \left[\frac{1 - e^{iNld}}{1 - e^{ild}} \right]$$

$$= \text{R.P. of } aR\, e^{i\omega t} \frac{\sin(al/2)}{al/2} \left[\frac{1 - \cos(Nld) - i\sin(Nld)}{1 - \cos(ld) - i\sin(ld)} \right].$$

So the intensity at that particular point will be

$$I = Y^* Y = a^2R^2 \frac{\sin^2(al/2)}{(al/2)^2} \frac{2(1 - \cos Nld)}{2(1 - \cos ld)}$$

$$= A_0^2 \frac{\sin^2\alpha}{\alpha^2} \frac{\sin^2 N\beta}{\sin^2\beta}$$

$A_0 = aR$

$\alpha = \frac{al}{2}$

$\beta = \frac{ld}{2}$

$$= I_1 \times I_2$$

$I_1 \to$ Diffraction

$I_2 \to$ Interference

Here, the first factor $I_1 = A_0^2 \frac{\sin^2\alpha}{\alpha^2}$ gives the intensity distribution in the diffraction pattern due to single slit and $I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$ gives the interference pattern produced by the beams coming from N number of slits.

Condition for Principal Maxima:

When $\beta = \pm n\pi$ with $n = 1, 2, 3, \ldots$ we see that $I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$ gives indeterminate form, but

$$\lim_{\beta \to n\pi} \frac{\sin N\beta}{\sin\beta} = \lim_{\beta \to n\pi} \frac{\frac{d}{d\beta}(\sin N\beta)}{\frac{d}{d\beta}(\sin\beta)} = \lim_{\beta \to n\pi} \frac{\cos N\beta \times N}{\cos\beta} = \pm N.$$

So resultant intensity is $I = A_0^2 \frac{\sin^2\alpha}{\alpha^2} N^2$ which is very large as N is large. So the condition for principal maxima is

$$\beta = n\pi \quad \text{or} \quad \frac{\ell d}{2} = n\pi \quad \text{or} \quad \frac{\cancel{2}\pi}{\lambda}(\sin i \pm \sin\theta) \times \frac{d}{\cancel{2}} = n\cancel{\pi}$$

$$\text{or} \quad \boxed{d(\sin i \pm \sin\theta) = n\lambda}, \quad n = 0, \pm 1, \pm 2, \ldots$$

These are the position of maxima and the $\pm$ sign indicates that there are two principal maxima of same order that lie on either side of 0^{th} order maxima.

Condition for Secondary Maxima & Minima:

$I_2 = \frac{\sin^2 N\beta}{\sin^2\beta}$ will be extremum (minimum or maximum) when

$$\frac{dI_2}{d\beta} = 0 \quad \text{and} \quad \frac{d^2 I_2}{d\beta^2} = \text{positive (minimum)}$$
$$= \text{negative (maximum)}$$

$$\text{So } \ln I_2 = 2\ln\frac{\sin N\beta}{\sin\beta} = 2(\ln \sin N\beta - \sin\beta).$$

Differentiating, $\frac{1}{I_2}\frac{dI_2}{d\beta} = 2(N\cot N\beta - \cot\beta)$

or $\frac{dI_2}{d\beta} = 2I_2(N\cot N\beta - \cot\beta)$.

and $\frac{d^2I_2}{d\beta^2} = 2\frac{dI_2}{d\beta}(N\cot N\beta - \cot\beta) + 2I_2(-N^2\operatorname{cosec}^2 N\beta + \operatorname{cosec}^2\beta)$

$= 2\frac{dI_2}{d\beta}(N\cot N\beta - \cot\beta) + 2I_2(1-N^2+\cot^2\beta - N^2\cot^2 N\beta)$

$\frac{dI_2}{d\beta} = 0$ when (a) $I_2 = 0$ that means when $\sin N\beta = 0$

but $\sin\beta \neq 0$ so that the factor $\frac{\sin N\beta}{\sin\beta}$ becomes zero. So for

minima, $N\beta = m\pi$, $m = \pm1, \pm2, \pm3, \ldots$

or $N\frac{2\cancel{\pi}}{\lambda}\frac{1}{\cancel{2}}d(\sin i \pm \sin\theta) = m\cancel{\pi}$

or $d(\sin i \pm \sin\theta) = \frac{m\lambda}{N}$ (Condition for secondary minima)

Point to realize is for $m = 0, N, 2N, \ldots$ $\sin\beta$ is also zero so that we obtain principal maxima. There are $(N-1)$ minima between two consecutive principal maxima.

(b) If however $N\cot N\beta - \cot\beta = 0$ then we obtain $\frac{d^2I_2}{d\beta^2} < 0$. So this condition yields the secondary maxima

so $N\cot N\beta - \cot\beta = 0 \Rightarrow N\tan\beta = \tan N\beta$.

or $N^2\frac{\sin^2\beta}{\cos^2\beta} = \frac{N^2\sin^2 N\beta}{N^2\cos^2 N\beta}$ or $\frac{\sin^2 N\beta}{\sin^2\beta} = N^2\frac{\cos^2 N\beta}{\cos^2\beta}$

or $\frac{\sin^2 N\beta}{\sin^2\beta} = N^2\frac{1}{\sec^2 N\beta\cos^2\beta} = N^2\frac{1}{(1+\tan^2 N\beta)\cos^2\beta} = \frac{N^2}{(1+N^2\tan^2\beta)\cos^2\beta}$ (as $N\tan\beta = \tan N\beta$)

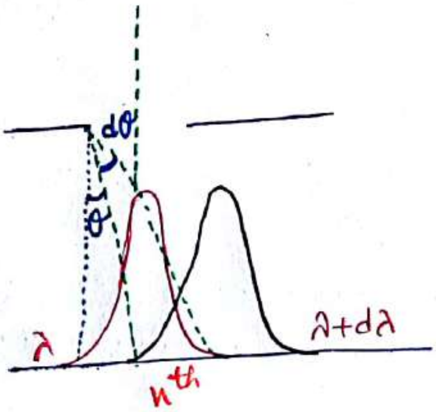
$= \frac{N^2}{\cos^2\beta + N^2\sin^2\beta} = \frac{N^2}{1+(N^2-1)\sin^2\beta}$.

∴ Intensity of secondary maxima $I_{SM} = I_1 \dfrac{N^2}{1+(N^2-1)\sin^2\beta}$

$$= \frac{I_{PM}}{1+(N^2-1)\sin^2\beta}$$ where intensity of principal maxima $= N^2$.

So as N increases, $I_{SM} << I_{PM}$, so secondary maxima will not be visible with a grating with 15000 lines/inch.

Width of the Principal Maxima :



To calculate the width of the n^{th} principal maxima, let θ and $\theta + d\theta$ be the angles of diffraction for the peak of the n^{th} principal maxima & the first minima adjacent to it. So the angular width of the principal maxima is $2d\theta$.

Now, the intensity of the ray diffracted at an angle θ is

$$I = I_1 \frac{\sin^2 N\beta}{\sin^2\beta}$$ where $\beta = \dfrac{\pi d \sin\theta}{\lambda}$ for normal incidence.

so that the amplitude is $A = \sqrt{I_1} \dfrac{\sin\left(\frac{N\pi d \sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d \sin\theta}{\lambda}\right)}$

The amplitude of the secondary minimum will be

$$A' = \sqrt{I_1} \frac{\sin\left(\frac{N\pi d}{\lambda} \sin(\theta + d\theta)\right)}{\sin\left(\frac{\pi d}{\lambda} \sin(\theta + d\theta)\right)}$$

[$\sin d\theta \simeq d\theta$
$\cos d\theta \simeq 1$
to first approximation]

$$= \sqrt{I_1} \frac{\sin\left[\frac{N\pi d}{\lambda}\sin\theta + \frac{N\pi d}{\lambda}\cos\theta\, d\theta\right]}{\sin\left[\frac{\pi d}{\lambda}\sin\theta + \frac{\pi d}{\lambda}\cos\theta\, d\theta\right]}$$

Substituting the condition for the principal maxima $d\sin\theta = n\lambda$,

$$A' = \sqrt{I_1} \frac{\sin[Nn\pi + Nn\pi \cot\theta\, d\theta]}{\sin[n\pi + n\pi \cot\theta\, d\theta]} = \sqrt{I_1} \frac{\sin(Nn\pi \cot\theta\, d\theta)}{\sin(n\pi \cot\theta\, d\theta)}$$

first secondary minima for a given λ could be obtained when $A' = 0$, i.e. $Nn\pi \cot\theta \, d\theta = \pi$. $\therefore d\theta = \frac{1}{Nn\cot\theta}$

$\therefore$ The width of the principal maxima $= 2d\theta = \frac{2}{Nn\cot\theta}$.

Here, as θ increases, $\cot\theta$ decreases. Therefore the width of the principal maxima increases for higher order number.

Q. A parallel beam of Sodium light is allowed to be incident normally on a plane grating having 4250 lines/cm and a 2^{nd} order spectral line is observed to be deviated through 30°. Calculate the wavelength of the spectral line.

$n = 2$, $\theta = 30^\circ$, $N = 4250$, so $d = a+b = \frac{1}{4250}$ cm

So from the condition of secondary maxima $d\sin\theta = n\lambda$ we have

$$\frac{1}{4250} \sin 30 = 2\lambda \quad \therefore \lambda = 5882 \times 10^{-8} \text{ cm} = 5882 \text{ Å}.$$

Q. In a plane transmission grating the angle of diffraction for second order maxima for wavelength 5×10^{-5} cm is 30°. Calculate the number of lines in 1 cm of the grating surface.

If N lines/cm exists then Grating element $d = a+b = \frac{1}{N}$ cm.

$n = 2$, $\theta = 30^\circ$, $\lambda = 5 \times 10^{-5}$ cm. From $d\sin\theta = n\lambda$ we have

$$\frac{1}{N} \sin 30 = 2 \times 5 \times 10^{-5} \quad \therefore N = 5000/\text{cm}.$$

Q. A wire grating is made of 200 wires/cm placed at equal distance apart. The diameter of each wire is 0.025 mm. Calculate the angle of diffraction for the 3^{rd} order spectrum and also find the absent spectra if any. $\lambda = 6000$ Å.

Grating element $d = a+b = \frac{1}{200}$ cm $= 0.005$ cm, $n = 3$.

$\lambda = 6000 \text{ Å} = 6000 \times 10^{-8}$ cm.

$0.005 \sin\theta = 3 \times 6000 \times 10^{-8}$ or $\theta = 2^\circ 4'$

Now b = width of opacity = diameter of wire = 0.0025 cm.

So width of transparency $a = d - b = 0.005 - 0.0025 = 0.0025$ cm.

Order of absent spectrum $n = \frac{d}{a} = \frac{0.005}{0.0025} = 2$. So the second order spectrum will be missing.

CW Show that in a diffraction grating with grating element 1.5×10^{-6} m and light of wavelength 550 nm, third and higher order principal maxima are not visible.

Grating element $d = a+b = 1.5 \times 10^{-6}$ m, $\lambda = 5500 \times 10^{-10}$ m.

As $\sin 90^\circ = 1$ so the maximum angle of diffraction is 90° and let n be the maximum number of order of spectrum that can be observed

So from $d\sin\theta = n\lambda$, $n = \frac{d}{\lambda} = \frac{1.5 \times 10^{-6}}{5500 \times 10^{-10}} = 2.72 < 3$.

Thus, only 2nd order will be visible. No 3rd or higher order is possible.

CW If we use white light source with a diffraction grating, with 15000 lines/inch, show that the 2nd and 3rd order spectra overlap. Given $\lambda_{violet} = 4000 Å$ and $\lambda_{red} = 7000 Å$ in the white light beam.

Grating element $d = a+b = \frac{1}{15000}$ inch $= \frac{2.54}{15000}$ cm $= 1.69 \times 10^{-6}$ m.

$\lambda_{violet} = 4000 \times 10^{-8}$ cm $= 4 \times 10^{-7}$ m.

$\lambda_{red} = 7000 \times 10^{-8}$ cm $= 7 \times 10^{-7}$ m.

In the 2nd order spectrum, $\sin\theta^{(2)}_{violet} = \frac{2 \times \lambda_{violet}}{d} = \frac{2 \times 4 \times 10^{-7}}{1.69 \times 10^{-6}} = 0.4734$

So $\theta^{(2)}_{violet} = 28.2^\circ$.

and $\sin\theta^{(2)}_{red} = \frac{2 \times \lambda_{red}}{d} = \frac{2 \times 7 \times 10^{-7}}{1.69 \times 10^{-6}} = 0.8284$. $\therefore \theta^{(2)}_{red} = 55.9^\circ$

In the 3rd orders spectrum, $\sin\theta^{(3)}_{violet} = \frac{3 \times 4 \times 10^{-7}}{1.69 \times 10^{-6}} = 0.71$