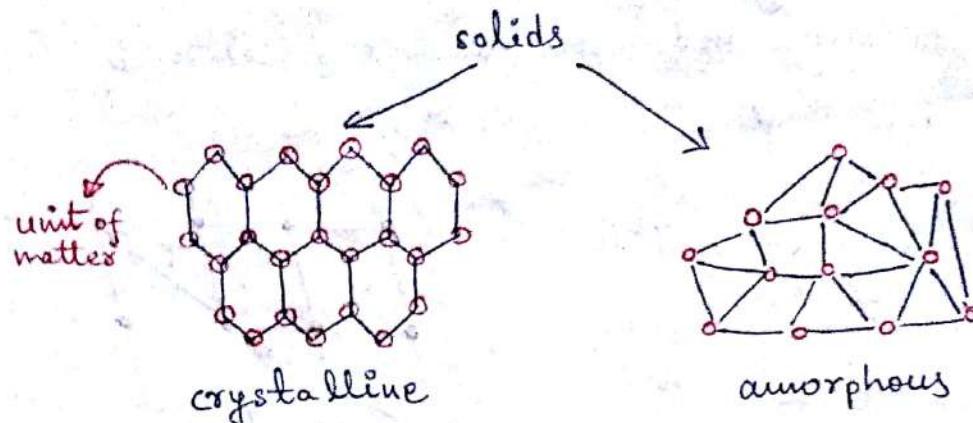


# SOLID STATE PHYSICS

Crystal structure, direct lattice & (un) holy grail



(i) Amorphous solid : no order in arrangement of unit of matter (atoms, molecules etc). XRD shows "liquid like" property.

what's "solid"?  $\rightarrow$  elasticity

$$\text{But } \frac{1}{2} K \bar{x}^2 = \frac{1}{2} k_B T \quad (\text{Equipartition theorem})$$

$$\therefore \bar{x}^2 = \frac{k_B T}{K} = \frac{k_B T}{k l}$$

so if  $K \rightarrow 0$ ,  $\bar{x}^2 \rightarrow \infty$

amorphous solids

$$\begin{aligned} \text{stress } \sigma &= K \frac{x}{l} \\ F &= -l^2 K \frac{x}{l} \\ &= -K l x = -K x, \end{aligned}$$

$x$  = displacement  
 $l$  = length of lattice

"rigidity"

highly viscous, supercooled liquids.

Example pitch, plastic, silicate glass.

**SALIENT**

**FEATURES**

- (i) molecular motion is irregular but distance is more or less same with elastic solid.
- (ii) no regular shape  $\rightarrow$  conductivity, elasticity, tensile strength is isotropic
- (iii) no long range order. short range / medium range order possible.
- (iv) no sharp melting point.

There are polycrystalline substances which are composed of many small domains/regions of single crystals. Crystalline substances are distinguished from amorphous solids by their anisotropic behaviour (direction dependent).

Ideal crystal: infinite repetition of identical structure in space.

Periodic arrangement of unit (atoms, molecules, ions) in a crystal is called the lattice, defined by three fundamental translation vectors  $\vec{a}, \vec{b}, \vec{c}$ . / basis vectors

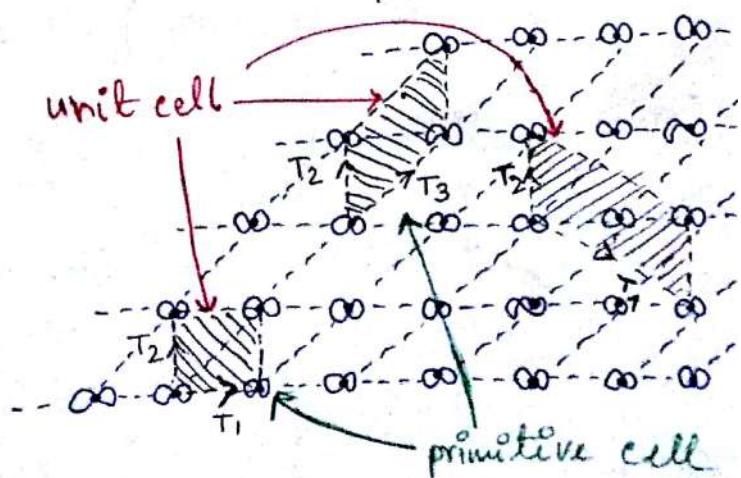
$$\text{Atomic position vector } \vec{r}' = \vec{r} + \vec{t} \\ = \vec{r} + n_1 \vec{a} + n_2 \vec{b} + n_3 \vec{c}$$

where  $n_1, n_2, n_3$  are integers.

### primitive lattice & Unit cell

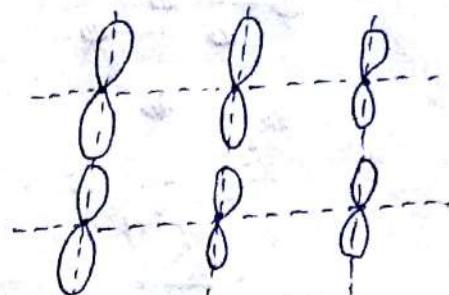
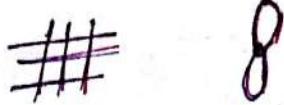
Unit cell is volume from which entire crystal can be constructed by translational repetition. (OABC parallelogram)

primitive cell is a type of unit cell that contain 1 lattice point at corners & minimum in volume =  $|(\vec{a} \times \vec{b}) \cdot \vec{c}|$



Basis In crystal structure, every lattice point is associated with an unit assembly of atoms/molecules/ions. This unit is called basis.

crystal structure = lattice + basis



Basis can contain even hundreds & thousands of molecules.

A translation operation leaves the crystal invariant.

$$f(\vec{r}) = f(\vec{r} + \vec{T})$$

physical significance

number density  
(point mass atom)

$$n(\vec{r}) = \sum_{\vec{T}} \delta(\vec{r} - \vec{T})$$

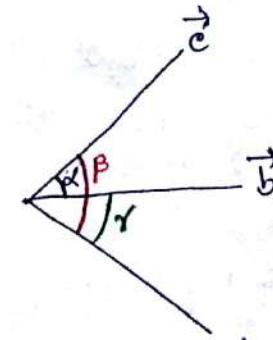
$$\rho(\vec{r}) = \sum_{\vec{T}, \alpha} m_{\alpha} \delta(\vec{r} - \vec{T} - \vec{c}_{\alpha})$$

where  $m_{\alpha}$  is mass of atom at lattice site  $\vec{c}_{\alpha}$ .

$$\rho(\vec{r}) = \rho(\vec{r} + \vec{T})$$

for cubic structure  $|\vec{a}| = |\vec{b}| = |\vec{c}|$

$$\alpha = \beta = \gamma = \frac{\pi}{2}$$

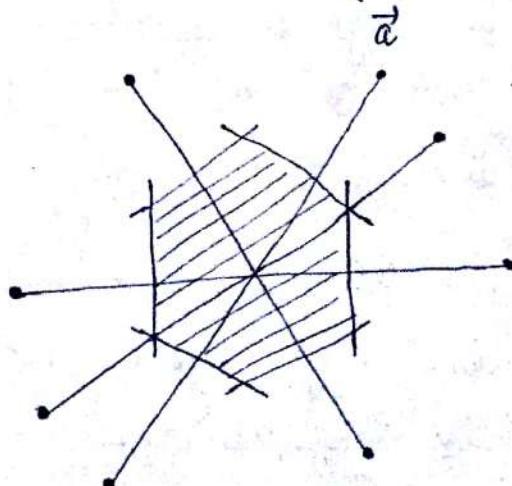


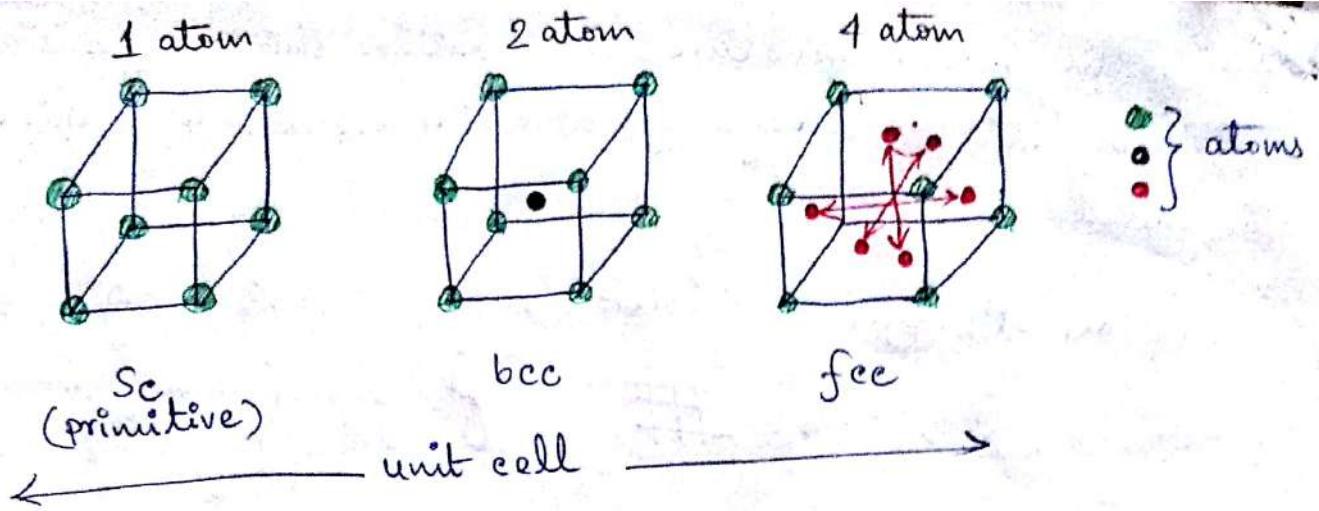
Wigner-Sielz primitive cell

1. Draw lines connecting nearby lattice points.

2. Draw planes/lines at midpoint of line & perpendicular to it.

This is ws primitive cell.





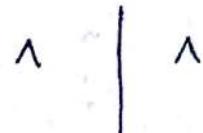
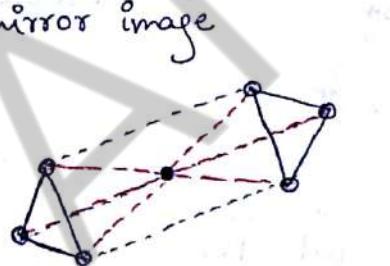
### Elements of symmetry

A symmetry operation transforms the crystal to itself.

- (i) Translation  $\vec{r}' = \vec{r} + \vec{T}$
  - (ii) Rotation  $n\phi = 2\pi$   
 $n =$  multiplicity of rotation axis.  
= 1, 2, 3, 4, ~~5~~, 6.
  - (iii) Reflection mirror image
  - (iv) Inversion
- A 2D square lattice has 4-fold rotational symmetry.

$$\xrightarrow{\quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad} \text{motif}$$

$\phi = 60^\circ$   
 $= \frac{2\pi}{6}$



Symmetry operations performed about a point / line are called point group symmetry. 3 types of point group (i) plane of symmetry (reflection), (ii) axis of symmetry (rotation), (iii) centre of symmetry (inversion)

5-fold rotational symmetry : quasicrystals.

But why 5-fold rotational symmetry is not permissible in crystal structure?

$$AB = a = |\vec{a}| = |\vec{r}_1 - \vec{r}_2| \\ = AA' = BB'$$

Suppose  $A'B' = q|\vec{a}|$  ( $q = \text{integer}$ )

$$\angle A'AC' = \angle D'BB' = \theta - \pi/2$$

$$A'C' = D'B' = |\vec{a}| \cos \theta$$

$$\therefore A'B' = |\vec{a}| + |\vec{a}| \cos \theta + |\vec{a}| \cos \theta$$

$$q|\vec{a}| = |\vec{a}| + 2|\vec{a}| \cos \theta \quad \Rightarrow |2\cos \theta| = \left| \frac{q-1}{2} \right|$$

as  $\cos \theta < 1$ , allowed values of  $q$  are  $-1, 0, 1, 2, 3$

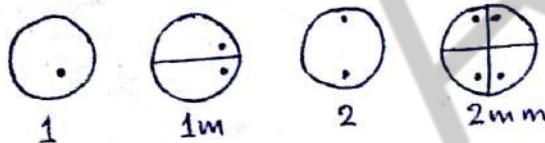
$$\left\{ q=-1, \theta = \frac{2\pi}{2} \right\}, \left\{ q=0, \theta = \frac{2\pi}{3} \right\}, \left\{ q=1, \theta = \frac{2\pi}{4} \right\}$$

$$\left\{ q=2, \theta = \frac{2\pi}{6} \right\}, \left\{ q=3, \theta = \frac{2\pi}{1} \right\}.$$

so,  $n = 1, 2, 3, 4, 6 \xrightarrow{\text{hexad}} \begin{matrix} \downarrow \\ \text{diad} \end{matrix} \xrightarrow{\text{triad}} \text{telrad}$

point group & space group

[translation, rotation, reflection] (point) point group.

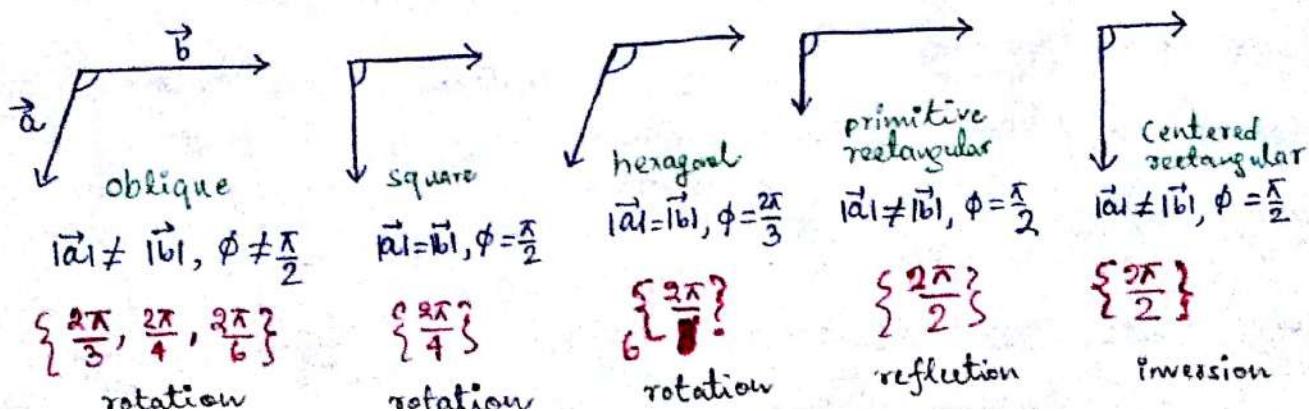


Total 32 point groups.

Group of all symmetry elements is space group.

In 2D: 17, 3D: 230.

Bravais lattice:  $|\vec{a}|, |\vec{b}|, \phi$  5 combination : symmetry operations are maintained.



# 3D lattice types

# 14 Bravais lattices

<u>Class</u>	Type & number	Angle	length of primitive cell
Cubic	P, F, I	$\alpha = \beta = \gamma = 90^\circ$	$a = b = c = \frac{\sqrt{2}}{2}$ cubic
Tetragonal	P, I	$\alpha = \beta = 90^\circ$	$a = b \neq c$ rectangular
Hexagonal	P	$\alpha = \beta = 90^\circ, \gamma = 120^\circ$	$a = b \neq c$
Rhombohedral/ Trigonal	R	$\alpha = \beta = 90^\circ, \gamma < 120^\circ$	$a = b = c$
Orthorhombic	P, F, I, C	$\alpha = \beta = \gamma = 90^\circ$	$a \neq b \neq c$
Monoclinic	P, C	$\alpha = \gamma = 90^\circ \neq \beta$	$a \neq b \neq c$
Triclinic	P	$\alpha \neq \beta \neq \gamma$	$a \neq b \neq c$ Biaxial crystal

## Atoms per unit cell

- (i) Eight corner atoms in cubic unit cell  $\frac{1}{8}$ th atom
- (ii) Six face atoms in unit cell  $\frac{1}{2}$ th atom.
- (iii) If on edge then shared between 4 unit,  $\frac{1}{4}$ th atom
- (iv) If inside cell, then (of course) 1 atom as whole.

## Simple cubic cell (sc)

$$\# \text{ of atoms / unit cell} = \frac{8}{8} = 1.$$

## Body centered cubic cell (bcc)

$$\# \text{ of atoms / unit cell} = \frac{8}{8} + 1 \frac{1}{2} = 2$$

## Face centered cubic cell (fcc)

$$\# \text{ of atoms / unit cell} = \frac{8}{8} + \frac{6}{2} = 4$$

## Coordination Number

In crystal lattice, the number of nearest neighbours of an atom is called coordination no.

sc cell, coord. no. = 6.

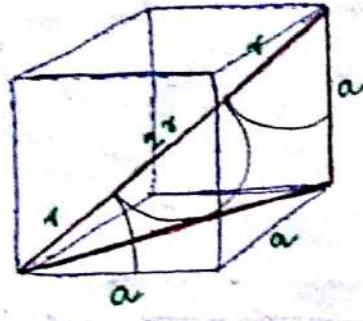
bcc cell, coord no. = 8

fcc cell, coord no. =  $4 \times 3 = 12$

1 atom in  $xy, yz, xz$  plane

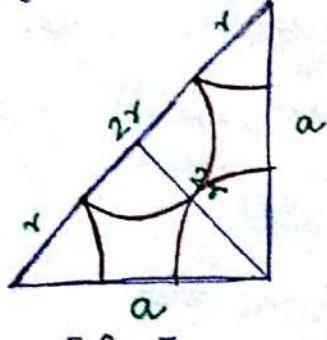
3

Atomic radius Distance between centre of two touching atoms.



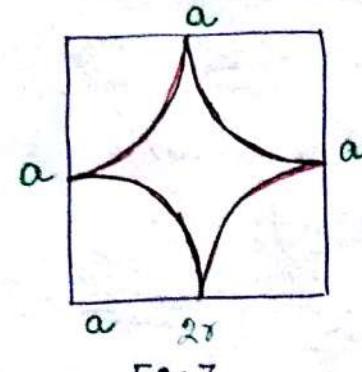
[bcc]

$$(4r)^2 = (\sqrt{2}a)^2 + a^2 \\ \text{or } r = \frac{\sqrt{3}}{4}a$$



[fcc]

$$(4r)^2 = 2a^2 \\ \text{or } r = \frac{a}{2\sqrt{2}}$$



[sc]

$$2r = a \\ \text{or } r = \frac{a}{2}$$

Atomic packing fraction/factor / relative packing density

$$\text{P.F. (f)} = \frac{\text{volume of atoms in unit cell}}{\text{volume of unit cell.}}$$

[bcc] 2 atoms / unit cell,  $r = \frac{\sqrt{3}}{4}a$

$$\therefore \text{vol. of atoms} = 2 \times \frac{4}{3}\pi r^3, \text{ vol. of unit cell} = a^3.$$

$$\therefore f = \frac{2 \times \frac{4}{3}\pi \times \left(\frac{\sqrt{3}}{4}a\right)^3}{a^3} = \frac{\sqrt{3}\pi}{8} = 68\%.$$

Example: Barium, chromium, sodium, iron, caesium chloride

[fcc] 4 atoms / unit cell,  $r = \frac{a}{2\sqrt{2}}$

$$\therefore f = \frac{4 \times \frac{4}{3}\pi \times \left(\frac{a}{2\sqrt{2}}\right)^3}{a^3} = \frac{\pi}{3\sqrt{2}} = 74\%. \quad \begin{array}{l} \text{nickel} \\ \text{example: barium, copper, aluminium, lithium,} \\ \text{chromium, calcium, } \end{array}$$

[sc] 1 atom / unit cell,  $r = \frac{a}{2}$

$$\therefore f = \frac{\frac{4}{3}\pi \left(\frac{a}{2}\right)^3}{a^3} = \frac{\pi}{6} = 52\%.$$

Example: polonium, potassium chloride

- HW 1. Primitive translation vector of hcp lattice's  $\vec{a} = \frac{\sqrt{3}}{2}a\hat{i} + \frac{a}{2}\hat{j}$ ,  $\vec{b} = -\frac{\sqrt{3}}{2}a\hat{i} + \frac{a}{2}\hat{j}$ ,  $\vec{c} = c\hat{k}$ . Compute the volume of the primitive cell.

2. Show that for a fcc crystal structure, lattice constant is  $a = \left(\frac{4M}{\rho N}\right)^{\frac{1}{3}}$  where M is the gram molecular weight of molecules at lattice points,  $\rho$  is the density & N is Avogadro's number.

### NaCl structure

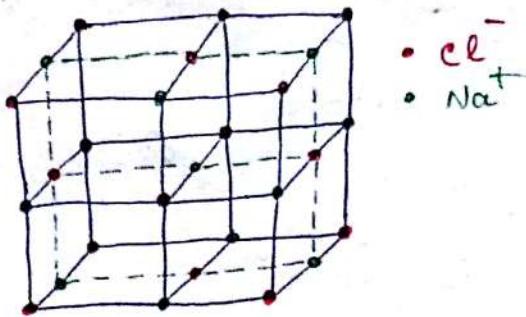
ionic crystal  $\text{Na}^+$  &  $\text{Cl}^-$ , fcc bravais lattice

Na  $(0,0,0)$   $(\frac{1}{2}, \frac{1}{2}, 0)$   $(\frac{1}{2}, 0, \frac{1}{2})$   $(0, \frac{1}{2}, \frac{1}{2})$

Cl  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$   $(0, 0, \frac{1}{2})$   $(0, \frac{1}{2}, 0)$   $(\frac{1}{2}, 0, 0)$

1 NaCl molecule in unit cube.

$\text{Na}^+(0,0,0) + \text{Cl}^-(\frac{a}{2}, 0, 0) \rightarrow$  6 nearest neighbour (coordination number).



Miller indices To designate the position & orientation of a crystal plane according to following rule:

(a) In terms of lattice constant, find the intercepts of the plane on crystal axes

$\vec{a}, \vec{b}, \vec{c}$  (primitive or nonprimitive)

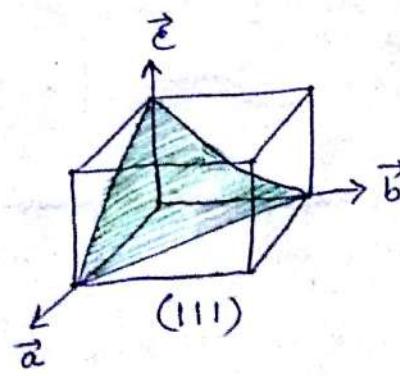
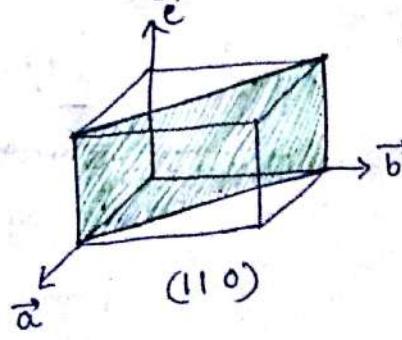
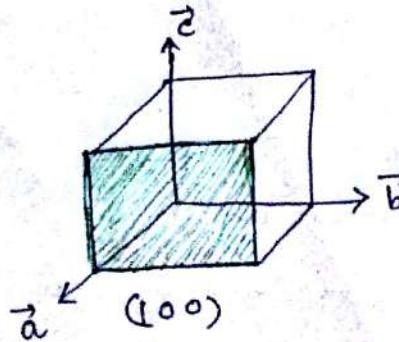
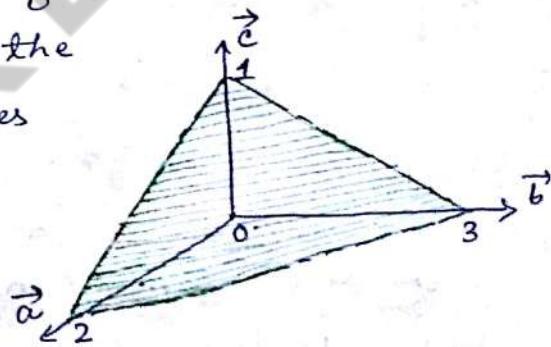
$(2,0,0), (0,3,0), (0,0,1) \rightarrow 2a, 3b, c.$

(b) Take reciprocals of them & reduce to smallest 3 integers,

Denote with  $(h, k, l)$

So  $2a, 3b, c \xrightarrow{\text{reciprocal}} \frac{1}{2}, \frac{1}{3}, 1 \xrightarrow{\text{smallest}} 3, 2, 6.$

Miller index is  $(3, 2, 6)$  plane.



If plane cuts negative side of axis, Miller index  $(h, k, l)$   
(say  $-\vec{b}$ )

6-faces of cubic crystal, Miller index  $(1, 0, 0), (0, 1, 0), (0, 0, 1)$   
because through rotation, all faces  $(\bar{1}, 0, 0), (0, \bar{1}, 0), (\bar{0}, 0, \bar{1})$  are equivalent & written in  $\S \S$ .

So  $(2, 0, 0)$  plane intercepts on  $\vec{a}, \vec{b}, \vec{c}$  are  $\frac{1}{2}a, \infty, \infty$ . & parallel  
(Miller index) to  $(1, 0, 0)$  &  $(\bar{1}, 0, 0)$  plane.

Indices of a direction  $[h, k, l]$  & direction is perpendicular to  
plane  $(h, k, l)$ .  $\vec{a}$  axis =  $[1, 0, 0]$ ,  $-\vec{b}$  axis =  $[0, \bar{1}, 0]$   
body diagonal =  $[1, 1, 1]$

\*

### Spacing of planes in sc lattice

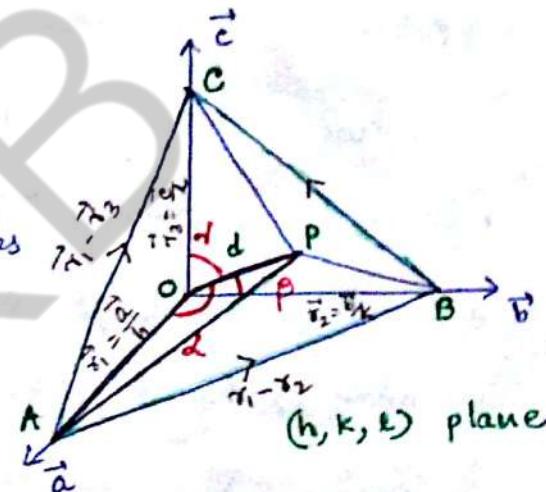
simple unit cell  $\vec{a} \perp \vec{b} \perp \vec{c}$  &  
a plane  $(h, k, l)$  (Miller index).

Intercepts  $a/h, b/k, c/l$  on  $\vec{a}, \vec{b}, \vec{c}$  axes

$OP \perp (h, k, l)$  plane &  $OP = d$ .

I  $\angle AOP = \alpha, \angle BOP = \beta, \angle COP = \gamma$ .

II  $\angle APO = \angle BPO = \angle CPO = 90^\circ$ .



$$\frac{OP}{OA} = \cos \alpha \quad \text{or} \quad OP = OA \cos \alpha \quad \text{or} \quad d = \frac{a}{h} \cos \alpha \quad \text{or} \cos \alpha = \frac{dh}{a}$$

$$\text{Similarly } \cos \beta = \frac{dk}{b}, \cos \gamma = \frac{dl}{c}.$$

Law of direction cosines,  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ .

$$\cos^2 d \left( \frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} \right) = 1.$$

$$\therefore d = \frac{1}{\sqrt{\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}}}$$

If cubic lattice,  $a=b=c$ ,  $d = \frac{a}{\sqrt{h^2+k^2+l^2}}$

$$d_{100} = \frac{a}{\sqrt{1+0+0}} = a, \quad d_{110} = \frac{a}{\sqrt{1+1+0}} = \frac{a}{\sqrt{2}}, \quad d_{111} = \frac{a}{\sqrt{1+1+1}} = \frac{a}{\sqrt{3}}$$

### Spacing of planes in bcc lattice

One atom at each corner + one atom at cube centre.  
(portion) (whole)

$\therefore d_{100} = \frac{a}{2}$  as additional  $(1,0,0)$  is there halfway between  $(100)$  plane of sc.

$d_{110} = d_{110}^{\text{sc}} = \frac{a}{\sqrt{2}}$ . but  $d_{111} = \frac{1}{2} \frac{a}{\sqrt{3}}$  as  $(1,1,1)$  plane lies midway of  $(111)$  plane of sc.

### Spacing of planes in fcc lattice

One atom at each corner + one atom at each face.  
(portion) (portion)

$\therefore d_{100} = \frac{a}{2}$  as additional  $(1,0,0)$  is there halfway between  $(1,00)$  plane of sc.

But  $d_{110} = \frac{1}{2} \frac{a}{\sqrt{2}}$  as additional set of  $(110)$  is there halfway between  $(1,1,0)$  plane.

$d_{111} = \frac{a}{\sqrt{3}}$  as centre of all face plane without new plane.

$$\textcircled{*} \quad \vec{r}_1 = \vec{a}/h, \vec{r}_2 = \vec{b}/k, \vec{r}_3 = \vec{c}/l.$$

$h\vec{a} + k\vec{b} + l\vec{c}$  represents  $[h, k, l]$

$$\text{Now } (\vec{r}_1 - \vec{r}_2) \cdot (h\vec{a} + k\vec{b} + l\vec{c}) = \left(\vec{a}/h - \frac{\vec{b}}{k}\right) \cdot (h\vec{a} + k\vec{b} + l\vec{c}) \\ = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 0. \quad (\text{as } |a| = |b|)$$

$$\text{Similarly } (\vec{r}_1 - \vec{r}_3) \cdot (h\vec{a} + k\vec{b} + l\vec{c}) = 0 \quad (\text{as } |a| = |c|)$$

As vectors  $\vec{r}_1 - \vec{r}_2$  &  $\vec{r}_1 - \vec{r}_3$  lie in  $(h, k, l)$  plane, so  $[h, k, l]$  is perpendicular to plane  $(h, k, l)$ .

Reciprocal lattice To represent slope & interplanar spacing<sup>5</sup> of crystal plane, each set of parallel plane in a space lattice is represented by normals of planes with length =  $\frac{1}{\text{interplanar spacing}}$  points marked at ends.

points form regular arrangement  $\rightarrow$  reciprocal lattice

for  $\vec{a}, \vec{b}, \vec{c}$ , we describe reciprocal basis vectors  $\vec{a}^*, \vec{b}^*, \vec{c}^*$

(primitive) such that  $\vec{a} \cdot \vec{a}^* = 2\pi, \vec{b} \cdot \vec{a}^* = 0, \vec{c} \cdot \vec{a}^* = 0$   
 $\vec{a} \cdot \vec{b}^* = 0, \vec{b} \cdot \vec{b}^* = 2\pi, \vec{c} \cdot \vec{b}^* = 0$   
 $\vec{a} \cdot \vec{c}^* = 0, \vec{b} \cdot \vec{c}^* = 0, \vec{c} \cdot \vec{c}^* = 2\pi.$

$$\vec{a}^* = 2\pi \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot \vec{b} \times \vec{c}}, \vec{b}^* = 2\pi \frac{\vec{c} \times \vec{a}}{\vec{a} \cdot \vec{b} \times \vec{c}}, \vec{c}^* = 2\pi \frac{\vec{a} \times \vec{b}}{\vec{a} \cdot \vec{b} \times \vec{c}}$$

reciprocal lattice vector  $\vec{r}^* = h\vec{a}^* + k\vec{b}^* + l\vec{c}^*$

properly (i) reciprocal lattice is normal to lattice plane of direct crystal lattice-

$$\vec{r}^* \cdot (\vec{r}_1 - \vec{r}_2) = (h\vec{a}^* + k\vec{b}^* + l\vec{c}^*) \cdot \left( \frac{\vec{a}}{h} - \frac{\vec{b}}{k} \right) = 0.$$

Similarly  $\vec{r}^* \cdot (\vec{r}_1 - \vec{r}_3) = 0$ .

(ii) direct lattice is reciprocal of reciprocal lattice.

sc = self-reciprocal.

bcc  $\leftrightarrow$  fcc reciprocal of each other.

Definition of R.L.  $\vec{r} = n_1 \vec{a} + n_2 \vec{b} + n_3 \vec{c}$  direct lattice vector &

say  $\vec{k}$  constitutes a plane wave  $e^{i\vec{k} \cdot \vec{r}}$  which may not have the periodicity of Bravais lattice but  $\vec{R}$  has that periodicity.

$$e^{i\vec{k} \cdot (\vec{r} + \vec{T})} = e^{i\vec{k} \cdot \vec{r}} \quad \text{or} \quad e^{i\vec{k} \cdot \vec{T}} = 1$$

$$\vec{R} = k_1 \vec{a}^* + k_2 \vec{b}^* + k_3 \vec{c}^* =$$

$$\therefore \vec{R} \cdot \vec{T} = 2\pi(k_1 n_1 + k_2 n_2 + k_3 n_3)$$

If  $e^{i\vec{k} \cdot \vec{T}} = 1$ , then  $\vec{R} \cdot \vec{T}$  must be  $2\pi \times$  integer  $\Rightarrow k_1, k_2, k_3$  integers

So from  $\vec{K}$  only  $\vec{K}$  which is linear combination of  $\vec{a}^*, \vec{b}^*, \vec{c}^*$  with integral coefficient makes  $\vec{K}$  a reciprocal lattice vector.

### Reciprocal of reciprocal lattice

As by construction, reciprocal lattice is a "Browais lattice", reciprocal gives back the direct lattice.

HW Define  $\vec{a}^{**} = 2\pi \frac{\vec{b}^* \times \vec{c}^*}{\vec{a}^* \cdot \vec{b}^* \times \vec{c}^*}$ , ~~but that's not true~~

$$\vec{b}^{**} = 2\pi \frac{\vec{c}^* \times \vec{a}^*}{\vec{a}^* \cdot \vec{b}^* \times \vec{c}^*}, \quad \vec{c}^{**} = 2\pi \frac{\vec{a}^* \times \vec{b}^*}{\vec{a}^* \cdot \vec{b}^* \times \vec{c}^*} \text{ as three}$$

vectors generated by primitive vectors  $\vec{a}^*, \vec{b}^*, \vec{c}^*$ . Check first,  $\vec{a}^* \cdot \vec{b}^* \times \vec{c}^* = \frac{(2\pi)^3}{\vec{a} \cdot \vec{b} \times \vec{c}}$  & then show that  $\vec{a}^{**} = \vec{a}$ .

$$\vec{b}^{**} = \vec{b}, \quad \vec{c}^{**} = \vec{c}.$$

### Reciprocal of sc lattice

$$\vec{a} = \hat{a}\vec{i}, \quad \vec{b} = \hat{b}\vec{j}, \quad \vec{c} = \hat{c}\vec{k}.$$

$$\therefore \vec{a}^* = 2\pi \frac{\hat{b}\vec{j} \times \hat{c}\vec{k}}{\hat{a}\vec{i} \cdot (\hat{b}\vec{j} \times \hat{c}\vec{k})} = 2\pi \frac{\hat{b}\hat{c}}{\hat{a}\hat{b}\hat{c}} \vec{i} = \frac{2\pi}{a} \vec{i}$$

$$\vec{b}^* = 2\pi \frac{\hat{c}\vec{k} \times \hat{a}\vec{i}}{\hat{a}\vec{i} \cdot (\hat{b}\vec{j} \times \hat{c}\vec{k})} = \frac{2\pi}{b} \vec{j} = \frac{2\pi}{a} \vec{j} \quad (a=b=c)$$

$$\vec{c}^* = 2\pi \frac{\hat{a}\vec{i} \times \hat{b}\vec{j}}{\hat{a}\vec{i} \cdot (\hat{b}\vec{j} \times \hat{c}\vec{k})} = \frac{2\pi}{c} \vec{k} = \frac{2\pi}{a} \vec{k}.$$

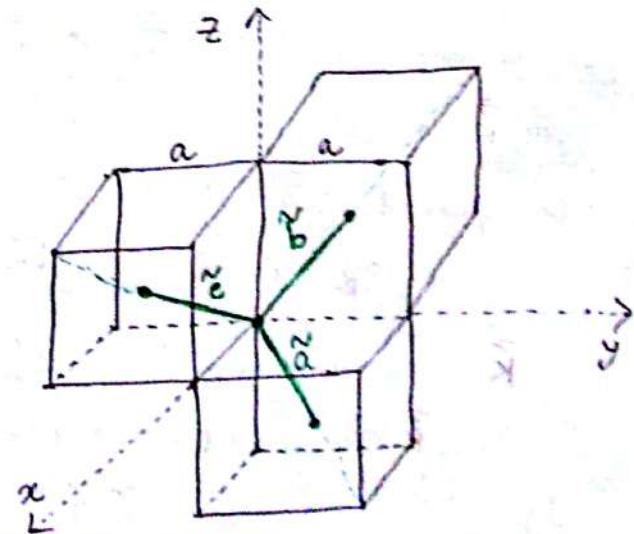
Lattice constant =  $2\pi/a$ .

### Reciprocal of bcc lattice

$$\vec{a} = \frac{a}{2}(\hat{i} + \hat{j} - \hat{k})$$

$$\vec{b} = \frac{a}{2}(-\hat{i} + \hat{j} + \hat{k})$$

$$\vec{c} = \frac{a}{2}(\hat{i} - \hat{j} + \hat{k}).$$



$$\therefore \text{volume of primitive cell} = \vec{a} \cdot \vec{b} \times \vec{c} = a^3/2.$$

$$\therefore \vec{a}^* = 2\pi \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot \vec{b} \times \vec{c}} = \frac{2\pi}{a} (\hat{i} + \hat{j}),$$

$$\vec{b}^* = 2\pi \frac{\vec{c} \times \vec{a}}{\vec{a} \cdot \vec{b} \times \vec{c}} = \frac{2\pi}{a} (\hat{j} + \hat{k}).$$

$$\vec{c}^* = 2\pi \frac{\vec{a} \times \vec{b}}{\vec{a} \cdot \vec{b} \times \vec{c}} = \frac{2\pi}{a} (\hat{i} + \hat{k}).$$

Reciprocal of fcc lattice

$$\vec{a} = \frac{a}{2} (\hat{i} + \hat{j}), \quad \vec{b} = \frac{a}{2} (\hat{j} + \hat{k})$$

$$\vec{c} = \frac{a}{2} (\hat{i} + \hat{k})$$

$$\text{volume of primitive cell} = \vec{a} \cdot \vec{b} \times \vec{c} = a^3/4.$$

$$\text{and } \vec{a}^* = \frac{2\pi}{a} (\hat{i} + \hat{j} - \hat{k}), \quad \vec{b}^* = \frac{2\pi}{a} (-\hat{i} + \hat{j} + \hat{k}), \quad \vec{c}^* = \frac{2\pi}{a} (\hat{i} - \hat{j} + \hat{k})$$

$\therefore$  Reciprocal bcc lattice vectors = primitive fcc lattice vectors

Reciprocal fcc lattice vectors = primitive bcc lattice vectors

## Crystal diffraction

Why use x-ray for crystallography?

Atomic spacing (say for NaCl) is  $2.8 \text{ \AA}$ . When x-ray is produced by accelerating electrons through a potential difference V,

$$eV = h\nu = \frac{hc}{\lambda} \quad \text{or} \quad \lambda = \frac{hc}{eV} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{1.6 \times 10^{-19} \times 10^4} \quad (\text{say } V = 10 \text{ kV}) \\ = 1.24 \text{ \AA}$$

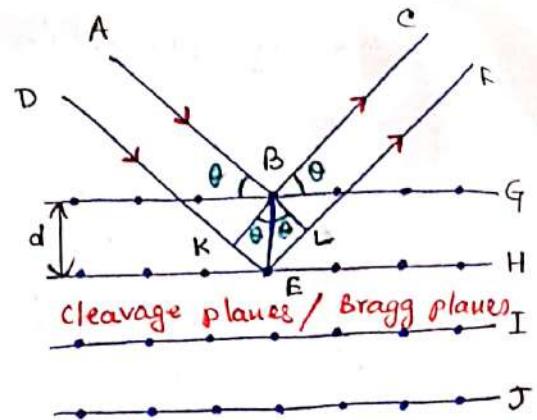
$\lambda_{x\text{-ray}} \approx a$  (elastic scattering without change in  $\lambda$ )

$\lambda_{\text{visible/UV}} \gg a$  (reflection or refraction)

$\lambda_{x\text{-ray}} \ll a$  (small angle diffraction)

## Bragg's law for crystal diffraction

Maximum intensity from reflected beam (waves) from two different atomic planes (cleavage planes) with path difference equal to integral multiple of  $\lambda_{\text{x-ray}}$ .



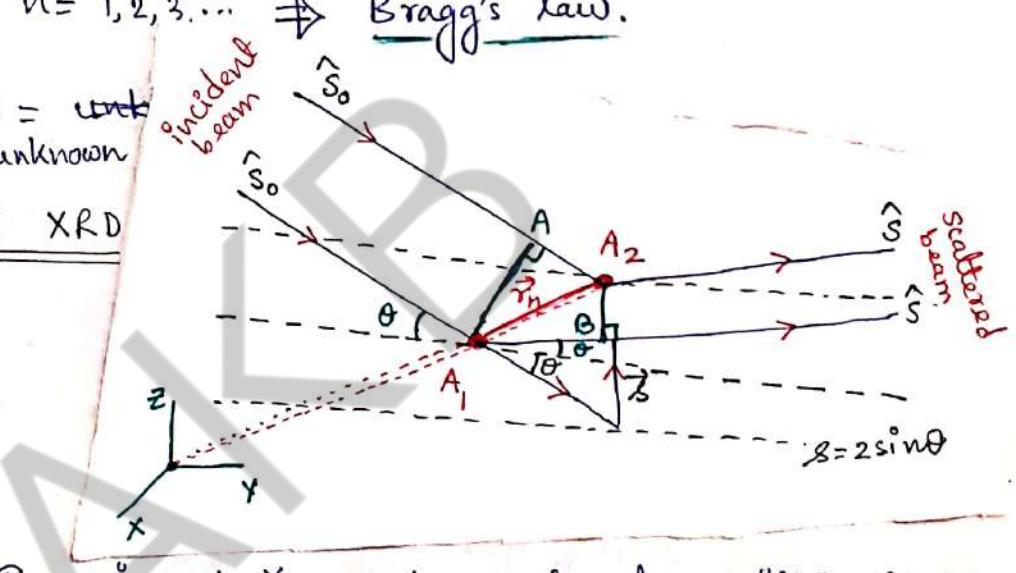
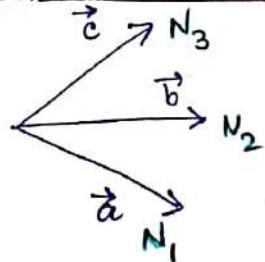
Path difference between ray [AB, BC] & [DE, EF] is  $KE + EL$

$= d \sin \theta + d \sin \theta = 2d \sin \theta$ . So for constructive interference,

$$2d \sin \theta = n\lambda, \quad n = 1, 2, 3, \dots \Rightarrow \text{Bragg's law.}$$

$\lambda, \theta$  known,  $d$  unknown

Lau's equation of XRD



Assumptions : (a) The primary X-ray beam travels within the crystal at the speed of light. (b) Each scattered wavelet travels through the crystal without getting scattered.

Say  $N_1$  number of points along direction  $\vec{a}$

$N_2$  number of points along direction  $\vec{b}$

$N_3$  number of points along direction  $\vec{c}$

Total  $N = N_1 N_2 N_3$  points in the crystal lattice.

Path difference between two x-rays is  $d = \vec{r}_n \cdot \vec{s} - \vec{r}_n \cdot \vec{s}_0 = \vec{r}_n \cdot \vec{s}$

$\therefore$  Phase difference is  $\frac{2\pi}{\lambda} d = \frac{2\pi}{\lambda} \vec{r}_n \cdot \vec{s} = k \vec{r}_n \cdot \vec{s}$

remember:  $\vec{s}, \vec{s}_0$  unit vector,  $|\vec{s}| = \beta = a \sin \theta$ ,  $\vec{r}_n = n^{\text{th}}$  lattice point from origin =  $\vec{r} = n_1 \vec{a} + n_2 \vec{b} + n_3 \vec{c}$ .

$$\vec{r}_n = \vec{T}$$

If  $y$  is the displacement of the scattered wave from origin at a distance  $R$  at time  $t$  with amplitude  $A_0$ , then

$$y_0 = \frac{A_0}{R} e^{i\omega t}. \therefore \text{displacement from } \vec{r}_n \text{ is}$$

$$y = \frac{A_0}{R} e^{i\omega t} e^{iK\vec{r}_n \cdot \vec{R}}$$

$\therefore$  Total displacement due to the whole Bravais lattice is

$$Y = \sum \frac{A_0}{R} e^{i\omega t} e^{iK\vec{r}_n \cdot \vec{R}}$$

$$= \sum_{\substack{\text{all points} \\ n_1=0}}^{N_1-1} \sum_{\substack{n_2=0 \\ n_2=0}}^{N_2-1} \sum_{\substack{n_3=0 \\ n_3=0}}^{N_3-1} e^{iK[(n_1\vec{a} + n_2\vec{b} + n_3\vec{c}) \cdot \vec{R}]} \frac{A_0}{R} e^{i\omega t}$$

$$= \frac{A_0}{R} e^{i\omega t} \sum_{n_1=0}^{N_1-1} e^{iKn_1\vec{a} \cdot \vec{R}} \sum_{n_2=0}^{N_2-1} e^{iKn_2\vec{b} \cdot \vec{R}} \sum_{n_3=0}^{N_3-1} e^{iKn_3\vec{c} \cdot \vec{R}}$$

$$\text{Now } \sum_{n_1=0}^{N_1-1} e^{iKn_1\vec{a} \cdot \vec{R}} = 1 + e^{iK\vec{a} \cdot \vec{R}} + e^{i2K\vec{a} \cdot \vec{R}} + \dots + e^{i(N_1-1)K\vec{a} \cdot \vec{R}}$$

$$= \frac{1 - e^{iN_1(\vec{a} \cdot \vec{R})K}}{1 - e^{i(\vec{a} \cdot \vec{R})K}}$$

$$\therefore \left( \sum_{n_1=0}^{N_1-1} e^{iKn_1\vec{a} \cdot \vec{R}} \right) \left( \sum_{n_1=0}^{N_1-1} e^{iKn_1\vec{a} \cdot \vec{R}} \right)^* = \frac{1 - e^{iN_1(\vec{a} \cdot \vec{R})K}}{1 - e^{-iN_1(\vec{a} \cdot \vec{R})K}} \times \frac{1 - e^{-iN_1(\vec{a} \cdot \vec{R})K}}{1 - e^{iN_1(\vec{a} \cdot \vec{R})K}}$$

$$= \frac{1 - \cos \{ N_1(\vec{a} \cdot \vec{R})K \} + i \sin \{ N_1(\vec{a} \cdot \vec{R})K \}}{1 - \cos \{ (\vec{a} \cdot \vec{R})K \} - i \sin \{ (\vec{a} \cdot \vec{R})K \}} \times$$

$$\frac{1 - \cos \{ N_1(\vec{a} \cdot \vec{R})K \} + i \sin \{ N_1(\vec{a} \cdot \vec{R})K \}}{1 - \cos \{ (\vec{a} \cdot \vec{R})K \} + i \sin \{ (\vec{a} \cdot \vec{R})K \}}$$

$$= \frac{(1 - \cos \{ N_1(\vec{a} \cdot \vec{R})K \})^2 + (\sin \{ N_1(\vec{a} \cdot \vec{R})K \})^2}{(1 - \cos \{ (\vec{a} \cdot \vec{R})K \})^2 + (\sin \{ (\vec{a} \cdot \vec{R})K \})^2}$$

$$= \frac{1 - \cos \{ N_1(\vec{a} \cdot \vec{R})K \}}{1 - \cos \{ (\vec{a} \cdot \vec{R})K \}} = \frac{\sin^2 \frac{N_1(\vec{a} \cdot \vec{R})K}{2}}{\sin^2 \frac{(\vec{a} \cdot \vec{R})K}{2}} = \frac{\sin^2 (N_1 \psi_1)}{\sin^2 (\psi_1)}$$

where  $\Psi_1 = \frac{1}{2} K \vec{a} \cdot \vec{s}$ .

$$\therefore \text{Total intensity } I = YY^* = \left(\frac{|A_0|}{R}\right)^2 \frac{\sin^2(N_1 \Psi_1)}{\sin^2 \Psi_1} \frac{\sin^2(N_2 \Psi_2)}{\sin^2 \Psi_2} \frac{\sin^2(N_3 \Psi_3)}{\sin^2 \Psi_3}$$

$$\Psi_1 = \frac{1}{2} K \vec{a} \cdot \vec{s} = \frac{1}{2} K |\vec{a}| |\vec{s}| \cos d = \frac{1}{2} \frac{2\pi}{\lambda} a 2 \sin \theta \cos d = \frac{2\pi a \sin \theta \cos d}{\lambda}$$

similarly  $\Psi_2 = \frac{1}{2} K \vec{b} \cdot \vec{s} = \frac{2\pi b \sin \theta \cos \beta}{\lambda}$ ,

$$\Psi_3 = \frac{1}{2} K \vec{c} \cdot \vec{s} = \frac{2\pi c \sin \theta \cos \gamma}{\lambda}$$

[Notice the analogy of  $\vec{s}$  with  $[h, k, l]$  plane with angles  $d, \gamma, \beta$ ]

In  $\lim_{\Psi_1 \rightarrow h\pi}$ ,  $\frac{\sin^2(N_1 \Psi_1)}{\sin^2 \Psi_1}$  is maximum  $= N_1^2$

Similarly  $\lim_{\Psi_2 \rightarrow k\pi} \frac{\sin^2(N_2 \Psi_2)}{\sin^2 \Psi_2} = N_2^2$ ,  $\lim_{\Psi_3 \rightarrow l\pi} \frac{\sin^2(N_3 \Psi_3)}{\sin^2 \Psi_3} = N_3^2$

$$\text{Then } I_{\max} = \left(\frac{|A_0|}{R}\right)^2 N_1^2 N_2^2 N_3^2 = \frac{|A_0|^2}{R^2} N^2$$

$$\therefore \frac{2\pi a \sin \theta \cos d}{\lambda} = h\pi,$$

$$\frac{2\pi b \sin \theta \cos \beta}{\lambda} = k\pi,$$

$$\frac{2\pi c \sin \theta \cos \gamma}{\lambda} = l\pi,$$

$$2a \sin \theta \cos d = h\lambda$$

$$2b \sin \theta \cos \beta = k\lambda$$

$$2c \sin \theta \cos \gamma = l\lambda$$

"Lau equations".

Bragg's law from Lau equations

from Lau equation, direction

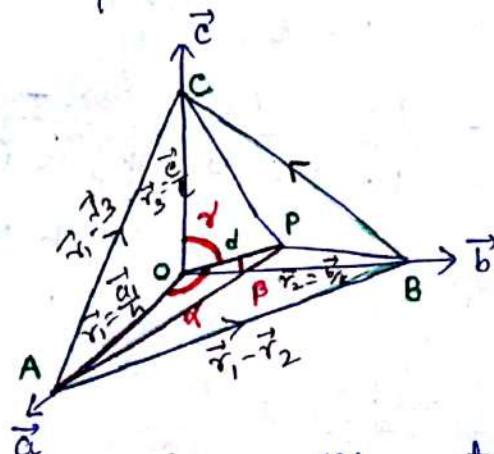
cosines of  $\vec{s}$  are

$$\cos d = \frac{h\lambda}{2a \sin \theta}, \cos \beta = \frac{k\lambda}{2b \sin \theta},$$

$$\cos \gamma = \frac{l\lambda}{2c \sin \theta}.$$

But also see that if  $(h, k, l)$  is a miller plane with equation

$$\frac{x}{a/h} + \frac{y}{b/k} + \frac{z}{c/l} = 1 \quad \text{then} \quad \frac{a}{h} \cos d = \frac{b}{k} \cos \beta = \frac{c}{l} \cos \gamma = d.$$



∴ The direction cosines of  $\vec{s}$  are also proportional to  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ , so the X-ray is diffracted from  $\vec{s}_0$  to  $\vec{s}$  by the miller plane  $(h, k, l)$ .

$$\therefore d = \frac{a}{h} \cos\alpha = \frac{a}{h} \frac{h\lambda}{2bs\sin\theta} = \frac{\lambda}{2s\sin\theta}$$

$$= \frac{b}{k} \cos\beta = \frac{b}{k} \frac{k\lambda}{2bs\sin\theta} = \frac{\lambda}{2s\sin\theta}$$

$$= \frac{c}{l} \cos\gamma = \frac{c}{l} \frac{l\lambda}{2cs\sin\theta} = \frac{\lambda}{2s\sin\theta}$$

Note that  $h, k, l$  of Laue equation aren't necessarily identical with Miller indices but may contain a common factor  $n$ .

$$\therefore 2ds\sin\theta = n\lambda$$

with  $d$  = adjacent interplanar spacing with Miller indices

$$\frac{h}{n}, \frac{k}{n} \text{ & } \frac{l}{n}$$

### Interpretation of Laue's equation in reciprocal lattice

Reciprocal lattice vector  $\vec{r}^* = \vec{q} = \vec{h}\vec{a}^* + \vec{k}\vec{b}^* + \vec{l}\vec{c}^*$   
magnitude = reciprocal of spacing of  $(h, k, l)$  planes of direct lattice.

direction = perpendicular to  $(h, k, l)$  plane.

$$\vec{G} \cdot \vec{a} = \vec{r}^* \cdot \vec{a} = 2\pi h \quad \left. \right\}$$

$$\vec{G} \cdot \vec{b} = \vec{r}^* \cdot \vec{b} = 2\pi k \quad \left. \right\}$$

$$\vec{G} \cdot \vec{c} = \vec{r}^* \cdot \vec{c} = 2\pi l \quad \left. \right\}$$

From Laue equation,  $\Psi_1 = \frac{1}{2} \vec{k} \vec{a} \cdot \vec{s} = h\pi \Rightarrow \frac{1}{2} \frac{2\pi}{\lambda} \vec{s} \cdot \vec{a} = h\pi$

$$\Rightarrow \frac{2\pi \vec{s}}{\lambda} \cdot \vec{a} = 2\pi h.$$

Similarly from  $\Psi_2$  &  $\Psi_3$ ,  $\frac{2\pi \vec{s}}{\lambda} \cdot \vec{b} = 2\pi k$ ,  $\frac{2\pi \vec{s}}{\lambda} \cdot \vec{c} = 2\pi l \quad \left. \right\}$

Comparing,

$$\boxed{\vec{r}^* = \vec{G} = \frac{2\pi \vec{s}}{\lambda}}$$

## Ewald's construction

Geometrical construction to obtain a relation between wave vector  $\vec{K}$  & the direction of incident X-ray using the reciprocal lattice & deducing Bragg's law in vectorial form.

$$\vec{K} = \frac{2\pi}{\lambda} \text{ (magnitude), direction}$$

along X-ray beam from O & terminating at point A.

From O with radius  $k = \frac{2\pi}{\lambda}$ , draw a sphere (reflex sphere).

Suppose it intersects B, then  $\vec{AB}$  represents reciprocal vector  $\vec{G}$  &  $G \perp OC$  (direct lattice plane)

$$G = \frac{2\pi n}{d}$$

$\vec{K}'$  = diffracted (reflected) wave vector, with  $|\vec{K}| = |\vec{K}'|$

So magnitude is same, only direction changes.

$$\vec{K}' = \vec{K} + \vec{G}$$

$$\cancel{|\vec{K}'|^2} = (\vec{K} + \vec{G}) \cdot (\vec{K} + \vec{G}) = \cancel{|\vec{K}|^2} + 2\vec{K} \cdot \vec{G} + \vec{G} \cdot \vec{G}$$

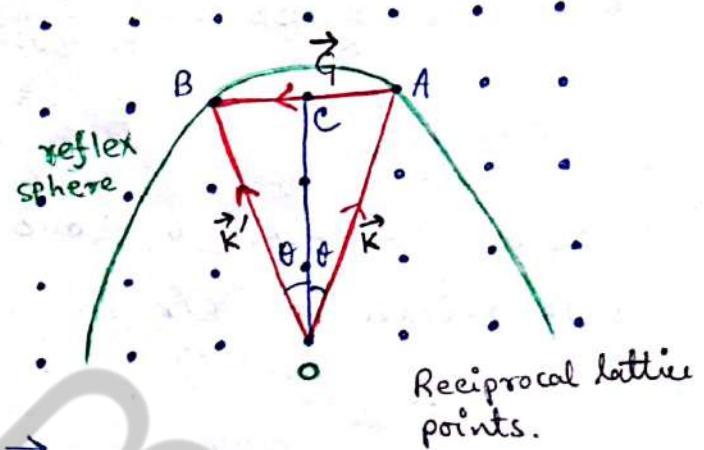
$$\therefore (\vec{K} + \frac{\vec{G}}{2}) \cdot \vec{G} = 0 \Rightarrow \text{Bragg's law (vectorial form) in reciprocal lattice.}$$

Notice that  $AC = OA \sin\theta = CB$ .

$$\therefore AB = 2OA \sin\theta = 2K \sin\theta = 2 \frac{2\pi}{\lambda} \sin\theta$$

$$\therefore G = \frac{4\pi}{\lambda} \sin\theta \quad \therefore \frac{2\pi n}{d} = \frac{4\pi}{\lambda} \sin\theta$$

$$\therefore 2dsin\theta = n\lambda$$



Reciprocal lattice points.

CW 1. Calculate wavelength & speed of neutron beam, where spacing between successive (100) planes is  $3.84 \text{ \AA}$ , grazing angle is  $30^\circ$  & order of Bragg reflection = 1.

Bragg's Law  $2ds\sin\theta = n\lambda$ ,

$$d = 3.84 \times 10^{-10} \text{ m}, \theta = 30^\circ, n=1 \quad \therefore 2 \times 3.84 \times 10^{-10} \times \frac{1}{2} = \lambda$$

$$\therefore \lambda = 3.84 \text{ \AA}.$$

Using de-Broglie relation  $\lambda = \frac{h}{p} = \frac{h}{mv}$

$$v = \frac{h}{m\lambda} = \frac{6.62 \times 10^{-34} \text{ Js}}{1.67 \times 10^{-27} \text{ kg} \times 3.84 \times 10^{-10} \text{ m}} = 1.03 \times 10^3 \text{ m/s}$$

$$= 1.03 \text{ km/s.}$$

2. X-ray of wavelength  $1.24 \text{ \AA}$  is reflected by cubic crystal KCl.

Calculate the interplanar distance for (100), (110) & (111) planes.

Given density of KCl =  $1.98 \times 10^3 \text{ kg/m}^3$ , molecular weight  $74.5 \text{ kg}$ .

Avogadro's no.  $N = 6.023 \times 10^{26} \text{ kg/mole.}$

for cubic crystal,  $a = \left(\frac{NM}{\rho N_A}\right)^{1/3}$ .

$$\text{for KCl, } n=4, \quad a = \left(\frac{4 \times 74.5}{1.98 \times 10^3 \times 6.023 \times 10^{26}}\right)^{1/3} = 6.3 \times 10^{-10} \text{ m} = 6.3 \text{ \AA}$$

$$\therefore d_{100} = \frac{a}{\sqrt{1^2+0^2+0^2}} = \frac{6.3 \text{ \AA}}{2} = \frac{a}{\sqrt{1^2+1^2+0^2}} = \frac{a}{\sqrt{2}} = \frac{a}{2} = \frac{6.3 \text{ \AA}}{2} = 3.15 \text{ \AA}.$$

$$d_{111} = \frac{a}{\sqrt{1^2+1^2+1^2}} = \frac{a}{\sqrt{3}} = 3.63 \text{ \AA}.$$

(remember KCl is fcc).

3.(a) Calculate the Bragg angle for x-rays with  $\lambda = 1.54 \text{ \AA}$  in different orders 1, 2, 3 if interplanar spacing is  $2.67 \text{ \AA}$ . (b) If Bragg glancing angle is  $15^\circ$  for 1<sup>st</sup> order, then calculate glancing angles for 2<sup>nd</sup> & 3<sup>rd</sup> order spectrum?

$$2ds\sin\theta = n\lambda.$$

$$\lambda = 1.54 \times 10^{-10} \text{ m}, \quad d = 2.67 \times 10^{-10} \text{ m}.$$

$$n=1 \text{ (1st order)} \quad 2d \sin \theta_1 = \lambda$$

$$\theta_1 = \sin^{-1} \left[ \frac{\lambda}{2d} \right] = \sin^{-1} \left[ \frac{1.54 \times 10^{-10}}{2 \times 2.67 \times 10^{-10}} \right] = 16.76^\circ.$$

$$n=2 \text{ (2nd order)} \quad \theta_2 = \sin^{-1} \left[ \frac{2\lambda}{2d} \right] = 35.22^\circ.$$

$$n=3 \text{ (3rd order)} \quad \theta_3 = \sin^{-1} \left[ \frac{3\lambda}{2d} \right] = 59.9^\circ.$$

$$(b) \quad 2d \sin \theta_1 = \lambda, \quad \theta_1 = 15^\circ \quad \therefore \cancel{2d} = \frac{\lambda}{2 \sin 15^\circ} = 0.2588$$

$$\text{So for 2nd order, } \sin \theta_2 = 2 \frac{\lambda}{2d} = 2 \times 0.2588 = 0.5176$$

$$\theta_2 = 31.17^\circ.$$

$$\text{for 3rd order, } \sin \theta_3 = 3 \frac{\lambda}{2d} = 3 \times 0.2588 = 0.7769$$

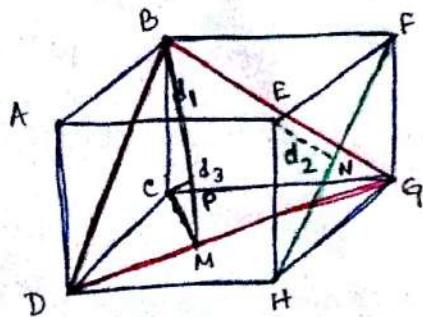
$$\theta_3 = 50.93^\circ.$$

HW 1. Molecular weight of rock salt ( $\text{NaCl}$ ) crystal is 52.5 kg/kilomole & density  $2.16 \times 10^3 \text{ kg/m}^3$ . Calculate grating spacing  $d_{100}$  of rock salt. Using that, calculate  $\lambda$  of X-rays in 2nd order if angle of diffraction is  $26^\circ$ .

2. If X-rays with  $\lambda = 0.5 \text{ \AA}$  is diffracted at  $5^\circ$  in 1st order, what is the spacing between adjacent planes of a crystal? At ~~what~~ what angle will 2nd maximum occur?

3. Bragg angle for 1st order reflection from (111) plane of a crystal is  $60^\circ$ , when  $\lambda = 1.8 \text{ \AA}$ . Calculate interatomic spacing.

### Determination of crystal structure



$d$  is to be calculated for given X-ray ( $\lambda$ ) by using different plane.

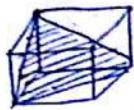
ABFE & CGHD.

$d_1$  distance apart.  $\rightarrow$  Total 6 faces (100) plane.



Diagonal plane BFHD inclined at  $\pi/4$  to (100) planes

$d_2$  is interplanar spacing  $\frac{d_2}{d_1} = \sin 45^\circ = \frac{1}{\sqrt{2}}$   $\therefore d_2 = \frac{d_1}{\sqrt{2}}$ .  
 (110) plane.



BGD plane. Here  $CM \perp DG$  &  $BM$  joined to obtain right-angle triangle BCM.  $CM = d_2$

$$BM = \sqrt{d_1^2 + d_2^2} \quad CP = d_3,$$

$$\sin B = \frac{d_3}{d_1} = \frac{d_2}{\sqrt{d_1^2 + d_2^2}}$$

$$\therefore d_3 = \frac{d_1 d_2}{\sqrt{d_1^2 + d_2^2}} = \frac{d_1}{\sqrt{3}} \quad (\text{substitute } d_2 = \frac{d_1}{\sqrt{2}}).$$

These are (111) planes.

$$\therefore \frac{1}{d_1} : \frac{1}{d_2} : \frac{1}{d_3} = 1 : \sqrt{2} : \sqrt{3}$$

Bragg found for KCl crystal for 1<sup>st</sup> order reflection

$$\theta_1 \text{ (from (100) plane)} = 5.22^\circ \quad \theta_3 \text{ (from (111) plane)} = 9.05^\circ.$$

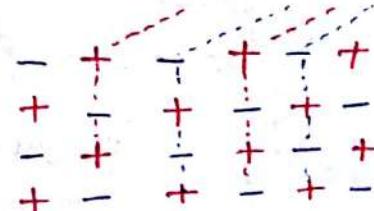
$$\theta_2 \text{ (from (110) plane)} = 7.30^\circ$$

$$\text{as } \frac{1}{d} = \frac{2 \sin \theta}{\lambda} \quad \therefore \frac{1}{d_1} : \frac{1}{d_2} : \frac{1}{d_3} = \sin 5.22 : \sin 7.30 : \sin 9.05 \\ = 0.0910 : 0.1272 : 0.1570 \\ = 1 : 1.40 : 1.73 = 1 : \sqrt{2} : \sqrt{3}.$$

So KCl has cubic crystal symmetry.

### NaCl crystal

8 ions at corner  $\rightarrow 4 \text{ Na}^+, 4 \text{ Cl}^-$



$\therefore$  Each ion of NaCl is shared between two adjacent cube & unit cell contain half a molecule of NaCl.

$$\text{mass of unit cell} = \frac{M}{2N} = \frac{23 + 35.5}{2 \times 6.023 \times 10^{26}} \text{ kg.}$$

$$\text{density of NaCl} = 2.17 \times 10^3 \text{ kg/m}^3.$$

$$\therefore \text{volume } d^3 = \frac{58.5}{2 \times 6.023 \times 10^{26} \times 2.17 \times 10^3} \quad \therefore d = 2.814 \text{ \AA}.$$

Now verify Bragg's law for different order of diffraction.

1<sup>st</sup> order,  $n=1$ ,  $\theta = 11.8^\circ$

$$\lambda = \frac{2d \sin \theta}{n} = 2 \times 2.814 \times 10^{-10} \times \sin 11.8^\circ \\ = 1.12 \text{ \AA}$$

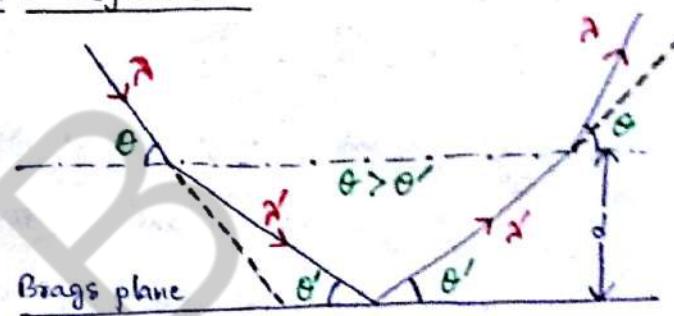
2<sup>nd</sup> order,  $n=2$ ,  $\theta = 23.5^\circ$ ,  $\lambda = \frac{2d \sin \theta}{2} = \frac{2.814 \times 10^{-10}}{2} \times \sin 23.5^\circ \\ = 1.12 \text{ \AA}$

3<sup>rd</sup> order,  $n=3$ ,  $\theta = 36^\circ$ ,  $\lambda = \frac{2d \sin \theta}{3} = \frac{2}{3} \times 2.814 \times 10^{-10} \times \sin 36^\circ \\ = 1.12 \text{ \AA}$

∴ Diffraction from NaCl crystal verified Bragg's law.

### Modification of Bragg's law due to refraction

Refraction of X-rays due to change in wavelength & angle of incidence because of the refractive index of the crystal.



Bragg's equation  $n\lambda' = 2d \sin \theta'$

Using Snell's law, the refractive index is  $\mu = \frac{\lambda}{\lambda'} = \frac{\cos \theta}{\cos \theta'}$

$$\therefore n \frac{\lambda}{\mu} = 2d \sqrt{1 - \frac{\cos^2 \theta}{\mu^2}}$$

$$\text{or } n\lambda = 2d \sqrt{\mu^2 - \cos^2 \theta} = 2d \sqrt{\sin^2 \theta - (1 - \mu^2)} = 2d \sin \theta \sqrt{1 - \frac{1 - \mu^2}{\sin^2 \theta}}$$

$$\approx 2d \sin \theta \left(1 - \frac{1 - \mu^2}{2 \sin^2 \theta}\right)$$

$$[1 - \mu^2 = (1 + \mu)(1 - \mu)]$$

$$\approx 2d \sin \theta \left(1 - \frac{2(1 - \mu)}{2 \sin^2 \theta}\right)$$

$$\approx 2(1 - \mu) \text{ as } \mu \approx 1$$

$$\approx 2d \sin \theta \left(1 - (1 - \mu) \frac{4\theta^2}{n^2 \lambda^2}\right)$$

$$[2d \sin \theta = n\lambda \text{ or } \frac{1}{\sin^2 \theta} = \frac{4d^2}{n^2 \lambda^2}]$$

$$n\lambda = 2d \sin \theta \left[1 - \frac{4d^2(1 - \mu)}{n^2 \lambda^2}\right]$$

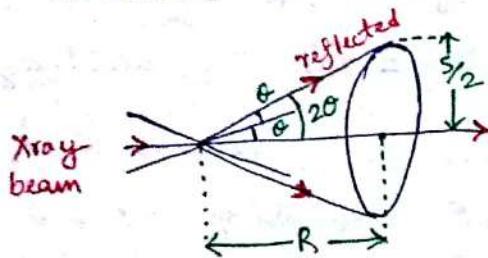
Karpest! The correction term  $\frac{4d^2(1 - \mu)}{n^2 \lambda^2}$  is small & becomes more small as "n" increases.

## Powder Method of XRD

$2ds\sin\theta = n\lambda$ ,  $d$  &  $\theta$  varies for fixed  $\lambda$ . Powdered specimen is kept in a thin capillary tube on a movable mount at the centre of a cylindrical camera.

For arbitrary orientation, some planes satisfy Bragg reflection  $2ds\sin\theta = n\lambda$ .

They lie on a conical section with semi-vertical angle  $2\theta$ . Other cones arise due to other set of planes. Cones intersect X-ray film in concentric rings with sharp centre. Specimen is rotated to ensure all possible planes to face the X-rays.



$S$  = distance between diffracted lines  
 $R$  = radius of the film

$$\frac{S}{2R} = 2\theta \text{ or } \theta = \frac{S}{4R} \quad d \sin\theta \approx d$$

$$\text{so that } 2ds\sin\theta = \lambda \quad (\text{for } n=1)$$

$$\approx 2d\theta = \lambda$$

$$\text{or } 2d \frac{S}{4R} = \lambda \Rightarrow d = \frac{2R\lambda}{S}$$

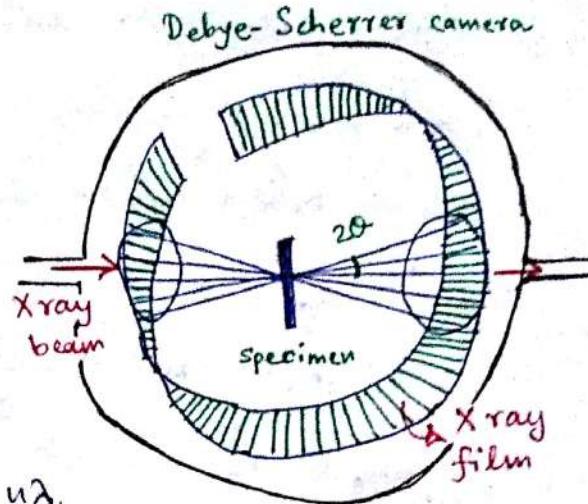
from known (measured)  $R, S, \lambda$ , interplanar spacing  $d$  is calculated.

## Brillouin Zones

We have learned that all  $\mathbf{k}$  values for which the reciprocal lattice points intersect the Ewald sphere are Bragg reflected. Brillouin zone is the locus of all these  $\mathbf{k}$  values in the reciprocal lattice which are Bragg reflected.

### Brillouin zones for sc lattice in 2D

Primitive translation vectors  $\vec{a} = \hat{a}\hat{i}$ ,  $\vec{b} = \hat{a}\hat{j}$ ,  $\vec{c} = \hat{a}\hat{l}$   
corresponding translation vector in reciprocal lattice  $\vec{a}^* = \frac{2\pi}{a}\hat{i}$ ,  $\vec{b}^* = \frac{2\pi}{a}\hat{j}$



so that reciprocal lattice vector  $\vec{G} = h\hat{a}^* + k\hat{b}^*$   
 $= \frac{2\pi}{a}(h\hat{i} + k\hat{j})$ .  
 (h, k are integers)

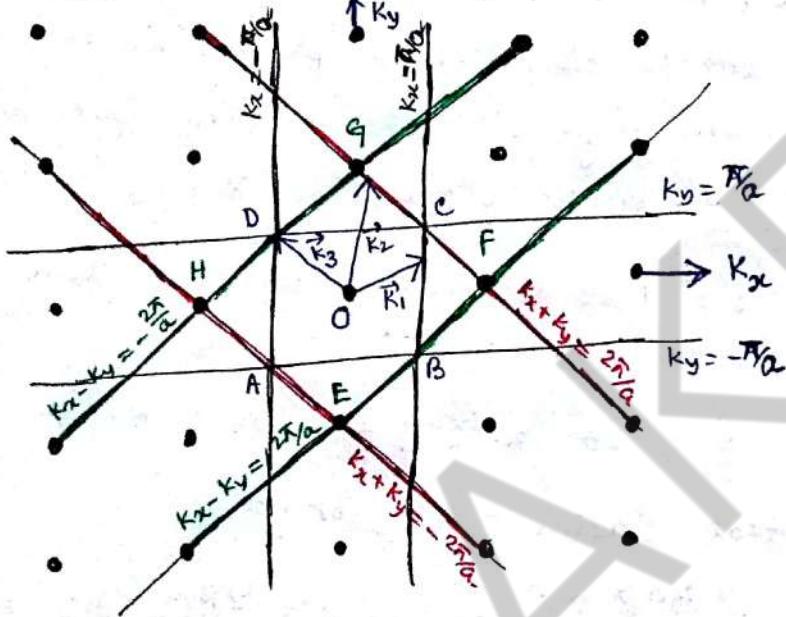
$\vec{R} = k_x\hat{i} + k_y\hat{j}$ . So from Bragg's vectorial condition

$$2\vec{R} \cdot \vec{G} + G^2 = 0$$

$$\text{or } \frac{4\pi}{a}(hK_x + kK_y) + \frac{4\pi^2}{a^2}(h^2 + k^2) = 0$$

$$\text{or } hK_x + kK_y = -\frac{\pi}{a}(h^2 + k^2)$$

For all h, k values, we can obtain R.



If  $h = \pm 1, k = 0$  then

$$K_x = \pm \frac{\pi}{a} \quad (K_y \text{ arbitrary})$$

If  $h = 0, k = \pm 1$ , then

$$K_y = \pm \frac{\pi}{a} \quad (K_x \text{ arbitrary})$$

All  $\vec{R}$  (for example  $\vec{K}_1, \vec{K}_2, \vec{K}_3$ ) originating from O & terminating on these parallel lines are Bragg reflected.

$$\text{If } h = \pm 1, k = \pm 1 \text{ then } \pm K_x \pm K_y = \frac{2\pi}{a}.$$

Region enclosed by such lines are the Brillouin zones.

ABCD is the first Brillouin zone & EFGH is the second Brillouin zone.

Brillouin zone boundary represent loci of  $\vec{R}$  that obey Bragg's law, meaning they're the reflecting planes.  $ABCD \Rightarrow 2dsin\theta = \lambda$ .

$EFGH \Rightarrow 2dsin\theta = 2\lambda$  & so on.

$$\text{In 3D, } hK_x + kK_y + lK_z = -\frac{\pi}{a}(h^2 + k^2 + l^2)$$

with cubes represent Brillouin zone.

## Brillouin zones of the fcc lattice

primitive translation vectors of fcc lattice are

$$\vec{a} = \frac{a}{2}(\hat{i} + \hat{j}), \vec{b} = \frac{a}{2}(\hat{j} + \hat{k}), \vec{c} = \frac{a}{2}(\hat{k} + \hat{i}) \text{ & primitive}$$

translation vectors in reciprocal space are

$$\vec{a}^* = \frac{2\pi}{a}(\hat{i} + \hat{j} - \hat{k}), \vec{b}^* = \frac{2\pi}{a}(-\hat{i} + \hat{j} + \hat{k}), \vec{c}^* = \frac{2\pi}{a}(\hat{i} - \hat{j} + \hat{k})$$

$$\begin{aligned}\therefore \vec{q} &= h\vec{a}^* + k\vec{b}^* + l\vec{c}^* \\ &= \frac{2\pi}{a}[(h-k+l)\hat{i} + (h+k-l)\hat{j} + (-h+k+l)\hat{k}]\end{aligned}$$

To make shortest  $\vec{q}$ , we can use 8 combinations

$$\vec{q} = \frac{2\pi}{a}(\pm\hat{i} \pm \hat{j} \pm \hat{k})$$

first zone boundary is determined by the 8 planes  $\perp q$  at their midpoint. But the corners of the octahedron are truncated by planes which are perpendicular bisector of 6 reciprocal lattice vector  $\frac{2\pi}{a}(\pm 2\hat{i}), \frac{2\pi}{a}(\pm 2\hat{j}), \frac{2\pi}{a}(\pm 2\hat{k})$ . So first Brillouin zone is truncated octahedron, which is also the primitive unit cell of bcc lattice.

## Brillouin zones of bcc lattice

primitive translation vectors of bcc lattice are

$$\vec{a} = \frac{a}{2}(\hat{i} + \hat{j} - \hat{k}), \vec{b} = \frac{a}{2}(-\hat{i} + \hat{j} + \hat{k}), \vec{c} = \frac{a}{2}(\hat{i} - \hat{j} + \hat{k}) \text{ &}$$

primitive translation vectors of reciprocal lattice are

$$\vec{a}^* = \frac{2\pi}{a}(\hat{i} + \hat{j}), \vec{b}^* = \frac{2\pi}{a}(\hat{j} + \hat{k}), \vec{c}^* = \frac{2\pi}{a}(\hat{k} + \hat{i}).$$

$$\vec{q} = \frac{2\pi}{a}[(h+k)\hat{i} + (h+l)\hat{j} + (k+l)\hat{k}] \text{ & shortest } \vec{q} \text{ are}$$

the 12 vectors,  $\vec{q} = \frac{2\pi}{a}(\pm\hat{i} \pm \hat{j})$

$$= \frac{2\pi}{a}(\pm\hat{j} \pm \hat{k})$$

$$= \frac{2\pi}{a}(\pm\hat{k} \pm \hat{i})$$

First Brillouin zone is volume by normal bisector of 12 vectors  
 $\Rightarrow$  rhombic dodecahedron.

## Structure of Solids - Fermi's Golden Rule approach

In scattering experiment, transition rate between incident plane wave state  $|K\rangle$  with momenta  $\hbar\vec{K}$  to outgoing (scattered) plane wave state  $|K'\rangle$  with momenta  $\hbar\vec{K}'$  is measured.

If the scattering medium interaction potential is  $U(\vec{x})$ , then from Fermi's golden rule, the transition rate per unit time is

$$\Gamma_{KK'} = \langle K|U|K' \rangle = \int d^d x' e^{-i\vec{K} \cdot \vec{x}'} U(\vec{x}') e^{i\vec{K}' \cdot \vec{x}'}, \quad d=1,2,3$$

so that differential cross-section per unit solid angle is

$$\frac{d^2\sigma}{d\Omega} \sim \frac{2\pi}{\hbar} |\langle K|U|K' \rangle|^2 \delta(E_K - E_{K'}). \quad \text{This is true for any}$$

arbitrary arrangement of atoms/lattice points (periodic/aperiodic).

But for a periodic crystal

$$\vec{x}' = \vec{x} + \vec{T} \quad \text{and} \quad U(\vec{x} + \vec{T}) = U(\vec{x})$$

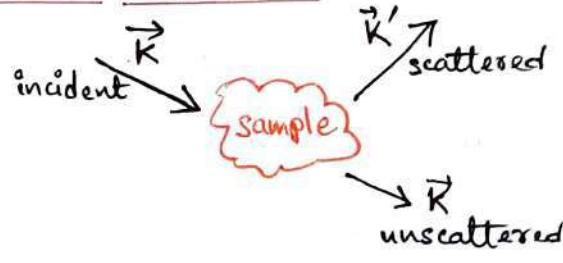
so that

$$\begin{aligned} \Gamma_{KK'} &= \int d^d \vec{x}' e^{-i(\vec{K}-\vec{K}') \cdot \vec{x}'} U(\vec{x}') \\ &= \sum_{\vec{T}} \int_{\text{unit cell}} d^d \vec{x} e^{-i(\vec{K}-\vec{K}') \cdot (\vec{x} + \vec{T})} U(\vec{x} + \vec{T}) \\ &= \left[ \sum_{\vec{T}} e^{-i(\vec{K}-\vec{K}') \cdot \vec{T}} \right] \left[ \int_{\text{unit cell}} d^d \vec{x} e^{-i(\vec{K}-\vec{K}') \cdot \vec{x}} U(\vec{x}) \right] \\ &= \left[ \sum_{\vec{T}} e^{-i\vec{G} \cdot \vec{T}} \right] \int_{\text{unit cell}} d^d \vec{x} e^{-i\vec{G} \cdot \vec{x}} U(\vec{x}) \quad (\because \vec{K}-\vec{K}' = \vec{G}) \\ &= N_C S_G \end{aligned}$$

where  $N_C = \# \text{ of cells in lattice}$

and  $S_G$  is known as the structure factor (geometrical). This is the scattering amplitude and square of this is the scattering intensity.

The atomic scattering factor (form factor) is defined as the ratio of the radiation amplitude scattered by the actual electron distribution in an atom to that scattered by a single electron at a point.



$$f = \frac{\text{Amplitude of radiation scattered from an atom}}{\text{Amplitude of radiation scattered from an electron}} \cdot Z$$

where  $Z = \text{atomic number} = \# \text{ of electrons present in the atom.}$

Due to weak interaction with X-rays with atomic nuclei, the scattering due to nuclei is neglected compared with scattering due to electrons.

Let us consider spherically symmetric electron distribution of atom and we want to calculate scattering from charge  $\rho(\vec{r}) dV$  at a distance  $\vec{r}$  from origin and electron at origin O. We

know that the phase difference between incident & scattered X-ray is  $\frac{2\pi}{\lambda} \vec{r} \cdot \vec{s}$ . So if the X-ray scattering along  $\vec{k}'$  from the electron at O is  $A e^{i(kx - wt)}$  then the scattered intensity from  $\rho(\vec{r}) dV$  will be

$$A e^{i[(kx - wt) + \frac{2\pi}{\lambda} (\vec{r} \cdot \vec{s})]} \rho(\vec{r}) dV$$

$"\Phi_r"$

$$\therefore df = \frac{\text{Amplitude of radiation scattered by } \rho(\vec{r}) dV}{\text{Amplitude of radiation scattered by electron at O}}$$

$$= \frac{A e^{i[(kx - wt) + \phi_r]} \rho(\vec{r}) dV}{A e^{i(kx - wt)}} = e^{i \frac{2\pi}{\lambda} (\vec{r} \cdot \vec{s})} \rho(\vec{r}) dV$$

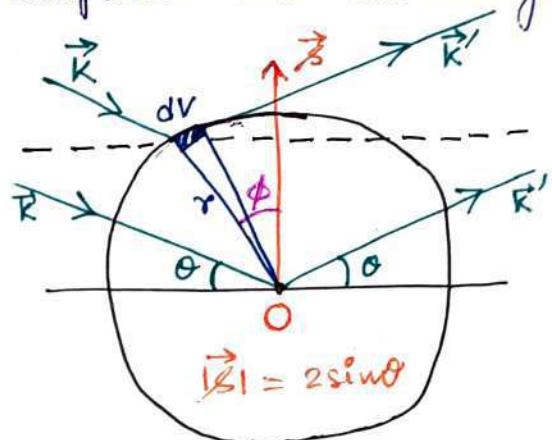
$$\text{Now } dV = 2\pi r^2 \sin\phi d\phi dr, \quad \frac{2\pi}{\lambda} (\vec{r} \cdot \vec{s}) \rho(\vec{r}) dr = \frac{2\pi}{\lambda} r s \cos\phi$$

$$= \frac{2\pi}{\lambda} r s \sin\phi \cos\phi = \frac{4\pi}{\lambda} r \sin\phi \cos\phi = \mu r \cos\phi$$

$$\therefore \text{Atomic form factor } f = \int df = \int_{r=0}^{\infty} \int_{\phi=0}^{\pi} e^{i\mu r \cos\phi} \rho(r) 2\pi r^2 \sin\phi d\phi dr$$

$$\text{Now } \int_{\phi=0}^{\pi} e^{i\mu r \cos\phi} \sin\phi d\phi = \frac{2 \sin \mu r}{\mu r}; \quad \lim_{\phi \rightarrow 0} \mu \rightarrow 0 \text{ & } \frac{\sin \mu r}{\mu r} \rightarrow 1.$$

$$\therefore f = \int_{r=0}^{\infty} 4\pi r^2 \rho(r) \frac{\sin \mu r}{\mu r} dr \quad \lim_{\mu \rightarrow 0} \int_{r=0}^{\infty} 4\pi r^2 \rho(r) dr = Z = \# \text{ of electrons in atom}$$



For atoms containing large number of electrons (beyond Rubidium) using Hartree-Fock approximation or from Thomas-Fermi model  $\rho(\vec{r})$  can be obtained that match experiments fairly well.

### Geometrical Structure Factor

$$\text{Geometrical Structure factor} = \frac{\text{Amplitude of radiation scattered by unit cell}}{\text{Amplitude of radiation scattered by electron at origin.}}$$

The position of  $j^{\text{th}}$  atom in unit cell is

$$\vec{r}_j = x_j \vec{a} + y_j \vec{b} + z_j \vec{c}, \quad 0 \leq [x_j, y_j, z_j] \leq 1.$$

for a given (hkl) reflection the scattering amplitude is

$$F_{hkl} = \sum_j f_j e^{i\phi_j} = \sum_j f_j e^{i\left(\frac{2\pi}{\lambda}\right)(\vec{r}_j \cdot \vec{s})}, \quad \text{Here } f_j \text{ is the atomic}$$

form factor,  $\phi_j$  is the phase difference between scattering from  $j^{\text{th}}$  atom of the unit cell to the scattering from origin and  $\sum$  is over all the atoms in the unit cell.

$$\text{from } \vec{a} \cdot \vec{s} = 2a \sin \theta \cos \alpha = h\lambda, \quad \vec{b} \cdot \vec{s} = 2b \sin \theta \cos \beta = k\lambda$$

$$\text{and } \vec{c} \cdot \vec{s} = 2c \sin \theta \cos \gamma = l\lambda \quad (\text{Laue equations})$$

$$\begin{aligned} F_{hkl} &= \sum_j f_j e^{i\frac{2\pi}{\lambda}[(x_j \vec{a} + y_j \vec{b} + z_j \vec{c}) \cdot \vec{s}]} \\ &= \sum_j f_j e^{i\frac{2\pi}{\lambda}[x_j(h\lambda) + y_j(k\lambda) + z_j(l\lambda)]} \\ &= \sum_j f_j e^{i2\pi(hx_j + ky_j + lz_j)} = f \sum_j e^{i2\pi(hx_j + ky_j + lz_j)} \end{aligned}$$

$$\therefore \text{Geometrical structure factor} = \frac{F_{hkl}}{f} = \sum_j e^{i2\pi(hx_j + ky_j + lz_j)}$$

$$\begin{aligned} \text{Scattering intensity } I &= F_{hkl}^* F_{hkl} = \left[ \sum_j f_j \cos 2\pi(hx_j + ky_j + lz_j) \right]^2 \\ &\quad + \left[ \sum_j f_j \sin 2\pi(hx_j + ky_j + lz_j) \right]^2. \end{aligned}$$

GSF for sc lattice There is only 1 atom per unit cell

which can be assumed to be at origin ( $x_j = y_j = z_j = 0$ )

$\therefore F_{hkl} = f$ . &  $I = f^2$ . So all lines satisfying Bragg's law will be visible provided the atomic form factor  $f$  is large enough.

GSF for bcc lattice There are two atoms per unit cell, one

at origin ( $x_1 = y_1 = z_1 = 0$ ) & other at body diagonal ( $x_2 = y_2 = z_2 = \frac{1}{2}$ ), so that

$$F_{hkl} = f \sum_s e^{i2\pi(hx_s + ky_s + lz_s)} \\ = f[1 + e^{i\pi(h+k+l)}].$$

So for a reflection where  $h+k+l = 2n$  (even) we have

$$F_{hkl}^{\text{even}} = f(1+1) = 2f, I = 4f^2. \text{ for } h+k+l = 2n+1 \text{ (odd)},$$

$F_{hkl}^{\text{odd}} = f(1-1) = 0$  &  $I = 0$ . Therefore  $(100), (111), (210)$  etc reflections will be missing in first order, but  $(200)$  will be present. This is the case with Metallic Sodium.

GSF for fcc lattice There are 4 atoms per unit cell at

$(0,0,0), (\frac{1}{2},0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2},0), (0,\frac{1}{2},\frac{1}{2})$ , so that

$$F_{hkl} = f[1 + e^{i\pi(h+k)} + e^{i\pi(k+l)} + e^{i\pi(h+l)}].$$

If  $h, k, l$  all even/odd,  $F_{hkl} = 4f$ ,  $I = 16f^2$ , while if  $h, k, l$  = mix of even & odd,  $F_{hkl} = 0$ ,  $I = 0$ . So no reflection can occur for which  $(h, k, l)$  are partly even & partly odd.

### systematic absences of scattering

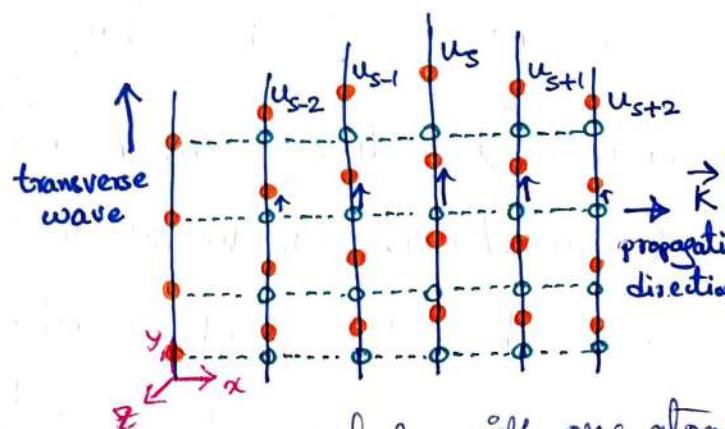
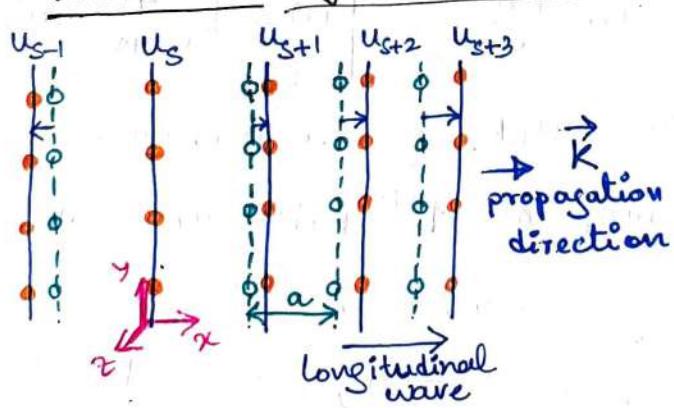
sc	all $(h,k,l)$ allowed
bcc	$h+k+l$ must be even
fcc	$h, k, l$ must be all odd or all even

## Crystal Vibrations

A solid is comprised of atoms/molecules that are closely packed & there is a competition between attractive long-ranged electrostatic force & short-ranged repulsive (mostly steric or soft) force, due to which atoms/molecules remain fixed to their equilibrium position, yet due to interatomic vibration, vibrate about their mean position. These are known as lattice vibrations. There exists various elementary excitations in solids, e.g. EM wave (photon), elastic wave (phonon), collective electron wave (plasmon), magnetization wave (magnon), electron interacting with elastic deformation (polaron), polarization wave (exciton) & so on.

To closest representation, crystals are periodic atoms connected with elastic springs. Dynamics of one atom/molecule is coupled to other atoms/molecules & the lattice will freely vibrate in normal mode. Thermal properties of solids (e.g. thermal conductivity, specific heat) are attributed to this kind of vibration.

### Vibrations of a 1D monoatomic chain



Let us consider the elastic vibrations of a crystal with one atom in the primitive unit cell. We want to find the frequency of

an elastic wave as a function of  $\vec{k}$  & elastic constants. When an elastic wave propagates along cube edge ([100]), face diagonal ([110]) and body diagonal ([111]), then entire plane of atoms move in phase with displacements in  $\parallel$  &  $\perp$  direction of  $\vec{k}$ .

The problem becomes 1D as we can define displacement of s-plane from its equilibrium position as  $u_s$  & for every  $\vec{k}$  there will be 3 modes, 1 for longitudinal polarization & 2 for transverse polarization.

At low temperature, elastic energy is harmonic in displacement so that force exerted on s-plane by  $s \pm p$  plane ( $p=1$  corresponds to nearest neighbour plane only) is proportional to  $u_{s \pm p} - u_s$ , i.e.

$$F_s = c(u_{s+1} - u_s) + c(u_{s-1} - u_s) = m \frac{d^2 u_s}{dt^2}, \quad c = \text{force constant}$$

$$m \ddot{u}_s = c(u_{s+1} + u_{s-1} - 2u_s) \quad m = \text{mass of atom}$$

The periodic travelling wave solution of the displacement is

$$u_{s \pm 1} = u e^{i(\omega t + (s \pm 1)ka)}, \quad a = \text{interplanar spacing},$$

$$u_s = u e^{i(\omega t + ska)} \quad k = \text{wave vector},$$

$$\therefore -\omega^2 m u e^{i(\omega t + ska)} = c u e^{i\omega t} [e^{i(s+1)ka} + e^{i(s-1)ka} - 2e^{isk}]$$

$$\text{or } \omega^2 m = -c [e^{ika} + e^{-ika} - 2] = c[2 - 2 \cos ka]$$

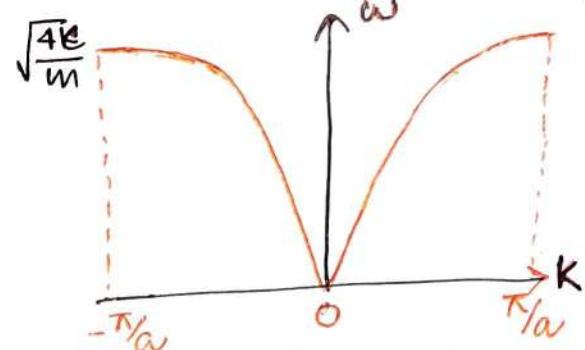
$$\text{or } \omega^2 = \frac{2c}{m} (1 - \cos ka) = \frac{4c}{m} \sin^2 \frac{ka}{2} \Rightarrow \boxed{\omega = \pm \sqrt{\frac{4c}{m} \sin^2 \frac{ka}{2}}}$$

This is the dispersion relation  $\omega = \omega(k)$  for a monoatomic linear chain. We know that boundary of 1st Brillouin zone is at  $K = \pm \frac{\pi}{a}$ , at which slope of  $\omega$  vs.  $K$  is

$$\left. \frac{d\omega}{dK} \right|_{K=\pm\frac{\pi}{a}} = \pm \sqrt{\frac{4c}{m}} \frac{a}{2} \cos \frac{ka}{2} \Big|_{K=\pm\frac{\pi}{a}} \sim \cos \frac{\pi}{2} = 0, \text{ means } K = \pm \frac{\pi}{a}$$

is an inflection point (change of direction of curvature)

Energy of elastic wave is quantised & the quantum of energy is known as "phonon". Phonon wave propagates in crystal at speed of sound.



To find out the range of  $K$  that are physically significant notice that  $\frac{u_{s+1}}{u_s} = \frac{ue^{i[\omega t + (s+1)ka]}}{ue^{i[\omega t + ska]}} = e^{ika}$  that defines the range of  $K$  as  $-\pi \leq ka \leq \pi$  or  $-\frac{\pi}{a} \leq K \leq \frac{\pi}{a}$  ( $1^{\text{st}}$  BZ) & by a linear transformation, any other value of  $K' = K \pm \frac{2\pi n}{a}$ , that lie outside the  $1^{\text{st}}$  BZ can be mapped back to  $1^{\text{st}}$  BZ as  $\frac{u_{s+1}}{u_s} = e^{ik'a} = e^{ika} e^{\pm i2\pi n} = e^{ika}$ . So the displacement can always be described by  $K$  within  $1^{\text{st}}$  BZ by adding/subtracting  $\frac{2\pi n}{a}$  which is the reciprocal lattice vector  $G$  (where  $d = a$  the interplanar spacing for  $[100]$  plane). At  $1^{\text{st}}$  BZ boundary,  $u_s$  represent a standing wave. This can be realized as follows.

Dispersion relation  $\omega = \pm \sqrt{\frac{4E}{m}} \sin \frac{Ka}{2}$ . If instead of individual atoms, we consider an entire plane, then linear density along line is  $\rho = \frac{m}{a}$ , so  $\omega = \pm \sqrt{\frac{4E}{\rho a^2}} \sin \frac{Ka}{2} = \pm \frac{2v_0}{a} \sin \frac{Ka}{2}$ . If  $E = \text{elastic stiffness} = Ca$ , then  $\omega = \pm \sqrt{\frac{4E}{a^2 \rho}} \sin \frac{Ka}{2} = \pm \frac{2v_0}{a} \sin \frac{Ka}{2}$  where  $v_0 = \sqrt{\frac{E}{\rho}}$  is the longitudinal velocity of wave along line.

(i) Low-frequency, long wavelength limit:

$$\lim_{K \rightarrow 0} \sin \frac{Ka}{2} = \frac{Ka}{2}, \text{ so } \omega = \pm \frac{2v_0}{a} \frac{Ka}{2} = \pm v_0 K$$

So phase velocity  $v_p = \frac{\omega}{K} = \pm v_0$  which is same as group velocity  $v_g = \frac{dv}{dk} = \pm v_0$ .

This is true for a homogeneous line for which the atomic nature of linear chain has no importance.

(ii) High-frequency, short-wavelength limit :

from dispersion relation  $v_g = \frac{dw}{dk} = \pm v_0 \cos \frac{ka}{2}$  and  $v_p = \frac{\omega}{k}$   
 $= \frac{2v_0}{ka} \sin \frac{ka}{2}$ . While both group & phase velocity is function  
of  $k$ , the crystal medium is "dispersive", as in dispersion of light.  
the refractive index of medium is a function of frequency.

(iii) Frequency maxima  $\omega = \sqrt{\frac{4c}{m}}$  :  $\sin \frac{ka}{2} = \pm 1 \Rightarrow \frac{ka}{2} = \pm \frac{\pi}{2}$   
 $(\text{or } k = \pm \frac{\pi}{a})$

$$\Rightarrow \frac{2\pi}{\lambda} \frac{a}{2} = \pm \frac{\pi}{2} \Rightarrow \underline{\lambda = 2a}.$$

$\therefore v_p = \frac{\omega}{k} = \frac{2v_0}{ak} \sin \frac{ka}{2} = \pm \frac{2v_0}{a \frac{\pi}{a}} \sin \frac{\pi}{2} = \pm \frac{2v_0}{\pi}$  and  
 $v_g = \frac{dw}{dk} = \pm v_0 \cos \frac{ka}{2} = \pm v_0 \cos \frac{\pi}{2} = 0$ . So the group velocity,  
that represents the velocity of wave, energy transmission is  
zero represents a standing wave.

This is similar to 1<sup>st</sup> order Bragg reflection at normal  
incidence ( $\theta = \pi/2$ ,  $n=1$ ) so that Bragg's law  $2d \sin \theta = n\lambda$  becomes  
 $2d = \lambda$ .\* Moreover, the linear chain can propagate waves  $\omega \leq \frac{2v_0}{a}$   
(cutoff frequency), so the monoatomic lattice acts as a "low  
pass filter". For typical crystal with  $a = 3 \text{ \AA}$ ,  $\omega = 2 \times 10^{13} \text{ Hz}$ .  
[\* When Bragg condition is satisfied, a travelling wave cannot  
propagate in a lattice but through successive reflections sets up a  
standing wave.]

(iv) Calculation of force constant (c) : Long range interaction  
between ion to ion through the conduction electron sea  
happen in metals, where planes of atoms separate

by nearly 20 planes are found to interact. In such case to derive dispersion relation, one has to consider displacement  $u_{S+p}$  of the  $p^{\text{th}}$  plane from the reference  $u_S$  plane displacement, so that

$$\omega^2 = \frac{2}{m} \sum_{p>0} C_p (1 - \cos pka) \Rightarrow m\omega^2 \cos rka = 2 \sum_{p>0} C_p (1 - \cos pka) \cos rka$$

$$\therefore m \int_{-\pi/a}^{\pi/a} \omega^2 \cos rka = 2 \sum_{p>0} C_p \int_{-\pi/a}^{\pi/a} \cos rka (1 - \cos pka) dk \\ = 2 \sum_{p>0} C_p \left[ \int_{-\pi/a}^{\pi/a} \cos rka dk - \int_{-\pi/a}^{\pi/a} \cos rka \cos pka dk \right]$$

$$\text{Now } \int_{-\pi/a}^{\pi/a} \cos rka dk = \frac{1}{ra} \sin rka \Big|_{-\pi/a}^{\pi/a} = \frac{2}{ra} \sin r\pi = 0 \quad (r = \text{integer})$$

$$l - 2 \int_{-\pi/a}^{\pi/a} \cos rka \cos pka dk = - \int_{-\pi/a}^{\pi/a} [\cos(r-p)ka + \cos(r+p)ka] dk$$

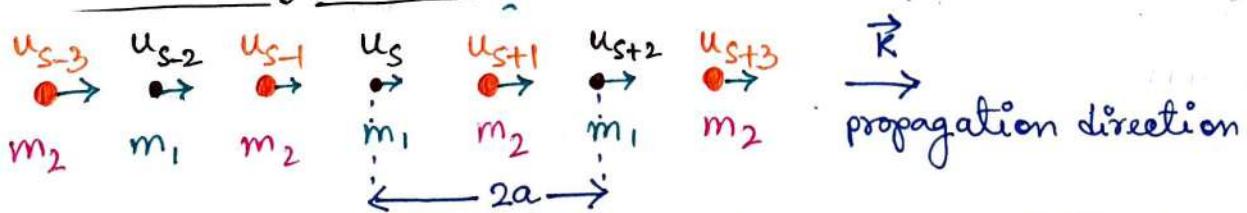
$$= - \frac{1}{a(r-p)} \sin(r-p)ka \Big|_{-\pi/a}^{\pi/a} + \frac{1}{a(r+p)} \sin(r+p)ka \Big|_{-\pi/a}^{\pi/a} \quad (r+p = \text{integer}) \\ = - \frac{2}{a(r-p)} \sin(r-p)\pi + \frac{2}{a(r+p)} \sin(r+p)\pi$$

$$r=p \quad - \frac{2\pi}{a} s_{rp} \quad (\text{Using L'Hospital rule})$$

$$\therefore m \int_{-\pi/a}^{\pi/a} \omega^2 \cos rka = - \frac{2\pi}{a} C_p s_{rp}$$

$$C_p = - \frac{ma^2}{2\pi} \int_{-\pi/a}^{\pi/a} \omega^2 \cos pka dk$$

### Vibrations of a 1D linear diatomic Chain



Let us consider the elastic vibrations of a diatomic crystal (two atoms per primitive basis), e.g. NaCl, KBr, Diamond etc., for which the atomic arrangements are shown above. For ideal arrangement, force constant  $C$  is same. Let  $u_{S+p}$  with  $p = \text{even}$  is the displacement of  $m_1$  atoms with amplitude  $u$  and for  $p = \text{odd}$  is the

displacement of  $m_2$  atoms with amplitude  $v$  with  $p$  atoms in the primitive cell. Suppose  $m_2 > m_1$  & distance between nearest neighbours is  $a$ , so that the repeat distance is  $2a$ . Elastic waves propagate along body diagonal ( $[111]$ ) in NaCl & cube edge ( $[100]$ ) in NaCl so that single plane of vibration contains only one type of atom & each plane interacts only with nearest-neighbour plane.

The equation of motion is  $m_1 \frac{d^2 u_s}{dt^2} = c(u_{s+1} + u_{s-1} - 2u_s) \&$   
 $m_2 \frac{d^2 u_{s+1}}{dt^2} = c(u_{s+2} + u_s - 2u_{s+1})$

The travelling wave solution on alternate planes are

$$u_s = u e^{i(\omega t - ska)} \neq u_{s+2} = u e^{i(\omega t - (s+2)ka)} \& \\ u_{s+1} = v e^{i(\omega t - (s+1)ka)}, \text{ with } k = \text{wave vector of that particular mode of vibration - same for } m_1 \& m_2 \text{ as they both participate in wave motion. Substituting in E.O.M. we have}$$

$$-\omega^2 m_1 u = c v (e^{ika} + e^{-ika}) - 2cu \&$$

$$-\omega^2 m_2 v = c u (e^{ika} + e^{-ika}) - 2cv$$

$$\text{or. } (2c - m_1 \omega^2) u - 2c \cos ka v = 0 \& \quad \text{or. } (2c - m_2 \omega^2) v - 2c \cos ka u = 0. \quad \text{This pair of homogeneous equations will have a solution provided that determinant of}$$

$$\text{the coefficient vanishes, or, } \begin{vmatrix} (2c - m_1 \omega^2) & -2c \cos ka \\ -2c \cos ka & (2c - m_2 \omega^2) \end{vmatrix} = 0$$

$$\text{or. } (2c - m_1 \omega^2)(2c - m_2 \omega^2) - 4c^2 \cos^2 ka = 0$$

$$\text{or. } m_1 m_2 \omega^4 - 2c(m_1 + m_2) \omega^2 + 4c^2(1 - \cos^2 ka) = 0$$

$$\text{or. } \omega^4 - 2c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \omega^2 + \frac{4c^2}{m_1 m_2} \sin^2 ka = 0$$

This equation is true for two positive values of  $\omega^2$  as

$$\omega_{\pm}^2 = c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \pm c \left[ \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^2 - \frac{4}{m_1 m_2} \sin^2 ka \right]^{1/2}$$

so that for a given  $K$ , there are two  $\omega$  (namely  $\omega_+$  &  $\omega_-$ ). To remind, dispersion relation for 1D linear monoatomic lattice was  $\omega^2 = \frac{4c}{m} \sin^2 \frac{ka}{2}$ , so for a given  $K$ , only one positive  $\omega$  was possible. Here, instead, for each polarization mode in a given propagation direction, dispersion relation develops 2 branches, known as the "acoustical" (longitudinal (LA) & transverse (TA)) and "optical" (longitudinal (LO) & transverse (TO) modes) branches. With  $p$  atoms in primitive cell, 3p branches to the dispersion relation, there are 3 acoustical branch (1 LA, 2 TA) and 3p-3 optical branches (1 LO, 2 TO for Germanium/KBr with total 6 branches to dispersion relation as  $p=2$ ).

### Analysis of different branches of the dispersion relation curve

$$\omega_+^2 = c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + c \left[ \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^2 - \frac{4}{m_1 m_2} \sin^2 ka \right]^{1/2} \rightarrow \text{Optical branch}$$

$$\omega_-^2 = c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - c \left[ \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^2 - \frac{4}{m_1 m_2} \sin^2 ka \right]^{1/2} \rightarrow \text{Acoustical branch}$$

(i) Optical branch ( $\omega_+$ ):  $\lim_{K \rightarrow 0} \sin ka \rightarrow 0$  so that

$$\omega_+^2 = c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \Rightarrow \omega_+ = \left[ c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \right]^{1/2}$$

$$\lim_{K \rightarrow K_{2a}} \sin ka \rightarrow 1, \text{ so } \omega_+^2 = c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + c \left[ \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^2 - \frac{4}{m_1 m_2} \right]^{1/2}$$

$$= c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + c \left( \frac{1}{m_1} - \frac{1}{m_2} \right) = \frac{2c}{m_1}$$

$$\therefore \omega_+ = \sqrt{\frac{2c}{m_1}}$$

(ii) Acoustical branch ( $\omega_-$ ):  $\lim_{K \rightarrow 0} \sin ka \rightarrow ka$  otherwise  $\omega_- = 0$

$$\text{So that } \omega_-^2 = c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \left[ 1 - \frac{4K^2 a^2}{m_1 m_2} \left( \frac{m_1 m_2}{m_1 + m_2} \right)^2 \right]^{1/2}$$

$$= c \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \left[ 1 - \left\{ 1 - \frac{m_1 m_2}{(m_1 + m_2)^2} 4K^2 a^2 \right\}^{1/2} \right]$$

$$\approx c\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \left[1 - k^2 + \frac{m_1 m_2}{(m_1 + m_2)^2} 2k^2 a^2 + \dots\right] \text{ neglecting higher powers}$$

$$= \frac{2c}{m_1 + m_2} k^2 a^2$$

$$\therefore \omega_- = ka \sqrt{\frac{2c}{m_1 + m_2}}$$

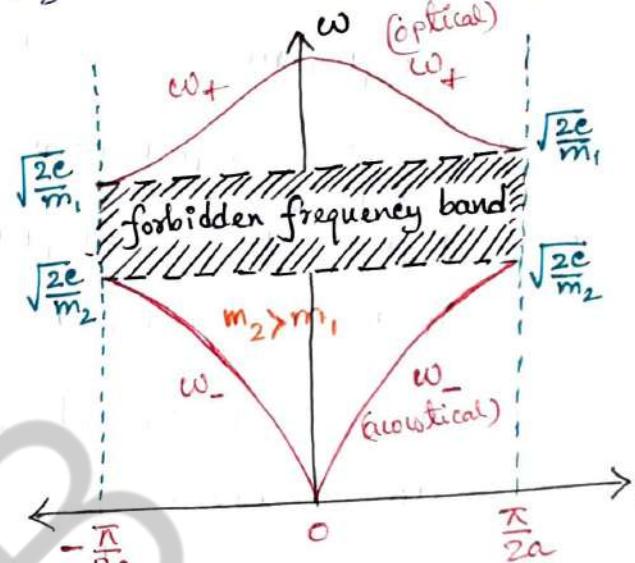
$$\text{for } \lim_{K \rightarrow \infty} \sin ka \rightarrow 1, \quad \omega_-^2 = c\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - c\left[\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 - \frac{4}{m_1 m_2}\right]$$

$$= c\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - c\left(\frac{1}{m_1} - \frac{1}{m_2}\right) = \frac{2c}{m_2}$$

$$\therefore \omega_- = \sqrt{\frac{2c}{m_2}}$$

Figure beside displays the dispersion relation along with the two frequency branches. We note the following features:

- (i) The frequency range of propagation is split into two branches - acoustic branch  $\omega_-$  (lower) & optical branch  $\omega_+$  (upper).



We recall the equation of motion ①

$$(2c - m_1 \omega^2)u - 2c \cos ka v = 0, \quad (2c - m_2 \omega^2)v - 2c \cos ka u = 0$$

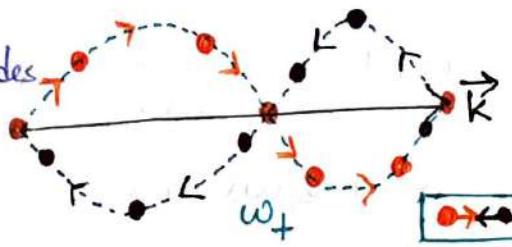
$$\text{For } \omega_+, \lim_{K \rightarrow 0} \cos ka \rightarrow 1, \text{ so } (2c - m_1 \omega_+^2)u - 2cv = 0 \quad \& \quad (2c - m_2 \omega_+^2)v - 2cu = 0.$$

$$\therefore \frac{u}{v} = \frac{2c}{2c - m_1 \omega_+^2} = \frac{2c - m_2 \omega_+^2}{2c} \quad \begin{aligned} -m_1 \omega_+^2 u &= 2c(v - u) \\ -m_2 \omega_+^2 v &= 2c(u - v) \end{aligned} \quad \text{so}$$

$$\frac{u}{v} = \frac{(v - u)/m_1}{(u - v)/m_2} = -\frac{m_2}{m_1}. \quad \text{This means that the two atoms}$$

move in opposite directions & their amplitudes are inversely proportional to their masses so that the center of mass of the unit cell does not move. For equal masses too they move in opposite direction & for unequal masses as  $m_2 > m_1$ ,  $u > v$ .

For  $\omega_-$ ,  $\lim_{K \rightarrow 0} \cos ka = 1 - \frac{k^2 a^2}{2}$  and then the equation of motion ① is



$$(2c - m_1 \omega_-^2)u - 2c(1 - \frac{k^2 a^2}{2})v = 0, \quad (2c - m_2 \omega_-^2)v - 2c(1 - \frac{k^2 a^2}{2})u = 0$$

$$\therefore -m_1 \omega_-^2 u = 2c(1 - \frac{k^2 a^2}{2})v - 2cu$$

+  $m_2 \omega_-^2 v = 2c(1 - \frac{k^2 a^2}{2})u - 2cv$ . Adding and substituting

$$\omega_- = ka \sqrt{\frac{2c}{m_1 + m_2}} \text{ (as obtained earlier), we get.}$$

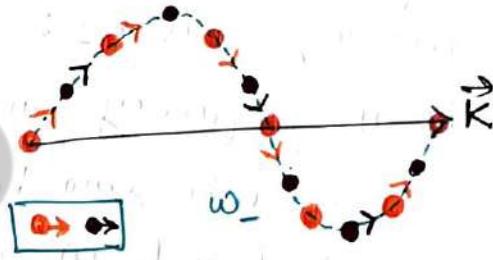
$$+ \omega_-^2 (m_1 u + m_2 v) = + k^2 a^2 (u + v)c$$

$$\text{or } k^2 a^2 \frac{2c}{m_1 + m_2} (m_1 u + m_2 v) = k^2 a^2 (u + v)c$$

$$\therefore 2m_1 u + 2m_2 v = (m_1 + m_2)(u + v) \text{ or } m_1 u + m_2 v = m_1 v + m_2 u$$

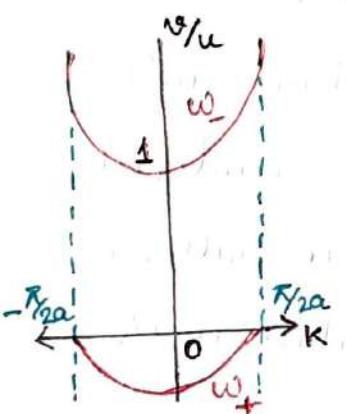
$$\text{or } (m_1 - m_2)u = (m_1 - m_2)v \text{ or } \frac{u}{v} = +1. \text{ This signifies that}$$

in the acoustical branch even if the masses are different then also two atoms move in same direction with equal amplitude. Because of this, the center of mass moves just like pressure wave (sound). Monoatomic crystal subjected to a force makes the atoms to move in same direction, by the vibrations of the acoustical branch. On the other hand, movement of neighbouring atoms can be achieved using a light source that due to its EM nature will vibrate the optical branch. Hence these names are justified. Diamond & hcp lattices exhibit optical branch.



iii) As drawn in dispersion curve, that wave like solutions do not exist for  $\sqrt{\frac{2c}{m_2}} \leq \omega \leq \sqrt{\frac{2c}{m_1}}$ , so there is a forbidden frequency band for unequal mass atoms ( $m_1 \neq m_2$ ) for which frequencies cannot propagate. As the atomic masses are larger ( $\frac{m_2}{m_1} \gg 1$ ), the wider the forbidden frequency band between acoustical & optical branch. Only for ~~monoatomic~~ equal mass atoms ( $m_1 = m_2$ ), the band vanishes. The 1st Brillouin zone is with wavevector  $-\frac{\pi}{2a} \leq k \leq \frac{\pi}{2a}$ . So the

Smallest possible wavelength of this zone is  $4a$  corresponding to the largest wavevector  $K = \frac{\pi}{2a}$  at the zone boundary. As  $m_2$  increases more, the optical branch flattens & acoustical branch goes downward. At  $m_2 \rightarrow \infty$ , optical branch becomes parallel to K-axis (flat) & acoustic branch disappears. As  $m_1$  decreases, optical branch moves upward (acoustical branch is unaffected). At  $m_1 \rightarrow 0$ , optical branch disappears as in monoatomic lattice vibration, with lattice constant  $2a$ .



For all wave vector  $K$ , the amplitude variation for both branches is shown beside. For  $K = \pm \frac{\pi}{2a}$  we know  $w_+ = \sqrt{\frac{2c}{m_1}}$ , from equation ①,  $\frac{v}{u} = 0$  and for  $w_- = \sqrt{\frac{2c}{m_2}}$   $\frac{v}{u} = \infty$ . So in optical branch,  $v=0$  means all the heavier atoms  $m_2$  are at rest, while in acoustical branch  $u=0$  means all the lighter atoms  $m_1$  are at rest. So only one of the sublattices, lighter atoms of optical branch & heavier atoms in acoustical branch oscillates at the BZ boundary. They both represent standing wave with  $\frac{\pi}{2}$  phase difference. Both the sublattices vibrate for  $K < \frac{\pi}{2a}$ .

### Phonons - quantised lattice vibrations

According to quantum mechanics, energy of an elastic wave with angular frequency of vibration  $\omega$  with the mode is excited to a quantum number  $n$  is  $E = (n + \frac{1}{2})\hbar\omega$  or the mode is occupied with  $n$  phonons. The zero point energy of the mode is  $\frac{1}{2}\hbar\omega$ . Due to equivalence of photons (EM wave quanta) and phonons (elastic wave quanta) to harmonic oscillator, energy of phonon is quantised with each phonon having energy  $\hbar\omega$ . and velocity of propagation of sound. Vibration spectrum of phonon

is  $10^4$  -  $10^{12}$  Hz, with the lower frequency side is in the acoustic region (sound waves in crystal lattice) and higher side in the infra-red region (similar to thermally excited photons in blackbody radiation). Like photons, phonons are spin-zero wave satisfying

Bose-Einstein statistics  $\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}$  (indistinguishable)

Despite no direct evidence of quantization of phonons, the following experimental inferences predict the existence:

- (a) Lattice heat capacity is zero at the zero temperature limit which can be explained only if lattice vibrations is quantized &
- (b) The change of momentum/energy during inelastic scattering of x-ray or neutron beam on crystal is associated with gain/loss of phonons.

phonon amplitude: To compute the quantized mean square phonon amplitude, consider the standing wave mode

$u = u_0 \cos kx \cos \omega t$  where  $u$  is the displacement of a volume element from its equilibrium position  $x$  in the crystal. The time-averaged energy  $\bar{E} = (n + \frac{1}{2})\hbar\omega = \frac{1}{2}(n + \frac{1}{2})\hbar\omega + \frac{1}{2}(n + \frac{1}{2})\hbar\omega = \bar{E}_{KE} + \bar{E}_{PE}$ . KE density with mass density  $\rho$  is  $\frac{1}{2}\rho(\frac{\partial u}{\partial t})^2$

∴ Total KE of crystal at volume  $V = \frac{1}{4}\rho V \omega^2 u_0^2 \sin^2 \omega t$

$$\therefore \bar{E}_{KE} = \frac{1}{8} \rho V \omega^2 u_0^2 = \frac{1}{2}(n + \frac{1}{2})\hbar\omega \Rightarrow u_0^2 = 4(n + \frac{1}{2}) \frac{\hbar}{\rho V \omega}$$

$$\therefore u_0 = \sqrt{\frac{4(n + \frac{1}{2})\hbar}{\rho V \omega}}$$

phonon momentum: A phonon of wavevector  $\vec{k}$  due to its particle nature as well interacts/collides with photons, neutrons, electrons as if it had a momentum  $\hbar\vec{k}$ . This momentum is

called "crystal momentum", while a phonon on a lattice really has no momentum in the physical sense. This is because a phonon coordinate (except for  $K=0$ ) involves relative atomic coordinates, e.g. in  $H_2$  molecule, internuclear vibrational coordinate  $\vec{r}_1 - \vec{r}_2$  doesn't carry linear momentum, while the CM coordinate  $\frac{\vec{r}_1 + \vec{r}_2}{2}$  carries linear momentum although this corresponds to  $K=0$ .

Normal Process: From Fermi's golden rule we know that in crystals there is wavevector selection rule for allowed transitions between quantum states. Bragg's law defines the elastic scattering of an x-ray photon by a crystal with wavevector selection rule

$$\vec{k}' = \vec{k} + \vec{q}, \quad \vec{k}' = \text{wavevector of scattered photon},$$

$$\omega \vec{h}\vec{k}' = \omega \vec{h}\vec{k} + \omega \vec{h}\vec{q} \quad \vec{k} = \text{wavevector of incident photon}$$

$$\vec{q} = \text{reciprocal lattice vector.}$$

In this process the crystal recoils with a momentum  $\omega \vec{h}\vec{q}$  without changing the wavelength of the incident photon i.e.  $|\vec{k}'| = |\vec{k}|$ . This is called a "Normal process" (N-process) where no phonon is created or absorbed.

Umklapp Process: In N-process total wavevector of interacting waves is conserved, or crystal momentum is conserved which is the momentum modulo the reciprocal lattice vector. Note that this is not strictly due to Noether's theorem that deals with conservation of momentum due to translational invariance of space. Crystal momentum conservation results from space being invariant under translations of lattice constant  $a$ , giving momentum conserved modulo  $\frac{2\pi}{a}$ .

In an ~~el~~ inelastic scattering of a photon by a crystal, frequency of the incident photon changes & a new photon is emitted along with a phonon of wavevector  $\vec{K}$ , so that according to wavevector conservation,

$$\text{law, } \vec{K}' + \vec{R} = \vec{K} + \vec{q} \text{ (emission)}$$

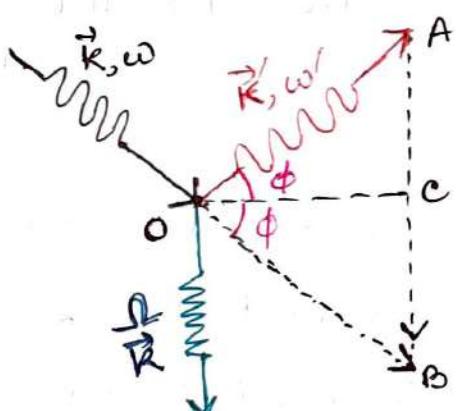
$$\vec{K}' = \vec{K} + \vec{R} + \vec{q} \text{ (absorption)}$$

Such processes in which wavelength of the photon changes is called "umklapp process" (U-process). The energy conservation yields  $\hbar\omega' + \hbar\Omega = \hbar\omega$  (emission)

$$\hbar\omega' = \hbar\omega + \hbar\Omega \text{ (absorption). To note, emission/absorption}$$

of photon by scattering of a phonon is not possible, although they have mechanical similarity with photons (wave-particle duality, energy quanta, indistinguishable particle nature obeying BE statistics). Due to sound wave nature of phonons, they're very different than EM natures photons.

### Inelastic Scattering of photons by phonons



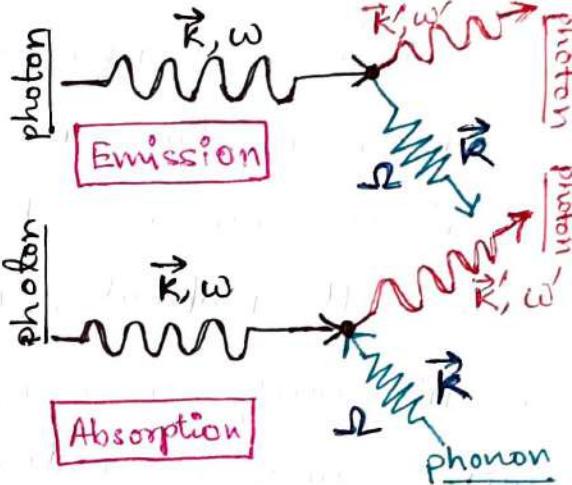
Let us consider a photon with wavevector  $K$  and angular frequency  $\omega$  is incident on a crystal with refractive index  $n$ . The velocity of the photon then is  $\frac{c}{n}$  and from  $v = \lambda f$  we have  $\frac{c}{n} = \frac{\omega}{2\pi\lambda} = \frac{\omega}{K}$ , so  $K = \frac{\omega n}{c}$ .

As the refractive index of the crystal changes due to change in concentration of atoms by the elastic strain field of the sound waves, the photon interacts with phonon to get inelastically scattered. Suppose the magnitude of phonon wavevector  $R$  is comparable to incident photon wavevector  $K$  and the phonon velocity  $v$  is constant, then

$$v = \lambda f = \frac{\lambda}{2\pi} \frac{2\pi}{K} = \frac{\lambda}{K}, \quad \text{or} \quad \lambda = vK$$

while  $c_n$  (light velocity in medium)  $\gg v$  (velocity of sound)

$$\text{so } c_n K \gg v K \quad (\text{as } K \approx \lambda) \quad \text{or} \quad \omega \gg \lambda$$



Using this in energy conservation  $\hbar\omega' \approx \hbar\omega$  or  $\omega' \approx \omega$  and  $k' = \frac{\omega'n}{c} \approx \frac{\omega n}{c} = k$ . In the isosceles triangle OAB, we have  $OA = OB$  and  $AB = |\vec{R}| = 2k \sin\phi = \frac{2\omega n}{c} \sin\phi$ . So  $\Omega = \omega R = \frac{2\omega n}{c} \sin\phi$ . This is the wavevector & angular frequency of the emitted phonon, that matches the experiment on quartz/sapphire crystal directed to a visible laser light source to produce phonons in the microwave range.

### Thermal properties of Phonons

Next we discuss on the heat capacity of a phonon gas. When defining 1st law of TD we have already discussed the definition of Molar specific heat at constant volume  $C_V = \left(\frac{d\mathcal{E}}{dT}\right)_V$  and at constant pressure  $C_P = \left(\frac{d\mathcal{E}}{dT}\right)_P$ . In solids, the increase in internal energy takes place due to vigorous vibrations of (nearly) free atoms about their equilibrium positions and due to thermal excitation of electrons to higher energy states (metals & semiconductors).

$$\therefore U = U_{\text{lattice}} + U_{\text{electron}} \quad \text{and we have heat capacity}$$

$$C_V = \left(\frac{dU}{dT}\right)_V = \frac{dU_{\text{lattice}}}{dT} + \frac{dU_{\text{electron}}}{dT} = \underbrace{C_{\text{lattice}}}_{\substack{\uparrow \\ \text{Lattice heat capacity}}} + C_{\text{electron}}$$

$$\approx C_{\text{lattice}}$$

while the basic assumption is no free electrons are present, so that the main contribution to specific heat is due to the excitations of thermal vibrations of the lattice. At room temperature, the heat capacity is  $\sim 3R$  where  $R = Nk_B = \# \text{ of atoms in solid} \times$  Boltzmann's constant  $\sim 6 \text{ cal/mole-K}$ . As one approaches  $\lim_{T \rightarrow 0K}$ ,  $C_{\text{lattice}}$  sharply decreases by following  $T^3$ -law (insulators),  $T^{-1}$ -law (metals)

for superconductor the decrement is faster. In magnetoo solids,  $C_{lattice}$  increases near the Curie temperature when the magnetic moments are ordered.

Classical Theory: A crystal can be considered as a 3D harmonic oscillator where due to strong cohesive energy atoms are arranged in periodic manner, freely vibrate about equilibrium position. Thermal energy increases the vibrational energy (internal energy) & so they can be termed as  $N$  independent oscillators with 3 d.o.f. or  $3N$  1D harmonic oscillators. with natural frequency  $\omega_0$ , so that

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 \text{ is the total energy.}$$

While the energy distribution of the oscillators obey Gibbs distribution the average energy of each harmonic oscillator is

$$\begin{aligned} \bar{E} &= \frac{\int e E e^{-\beta E} dE}{\int e^{-\beta E} dE} = \frac{\int \left( \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 \right) e^{-\frac{\beta p^2}{2m}} e^{-\frac{m\omega_0^2x^2\beta}{2}} dp dx}{\int e^{-\frac{\beta p^2}{2m}} e^{-\frac{m\omega_0^2x^2\beta}{2}} dp dx} \\ &= \frac{\int \frac{p^2}{2m} e^{-\frac{\beta p^2}{2m}} dp}{\int e^{-\frac{\beta p^2}{2m}} dp} + \frac{\int \frac{1}{2}m\omega_0^2x^2 e^{-\frac{m\omega_0^2x^2\beta}{2}} dx}{\int e^{-\frac{m\omega_0^2x^2\beta}{2}} dx} \\ &= \frac{1}{2m} \frac{\frac{1}{4}\sqrt{\pi}(2mK_B T)^3}{\frac{1}{2}\sqrt{\pi}(2mK_B T)} + \frac{1}{2}m\omega_0^2 \frac{\frac{1}{4}\sqrt{\pi}\left(\frac{2K_B T}{m\omega_0^2}\right)^3}{\frac{1}{2}\sqrt{\pi}\left(\frac{2K_B T}{m\omega_0^2}\right)} \\ &= \frac{1}{2}K_B T + \frac{1}{2}K_B T = K_B T \end{aligned}$$

$$\boxed{\begin{aligned} \beta &= \frac{1}{K_B T} \\ \int_0^\infty x^2 e^{-ax^2} dx &= \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \\ \int_0^\infty e^{-ax^2} dx &= \frac{1}{2}\sqrt{\frac{\pi}{a}} \end{aligned}}$$

So total vibrational energy  $E = 3N\bar{E} = 3NK_B T$  & heat capacity  $C_V = \left(\frac{\partial E}{\partial T}\right)_V = 3NK_B = 3R = 5.96 \text{ cal/moleK}$ . So the molar specific heat of all solids is constant & independent of temperature & frequency. This is called Dulong & Petit's law which fails at the low temperature limit. due to classical nature (energy ranging continuously from 0 to  $\infty$ ).

## Discrepancy removal in Einstein's Theory

In 1911 using Planck's quantum theory by replacing classical harmonic oscillators with quantum harmonic oscillators, so that discrete energy values are feasible, resolved the low temperature discrepancy of Dulong-Petit's law. The basic assumptions were

- (i) atoms are identical independent quantum harmonic oscillators with discrete energy levels  $E_n = (n + \frac{1}{2})\hbar\omega_0$ ,  $n = 0, 1, 2, 3$  quantum numbers. There are  $3N$  quantum oscillators.
- (ii) The atomic oscillators are distinguishable & obey Gibb's distribution in energies, all atoms vibrate in identical environments & have the same natural frequency.

So average energy of an oscillator  $\bar{E} = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}$

$$\begin{aligned}
 &= \frac{\sum_{n=0}^{\infty} (n + \frac{1}{2})\hbar\omega_0 \exp[-(n + \frac{1}{2})\beta\hbar\omega_0]}{\sum_{n=0}^{\infty} \exp[-(n + \frac{1}{2})\beta\hbar\omega_0]} \\
 &= \hbar\omega_0 \frac{\sum_{n=0}^{\infty} (n + \frac{1}{2}) \exp[(n + \frac{1}{2})x]}{\sum_{n=0}^{\infty} \exp[(n + \frac{1}{2})x]} = \hbar\omega_0 \left[ \frac{\frac{1}{2}e^{x\frac{1}{2}}}{e^{x\frac{1}{2}}} + \frac{3}{2}e^{x\frac{3}{2}} + \frac{5}{2}e^{x\frac{5}{2}} + \dots \right] \\
 &= \hbar\omega_0 \frac{d}{dx} \ln(e^{x\frac{1}{2}} + e^{x\frac{3}{2}} + e^{x\frac{5}{2}} + \dots) \\
 &= \hbar\omega_0 \frac{d}{dx} \ln[e^{x\frac{1}{2}} (1 + e^x + e^{2x} + \dots)] \\
 &= \hbar\omega_0 \frac{d}{dx} \left[ \frac{x}{2} + \ln(1 + e^x + e^{2x} + \dots) \right] = \hbar\omega_0 \left[ \frac{1}{2} + \frac{1}{e^{x\frac{1}{2}} - 1} \right] \\
 &= \hbar\omega_0 \left[ \frac{1}{2} + \frac{e^x}{1 - e^x} \right] = \hbar\omega_0 \left[ \frac{1}{2} + \frac{1}{e^{\hbar\omega_0\beta} - 1} \right] \\
 &= \frac{1}{2}\hbar\omega_0 + \frac{\hbar\omega_0}{e^{\hbar\omega_0\beta} - 1}
 \end{aligned}$$

$$\begin{aligned}
 x &= -\frac{\hbar\omega_0}{k_B T} \\
 &= -\beta\hbar\omega_0
 \end{aligned}$$

Note that at  $T = 0K$ ,  $\bar{E} = \frac{1}{2}\hbar\omega_0$  so each atom possesses vibrational energy that isn't zero but contribution to  $C_V$  is zero. Also  $\bar{E}$  is a function of  $\omega_0$  and  $\beta$ .

So the energy (total internal) of the crystal is

**zero point energy**

$$E = 3N\bar{E} = \frac{3}{2}N\hbar\omega_0 + \frac{3N\hbar\omega_0}{e^{\beta\hbar\omega_0} - 1} \quad \text{so that heat capacity is}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = 3NK_B \left(\frac{\hbar\omega_0}{K_B T}\right)^2 \frac{e^{\hbar\omega_0/K_B T}}{(e^{\hbar\omega_0/K_B T} - 1)^2} = 3R \left(\frac{\theta_E}{T}\right)^2 \frac{e^{\theta_E/T}}{(e^{\theta_E/T} - 1)^2}$$

where  $\theta_E = \frac{\hbar\omega_0}{K_B}$  is the characteristic Einstein temperature. The frequency  $\nu$  of the atomic oscillator corresponding to the Einstein temperature  $\theta_E$  is known as Einstein frequency,  $\nu = \frac{k_B \theta_E}{\hbar}$ .

Predictions and comparison with experimental results:

(i) High temperature limit ( $K_B T \gg \hbar\omega_0$ ) ( $T \gg \theta_E$ ):

$$\text{At large temperature } \bar{E} = \frac{1}{2}\hbar\omega_0 + \frac{\hbar\omega_0}{e^{\hbar\omega_0/K_B T} - 1} \approx \frac{1}{2}\hbar\omega_0 + \frac{\hbar\omega_0}{1 + \frac{\hbar\omega_0}{K_B T}} = \frac{1}{2}\hbar\omega_0 + K_B T \text{ which agrees with the classical picture.}$$

$$C_V = 3R \left(\frac{\theta_E}{T}\right)^2 \frac{e^{\theta_E/T}}{(e^{\theta_E/T} - 1)^2} \approx 3R \left(\frac{\theta_E}{T}\right)^2 \frac{1}{(1 + \frac{\theta_E}{T} - 1)^2} = 3R \left(\frac{\theta_E}{T}\right)^2 \left(\frac{T}{\theta_E}\right)^2$$

$= 3R$  which is the Dulong-Petit's law.

(ii) Low temperature limit ( $K_B T \ll \hbar\omega_0$ ) ( $T \ll \theta_E$ ):

$$\text{At low temperature } \bar{E} = \frac{1}{2}\hbar\omega_0 + \frac{\hbar\omega_0}{e^{\hbar\omega_0/K_B T} - 1} \approx \frac{1}{2}\hbar\omega_0 + \frac{\hbar\omega_0}{e^{-\hbar\omega_0/K_B T}}$$

$= \frac{1}{2}\hbar\omega_0 + \hbar\omega_0 e^{-\hbar\omega_0/K_B T}$ . So the average vibrational energy decreases exponentially with decreasing temperature.

$$C_V = 3R \left(\frac{\theta_E}{T}\right)^2 \frac{e^{\theta_E/T}}{(e^{\theta_E/T} - 1)^2} \approx 3R \left(\frac{\theta_E}{T}\right)^2 e^{-\theta_E/T}$$

A plot of  $C_V$  vs  $\frac{T}{\theta_E}$  for Diamond at  $\theta_E = 1320\text{K}$  fits the experimental results over a wide range of temperature except at very-low liquid helium temperature where  $C_V \propto e^{-\theta_E/T}$  breaks down and follows  $C_V \propto T^3$ . Experimentally, Einstein temperature can be easily measured using  $\omega_0$  (for monoatomic basis  $\omega_0 = \sqrt{\frac{4c}{m}}$ ) by substituting the elastic

constant. For most metals,  $100K \leq \theta_E \leq 200K$ .

### Discrepancy removal in Debye's theory

In 1912, Debye noticed the factual oversimplification in Einstein's theory of the independence of quantum oscillators. The basic flaw was that the vibrational motion of the crystal was taken to be same as natural vibrational frequency of a single atom  $\omega_0$ . Debye proposed that the crystals can propagate from low frequency (sound waves) to high frequency (infrared absorption) elastic waves having a number of modes of vibration. The number of vibrational modes per unit frequency range is called the density of modes  $D(\nu)$  that we determine next.

In 1D if we have a continuous string of length  $L$  with two ends fixed, then the solution of wave equation  $u'' = \frac{1}{v^2} u$  corresponds to a standing wave  $u(x,t) = A \sin\left(\frac{n\pi x}{L}\right) \cos 2\pi\nu_n t$ . Substituting this back, we get  $\frac{n^2\pi^2}{L^2} = \frac{1}{v^2} 4\pi^2 n^2 \Rightarrow \nu_n = \frac{nv}{2L}$   $= \frac{\nu}{2L/n} = \frac{\nu}{\lambda_n}$  where  $\lambda_n = \frac{2L}{n}$ . This means that frequency of the string have integral multiple of  $\frac{\nu}{2L}$ . While  $n = \frac{2L}{\lambda} \nu_n$  we have  $dn = \frac{2L}{\lambda} d\nu = D(\nu) d\nu$ . This is the number of vibrational modes. In 3D, the wave equation is  $\nabla^2 u = \frac{1}{v^2} u$  that corresponds to a fixed-face cube of size  $L$ . The standing wave solutions are  $u(\vec{r},t) = A \sin\left(\frac{n_x\pi x}{L}\right) \sin\left(\frac{n_y\pi y}{L}\right) \sin\left(\frac{n_z\pi z}{L}\right) \cos 2\pi\nu t$  with  $n_x, n_y, n_z \geq 1$ . Substituting back we have

$$\frac{\pi^2}{L^2} (n_x^2 + n_y^2 + n_z^2) = \frac{4\pi^2\nu^2}{v^2} \Rightarrow n_x^2 + n_y^2 + n_z^2 = \frac{4L^2\nu^2}{v^2}$$

To determine the possible modes of vibration  $D(\nu) d\nu$  we consider a network of points whose coordinates are  $(n_x, n_y, n_z)$ . The radius

vector  $R_n$  from origin to any point is  $R_n^2 = n_x^2 + n_y^2 + n_z^2$

so  $R_n = \frac{2L}{v} n$  or  $dR = \frac{2L}{v} dv$ . As each point occupies on an average a unit volume in the integer space, so the number of points in the spherical shell between  $R$  &  $R+dr$  is equal to the volume of one octant of a spherical shell,

$$\text{So Volume of one octant of spherical shell} = \frac{1}{8} 4\pi R^2 dr \\ = D(r) dr$$

Substituting  $R$  &  $dr$ , we obtain

$$D(r) dr = \frac{4\pi}{8} \frac{4L^2 r^2}{v^2} \frac{2L}{v} dr = \frac{4\pi L^3}{v^3} r^2 dr = \frac{4\pi V}{v^3} r^2 dr$$

while there are one longitudinal wave & two transverse waves (having two  $\perp$  direction of polarisation) in an elastic solid, we have the total number of vibrational modes as

$$D(r) dr = 4\pi V \left( \frac{1}{v_e^3} + \frac{2}{v_t^3} \right) r^2 dr$$

### Debye Approximation

Debye assumed that the crystal can be regarded as a continuum as long as wavelength of elastic wave is large compared to interatomic spacing. Also while the total number of vibrational modes are limited to  $3N$ , so there is an upper cutoff to the frequency  $v_D$  (Debye frequency) of the elastic wave. Therefore

$$\int_0^{v_D} D(r) dr = 3N \quad \Rightarrow \quad \int_0^{v_D} 4\pi V \left( \frac{1}{v_e^3} + \frac{2}{v_t^3} \right) r^2 dr = 3N$$

$$\text{or } v_D^3 = \frac{9N}{4\pi V} \left( \frac{1}{v_e^3} + \frac{2}{v_t^3} \right)^{-1}$$

This is not a good approximation

because taking  $v_e \approx v_t \sim 10^3$  m/s &  $\frac{N}{V} = 10^{28}/m^3$ ,  $v_D \sim 10^{13} s^{-1}$  that corresponds to wavelength  $\sim 1\text{\AA}$  = interatomic spacing, so the validity of continuum theory is doubtful. Also  $v_e$  &  $v_t$  is not always independent of the wavelength.

Neglecting the zero point energy that doesn't contribute to  $C_V$ , we know that the average vibrational energy of an oscillator with frequency  $\nu$  at  $T$  is known from Planck's theory

$\bar{E} = \frac{h\nu}{e^{h\nu/k_B T} - 1}$ . We can associate with each vibrational mode a quantum oscillator so that the

vibrational energy of the crystal is  $E = \int_{\nu_0}^{\nu_D} \bar{E} D(\nu) d\nu$

$$= \int_0^{\nu_D} 4\pi \nu \left( \frac{1}{\nu_L^3} + \frac{2}{\nu_t^3} \right) \frac{h\nu^3}{e^{h\nu/k_B T} - 1} d\nu = 4\pi h\nu \left( \frac{1}{\nu_L^3} + \frac{2}{\nu_t^3} \right) \int_0^{\nu_D} \frac{\nu^3 d\nu}{e^{h\nu/k_B T} - 1}$$

$$= \frac{9Nh}{\nu_D^3} \int_0^{\nu_D} \frac{\nu^3 d\nu}{e^{h\nu/k_B T} - 1} = \frac{9Nh}{\nu_D^3} \left( \frac{k_B T}{h} \right)^4 \int_0^{x_m} \frac{x^3 dx}{e^x - 1} \quad \left[ \frac{h\nu}{k_B T} = x, \frac{h\nu_D}{k_B T} = x_m \right] \\ = 9N \left( \frac{k_B T}{h\nu_D} \right)^3 k_B T \int_0^{x_m} \frac{x^3 dx}{e^x - 1} = 9N k_B T \left( \frac{T}{\theta_D} \right)^3 \int_0^{\theta_D/T} \frac{x^3 dx}{e^x - 1}$$

So the specific heat is  $C_V = \left( \frac{\partial E}{\partial T} \right)_V = 9N k_B \left( \frac{T}{\theta_D} \right)^3 \int_0^{\theta_D/T} \frac{e^x x^4}{(e^x - 1)^2} dx = 3R \left( \frac{\theta_D}{T} \right) F_D$

where  $F_D$  is called the Debye function  $= 3 \left( \frac{T}{\theta_D} \right)^4 \int_0^{\theta_D/T} \frac{e^x x^4}{(e^x - 1)^2} dx$

and  $\theta_D = \frac{h\nu_D}{k_B}$  is the Debye temperature.

Predictions and comparison with experimental results:

(i) High temperature limit ( $k_B T \gg h\nu$ ) ( $T \gg \theta_D$ ):

At high temperature  $e^x - 1 \approx 1 + x - 1 = x$  so that

$$E = 9N k_B T \left( \frac{T}{\theta_D} \right)^3 \int_0^{\theta_D/T} x^2 dx = 3N k_B T = 3RT \text{ so that the}$$

specific heat  $C_V = \frac{\partial E}{\partial T} = 3R$  (Dulong-Petit's law). So the quantum considerations are insignificant at high temperatures.

(ii) Low temperature limit ( $k_B T \ll h\nu$ ) ( $T \ll \theta_D$ ):

while  $\theta_D \gg T$ ,  $x_m = \frac{\theta_D}{T} \rightarrow \infty$ . so that mean vibrational energy

$$E = 9RT \left(\frac{T}{\theta_D}\right)^3 \int_0^\infty \frac{x^3 dx}{e^x - 1} = 9RT \left(\frac{T}{\theta_D}\right)^3 \frac{\pi^4}{15} = \frac{3\pi^4}{5} R \frac{T^4}{\theta_D^3}$$

noticeable similarity is  $E \propto T^4$  (Stefan's law in Blackbody radiation of photons) which is only true for phonons at small temperatures only. Now the heat capacity becomes

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{12\pi^4}{5} R \left(\frac{T}{\theta_D}\right)^3. \text{ So } C_V \propto T^3 \text{ (Debye-T}^3\text{ law)}$$

and it holds for  $T \leq 0.1\theta_D$ . For Silver with  $\theta_D = 225K$ , Debye model remarkably matches with experiment. The Debye model yields a larger specific heat compared to Einstein's model as Debye model considers the low frequency modes which at low temperature have higher vibrational energy (or more  $C_V$ ). At very low temperature,  $C_V$  is relatively insensitive to the variations of the density of modes.

### Limitations of the Debye Model

1. Because it's a continuum model valid for long-wavelength, low frequencies only in the solid, it neglects the dispersion of waves.
2. While solid is treated as elastic continuum, it can have infinite number of frequencies but Debye's theory restricts to  $3N$  modes.
3.  $\nu_D$  is assumed equal for  $\nu_e$  and  $\nu_t$  which is very unlikely.
4. At  $< 0.1\theta_D$  the theory breaks down & not only that  $\theta_D = \frac{h\nu_D}{k_B}$  varies upto 10% with temperature.
5. Debye theory is inapplicable to non-identical atoms crystals, e.g., NaCl, CsCl etc.
6. Debye theory completely neglects interatomic interaction & electronic contribution to heat capacity.

## Crystal Bonding

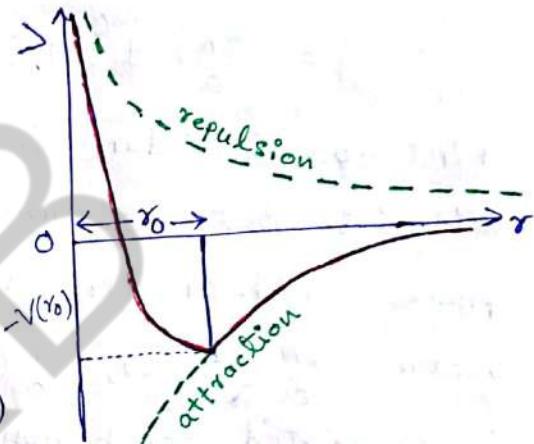
The ability to hold the atoms/ions together is called bonding. Atoms vibrate in lattice & nucleus due to heavy mass is almost at rest. So electrostatic interaction happen between electron cloud & distribution of electron leads to 5 types of bonding due to (a) "attractive force" of negatively charged electron cloud of one atom with positive nuclear charge of other atom (b) "repulsive force" of overlapping negatively charged electron clouds & positively charged nucleus of two atoms.

"spring effect"  $\rightarrow$  attraction  $\rightarrow$  repulsion.

$$F = - \frac{dV}{dr}$$

attractive force = negative potential.

repulsive force = positive potential.



cohesive / binding energy  $V(r_0)$  (negative)  
dissociation energy  $-V(r_0)$  (positive)

Cohesive energy of a solid is the energy that will be given out in forming a crystal by bringing neutral atoms from  $\infty$  to equilibrium separation  $r_0$ .

Suppose  $V_{\text{attractive}} \propto r^{-m}$  &  $V_{\text{repulsive}} \propto r^{-n}$

$$\therefore \text{Cohesive energy } V = V_{\text{attractive}} + V_{\text{repulsive}} = -Ar^{-m} + Br^{-n}$$

$$\text{& force } F = -\frac{dV}{dr} = mA\gamma^{-(m+1)} - nB\gamma^{-(n+1)}$$

$$\text{at } r = r_0, F = 0 = mA\gamma_0^{-(m+1)} - nB\gamma_0^{-(n+1)}.$$

$$\text{or } \gamma_0^{m-n} = \frac{A}{B} \frac{m}{n}$$

$$\begin{aligned} \text{Then equilibrium potential energy } V(r_0) &= -Ar_0^{-m} + Br_0^{-n} \\ &= -A\gamma_0^{-m} \left(1 - \frac{B}{A}\gamma_0^{m-n}\right) = -A\gamma_0^{-m} \left(1 - \frac{m}{n}\right). \end{aligned}$$

For  $V$  to be minimum, it must be concave upwards curvature,

$$\left. \frac{d^2V}{dr^2} \right|_{r=r_0} > 0 \quad \text{or} \quad \left[ -m(m+1)A r^{-(m+2)} + n(n+1)B r^{-(n+2)} \right]_{r=r_0} > 0$$

$$\text{or} \quad -m(m+1) + n(n+1) \frac{B}{A} r_0^{m-n} > 0$$

$$\text{or} \quad -m(m+1) + n(n+1) \frac{m-n}{m} \frac{B}{A} > 0$$

$$\text{or} \quad n-m > 0 \quad \text{or} \quad n > m.$$

Thus to form a chemical bond, we always need repulsive force be of shorter range than attractive force.

Types of bonding 5 types of bonding exist

(a) Ionic bond (transfer of valence electron): NaCl, LiF.

(b) Covalent bond (sharing of valence electrons): Diamond, SiC.

(c) Metallic bond (free nature of valence electron): Cu, Ag, Fe

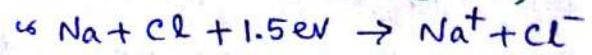
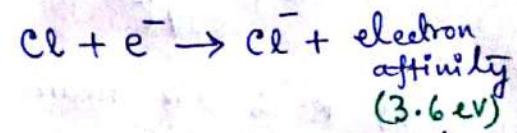
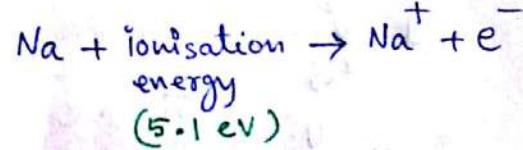
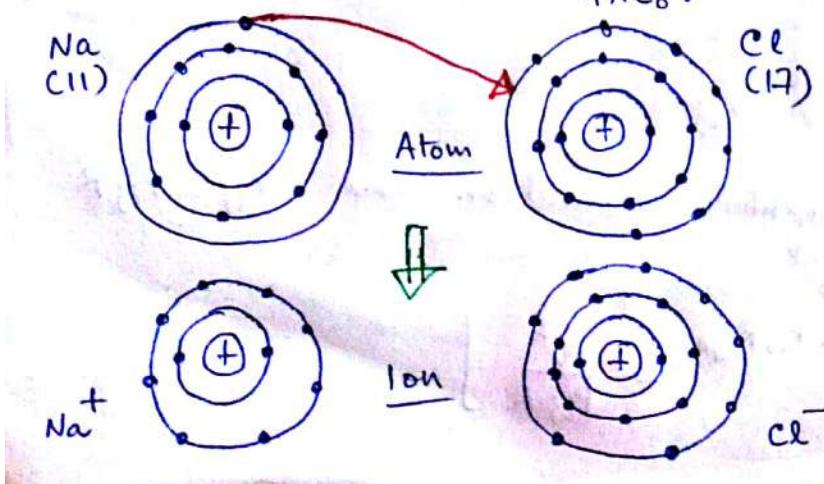
(d) Hydrogen bond ( $V \propto -r^{-2}$ ): Ice

(e) van der Waal's bonding (dipole-dipole interaction)

Ionic / Electrovalent Bonding

Transfer of electrons from an electropositive element to electronegative element, to create  $+$   $-$  ion. Electronegative element of large electron affinity accomodate extra added electron to complete outermost valence orbit to stabilize. Oppositely charged ions

$$\text{attract } V_{\text{attraction}} = -\frac{Z_1 Z_2 e^2}{4\pi \epsilon_0 r}$$



$$Z_1 = Z_2 = 1.$$

$$\text{So potential energy } V = -\frac{e^2}{4\pi\epsilon_0 r_0} = \frac{-(1.6 \times 10^{-19})^2}{4\pi \times 8.85 \times 10^{-12} \times 2.4 \times 10^{-10}} \\ = -\frac{9.6 \times 10^{-19}}{1.6 \times 10^{-19}} \text{ eV} = -6 \text{ eV.}$$

$$\text{So net energy released} = 5.1 - 3.6 - 6 = -1.5 \text{ eV.}$$

Cohesive energy Binding energy calculated by Born & Madelung in 1910 extended by Mayer.

assumptions : (a) Ionic crystals are formed by positive & negative ions with spherical charge distribution. (b) force of attraction depends on inter-ionic distance & isotropic (orientation independent), (c) Electrostatic interaction (Madelung energy  $V_a = -\frac{\alpha q^2}{4\pi\epsilon_0 r}$ ,  $\alpha$  = Madelung constant) contributes to cohesive energy

According to Born-Madelung theory interaction energy  $U_i$  on ion  $i$  due to all  $j$  other ions,  $U_i = \sum_{j \neq i} U_{ij}$

$U_i$  consists of two parts:

1. Short range central field repulsive potential  $\beta r_{ij}^{-n}$  between + & - ions which was modified by  $\lambda e^{-r_{ij}/\rho}$ ,  $\lambda$  = strength,  $\rho$  = range of interaction (screened Coulomb)
2. Attractive or repulsive long ranged coulomb force with energy  $\pm \frac{q^2}{r_{ij}}$

$$\therefore U_i = \sum_{j \neq i} \left[ \lambda e^{-r_{ij}/\rho} \pm \frac{q^2}{r_{ij}} \right]$$

If  $R$  is the nearest neighbour separation then  $r_{ij} = p_{ij} R$  where  $p_{ij}$  is a dimensionless quantity.

$$\text{Then } U_i = \sum_{j \neq i} \left[ \lambda e^{-p_{ij} R/\rho} \pm \frac{q^2}{p_{ij} R} \right]$$

$$= Z \lambda e^{-\frac{R}{\rho}} \pm \sum_{j \neq i} \frac{q^2}{p_{ij} R} = Z \lambda e^{-\frac{R}{\rho}} - \frac{\alpha q^2}{R}$$

where  $Z$  is number of nearest neighbours of  $i^{th}$  ion &  $\alpha = \pm \sum_{j \neq i} \frac{1}{p_{ij}}$

is called Madelung constant

If the crystal contain  $2N$  ions or  $N$  molecules, then

$$U_{\text{total}} = N U_i = N \left[ Z \lambda e^{-\frac{R}{\rho}} - \frac{\alpha q^2}{R} \right]$$

at equilibrium distance  $R = R_0$ ,  $\frac{dU_{\text{total}}}{dR} = 0$

$$\therefore -\frac{Z \lambda}{\rho} e^{-\frac{R_0}{\rho}} + \frac{\alpha q^2}{R_0^2} = 0$$

$$e^{-\frac{R_0}{\rho}} = \frac{\rho \alpha q^2}{Z \lambda R_0^2}$$

$$\therefore U_{\text{total}} = N \left[ Z \lambda \frac{\rho \alpha q^2}{Z \lambda R_0^2} - \frac{\alpha q^2}{R_0} \right] = -\frac{N \alpha q^2}{R_0} \left( 1 - \frac{R_0}{R} \right)$$

Madelung energy

contribution from short range repulsion

$$U_i = -\frac{\alpha q^2}{R_0} \left( 1 - \frac{R_0}{R} \right)$$

or in SI units,

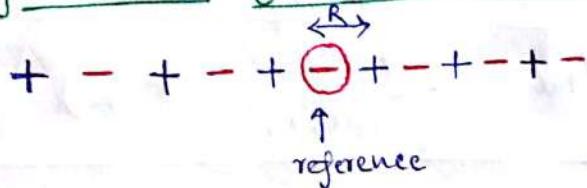
$$U_i = -\frac{\alpha q^2}{4\pi\epsilon_0 R_0} \left( 1 - \frac{R_0}{R} \right)$$

as  $\frac{R_0}{R} \rightarrow 0$  repulsive interaction is very short range.

In  $\alpha = \sum \pm \frac{1}{p_{ij}}$ ,  $+$  is used for +ive ion & - for -ive ion if  $j^{th}$  ion is -ive. & we consider repulsive

interaction effective for nearest neighbours only.

Madelung constant for a 1D lattice



$$\alpha = \sum_{j \neq i} \pm \frac{1}{p_{ij}}, \quad \frac{\alpha}{R} = \sum_{j \neq i} \pm \frac{1}{p_{ij} R} = \pm \sum_{j \neq i} \frac{1}{r_{ij}}$$

$$\therefore \frac{\alpha}{R} = 2 \left[ \frac{1}{R} - \frac{1}{2R} + \frac{1}{3R} - \frac{1}{4R} + \dots \right]$$

$$\alpha = 2 \left[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \right]$$

↑ due to both side of reference ion

$$\text{but } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\text{If } x=1, \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$\therefore \alpha = 2 \ln 2 = 1.38$$

### Madelung constant for NaCl crystal

Nearest neighbour to -ive (reference) ion = 6 +ive ions with  
 $p_{ij} = p = 1$ . 12 -ive ions at  $p = \sqrt{2}$ . 8 +ive ions at  $p = \sqrt{3}$ .  
 6 -ive ions at  $p = \sqrt{4}$  & so on

$$\alpha = \frac{6}{1} - \frac{12}{\sqrt{2}} + \frac{8}{\sqrt{3}} - \frac{6}{\sqrt{4}} + \dots = -1.748$$

Bigger  $\alpha$ , more cohesive energy, greater stability of structure

### Cohesive energy for repulsive core potential $\propto r^{-n}$

$$V_i = V_{\text{attractive}} + V_{\text{repulsive}} = -\frac{\alpha q^2}{4\pi\epsilon_0 r} + \frac{B}{r^n}$$

$$\text{at } r = r_0, V \text{ is minimum, } \left. \frac{dV}{dr} \right|_{r=r_0} = 0 = \frac{dq^2}{4\pi\epsilon_0 r_0^{n+1}} - \frac{nB}{r_0^{n+1}}$$

$$\propto B = \frac{\alpha q^2 r_0^{n-1}}{4\pi\epsilon_0^n}$$

$$\therefore V_i = -\frac{\alpha q^2}{4\pi\epsilon_0 r_0} \left( 1 - \frac{1}{n} \right)$$

$$\propto \text{for } 2N \text{ molecules, } V_{\text{tot}} = -\frac{N\alpha q^2}{4\pi\epsilon_0 r_0} \left( 1 - \frac{1}{n} \right)$$

$$\text{for NaCl, } \alpha = 2 \ln 2,$$

$$V_{\text{tot}} = -\frac{2Nq^2 \ln 2}{4\pi\epsilon_0 r_0} \left( 1 - \frac{1}{n} \right).$$

## Bulk modulus of ionic crystals

Volume strain =  $\frac{dV}{V}$ , change in pressure  $dP$ , Bulk modulus

$$B = - \left. \frac{dP}{dV/V} \right|_{R=R_0}, \text{ Using 1st law of thermodynamics, } d\mathcal{G} = dU + PdV$$

$$\text{or } \frac{dU}{dV} = -P \quad (d\mathcal{G}=0) \quad \text{or } \frac{d^2U}{dV^2} = - \frac{dP}{dV}$$

$$\therefore B = \left. \sqrt{\frac{d^2U}{dV^2}} \right|_{R=R_0}$$

Volume occupied by  $\frac{1}{2}$  molecule  $\rightarrow R_0^3$

Volume occupied by 1 molecule  $\rightarrow 2R_0^3$

Volume occupied by  $N$  molecule  $\rightarrow 2NR_0^3$  (2N ions)

Volume of unit cell  $\rightarrow (2R_0)^3 = 8R_0^3$  because  $a = 2R_0$

$$V = 2NR_0^3, \frac{dV}{dR} = 6NR_0^2 \text{ and } \left. \frac{dU}{dR} \right|_{R=R_0} = 0$$

$$\begin{aligned} \therefore \frac{d^2U}{dV^2} &= \frac{d}{dV} \left( \frac{dU}{dV} \right) = \frac{d}{dV} \left( \frac{dU}{dR} \cdot \frac{dR}{dV} \right) = \frac{d}{dV} \left( \frac{dU}{dR} \right) \cdot \frac{dR}{dV} + \frac{dU}{dR} \cdot \frac{d^2R}{dV^2} \\ &= \frac{d}{dR} \left( \frac{dU}{dR} \right) \frac{dR}{dV} \cdot \frac{dR}{dV} + \frac{dU}{dR} \cdot \frac{d^2R}{dV^2} = \frac{d^2U}{dR^2} \cdot \left( \frac{dR}{dV} \right)^2 + \frac{dU}{dR} \cdot \frac{d^2R}{dV^2} \end{aligned}$$

$$\therefore \left. \frac{d^2U}{dV^2} \right|_{R=R_0} = \left. \frac{d^2U}{dR^2} \cdot \left( \frac{dR}{dV} \right)^2 \right|_{R=R_0} = \left. \frac{1}{(6NR_0^2)^2} \frac{d^2U}{dR^2} \right|_{R=R_0}$$

$$\therefore B = \left. \sqrt{\frac{d^2U}{dV^2}} \right|_{R=R_0} = \left. 2NR_0^3 \frac{1}{36N^2R_0^4} \frac{d^2U}{dR^2} \right|_{R=R_0} = \left. \frac{1}{18NR_0} \frac{d^2U}{dR^2} \right|_{R=R_0}$$

We learned that  $U_{\text{total}} = N \left[ ZA e^{-R/\rho} - \frac{\alpha q^2}{R} \right]$

$$\therefore \frac{dU_{\text{total}}}{dR} = - \frac{NZA}{\rho^2} e^{-R/\rho} + \frac{N\alpha q^2}{R^2}$$

$$\frac{d^2U_{\text{total}}}{dR^2} = \frac{NZA}{\rho^2} e^{-R/\rho} - \frac{2N\alpha q^2}{R^3}, \text{ also } e^{-R_0/\rho} = \frac{\rho^2 \alpha q^2}{Z A R_0^2}$$

$$\therefore B = \frac{1}{18NR_0} \left[ \frac{NZA}{\rho^2} e^{-R_0/\rho} - \frac{2N\alpha q^2}{R_0^3} \right] = \frac{1}{18NR_0} \left[ \frac{NZA}{\rho^2} \frac{\rho^2 \alpha q^2}{Z A R_0^2} - \frac{2N\alpha q^2}{R_0^3} \right]$$

$$B = \frac{\alpha q^2}{18R_0^4} \left( \frac{R_0}{\rho} - 2 \right)$$

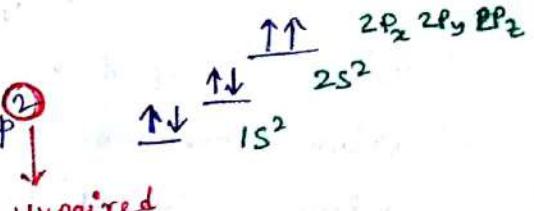
From  $B$  &  $R_0$ , range of repulsive interaction can be calculated.

## Covalent Bonding

Equal sharing of electrons between neighbouring atom with incomplete outermost shell. Unlike isotropic bonds in ionic, these are directional, due to electron's restricted orbital motion.

Covalent bond can happen due to overlap of s-orbital with opposite spin paired electrons (like  $H_2$ ) or hybrid bonding due to overlapping s & p orbitals.

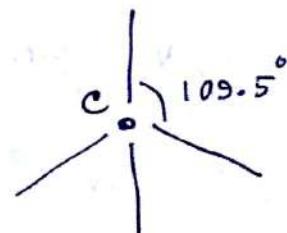
for single carbon atom  $C_6 = 1S^2 2S^2 2P^2$



But when more carbon atom comes close

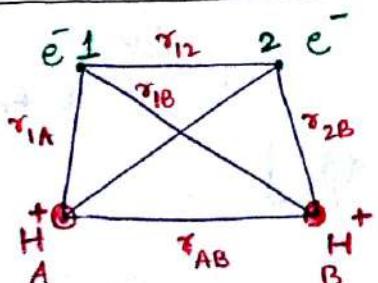


So 4-bonds can form with bond angle  $109.5^\circ$  in a regular tetrahedron using 4 unpaired electrons in 2s, 2p orbital.



( $sp^3$  hybridization)  $\Rightarrow$  Diamond, Si, Ge etc.

## Heitler-London theory

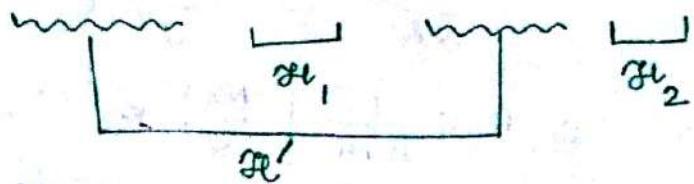


A & B are  $H^+$  nucleus (hydrogen atom) with two electrons 1 & 2 in  $1S^1$  orbital.

$\Psi_{1A}, \Psi_{2B}$  are eigenfunction of A, B atoms.

Total Hamiltonian is  $\mathfrak{H} = \frac{e^2}{r_{AB}} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1A}} - \frac{e^2}{r_{1B}} - \frac{e^2}{r_{2A}} - \frac{e^2}{r_{2B}}$

$\mathfrak{H}_1, \mathfrak{H}_2$  = potential energy of electron 1 & 2 (without overlap)



$\mathfrak{H}'$  = exchange potential (interaction)

When there is no spin-orbit coupling, A & B are far apart

$$\Psi(\vec{r}_1 \vec{s}_1; \vec{r}_2 \vec{s}_2) = \underbrace{\psi_{1A}}_{\text{space}} \underbrace{\psi_{2B}}_{\text{spin}} \phi_{1A} \phi_{2B} \quad \text{with Pauli exclusion principle}$$

$$\text{principle } \Psi(\vec{r}_1 \vec{s}_1; \vec{r}_2 \vec{s}_2) = -\Psi(\vec{r}_2 \vec{s}_2; \vec{r}_1 \vec{s}_1).$$

We can write the Schrödinger equation for the two electron system  $-\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2)\Psi + (\mathcal{H}_1 + \mathcal{H}_2)\Psi = E\Psi \quad (\text{no overlap})$

$$\therefore \nabla_1^2 \psi_{1A} + \frac{2m}{\hbar^2}(E_1 - \mathcal{H}_1)\psi_{1A} = 0 \quad \left. \begin{array}{l} \text{H-atom solution in radial part, spherical polar \& azimuthal part} \\ \nabla_2^2 \psi_{2B} + \frac{2m}{\hbar^2}(E_2 - \mathcal{H}_2)\psi_{2B} = 0 \end{array} \right\}$$

$$\psi_{1A} \sim \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r_{1A}}{a_0}}, \quad \psi_{2B} \sim \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r_{2B}}{a_0}}, \quad E_1 = E_2 = \frac{e^2}{a_0}$$

$[a_0 = \text{Bohr orbit radius}]$

For 2-H atom  $\Psi = \psi_{1A}\psi_{2B}$  is the wavefunction &  $|\Psi|^2$  is probability of finding both electrons.

But they are indistinguishable, so due to exchange degeneracy  $\psi_{2A}\psi_{1B}$  is also a wavefunction.

Superpositions are also wavefunction  $\Psi_s$  (symmetric),  $\Psi_A$  (antisymmetric)

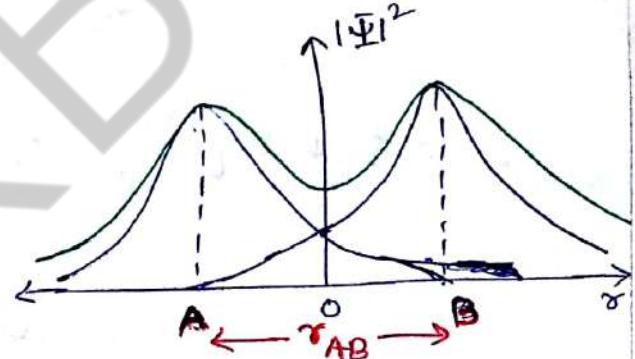
$$\Psi_s = \frac{1}{\sqrt{2+2S}} (\psi_{1A}\psi_{2B} + \psi_{2A}\psi_{1B}),$$

$$\Psi_A = \frac{1}{\sqrt{2-2S}} (\psi_{1A}\psi_{2B} - \psi_{2A}\psi_{1B})$$

$$\therefore H = E_0 + E_0 + \mathcal{H}' = 2E_0 + \mathcal{H}'$$

$$\text{Exchange interaction : } E' = \frac{\int \Psi^* \mathcal{H}' \Psi d^3r}{\int \Psi^* \Psi d^3r}$$

$$\text{with } \Psi = \frac{1}{\sqrt{2\pm 2S}} (\psi_{1A}\psi_{2B} \pm \psi_{2A}\psi_{1B}).$$



$$S = \iint (\psi_{1A}\psi_{2B})^* (\psi_{2A}\psi_{1B}) d\mathbf{r}_1 d\mathbf{r}_2$$

$$= \iint (\psi_{2A}\psi_{1B})^* (\psi_{1A}\psi_{2B}) d\mathbf{r}_1 d\mathbf{r}_2$$

= overlap integral  $\leq 1$

Remove normalization factors as they cancel from numerator & denominator,

$$\begin{aligned} & \iint (\Psi_{1A} \Psi_{2B} \pm \Psi_{2A} \Psi_{1B})^* (\Psi_{1A} \Psi_{2B} \mp \Psi_{2A} \Psi_{1B}) dr_1 dr_2 \\ = & \iint \Psi_{1A}^* \Psi_{2B}^* \cancel{\Psi_{1A} \Psi_{2B}} \overset{1}{dr_1 dr_2} + \iint \Psi_{2A}^* \Psi_{1B}^* \cancel{\Psi_{2A} \Psi_{1B}} \overset{1}{dr_1 dr_2} \\ \pm 2 \iint \Psi_{1A}^* \Psi_{2B}^* \Psi_{2A} \Psi_{1B} dr_1 dr_2 & = 2 \pm 2 S \end{aligned}$$

and  $\iint (\Psi_{1A} \Psi_{2B} \pm \Psi_{2A} \Psi_{1B})^* \mathcal{S}' (\Psi_{1A} \Psi_{2B} \mp \Psi_{2A} \Psi_{1B}) dr_1 dr_2$

$$= \iint \underbrace{\Psi_{1A}^* \Psi_{2B}^* \mathcal{S}' \Psi_{1A} \Psi_{2B} dr_1 dr_2}_{\text{Coulomb integral}} + \iint \Psi_{2A}^* \Psi_{1B}^* \mathcal{S}' \Psi_{2A} \Psi_{1B} dr_1 dr_2$$

$$\pm \iint [\Psi_{1A}^* \Psi_{2B}^* \mathcal{S}' \Psi_{2A} \Psi_{1B} + \Psi_{2A}^* \Psi_{1B}^* \mathcal{S}' \Psi_{1A} \Psi_{2B}] dr_1 dr_2$$

$2 \leftrightarrow 1$  interchange

$$= 2 \xi \pm 2 \eta , \quad \xi = \iint \Psi_{1A}^* \Psi_{2B}^* \mathcal{S}' \Psi_{1A} \Psi_{2B} dr_1 dr_2$$

= same atom interaction.  $< 0$

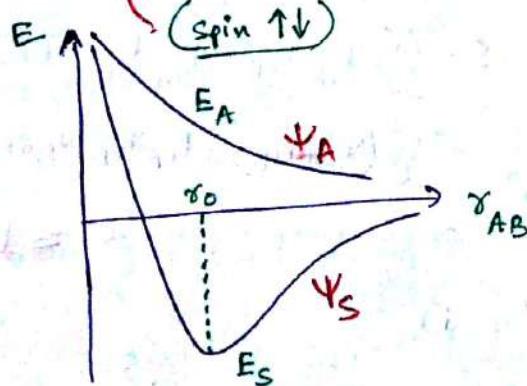
$$\eta = \iint \Psi_{1A}^* \Psi_{2B}^* \mathcal{S}' \Psi_{1B} \Psi_{2A} dr_1 dr_2$$

= exchange interaction  $< 0$

$\therefore$  Energy eigenvalues are

$$E_S = 2E_0 + \frac{\xi + \eta}{1 + S} , \quad E_A = 2E_0 + \frac{\xi - \eta}{1 - S}$$

$(\text{spin } \uparrow\downarrow)$

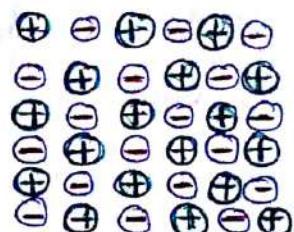


$\Psi_A$  cannot form a bond because  $E > 0$

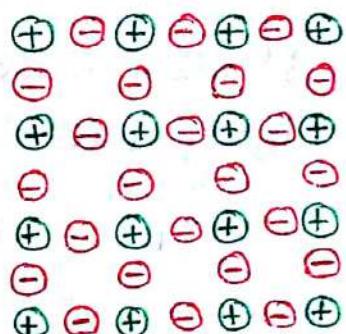
$\Psi_S$  can form the covalent bond.

So two electrons of  $\uparrow\downarrow$  pair up to form a bond due to "exchange interaction".

## Metallic Bonding

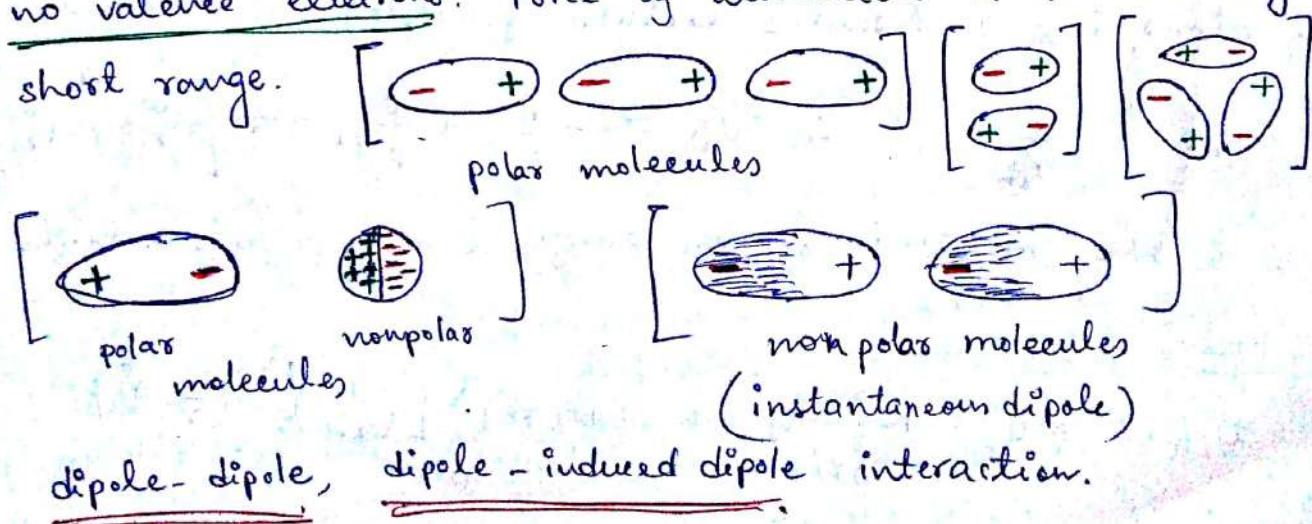


Ionisation energy is low and high electrical conductivity. They have vacant valency orbitals & have very few valency electrons compared to the number of valency orbitals. These electrons in the outermost orbit are loosely bound (conduction/free electrons) & 1 or 2 can be detached from parent atom due to attraction of neighbouring atom cores. These mobile electrons move from one kernel (positively charged atom) to other in metal lattice, & in the process bind two or more kernels together, by electrostatic interaction (partial). Because electrons are delocalized, do not have directional polarity & weak than covalent bond. They are easy to shear, opaque & lustrous appearance because they radiate light energy of different frequency.



## Vander Waal's bonding in molecular crystals

Inert gases attract with weak attractive force, although their outermost electron orbits are completely filled so they have no valence electrons. Force of attraction  $\propto r^{-7}$  so very short range.



These bonds are around 0.1 eV/bond & break by temperature fluctuations at room temperature. As temperature is reduced, vanderwaal's force dominate & matter transform from gas to liquid or solid.

### Binding energy of Inert gas crystals

Dipole-dipole interaction produces a weak attractive force.

Fluctuation of charge distribution on  $j^{\text{th}}$  atom induces instantaneous dipole moment  $\vec{P}_j$  on  $i^{\text{th}}$  atom. This produces field  $\vec{E}$  at centre of the  $j^{\text{th}}$  atom

$$E_i = \frac{2P_i}{r_{ij}^3}$$

$$\text{Instantaneous dipole moment } P_j = \alpha E_i = \frac{2\alpha P_i}{r_{ij}^3}$$

$\therefore$  Potential energy of the dipole moments is

$$U_{\text{at}} = - \frac{2P_i P_j}{r_{ij}^3} = - \frac{4\alpha P_i^2}{r_{ij}^6} \quad \alpha = \frac{1}{r_{ij}^6} \rightarrow \begin{array}{l} \text{short range} \\ \downarrow \text{attractive} \end{array}$$

Repulsive interaction is due to overlap of electron clouds of atoms  $i$  &  $j$  (Pauli's exclusion principle)

$$U_{\text{rep}} = \frac{c}{r_{ij}^{12}} \quad \therefore U_{ij} = - \frac{B}{r_{ij}^6} + \frac{c}{r_{ij}^{12}} = 4\epsilon \left[ \left( \frac{\sigma}{r_{ij}} \right)^{12} - \left( \frac{\sigma}{r_{ij}} \right)^6 \right]$$

$\epsilon$  is the magnitude of the energy &  $\sigma$  represents extent.

### Cohesive energy

$$U_{\text{tot}} = N U_i = N \sum_{j \neq i} U_{ij} = \frac{1}{2} N (4\epsilon) \sum_j \left[ \left( \frac{\sigma}{r_{ij}} \right)^{12} - \left( \frac{\sigma}{r_{ij}} \right)^6 \right]$$

substituting  $\pi_{ij} = \rho_{ij} R$

$$U_{\text{tot}} = 2NE \left[ \sum_j \left( \frac{\sigma}{\rho_{ij}R} \right)^{12} - \sum_j \left( \frac{\sigma}{\rho_{ij}R} \right)^6 \right] = 2NE \left[ 12.131 \left( \frac{\sigma}{R} \right)^{12} - 14.459 \left( \frac{\sigma}{R} \right)^6 \right]$$

for fcc crystal.

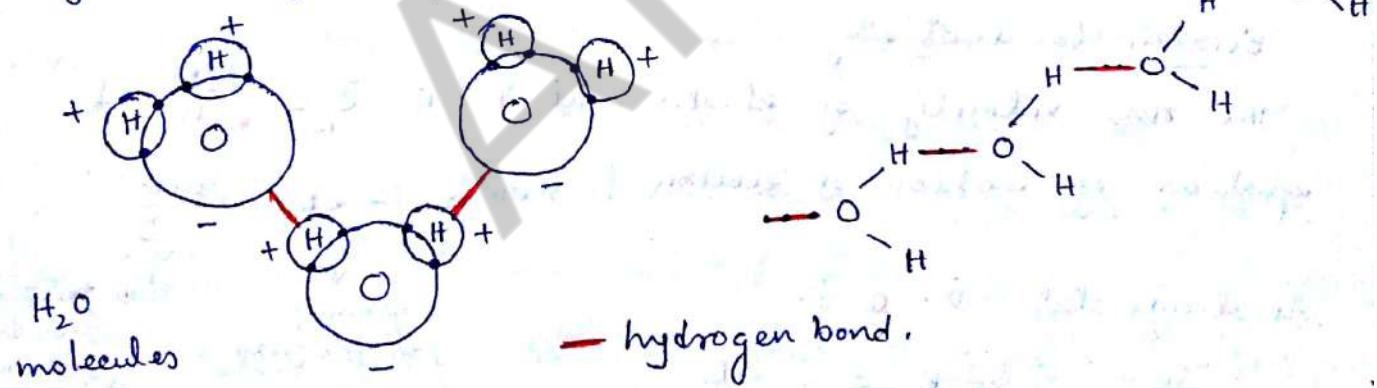
$$\left. \frac{dU_{\text{tot}}}{dR} \right|_{R=R_0} = 0 \Rightarrow \frac{R_0}{\sigma} = 1.09.$$

$$\text{and } U_{\text{tot}} = 2NE \left[ 12.131 \left( \frac{\sigma}{R_0} \right)^{12} - 14.459 \left( \frac{\sigma}{R_0} \right)^6 \right] = -8.6NE$$

This is the cohesive energy of the inert gas crystal at absolute zero temperature & zero pressure.

### Hydrogen Bonding

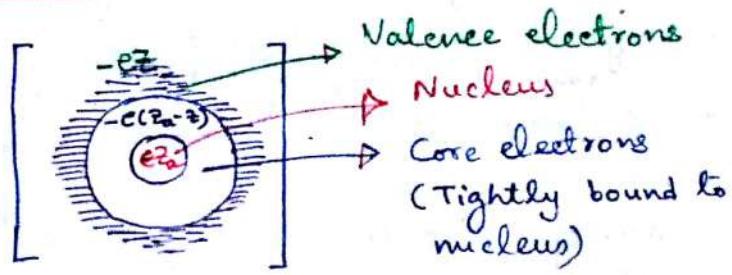
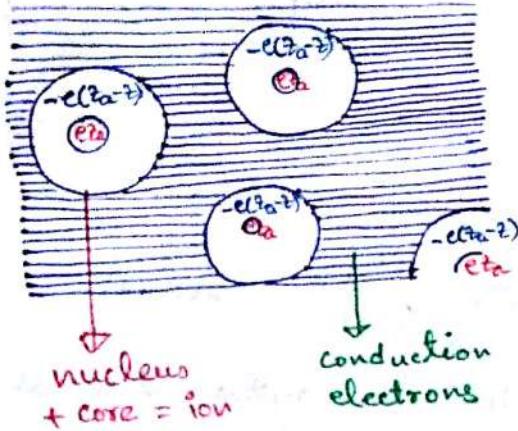
A special type of van der Waal bond happens between hydrogen atom, which has 1 electron loses to other atom leaving behind a poorly shielded proton. The proton ( $H^+$  ion) has two negative ions to attract & its radius is  $10^{-15} m$ , so the VW force is very strong & varies as  $V(r) \propto -1/r^2$ .



Interaction between oppositely charged ends of permanently polarized molecules with having H-atom is called the Hydrogen bond.

VW $\sim 2-7 \text{ kJ/mole}$	Metallic $\sim 20-100 \text{ kJ/mole}$	Cohesive Energy
H-bond $\sim 10 \text{ kJ/mole}$	Covalent $\sim 170-244 \text{ kJ/mole}$	
	Ionic $\sim 184 \text{ kJ/mole}$	

## Free Electron Theory of Metals



Doude-Lorentz theory (Classical free electron theory) :

Dense metallic electron gas  $\rightarrow$  kinetic theory of neutral dilute gas. collisions of electron with other electron & ions is neglected, and under external field they move in straight line with Newton's law. Electron-electron interaction is neglected (independent electron approximation), electron-ion interaction is also neglected (free electron approximation). Their speed distribution is Maxwellian & their collisions are elastic.

### Electric Conductivity

The rms velocity of electron at  $T$  is  $\bar{v}_{\text{rms}} = \sqrt{\frac{3kT}{m}}$ .

equation of motion of electron  $m\ddot{v} = -eE - \frac{mv}{\tau}$

At steady state  $\dot{v} = 0 \Rightarrow$

$$v_d = -\frac{e\tau}{m} E = -\mu E$$

$v_d$  = drift velocity,  $\mu$  = mobility

$$\begin{aligned} \text{Current density } J &= -nev_d \quad (-ne = \text{charge/unit volume}, \\ &\qquad n = \text{no. density of conduction} \\ &\qquad \text{electron}) \\ &= -ne\left(-\frac{e\tau}{m} E\right) \\ &= \frac{ne^2\tau}{m} E = \sigma E \end{aligned}$$

$$\therefore \text{Electric conductivity } \sigma = \frac{ne^2\tau}{m} \propto n$$

$$= (ne/m) \times e^2 \tau_m = ne\mu$$

$$\text{resistivity } \rho = \sigma^{-1} = \frac{1}{ne\mu} = \frac{m}{ne^2\sigma}$$

In metals,  $n \approx \text{constant}$ ,  $\mu$  decreases with temperature, so as  $\sigma$ .

In semiconductors,  $n$  exponentially increases with temperature,  $\sigma$  increases.

In insulators,  $n \approx \text{constant}$ ,  $\mu$  increases exponentially  $\rightarrow$  dielectric breakdown

Wiedemann-Franz law (metals)  $\frac{K}{\sigma} \propto T$ ,  $K = \text{thermal conductivity}$   
 $\sigma = \text{electrical conductivity}$

Good conductor of electricity are also good conductors of heat & the ratio  $K/\sigma T = \frac{\pi^2 k_B^2}{3e^2} = \text{Lorentz number}$

Although free electron theory explains WF law & validate's ohm's law, low temperature behaviour,  $\rho \propto T$  etc cannot be obtained.

### Sommerfeld's free electron theory

Despite the success of Drude-Lorentz classical electron theory to explain WF law, difficulties were

- (a) Why Debye theory of lattice specific heat that ignores electronic specific heat is accurately "valid" for metals?
- (b) Paramagnetism of metals does "not" obey Curie law ( $\chi \propto \frac{1}{T}$ ) & is independent of temperature. But a gas of electrons, each of which are tiny magnet must exhibit large magnetic susceptibility.
- (c) Hall effect of some divalent metal is positive, meaning that the charge carriers are "positive".
- (d) Electric resistivity has a temperature variation  $\rho = \rho_0(1 + \alpha T)$  that can't be explained from classical electron gas model.
- (e) Certain metals (tin, mercury etc) having poor electrical conductivity becomes superconductor at very low temperature but alkali & noble metals (Ag, Au, Pt etc) do not show superconductivity.

Schrödinger equation for free electron gas in 3D.

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi \Leftrightarrow \nabla^2 \psi + \frac{2m}{\hbar^2} E\psi = 0$$

Plane wave solution  $\psi(\vec{r}) = Ae^{i\vec{k} \cdot \vec{r}}$ ,  $\nabla^2 \psi = -k^2 \psi$ .

$$\therefore -k^2 \psi + \frac{2m}{\hbar^2} E\psi = 0 \Leftrightarrow E = \frac{\hbar^2 k^2}{2m}$$

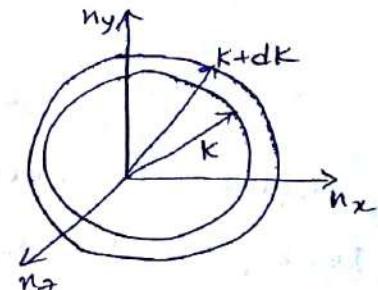
where  $k^2 = \left(\frac{2\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$ ,  $n_x, n_y, n_z$  integers

because  $\psi(\vec{r}) = \psi(\vec{r} + \vec{L})$ . Each set of  $(n_x, n_y, n_z)$  gives a stationary state of an electron inside the metal. In k-space, number of possible states of integers within  $K$  &  $K+dK$ , each of which gives rise to one state of electron

$$\therefore L^3 D(K) = \frac{4\pi K^2 dK}{(2\pi/L)^3}$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$dK = \frac{m dE \hbar}{\hbar^2 \sqrt{2mE}}$$



$$= \frac{4\pi L^3}{8\pi^3} \frac{2mE}{\hbar^2} \frac{1}{\hbar} \sqrt{\frac{m}{2}} E^{1/2} dE$$

$$= \frac{1}{\hbar} \sqrt{\frac{m}{2}} E^{1/2} dE$$

$$= L^3 2\pi \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE$$

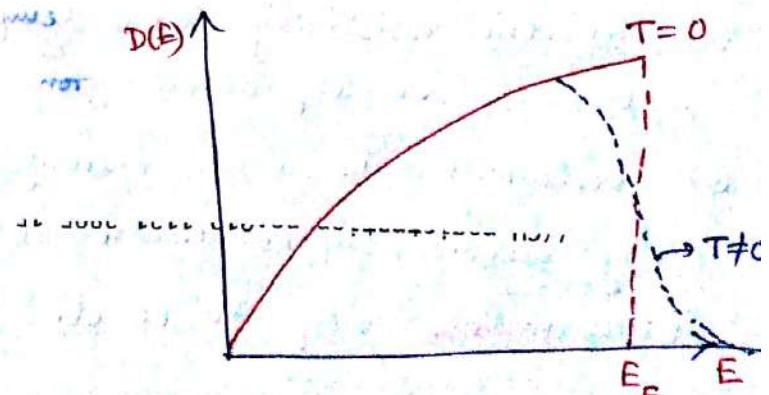
$$\therefore D(E) = 2\pi \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE. \quad [\text{no. of states/unit volume}]$$

Using Pauli's exclusion principle, since electrons are spin  $\pm \frac{1}{2}$  so each energy state will have 2-fold degeneracy.

$$\therefore D(E) = 2 \times 2\pi \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE.$$

$$D(E) = 4\pi \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE$$

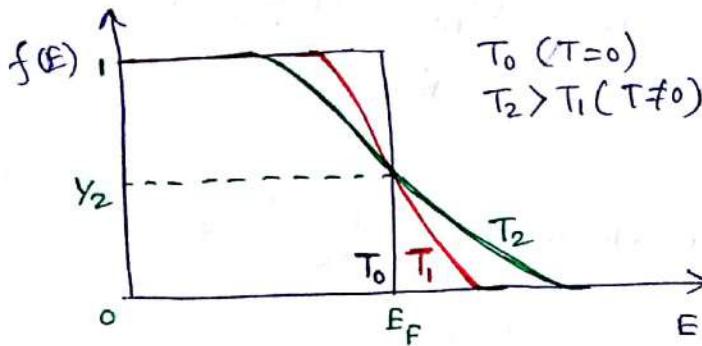
This is valid at  $T=0K$ .



At a finite temperature  $T$ , the probability that an electron occupies a state with energy  $E$  is given by Fermi-Dirac function

$$f(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1} \quad \text{where } E_F = \mu = \frac{\partial G}{\partial N} = \text{chemical potential}$$

$$\text{at } T=0, \quad f(E) = 0 \quad \text{if } E > E_F \\ = 1 \quad \text{if } E < E_F \quad \text{at } T \neq 0, \quad f(E) = \frac{1}{2} \quad \text{if } E = E_F$$



at  $T=0$ , Below Fermi energy all states are filled & above states are empty. at  $T \neq 0$ , at Fermi energy half of the states are filled.

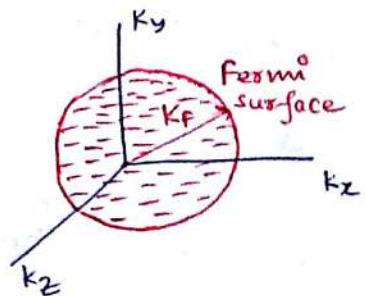
$$E_F = \frac{\hbar^2 k_F^2}{2m} \quad \text{and in K-space, Fermi-sphere } S_F = 4\pi k_F^2$$

$$\text{Now } \int_0^\infty f(E) D(E) dE = n$$

$$\text{at } T=0, \quad f(E) = 1 \text{ till } E=E_F$$

$$\therefore \int_0^{E_F} f(E) D(E) dE = n.$$

$$\therefore 4\pi \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{E_F} E^{1/2} dE = n. \quad \boxed{E_F = \frac{\hbar^2}{2m} \left(\frac{3n}{8\pi}\right)^{2/3}} \propto n^{2/3}$$



at  $T \neq 0$  for  $E_F \gg k_B T$ ,  $E_F(T)$  is given by Sommerfeld equation

$$E_F(T) = E_F \left[ 1 - \frac{\pi^2}{12} \frac{(k_B T)^2}{E_F^2} \right]$$

$$\text{from } E_F = \frac{\hbar^2 k_F^2}{2m}, \quad k_F^2 = \frac{2m}{\hbar^2} \frac{h^2}{2m} \left(\frac{3n}{8\pi}\right)^{2/3} = (3n\pi^2)^{2/3}$$

Fermi wave vector

$$\boxed{k_F = (3\pi^2 n)^{1/3}}$$

Fermi velocity

$$\boxed{v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m} (3\pi^2 n)^{1/3}}$$

$$E_F = \frac{3}{2} k_B T_F \Rightarrow \text{Fermi temperature } T_F = \frac{2E_F}{3k_B} = \frac{2\hbar^2}{6k_B m} (3\pi^2 n)^{2/3}$$

Substituting values &  $n = 10^{22}/\text{cc}$ ,  $T_F \approx 39,000 \text{ K}$ .

Average energy of the electron at  $T=0 \text{ K}$  is

$$\begin{aligned} \bar{E} &= \frac{1}{n} \int_0^{E_F} E D(E) dE = \frac{1}{n} 4\pi \left(\frac{2m}{h^2}\right)^{3/2} \int_0^{E_F} E^{3/2} dE \\ &= \frac{4\pi}{n} \left(\frac{2m}{h^2}\right)^{3/2} \frac{2}{5} E_F^{5/2} = \frac{1}{n} 4\pi \left(\frac{2m}{h^2}\right)^{3/2} E_F^{2/2} \underbrace{\frac{2}{5}}_{\text{Red}} E_F \end{aligned}$$

$$\boxed{\bar{E} = \frac{3}{5} E_F}$$

Average speed of electron at  $T=0 \text{ K}$  is  $\bar{v} = \frac{1}{n} \int_0^{v_F} v dn$

$$\text{Now } v_F = \frac{\hbar}{m} (3\pi^2 n)^{1/3} \text{ or } n = \frac{1}{3\pi^2} \left(\frac{m v_F}{\hbar}\right)^3$$

If all velocity are below  $v_F$  & then  $n = \frac{1}{3\pi^2} \left(\frac{m}{\hbar}\right)^3 v^3$

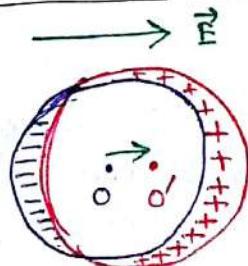
& number density of states between  $v$  &  $v+dv$  is

$$dn = \frac{1}{3\pi^2} \left(\frac{m}{\hbar}\right)^3 3v^2 dv$$

$$\therefore \bar{v} = \frac{1}{n} \int_0^{v_F} \frac{1}{3\pi^2} \left(\frac{m}{\hbar}\right)^3 3v^4 dv = \frac{1}{n} \frac{1}{3\pi^2} \left(\frac{m}{\hbar}\right)^3 \frac{3}{4} v_F^4$$

$$= \frac{1}{n} \frac{1}{3\pi^2} \left(\frac{m}{\hbar}\right)^3 v_F^3 \underbrace{\frac{3}{4} v_F}_v \quad \text{or} \quad \boxed{\bar{v} = \frac{3}{4} v_F}$$

### Sommerfeld's free electron theory & conductivity



Concept of Fermi surface  $S_F = 4\pi k_F^2$  introduced by Sommerfeld changes the notion of conduction in metals. When an electric field is switched on, movement of the Fermi surface gives a

displacement of the centre of Fermi surface. This displacement is equivalent to creation of electrons on one side & positive charges

on the other. Like each electron have velocity  $\vec{v} = \frac{e\vec{k}}{m}$ , then equation of motion of each electron in Fermi surface under steady field is  $\frac{d\vec{p}}{dt} = e\vec{E}$ . So in the absence of any resistive force, the Fermi surface will move at constant rate in K space.

$$\int_{\vec{K}(0)}^{\vec{K}(t)} e dk = \int_0^t e\vec{E} dt \quad \text{or} \quad \delta \vec{K} = \frac{e\vec{E}}{m} t = \frac{e\vec{E}}{m} \left( \frac{\lambda}{v_F} \right)$$

where  $\lambda$  is mean free path or distance between two ions. But the collision of electron with impurity ions, imperfection & phonons (lattice vibrations) will create a restoring force.

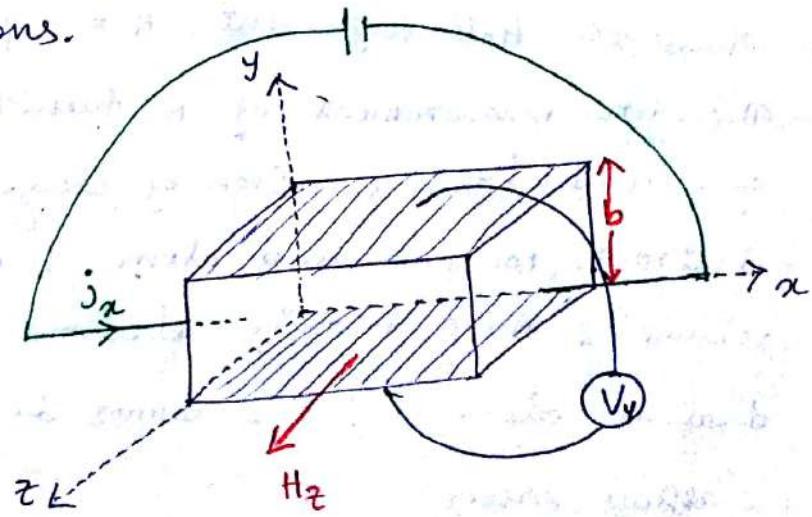
Thus in steady state,  $\vec{J} = ne\vec{v}_s$  where  $\vec{v}_s = \frac{e\vec{E}}{m}$ .

$$= \frac{neh}{m} \delta \vec{K} = \frac{neh}{m} \frac{e\vec{E}\lambda}{mv_F} = \frac{ne^2 \lambda}{mv_F} \vec{E} = \sigma \vec{E}$$

This expression is identical to Drude's free electron theory but it destroys the notion of classical theory that all free electrons are conduction electrons. In Sommerfeld theory only few electrons that lie in the vicinity of the Fermi surface are the conduction electrons.

### Hall effect

In 1879, Hall discovered that if a uniform magnetic field  $H_z$  is applied in z-direction normal to the direction of a steady current flow  $i_x$  in a rectangular slab, then a transverse electrical potential difference develops in the y direction.



$$V_y \propto j_x$$

$$\propto H_z$$

$$\propto b$$

↓      ↓      ↓

Hall      Hall      thickness of  
voltage   coefficient   slab

Hall effect can be explained by simple classical theory. Current flows in  $x$  direction &  $H_z$  is applied in  $z$  direction. Thus the Lorentz force exerted on an electron in the slab is  $F_y = -ev_x H_z$  where  $v_x$  is uniform drift velocity, so electrons are deflected in  $y$  direction. Because electrons are deposited near the surface, a potential difference in  $y$  direction is developed until the Hall electric field  $E_y$  stops further deflection of electrons.

$$E_y = \frac{V_y}{b} \quad \text{and the force } F_y = -eE_y = -\frac{ev_y}{b}$$

$$\text{Equating, } -ev_x H_z = -\frac{ev_y}{b} \Rightarrow v_x = \frac{V_y}{bH_z}$$

$$\therefore \text{Current density } j_x = -nev_x = -ne \frac{V_y}{bH_z}$$

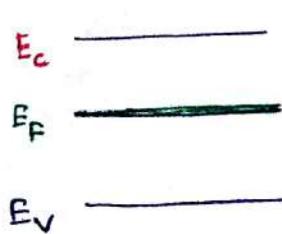
$$\text{or } V_y = \underbrace{\left(-\frac{1}{ne}\right)}_R j_x H_z b.$$

Thus the Hall coefficient  $R = -\frac{1}{ne}$  is  $\propto \frac{1}{n}$  and  $-\frac{1}{e}$ .

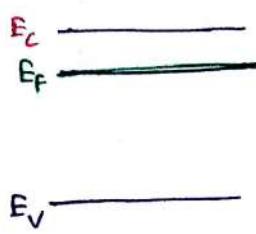
Therefore measurement of  $R$  furnishes two important characteristics of a conductor: (1) sign of charge carrier, (2) density of electrons. For monovalent atoms,  $n$  is number of electrons/unit volume &  $R < 0$  meaning electrons are carriers. But for certain divalent atom,  $R > 0$  & cannot be explained only by the classical theory.

Read about 1D crystal's density of states,  $\bar{E} = \frac{1}{3} E_F$  from any standard book.

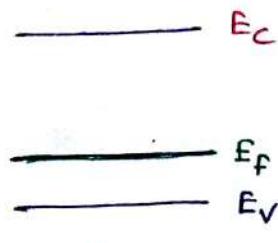
## Position of Fermi level



Intrinsic semiconductor



n-type material



p-type material

For intrinsic semiconductor  
concentration of electrons in conduction band = concentration of holes in valence band & so

E<sub>F</sub> lies at middle of band gap.

for n-type material concentration of electrons in conduction band > hole concentration in valence band. E<sub>F</sub> lies near to E<sub>C</sub>. This is opposite in p-type material & E<sub>F</sub> lies near to E<sub>V</sub>. Whenever a PN junction diode is formed, barrier potential V<sub>B</sub> is  $eV_B = E_{F_n} - E_{F_p}$ .

HW 1. In Sodium, free electrons per cubic metre are  $2.5 \times 10^{28}$ . Calculate the Fermi energy & Fermi velocity. You can use  $h = 6.625 \times 10^{-34} \text{ Js}$ ,  $m = 9.1 \times 10^{-31} \text{ kg}$ .

2. Consider silver in the metallic state with one free electron per atom. Calculate the Fermi energy. Given density of silver is  $10.5 \text{ gm/cm}^3$  & atomic weight 108,  $N = 6.02 \times 10^{23} / \text{gm atom}$

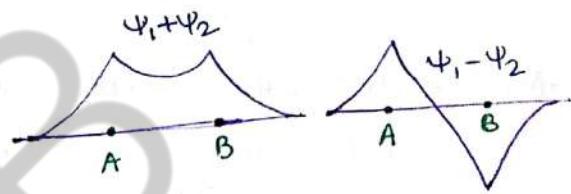
3. Aluminium metal crystallites form fcc structure. If each atom contributes single electron as free electron & lattice constant  $a$  is  $4\text{\AA}$ , treating conduction electrons as free electron Fermi gas, calculate Fermi energy E<sub>F</sub>, Fermi vector k<sub>F</sub> & total K.E. per unit volume at T=0K.

## Band Theory of Solids

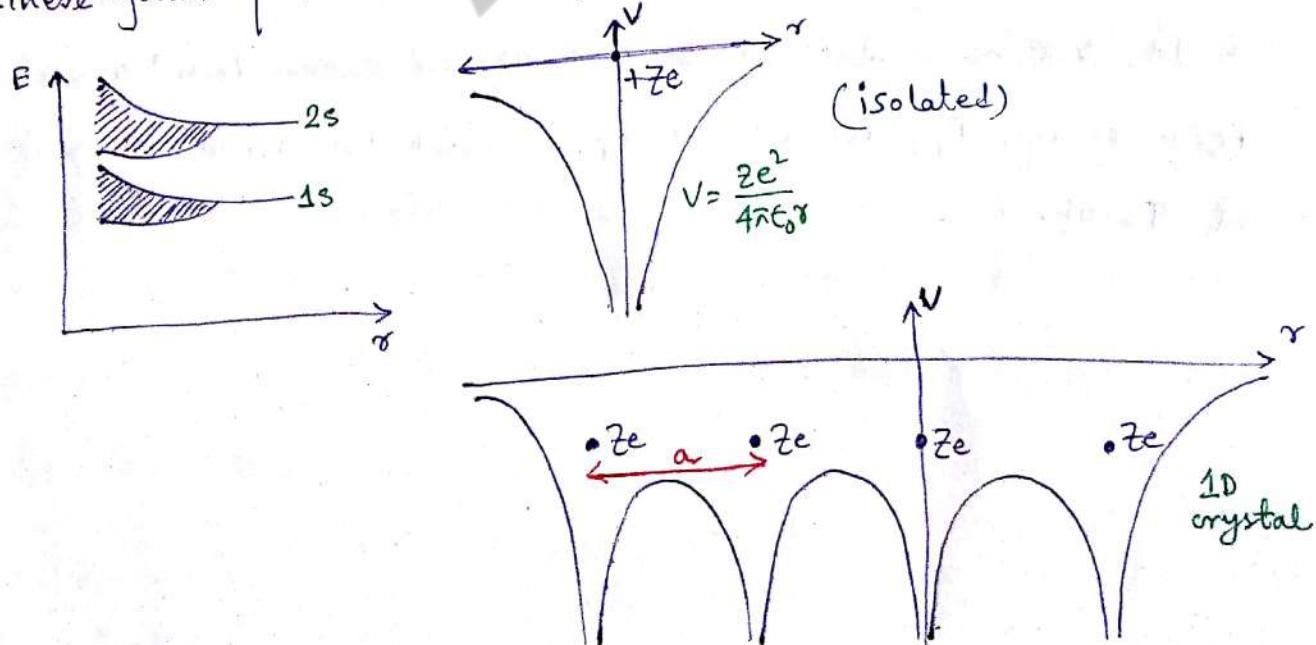
### Formation of Energy Bands

In solids, a periodic potential is formed by the nucleus and other electron cloud. Motion of an electron in periodic potential can be represented by Schrödinger equation, whose solution gives energy states. These states are filled with electrons according to Pauli's exclusion principle & all states are not accessible but bands of energies separated by forbidden energies are possible.

When two atoms are brought close, single energy level splits into a pair of levels. If  $\Psi_1$  &  $\Psi_2$  are electronic wave functions, then due to overlap, resultant wave function is  $(\Psi_1 \pm \Psi_2)$ , for symmetric case, electron can remain midway to A-B but for antisymmetric it cannot, so there is a difference in energy between  $(\Psi_1 + \Psi_2)$  &  $(\Psi_1 - \Psi_2)$ .



Similarly when N no. of atoms are brought together, each energy state splits into N energy states whose separation is very small. These form quasi-continuous energy band.



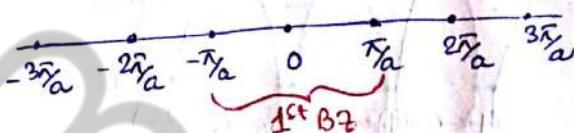
Origin of band structure This can be understood from nearly free electron model where electrons move in a periodic kernel in 1D. Energy of a free electron  $E_K = \frac{\hbar^2 k^2}{2m}$  &  $\Psi_K(x) = e^{ikx}$

Low energy electrons can freely travel as their  $\lambda \gg a$ . High energy electrons almost near to fermi energy have  $\lambda \approx a$ , & suffer diffraction like x-rays in crystal surface. Electron with deBroglie wavelength  $\lambda$  is bragg reflected,  $2a \sin\theta = n\lambda$ .

$$\text{or } K = \frac{2\pi}{\lambda} \cdot \text{ after substitution, } K = \pm \frac{n\pi}{a \sin\theta}. \text{ For 1D}$$

$$\text{lattice } \Theta = \frac{\pi}{2} \text{ or } K = \pm \frac{n\pi}{a}.$$

Thus a line representing  $K$  values is divided by energy discontinuities into segments of length  $\pm \pi/a$  which are the Brillouin zones.

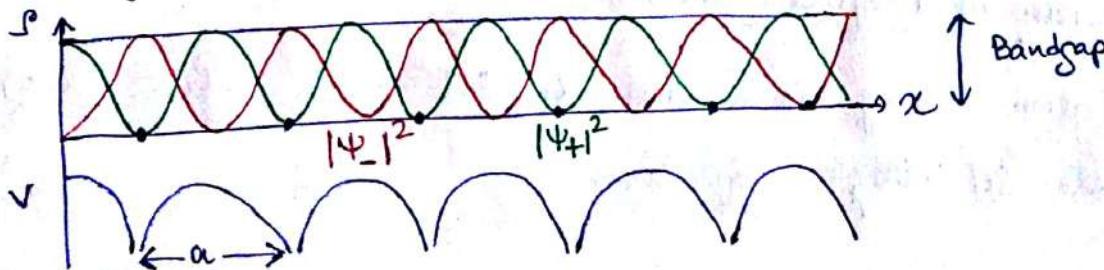


At the boundary  $K = \pm \frac{\pi}{a}$ , electron wave functions are not traveling waves  $e^{i\pi x/a}$  &  $e^{-i\pi x/a}$  but are standing waves, due to reflection. Two types of standing wave can form

$$\Psi_+ = e^{i\pi x/a} + e^{-i\pi x/a} = 2 \cos\left(\frac{\pi x}{a}\right) \quad [x \rightarrow -x, \Psi_+ \rightarrow \Psi_+]$$

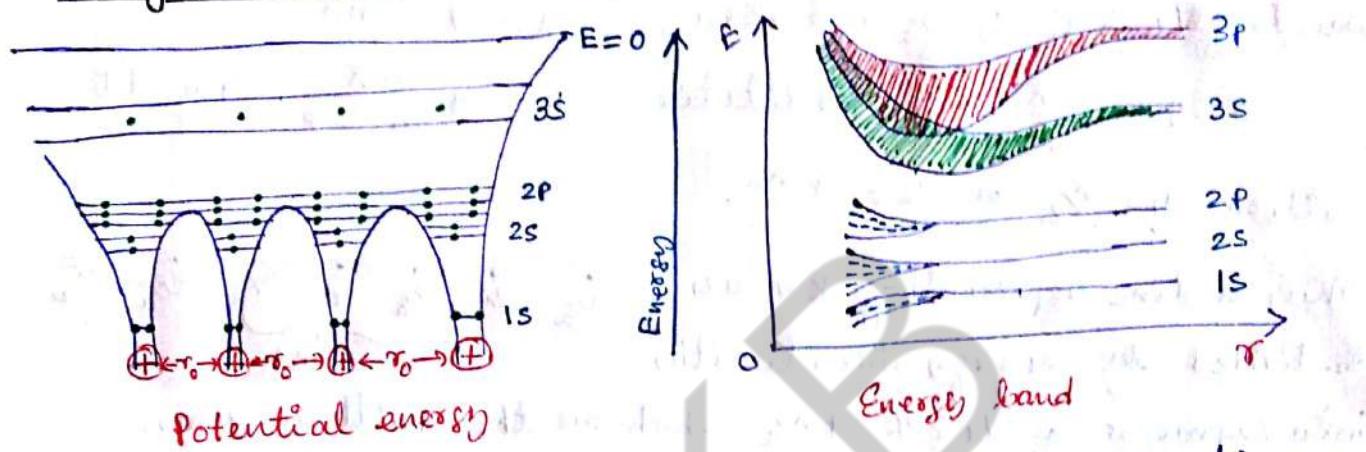
$$\Psi_- = e^{i\pi x/a} - e^{-i\pi x/a} = 2i \sin\left(\frac{\pi x}{a}\right). \quad [x \rightarrow -x, \Psi_- \rightarrow -\Psi_-]$$

In quantum mechanics, probability density of electron is  $\rho = \Psi^* \Psi$  and for traveling wave  $\Psi = e^{\pm iKx}$ ,  $\rho = 1$  so that electron charge density  $e|\Psi|^2 = \text{constant}$ . However for standing wave, charge density isn't constant but  $e|\Psi_+|^2 \propto \cos^2\left(\frac{\pi x}{a}\right)$ ,  $e|\Psi_-|^2 \propto \sin^2\left(\frac{\pi x}{a}\right)$



for  $x=0, a, 2a, \dots$   $\cos^2 \frac{\pi x}{a} = 1$ , so its maximum at the kernel core and thus negative electron charge density lowering P.E. of Kernel. For  $x=\frac{a}{2}, \frac{3a}{2}, \dots$ ,  $\sin^2 \frac{\pi x}{a} = 1$ , so its maximum in midway between Kernel & increasing the P.E. w.r.t. to travelling wave. So  $E_g$  is the difference of two energies.

### Energy bands in Sodium crystal



Valence orbital overlap for 3s orbital to form quasi continuous energy band. Empty 3p level also spreads, so as 2s & 2p with decrease of  $r$ . When  $r=r_0 = 0.367 \text{ nm}$ , 3s & 3p states overlap.

In metals, band overlap happens but in other materials they are separated by a band gap  $E_g$  & the energy of highest filled level is Fermi energy  $E_F$ . At OK, levels upto  $E_F$  is filled & those above are empty. Using Pauli's principle, each s-band having  $N$  atoms can accomodate  $2N$  electrons. If highest s-band is fully filled then electron drift using external force is stopped & such solids are called insulators.

### Bloch Theorem or Floquet's theorem

for an electron moving in a 1D potential with  $V(x) = V(x+a)$  is given by the Schrödinger equation

$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] \psi = 0$ . Bloch theorem states that the solution of the Schrödinger equation for a periodic potential is

$\boxed{\psi(x) = e^{\pm ikx} u_K(x)}$  where  $u_K(x) = u_K(x+a)$ , which are plane waves  $e^{\pm ikx}$  modulated by  $u_K(x)$  with the periodicity of the lattice.  $\psi(x)$  is the Bloch wave or Bloch function.

### Proof of Bloch's theorem

If  $f(x)$  and  $g(x)$  are two real, independent solutions of Schrödinger equation  $\psi(x) = A f(x) + B g(x)$ . As  $V(x) = V(x+a)$ ,  $f(x+a)$  and  $g(x+a)$  are also solutions.

$$\therefore f(x+a) = \alpha_1 f(x) + \alpha_2 g(x), \quad g(x+a) = \beta_1 f(x) + \beta_2 g(x).$$

$$\psi(x+a) = A f(x+a) + B g(x+a)$$

$$= (A\alpha_1 + B\beta_1) f(x) + (A\alpha_2 + B\beta_2) g(x)$$

$$= \lambda A f(x) + \lambda B g(x) = \lambda \psi(x). \text{ where we have}$$

chosen  $A\alpha_1 + B\beta_1 = \lambda A$ ,  $A\alpha_2 + B\beta_2 = \lambda B$ . with  $\lambda$  a constant.

Now this gives nonzero values of  $A$  and  $B$  if determinant of coefficient is zero

$$\begin{vmatrix} \alpha_1 - \lambda & \beta_1 \\ \alpha_2 & \beta_2 - \lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - (\alpha_1 + \beta_2)\lambda + \alpha_1\beta_2 - \alpha_2\beta_1 = 0 \quad (1)$$

[As  $f(x)$  &  $g(x)$  are solution of Schrödinger equation,

$$\left\{ \frac{d^2f(x)}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] f(x) = 0 \right\}, \quad \left\{ \frac{d^2g(x)}{dx^2} + \frac{2m}{\hbar^2} [E - V(x)] g(x) = 0 \right\}$$

$\cancel{g(x) \times} \quad . \quad \text{Subtract}$

$$f(x) \frac{d^2g(x)}{dx^2} - g(x) \frac{d^2f(x)}{dx^2} = 0 \quad \text{or} \quad f(x) \frac{dg(x)}{dx} - g(x) \frac{df(x)}{dx} = \text{constant}$$

$\cancel{\text{constant}} \quad \approx W(x)$

$$\therefore W(x+a) = f(x+a) \frac{dg(x+a)}{dx} - g(x+a) \frac{df(x+a)}{dx}$$

$$= f(x+a) \left[ \beta_1 \frac{df(x)}{dx} + \beta_2 \frac{dg(x)}{dx} \right] - g(x+a) \left[ \alpha_1 \frac{df(x)}{dx} + \alpha_2 \frac{dg(x)}{dx} \right]$$

$$= [\alpha_1 f(x) + \alpha_2 g(x)] \left[ \beta_1 \frac{df(x)}{dx} + \beta_2 \frac{dg(x)}{dx} \right] - [\beta_1 f(x) + \beta_2 g(x)] \left[ \alpha_1 \frac{df(x)}{dx} + \alpha_2 \frac{dg(x)}{dx} \right]$$

$$= (\alpha_1 \beta_2 - \alpha_2 \beta_1) \left[ f(x) \frac{dg(x)}{dx} - g(x) \frac{df(x)}{dx} \right] = (\alpha_1 \beta_2 - \alpha_2 \beta_1) w(x)$$

But  $w(x+a) = w(x) = \text{constant}$ ,  $\therefore \underline{\alpha_1 \beta_2 - \alpha_2 \beta_1 = 1}$

Eq. ① becomes,  $\lambda^2 - (\alpha_1 + \beta_2)\lambda + 1 = 0$ . Here  $\alpha_1 + \beta_2$  is a function of energy E & we have two roots  $\lambda_1$  &  $\lambda_2$  or two functions  $\psi_1(x)$  and  $\psi_2(x)$  with  $\psi_1(x+a) = \lambda \psi_1(x)$  &  $\lambda_1 \lambda_2 = 1$ .

Special cases  $(\alpha_1 + \beta_2)^2 < 4$ ,  $\lambda^2 - (\alpha_1 + \beta_2)\lambda + 1 = 0$  have complex roots, & conjugate to each other.  $\lambda = e^{\pm ika}$

$\psi(x+a) = e^{\pm ika} \psi(x)$  which is of the Bloch form  $\Psi(x) = e^{\pm ika} u_k(x)$

$$= e^{\pm ika(x+a)} u_k(x+a) = e^{\pm ika} e^{\pm ika x} u_k(x+a) = e^{\pm ika} \psi(x)$$

$$= \lambda \psi(x)$$

Bloch theorem hence proved.

Special cases  $(\alpha_1 + \beta_2)^2 > 4$ ,  $\lambda^2 - (\alpha_1 + \beta_2)\lambda + 1 = 0$  have real roots  $\lambda_1 = e^{\mu a}$ ,  $\lambda_2 = e^{-\mu a}$ ,  $\mu = \text{real}$ . & corresponding Schrödinger equation,  $\psi_1(x) = e^{\mu x} u(x)$ ,  $\psi_2(x) = e^{-\mu x} u(x)$

Although mathematically valid, these are forbidden wavefunctions as they're not bounded. at  $\pm \infty$ , both diverge.

The allowed roots  $e^{\pm ika}$  and forbidden roots  $e^{\pm \mu a}$  are functions of  $(\alpha_1 + \beta_2)$  and hence energy. So energy spectrum of electron moving in periodic potential consists of allowed & forbidden energy regions or bands.

Kronig-Penney Model [ Energy spectrum of electron consists of a allowed energy bands separated by forbidden region ]

In free electron theory the assumption is valence electrons see zero potential but this isn't true with ionic & covalent bond as electrons are localized near the nuclei, that gives periodically varying potential. whose solution from Schrödinger equation is very hard.

Instead Kronig & Penney solved it using simpler 1D potential of sharp ~~and~~ edges with periodicity  $a+b$ . whose Schrödinger equation is

$$\begin{cases} V=0, & 0 < x < a \\ = V_0, & -b < x < 0 \end{cases}$$

$$\left. \begin{aligned} \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi &= 0, & 0 < x < a \\ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} (E - V_0)\psi &= 0, & -b < x < 0 \end{aligned} \right\} \quad \begin{aligned} ① \\ V(x) = V(x + a+b). \end{aligned}$$

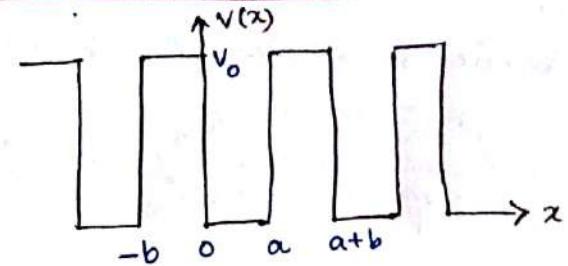
According to Bloch theorem, solution of wave equation for periodic potential will be plane wave modulated by the lattice periodicity.  $\psi(x) = u_K(x) e^{ikx}$  with  $u_K(x) = u_K(x+a+b)$ .

By substituting  $\psi(x)$  in equation ① and substituting the boundary condition:  $(u_1)_{x=0} = (u_2)_{x=0}$      $(u_1)_{x=a} = (u_2)_{x=b}$

$$\left( \frac{du_1}{dx} \right)_{x=0} = \left( \frac{du_2}{dx} \right)_{x=0} \quad \left( \frac{du_1}{dx} \right)_{x=a} = \left( \frac{du_2}{dx} \right)_{x=b}$$

one gets four equations & to get nonzero coefficients the  $4 \times 4$  determinant vanish. from that K-P obtained,

$$\frac{\beta^2 + \alpha^2}{2\alpha\beta} \sinh \beta b \sin da + \cosh \beta b \cos da = \cos k(a+b)$$



To simplify K-P considered when  $V_0 \rightarrow \infty$  and  $b \rightarrow 0$ ,  $V_{0b} \rightarrow$  finite or potential barriers become S-functions.  $V_{0b}$  is known as barrier strength. As  $b \rightarrow 0$ ,  $\sinh \beta b \rightarrow \beta b$ ,  $\cosh \beta b \rightarrow 1$ , and

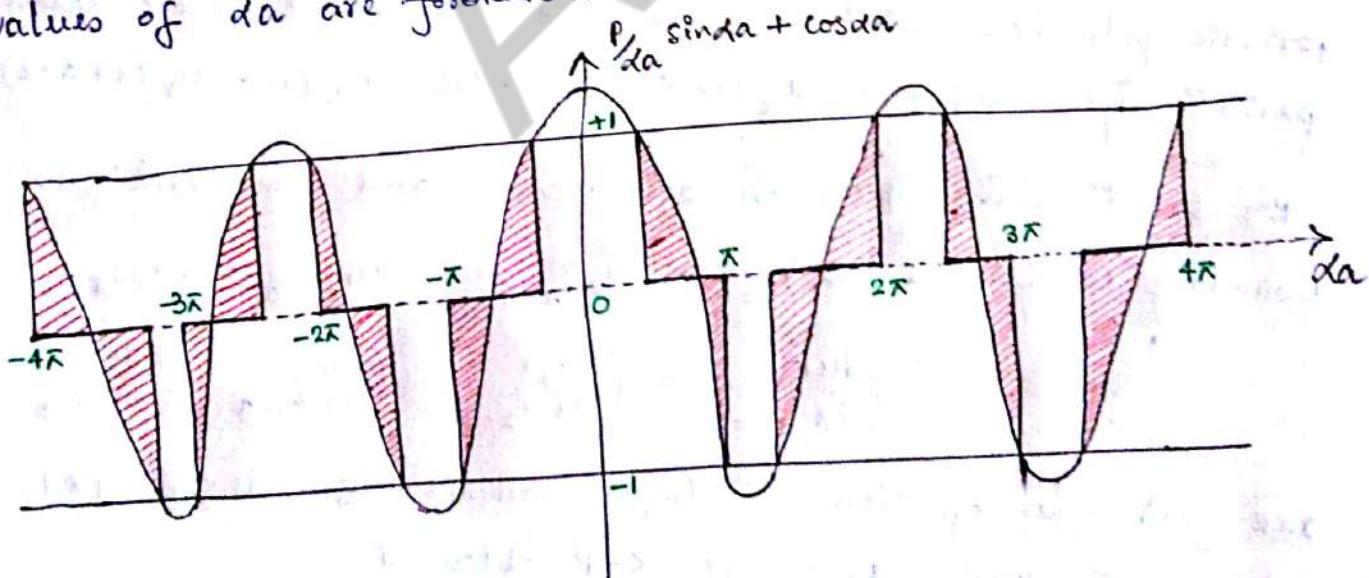
$$\frac{\beta^2 + \alpha^2}{2\alpha\beta} = \frac{mV_0}{\alpha\beta\hbar^2} \quad \therefore \quad \frac{mV_{0b}}{\alpha\hbar^2} \sinh \alpha + \cos \alpha = \cos K\alpha$$

$$\therefore P \frac{\sinh \alpha}{\alpha} + \cos \alpha = \cos K\alpha \quad \text{where } P = \frac{mV_0 ab}{\hbar^2}$$

When  $P$  is increased, the area of potential barrier is increased and the electron is bound more strongly to a potential well.  $P \rightarrow 0$  means barrier is very weak & the electrons become free electrons.

$$\therefore \lim_{P \rightarrow 0}, \alpha = K\alpha \quad \therefore \alpha^2 = K^2 = \frac{2mE}{\hbar^2} \quad \therefore E = \frac{\hbar^2 K^2}{2m}$$

As  $\cos K\alpha$  is bound between +1 and -1, LHS should take values of  $\alpha$  for which it lies between +1 & -1.  $\therefore$  Such  $\alpha$  represent wave like solutions  $\psi(x) = e^{ikx} u_k(x)$ . Other values of  $\alpha$  are forbidden.



as  $K\alpha \rightarrow [0, \pi]$ ,  $\cos K\alpha \rightarrow [+1, -1]$  allowed boundaries  $\cos K\alpha = \pm 1$

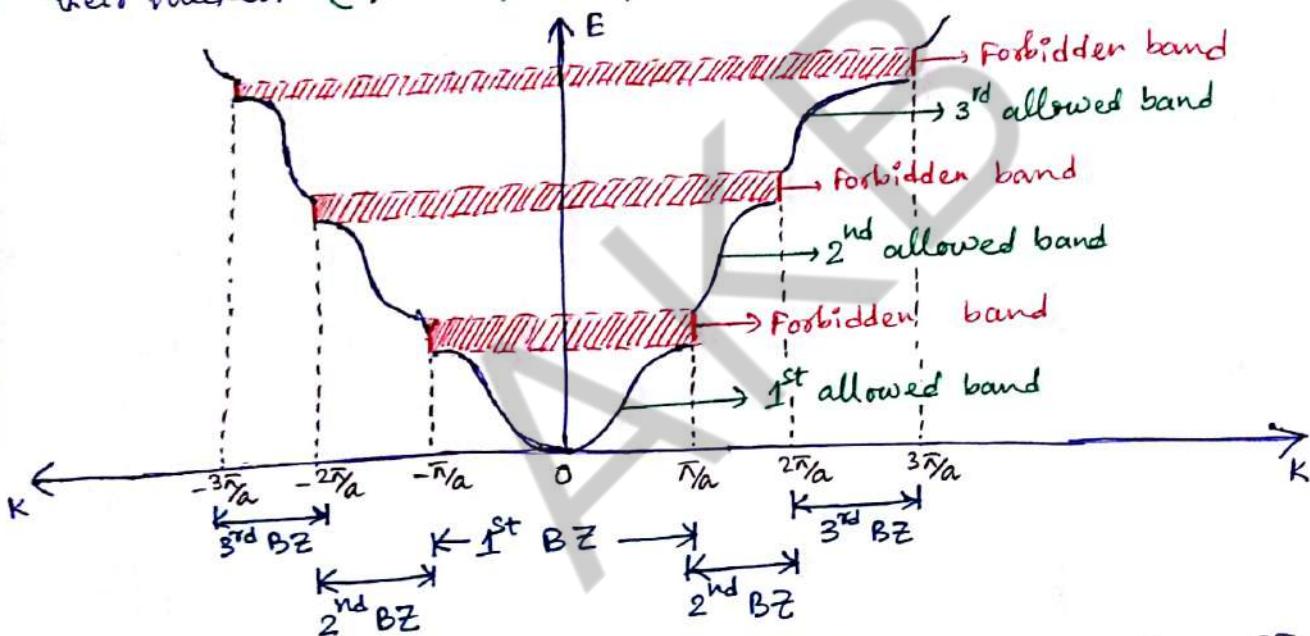
$$\therefore K\alpha = n\pi \quad \therefore K = \frac{n\pi}{\alpha}.$$

As  $\alpha$  increases,  $p \frac{\sin \alpha}{\alpha}$  decreases, so the width of allowed energy bands ~~decrease~~ increases & forbidden energy regions become narrower. As  $p$  increases, width of allowed energy bands decreases and for  $p \rightarrow \infty$ , they are infinitely thin & independent of  $K$ .

for  $p \rightarrow \infty$ , allowed  $\alpha$  are points,  $\alpha = \pm n\pi$

$$\Rightarrow \alpha^2 = \frac{n^2 \pi^2}{a^2} = \frac{2mE}{\hbar^2} \quad \therefore E = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \neq E(K)$$

Energy level is discrete & the electron is completely bound to their nuclei. (line spectrum) for  $P=0$  (quasi-continuous)



$E = \frac{\hbar^2 k^2}{2m}$  will now have discontinuities at  $k = \pm \frac{\pi}{a}, \pm \frac{2\pi}{a}, \pm \dots$

These  $K$  values define the 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, ... etc. Brillouin zones (BZ)

The curves (bands) are horizontal at bottom & top, parabolic near top & bottom with curvature in opposite direction, within a band, energy is periodic in  $K$ . as  $\cos(K + \frac{2\pi n}{a})a = \cos Ka$  &

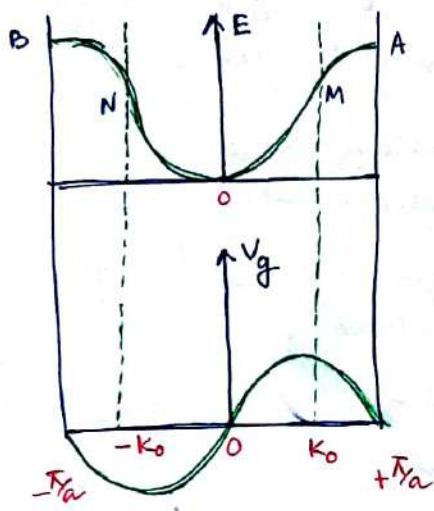
$$p \frac{\sin \alpha}{\alpha} + \cos \alpha = \cos Ka \text{ eqn. remains same.}$$

## Variation of Energy & velocity with wavevector

According to deBroglie an electron moving with a velocity  $v$  is equivalent to a wave packet moving with group velocity = particle velocity  $v_g = \frac{d\omega}{dk}$ . As energy of particle is  $E = \hbar\omega$  or  $\omega = \frac{E}{\hbar}$

$$\therefore v_g = \frac{d\omega}{dk} = \frac{1}{\hbar} \frac{dE}{dk}. \text{ Now we have for free electron } E = \frac{\hbar^2 k^2}{2m}.$$

$$\therefore \frac{dE}{dk} = \frac{\hbar^2}{m} k, \quad \therefore v_g = \frac{1}{\hbar} \frac{dE}{dk} = \frac{\hbar}{m} k = \frac{p}{m}.$$



from band theory  $E \propto k^2$  but varies as shown. Curve is symmetric about OE axis with points of inflection at M & N where  $\frac{dE}{dk} = \text{maximum}$ , so as  $v$  (as  $v = \frac{1}{\hbar} \frac{dE}{dk}$ ). At points A, O, B slope  $\frac{dE}{dk} = 0$ . Similarly variation of  $v$  shows that at  $k=0$ , i.e. velocity of electron is zero. So the velocity is zero at bottom & top of Brillouin zone. At inflection point  $\pm k_a$  velocity is maximum (free electron velocity)

## Effective mass of an electron

The electrons in a crystal are not free but interact with the periodic potential of the lattice. So effective mass is introduced so that that can be taken as free carriers of charge (electron or hole) in our calculation.

If electron moves distance  $dx$  by electric field  $E$  in  $dt$  time, then  $dE = eE dx = eE v dt$  where  $v = \frac{dx}{dt} = \text{velocity}$

$$\text{Now } v = \frac{1}{\hbar} \frac{dE}{dk} \quad \therefore dE = \frac{eE}{\hbar} \frac{dE}{dk} dt \quad \text{or} \quad \frac{dk}{dt} = \frac{eE}{\hbar}$$

$$\text{Now } \hbar k = p \quad \text{or} \quad \hbar \frac{dk}{dt} = \frac{dp}{dt} = f = eE$$

$$\text{Now } \alpha = \frac{dv}{dt} = \frac{1}{\hbar} \frac{d^2 E}{dk dt} = \frac{1}{\hbar} \frac{d^2 E}{dk^2} \frac{dk}{dt} = \frac{1}{\hbar} \frac{d^2 E}{dk^2} \frac{E}{\hbar}$$

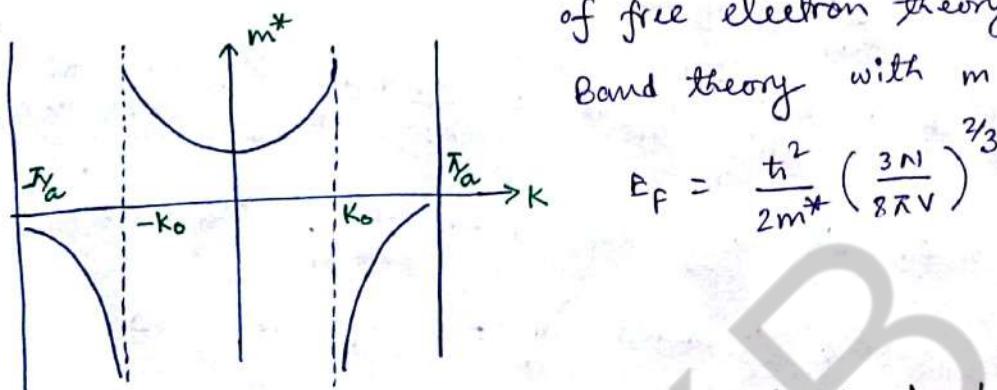
$$\text{or } \frac{\alpha}{F} = \frac{1}{\hbar^2} \frac{d^2 F}{dk^2} = \frac{1}{m^*} \quad (\text{using } F = m^* \alpha)$$

$m^* = \frac{\hbar^2}{\frac{d^2 E}{dk^2}}$ . The ratio of rest mass of free electron  $m$  to

effective mass in crystal in  $K$ -state is  $f_K = \frac{m}{m^*} = \frac{m}{\hbar^2} \frac{d^2 E}{dk^2}$  which

determines the extent to which electron can be thought free. Results

of free electron theory can be applied to  
Band theory with  $m$  replaced by  $m^*$ .



$$E_F = \frac{\hbar^2}{2m^*} \left( \frac{3N}{8\pi V} \right)^{2/3}$$

### Valence & conduction band ; forbidden band

The highest filled energy band which includes electrons stored in covalent bonds or electrons transferred in ionic bonds is known as valence band, denoted by  $E_V$ . When the number of valence electrons in one atom is less than the number of electrons to fill the outer orbit of other atom in solid, valence electrons are free to move to form free electron gas. A band of energy from 0 to  $E_F$  is formed known as conduction band, denoted by  $E_C$ . The forbidden energy region where no electron can remain between  $E_V$  &  $E_C$  is the forbidden band, denoted by  $E_g$ .

We can distinguish conductors (metals), insulators, semiconductors on the basis of band theory.  $f_K = \frac{m}{m^*} = \frac{m}{\hbar^2} \frac{d^2 E}{dk^2}$  that measures how much electrons can take part in electric conduction.

Now for a 1-D lattice of periodicity  $L$ ,  $\Psi(x+L) = \Psi(x)$

$$\therefore e^{ik(x+L)} u_k(x+L) = e^{ikx} u_k(x)$$

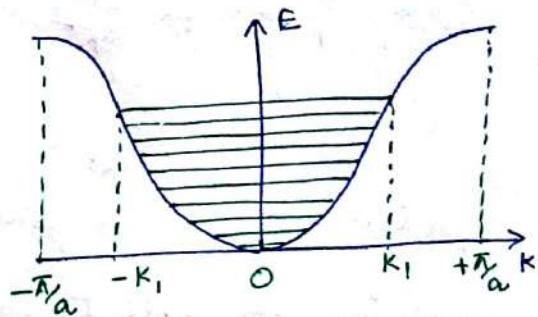
Because of periodicity  $u_k(x+L) = u_k(x)$

$$\therefore e^{ik(x+L)} = e^{ikx} \Rightarrow k = \frac{2\pi n}{L} \Rightarrow dk = \frac{2\pi}{L} dn, dn = \frac{L}{2\pi} dk$$

$dn$  is the number of possible states between  $k$  &  $k+dk$ . Since two electrons occupy each state, effective no. of free electrons in shaded region is

$$N_{\text{eff}} = 2 \int_{-k_1}^{k_1} f_k dn = 2 \int_{-k_1}^{k_1} \frac{m}{\hbar^2} \frac{d^2 E}{dk^2} \frac{L}{2\pi} dk$$

$$= \frac{mL}{\pi \hbar^2} \int_{-k_1}^{k_1} \frac{d^2 E}{dk^2} dk = \frac{2mL}{\pi \hbar^2} \int_0^{k_1} \frac{d^2 E}{dk^2} dk = \frac{2mL}{\pi \hbar^2} \left[ \frac{dE}{dk} \right]_0^{k_1} = \frac{2mL}{\pi \hbar^2} \left( \frac{dE}{dk} \right)_{k_1}$$



as  $\frac{dE}{dk}$  at  $k=0 = 0$ . So  $N_{\text{eff}}$  depends on  $(\frac{dE}{dk})_{k_1}$ . When the band

is completely full so atop the band  $\frac{dE}{dk} = 0 \therefore N_{\text{eff}} = 0$ .

CW Dispersion relation for a 1D crystal of lattice constant  $a$  is  
 $E(k) = E_0 - \alpha - 2\beta \cos ka$  where  $E_0, \alpha, \beta$  constants. Find out the effective mass of the electron at the bottom & top of the band.

$$\text{We know } m^* = \frac{\hbar^2}{\frac{d^2 E}{dk^2}} = \frac{\hbar^2}{2\beta a^2 \cos ka}$$

Atop the band  $\cos ka = \cos \pi = -1$ , bottom of band,  $\cos ka = \cos 0 = 1$

$$\therefore m_{\text{top}}^* = -\frac{\hbar^2}{2\beta a^2}, m_{\text{bottom}}^* = \frac{\hbar^2}{2\beta a^2}$$