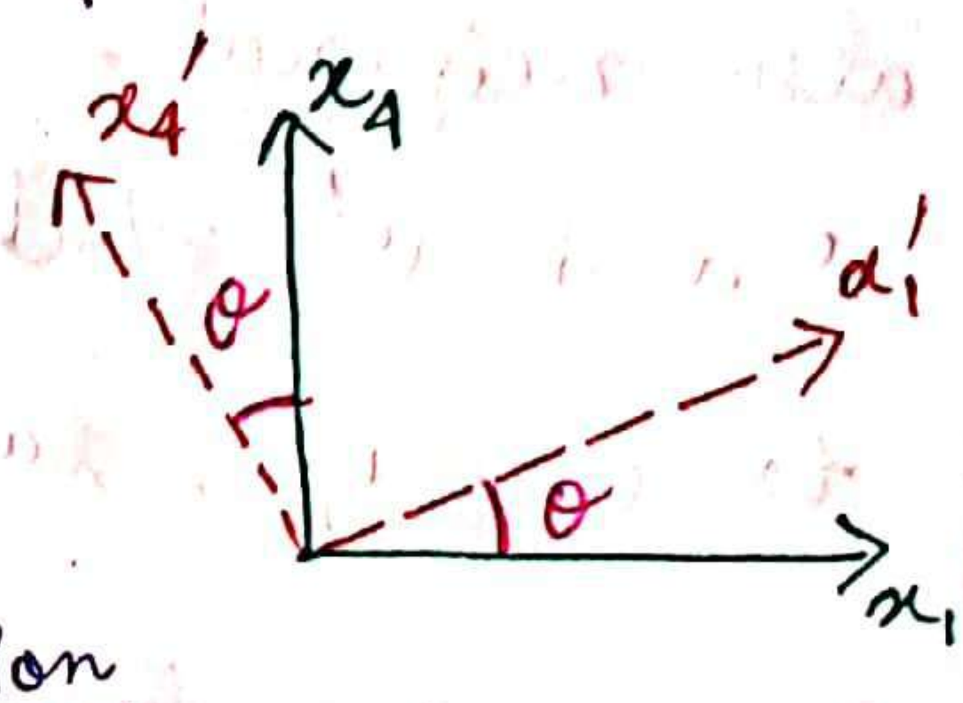
Mathematical Properties of the Space-time ; 4D Formulation of Poincare' & Minkowski

We have already seen that space-time interval $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$ remains invariant under LT. If we rewrite $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$ then $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. To note the similarity of this with rotations: (a) Orthogonal group in 2D $O(2)$ is rotation in 2D that leaves x^2+y^2 invariant, $x' = x\cos\phi - y\sin\phi$, $y' = x\sin\phi + y\cos\phi$, $0 \le \phi \le 2\pi$, then $x'^2 + y'^2 = x^2 + y^2$; (b) Orthogonal group in 3D $O(3)$ that leave $x^2 + y^2 + z^2$ invariant. Poincare noticed in 1906 that if we rewrite LT as $x_1' = \gamma(x_1 + i\beta x_4)$, $x_2' = x_2$

$x_4' = \gamma(x_4 - i\beta x_1)$, $x_3' = x_3$

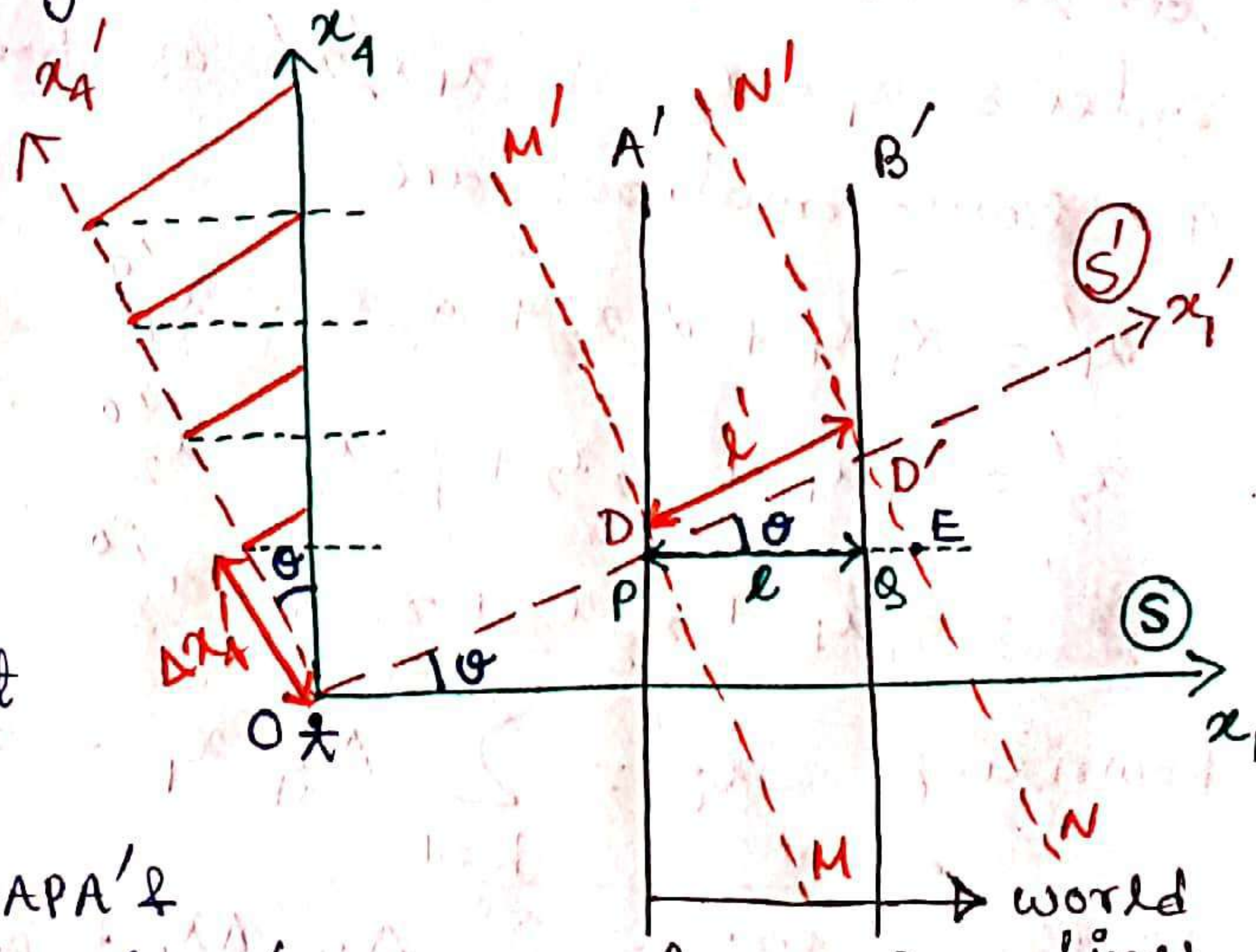


with $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$, then in 1+1 dimension this corresponds to rotation, with $x_1' = x_1\cos\theta + x_4\sin\theta$, $x_4' = -x_1\sin\theta + x_4\cos\theta$

so that $\cos\theta = \gamma$ and $\sin\theta = i\gamma\beta$ to yield $\tan\theta = i\beta \Rightarrow \theta = \tan^{-1}(i\beta)$

So LT is equivalent to rigid rotation of Cartesian axes in a 4D world with imaginary angle $\tan^{-1}(iv/c)$.

We can obtain Einstein's results (length contraction, time dilation, velocity addition) from Poincare's geometrical form. Suppose that a rod of length l with end points P & Q is at rest in S-frame, so their world lines APA' & BQB' are parallel to x_4 axis.



For an observer at S' frame, world points on x_1' axis represent simultaneous events ($x_4' = 0$), so point of intersection of x_1' with APA' & BQB' (points D & D'), are simultaneous to S'-frame observer. So $l' = DD'$ & $l = PQ$ & from geometry $l' = l\sec\theta = l/\gamma$. If now rod was static in S' frame then MDM' & ND'N' are the world lines & observer-S will find $l = DE = DD'\sec\theta$

$= l'/\gamma$. So length contraction is reciprocal.

For S-frame observer at O, world line of his clock is Ox_4 & it's projection on x_4' axis will define interval for S'-frame clock observer. So $\Delta x_4' = \Delta x_4 \cos\theta = \gamma \Delta x_4$

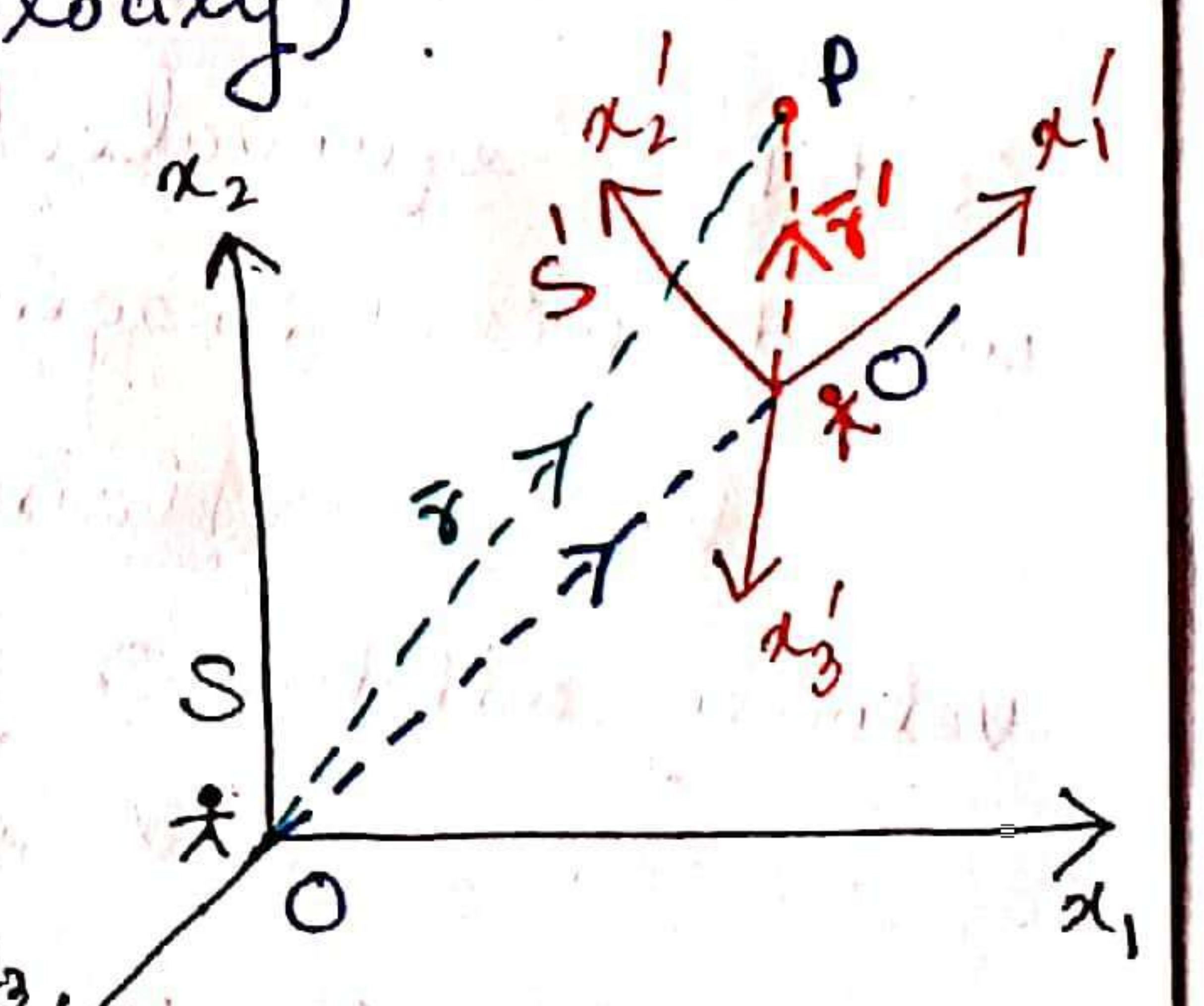
or $ic\Delta t' = \gamma\, ic\Delta t \Rightarrow \Delta t' = \gamma \Delta t$.

Similar argument can be given that for a stationary clock at S' frame, S-frame observer will find $\Delta t = \gamma \Delta t'$, so time dilation is also reciprocal. If we perform 2 LT one after another with velocity v and v' which is equivalent to rotations θ & θ', so that

$\tan\theta = i\beta$, $\tan\theta' = i\beta'$, so $\tan(\theta+\theta') = \dfrac{\tan\theta + \tan\theta'}{1-\tan\theta\tan\theta'} = \dfrac{i(\beta+\beta')}{1+\beta\beta'}$

If these transformations produces a velocity v'' corresponding to an angle θ'' then $\tan\theta'' = i\beta'' = \dfrac{i(\beta+\beta')}{1+\beta\beta'} \Rightarrow \beta'' = \dfrac{\beta+\beta'}{1+\beta\beta'} \Rightarrow$

$$v'' = \frac{v+v'}{1+vv'/c^2} \quad \text{(addition law of velocity)}.$$



3-Vectors in O(3)

According to Poincare's formulation, let us consider 2 Cartesian coordinate systems $x_1x_2x_3$ & $x_1'x_2'x_3'$ as shown. The transformations are

$$x_1' = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 - x_1'^{0}$$
$$x_2' = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 - x_2'^{0}$$
$$x_3' = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 - x_3'^{0}$$

$\alpha_i, \beta_i, \gamma_i \rightarrow$ direction cosines

$x_i'^{0} \rightarrow x_i'$ component of OO'.

Rewriting, $x_i' = \sum_{k=1}^{3} \Lambda_i^{\ k} x_k - x_i'^{0}$, where $\Lambda_1^{\ 1} = \alpha_1$, $\Lambda_1^{\ 2} = \alpha_2$ etc.

so that $\boxed{\Delta x_i' = \sum_{k=1}^{3} \Lambda_i^{\ k} \Delta x_k}$ and $\boxed{\Delta x_k = \sum_{i=1}^{3} \Lambda_k'^{\ i} \Delta x_i'}$ with $|\Lambda_i^{\ k}| \neq 0$

$\Longleftrightarrow$ Orthogonal transformation (OT)

Here $\Lambda_i^{\ k} = \dfrac{\partial x_i'}{\partial x_k}$ and $\Lambda_k'^{\ i} = \dfrac{\partial x_k}{\partial x_i'}$. For OT, $\Lambda_i^{\ k} = \Lambda_k'^{\ i}$

Using Pythagoras theorem, we have norm of Δx_i

$$\sum_{i=1}^{3} \Delta x_i^2 = \sum_{i=1}^{3} \Delta x_i'^2 \quad \Rightarrow \quad \sum_{i=1}^{3} \Delta x_i \Delta x^i = \sum_{i=1}^{3} \Delta x_i'^2$$

$$\Rightarrow \sum_{i=1}^{3}\sum_{K=1}^{3} \Lambda'_i{}^K \Delta x'_K \sum_{l=1}^{3} \Lambda'^i{}_l \Delta x'^l = \sum_{i=1}^{3} \Delta x_i'^2$$

$$\Rightarrow \sum_{i,K,l=1}^{3} \Lambda'_i{}^K \Lambda'^i{}_l \Delta x'_K \Delta x'^l = \sum_{i=1}^{3} \Delta x'_i \Delta x'^i$$ can be true only if

$$\sum_{i=1}^{3} \Lambda'_i{}^K \Lambda'^i{}_l = \delta^K{}_l \quad — (1)$$

Again using OT, $\Delta x_K = \sum_{i=1}^{3} \Lambda'_K{}^i \Delta x'_i = \sum_{i,l=1} \Lambda'_K{}^i \Lambda_i{}^l \Delta x_l$ can be true only if

$$\sum_{i=1}^{3} \Lambda'_K{}^i \Lambda_i{}^l = \delta_K{}^l \quad — (2)$$

$$(1) \times \sum_{l=1}^{3} \Lambda^l{}_m \Rightarrow \sum_{i,l=1}^{3} \Lambda'_i{}^K \underbrace{\Lambda'^i{}_l \Lambda^l{}_m} = \sum_{i=1}^{3} \Lambda'_i{}^K \delta^i{}_m = \Lambda'_m{}^K$$

$$\Rightarrow \sum_{i,l=1}^{3} \underbrace{\Lambda'_i{}^K \Lambda'^i{}_l} \Lambda^l{}_m = \sum_{l=1}^{3} \delta^K_l \Lambda^l_m = \Lambda_m{}^K.$$

* $\therefore \boxed{\Lambda'_m{}^K = \Lambda_m{}^K}$ for OT. Equation (1) and (2) takes the form

$$\sum_{i=1}^{3} \Lambda_i{}^K \Lambda^i{}_l = \delta^K{}_l \quad \text{and} \quad \sum_{i=1}^{3} \Lambda_K{}^i \Lambda_i{}^l = \delta_K{}^l.$$

Taking determinant to find the transformation coefficients of OT,

$$\det \sum_{i=1}^{3} \Lambda_K{}^i \Lambda_i{}^l = \det \delta_K{}^l$$

Schwarz inequality $\det(AB) \le \det(A)\det(B)$

or, $(\det \Lambda_K{}^i)(\det \Lambda_i{}^l) = \det \delta_K{}^l$ or $(\det \Lambda_K{}^i)^2 = 1 \quad \because K = l.$

$\Rightarrow \det \Lambda_K{}^i = \pm 1$ or $e^{\pm i\theta}$

In $O(2)$ we have $x_1' = x_1 \cos\theta + x_2 \sin\theta$, $x_2' = -x_1 \sin\theta + x_2 \cos\theta$,

so $\det \Lambda = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$ (rotation) $\rightarrow O^+(3)$ proper OT.

If $x_1' = x_1$, $x_2' = x_2$, $x_3' = -x_3$ then $\det \Lambda = -1$ (reflection)

In $O(3)$ by solving the eigenvalue equation $\Lambda u = \lambda u$, we have

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \text{ so that } V^{-1}\Lambda V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

* By identical argument, one can easily check that $\Lambda'^K{}_m = \Lambda^K{}_m$.

Vectors, Tensors, Metric & LT

The group of transformations that leave $s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$ invariant is called homogeneous Lorentz group, that contains rotation & LT. On the contrary, the group of transformations that leave invariant $s^2(x,y) = (x_0-y_0)^2 - (x_1-y_1)^2 - (x_2-y_2)^2 - (x_3-y_3)^2$ is called inhomogeneous Lorentz group (Poincaré group).

Einstein's 1st postulate dictates covariance, i.e. invariance in form under the transformations of homogeneous Lorentz group. In 4 dimensional non-Euclidean vector space, suppose a transformation between coordinates is defined as $x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3)$, $\alpha = 0,1,2,3$.

(a) Scalar (Tensor of rank 0) → interval s^2 → Lorentz scalar

(b) Vectors (Tensor of rank 1) → Contravariant vector $A^\alpha = (A^0, A^1, A^2, A^3)$

↳ Covariant vector $B_\alpha = (B_0, B_1, B_2, B_3)$

Transformation rule

$$\begin{cases} A'^\alpha = \dfrac{\partial x'^\alpha}{\partial x^\beta} A^\beta = \dfrac{\partial x'^\alpha}{\partial x^0} A^0 + \dfrac{\partial x'^\alpha}{\partial x^1} A^1 + \dfrac{\partial x'^\alpha}{\partial x^2} A^2 + \dfrac{\partial x'^\alpha}{\partial x^3} A^3 \\ B'_\alpha = \dfrac{\partial x^\beta}{\partial x'^\alpha} B_\beta = \dfrac{\partial x^0}{\partial x'^\alpha} B_0 + \dfrac{\partial x^1}{\partial x'^\alpha} B_1 + \dfrac{\partial x^2}{\partial x'^\alpha} B_2 + \dfrac{\partial x^3}{\partial x'^\alpha} B_3 \end{cases}$$

(Einstein's summation convention)

(c) 2nd rank Tensors (Tensor of rank 2) → Contravariant tensor $F'^{\alpha\beta} = \dfrac{\partial x'^\alpha}{\partial x^\gamma} \dfrac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$

→ Covariant tensor $G'_{\alpha\beta} = \dfrac{\partial x^\gamma}{\partial x'^\alpha} \dfrac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta}$

→ Mixed tensor $H'^\alpha{}_\beta = \dfrac{\partial x'^\alpha}{\partial x^\gamma} \dfrac{\partial x^\delta}{\partial x'^\beta} H^\gamma{}_\delta$

Scalar product $B \cdot A = B_\alpha A^\alpha = B^\alpha A_\alpha =$ covariant vector × contravariant vector or vice versa.

Invariance of scalar product (contraction)

$$B' \cdot A' = \underbrace{\frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\gamma} B_\beta A^\gamma}_{B_\alpha A^\alpha} = \frac{\partial x^\beta}{\partial x^\gamma} B_\beta A^\gamma = \delta^\beta{}_\gamma B_\beta A^\gamma = B \cdot A.$$

The interval (infinitesimal) $ds^2 = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}$ defines the norm of vector space of special relativity & is a special case of general differential length element $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ with $g_{\alpha\beta} = g_{\beta\alpha}$ is called the "Metric tensor." In SR, $g_{\alpha\beta}$ is diagonal with the form

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} ; \quad g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^{\ \beta} \begin{array}{l} = 0 \ (\alpha \neq \beta) \\ = 1 \ (\alpha = 0,1,2,3) \end{array}$$

Covariant 4-vector x_α can be obtained from contravariant x^β by contracting with $g_{\alpha\beta}$ & inverse transformation likewise

$$x_\alpha = g_{\alpha\beta} x^\beta, \quad x^\alpha = g^{\alpha\beta} x_\beta$$

In LT, we have $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, $t' = \gamma(t - \frac{v}{c^2}x)$. If we define $A_1 = x$, $A_2 = y$, $A_3 = z$ and $ct = A_0$, then

$$A_1' = \gamma(A_1 - \tfrac{v}{c}A_4), \ A_2' = A_2, \ A_3' = A_3, \ A_0' = \gamma(A_0 - \tfrac{v}{c}A_1) \text{ and}$$

$$A^{1'} = \gamma(A^{1'} + \tfrac{v}{c}A^4), \ A^{2'} = A^2, \ A^{3'} = A^3, \ A^{0'} = \gamma(A^0 + \tfrac{v}{c}A^1).$$

So $A^\alpha = (A^0, \vec{A})$, $A_\alpha = (A^0, -\vec{A})$, $A^0 = A_0$, $A_1 = -A^1$, $A_2 = -A^2$, $A_3 = -A^3$.

Scalar product of 2 4-vectors $B.A = B_\alpha A^\alpha = B^0 A^0 - \vec{B}.\vec{A}$.

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right) = (\partial^0, -\vec{\partial}^i) \ ; \quad \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta}$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) = (\partial_0, \vec{\partial}_i) \quad \text{(covariant)}.$$

4-divergence of 4-vector A is invariant

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla}.\vec{A} \quad \text{(continuity equation)}$$

$$\Box^2 = \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2 \quad \text{(operator of the wave equation)}$$

A 4-tensor of 2nd rank is a set of 16 quantities A^{ik} that transform as product of components of 2 4-vectors.

$A_{00} = A^{00}$, $A_{01} = -A^{01}$, $A_{11} = A^{11}$, ... , $A_0^{\ 0} = A^{00}$, $A_0^{\ 1} = A^{01}$, $A_1^{\ 0} = -A^{01}$, $A_1^{\ 1} = -A^{11}$

Under purely spatial transformations, 9 quantities $A^{11}, A^{12}, \ldots$ form a 3-tensor. The 3-components A^{01}, A^{02}, A^{03} & A^{10}, A^{20}, A^{30} constitute 3D vectors, A^{00} is 3D scalar.

Tensors $\delta_i^{\ k}$, g_{ik}, g^{ik} and ϵ^{iklm} (4th rank antisymmetric unit tensor) are special as their components are invariant in all coordinate systems.

$\epsilon^{0123} = -\epsilon_{0123} = 1$, $\epsilon^{iilm} = 0$ (as antisymmetric) & $\epsilon^{iklm}\epsilon_{iklm} = -24$ (as total 4! components). This is a "pseudotensor" as reflection cannot be transformed into rotation. If A^{ik} is an antisymmetric tensor then pseudotensor $A^{*ik} = \frac{1}{2}\epsilon^{iklm} A_{lm}$ are dual to each other. Similarly if A^i is vector then $\epsilon^{iklm} A_m$ is an antisymmetric pseudotensor of rank 3. $A^{ik} A^*_{ik}$ is a "pseudoscalar." Any completely antisymmetric tensor of rank equal to number of space dimensions is invariant under rotations of the coordinate system. Thus ϵ^{iklm} is unchanged under rotation of 4D coordinate & ϵ_{ijk} is unchanged by rotations of 3D axes.

Under a reflection (change of sign of all coordinates), components of vector changes sign are called "polar vector." Components of a vector which is a crossproduct of 2 polar vector is called "axial vector," that do not change sign under inversion. Scalar product of polar & axial vector is a "pseudoscalar," that changes sign under inversion. An axial vector is a "pseudovector," dual to antisymmetric tensor.

$$C_i = (A \times B)_i = \frac{1}{2}\epsilon_{ijk} C_{jk} \quad \text{where } C_{jk} = A_j B_k - A_k B_j$$

Matrix representation of Proper homogeneous Lorentz Group's (PHLG)

Infinitesimal generators S_i, K_i

In special relativity to stick to real coordinates (instead of complex) the metric tensor $g_{\mu\nu}$ is introduced [which is a 4x4 matrix = diag(1, −1, −1, −1)] so that the Lorentz scalar $c^2t^2 - x_i^2$ can be written as $g_{\mu\nu}x^\mu x^\nu$.

Suppose the coordinates x^0, x^1, x^2, x^3 are the components of a contravariant 4-vector which form a column vector $x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ and the covariant 4-vector

$$x_\nu = \underset{(gx)}{g_{\mu\nu} x^\mu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}.$$ The scalar product (without indices) is defined as

$$a \cdot b = a_\mu b^\mu = (a, gb) = (ga, b) = a^T g b$$ where $(a,b) \equiv a^T b$ with $g^T = g$, $g^2 = I$ (4x4 unit matrix). What we require is a group of linear transformation of the coordinates $x' = \Lambda x$ such that norm (x, gx) is invariant, meaning

$\therefore x'^T g\, x' = x^T g x$ or, $x^T \Lambda^T g \Lambda x = x^T g x$

To satisfy this relation for all x, $\boxed{\Lambda^T g \Lambda = g}$ — ①. This is just a generalization of the 3D rotations of Cartesian axes $\Lambda^T \Lambda = \mathbb{I}$ (3x3 identity matrix) in 4D space. If we take determinant of ①, then $\det(\Lambda^T g \Lambda) = \det g\,(\det\Lambda)^2 = \det(g)$.

As $\det g = -1$, this can be true if $(\det\Lambda)^2 = 1$ or $\det\Lambda = \pm 1$.

$\therefore$ There are 2 classes of transformation: proper homogeneous Lorentz transformation (PHLG) (6 parameters) with $\det\Lambda = 1$ and improper homogeneous Lorentz transformation (PHIG) (10 parameters) with $\det\Lambda = -1$. Two counter examples of PHIG is $\Lambda = g$ (space inversion), $\det\Lambda = -1$ & $\Lambda = -\mathbb{I}$ (space & time inversion), $\det\Lambda = -1$.

Because ① is symmetric under transpose, $4^2 = 16$ parameters of Λ are not independent but $16 - (1+2+3) = 10$ linearly independent equations are obtained from ①. This means that PHLG is a 6 parameter group having 3 Euler angles (rotation) & 3 relative velocity components of $\vec{\beta} = \frac{\vec{v}}{c}$. We want to know the form (matrix) of the most general element of PHLG.

Let L is a 4x4 real traceless matrix such that $\Lambda = e^L$. This choice gives us $\det\Lambda = +1 = \det e^L = e^{\mathrm{Tr}\,L} = e^0 = +1$.

Now $\Lambda^T = e^{L^T}$, so from ① we have $\Lambda^T g \Lambda = g$ ($\times g$ on the left, $\times \Lambda^{-1}$ on the right)

or $g\Lambda^T g \underbrace{\Lambda\Lambda^{-1}}_{\mathbb{I}} = gg\Lambda^{-1} = \Lambda^{-1}$

or $g\Lambda^T g = \Lambda^{-1} = g e^{L^T} g = e^{gL^Tg} = e^{-L}$

This means $gL^Tg = -L$ or $\boxed{(gL)^T = -gL}$ — ②

So if $L = \begin{pmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix}$ then $gL = \begin{pmatrix} L_{00} & -L_{01} & -L_{02} & -L_{03} \\ L_{10} & -L_{11} & -L_{12} & -L_{13} \\ L_{20} & -L_{21} & -L_{22} & -L_{23} \\ L_{30} & -L_{31} & -L_{32} & -L_{33} \end{pmatrix}$

② $\Rightarrow \begin{pmatrix} L_{00} & L_{10} & L_{20} & L_{30} \\ -L_{01} & -L_{11} & -L_{21} & -L_{31} \\ -L_{02} & -L_{12} & -L_{22} & -L_{32} \\ -L_{03} & -L_{13} & -L_{23} & -L_{33} \end{pmatrix} = \begin{pmatrix} -L_{00} & L_{01} & L_{02} & L_{03} \\ -L_{10} & L_{11} & L_{12} & L_{13} \\ -L_{20} & L_{21} & L_{22} & L_{23} \\ -L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix}$, This can be only true if

$L_{00} = L_{11} = L_{22} = L_{33} = 0$, $L_{01} = L_{10}$, $L_{02} = L_{20}$, $L_{03} = L_{30}$ and

$L_{ij} = -L_{ji}$ $(i \neq j)$, meaning

$$L = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

→ 4×4 symmetric matrix (boost)

(boost)

→ 3×3 antisymmetric spatial matrix (rotations)

$\equiv \{S_i, K_i\}$ matrices are the infinitesimal generators of PHLG. $= -\vec{\omega}\cdot\vec{S} - \vec{\xi}\cdot\vec{K}$

Rotations

$$S_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix},\quad S_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix},\quad S_3 = \begin{pmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix} \;\&$$

$$S_1^2 = \begin{pmatrix} 0&&&0\\&0&&\\&&-1&\\0&&&-1 \end{pmatrix},\quad S_2^2 = \begin{pmatrix} 0&&&0\\&-1&&\\&&0&\\0&&&-1 \end{pmatrix},\quad S_3^2 = \begin{pmatrix} 0&&&0\\&-1&&\\&&-1&\\0&&&0 \end{pmatrix}.$$

Boosts

$$K_1 = \begin{pmatrix} 0&1&0&0\\1&&&\\0&&0&\\0&&& \end{pmatrix},\quad K_2 = \begin{pmatrix} 0&0&1&0\\0&&&\\1&&0&\\0&&& \end{pmatrix},\quad K_3 = \begin{pmatrix} 0&0&0&1\\0&&&\\0&&0&\\1&&& \end{pmatrix} \;\&$$

$$K_1^2 = \begin{pmatrix} 1&&&0\\&1&&\\&&0&\\0&&&0 \end{pmatrix},\quad K_2^2 = \begin{pmatrix} 1&&&0\\&0&&\\&&1&\\0&&&0 \end{pmatrix},\quad K_3^2 = \begin{pmatrix} 1&&&0\\&0&&\\&&0&\\0&&&1 \end{pmatrix}.$$

If we consider a unit vector $\hat{\epsilon} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$ and $\vec{S} = \hat{i} S_1 + \hat{j} S_2 + \hat{k} S_3$, then $\hat{\epsilon}\cdot\vec{S} = \frac{1}{\sqrt{3}}(S_1 + S_2 + S_3)$

$$= \frac{1}{\sqrt{3}}\begin{pmatrix} 0&0&0&0\\0&0&-1&1\\0&1&0&-1\\0&-1&1&0 \end{pmatrix}$$

$$\therefore (\hat{\epsilon}\cdot\vec{S})^3 = \frac{1}{3\sqrt{3}}\begin{pmatrix} 0&0&0&0\\0&0&3&-3\\0&-3&0&3\\0&3&-3&0 \end{pmatrix} = -\frac{1}{\sqrt{3}}\begin{pmatrix} 0&0&0&0\\0&0&-1&1\\0&1&0&-1\\0&-1&1&0 \end{pmatrix} = -(\hat{\epsilon}\cdot\vec{S})$$

Similarly $\hat{\epsilon}' = \frac{1}{\sqrt{3}}(\hat{i}' + \hat{j}' + \hat{k}')$ and $\vec{K} = \hat{i} K_1 + \hat{j} K_2 + \hat{k} K_3$, $(\hat{\epsilon}'\cdot\vec{K})^3 = +(\hat{\epsilon}'\cdot\vec{K})$. So any power of these matrices can be expressed as multiple of the matrix or square. Furthermore, following commutation relation hold,

$$[S_i, S_j] = \epsilon_{ijk} S_k \quad \text{(angular momentum)}$$
$$[S_i, K_j] = \epsilon_{ijk} K_k \quad \text{(K transforms as vector under rotation)}$$
$$[K_i, K_j] = -\epsilon_{ijk} S_k \quad \text{(boosts do not commute, like rotation)}$$

Lie algebra

Physical Significance of L:

An infinitesimal LT and RLT can be defined as

$$x'^i = (g^{ij} + \epsilon^{ij}) x_j \quad \& \quad x^i = (g^{ij} + \epsilon'^{ij}) x'_j, \qquad \epsilon^{ij}, \epsilon'^{ij} \text{ are infinitesimal.}$$

Substituting one into the other we obtain

$$x'^i = (g^{ij} + \epsilon^{ij}) g_{ij} x^i = (g^{ij} + \epsilon^{ij}) g_{ij} (g^{ij} + \epsilon'^{ij}) x'_j$$

$$\Rightarrow x'^i = (g^{ij} + \epsilon^{ij})(\mathbb{I} + g_{ij}\epsilon'^{ij}) x'_j = (g^{ij} + \epsilon^{ij})(\mathbb{I} + g_{ij}\epsilon'^{ij}) g_{ij} x'^i$$

$$\therefore (g^{ij} + \epsilon^{ij})(g_{ij} + g_{ij}\epsilon'^{ij} g_{ij}) = \mathbb{I}.$$

$$\Rightarrow \mathbb{I} + \epsilon^{ij} g_{ij} + \epsilon'^{ij} g_{ij} + \cancel{O(\epsilon\epsilon')} = \mathbb{I}, \qquad \therefore \underline{\epsilon^{ij} = -\epsilon'^{ij}}$$

Again, to preserve the norm, we must have $x'^T g x' = x^T g x$

$$\text{Now } x'^{iT} g^{ij} x'^i = [(g^{ij} + \epsilon^{ij}) x_j]^T g^{ij} (g^{ij} + \epsilon^{ij}) x_j$$

$$= [(g^{ij} + \epsilon^{ij}) g_{ij} x^i]^T g^{ij} (g^{ij} + \epsilon^{ij}) g_{ij} x^i \qquad [\because g \,\&\, \epsilon \text{ commute}]$$

$$= (g_{ij} x^i)^T (g^{ij} + \epsilon^{ij})^T \underbrace{g^{ij} g_{ij}}_{\mathbb{I}} (g^{ij} + \epsilon^{ij}) x^i$$

$$= x^{iT} g_{ij} (g^{ij} + \epsilon^{ji})(g^{ij} + \epsilon^{ij}) x^i = x^{iT} g_{ij} x^i \text{ only if}$$

$$(g^{ij} + \epsilon^{ji})(g^{ij} + \epsilon^{ij}) = \mathbb{I} = \mathbb{I} + g^{ij}\epsilon^{ij} + g^{ij}\epsilon^{ji} + \cancel{O(\epsilon^2)}$$

$$\therefore \underline{\epsilon^{ij} = -\epsilon^{ji}}$$

Again we have $\epsilon^{ij} g_{ij} + \epsilon'^{ij} g_{ij} = 0.$

$$\text{So } \epsilon'^{ij} g_{ij} = -\epsilon^{ij} g_{ij} = \epsilon^{ji} g_{ij}$$

$$\Rightarrow -\epsilon^{ij} g_{ij} = \epsilon^{ji} g_{ij} \quad \Rightarrow \quad \underline{(\epsilon g)^T = -\epsilon g.}$$

Hence L matrix is the ϵ matrix that represents an infinitesimal Lorentz transformation.

$$L = -\underbrace{\vec{\omega}\cdot\vec{S}}_{\text{(rotation)}} - \underbrace{\vec{\xi}\cdot\vec{K}}_{\text{(boost)}} \quad \text{and} \quad \Lambda = e^{L} = e^{-(\vec{\omega}\cdot\vec{S} - \vec{\xi}\cdot\vec{K})}$$

For a boost in arbitrary direction $\Lambda = e^{-\vec{\xi}\cdot\vec{K}} = e^{-\hat{\beta}\cdot\vec{K}\tanh^{-1}\beta}$

where the boost vector $\vec{\xi} = \hat{\beta}\tanh^{-1}\beta$ with $\beta = \frac{v}{c}$ and $\hat{\beta}$ is a unit vector in the direction of the inertial frames.

In the hyperbolic form $\gamma = \cosh\xi$, $\gamma\beta = \sinh\xi$ so that $\beta = \tanh\xi$

$$\Lambda = e^{-\xi(\hat\beta\cdot\vec K)} = 1 - (\hat\beta\cdot\vec K)\xi + (\hat\beta\cdot\vec K)^2\frac{\xi^2}{2!} - (\hat\beta\cdot\vec K)^3\frac{\xi^3}{3!} + (\hat\beta\cdot\vec K)^4\frac{\xi^4}{4!} - \dots$$

$$= 1 - (\hat\beta\cdot\vec K)\left(\xi + \frac{\xi^3}{3!} + \frac{\xi^5}{5!} + \cdots\right) + (\hat\beta\cdot\vec K)^2\left(1 + \frac{\xi^2}{2!} + \frac{\xi^4}{4!} + \cdots\right) - (\hat\beta\cdot\vec K)^2$$

$$= 1 - (\hat\beta\cdot\vec K)\sinh\xi + (\hat\beta\cdot\vec K)^2(\cosh\xi - 1)$$

Now $\hat\beta = \frac{\beta_1}{\beta}\hat i + \frac{\beta_2}{\beta}\hat j + \frac{\beta_2}{\beta}\hat k$, $\vec K = K_1\hat i + K_2\hat j + K_3\hat k$, so that

$$\hat\beta\cdot\vec K = \frac{\beta_1}{\beta}K_1 + \frac{\beta_2}{\beta}K_2 + \frac{\beta_3}{\beta}K_3 = \begin{pmatrix} 0 & \beta_1/\beta & \beta_2/\beta & \beta_3/\beta \\ \beta_1/\beta & & & \\ \beta_2/\beta & & 0 & \\ \beta_3/\beta & & & \end{pmatrix}$$ and

$$(\hat\beta\cdot\vec K)^2 = \begin{pmatrix} 0 & \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \\ \frac{\beta_1}{\beta} & 0 & 0 & 0 \\ \frac{\beta_2}{\beta} & 0 & 0 & 0 \\ \frac{\beta_3}{\beta} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \\ \frac{\beta_1}{\beta} & 0 & 0 & 0 \\ \frac{\beta_2}{\beta} & 0 & 0 & 0 \\ \frac{\beta_3}{\beta} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \overset{=1}{\frac{\beta_1^2+\beta_2^2+\beta_3^2}{\beta^2}} & 0 & 0 & 0 \\ 0 & \frac{\beta_1^2}{\beta^2} & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_1\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_2^2}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_3}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} & \frac{\beta_3^2}{\beta^2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta_1^2}{\beta^2} & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_1\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_2^2}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_3}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} & \frac{\beta_3^2}{\beta^2} \end{pmatrix}$$

$$\therefore \Lambda = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} - \gamma\beta\begin{pmatrix} 0 & \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \\ \frac{\beta_1}{\beta} & & & \\ \frac{\beta_2}{\beta} & & 0 & \\ \frac{\beta_3}{\beta} & & & \end{pmatrix} + (\gamma - 1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta_1^2/\beta^2 & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_1\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_2}{\beta^2} & \beta_2^2/\beta^2 & \frac{\beta_2\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_3}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} & \frac{\beta_3^2}{\beta^2} \end{pmatrix}$$

$$\text{or } \boxed{\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1 + \frac{(\gamma-1)\beta_1^2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1 + \frac{(\gamma-1)\beta_2^2}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1 + \frac{(\gamma-1)\beta_3^2}{\beta^2} \end{pmatrix}}$$

To remind, $X' = \Lambda X$, where $X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ \vec x \end{pmatrix}$.

The same matrix obtained without 4-vector notation in Generalized 3D LT with $X = \begin{pmatrix} \vec x \\ t \end{pmatrix}$.

Pure Boost along x-direction

We have from ① $\Lambda = e^{-\xi(\hat{\beta}\cdot\vec{K})} = 1 - (\hat{\beta}\cdot\vec{K})\sinh\xi + (\hat{\beta}\cdot\vec{K})^2(\cosh\xi - 1)$

If $\hat{\beta} = \hat{i}$, $\Lambda = 1 - K_1 \sinh\xi + K_1^2(\cosh\xi - 1)$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} - \sinh\xi \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & 0 & \\ 0 & & & \end{pmatrix} + \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}(\cosh\xi - 1)$$

$$= \begin{pmatrix} \cosh\xi & -\sinh\xi & & 0 \\ -\sinh\xi & \cosh\xi & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

which is the same as rotation matrix in hyperbolic form encountered earlier.

$\therefore\ ct' = \cosh\xi\, ct - x\sinh\xi$

$x' = -\sinh\xi\, ct + x\cosh\xi$

Pure Rotation along x-direction

Here $L = -\vec{\omega}\cdot\vec{S}$ and $\Lambda = e^{-\vec{\omega}\cdot\vec{S}} = e^{-\theta(\hat{\omega}\cdot\vec{S})}$

If $\hat{\omega} = \hat{i}$, $\Lambda = e^{-\theta S_1} = 1 - \theta S_1 + \frac{(\theta S_1)^2}{2!} - \frac{(\theta S_1)^3}{3!} + \frac{(\theta S_1)^4}{4!} - \frac{(\theta S_1)^5}{5!} + \dots$

Now $S_1^3 = -S_1$, so $S_1^4 = -S_1^2$, $S_1^6 = S_1^3 \cdot S_1^3 = S_1^2$, $S_1^5 = S_1^3 \cdot S_1^2 = -S_1 \cdot S_1^2$
$= -S_1^3 = S_1$

$$\therefore\ \Lambda = 1 - \theta S_1 + \frac{\theta^2 S_1^2}{2!} + \frac{\theta^3 S_1}{3!} - \frac{\theta^4 S_1^2}{4!} - \frac{\theta^5 S_1}{5!} + \dots$$

$$= 1 - S_1\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) - S_1^2\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + S_1^2$$

$$= 1 - S_1 \sin\theta - S_1^2(\cos\theta - 1).$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\sin\theta - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ 0 & & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{cases} y' = y\cos\theta + z\sin\theta \\ z' = -y\sin\theta + z\cos\theta \end{cases}$$

Position & momentum 4-vector

In general convention to covariant & contravariant vector, we define $x^\alpha = (x^0, \bar{x})$ & $x_\alpha = (x^0, -\bar{x})$ and line element $x^\alpha x_\alpha = (ct)^2 - \bar{x}^2$

where $x^0 = ct = x_0$

Now LT, $ct' = \gamma(ct - \frac{v}{c}x)$, $y' = y$
$x' = \gamma(x - \frac{v}{c}ct)$, $z' = z$ can be written as

$x^{0'} = \gamma(x^0 - \frac{v}{c}x^1)$, $x^{1'} = \gamma(x^1 - \frac{v}{c}x^0)$, $x^{2'} = x^2$, $x^{3'} = x^3$

We can define 4-velocity as $V^\mu = \frac{dx^\mu}{d\tau}$ where τ is the proper time, so that $dt = \frac{d\tau}{\sqrt{1-v^2/c^2}} = \gamma d\tau$.

So $V^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} = c\gamma$ and $V^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma v^i$

are the contravariant components & $V_0 = c\gamma$ and $V_i = -\gamma v_i$ are the covariant components of the 4-velocity.

$\therefore V_\mu V^\mu = c^2\gamma^2 - v^2\gamma^2 = \frac{c^2 - v^2}{1 - v^2/c^2} = c^2 > 0$ (timelike)

For a timelike vector, we can always do LT $\Lambda V = V'$ so that $V' \equiv (V'^0, \vec{0})$ has no space components. In a similar fashion if $V_\mu V^\mu < 0$ (spacelike) then in a LT we can find $V'^\mu = (0, \vec{V})$ without any time component. If $V_\mu V^\mu = 0$ (light like), e.g. photon in free space having energy momentum 4-vector $K^\mu = (K^0, \hbar\vec{k})$ with $K^0 = \frac{h\nu}{c} = \frac{E}{c}$ $= \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$ so that $K_\mu K^\mu = K^{0^2} - \hbar k^2 = 0$.

So we define 4-momentum as $p^\mu = (p^0, p^i)$ so that

$p^0 = m_0 V^0 = m_0 c\gamma = \frac{m_0 c^2 \gamma}{c} = \frac{E}{c}$ where $E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = m_0 c^2 \gamma$

& $p^i = m_0 V^i = m_0 v^i \gamma = m v^i$, where $m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0\gamma$.

$\therefore p^\mu = (m_0\gamma c, m_0\gamma v^i) = (E/c, \bar{p})$ contravariant 4-vector &

$p_\mu = (\frac{E}{c}, -\bar{p})$ covariant 4-vector.

So the invariance of momentum & energy conservation are not separate, & we've already seen $p_\mu p^\mu$ as a Lorentz scalar.

In a similar fashion we define 4-dimensional Minkowski force $f^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}(m_0 V^\mu)$. (Newton's law) (covariant form)

where $f^0 = \frac{d}{d\tau}(m_0 V^0) = \frac{dt}{d\tau}\frac{d}{dt}(m_0 V^0) = \gamma\frac{d}{dt}(m_0 c\gamma)$ and

$f^i = \frac{d}{d\tau}(m_0 V^i) = \frac{dt}{d\tau}\frac{d}{dt}(m_0 V^i) = \gamma\frac{d}{dt}(m_0 v^i \gamma)$

for a n-dimensional vector of constant magnitude ($A^i A_i$ = constant) it's derivative is orthogonal ($A_i \frac{dA^i}{d\tau} = 0$). Here we notice that

$$V^\mu f_\mu = V_\mu f^\mu = V_\mu \frac{d}{d\tau}(m_0 V^\mu) = 0 \Rightarrow V_0 f^0 + V_i f^i = 0.$$

or $c\gamma \, \gamma \frac{d}{dt}(m_0 c\gamma) - \gamma v_i \gamma \frac{d}{dt}(m_0 v^i \gamma) = 0$ $\quad [\because V_0 = V^0 = c\gamma, \; V_i = -V^i = -\gamma v^i]$

or $v_i \frac{d}{dt}(m_0 v^i \gamma) = \frac{d}{dt}(m_0 c^2 \gamma)$.

or $v_i \underset{\substack{\downarrow \\ \text{3D momentum}}}{\frac{d}{dt}(p^i)} \equiv v_i \underset{\substack{\downarrow \\ \text{3D force}}}{f^i} \equiv \boxed{\frac{dE}{dt} = \frac{d}{dt}(m_0 c^2 \gamma)}$, where rate of change of total kinetic energy of a particle is the rate at which force does work on the particle $= \vec{v}.\vec{F}$. Integrating over time above equation,

$E = m_0 c^2 \gamma + E_0$, E_0 = integration constant.

If $E = E_K$ = kinetic energy of the particle then when $v = 0$ or $\gamma = 1$ then $E_K = 0$. $\therefore 0 = m_0 c^2 + E_0$ or $E_0 = -m_0 c^2$

$\therefore E_K = m_0 c^2 \gamma - m_0 c^2 = mc^2 - m_0 c^2$ or $\boxed{E = E_K + m_0 c^2 = mc^2}$

This is Einstein's mass-energy relationship. As $E_K = (m - m_0)c^2$, so the KE is c^2 times gain in mass due to motion. The rest mass energy $m_0 c^2$ is due to the thermal motion.

$$P_\mu P^\mu = \frac{E^2}{c^2} - p^2 = \frac{m^2 c^4}{c^2} - p^2 = m_0^2 \gamma^2 c^2 - m_0^2 \gamma^2 v^2 = m_0^2 c^2 > 0$$

(timelike)

(Energy-Momentum 4-Lorentz scalar)

From the covariant formulation of Newton's law, we will show next that the force is parallel to acceleration only when velocity is either parallel or perpendicular to acceleration.

We have already seen above that $\frac{dE}{dt} = \frac{d}{dt}(m_0 c^2 \gamma)$ so that

$$f^0 = \gamma \frac{d}{dt}(m_0 c\gamma) = \frac{\gamma}{c}\frac{dE}{dt} = \frac{\gamma}{c}\vec{v}.\vec{F} \quad \& \qquad v^2 = v_i v^i$$

$$f^i = \gamma \frac{d}{dt}(m_0 v^i \gamma) = \gamma \frac{dp^i}{dt} = \gamma F^i \quad \text{—(A)} \qquad \left[\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\right]$$

$$= \gamma \frac{d}{dt}\left(\frac{m_0 v^i}{\sqrt{1 - v^2/c^2}}\right) = \frac{m_0 a^i}{1 - v^2/c^2} + \frac{m_0 v^i v^j \frac{dv_j}{dt}}{c^2(1 - v^2/c^2)^2} = \frac{m_0 a^i}{1 - v^2/c^2} + \frac{m_0 v^i v^j a_j}{c^2(1 - v^2/c^2)^2}$$

So if $\overset{\circ}{f^i} \parallel \overset{\circ}{a^i}$, then 2^{nd} term in RHS is zero, which can happen if $v^j a_j = \vec{v}\cdot\vec{a} = 0$ or velocity is $\perp$ to acceleration. If $\vec{v} \parallel \vec{a}$ then

$a_j = c' v_j$ where c' is a scalar, so that

$$\overset{\circ}{f^i} = \frac{m_0 \overset{\circ}{a^i}}{1 - v^2/c^2} + \frac{m_0 v^i v^j c' v_j / c^2}{(1 - v^2/c^2)^2} = \frac{m_0 \overset{\circ}{a^i}}{1 - v^2/c^2} + \frac{m_0 (\overbrace{c' v^i}^{= \overset{\circ}{a^i}})(\overbrace{v^j v_j}^{= v^2})/c^2}{(1 - v^2/c^2)^2}$$

$$= \frac{m_0 \overset{\circ}{a^i}}{1 - v^2/c^2} + \frac{m_0 \overset{\circ}{a^i} v^2/c^2}{(1 - v^2/c^2)^2} = \frac{m_0 \overset{\circ}{a^i}}{(1 - v^2/c^2)^2}.$$ Using (A) we have

<u>CASE 1</u> $(f \parallel a ;\ \vec{v} \perp \vec{a})$ $\quad \gamma \overset{\circ}{F^i} = \overset{\circ}{f^i} = \frac{m_0 \overset{\circ}{a^i}}{1 - v^2/c^2} = \gamma^2 m_0 \overset{\circ}{a^i}$

or $\overset{\circ}{F^i} = \gamma m_0 \overset{\circ}{a^i} = m_{\text{transverse}}\, \overset{\circ}{a^i}$ where <u>transverse mass $= \gamma m_0$</u>

<u>CASE 2</u> $(f \parallel a ;\ \vec{v} \parallel \vec{a})$ $\quad \gamma \overset{\circ}{F^i} = \frac{m_0 \overset{\circ}{a^i}}{(1 - v^2/c^2)^2} = m_0 \gamma^4 \overset{\circ}{a^i}$

or $\overset{\circ}{F^i} = \gamma^3 m_0 \overset{\circ}{a^i} = m_{\text{longitudinal}}\, \overset{\circ}{a^i}$ where <u>longitudinal mass $= \gamma^3 m_0$</u>

In general, $\gamma \overset{\circ}{F^i} = \gamma^2 (m_0 \overset{\circ}{a^i}) + \gamma^4 (m_0 v^i v^j a_j / c^2)$

or $$\boxed{\overset{\circ}{F^i} = \gamma m_0 \overset{\circ}{a^i} + \gamma^3 m_0 v^i v^j \frac{a_j}{c^2}}, \quad (\vec{F} \nparallel \vec{a})$$