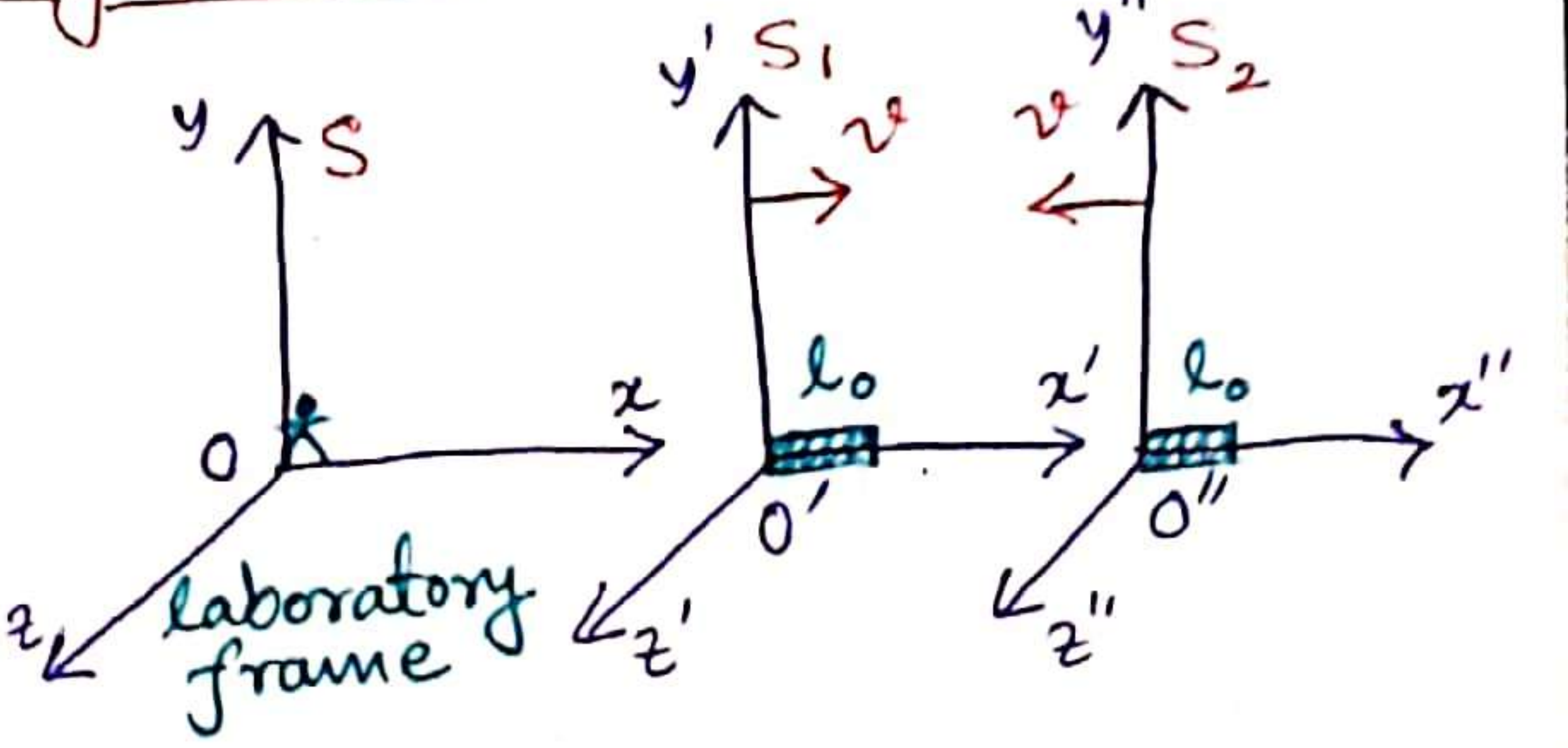
Special Relativity Assignment-1



1. (a) Let S be the laboratory frame from which observer at O watches the two rods with proper length l_0 fixed to their respective inertial frame S_1 and S_2.

Velocity of S_2 frame w.r.t. S frame $w = -v$,
velocity of S_1 frame w.r.t. S frame $u_1 = v$ and
velocity of S_2 frame w.r.t. S_1 frame be u_2.
Using Einstein's velocity addition theorem, we have $w = \dfrac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}}$

or $-v = \dfrac{v + u_2}{1 + \frac{v u_2}{c^2}}$ or $-v - \dfrac{v^2 u_2}{c^2} = v + u_2$

or $u_2 (1 + v^2/c^2) = -2v$ or $u_2 = -\dfrac{2v}{1+\beta^2}$ where $\beta = \dfrac{v}{c}$.

∴ Length of each rod in reference frame S_1 or S_2 is

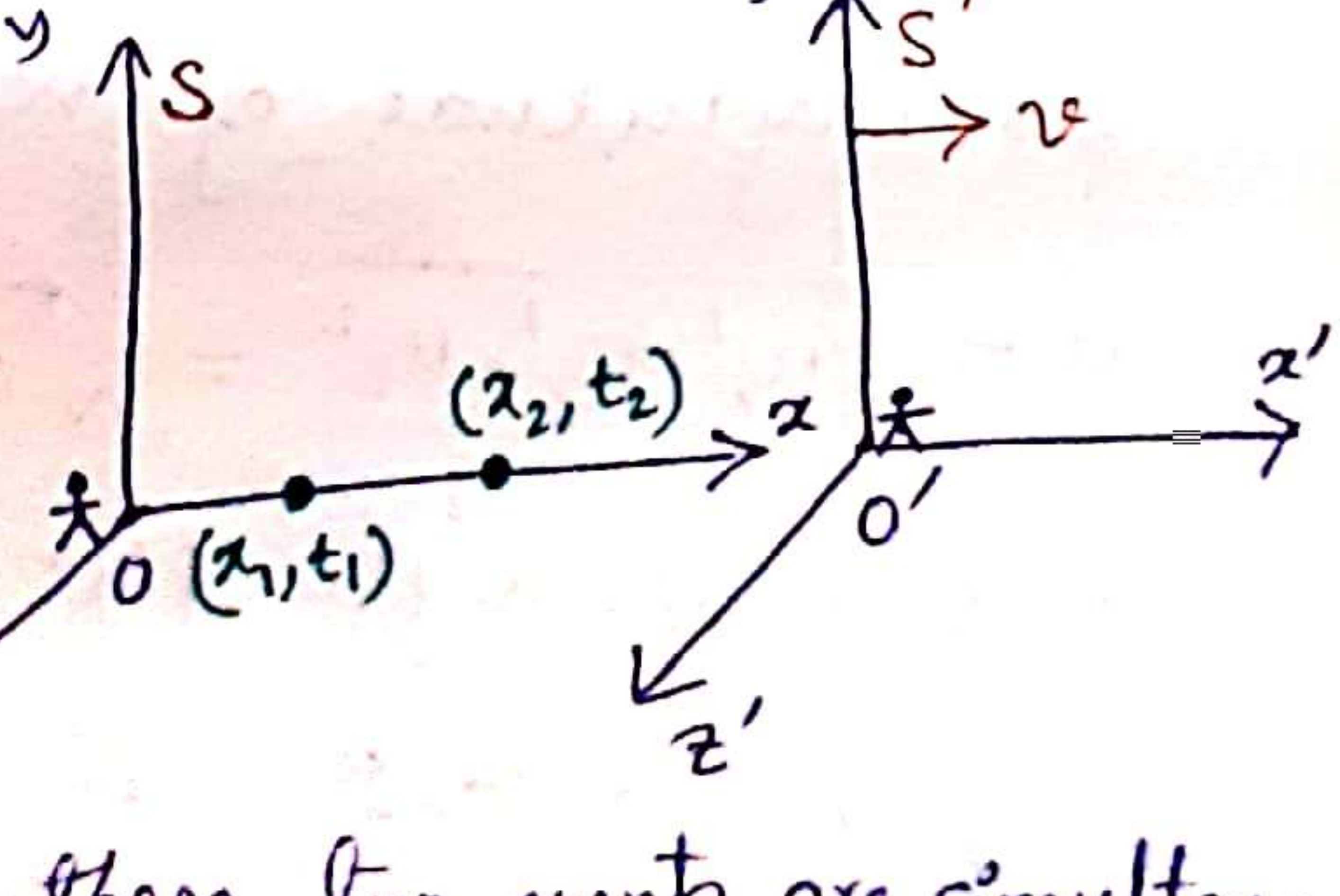
$$l = l_0\sqrt{1 - u_2^2/c^2} = l_0\sqrt{1 - \frac{4v^2}{(1+\beta^2)^2 c^2}} = l_0\sqrt{1 - \frac{4\beta^2}{(1+\beta^2)^2}}$$

$$\text{or, } l = l_0\sqrt{\frac{(1+\beta^2)^2 - 4\beta^2}{(1+\beta^2)^2}} = l_0\sqrt{\frac{(1-\beta^2)^2}{(1+\beta^2)^2}} = l_0\left(\frac{1-\beta^2}{1+\beta^2}\right).$$

(b) Given for the rockets, $v = c/2$ or $\beta = 0.5$. So length of each rocket as seen by the other is

$$l = l_0 \frac{1-\beta^2}{1+\beta^2} = l_0 \frac{1-\frac{1}{4}}{1+\frac{1}{4}} = \underline{\underline{\frac{3}{5} l_0}}.$$

(c) In frame S, the spacetime coordinates of two events are



Event 1: $x_1 = x_0$, $t_1 = \dfrac{x_0}{c}$, $y_1 = z_1 = 0$

Event 2: $x_2 = 2x_0$, $t_2 = \dfrac{x_0}{2c}$, $y_2 = z_2 = 0$

while given that ~~to~~ S' frames observer these two events are simultaneous

So $t_1' = t_2'$. Using time dilation expression, we have

$$t_1' = \gamma(t_1 - \frac{vx_1}{c^2}) = \gamma(\frac{x_0}{c} - \frac{vx_0}{c^2}) \quad \text{and}$$

$$t_2' = \gamma(t_2 - \frac{vx_2}{c^2}) = \gamma(\frac{x_0}{2c} - \frac{2vx_0}{c^2}).$$ and so to satisfy the condition

$$\cancel{\gamma}(\frac{x_0}{c} - \frac{vx_0}{c^2}) = \cancel{\gamma}(\frac{x_0}{2c} - \frac{2vx_0}{c^2}) \quad \text{or} \quad \frac{\cancel{x_0}}{2c} = -\frac{v\cancel{x_0}}{c^2} \quad \text{or} \quad \boxed{v = -\frac{c}{2}}$$

As anticipated for simultaneous events, events are space like.

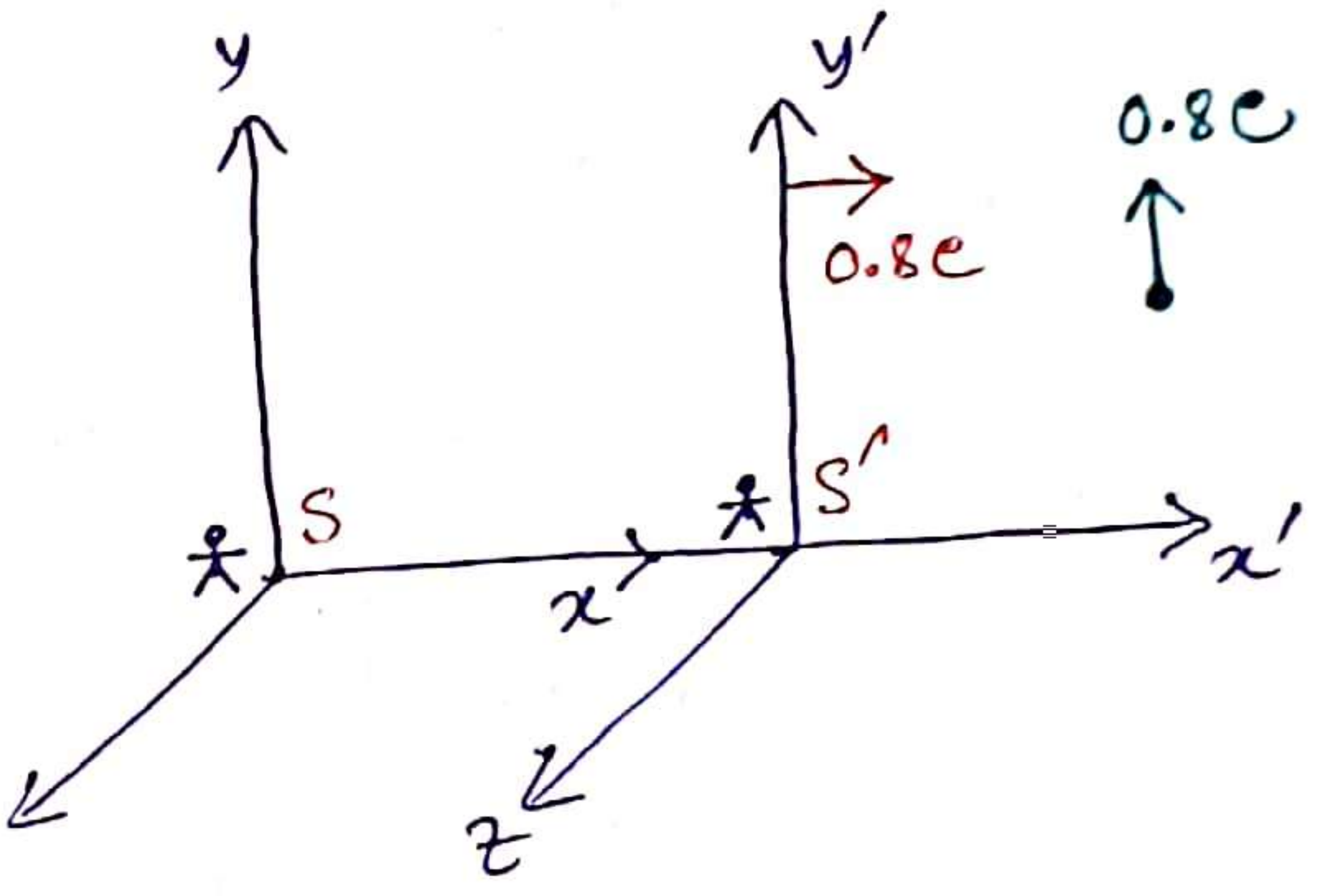
Now substituting back the value in time, we have

$$t_1' = \gamma(\frac{x_0}{c} - \frac{vx_0}{c^2}) = \gamma(\frac{x_0}{c} + \frac{x_0}{2c}) = \frac{1}{\sqrt{1-\frac{c^2}{4c^2}}}\frac{3x_0}{2c} = \frac{\cancel{2}}{\sqrt{3}}\frac{3x_0}{\cancel{2}c} = \frac{\sqrt{3}x_0}{c}$$

$$\text{and } t_2' = \gamma(\frac{x_0}{2c} - \frac{2vx_0}{c^2}) = \gamma(\frac{x_0}{2c} + \frac{\cancel{2}x_0}{c^2}\frac{c}{\cancel{2}}) = \gamma(\frac{x_0}{2c} + \frac{x_0}{c}) = t_1'$$

So $\boxed{t = \frac{\sqrt{3}x_0}{c}}$

2. (a) As depicted in the diagram beside, for S' frame observer we have $u_x' = 0$, $u_y' = 0.8c$, $u_z' = 0$



Using the velocity transformation formula, we have for S frame observer

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = v, = 0.8c \qquad u_y = \frac{u_y'\sqrt{1 - v^2/c^2}}{1 + \frac{u_x' v}{c^2}} = 0.8c\sqrt{1-(0.8)^2} = 0.48c.$$

$$u_z = \frac{u_z'\sqrt{1 - v^2/c^2}}{1 + \frac{u_x' v}{c^2}} = 0.$$

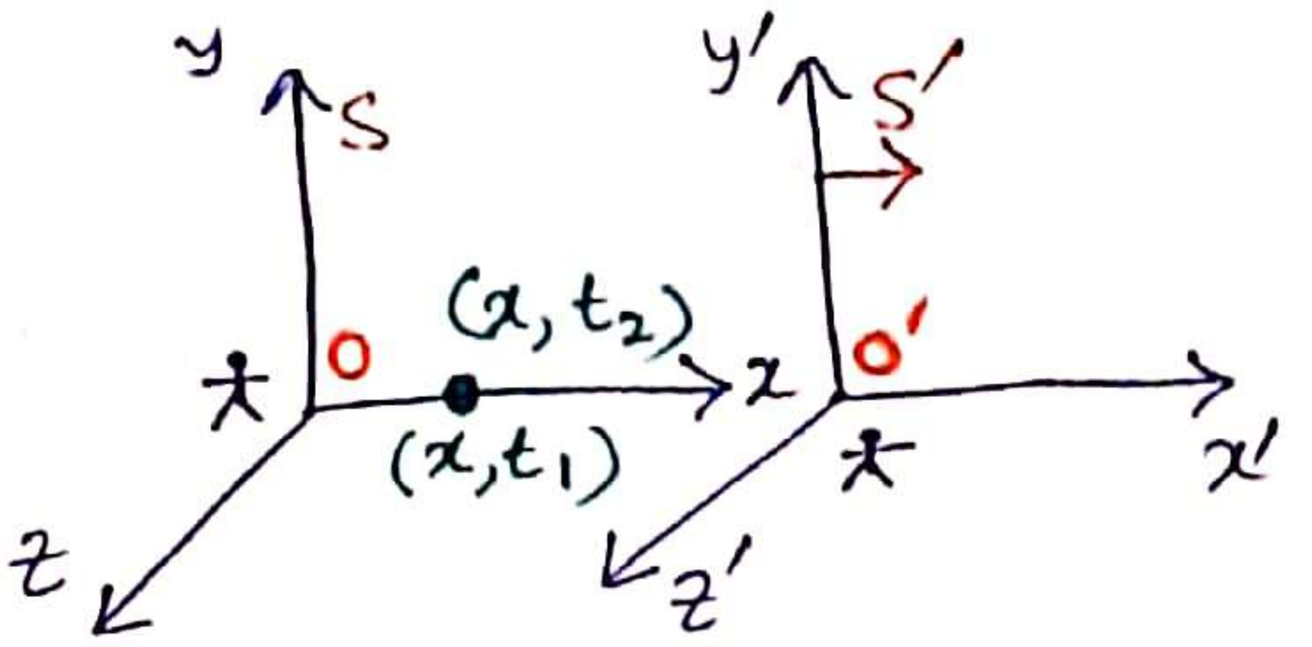
So magnitude of velocity to S-frame observer is

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{0.8^2 + 0.48^2}\,c = \underline{0.933c}$$

If θ is the angle that u makes with x-axis then

$$u_x = u\cos\theta \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{u_x}{u}\right) = \cos^{-1}\left(\frac{0.8}{0.933}\right) = \underline{30.96^\circ}$$

(b) Let the two events occur in S-frame at (x,t_1) and (x,t_2) space-time point so that $t_2 - t_1 = 4$ sec. In S' frame, $t_2' - t_1' = 6$ sec. So using LT, we have



$$t_2' = \gamma\left(t_2 - \frac{vx}{c^2}\right) \text{ and } t_1' = \gamma\left(t_1 - \frac{vx}{c^2}\right) \text{ so that}$$

$$\Rightarrow t_2' - t_1' = \gamma(t_2 - t_1) \Rightarrow 6/4 = \gamma \Rightarrow \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{3}{2}$$

$$\Rightarrow 1 - \frac{v^2}{c^2} = \frac{4}{9} \Rightarrow \frac{v^2}{c^2} = \frac{5}{9} \Rightarrow \beta = \frac{v}{c} = \frac{\sqrt{5}}{3}.$$

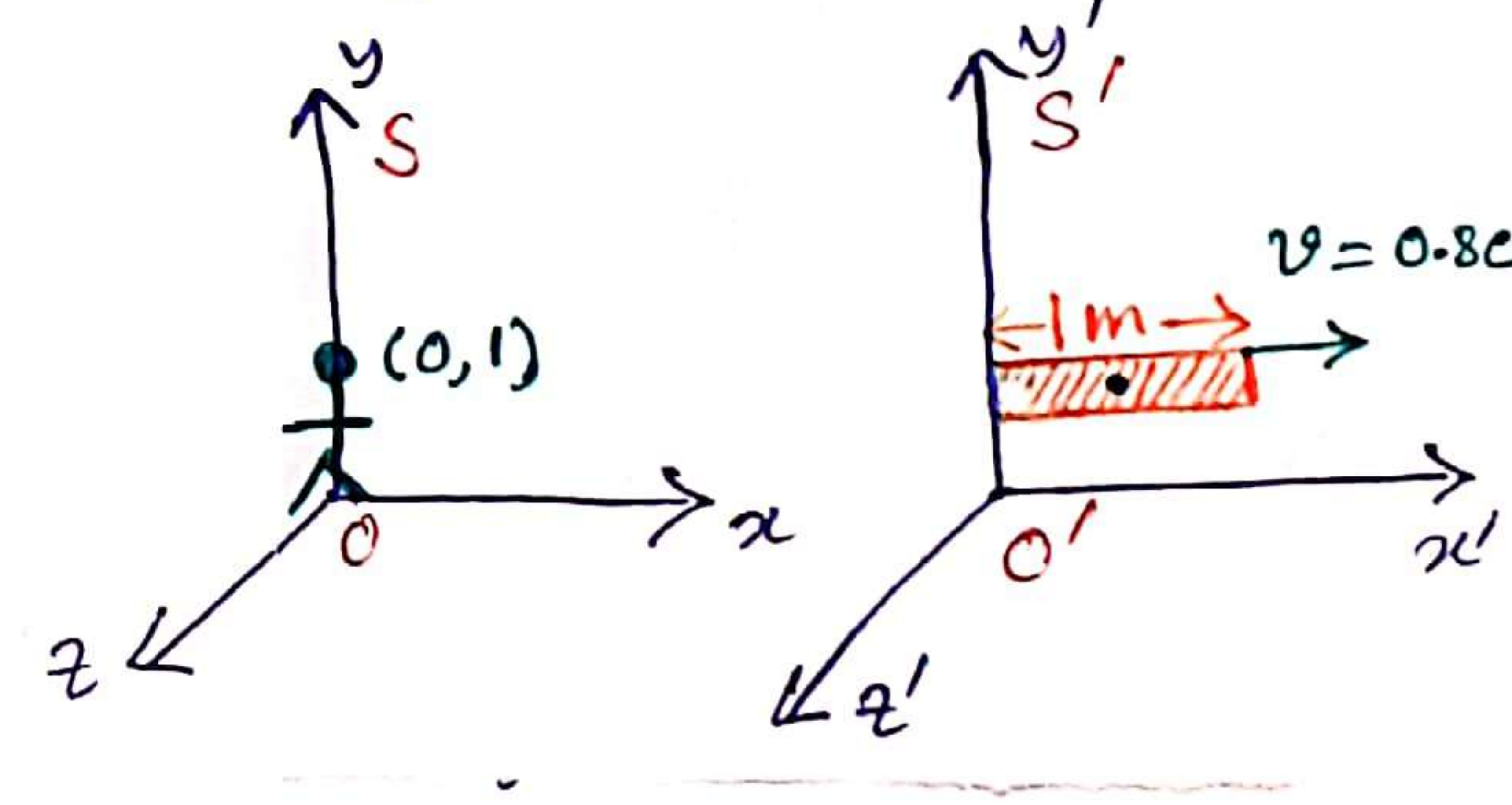
So if in the s' frame these two events have space-time coordinate (x_1', t_1') and (x_2', t_2') then again from L.T. we have

$$x_2' = \gamma(x_2 - vt_2) \text{ and } x_1' = \gamma(x_1 - vt_1) \text{ so that}$$

$$x_2' - x_1' = \gamma[-v(t_2 - t_1)] = \frac{3}{2}\left(-\frac{\sqrt{5}}{3}c \times 4\right) = -2\sqrt{5}\,c.$$

∴ Spatial distance of these events in s' frame is $2\sqrt{5}\,c$.

5 → (a) The proper length (rest) of the stick is $l_0 = 1$ m. in the s' frame. Due to length contraction, observer at S frame measures the length of the meter stick to be $l = \frac{l_0}{\gamma}$

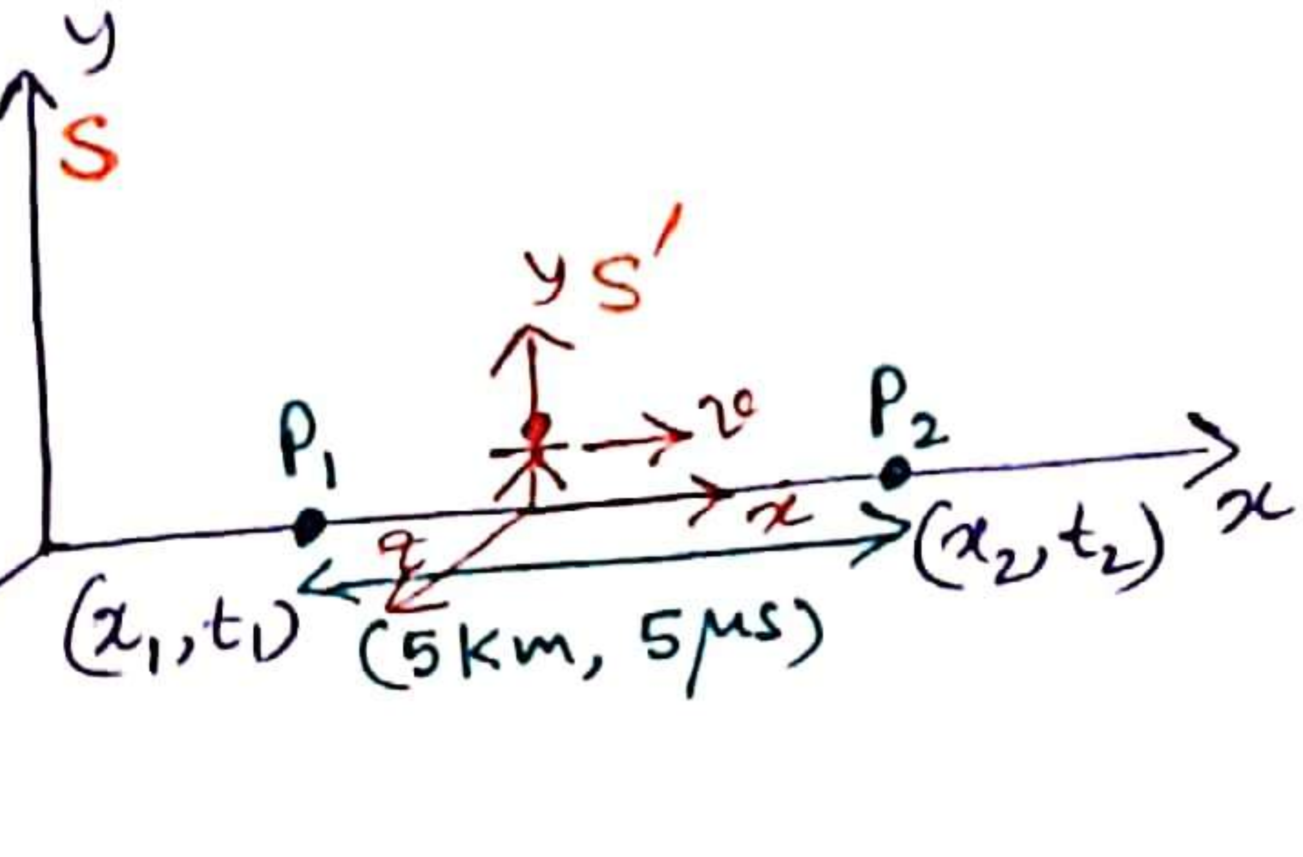


$$\Rightarrow l = l_0\sqrt{1 - v^2/c^2} = 1\sqrt{1 - 0.8^2} = 0.6 \text{ metre.}$$ So the midpoint of the scale appears to be $l/2 = 0.3$ m, because $t' = 0$.

∴ Distance traveled by the light coming from midpoint of the scale $L = \frac{1}{2}\text{m} + 0.3\text{m} = 0.8$ m due to velocity $v = 0.8c$, so

$$\text{time measured by observer} = \frac{L}{v} = \frac{0.8}{0.8c} = \frac{1}{3 \times 10^8} = 0.33 \times 10^{-8} \text{ sec.}$$

2.(c) Let two sources P_1 and P_2 at space time point (x_1, t_1) & (x_2, t_2) which are 5 Km & 5 μs apart in an inertial frame S sends signal.



$\therefore x_2 - x_1 = 5\text{ km},\ t_2 - t_1 = 5\,\mu s.$

For the moving observer in S' frame, using L.T. we have

$x_2' = \gamma(x_2 - vt_2),\quad x_1' = \gamma(x_1 - vt_1)$

$t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right),\quad t_1' = \gamma\left(t_1 - \frac{vx_1}{c^2}\right).$ While in S' frame the signals are simultaneous, so $t_1' = t_2'$

$\therefore \gamma\left(t_2 - \frac{vx_2}{c^2}\right) = \gamma\left(t_1 - \frac{vx_1}{c^2}\right) \quad \Rightarrow t_2 - t_1 = \frac{v}{c^2}(x_2 - x_1)$

$\Rightarrow v = \frac{c^2(t_2 - t_1)}{x_2 - x_1} = \frac{(3\times10^8)^2 \times 5\times10^{-6}}{5\times10^3}\text{ m/s} = \underline{\underline{9\times10^7\text{ m/s}}}.$

(3) (a) Suppose m_0 is the rest mass of the particle and v is the velocity at which its mass increases by $f\%$.

So if $m_0 = 100$ unit then $m = (100+f)$ unit.

From the relativistic mass formula, we know $m = \gamma m_0$

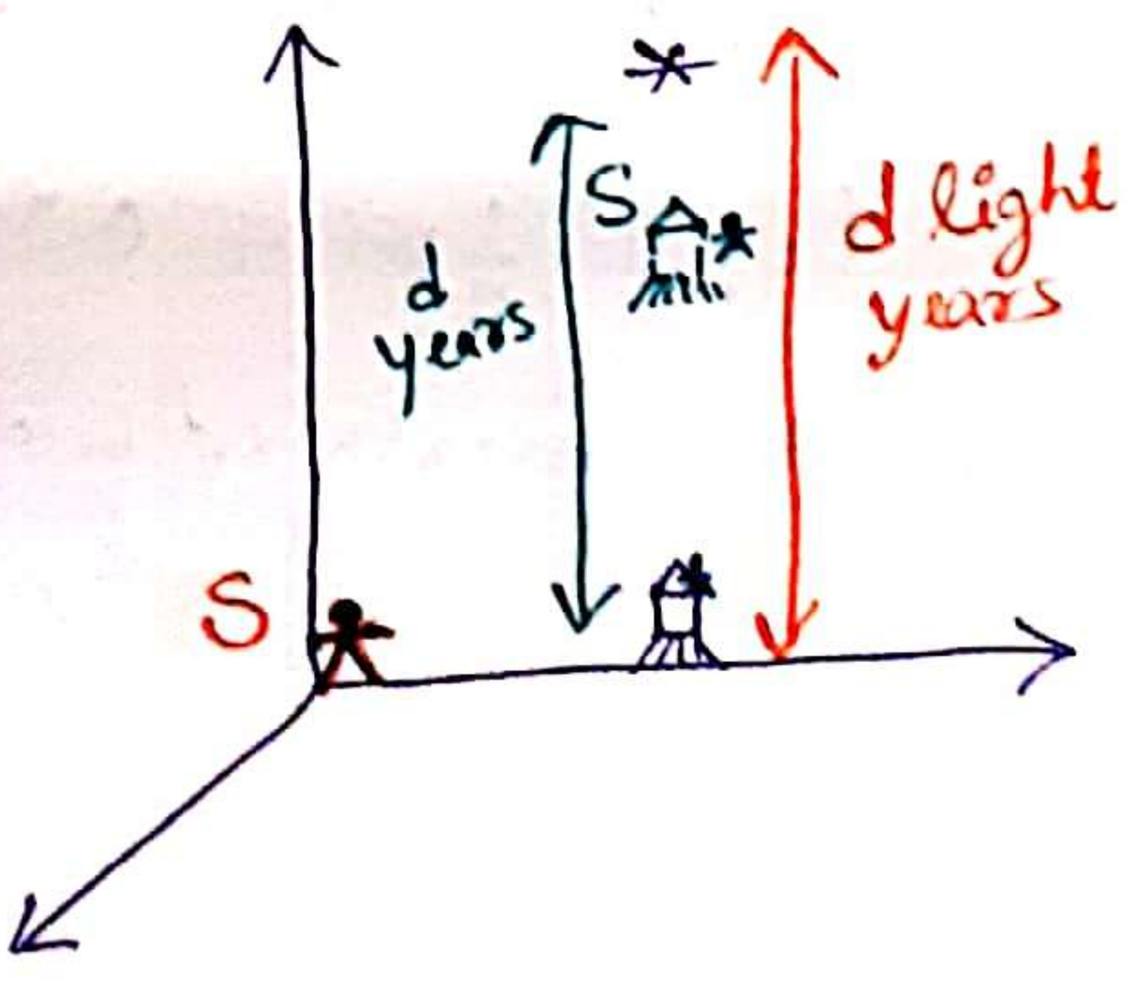
$\Rightarrow \frac{m_0}{m} = \sqrt{1 - v^2/c^2} \quad \Rightarrow \left(\frac{100}{100+f}\right)^2 = 1 - \frac{v^2}{c^2}$

$\Rightarrow v^2 = c^2\left[1 - \left(\frac{100}{100+f}\right)^2\right] = \left(1 + \frac{100}{100+f}\right)\left(1 - \frac{100}{100+f}\right)c^2$

$= \left(\frac{200+f}{100+f}\right)\left(\frac{f}{100+f}\right)c^2$

$\therefore v = \underline{\frac{\sqrt{(200+f)f}}{100+f}\,c}.$

(b) Suppose the rocket is moving with velocity v. If there are K seconds in a year then to Earth observer, distance of star = d light years = dcK metre



Time taken as measured in Earth's frame $= \frac{dck}{v}$ sec.

In rocket's frame, time measured $= d$ years $= dk$ sec. But in this frame, end occur at the same point, so $d\bar{x}' = 0$.

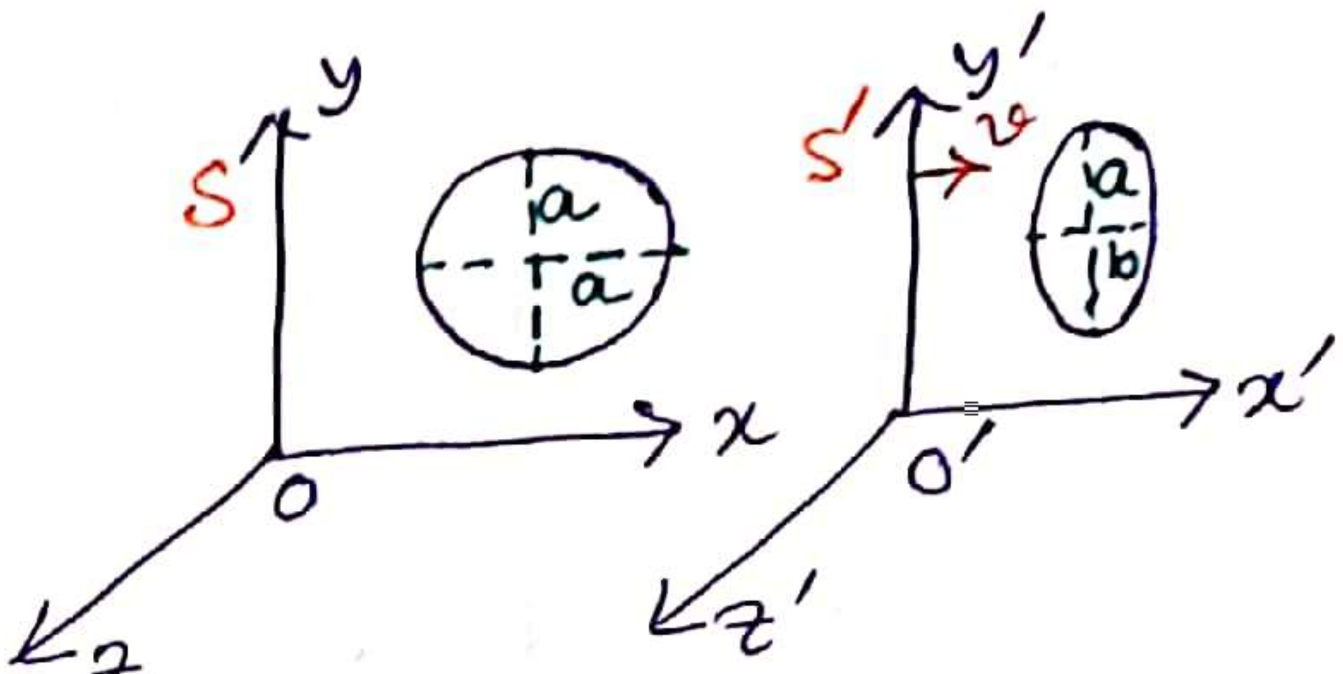
while the space-time interval is a Lorentz invariant, so

$$c^2t^2 - \bar{x}^2 = c^2t'^2 - \bar{x}'^2$$

$$\Rightarrow \left(c\frac{dck}{v}\right)^2 - (dck)^2 = (cdk)^2 \Rightarrow \frac{c^2}{v^2} = 2 \Rightarrow v = \frac{c}{\sqrt{2}}.$$

∴ The speed of spaceship relative to Earth is $c/\sqrt{2}$.

(c) In S-frame, equation for circle is



$x^2 + y^2 = a^2$, meaning the circle disects at $(a, 0)$ and $(0, a)$ at x-y axis.

while in S' frame due to length contraction, the circle disects x-axis at $(\frac{a}{\gamma}, 0)$ but y-axis disection remain unchanged at $(0, a)$.

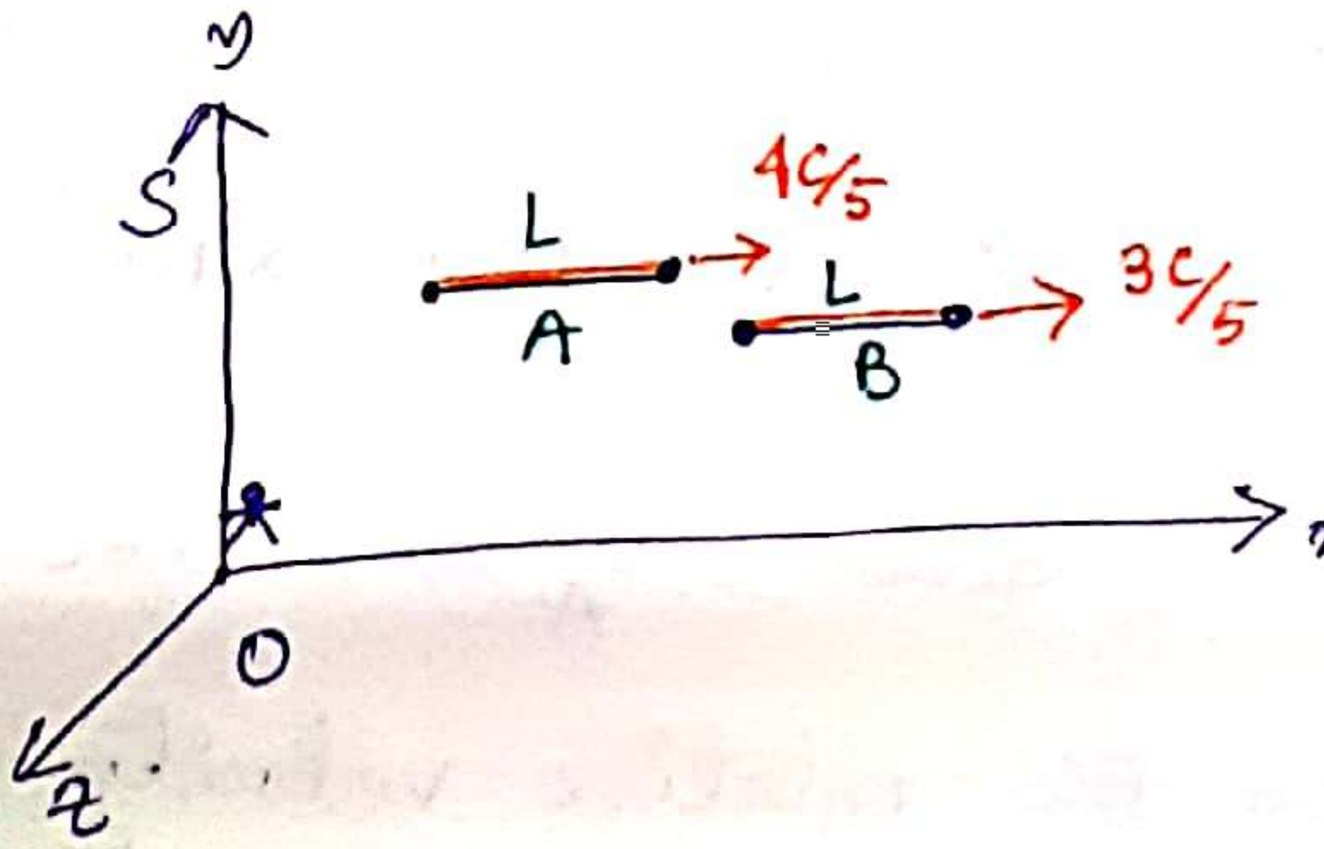
So equation of circle in S' frame is $\frac{x^2}{a^2/\gamma^2} + \frac{y^2}{a^2} = 1$

$\Rightarrow \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ which is equation of an ellipse.

Eccentricity of the ellipse $e = \sqrt{1 - b^2/a^2} = \sqrt{1 - \frac{a^2}{\gamma^2}\frac{1}{a^2}} = \sqrt{1 - \frac{1}{\gamma^2}}$

$$= \sqrt{1 - (1 - v^2/c^2)} = \frac{v}{c}.$$

(4) (a) Given, speed of train A $= \frac{4c}{5}$



and that of B $= \frac{3c}{5}$ & they move as depicted beside.

$$\therefore \gamma_A = \frac{1}{\sqrt{1 - v_A^2/c^2}} = \frac{1}{\sqrt{1 - 16/25}} = \frac{5}{3} \ \&$$

$$\gamma_B = \frac{1}{\sqrt{1 - v_B^2/c^2}} = \frac{1}{\sqrt{1 - 9/25}} = \frac{5}{4}.$$

According to LT, Length contraction seen by observer at S-frame will be, Length of train A $= \frac{L}{\gamma_A} = \frac{3}{5}L$, Length of train B $= \frac{L}{\gamma_B} = \frac{4}{5}L$.

If time taken for A to overtake B be t then it has to travel a total distance $= \frac{3}{5}L + \frac{4}{5}L = \frac{7L}{5}$

$$\therefore \frac{4}{5}ct - \frac{3}{5}ct = \frac{7L}{5}$$

$$\therefore \frac{ct}{5} = \frac{7L}{5} \quad \therefore t = \underline{\underline{\frac{7L}{c}}} \text{ unit.}$$

This much time is required for the overtake to happen.

(b) Lifetime of the muon at rest $t = 2 \times 10^{-6}$ s.

velocity of travel $= \frac{3}{5}c$. $\therefore \gamma' = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-9/25}} = \frac{5}{4}$.

Due to time dilation, lifetime of the travelling muon will be

$$t' = \gamma' t = \frac{5}{4} \times 2 \times 10^{-6} \text{ s} = \underline{2.5 \times 10^{-6} \text{ s}}.$$

(c) Half-life of the pions at rest $t_{1/2} = 1.77 \times 10^{-8}$ sec

Velocity of the collimated pion beam leaving the accelerator target $= 0.99c$. $\therefore \gamma' = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0.99^2}} = 7.089$

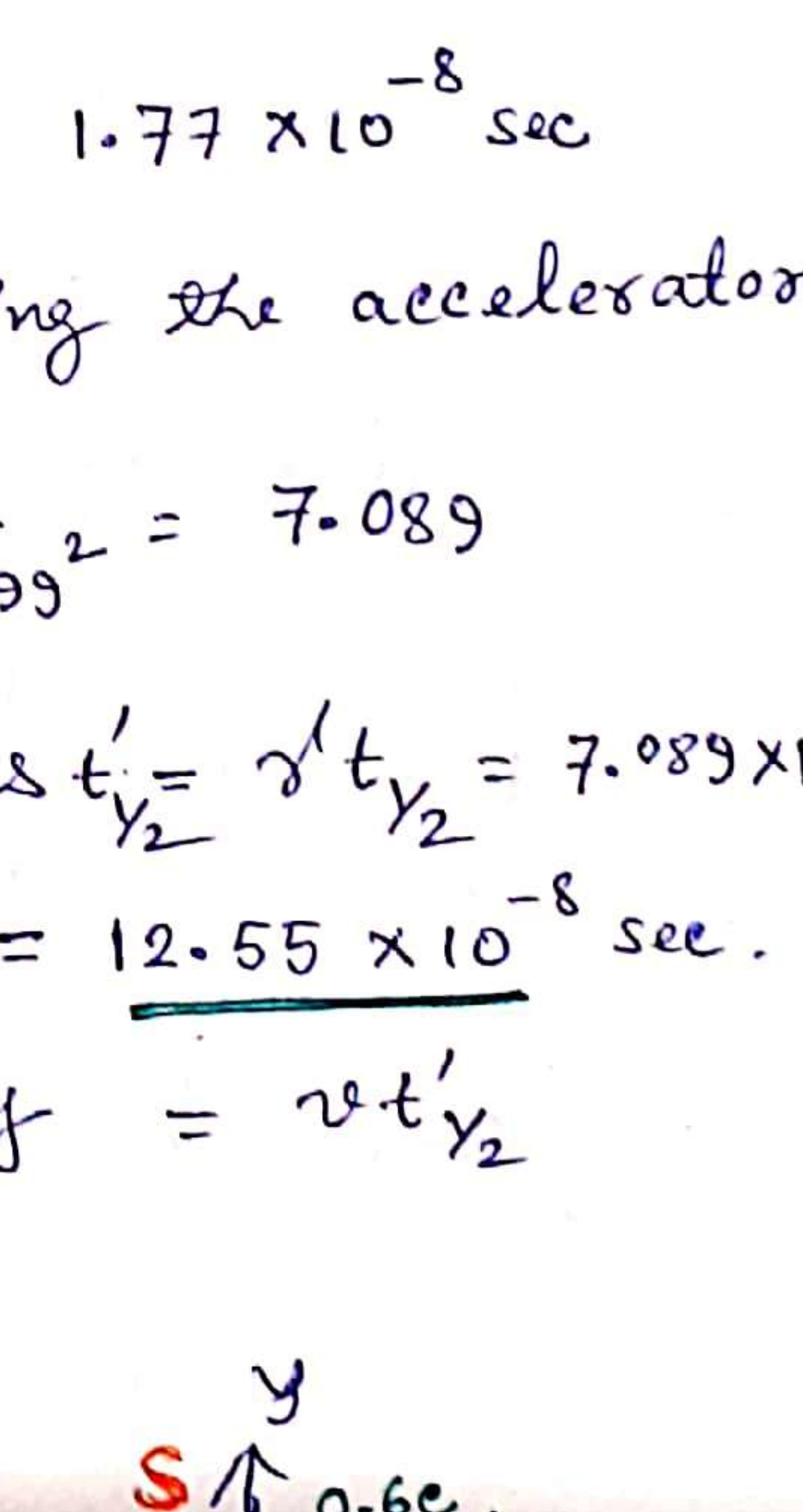
$\therefore$ Half-life of the travelling beam of pions $t'_{1/2} = \gamma' t_{1/2} = 7.089 \times 1.77 \times 10^{-8}$

$= \underline{12.55 \times 10^{-8}}$ sec.

Distance traveled by them in the laboratory $= v t'_{1/2}$

$= 0.99c \times 12.55 \times 10^{-8}$ m $= \underline{37.274 \text{ m}}$.

(5) (b) Given $v_A = -0.6c$ and $v_B = 0.6c$



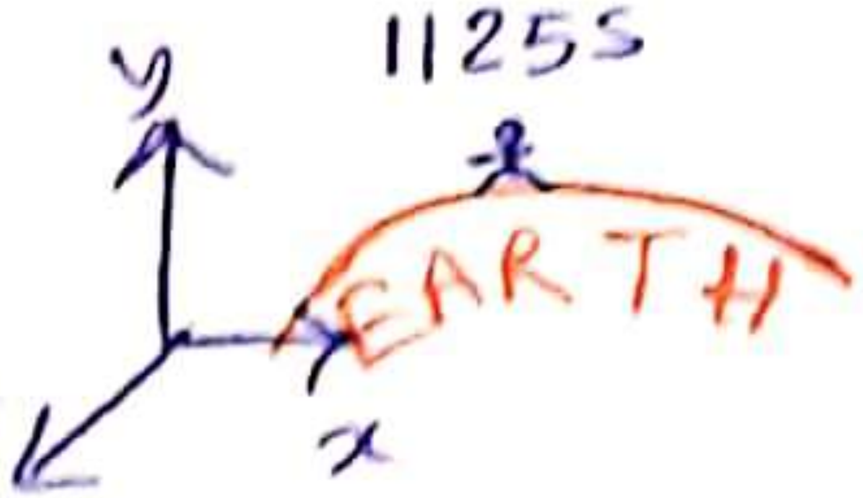
So the relative velocity of particle B w.r.t. particle A is

$$v_{AB} = \frac{v_A - v_B}{1 - \frac{v_A v_B}{c^2}} = \frac{-0.6c - 0.6c}{1 + (0.6)^2}$$

$$= -\frac{1.2c}{1.36} = \underline{-0.88c}.$$

Similarly, $v_{BA} = \underline{0.88c}$.

(c) We know, velocity of radio signal = c and it took 1125 sec to reach human observer on Earth, so distance of spaceship from Earth is 1125 c metre.



∴ According to Earth's observer, time taken for spaceship to return $t = \frac{1125c}{0.5c} = \underline{2250 \text{ sec}}$.

Now as velocity of spaceship = 0.5c, so $\gamma = \frac{1}{\sqrt{1-0.5^2}} = 1.155$.

So for the crew of spaceship this time measured will be the proper time t_0 (the time dilated measurements are done by Earth's observer).
$$t_0 = \frac{t}{\gamma} = \frac{2250}{1.155} = \underline{1948.6 \text{ sec}}.$$