Assignment - III (Poincarian Relativity)

1. (a) Transformation rule from $S(x, y, z)$ to $S'(x', y', z')$ is $x' = -x$, $y' = -y$, $z' = -z$. It is also given that A & B are two contravariant vector that transform according to the aforesaid prescription.

So, $A'^{\alpha} = \dfrac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} = -A^{\alpha} \Rightarrow \dfrac{\partial x'^{\alpha}}{\partial x^{\beta}} = -\delta^{\alpha}_{\beta}$ &

$B'^{\delta} = \dfrac{\partial x'^{\delta}}{\partial x^{\gamma}} A^{\gamma} = -A^{\gamma} \Rightarrow \dfrac{\partial x'^{\delta}}{\partial x^{\gamma}} = -\delta^{\delta}_{\gamma}$. This also means that covariant components change sign under transformation because one has to perform with contraction with $g_{\alpha\beta}$.

$A'_{\alpha} = A_{\alpha} \Rightarrow \dfrac{\partial x^{\beta}}{\partial x'^{\alpha}} = \delta^{\beta}_{\alpha}$, $B'_{\delta} = B_{\delta} \Rightarrow \dfrac{\partial x^{\gamma}}{\partial x'^{\delta}} = \delta^{\gamma}_{\delta}$.

So $\vec{C}' = \vec{A}' \times \vec{B}'$ in component form reads

$C'^{\alpha} = \epsilon^{\alpha\beta\gamma} A'_{\beta} B'_{\gamma} = \epsilon^{\alpha\beta\gamma} \delta^{\sigma}_{\beta} A_{\sigma} \delta^{\lambda}_{\gamma} B_{\lambda} = \epsilon^{\alpha\beta\gamma} A_{\beta} B_{\gamma} = C^{\alpha}$

So $\underline{c'^{\alpha} \neq -c^{\alpha}}$ and hence the cross product cannot be reckoned as a vector under this particular transformation rule.

(b) From transformation rules of tensors of different rank, we have $A'^{ij}_{K} = \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^K} A^{\alpha\beta}_{\gamma}$ and

$B'^{p}_{r} = \frac{\partial x'^p}{\partial x^\delta}\frac{\partial x^\sigma}{\partial x'^r} B^{\delta}_{\sigma}$ has to be satisfied.

Inner product of above $= A'^{ij}_{K} B'^{p}_{r} \delta^{p}_{K} = A'^{ij}_{K} B'^{K}_{r}$

$$= \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}\underbrace{\frac{\partial x^\gamma}{\partial x'^K}\frac{\partial x'^p}{\partial x^\delta}}\frac{\partial x^\sigma}{\partial x'^r} A^{\alpha\beta}_{\gamma} B^{\delta}_{\sigma} \delta^{p}_{K}$$

$$= \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}\underbrace{\frac{\partial x^\gamma}{\partial x'^K}\frac{\partial x'^K}{\partial x^\delta}}_{=\frac{\partial x^\gamma}{\partial x^\delta}}\frac{\partial x^\sigma}{\partial x'^r} A^{\alpha\beta}_{\gamma} B^{\delta}_{\sigma} = \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}\delta^{\gamma}_{\delta}\frac{\partial x^\sigma}{\partial x'^r} A^{\alpha\beta}_{\gamma} B^{\delta}_{\sigma}$$

$$= \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}\frac{\partial x^\sigma}{\partial x'^r} A^{\alpha\beta}_{\gamma} B^{\gamma}_{\sigma}.$$ After contracting $\alpha, \beta, \gamma, \sigma$ indices we are left with i, j, r indices and it transforms as a $\underline{\text{3rd rank}}$ mixed tensor with ij contravariant & r covariant indices.

(c) Let A^j denote the contravariant components in rectangular coordinates $x^1 = x$, $x^2 = y$ and $x^3 = z$ so that

$$A^1 = xy = x^1x^2, \quad A^2 = 2y - z^2 = 2x^2 - (x^3)^2, \quad A^3 = xz = x^1x^3.$$

Now $ds^2 = dx^2 + dy^2 + dz^2$, so $g_{11} = g_{22} = g_{33} = 1$, g (off diagonal) $= 0$.

$$\therefore g_{ij} = g^{ij} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$ and so the covariant components of A are $A_1 = g_{11}A^1 = x^1x^2$, $A_2 = g_{22}A^2 = 2x^2 - (x^3)^2$, $A_3 = g_{33}A^3 = x^1x^3$.

Let A'_K denote the covariant components in cylindrical coordinates $x'^1 = \rho$, $x'^2 = \phi$, $x'^3 = z$. Then $A'_K = \frac{\partial x^j}{\partial x'^K} A_j$.

The transformation equations between coordinate systems are

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$ translates as

$x^1 = x^{1\prime}\cos x^{2\prime}$, $x^2 = x^{1\prime}\sin x^{2\prime}$, $x^3 = x^{3\prime}$

$\therefore\ A_1' = \frac{\partial x^1}{\partial x^{1\prime}}A_1 + \frac{\partial x^2}{\partial x^{1\prime}}A_2 + \frac{\partial x^3}{\partial x^{1\prime}}A_3 = (\cos x^{2\prime})(x_1 x_2) + (\sin x^{2\prime})[2x^2 - (x^3)^2] + 0 = (\cos\phi)(\rho\cos\phi\,\rho\sin\phi) + (\sin\phi)[2\rho^2\cos^2\phi - z^2]$

$= \underline{3\rho^2\sin\phi\cos^2\phi - z^2\sin\phi}$

$A_2' = \frac{\partial x^1}{\partial x^{2\prime}}A_1 + \frac{\partial x^2}{\partial x^{2\prime}}A_2 + \frac{\partial x^3}{\partial x^{2\prime}}A_3 = (-x^{1\prime}\sin x^{2\prime})(\rho^2\sin\phi\cos\phi) + (x^{1\prime}\cos x^{2\prime})(2\rho^2\cos^2\phi - z^2) + 0 = -\rho\sin\phi(\rho^2\sin\phi\cos\phi) + \rho\cos\phi(2\rho^2\cos^2\phi - z^2)$

$= \underline{\rho^3\cos\phi(2\cos^2\phi - \sin^2\phi) - \rho z^2\cos\phi}.$

$A_3' = \frac{\partial x^1}{\partial x^{3\prime}}A_1 + \frac{\partial x^2}{\partial x^{3\prime}}A_2 + \frac{\partial x^3}{\partial x^{3\prime}}A_3 = 0 + 0 + A_3 = x^1x^3 = xz$

$= \underline{\rho z\cos\phi}.$

These are the required covariant components. [Note that getting contravariant components needs contraction with g in cylindrical coordinates where $g_{ij} \neq g^{ij}$, so you've to calculate that accordingly before contracting].

(d) If we assume the transformation of coordinates $x^{j\prime} = x^{j\prime}(x^1, x^2, \dots, x^N)$ where $x^1, \dots, x^N$ are a set of N coordinates, then $dx^{j\prime} = \frac{\partial x^{j\prime}}{\partial x^k}dx^k$ has to be true or dx^k is a contravariant vector. This can be seen also if we take any contravariant vector A^j with transformation rule $A^{j\prime} = \frac{\partial x^{j\prime}}{\partial x^k}A^k$ and then identify A^j with dx^j. While in three dimension dx^j corresponds to differential distance vector $d\vec{r}$, so $d\vec{r}$ transform like a contravariant vector.

The gradient of a scalar $\vec{\nabla}\phi$ in component form $\frac{\partial\phi}{\partial x_i}$ with $\phi(x^1, \dots, x^N) = \phi'(x^{1\prime}, \dots, x^{N\prime})$, so that the transformation rule is

$$\frac{\partial \phi'}{\partial x^{j'}} = \frac{\partial \phi}{\partial x^{j'}} = \frac{\partial x^k}{\partial x^{j'}} \frac{\partial \phi}{\partial x^k} . \quad \text{so } \frac{\partial \phi}{\partial x^k} \text{ transforms as } A'_j = \frac{\partial x^k}{\partial x^{j'}} A_k$$

which is a covariant vector.

(2)(a) By definition $A_j = g_{jk} A^k$, $A^k = g^{jk} A_j$, so that substituting one into another gives $A_j = g_{jk} g^{jk} A_j$ which can be true if $g_{jk} g^{jk} = \mathbb{I}$ is identity matrix of dimension A.

This implies $\boxed{g^{jk} = (g_{jk})^{-1}}$

In $1+2$ dimensional flat spacetime, line element is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{so that } g_{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}.$$

$$\det(g_{\mu\nu}) = 1 \quad \& \quad \operatorname{adj}(g_{\mu\nu}) = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}.$$

$$\therefore g^{jk} = g^{-1}_{jk} = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$$

(b) The coordinate system (a,b,c) is related to the Cartesian coordinate system via $x = bc$, $y = ca$, $z = ab$. and the metric tensor transformation is $g'_{ij} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^{i'}} \frac{\partial x^\beta}{\partial x^{j'}}$ where g'_{ij} is that of (a,b,c) system & $g_{\alpha\beta}$ that of cartesian system, which is $g_{\alpha\beta} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ since

$$dx^2 + dy^2 + dz^2 = ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta .$$

Now
$$g'_{aa} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial a} \frac{\partial x^\beta}{\partial a} = \underset{=0}{\frac{\partial x}{\partial a} \frac{\partial x}{\partial a}} + \underset{=c}{\frac{\partial y}{\partial a}} \underset{=c}{\frac{\partial y}{\partial a}} + \underset{=b}{\frac{\partial z}{\partial a}} \underset{=b}{\frac{\partial z}{\partial a}} = c^2 + b^2$$

$$g'_{ab} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial a} \frac{\partial x^\beta}{\partial b} = \underset{=0}{\frac{\partial x}{\partial a} \frac{\partial x}{\partial b}} + \underset{=c}{\frac{\partial y}{\partial a}} \underset{=0}{\frac{\partial y}{\partial b}} + \underset{=b}{\frac{\partial z}{\partial a}} \underset{=a}{\frac{\partial z}{\partial b}} = ab = g_{ba}$$

$$g'_{ac} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial a} \frac{\partial x^\beta}{\partial c} = \underset{=0}{\frac{\partial x}{\partial a}} \underset{=b}{\frac{\partial x}{\partial c}} + \underset{=c}{\frac{\partial y}{\partial a}} \underset{=a}{\frac{\partial y}{\partial c}} + \underset{=b}{\frac{\partial z}{\partial a}} \underset{=0}{\frac{\partial z}{\partial c}} = ac = g_{ca}$$

$$g'_{bb} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial b} \frac{\partial x^\beta}{\partial b} = \underset{=c}{\frac{\partial x}{\partial b}} \underset{=c}{\frac{\partial x}{\partial b}} + \underset{=0}{\frac{\partial y}{\partial b}} \underset{=0}{\frac{\partial y}{\partial b}} + \underset{=a}{\frac{\partial z}{\partial b}} \underset{=a}{\frac{\partial z}{\partial b}} = a^2 + c^2$$

$$g'_{bc} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial b} \frac{\partial x^\beta}{\partial c} = \underset{=c}{\frac{\partial x}{\partial b}} \underset{=b}{\frac{\partial x}{\partial c}} + \underset{=0}{\frac{\partial y}{\partial b}} \underset{=a}{\frac{\partial y}{\partial c}} + \underset{=a}{\frac{\partial z}{\partial b}} \underset{=0}{\frac{\partial z}{\partial c}} = bc = g_{cb}$$

$$g'_{cc} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial c}\frac{\partial x^\beta}{\partial c} = \underbrace{\frac{\partial x}{\partial c}}_{b}\underbrace{\frac{\partial x}{\partial c}}_{b} + \underbrace{\frac{\partial y}{\partial c}}_{a}\underbrace{\frac{\partial y}{\partial c}}_{a} + \underbrace{\frac{\partial z}{\partial c}}_{0}\underbrace{\frac{\partial z}{\partial c}}_{0} = a^2+b^2$$

$$\therefore g'_{ij} = \begin{pmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{pmatrix}$$

(c) Given the square of elementary length $ds^2 = g_{ij}\,dx^i dx^j = (dx^1)^2 + (dx^2)^2 + 4(dx^1)(dx^2)$, we find $g_{11} = 1$, $g_{22} = 1$, $g_{12} = 2$ & $g_{21} = 2$ $\therefore g_{ij} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. While $g_{ij}g^{ij} = \mathbb{I}_{2\times 2}$ so

$$g^{ij} = g_{ij}^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}.$$

(3)(a) A Lorentz transformation is defined as $A^{i'} = \Lambda^i_j A^j$ where Λ^i_j satisfies $\Lambda^\gamma_\mu g_{\gamma\delta}\Lambda^\delta_\nu = g_{\mu\nu}$ which in matrix form reads

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\frac{(\gamma-1)\beta_1^2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1+\frac{(\gamma-1)\beta_2^2}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1+\frac{(\gamma-1)\beta_3^2}{\beta^2} \end{pmatrix}, \quad \beta_i = \frac{v_i}{c}, \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

(symbols have their usual meaning.)

Now $A'_\nu B^{\nu'} = A^{\mu'} g_{\mu\nu} B^{\nu'} = \Lambda^\mu_\alpha A^\alpha g_{\mu\nu}\Lambda^\nu_\beta B^\beta = A^\alpha \underbrace{\Lambda^\mu_\alpha g_{\mu\nu}\Lambda^\nu_\beta}\, B^\beta = A^\alpha g_{\alpha\beta} B^\beta = A_\beta B^\beta = A_\nu B^\nu$. So the scalar product of two 4-vectors $A_\nu B^\nu$ is invariant under LT.

Similarly for $A'_{\mu\nu} B^{\mu\nu'} = A^{\alpha\beta'} g_{\mu\alpha} g_{\nu\beta} B^{\mu\nu'} =$

$$= A^{ij}\Lambda^\alpha_i \Lambda^\beta_j g_{\mu\alpha} g_{\nu\beta}\Lambda^\mu_k \Lambda^\nu_l B^{kl}$$

$$= A^{ij}\underbrace{\Lambda^\alpha_i g_{\mu\alpha}\Lambda^\mu_k}_{g_{ik}}\underbrace{\Lambda^\beta_j g_{\nu\beta}\Lambda^\nu_l}_{g_{jl}} B^{kl} = A^{ij} g_{ik} g_{jl} B^{kl} = A_{kl}B^{kl} = A_{\mu\nu}B^{\mu\nu}$$

Note that matrix operation with g is commutative solely because of its diagonal nature.

Similarly lowering the other two indices it is easy to show that $A^{\mu\nu} B_{\mu\nu}$ is LT invariant.

(b) In the Minkowski spacetime the time-like 4-vector is $u^\mu = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$

On contrary to proof, let $v^\mu = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be another time like 4-vector.

This means $u^\mu u_\mu > 0$ and $v^\mu v_\mu > 0$. or,

$$u^\mu g_{\mu\nu} u^\nu = u_0^2 - u_1^2 - u_2^2 - u_3^2 > 0. \; \& \quad \text{—} \; (1)$$

$$v^\mu g_{\mu\nu} v^\nu = v_0^2 - v_1^2 - v_2^2 - v_3^2 > 0.$$

Now if these vectors have then

to satisfy $u^\mu v_\mu = 0 = u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3$,

$$u_0 v_0 = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Squaring both sides, we have, $u_0^2 v_0^2 = u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 + 2u_1 v_1 u_2 v_2 + 2u_2 v_2 u_3 v_3 + 2u_1 v_1 u_3 v_3$

Since, u, v are time-like 4-vectors, we have $u_i^2 + v_i^2 \geq 2u_i v_i$

$$\therefore \; u_2^2 v_3^2 + u_3^2 v_2^2 \geq 2u_2 v_2 u_3 v_3.$$

$$\therefore \; u_0^2 v_0^2 \leq u_2^2 v_2^2 + u_3^2 v_3^2 + u_1^2 v_1^2 + u_1^2 v_2^2 + u_2^2 v_1^2 + u_2^2 v_3^2 + u_3^2 v_2^2 + u_1^2 v_3^2 + u_3^2 v_1^2 \quad \text{—} \; (2)$$

from (1) we have $u_0^2 v_0^2 > (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$

$$\text{or} \; u_0^2 v_0^2 > u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 + u_1^2 v_2^2 + u_2^2 v_1^2 + u_2^2 v_3^2 + u_3^2 v_2^2 + u_1^2 v_3^2 + u_3^2 v_1^2 \quad \text{—} \; (3)$$

While (2) & (3) both cannot be true, so the initial assumption on v^μ being time like vector is incorrect and both equal & greater than equal cannot be simultaneously true, so it cannot be light like. $\therefore$ v^μ has to be space like to have $u^\mu v_\mu = 0$.

Another way to prove this is if u^μ is time like then under a LT, $u^\nu = \Lambda^\nu{}_\mu u^\mu = (u^0, \vec{0})$ and if v^μ is spacelike then under a LT, $v^\nu = \Lambda^\nu{}_\mu v^\mu = (0, \vec{v})$, so that $u^\mu v_\mu = 0$.

We know that 4-momentum is $p^\mu = (p^0, p^i)$ and for photon $p_\mu p^\mu = 0$ (light like) $= \left(\frac{E}{c}, \vec{p}\right)$

$$\therefore -|p|^2 + E^2/c^2 = 0 \quad \text{or} \quad |p| = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar K.$$

This is the linear momentum of the photon.