PHSA CC-1-2 TH MECHANICS: Non-Inertial Systems

Instructor : AKB

Books : 1. An Introduction to Mechanics → Kleppner/Kolenkow (Tata McGraw Hill). ⇒ Good for problem solving

2. Theoretical Mechanics → M.R. Spiegel (Schaum Series) ⇒ Good to learn solved problems & for solving problems.

3. Feynman lectures on Physics (vol. 1) → Feynman/Leigton/Sands (Narosa) ⇒ Good for concept building from not so conventional thinking.

4. Berkeley Physics Course (vol 1) → Kittel/Knight/Ruderman/Helmholtz/Moyer (Tata Mc Grow Hill) ⇒ Very good book for concept development.

5. Fundamentals on Physics → Halliday/Resnick/Walker (John Wiley & Sons) ⇒ Less theoretic, more application oriented, good for practical knowledge.

Newton's law & inertial systems (recapitulation) ⇒

→ Describe the behaviour of point masses (where size of the body is small compared with the interaction distance)

→ Applies to particulate system and not suitable for continuous medium like fluid.

→ Interaction between two charged objects violates Newton's 3rd law as the interaction produced by the created electric fields is not instantaneously transmitted but propagates at the speed of light $c \sim 3\times10^8$ m/sec. Within the propagation time, violation occurs

1st law : $\vec{a} = 0$ when $\vec{F} = 0$

2nd law : $\vec{F} = m\vec{a}$, if $\frac{dm}{dt} = 0$ $(v \ll c)$

3rd law : $\vec{F}_{12} = -\vec{F}_{21}$ [unit $1N = 10^3 gm \times 10^2 cm/s^2 = 10^5$ dyn]

Newton's laws hold true (1st & 2nd law) only when observed in inertial reference frame, in which a body devoid of a force or torque is not accelerating, either at rest or moving at a constant speed. But suppose, if the reference frame is at rest on a rotating merry-go-round, one doesn't have zero acceleration in the absence of applied forces. One can stand still on the merry-go-round only by pushing some part or causing that part to exert a force $m\omega^2 r$ on someone toward the axis of rotation, $\omega$ = angular acceleration. Or suppose the reference frame is at rest in an aircraft that accelerates rapidly during takeoff, where someone is pressed back against the seat by the acceleration & someone is at rest relative to the airplane by the force exerted on someone by the back of the seat.

<u>Example</u>: <u>Ultracentrifuge</u>: Moving out of inertial frame of reference have enormous effect on practical applications, e.g. to increase acceleration of a molecule suspended in a liquid compared to acceleration due to gravity. $g$.

If the molecule rotates at 10 cm from the axis of rotation with 1000 revolutions/sec or $6 \times 10^4$ rpm, then angular velocity

$$\omega = 2\pi \times 10^3 \simeq 6 \times 10^3 \text{ rad/sec.}$$ & linear velocity

$$v = \omega r \simeq 6 \times 10^3 \times 10 \simeq 6 \times 10^4 \text{ cm/s}$$

$$a = \omega^2 r \simeq (6 \times 10^4)^2 \times 10 \simeq 4 \times 10^8 \text{ cm/s}^2, \quad g = 980 \text{ cm/s}^2$$

$$\therefore \frac{a}{g} \simeq \frac{4 \times 10^8}{980} \simeq 4 \times 10^5.$$ Due to such high acceleration, molecules having density different from surrounding fluid will

see a strong force to separate out from the fluid.

To a fixed frame (laboratory), molecule wants to remain at rest or move with constant speed in straight line & not dragged with high $\omega$. So to an observer at rest in the ultracentrifuge, molecule is exerted a "centrifugal" force $m\omega^2 r$ to pull it away from the axis of rotation.

If $m = 10^5 \times$ mass of proton $= 10^5 \times 1.7 \times 10^{-24} \simeq 2 \times 10^{-19}$ gm

then $F = ma = m\omega^2 r \simeq 2 \times 10^{-19} \times 4 \times 10^8 \simeq 8 \times 10^{-11}$ dyne.

Centrifugal force outward is balanced by the drag force by the surrounding liquid on the molecule. Due to density differance there will be stratification of layer, so that in the reference frame of the ultracentrifuge, centrifugal force is like an artificial gravity directed outward with increasing intensity with distance from axis.

Force measured in inertial frame is called true force. The Earth as a reference (inertial) frame is a good approximation, but not completely. A mass at rest on Earth surface at the equator experiences a centripetal acceleration $a = \frac{v^2}{R_e} = \omega_e^2 R_e$

Now $\omega_e = 2\pi f_e = \frac{2\pi}{T_e} = \frac{2\pi}{1 \text{ day}} = \frac{2\pi}{8.64 \times 10^4} = 7.3 \times 10^{-5}$ sec$^{-1}$

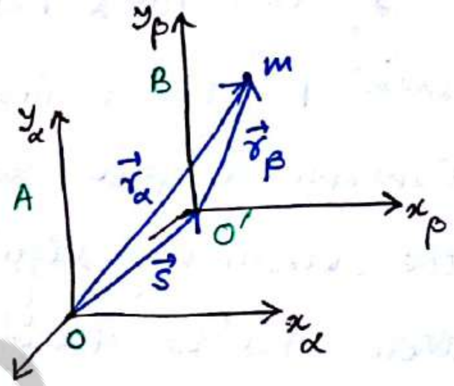
with $R_e = 6.4 \times 10^8$ cm, $a = (7.3 \times 10^{-5})^2 \times 6.4 \times 10^8 \simeq 3.4$ cm/s$^2$

As this is the force supplied to a point mass at equator, force necessary to hold the mass in equilibrium against gravity is $3.4\,m$ dynes less than that of $mg$. Rest of the variation in $g$ is due to the ellipsoidal shape & pole to equator variation is $5.2$ cm/s$^2$

Since 1 year $\simeq \pi \times 10^{7}$ sec, angular velocity of Earth about the Sun is $\omega \simeq \frac{2\pi}{\pi \times 10^{7}} \simeq 2\times 10^{-7}$ sec$^{-1}$. With $R \simeq 1.5\times 10^{13}$ cm, the centripetal acceleration of Earth about Sun is

$a = \omega^2 R \simeq (2\times 10^{-7})^2 \times 1.5\times 10^{13} \simeq 0.6$ cm/s$^2$ which is one order of magnitude smaller than the acceleration at equator due to the rotation of Earth.

## Galilean Transformation :




Let us consider two frames of reference A & B such that A is at rest & B moves with a constant velocity $\vec{v}$ with respect to A. We want to find the transformation that relates the coordinates $\vec{r}_\alpha$ & time $t_\alpha$ as measured from A frame to the coordinates $\vec{r}_\beta$ & time $t_\beta$ as measured from B. At $t=0$, both O & O' origins coincide. Suppose Newton's law is read on A & B as

$\vec{F}_\alpha = m\vec{a}_\alpha$ , $\vec{F}_\beta = m\vec{a}_\beta$. We know $\vec{F}_\alpha$ is inertial frame measured true force & seek a relation between $\vec{F}_\alpha$ & $\vec{F}_\beta$.

By construction $\vec{s} = \vec{v}t$, if we define a set of transformation

$$\boxed{\vec{r}_\alpha = \vec{r}_\beta + \vec{v}t, \quad t_\alpha = t_\beta}$$

then we see, by differentiation, $\vec{v}_\alpha = \vec{v}_\beta + \vec{v}$ & $\vec{a}_\alpha = \vec{a}_\beta$ as $\frac{d\vec{v}}{dt} = 0$. $\therefore$ $\vec{F}_\beta = m\vec{a}_\beta = m\vec{a}_\alpha = \vec{F}_\alpha$.

So the above set of transformation leads $\vec{F}_\beta$ to be also true force or B frame to be inertial. These are called the Galilean transformation, where axiomatically (without thinking much) we have considered $t_\alpha = t_\beta$ or time is independent of the frame of reference. This is incorrect if $v \approx c$ while $t_\beta = t_\alpha\sqrt{1-\frac{v^2}{c^2}}$

Similarly we assumed same scale is used in A & B for measuring distance, but near $v \approx c$ $L_\beta = L_\alpha\sqrt{1-v^2/c^2}$ which is known in Special theory of Relativity as "Lorentz contraction" of a moving rod. for practical purpose, say velocity of a satellite around Earth is 8 Km/s & so $v^2/c^2 \approx 10^{-9}$.

Similarly moving mass differs from rest mass as $m = m_0/\sqrt{1-v^2/c^2}$

Principle of relativity → laws of physics are same in all inertial systems. In Einstein's relativity, not Galilean but Lorentz transformation is valid.

## Uniformly Accelerating Systems (Non inertial):

Suppose now frame B accelerates at constant rate $\vec{A}$ w.r.t. inertial frame A. We label quantities in noninertial frame B with prime. As $\frac{d\vec{v}}{dt} = \vec{A} \neq 0$ now,

$$\vec{a} = \vec{a}' + \vec{A}$$

So in the accelerated system, the measured (apparent) force is

$$\vec{F}' = m\vec{a}' = m\vec{a} - m\vec{A} = \vec{F} - m\vec{A} = \vec{F}_{true} + \vec{F}_{fict}$$

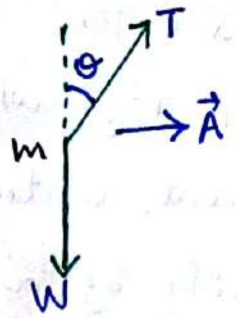
true force as measured in A

fictitious force $-m\vec{A}$

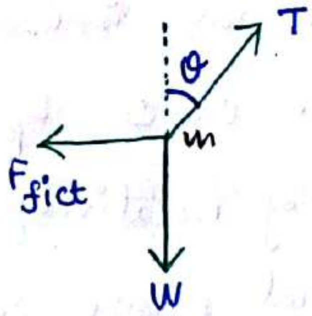
Fictitious force is oppositely directed and proportional to mass (just like Gravitational force). But origin of such force is not physical interaction, but acceleration of the coordinate system.

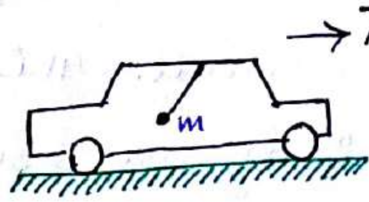
## Apparent force of Gravity

Laboratory frame




Accelerating frame




A small mass m hangs from a string in an accelerating car. Determine the static angle of the string with vertical & tension of the string.

Laboratory frame:

$T\cos\theta = W = mg$

$T\sin\theta = mA$

$\therefore T = m\sqrt{g^2+A^2}$

$\tan\theta = \frac{mA}{mg} = \frac{A}{g}$

Accelerating frame:

$T\cos\theta = W = mg$

$T\sin\theta = F_{fict} = mA$

$T = m\sqrt{g^2+A^2}$

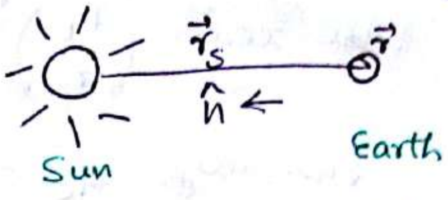
$\tan\theta = \frac{A}{g}$

## The Principle of Equivalence

The laws of physics in a uniformly accelerating system are identical to those in an inertial system after introducing a fictitious force $\vec{F}_{fict} = -m\vec{A}$, so $F_{fict} \propto m$ as gravitational force with $\vec{A} = -\vec{g}$. This two scenarios, one where a particle experiences local gravitational field $\vec{g}$, & where a particle in free space (no $\vec{g}$) uniformly accelerating at rate $\vec{A} = -\vec{g}$ are equivalent, but one cannot clearly distinguish these two scenarios → Mach's principle & Einstein's conjecture.

Real fields are local & at large distance they decrease while an accelerating coordinate system is nonlocal & extends uniformly throughout space. Only for small systems are the two indistinguishable.

## Tidal forces



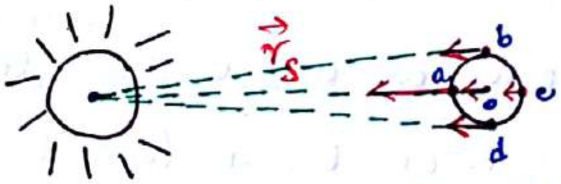

The Earth is in free fall toward the sun & according to "Principle of Equivalence" it should be impossible to observe Sun's gravitational force on Earth locally. Due to massive size, tidal effect (nonlocal) are observed.

Tides arise as Sun & Moon produce an apparent gravitational field that varies from point to point on Earth surface. If Earth accelerates toward the Sun at rate $\vec{G}_o$ then gravitational field of Sun at center of Earth is $\vec{G}_o = \frac{GM_S}{r_S^2}\hat{n}$

If $G(\vec{r})$ is the gravitational field of Sun on Earth surface then $\vec{F} = mG(\vec{r})$, so to an observer on Earth, apparent force is

$\vec{F}' = \vec{F} - m\vec{A} = mG(\vec{r}) - m\vec{G}_o$, so apparent field is

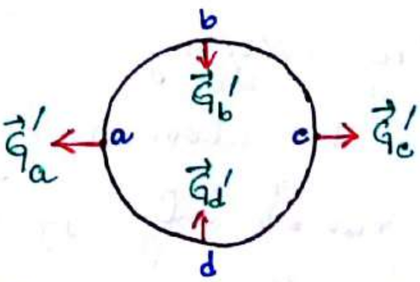
$G'(\vec{r}) = G(\vec{r}) - G_o$




True field $G(\vec{r})$

We notice at 4 points a, b, c, d having true field $\vec{G}_a, \vec{G}_b, \vec{G}_c, \vec{G}_d$ and at center O, $\vec{G}_o$ the following

$\vec{G}_o \approx \vec{G}_b \approx \vec{G}_d$ ; $\vec{G}_a \gg \vec{G}_o \gg \vec{G}_c$




Apparent field $G'(\vec{r}) = \vec{G} - \vec{G}_o$

(i) $\vec{G}_a'$ & $\vec{G}_c'$ :

Sun's field at a is $\vec{G}_a = \frac{GM_S}{(r_S - R_E)^2}$

where $\vec{r}_S - \vec{R}_E$ is distance between center of Sun to a. So apparent field is $\vec{G}_a' = \vec{G}_a - \vec{G}_o = \left[\frac{GM_S}{(r_S - R_E)^2} - \frac{GM_S}{r_S^2}\right]\hat{n} = \hat{n}\frac{GM_S}{r_S^2}\left[\frac{1}{(1-(R_E/r_S))^2} - 1\right]$

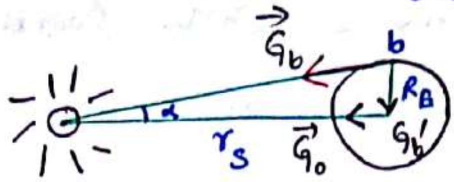
or $G_a' = G_o\left[\left(1 - \frac{R_E}{r_S}\right)^{-2} - 1\right] = G_o\left[\cancel{1} + \frac{2R_E}{r_S} + \dots - \cancel{1}\right] \approx 2G_o\frac{R_E}{r_S}$

All terms $\left(\frac{R_E}{r_s}\right)^n$ for $n \geq 2$ are neglected as $\frac{R_E}{r_s} = \frac{6.4 \times 10^3 \text{ Km}}{1.5 \times 10^8 \text{ Km}}$

Similarly, $G_c' = G_c - G_o = G_o\left[\left(1 + \frac{R_E}{r_s}\right)^{-2} - 1\right]$ $= 4.3 \times 10^{-5} \ll 1$

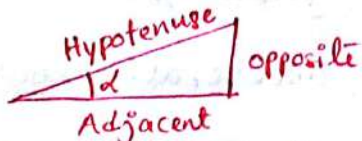
$\simeq -2G_o \frac{R_E}{r_s}$. $\vec{G}_a'$ & $\vec{G}_c'$ therefore point radially out.

(ii) $\vec{G}_b'$ & $\vec{G}_d'$ :




$\vec{G}_b$ & $\vec{G}_o$ are not parallel & angle between them $\alpha \approx \frac{R_E}{r_s} = 4.3 \times 10^{-5} \ll 1$. We Know

$\vec{G}_b = \vec{G}_b' + \vec{G}_o$ form $\perp^r$ triangle.




$\tan\alpha = \frac{G_b'}{G_o} \approx \alpha$ (for $\alpha \ll 1$, $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\alpha - \cancel{\frac{\alpha^3}{3!}} + \cancel{\frac{\alpha^5}{5!}} - \cdots}{1 + \cancel{\frac{\alpha^2}{2!}} - \cancel{\frac{\alpha^4}{4!}} + \cdots} = \alpha$)

$\sin\alpha = \frac{\text{Opposite}}{\text{Hypotenuse}}$

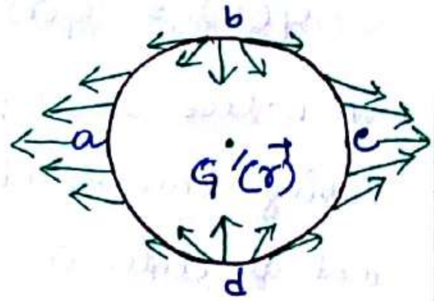
$\cos\alpha = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

$\tan\alpha = \frac{\text{Opposite}}{\text{Adjacent}}$

$\therefore G_b' = G_o\alpha \simeq G_o \frac{R_E}{r_s}$

By symmetry, $G_d'$ is equal & opposite to $G_b'$ & both of them point toward the center of Earth.

Force at a & c tend to lift the oceans & force at b & d tend to depress them. Thus we have 4 tides, 2 ebb & 2 flood of the tides everyday.




Although the above analysis correctly shows 4 tides but this is not the reason only. If we now consider Moon also & then force due to sun & moon is

$F_S = \frac{G M_E M_S}{r_S^2}$ So, $\frac{F_S}{F_M} = \frac{M_S}{M_M} \times \frac{r_M^2}{r_S^2} \sim 176$ as

$F_M = \frac{G M_E M_M}{r_M^2}$ $\left[\frac{r_S}{r_M} = 390,\ \frac{M_S}{M_M} = 2.68 \times 10^7.\right]$

So force due to Sun is 176 times stronger than that of Moon!

If we now consider the differential attraction between a & c point on Earth with hydrosphere (by any arbitrary mass m) due to Moon,

Attraction at a, $F_a = \frac{G M_M m}{(r_M - R_E)^2}$ & at c, $F_c = \frac{G M_M m}{(r_M + R_E)^2}$

So differential attraction $T_M = F_a - F_c$

$$= G M_M m \frac{(r_M + R_E)^2 - (r_M - R_E)^2}{(r_M - R_E)^2 (r_M + R_E)^2} = G M_M m \frac{4 r_M R_E}{(r_M^2 - R_E^2)^2}$$

$$= G M_M m \frac{4 r_M R_E}{r_M^4 \left(1 - \frac{R_E^2}{r_M^2}\right)^2} \simeq G M_M m \frac{4 R_E}{r_M^3} \quad \left(\text{as } \frac{r_E}{r_M} = 390\right)$$

$$\left(\frac{R_E}{r_M} = \frac{1}{60}\right)$$

Similarly $T_S = F_a - F_c \simeq G M_S m \frac{4 R_E}{r_S^3}$ for Sun.

$\therefore \frac{T_M}{T_S} = \frac{M_M}{M_S} \times \frac{r_S^3}{r_M^3} \sim 2.2$. $\therefore T_M = 2.2\, T_S$

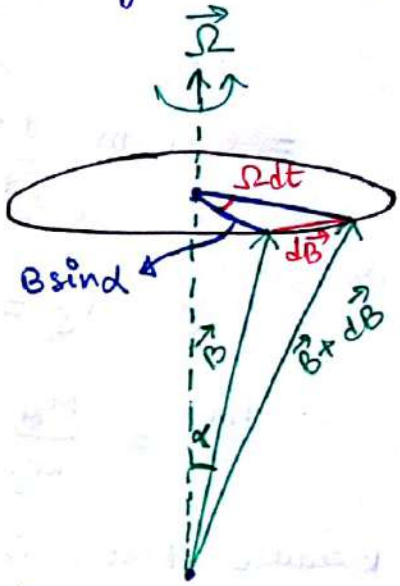
Because Moon is nearer to Earth, even though the actual attraction due to Moon is way smaller than the attraction of the sun, but due to differential attraction, tidal force is more prominent.

When natural frequency of oscillation of water coming in / flowing out matches with frequency of tidal waves due to coastal topography, large tides (e.g. Tsunami) are produced. The above example can produce tides of the order 2 feet only.

Not every sea (e.g. Mediterranean) has a tidal activity. As tidal bulge moves from east to west due to rotation of Earth, it so happens that Mediterranean sea has opening only to the west & so the tidal bulge cannot enter!

# Rotating Coordinate System

As we found in a linearly accelerating system by adding a nonphysical fictitious force $-m\vec{A}$ we could treat the problem in inertial system, we will derive next that by adding two fictitious force : centrifugal force & Coriolis force, motion in a rotating coordinate system can be treated as an inertial system. Focault pendulum & circular nature of weather system on surface of Earth can be explained.




## Rate of change of Rotating vector :

To find a relation between inertial & rotating system, suppose $\vec{B}$ rotates at rate $\Omega$ about an axis in direction $\vec{\Omega}$.

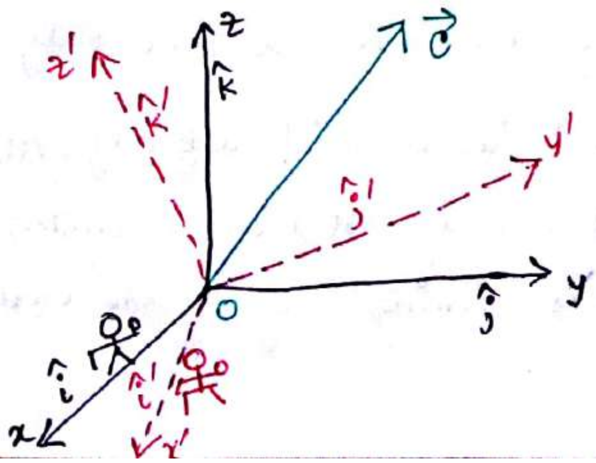
$\angle$ between $\vec{B}$ & $\vec{\Omega} = \alpha$.

In $dt$ time, tip of $\vec{B}$ sweeps a circular path of radius $B\sin\alpha$, so that $\vec{B}(t+dt) = \vec{B}(t) + d\vec{B}(t)$

where, $|d\vec{B}(t)| \simeq |B\sin\alpha \times \vec{\Omega}\, dt|$

$$\therefore \left|\frac{d\vec{B}}{dt}\right| = \lim_{dt\to 0} \frac{B\sin\alpha\, \Omega dt}{dt} = B\sin\alpha\,|\vec{\Omega}| = |\vec{\Omega}\times\vec{B}|$$

$d\vec{B} \perp \vec{B}$, $d\vec{B} \perp \vec{\Omega}$. So $\frac{d\vec{B}}{dt} = \vec{\Omega}\times\vec{B}$




Consider inertial frame $[xyz]$ & rotating frame $[x'y'z']$ at a rate $\vec{\Omega}$. $\left.\frac{d\vec{C}}{dt}\right|_{in}$ is the rate of change of $\vec{C}$ measured in $[xyz]$ frame & we want to calculate $\left.\frac{d\vec{C}}{dt}\right|_{rot}$

If $(\hat{i},\hat{j},\hat{k})$ and $(\hat{i}',\hat{j}',\hat{k}')$ are the base vectors in inertial & rotating frame, then

$$\vec{c} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k} = c_x'\hat{i}' + c_y'\hat{j}' + c_z'\hat{k}'$$

$$\therefore \left.\frac{d\vec{c}}{dt}\right|_{in} = \frac{d}{dt}\left(c_x'\hat{i}' + c_y'\hat{j}' + c_z'\hat{k}'\right)$$

$$= \left[\frac{dc_x'}{dt}\hat{i}' + \frac{dc_y'}{dt}\hat{j}' + \frac{dc_z'}{dt}\hat{k}'\right] + \left[c_x'\frac{d\hat{i}'}{dt} + c_y'\frac{d\hat{j}'}{dt} + c_z'\frac{d\hat{k}'}{dt}\right]$$

$$= \left.\frac{d\vec{c}}{dt}\right|_{rot} + \left[c_x'\vec{\Omega}\times\hat{i}' + c_y'\vec{\Omega}\times\hat{j}' + c_z'\vec{\Omega}\times\hat{k}'\right]$$

$$= \left.\frac{d\vec{c}}{dt}\right|_{rot} + \vec{\Omega}\times\left(c_x'\hat{i}' + c_y'\hat{j}' + c_z'\hat{k}'\right) = \left.\frac{d\vec{c}}{dt}\right|_{rot} + \vec{\Omega}\times\vec{c}$$

In operator notation, $\boxed{\left.\frac{d}{dt}\right|_{in} = \left.\frac{d}{dt}\right|_{rot} + \vec{\Omega}\times}$

## Velocity & Acceleration :

If $\vec{c}$ = position vector $\vec{r}$, then $\left.\frac{d\vec{r}}{dt}\right|_{in} = \left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\Omega}\times\vec{r}$

$$\therefore \boxed{\vec{v}_{in} = \vec{v}_{rot} + \vec{\Omega}\times\vec{r}}$$

$$\therefore \left.\frac{d\vec{v}_{in}}{dt}\right|_{in} = \left.\frac{d\vec{v}_{in}}{dt}\right|_{rot} + \vec{\Omega}\times\vec{v}_{in} = \left.\frac{d}{dt}(\vec{v}_{rot} + \vec{\Omega}\times\vec{r})\right|_{rot} + \vec{\Omega}\times(\vec{v}_{rot} + \vec{\Omega}\times\vec{r})$$

$$= \left.\frac{d\vec{v}_{rot}}{dt}\right|_{rot} + \vec{\Omega}\times\left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

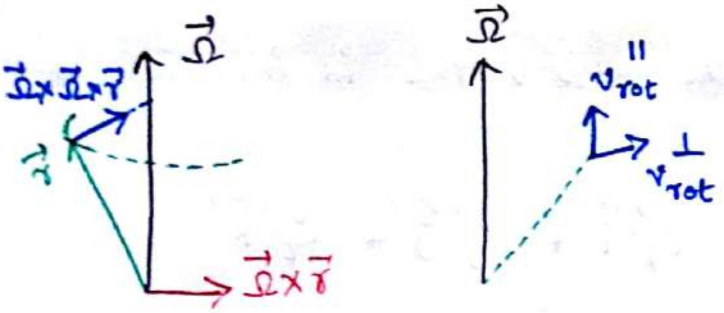
$$= \left.\frac{d\vec{v}_{rot}}{dt}\right|_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\therefore \boxed{\vec{a}_{in} = \vec{a}_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

$\therefore$ In rotating coordinate system

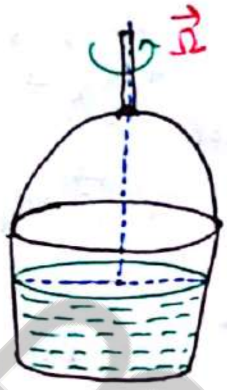
$$\vec{F}_{rot} = m\vec{a}_{rot} = m\vec{a}_{in} - 2m\vec{\Omega}\times\vec{v}_{rot} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

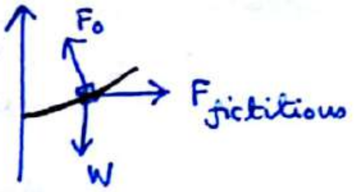
$$= \vec{F}_{in} + \left[\vec{F}_{coriolis} + \vec{F}_{centrifugal}\right] = \vec{F}_{in} + \vec{F}_{fictitious}$$

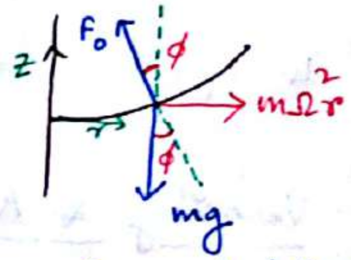



$\vec{F}_{centrifugal} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ is perpendicular to the axis of rotation & radially outward directed. In sketch, centripetal acceleration is radially inward. For $\vec{F}_{coriolis} = -2m\vec{\Omega}\times\vec{v}_{rot}$ & so only $v_{rot}^{\perp}$ contributes ( $\vec{\Omega}\times\vec{v}_{rot}^{\parallel} = 0$).

## Surface of a rotating Fluid




$F_0$ = contact force
$W = mg$ = weight
$F_{fictitious}$ = centrifugal force

To find the shape of the surface of a fluid on a bucket that is rotating with angular speed $|\vec{\Omega}|$, we consider in a coordinate system rotating with the bucket (so that the problem is static).

So on fluid meniscus, total force = 0

$$F_0 \cos\phi = W = mg$$

$$F_0 \sin\phi = m\Omega^2 r \quad \therefore \quad \tan\phi = \frac{\Omega^2 r}{g} = \frac{dz}{dr}$$

$$\text{or} \int_0^z dz = \int_0^r \frac{\Omega^2 r}{g}\, dr \quad \text{or} \quad z = \frac{\Omega^2}{2g} r^2 \rightarrow \text{Equation of surface}$$

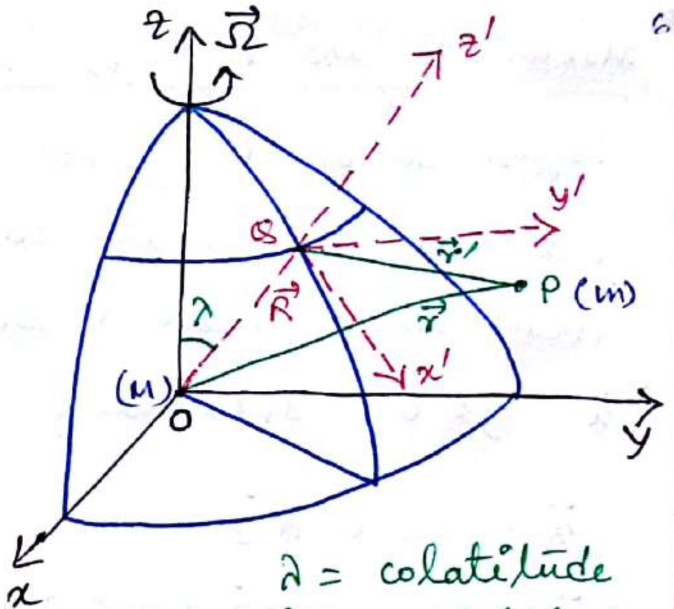
The surface is a paraboloid of revolution.

## Equation of motion of a particle relative to an observer on Earth's surface :

Suppose Earth is spherical with center at O rotating about z-axis with angular velocity $\vec{\Omega} = \Omega\hat{k}$ & its constant, $\dot{\vec{\Omega}} = 0 = \frac{d\vec{\Omega}}{dt}$. Also the frame can be taken inertial by neglecting Earth's rotation around the Sun.

Acceleration of Q relative to O is centripetal acceleration

$$\ddot{\vec{R}} = \frac{d^2\vec{R}}{dt^2} = \vec{\Omega}\times(\vec{\Omega}\times\vec{R})$$




$\lambda$ = colatitude

$90^\circ - \lambda$ = latitude

Newton's law of Gravitation,

$$\vec{F} = -\frac{GMm}{r^3}\vec{r} = m\frac{d^2\vec{r}}{dt^2}$$

Neglecting air resistance etc.

Now $\left.\frac{d^2\vec{r}}{dt^2}\right|_{in} = \left.\frac{d^2}{dt^2}(\vec{R}+\vec{r}')\right|_{in}$

$$= \left.\frac{d^2\vec{R}}{dt^2}\right|_{in} + \left.\frac{d^2\vec{r}'}{dt^2}\right|_{rot} + 2\vec{\Omega}\times\left.\frac{d\vec{r}'}{dt}\right|_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}') \quad [\text{as } \dot{\vec{\Omega}} = 0]$$

$$\therefore \left.\frac{d^2\vec{r}'}{dt^2}\right|_{rot} = \underbrace{-\frac{GM}{r^3}\vec{r} - \vec{\Omega}\times(\vec{\Omega}\times\vec{R})}_{\vec{g}} - 2\vec{\Omega}\times\vec{v}_{rot} - \cancel{\vec{\Omega}\times(\vec{\Omega}\times\vec{r}')}^{\,0} \qquad (1)$$

Near Earth's surface, contribution from $\vec{\Omega}\times(\vec{\Omega}\times\vec{r}') = 0$ & so,

$\left.\frac{d^2\vec{r}'}{dt^2}\right|_{rot} = g - 2\vec{\Omega}\times\vec{v}_{rot}$. Any other external force has to be added to this equation.

Now, $\vec{k} = (\hat{k}\cdot\hat{i}')\hat{i}' + (\hat{k}\cdot\hat{j}')\hat{j}' + (\hat{k}\cdot\hat{k}')\hat{k}'$

$$= -\sin\lambda\,\hat{i}' + 0\ (\hat{k}\perp\hat{j}') + \cos\lambda\,\hat{k}'$$

So, $\vec{\Omega} = \Omega\vec{k} = -\Omega\sin\lambda\,\hat{i}' + \Omega\cos\lambda\,\hat{k}'$

$$\therefore \vec{\Omega}\times\vec{v}_{rot} = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ -\Omega\sin\lambda & 0 & \Omega\cos\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix}$$

$$= \hat{i}'(-\Omega\cos\lambda\,\dot{y}') + \hat{j}'(\Omega\cos\lambda\,\dot{x}' + \Omega\sin\lambda\,\dot{z}') - \hat{k}'\Omega\sin\lambda\,\dot{y}'$$

Substituting in (1) & equating coefficients, we get

$$\ddot{x}' = 2\Omega\cos\lambda\,\dot{y}', \quad \ddot{y}' = -2\Omega(\dot{x}'\cos\lambda + \dot{z}'\sin\lambda), \quad \ddot{z}' = -g + 2\Omega\dot{y}'\sin\lambda$$

## Motion on the Rotating Earth

Suppose an object of mass m located at $x=y=0$, $z=h$ & at rest is dropped to the Earth's surface. Due to the Coriolis force straight line motion is turned into circular motion. At $t=0$, $\dot{x}=\dot{y}=0$. Integrating the equation of motion,

$$\ddot{x} = 2\Omega\cos\lambda\,\dot{y} \qquad \dot{x} = 2\Omega\cos\lambda\, y + c_1$$

$$\ddot{y} = -2(\Omega\cos\lambda\,\dot{x} + \Omega\sin\lambda\,\dot{z}) \qquad \dot{y} = -2(\Omega\cos\lambda\, x + \Omega\sin\lambda\, z) + c_2$$

$$\ddot{z} = -g + 2\Omega\sin\lambda\,\dot{y}$$

substituting boundary condition (B.C.), $\dot{x} = 2\Omega\cos\lambda\, y$

$$\dot{y} = -2(\Omega\cos\lambda\, x + \Omega\sin\lambda\, z) + 2\Omega\sin\lambda\, h$$

$$\therefore \ddot{z} = -g - \cancel{4\Omega^2}^{\to 0}\sin\lambda(\cos\lambda\, x + \sin\lambda(z-h)) \simeq -g$$

[as $\Omega^2 << g$ ]

or $\dot{z} = -gt + c_3$ & at $t=0$, $\dot{z}=0$ $\therefore \dot{z} = -gt$

$$\therefore \ddot{y} = (-2\Omega\cos\lambda)(2\Omega\cos\lambda\, y) + (-2\Omega\sin\lambda)(-gt)$$

$$= -\cancel{4\Omega^2}^{\to 0}\cos^2\lambda\, y + 2\Omega\sin\lambda\, gt \simeq 2\Omega\sin\lambda\, gt$$

$\therefore \dot{y} = \Omega g\sin\lambda\, t^2 + c_4$ & at $t=0$, $\dot{y}=0$ $\therefore c_4 = 0$

or $\dot{y} = \Omega g\sin\lambda\, t^2$ or $y = \frac{1}{3}\Omega g\sin\lambda\, t^3 + c_5$ & at $t=0$, $y=0$, $c_5=0$

or $y = \frac{1}{3}\Omega g\sin\lambda\, t^3$.

So after time $t$, object is deflected to east of the vertical

Again, $z = -\frac{1}{2}gt^2 + c_6$ & at $t=0$, $z=h$ $\therefore c_6 = h$.

$\therefore z = h - \frac{1}{2}gt^2$. $\therefore$ By time $h - \frac{1}{2}gt^2 = 0$, $z=0$

or $t = \sqrt{\frac{2h}{g}}$ object touches ground. The total deflection is

$$y = \frac{\Omega g}{3}\sin\lambda\left(\frac{2h}{g}\right)^{3/2} = \frac{\sqrt{8}}{3}\Omega g^{-1/2}\sin\lambda\, h^{3/2} = \sqrt{\frac{8h^3}{9g}}\,\Omega\sin\lambda.$$

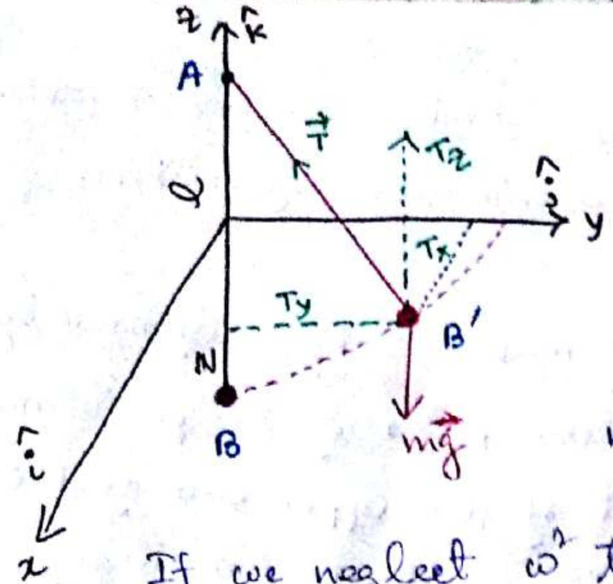
## The Foucault Pendulum

It is a simple device to conveniently detect the slowest rotation of the Earth and provides a direct experimental confirmation of the existence of the Coriolis force.

**Construction:** It consists of a heavy mass (28 kg) suspended by a large wire (70 metre), so that the time period of the pendulum is very long (17 seconds). The attachment of the upper end of the wire allows the pendulum to swing with equal freedom in any direction, so that the period of oscillation in any plane is exactly the same. A Foucault pendulum once set in oscillation continues to oscillate for a fairly long time, in a definite vertical plain.

**Working Principle:** The plain of oscillation is observed to precess (rotate) around the vertical axis within a period of several hours. If the pendulum is setup at the North pole of Earth, it will oscillate as a simple pendulum in a fixed vertical plain as in an inertial frame of reference. Since the Earth rotates from west to East with an angular velocity $\vec{\omega}$, to an observer on the surface of the Earth, plane of oscillation will appear to be turning from East to west (opposite direction) with angular velocity $\vec{\omega}$. It is not necessary that the pendulum to be mounted right at the North or South pole of the Earth. An apparent rotation of the plain of oscillation of the pendulum due to rotation of the Earth will be observed in any latitude on the Earth, except at the Equator.

In the Cartesian coordinate system, suppose from point A, the Focault pendulum of length $l$ is suspended. The tension in the string is given by,




$$\vec{T} = (\vec{T}\cdot\hat{i})\hat{i} + (\vec{T}\cdot\hat{j})\hat{j} + (\vec{T}\cdot\hat{k})\hat{k}$$
$$= T\cos\alpha\,\hat{i} + T\cos\beta\,\hat{j} + T\cos\gamma\,\hat{k}$$
$$= -T\frac{x}{l}\hat{i} - T\frac{y}{l}\hat{j} + T\frac{l-z}{l}\hat{k}$$

So the equation of motion of the bob is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{T} + m\vec{g} - 2m(\vec{\omega}\times\vec{v}) \cancel{- m\vec{\omega}\times(\vec{\omega}\times\vec{r})}^{\,0}$$

If we neglect $\omega^2$ term (last term) for simplicity, then,

$$m\frac{d^2x}{dt^2} = -T\frac{x}{l} + 2m\omega\dot{y}\cos\lambda$$

$$m\frac{d^2y}{dt^2} = -T\frac{y}{l} - 2m\omega(\dot{x}\cos\lambda + \dot{z}\sin\lambda)$$

$$m\frac{d^2z}{dt^2} = T\left(\frac{l-z}{l}\right) - mg + 2m\omega\dot{y}\sin\lambda$$

Now if we assume that the motion of the bob takes place in the XY plain, then $z = \dot{z} = \ddot{z} = 0$.

So, $0 = T - mg + 2m\omega\dot{y}\sin\lambda$ or $T = mg - 2m\omega\dot{y}\sin\lambda$.

$$\therefore \frac{d^2x}{dt^2} = -\frac{gx}{l} + \cancel{\frac{2\omega x}{l}\dot{y}\sin\lambda}^{\,0} + 2\omega\dot{y}\cos\lambda.$$

$$\frac{d^2y}{dt^2} = -\frac{gy}{l} + \cancel{\frac{2\omega y}{l}\dot{y}\sin\lambda}^{\,0} - 2\omega\dot{x}\cos\lambda.$$

The above nonlinear differential equation can be linearized in the limit, $x, y, \omega$ are small, so that $x\dot{y}\omega$ or $y\dot{y}\omega$ etc term can be neglected.

$$\therefore \ddot{x} = -\frac{gx}{l} + 2\omega\dot{y}\cos\lambda, \quad \ddot{y} = -\frac{gy}{l} - 2\omega\dot{x}\cos\lambda.$$

Suppose that initially the bob is in the YZ plain and is given a displacement from the z-axis of magnitude A. So initial condition is at $t=0$, $x = \dot{x} = 0$, $y = A$, $\dot{y} = 0$. Conveniently we put $K^2 = \frac{g}{l}$ and $\omega\cos\lambda = \alpha$; so that equations become,

$\ddot{x} = -k^2 x + 2\alpha \dot{y}$ and $\ddot{y} = -k^2 y + 2\alpha \dot{x}$. We can solve this linear second order coupled differential equation (See Prob. 6.20, Chapter-6, M.R. Spiegel) to get.

$$x = A \cos kt \sin \alpha t = A \cos\left(\sqrt{\tfrac{g}{l}}\, t\right) \sin(\omega t \cos\lambda)$$

$$y = A \cos kt \cos \alpha t = A \cos\left(\sqrt{\tfrac{g}{l}}\, t\right) \cos(\omega t \cos\lambda)$$ or in vector form

$\vec{r} = x\hat{i} + y\hat{j} = A \cos \sqrt{\frac{g}{l}}\, t\ \hat{n}$ where, $\hat{n} = \sin(\omega t \cos\lambda)\hat{i} + \cos(\omega t \cos\lambda)\hat{j}$ is a unit vector. The time period of $\cos\left(\sqrt{\frac{g}{l}}\, t\right)$ $\left[T = 2\pi\sqrt{\frac{l}{g}}\right]$ is very small compared to the time period of $\hat{n}$ $[T' = 2\pi/\omega\cos\lambda]$, so that $\hat{n}$ is a very slowly precessing (rotating) unit vector. Thus, physically the Foucault pendulum oscillates in a plane through the z-axis which is slowly rotating about the z-axis.

At $t = 0$, $\hat{n} = \hat{j}$, $y = A$. After $t = \frac{T'}{8} = \frac{2\pi}{8\omega\cos\lambda}$, so that $\hat{n} = \sin\frac{\pi}{4}\hat{i} + \cos\frac{\pi}{4}\hat{j}$, rotation of the plane is in the clockwise direction as observed from Earth's surface in the northern hemisphere (where $\cos\lambda > 0$) and counterclockwise direction in the southern hemisphere (where $\cos\lambda < 0$). This rotation of the plane was observed by Foucault in 1851.

Q. Calculate the latitude where the plane of vibration rotates once a day. $T' = \frac{2\pi}{\omega\cos\lambda}$ & $\frac{2\pi}{\omega} = 1$ day $= 24$ hrs.

$\therefore T' = 24$ hrs. $\therefore \cos\lambda = 1 \Rightarrow \lambda = 0^\circ$ i.e. at poles.