- **Contravariant Vector :**

(note the contraction)

$$\bar{A}^p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} A^q = \partial_q \bar{x}^p A^q$$

Transformation

$$\# \left\{ \begin{matrix} A^1, A^2, \ldots, A^N \\ x^1, x^2, \ldots, x^N \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \bar{A}^1, \bar{A}^2, \ldots, \bar{A}^N \\ \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N \end{matrix} \right\}$$

$$\boxed{\partial_i = \frac{\partial}{\partial x^i} \;;\; \frac{\partial G}{\partial x_i} = \partial^i G, \quad \frac{\partial G}{\partial x^i} = \partial_i G}$$

- **Covariant Vector :**

(note the contraction)

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q = \partial_p x^q A_q$$

Transformation

$$\# \left\{ \begin{matrix} A_1, A_2, \ldots, A_N \\ x^1, x^2, \ldots, x^N \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N \\ \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N \end{matrix} \right\}$$

Scalar $\rightarrow$ Tensor of Rank 0

Vector $\rightarrow$ Tensor of Rank 1

Vector $\rightarrow$ Tensor of Rank 1

RANK-2 TENSOR

• Contravariant Tensor :

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

(careful about contraction)

Transformation

$$\# \begin{Bmatrix} A^{qs} \\ x^1, x^2, \ldots, x^N \end{Bmatrix} \Rightarrow \begin{Bmatrix} \bar{A}^{pr} \\ \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N \end{Bmatrix}$$

• Covariant Tensor :

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}$$

Transformation

$$\# \begin{Bmatrix} A_{qs} \\ x^1, x^2, \ldots, x^N \end{Bmatrix} \Rightarrow \begin{Bmatrix} \bar{A}_{pr} \\ \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N \end{Bmatrix}$$

• Mixed Tensor :

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

$$\bar{A}_r{}^p = \frac{\partial x^s}{\partial \bar{x}^r}\frac{\partial \bar{x}^p}{\partial x^q} A_s{}^q$$

$$= \frac{\partial \bar{x}^p}{\partial x^q}\frac{\partial x^s}{\partial \bar{x}^r} A_s{}^q$$

Example: Balance laws (fluid):

Mass: $\partial_t \rho + \vec{\nabla} \cdot \vec{J}^{\text{mass}} = 0$

$$J_i^{\text{mass}} = -D_{ij}\partial^j \rho - E_{ijk}\partial^j \partial^k \rho - F_{ijkl}\partial^j \partial^k \partial^l \rho - \cdots$$

Linear momentum, energy, angular momentum

constitutive relation of the currents

$\Rightarrow$ Onsager-Casimir Reciprocity Relation. $\Rightarrow$ Nonequilibrium scalar → Projection = $\hat{a} \cdot \vec{b}$

pseudoscalar → sign change in parity inversion

$V = \vec{a} \cdot \underbrace{\vec{b} \times \vec{c}}_{\text{pseudovector}}$

vector (polar) → transforms under proper rotation

pseudovector (axial) → sign change in improper rotation (reflection)

$\tau = \vec{r} \times \vec{F}$, $\vec{L} = \vec{r} \times \vec{p}$, $\vec{\omega} = \vec{\nabla} \times \vec{v}$, $\tau = \vec{\mu} \times \vec{B}$, $\vec{B}$, $\vec{\Omega}$

polar vector × polar vector = axial vector / pseudovector

polar vector · pseudovector = pseudoscalar

tensor :

pseudotensor : sign change in parity inversion

Levicivita ϵ_{ij}, ϵ_{ijk}

dual tensor :