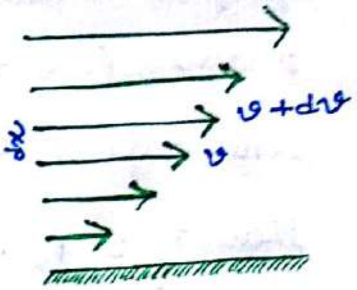
VISCOSITY



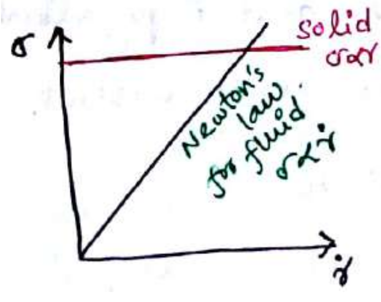
The central property that distinguishes a fluid from a solid is that, a fluid cannot sustain a "shear stress" for any period of time, meaning if a shear is applied to a fluid, it will move under the shear. Thicker liquids like paint, honey, cornstarch solution move less easily than fluids like water, glycerol or air. The measure of the ease with which a fluid yields is its viscosity.

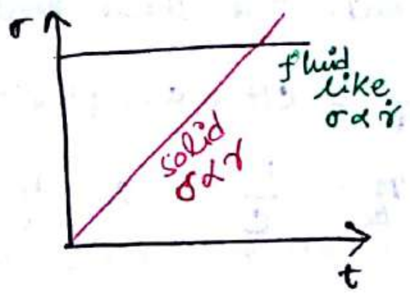
When a liquid is at rest, we do not observe any rigidity or shape elasticity in it but when the liquid is in orderly motion (not turbulent, but streamline), there comes into play a tangential stress between any two layers of the fluid, that are moving relative to each other. Difference in velocities between these two layers gives rise to internal friction, as a result of which the faster layer tends to accelerate the slower one & vice versa.

Newton found that for a fluid moving in parallel layers, the shearing stress at any point where the velocity gradient $\frac{\partial v_x}{\partial y}$ perpendicular to the direction of motion, the frictional force F_x is proportional to area of fluid layer & $\frac{\partial v_x}{\partial y}$.

$$\therefore \quad F_x \propto A \qquad \therefore \quad F_x = \eta A \frac{\partial v_x}{\partial y}$$

$$\propto \frac{\partial v_x}{\partial y} \qquad \text{or} \quad \frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y} \qquad \text{or} \quad \underbrace{\sigma}_{\text{stress}} = \underbrace{\eta}_{\text{coefficient of viscosity}} \underbrace{\dot{\gamma}}_{\text{strain rate}}$$





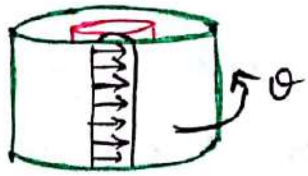
This is Newton's law of viscosity and fluid exhibiting such phenomena is called Newtonian fluid.

Compare with Hooke's law of elastic solid where stress $\propto$ strain ($\sigma = Y\gamma$, Y = Young's modulus) while for fluid stress $\propto$ strainrate ($\sigma = \eta\dot{\gamma}$, η = Coefficient of viscosity). If $A = 1$, $\frac{\partial v_y}{\partial x} = 1$ then $\eta = F_x$. So, coefficient of viscosity of a fluid is the tangential stress per unit velocity gradient. C.G.S. unit of η is "Poise". 1 Poise η means that a force of 1 dyne is required to maintain a velocity difference of 1 cm/sec between two layers of 1 square cm area which are 1 cm apart from each other.

Dimension of η: $\quad \eta = \dfrac{F/A}{\frac{\partial v_y}{\partial x}}$, $\therefore [\eta] = \dfrac{[MLT^{-2}]/[L]^2}{[LT^{-1}]/[L]}$

$$= [ML^{-1}T^{-1}]$$

Fugitive elasticity:



Other than sliding planes geometry, Couette flow can be generated by sandwiching a liquid between two concentric cylinders with inner (outer) one stationary. From consideration of shape elasticity, tangential stress $\sigma = n\theta$ where n = modulus of rigidity and θ is the angle of shear.

In the limit $\lim_{\theta \to 0}$, $\quad \sigma = n \tan\theta = n\frac{dy}{dx}$

From Newton's law of viscosity, $\sigma = \eta\frac{dv}{dx} = \eta\frac{d}{dx}\left(\frac{dy}{dt}\right)$

$$= \eta\frac{d}{dt}\left(\frac{dy}{dx}\right) = \eta\frac{d\theta}{dt}.$$

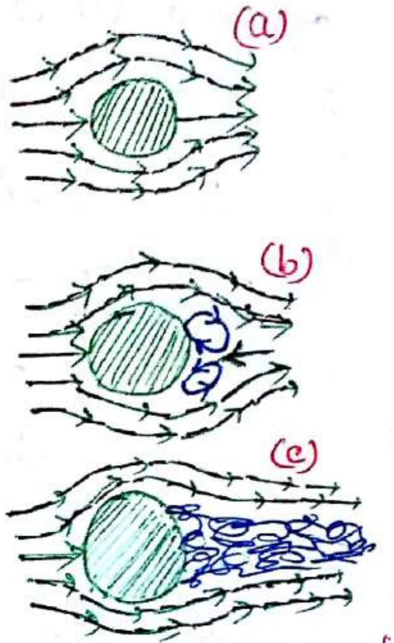
$\frac{d\theta}{dt}$ represents the rate at which the shear brakes and is proportional to the angle of shear. $\therefore \frac{d\theta}{dt} = c\theta$, c = proportionality constant.

$\therefore \sigma = \eta c\theta = n\theta \quad$ or $\quad \eta = \frac{n}{c} \quad \therefore \eta \propto n$

$\frac{1}{c}$ is the relaxation time of the medium fluid that measures the time taken by the shear to disappear, if force is taken off.

So the appearance of viscous force during streamline motion of a liquid is due to the existence of intermittent shear elasticity (fugitive elasticity). Maxwell stated that viscosity in a fluid is due to the existence of fugitive elasticity in it.

Streamline & Turbulent motion - Critical velocity & Reynold's number :



Viscous flow around a circular cylinder

If fluid flow is such that magnitude & direction of velocity at any point is always same then its called a steady streamline flow. As in panel (a), no two streamlines can cross each other & the tangent to a line at any point gives the flow direction. In turbulent flow as panel (b) & (c) velocity magnitude & direction changes in irregular manner in terms of eddies, vortices, zigzag motion.

A difference of pressure is maintained between ends for flow of liquid through a tube (Poiseuille's flow). The layer of liquid in contact with the wall of the tube is at rest ("no-slip" condition where both normal & tangential component of velocity is zero). Velocity of the layer increases towards the axis of the tube. The streamline flow is maintained when the velocity is below a certain limit known as "Critical velocity."

when critical velocity is reached, in different parts of the liquid no layer travel in a straight line along the tube & when velocity is further increased, streamline motion is completely lost (turbulent).

Reynolds using dimensional analysis showed that in Poiseuille flow, critical velocity v is related to the fluid density ρ, radius of tube r & coefficient of viscosity η.

Suppose, $v = Re\, \eta^x \rho^y r^z$ where Re = Reynold's number whose value is 1000 for narrow tubes. In general, for a liquid of high viscosity even for high velocity streamline motion is observed while for high density & wide bore of tube makes the motion turbulent.

Substituting dimensions, we have

$$[LT^{-1}] = [ML^{-1}T^{-1}]^x [ML^{-3}]^y [L]^z$$

$$= [M^{x+y}][L^{-x-3y+z}][T^{-x}]$$

Re is dimensionless number

Equating the powers of $[M]$, $[L]$ and $[T]$, we have

$$\left.\begin{aligned} x + y &= 0 \\ -x - 3y + z &= 1 \\ -x &= -1 \end{aligned}\right\} \quad x = 1,\ y = -1,\ z = -1$$

$$\therefore\ v = Re\, \eta \rho^{-1} r^{-1}$$

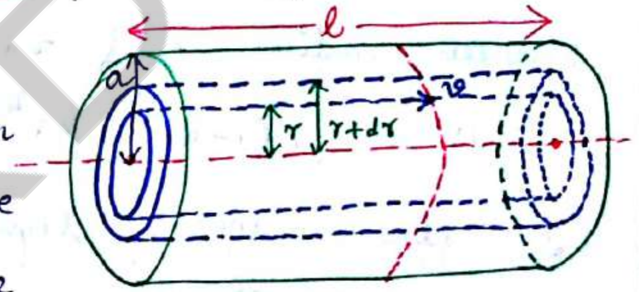
or $\boxed{Re = \dfrac{\rho v r}{\eta}}$

While discussing on motion of dry water, we will see that "vorticity" $\vec{\Omega} = \vec{\nabla} \times \vec{v}$ follows a simple kinetic equation

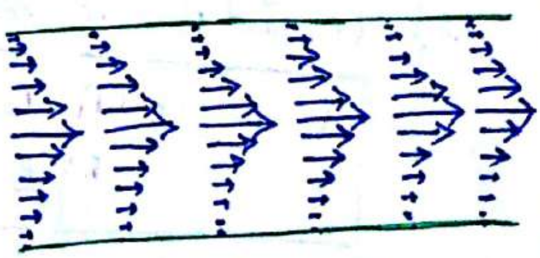
$$\frac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = \frac{1}{Re} \nabla^2 \vec{\Omega}.$$ This means that if we solve the flow problem for v_1 for a certain cylinder with radius r_1 & then ask about flow for a different radius r_2 for a different fluid with velocity v_2, the Reynold's number will be same means flows will appear same.

$\therefore\ Re = \dfrac{\rho_1 v_1 r_1}{\eta_1} = \dfrac{\rho_2 v_2 r_2}{\eta_2}$. So we can determine the flow of air past an airplane wing without building an airplane to try but instead make a model with velocity to yield same Re. We can only apply so provided we are dealing with "incompressible" liquid & not with "compressible" gas. Otherwise speed of sound in terms of "Mach number" has to be taken into account. $Ma = \dfrac{\text{speed of sound in fluid}}{\text{speed of sound in air}}$. So for velocities near the speed of sound or above, the flows are the same in two situations if both "Ma" & "Re" are same for both situations.

Poiseuille's equation for flow of liquid through a horizontal narrow tube:



Consider a horizontal streamline motion of a liquid through a narrow tube. The lines are parallel to the axis of tube (no radial flow). Pressure varies along the length of tube & due to no-slip, velocity of the liquid gradually decreases radially from the axis towards the wall of the tube.



When steady state flow is attained, let v be the velocity at a distance r from the axis of tube & velocity gradient is $\frac{dv}{dr}$. So the tangential stress is $\eta \frac{dv}{dr}$. This force acts over the unit area of surface of cylinder at r in a direction opposite to the pressure gradient. So the total resisting force over the surface of the liquid cylinder is $2\pi r l \, \eta \frac{dv}{dr}$.

If P is the pressure difference between the ends of the tube, then the active force is $p\pi r^2$ (As $p = \frac{F}{A} = \frac{F}{\pi r^2}$). This force tends to accelerate the liquid in cylinder and therefore in steady state, this accelerating force is balanced by the viscous retarding force,

$$p\pi r^2 = -\eta \frac{dv}{dr} 2\pi r l \quad \left(\because \frac{dv}{dr} < 0 \text{ as } v \text{ decrease with increasing } r \right)$$

$$\text{or} \quad r\,dr = -\frac{2\eta l}{p} dv$$

Integrating with the Boundary condition (B.C.) $v = 0$ at $r = a$,

$$\int_r^a r\,dr = -\frac{2\eta l}{p} \int_v^0 dv \Rightarrow \boxed{v = \frac{p}{4\eta l}(a^2 - r^2)}$$

This is an equation for parabola. Now if dV is the volume of liquid that flows through the cylindrical shell per unit time between radius r & $r+dr$ then

$$dV = [\pi(r+dr)^2 - \pi r^2]v = 2\pi r\,dr\,v = \frac{\pi p}{2\eta l}(a^2 - r^2)r\,dr$$

∴ Total volume of liquid passing through the tube per unit time is

$$V = \frac{\pi p}{2\eta l}\int_0^a (a^2 - r^2) r\,dr = \frac{\pi p}{2\eta l}\left(a^2\int_0^a r\,dr - \int_0^a r^3 dr \right)$$

$$= \frac{\pi p}{2\eta l}\left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \frac{\pi p a^4}{8\eta l}$$

If p_1 and p_2 are the pressure at two ends of the tube then

$$\boxed{V = \frac{\pi(p_1 - p_2)a^4}{8\eta l}}$$

This is known as Poiseuille's equation

<u>Correction to Poiseuille's formula</u>:

Poiseuille's equation $V = \frac{\pi p a^4}{8\eta l}$ is approximately true because two important factors are not taken into account. (a) pressure difference p is utilized partly in communicating kinetic energy to the liquid (b) acceleration of the liquid along the axis of the tube is neglected.

A finite acceleration at inlet of the tube becomes zero only after traversing a finite distance, so $1.64a$ is added to the length l. To find the kinetic energy correction, let's consider that p is the effective pressure difference that overcomes viscosity.

The workdone against viscous force per unit time is pV & Kinetic energy per unit time is $= \int_0^a \frac{1}{2} \underbrace{\rho\, 2\pi r dr\, v}_{mass} . v^2$

$$= \int_0^a \rho\pi v^3 r dr = \int_0^a \rho\pi\left(\frac{p}{4\eta l}\right)^3 (a^2-r^2)^3 r dr = \rho\pi\left(\frac{p}{4\eta l}\right)^3 \int_0^a (a^2-r^2)^3 r dr$$

$$= \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8}.$$ So total loss of energy $= pV + \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8}$

and this is equal to PV

$$\therefore \quad pV + \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8} = PV$$

$$\text{or} \quad pV + \frac{\rho}{\pi^2 a^4}\left(\frac{\pi p a^4}{8\eta l}\right)^3 = PV$$

$$\text{or} \quad pV + \frac{\rho V^3}{\pi^2 a^4} = PV \qquad \left(\because V = \frac{\pi p a^4}{8\eta l}\right)$$

$$\text{or} \quad p = P - \frac{\rho V^2}{\pi^2 a^4}.$$

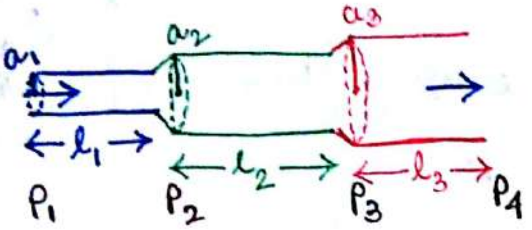
Experimentally it was found by Hagenbach, Couette & Wilberforce that correct form is $p = P - \frac{k\rho V^2}{\pi^2 a^4}$ with $k \approx 1$, differing for different scenario.

Corrected Poiseuille's formula is

$$\boxed{V = \frac{\pi a^4}{8\eta(l+1.64a)}\left(P - \frac{k\rho V^2}{\pi^2 a^4}\right)}$$

Flow of liquid through capillaries in series & parallel

Poiseuille's formula $V = \dfrac{P}{8\eta l/\pi a^4} = \dfrac{P}{R}$ can be compared with ohm's law for flow of electric current through a resistance $i = \dfrac{E}{R}$. So the rate of flow of the liquid V corresponds to current i, pressure difference P to the potential difference E and $\dfrac{8\eta l}{\pi a^4}$ to the resistance R.



Consider a series connection of three capillaries with radius a_1, a_2, a_3 & length l_1, l_2, l_3. Let P_1 & P_4 are the pressure at extreme ends & P_2, P_3 are pressure at the junctions. As there is no accumulation of the liquid at the junction, so V must be equal through all capillaries, just like current is same through any resistance connected in series.

$$\text{So,} \quad V = \frac{\pi(P_1 - P_2)a_1^4}{8\eta l_1} = \frac{\pi(P_2 - P_3)a_2^4}{8\eta l_2} = \frac{\pi(P_3 - P_4)a_3^4}{8\eta l_3}$$

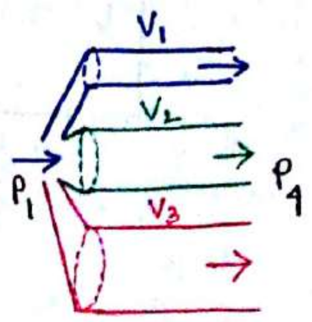
$$\therefore \left.\begin{aligned} P_1 - P_2 &= \frac{8\eta l_1 V}{\pi a_1^4} \\ P_2 - P_3 &= \frac{8\eta l_2 V}{\pi a_2^4} \\ P_3 - P_4 &= \frac{8\eta l_3 V}{\pi a_3^4} \end{aligned}\right\}$$

Summing, $P_1 - P_4 = \dfrac{8\eta V}{\pi}\left[\dfrac{l_1}{a_1^4} + \dfrac{l_2}{a_2^4} + \dfrac{l_3}{a_3^4}\right]$

$$\text{or } V = \frac{\pi P}{8\eta}\left[\frac{l_1}{a_1^4} + \frac{l_2}{a_2^4} + \frac{l_3}{a_3^4}\right]^{-1} \quad \text{— (1)}$$

where $P = P_1 - P_4$ is pressure difference across composite slab.

If we maintain $P = P_1 - P_4$ across ends of three capillaries connected in parallel, then volume of liquid flowing per unit time through them is $V = V_1 + V_2 + V_3 = \dfrac{\pi P a_1^4}{8\eta l_1} + \dfrac{\pi P a_2^4}{8\eta l_2} + \dfrac{\pi P a_3^4}{8\eta l_3}$

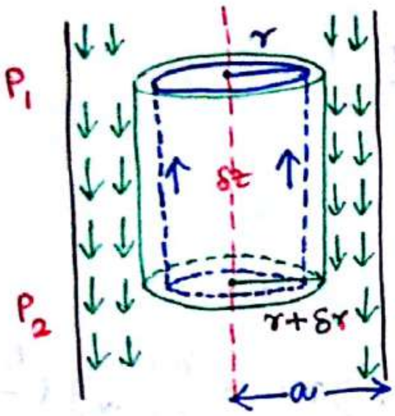


$$V = \frac{\pi P}{8\eta}\left(\frac{a_1^4}{l_1} + \frac{a_2^4}{l_2} + \frac{a_3^4}{l_3}\right) = P\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \quad (2)$$

Comparing (1) and (2) we recover the effective resistance for an equivalent series connected viscous flow as $R = R_1 + R_2 + R_3$ while for parallel connected, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ where

$$R_1 = \frac{8\eta l_1}{\pi a_1^4}, \quad R_2 = \frac{8\eta l_2}{\pi a_2^4}, \quad R_3 = \frac{8\eta l_3}{\pi a_3^4}.$$

Downward flow of a liquid through a vertical narrow tube:



Consider a vertical narrow tube through which a liquid flows steadily. Consider a cylindrical shell with radius r & $r+\delta r$ and length δz. So the viscous force on the inner shell wall in vertical upward direction is $2\pi r\,\delta z\,\eta\frac{dv}{dr}$.

Viscous force on the outer wall in vertical downward direction is $2\pi r\,\delta z\,\eta\frac{dv}{dr} + \frac{d}{dr}\left(2\pi r\,\delta z\,\eta\frac{dv}{dr}\right)\delta r$

Force due to pressure p on the upper annular flat surface of the element of the liquid cylinder in the vertical downward direction is $2\pi r\,\delta r\,p$ & similarly force due to pressure on the lower annular flat surface of the liquid cylinder in the upward direction is $2\pi r\,\delta r\left(p + \frac{dp}{dz}\delta z\right)$

At steady state, liquid acceleration $= 0$, resultant downward force is zero.

$$\therefore \; -\cancel{2\pi r\,\delta z\,\eta\frac{dv}{dr}} + \cancel{2\pi r\,\delta z\,\eta\frac{dv}{dr}} + \frac{d}{dr}\left(2\pi r\,\delta z\,\eta\frac{dv}{dr}\right)\delta r + \underbrace{2\pi r\,\delta r\,\delta z\,\rho g}$$

$$+ \cancel{2\pi r\,\delta r\,p} - 2\pi r\,\delta r\left(\cancel{p} + \frac{dp}{dz}\delta z\right) = 0$$

[Vertical downward force due to weight of cylindrical shell]

$$\text{or} \quad 2\pi\,\delta z\,\eta\frac{d}{dr}\left(r\frac{dv}{dr}\right) + 2\pi r\,\delta r\,\delta z\,\rho g - 2\pi r\,\delta r\frac{dp}{dz}\delta z = 0$$

or, $\eta \frac{d}{dr}\left(r\frac{dv}{dr}\right) + r\rho g = r\frac{dp}{dz}$ $[\because \eta \neq \eta(r)]$

or $\frac{dp}{dz} = \frac{\eta}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \rho g.$ or $-\frac{P_1 - P_2}{l} = \frac{\eta}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \rho g$

or $\left(\rho g + \frac{P_1 - P_2}{l}\right) r = -\eta \frac{d}{dr}\left(r\frac{dv}{dr}\right)$

Integrating with B.C. $r = a,\ v = 0$

$$\left(\rho g + \frac{P_1 - P_2}{l}\right)\frac{r^2}{2} + A = -\eta r \frac{dv}{dr}$$

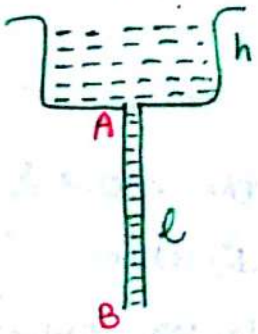
Integrating once again, $\left(\rho g + \frac{P_1 - P_2}{l}\right)\frac{r^2}{4} + A \ln r + B = -\eta v$

at $r = 0$, $v \neq \infty$ (remember $\ln 0 = -\infty$), A must be zero. & substituting $r = a$, $v = 0$. $B = -\left(\rho g + \frac{P_1 - P_2}{l}\right)\frac{a^2}{4}$

$$\therefore \eta v = \left(\rho g + \frac{P_1 - P_2}{l}\right)\left(\frac{a^2 - r^2}{4}\right)$$

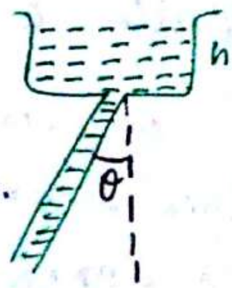
So the volume of liquid flowing per unit time is

$$V = \int_0^a 2\pi r\,dr\,v = \frac{\pi}{2\eta}\left(\rho g + \frac{P_1 - P_2}{l}\right)\int_0^a (a^2 - r^2)\,r\,dr = \frac{\pi a^4}{8\eta}\left(\rho g + \frac{P_1 - P_2}{l}\right)$$



In liquid viscometer, a narrow tube is connected to a liquid container. $P_1 = \pi + \rho g h$ is the pressure at inlet A and $P_2 = \pi$ is the pressure at outlet B where π is the barometric pressure.

$$\therefore V = \frac{\pi a^4}{8\eta}\left(\rho g + \frac{\rho g h}{l}\right) = \boxed{\frac{\pi a^4}{8\eta l}\rho g (l + h)}$$

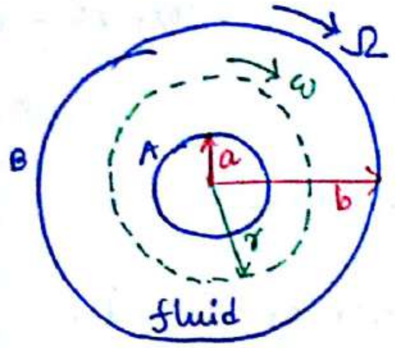


For inclined liquid viscometer, acceleration due to gravity g has $g\cos\theta$ component along vertical direction.

$$\text{So, } V = \frac{\pi a^4}{8\eta}\left(\rho g \cos\theta + \frac{P_1 - P_2}{l}\right) = \frac{\pi a^4}{8\eta}\left(\rho g \cos\theta + \frac{\rho g h}{l}\right)$$

$$= \boxed{\frac{\pi a^4}{8\eta l}\rho g (l\cos\theta + h)}$$

Torque on a cylinder immersed in a rotating fluid



A viscous liquid is filled within two coaxial cylinders A & B with cylinder B rotating about common axis with constant angular velocity Ω. The torque on A due to rotation B is required. The innermost layer of fluid at A is zero velocity (no-slip), a, b, l are the radii of A & B cylinder and length. Suppose the fluid at a distance r rotates with angular velocity ω. Its linear velocity is $r\omega$ and velocity gradient is

$$\frac{d}{dr}(r\omega) = r\frac{d\omega}{dr} + \cancel{\omega}^{\to 0}$$

↳ no viscosity effect.

∴ Viscous force on the side of fluid cylinder is

$$F = \eta\, 2\pi r l\, r\frac{d\omega}{dr}$$ & then the viscous torque $\Gamma = F \cdot r$

$$= 2\pi\eta l r^3 \frac{d\omega}{dr}$$

As the fluid rotates in steady state, torque on inner cylinder is in clockwise direction

$$\Gamma. \quad \text{or} \quad \Gamma\frac{dr}{r^3} = 2\pi\eta l\, d\omega$$

Integrating, $$\Gamma\int_a^b \frac{dr}{r^3} = 2\pi\eta l\int_0^{\Omega} d\omega$$

or $$\frac{\Gamma}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 2\pi\eta l\Omega \quad \text{or} \quad \boxed{\Gamma = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2}}$$

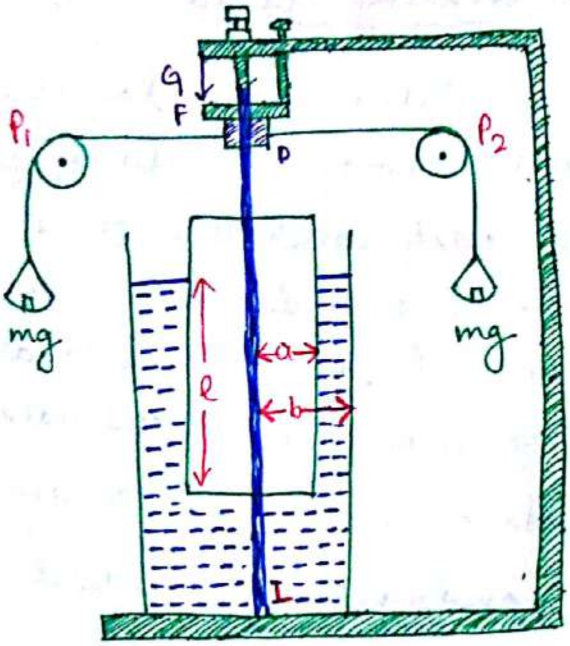
\# Suppose now A rotates with Ω & Γ is the clockwise torque

$$\Gamma = -2\pi\eta l r^3 \frac{d\omega}{dr}$$ ($\Gamma < 0$ as $\frac{d\omega}{dr} < 0$ as ω decreases with increasing r)

∴ $$-\Gamma\int_a^b \frac{dr}{r^3} = 2\pi\eta l\int_{\Omega}^{0} d\omega \quad \text{or} \quad \frac{\Gamma}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = -2\pi\eta l\Omega$$

or $$\boxed{\Gamma = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2}}$$

Searle's viscometer uses this technique for measurement of η of highly viscous liquids. Two weights rotate the inner



Searle's viscometer

cylinder by ball-bearing pulleys while the outer cylinder is fixed with the fluid in between. We know

$$\Gamma = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2} = mgd$$

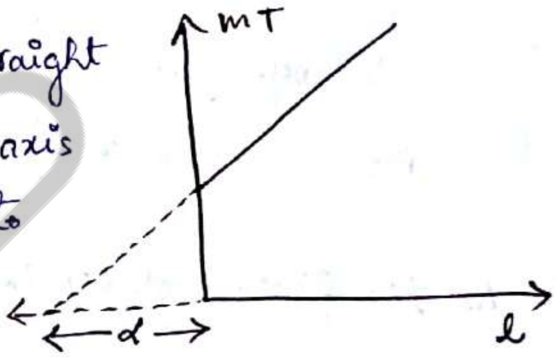
where d = diameter of drum D.

so $\eta = \frac{gd\,(b^2-a^2)}{4\pi\Omega a^2 b^2}\,\frac{m}{l}$ and as

$T = \frac{2\pi}{\Omega}$ is the time period of rotation

$$\eta = \frac{gd\,(b^2-a^2)}{8\pi^2 a^2 b^2}\,\frac{mT}{l}.$$

For a given liquid, $\frac{mT}{l}$ = constant



Though it should be a origin-passing straight line, but experimentally found to cut Y axis so that the above expression is modified to accomodate

$$\eta = \frac{gd\,(b^2-a^2)}{8\pi^2 a^2 b^2}\,\frac{mT}{l+\alpha}$$

Viscosity of high viscous liquids

Stoke's law : Viscous resisting force on a small sphere falling through a liquid of infinite extent is $F = 6\pi\eta a v$ where v = terminal velocity of the sphere, a = radius.

viscous retarding force = effective gravitational force.

$$\frac{4}{3}\pi a^3(\rho-\sigma)g = 6\pi\eta a v, \quad \begin{cases}\rho = \text{density of the sphere}\\ \sigma = \text{density of the liquid.}\end{cases}$$

or $$\eta = \frac{2}{9}\,\frac{a^2(\rho-\sigma)g}{v}$$

In practice in cylindrical vessel due to confinement, boundary effect due to wall & bottom of cylinder is corrected to yield

$$\eta = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{v(1+2.4\frac{a}{R})(1+3.3\frac{a}{h})}, \quad R = \text{radius of cylinder}, \; h = \text{height of liquid}.$$

Using dimensional analysis $F = 6\pi\eta a v$ can be deduced as

$F = K a^x \eta^y v^z$, K = dimensionless number.

$\therefore [MLT^{-2}] = [L]^x [ML^{-1}T^{-1}]^y [LT^{-1}]^z = M^y L^{x-y+z} T^{-y-z}$

Equating the powers of M, L, T, $\quad y=1, \; x-y+z=1, \; -y-z=-2$

$\therefore x = y = z = 1 \quad \therefore F = K\eta a v$

By solving $\eta \nabla^2 \vec{v} - \vec{\nabla} p = 0$ for a sphere in a liquid with incompressibility constraint $\vec{\nabla}\cdot\vec{v} = 0$, Stokes calculated $K = 6\pi$.

$\therefore F = 6\pi\eta a v$.

Observe that from $v = \frac{2}{9}\frac{ga^2(\rho-\sigma)}{\eta}$, $v \propto a^2$. For raindrop of radius $a = 10^{-3}$ cm falling through air with $\eta = 1.8\times10^{-4}$ poise

terminal velocity $v = \frac{2\times981\times10^{-6}}{9\times1.8\times10^{-4}} = 1.2$ cm/sec.

$\rho = 1, \; \sigma \to 0$ for air.

That is why raindrops fall with slow rate. $v \neq f(p)$ as $\eta \neq \eta(p)$. Notice also that its independent of mass.

Bigger raindrops fall rapidly through air as Stoke's law does not hold if $a > 0.01$ cm because of turbulence, where $F \propto \sqrt{v}$.

If $\sigma > \rho$ then $v < 0$ meaning if density of fluid > density of body, then body moves through the fluid in the upward direction. This is the reason, why air bubbles in water or in any other liquid rise up.

Equation of motion of a body falling through a viscous medium

Viscous force $\propto$ velocity (instantaneous), but now we have downward force mg due to gravity for sedimentation & no thermal (Brownian) force.

$$m\frac{d^2x}{dt^2} = mg - f = mg - \gamma\frac{dx}{dt}$$

or $\frac{d}{dt}\left(\frac{dx}{dt}\right) = g - \frac{\gamma}{m}\frac{dx}{dt}$ or $\frac{dv}{dt} = g - \frac{\gamma}{m}v$

or $\frac{dv}{g - \gamma/_m v} = dt$ or $-\frac{m}{\gamma}\frac{d(g - \gamma/_m v)}{g - \gamma/_m v} = dt$

or $\int \frac{d(g - \gamma/_m v)}{g - \gamma/_m v} = -\frac{\gamma}{m}\int dt + c$

or $\ln(g - \frac{\gamma}{m}v) = -\frac{\gamma}{m}t + c$. Now substitute the boundary condition, $t = 0$, $v = 0$ $\therefore c = \ln g$.

or $\ln\left(1 - \frac{\gamma v}{mg}\right) = -\gamma t/_m$ or $v = \frac{mg}{\gamma}(1 - e^{-\frac{\gamma}{m}t})$.

$\simeq \frac{mg}{\gamma}$ (if $\frac{\gamma t}{m} \gg$)

So the maximum (terminal) velocity is $v = mg/_\gamma$

Again, $\frac{dx}{dt} = \frac{mg}{\gamma}(1 - e^{-\gamma/_m t})$

Integrating, $x = \frac{mg}{\gamma}(t + \frac{m}{\gamma}e^{-\gamma t/_m}) + c'$. Again substitute the B.C. at $t = 0$, $x = 0$.

$\therefore$ $c' = -\frac{m^2 g}{\gamma^2}$

$\therefore$ $x = \frac{mg}{\gamma}(t + \frac{m}{\gamma}e^{-\gamma t/_m}) - \frac{m^2 g}{\gamma^2}$

$= \frac{mgt}{\gamma} + ge^{-\gamma t/_m} - \frac{m^2 g}{\gamma^2}$

Flow of Gas through a narrow tube

Unlike incompressible liquids (density is independent of pressure), gas is compressible (density $\propto$ pressure). So for a liquid, volume flowing through any cross-section in a given time is constant while for a gas, mass flowing through a crosssection in a given time is constant.

$\therefore \rho \propto P$ & ρV = constant or PV = constant (Boyle's law)

Let us consider an elemental length dx with pressure difference dP within the tube, & is small compared to the tube so that density variation within dx <<<< so that we can still write Poiseuille's equation for flowing liquid

$$V = -\frac{\pi a^4}{8\eta}\frac{dP}{dx}$$. It's –ive because $\frac{dP}{dx} < 0$ so that $V > 0$

If P_1 & P_2 are pressure of gas at inlet & outlet end with V_1 volume entering per unit time, then

$$P_1 V_1 = PV = -P\frac{\pi a^4}{8\eta}\frac{dP}{dx}$$

or $$\int_0^l P_1 V_1\, dx = -\frac{\pi a^4}{8\eta}\int_{P_1}^{P_2} P\,dP = -\frac{\pi a^4}{8\eta}\,\frac{P_2^2 - P_1^2}{2}$$

or $$P_1 V_1 l = \frac{\pi a^4 (P_1^2 - P_2^2)}{16\eta}$$ or $$P_1 V_1 = \frac{\pi a^4 (P_1^2 - P_2^2)}{16\eta l}$$

and $$\eta = \frac{\pi a^4 (P_1^2 - P_2^2)}{16 P_1 V_1 l}.$$ Here we have assumed "no-slip" or no relative motion between tube wall & adjacent gas layers, which breaks down at low pressure. The corrected form is

$$\eta = \frac{\pi a^4 (P_1^2 - P_2^2)}{16 P_1 V_1 l}\left(1 + \frac{4\lambda}{a}\right) - \frac{\rho V_1}{8\pi l}\left(K + \ln P_1/P_2\right)$$

λ = slipping coefficient (constant for gas) & K depends on apparatus.

Dependence of Viscosity on pressure & temperature

Liquids: η of liquids increases rapidly with pressure. However for water, glycerol, η decreases with pressure. η decreases rapidly with increase in temperature. For pure liquids $\eta = \frac{A}{(1+BT)^n}$ with A, B, n depending on nature of liquid. For liquid mixture there is no one recipe.

Gases: Using kinetic theory of gases & experiment, established is

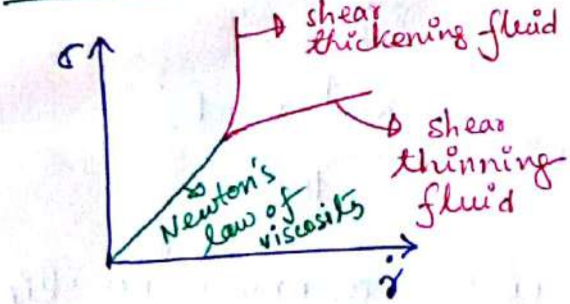
(i) At high pressures, η increases with pressure increment.
(ii) At moderate pressures, η is independent of pressure.
(iii) At low pressures, $\eta \propto P$.

(i) At high temperature, η rapidly increases with temperature & for Mercury $\eta = \alpha T^{1.6}$ (Kinetic theory $\eta = KT^{0.5}$).
(ii) At moderate temperature, $\eta = \eta_0 \frac{\alpha T^{0.5}}{1+\frac{S}{T}}$, η_0 is at $0^\circ C$

α, S = constants

(iii) At low temperature, no one formula agrees well.

Non-Newtonian liquids



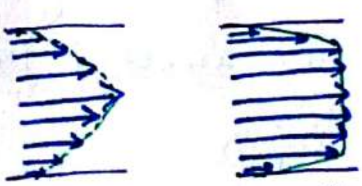
$\sigma \not\propto \dot{\gamma}$ thixotrophic fluid

sol $\leftrightarrow$ gel transition, both property of solid & fluid $\rightarrow$ Glass!!

Examples: shear thinning $\rightarrow$ "colloid"

e.g. paint, milk, blood.

Shear thickening: cornstarch solvent, molten chocolate.



Newtonian non-Newtonian

Displacement of fluid particles are usually non-affine even if Couette flow is established with linear flow profile.

PHSA CC-1-2 TH MECHANICS: Non-Inertial Systems

Instructor: AKB

Books: 1. An Introduction to Mechanics → Kleppner/Kolenkow (Tata McGraw Hill). ⇒ Good for problem solving

2. Theoretical Mechanics → M.R. Spiegel (Schaum Series) ⇒ Good to learn solved problems & for solving problems.

3. Feynman lectures on Physics (Vol. 1) → Feynman/Leigton/Sands (Narosa) ⇒ Good for concept building from not so conventional thinking.

4. Berkeley Physics Course (vol 1) → Kittel/Knight/Ruderman/Helmholtz/Moyer (Tata Mc Grow Hill) ⇒ Very good book for concept development.

5. Fundamentals on Physics → Halliday/Resnick/Walker (John Wiley & Sons) ⇒ Less theoretic, more application oriented, good for practical knowledge.

Newton's law & inertial systems (recapitulation) ⇒

→ Describe the behaviour of point masses (where size of the body is small compared with the interaction distance)

→ Applies to particulate system and not suitable for continuous medium like fluid.

→ Interaction between two charged objects violates Newton's 3rd law as the interaction produced by the created electric fields is not instantaneously transmitted but propagates at the speed of light $c \sim 3\times10^8$ m/sec. Within the propagation time, violation occurs

1st law: $\vec{a} = 0$ when $\vec{F} = 0$

2nd law: $\vec{F} = m\vec{a}$, if $\frac{dm}{dt} = 0$ $(v << c)$

3rd law: $\vec{F}_{12} = -\vec{F}_{21}$ [unit $1N = 10^3$ gm $\times 10^2$ cm/s^2 $= 10^5$ dyn]

Newton's laws hold true (1st & 2nd law) only when observed in inertial reference frame, in which a body devoid of a force or torque is not accelerating, either at rest or moving at a constant speed. But suppose, if the reference frame is at rest on a rotating merry-go-round, one doesn't have zero acceleration in the absence of applied forces. One can stand still on the merry-go-round only by pushing some part or causing that part to exert a force $m\omega^2 r$ on someone toward the axis of rotation, ω = angular acceleration. Or suppose the reference frame is at rest in an aircraft that accelerates rapidly during takeoff, where someone is pressed back against the seat by the acceleration. & someone is at rest relative to the airplane by the force exerted on someone by the back of the seat.

Example: Ultracentrifuge: Moving out of inertial frame of reference have enormous effect on practical applications, e.g. to increase acceleration of a molecule suspended in a liquid compared to acceleration due to gravity. g.

If the molecule rotates at 10 cm from the axis of rotation with 1000 revolutions/sec or 6×10^4 rpm, then angular velocity

$$\omega = 2\pi \times 10^3 \simeq 6 \times 10^3 \text{ rad/sec.}$$ & linear velocity

$$v = \omega r \simeq 6 \times 10^3 \times 10 \simeq 6 \times 10^4 \text{ cm/s}$$

$$a = \omega^2 r \simeq (6 \times 10^4)^2 \times 10 \simeq 4 \times 10^8 \text{ cm/s}^2, \quad g = 980 \text{ cm/s}^2$$

$\therefore \dfrac{a}{g} \simeq \dfrac{4 \times 10^8}{980} \simeq 4 \times 10^5$. Due to such high acceleration, molecules having density different from surrounding fluid will see a strong force to separate out from the fluid.

To a fixed frame (laboratory), molecule wants to remain at rest or move with constant speed in straight line & not dragged with high ω. So to an observer at rest in the ultracentrifuge, molecule is exerted a "centrifugal" force $m\omega^2 r$ to pull it away from the axis of rotation.

If $m = 10^5 \times$ mass of proton $= 10^5 \times 1.7 \times 10^{-24} \simeq 2 \times 10^{-19}$ gm

then $F = ma = m\omega^2 r \simeq 2 \times 10^{-19} \times 4 \times 10^8 \simeq 8 \times 10^{-11}$ dyne.

Centrifugal force outward is balanced by the drag force by the surrounding liquid on the molecule. Due to density differance there will be stratification of layer, so that in the reference frame of the ultracentrifuge, centrifugal force is like an artificial gravity directed outward with increasing intensity with distance from axis.

Force measured in inertial frame is called true force. The Earth as a reference (inertial) frame is a good approximation, but not completely. A mass at rest on Earth surface at the equator experiences a centripetal acceleration $a = \frac{v^2}{R_e} = \omega_e^2 R_e$

Now $\omega_e = 2\pi f_e = \frac{2\pi}{T_e} = \frac{2\pi}{1\text{ day}} = \frac{2\pi}{8.64 \times 10^4} = 7.3 \times 10^{-5}\ \text{sec}^{-1}$

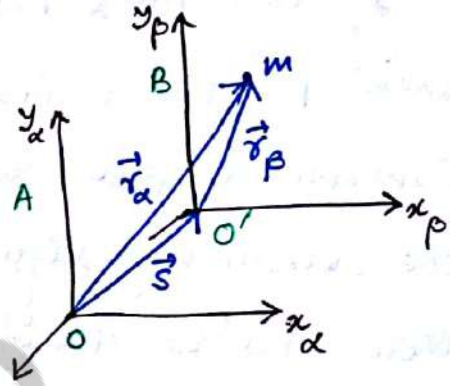
with $R_e = 6.4 \times 10^8$ cm, $a = (7.3 \times 10^{-5})^2 \times 6.4 \times 10^8 \simeq 3.4\ \text{cm/s}^2$

As this is the force supplied to a point mass at equator, force necessary to hold the mass in equilibrium against gravity is $3.4\,m$ dynes less than that of mg. Rest of the variation in g is due to the ellipsoidal shape & pole to equator variation is $5.2\ \text{cm/s}^2$

Since 1 year $\simeq \pi \times 10^7$ sec, angular velocity of Earth about the Sun is $\omega \simeq \frac{2\pi}{\pi \times 10^7} \simeq 2\times 10^{-7}$ sec^{-1}. With $R \simeq 1.5\times 10^{13}$ cm, the centripetal acceleration of Earth about Sun is

$a = \omega^2 R \simeq (2\times 10^{-7})^2 \times 1.5 \times 10^{13} \simeq 0.6$ cm/s^2 which is one order of magnitude smaller than the acceleration at equator due to the rotation of Earth.

Galilean Transformation :



Let us consider two frames of reference A & B such that A is at rest & B moves with a constant velocity $\vec{v}$ with respect to A. We want to find the transformation that relates the coordinates $\vec{r}_\alpha$ & time t_α as measured from A frame to the coordinates $\vec{r}_\beta$ & time t_β as measured from B. At $t=0$, both O & O' origins coincide. Suppose Newton's law is read on A & B as

$\vec{F}_\alpha = m\vec{a}_\alpha$, $\vec{F}_\beta = m\vec{a}_\beta$. We know $\vec{F}_\alpha$ is inertial frame measured true force & seek a relation between $\vec{F}_\alpha$ & $\vec{F}_\beta$.

By construction $\vec{s} = \vec{v}t$, if we define a set of transformation

$$\boxed{\vec{r}_\alpha = \vec{r}_\beta + \vec{v}t, \quad t_\alpha = t_\beta}$$

then we see, by differentiation, $\vec{v}_\alpha = \vec{v}_\beta + \vec{v}$ & $\vec{a}_\alpha = \vec{a}_\beta$ as $\frac{d\vec{v}}{dt} = 0$. $\therefore \vec{F}_\beta = m\vec{a}_\beta = m\vec{a}_\alpha = \vec{F}_\alpha$.

So the above set of transformation leads $\vec{F}_\beta$ to be also true force or B frame to be inertial. These are called the Galilean transformation, where axiomatically (without thinking much) we

have considered $t_\alpha = t_\beta$ or time is independent of the frame of reference. This is incorrect if $v \approx c$ while $t_\beta = t_\alpha\sqrt{1-\frac{v^2}{c^2}}$
Similarly we assumed same scale is used in A & B for measuring distance, but near $v \approx c$ $L_\beta = L_\alpha \sqrt{1-v^2/c^2}$ which is known in Special theory of Relativity as "Lorentz contraction" of a moving rod. for practical purpose, say velocity of a satellite around Earth is 8 Km/s & so $v^2/c^2 \approx 10^{-9}$.

Similarly moving mass differs from rest mass as $m = m_0/\sqrt{1-v^2/c^2}$
Principle of relativity → laws of physics are same in all inertial systems. In Einstein's relativity, not Galilean but Lorentz transformation is valid.

<u>Uniformly Accelerating Systems (Non inertial)</u>:

Suppose now frame B accelerates at constant rate $\vec{A}$ w.r.t. inertial frame A. We label quantities in noninertial frame B with prime. As $\frac{d\vec{v}}{dt} = \vec{A} \neq 0$ now,

$$\vec{a} = \vec{a}' + \vec{A}$$

So in the accelerated system, the measured (apparent) force is $\vec{F}' = m\vec{a}' = m\vec{a} - m\vec{A} = \vec{F} - m\vec{A} = \vec{F}_{true} + \vec{F}_{fict}$

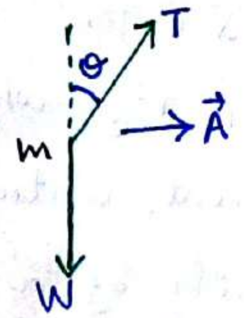
true force as measured in A

fictitious force $-m\vec{A}$

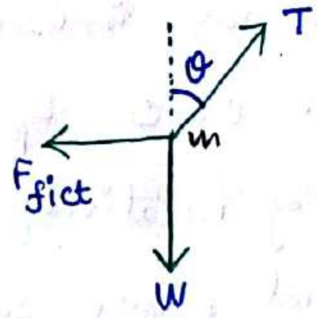
Fictitious force is oppositely directed and proportional to mass (just like Gravitational force). But origin of such force is not physical interaction, but acceleration of the coordinate system.

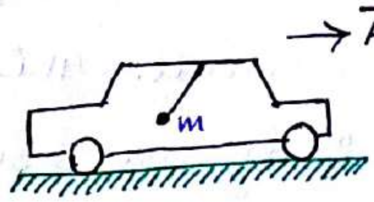
Apparent force of Gravity

Laboratory frame



Accelerating frame





A small mass m hangs from a string in an accelerating car. Determine the static angle of the string with vertical & tension of the string.

Laboratory frame:

$T\cos\theta = W = mg$

$T\sin\theta = mA$

$\therefore T = m\sqrt{g^2+A^2}$

$\tan\theta = \frac{mA}{mg} = \frac{A}{g}$

Accelerating frame:

$T\cos\theta = W = mg$

$T\sin\theta = F_{fict} = mA$

$T = m\sqrt{g^2+A^2}$

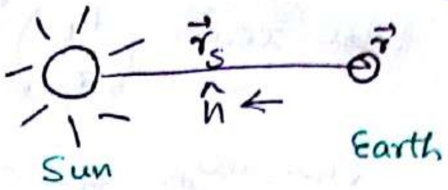
$\tan\theta = \frac{A}{g}$

The Principle of Equivalence

The laws of physics in a uniformly accelerating system are identical to those in an inertial system after introducing a fictitious force $\vec{F}_{fict} = -m\vec{A}$, so $F_{fict} \propto m$ as gravitational force with $\vec{A} = -\vec{g}$. This two scenarios, one where a particle experiences local gravitational field $\vec{g}$, & where a particle in free space (no $\vec{g}$) uniformly accelerating at rate $\vec{A} = -\vec{g}$ are equivalent, but one cannot clearly distinguish these two scenarios → Mach's principle & Einstein's conjecture.

Real fields are local & at large distance they decrease while an accelerating coordinate system is nonlocal & extends uniformly throughout space. Only for small systems are the two indistinguishable.

Tidal forces



The Earth is in free fall toward the sun & according to "Principle of Equivalence" it should be impossible to observe Sun's gravitational force on Earth locally. Due to massive size, tidal effect (nonlocal) are observed.

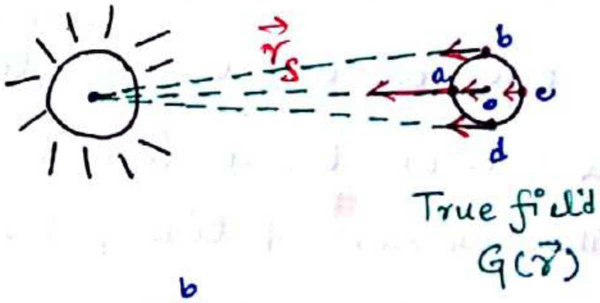
Tides arise as Sun & Moon produce an apparent gravitational field that varies from point to point on Earth surface. If Earth accelerates toward the Sun at rate $\vec{G}_0$ then gravitational field of Sun at center of Earth is $\vec{G}_0 = \frac{GM_S}{r_S^2}\hat{n}$

If $G(\vec{r})$ is the gravitational field of Sun on Earth surface then $\vec{F} = mG(\vec{r})$, so to an observer on Earth, apparent force is

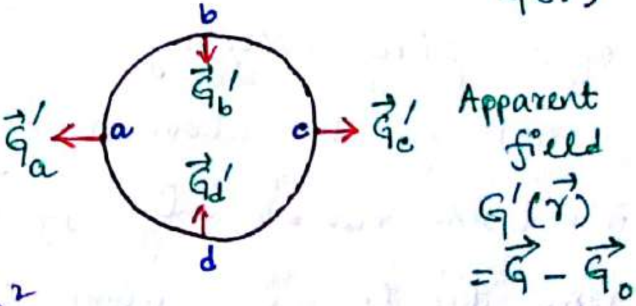
$\vec{F}' = \vec{F} - m\vec{A} = mG(\vec{r}) - m\vec{G}_0$, so apparent field is

$G'(\vec{r}) = G(\vec{r}) - G_0$

We notice at 4 points a, b, c, d having true field $\vec{G}_a, \vec{G}_b, \vec{G}_c, \vec{G}_d$ and at center O, $\vec{G}_0$ the following



$\vec{G}_0 \approx \vec{G}_b \approx \vec{G}_d$; $\vec{G}_a \gg \vec{G}_0 \gg \vec{G}_c$



(i) $\vec{G}_a'$ & $\vec{G}_c'$:

Sun's field at a is $\vec{G}_a = \frac{GM_S}{(r_S - R_E)^2}$

where $\vec{r}_S - \vec{R}_E$ is distance between center of Sun to a. So apparent field is $\vec{G}_a' = \vec{G}_a - \vec{G}_0 = \left[\frac{GM_S}{(r_S - R_E)^2} - \frac{GM_S}{r_S^2}\right]\hat{n} = \hat{n}\frac{GM_S}{r_S^2}\left[\frac{1}{(1-(R_E/r_S))^2} - 1\right]$

or $G_a' = G_0\left[\left(1 - \frac{R_E}{r_S}\right)^{-2} - 1\right] = G_0\left[\cancel{1} + \frac{2R_E}{r_S} + \dots - \cancel{1}\right] \approx 2G_0\frac{R_E}{r_S}$

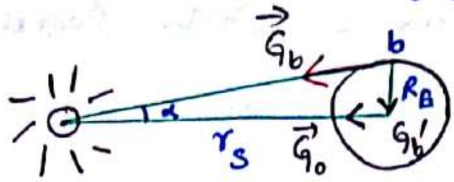
All terms $\left(\frac{R_E}{r_s}\right)^n$ for $n \geq 2$ are neglected as $\frac{R_E}{r_s} = \frac{6.4 \times 10^3 \text{ Km}}{1.5 \times 10^8 \text{ Km}}$

$= 4.3 \times 10^{-5} << 1$

Similarly, $G_c' = G_c - G_0 = G_0\left[\left(1 + \frac{R_E}{r_s}\right)^{-2} - 1\right]$

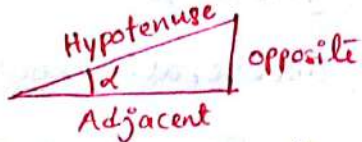
$\simeq -2G_0 \frac{R_E}{r_s}$. $\vec{G}_a'$ & $\vec{G}_c'$ therefore point radially out.

(ii) $\vec{G}_b'$ & $\vec{G}_d'$:



$\vec{G}_b$ & $\vec{G}_0$ are not parallel & angle between them $\alpha \approx \frac{R_E}{r_s} = 4.3 \times 10^{-5} << 1$. We Know

$\vec{G}_b = \vec{G}_b' + \vec{G}_0$ form $\perp^r$ triangle.



$\tan\alpha = \frac{G_b'}{G_0} \approx \alpha$ (for $\alpha << 1$,

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\alpha - \cancel{\frac{\alpha^3}{3!}} + \cancel{\frac{\alpha^5}{5!}} - \cdots}{1 + \cancel{\frac{\alpha^2}{2!}} - \cancel{\frac{\alpha^4}{4!}} + \cdots} = \alpha$)

$\sin\alpha = \frac{\text{Opposite}}{\text{Hypotenuse}}$

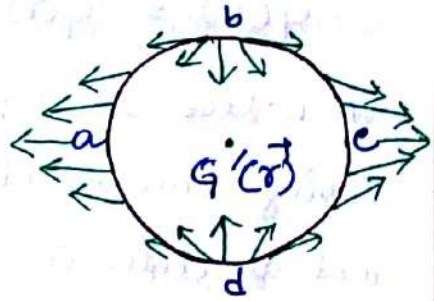
$\cos\alpha = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

$\tan\alpha = \frac{\text{Opposite}}{\text{Adjacent}}$

$\therefore G_b' = G_0\alpha \simeq G_0 \frac{R_E}{r_s}$

By symmetry, G_d' is equal & opposite to G_b' & both of them point toward the center of Earth.

Force at a & c tend to lift the oceans & force at b & d tend to depress them. Thus we have 4 tides, 2 ebb & 2 flood of the tides everyday.



Although the above analysis correctly shows 4 tides but this is not the reason only. If we now consider Moon also & then force due to sun & moon is

$F_S = \frac{G M_E M_S}{r_S^2}$ So, $\frac{F_S}{F_M} = \frac{M_S}{M_M} \times \frac{r_M^2}{r_S^2} \sim 176$ as

$F_M = \frac{G M_E M_M}{r_M^2}$ $\left[\frac{r_S}{r_M} = 390, \ \frac{M_S}{M_M} = 2.68 \times 10^7.\right]$

So force due to Sun is 176 times stronger than that of Moon!

If we now consider the differential attraction between a & c point on Earth with hydrosphere (by any arbitrary mass m) due to Moon,

Attraction at a, $F_a = \frac{G M_M m}{(r_M - R_E)^2}$ & at ~~b~~ c, $F_c = \frac{G M_M m}{(r_M + R_E)^2}$

So differential attraction $T_M = F_a - F_c$

$$= G M_M m \frac{(r_M + R_E)^2 - (r_M - R_E)^2}{(r_M - R_E)^2 (r_M + R_E)^2} = G M_M m \frac{4 r_M R_E}{(r_M^2 - R_E^2)^2}$$

$$= G M_M m \frac{4 r_M R_E}{r_M^4 \left(1 - \frac{R_E^2}{r_M^2}\right)^2} \simeq G M_M m \frac{4 R_E}{r_M^3} \quad \left(\text{as } \frac{r_S}{r_M} = 390\right)$$

$$\left(\frac{R_E}{r_M} = \frac{1}{60}\right)$$

Similarly $T_S = F_a - F_c \simeq G M_S m \frac{4 R_E}{r_S^3}$ for Sun.

$$\therefore \frac{T_M}{T_S} = \frac{M_M}{M_S} \times \frac{r_S^3}{r_M^3} \sim 2.2. \qquad \therefore T_M = 2.2\, T_S$$

Because Moon is nearer to Earth, even though the actual attraction due to Moon is way smaller than the attraction of the sun, but due to differential attraction, tidal force is more prominent.

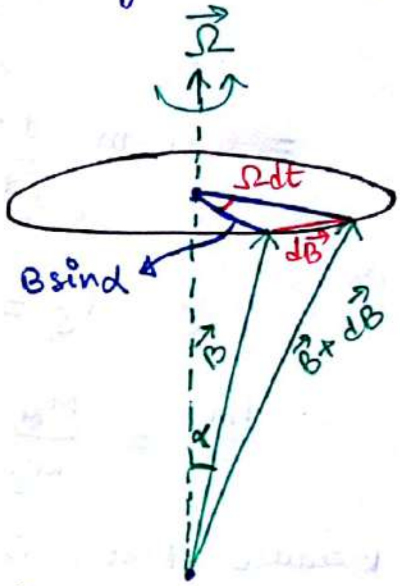
When natural frequency of oscillation of water coming in / flowing out matches with frequency of tidal waves due to coastal topography, large tides (e.g. Tsunami) are produced. The above example can produce tides of the order 2 feet only.

Not every sea (e.g. Mediterranean) has a tidal activity. As tidal bulge moves from east to west due to rotation of Earth, it so happens that Mediterranean sea has opening only to the west & so the tidal bulge cannot enter!

Rotating Coordinate System

As we found in a linearly accelerating system by adding a nonphysical fictitious force $-m\vec{A}$ we could treat the problem in inertial system, we will derive next that by adding two fictitious force : centrifugal force & Coriolis force, motion in a rotating coordinate system can be treated as an inertial system. Focault pendulum & circular nature of weather system on surface of Earth can be explained.

Rate of change of Rotating vector :



To find a relation between inertial & rotating system, suppose $\vec{B}$ rotates at rate Ω about an axis in direction $\vec{\Omega}$.

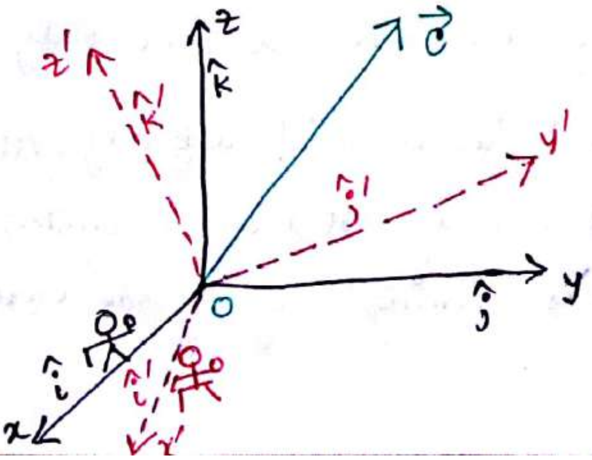
$\angle$ between $\vec{B}$ & $\vec{\Omega} = \alpha$.

In dt time, tip of $\vec{B}$ sweeps a circular path of radius $B\sin\alpha$, so that $\vec{B}(t+dt) = \vec{B}(t) + d\vec{B}(t)$

where, $|d\vec{B}(t)| \simeq |B\sin\alpha \times \vec{\Omega}dt|$

$$\therefore \left|\frac{d\vec{B}}{dt}\right| = \lim_{dt\to 0} \frac{B\sin\alpha\,\Omega dt}{dt} = B\sin\alpha|\vec{\Omega}| = |\vec{\Omega}\times\vec{B}|$$

$d\vec{B} \perp \vec{B}$, $d\vec{B} \perp \vec{\Omega}$. So $\frac{d\vec{B}}{dt} = \vec{\Omega}\times\vec{B}$



Consider inertial frame $[xyz]$ & rotating frame $[x'y'z']$ at a rate $\vec{\Omega}$. $\left.\frac{d\vec{C}}{dt}\right|_{in}$ is the rate of change of $\vec{C}$ measured in $[xyz]$ frame & we want to calculate $\left.\frac{d\vec{C}}{dt}\right|_{rot}$

If $(\hat{i}, \hat{j}, \hat{k})$ and $(\hat{i}', \hat{j}', \hat{k}')$ are the base vectors in inertial & rotating frame, then

$$\vec{c} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k} = c_x' \hat{i}' + c_y' \hat{j}' + c_z' \hat{k}'$$

$$\therefore \left.\frac{d\vec{c}}{dt}\right|_{in} = \frac{d}{dt}\left(c_x' \hat{i}' + c_y' \hat{j}' + c_z' \hat{k}' \right)$$

$$= \left[\frac{dc_x'}{dt} \hat{i}' + \frac{dc_y'}{dt} \hat{j}' + \frac{dc_z'}{dt} \hat{k}' \right] + \left[c_x' \frac{d\hat{i}'}{dt} + c_y' \frac{d\hat{j}'}{dt} + c_z' \frac{d\hat{k}'}{dt} \right]$$

$$= \left.\frac{d\vec{c}}{dt}\right|_{rot} + \left[c_x' \vec{\Omega} \times \hat{i}' + c_y' \vec{\Omega} \times \hat{j}' + c_z' \vec{\Omega} \times \hat{k}' \right]$$

$$= \left.\frac{d\vec{c}}{dt}\right|_{rot} + \vec{\Omega} \times \left(c_x' \hat{i}' + c_y' \hat{j}' + c_z' \hat{k}' \right) = \left.\frac{d\vec{c}}{dt}\right|_{rot} + \vec{\Omega} \times \vec{c}$$

In operator notation, $\boxed{\left.\frac{d}{dt}\right|_{in} = \left.\frac{d}{dt}\right|_{rot} + \vec{\Omega} \times}$

Velocity & Acceleration :

If $\vec{c} =$ position vector $\vec{r}$, then $\left.\frac{d\vec{r}}{dt}\right|_{in} = \left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\Omega} \times \vec{r}$

$$\therefore \boxed{\vec{v}_{in} = \vec{v}_{rot} + \vec{\Omega} \times \vec{r}}$$

$$\therefore \left.\frac{d\vec{v}_{in}}{dt}\right|_{in} = \left.\frac{d\vec{v}_{in}}{dt}\right|_{rot} + \vec{\Omega} \times \vec{v}_{in} = \left.\frac{d}{dt}\left(\vec{v}_{rot} + \vec{\Omega} \times \vec{r}\right)\right|_{rot} + \vec{\Omega} \times \left(\vec{v}_{rot} + \vec{\Omega} \times \vec{r} \right)$$

$$= \left.\frac{d\vec{v}_{rot}}{dt}\right|_{rot} + \vec{\Omega} \times \left.\frac{d\vec{r}}{dt}\right|_{rot} + \vec{\Omega} \times \vec{v}_{rot} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

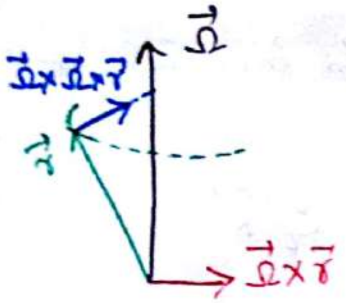
$$= \left.\frac{d\vec{v}_{rot}}{dt}\right|_{rot} + 2\vec{\Omega} \times \vec{v}_{rot} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

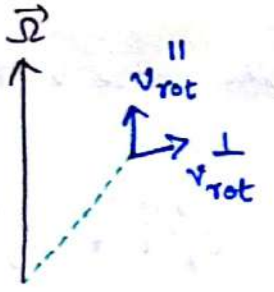
$$\therefore \boxed{\vec{a}_{in} = \vec{a}_{rot} + 2\vec{\Omega} \times \vec{v}_{rot} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})}$$

$\therefore$ In rotating coordinate system

$$\vec{F}_{rot} = m\vec{a}_{rot} = m\vec{a}_{in} - 2m\vec{\Omega} \times \vec{v}_{rot} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

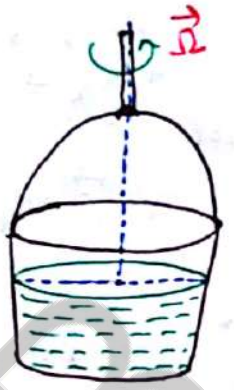
$$= \vec{F}_{in} + \left[\vec{F}_{coriolis} + \vec{F}_{centrifugal} \right] = \vec{F}_{in} + \vec{F}_{fictitious}$$

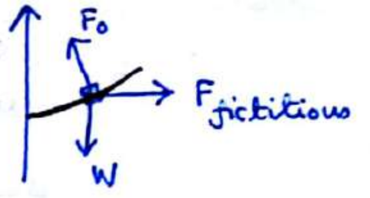


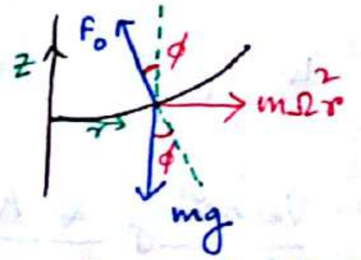


$\vec{F}_{centrifugal} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ is perpendicular to the axis of rotation & radially outward directed. In sketch, centripetal acceleration is radially inward. For $\vec{F}_{coriolis} = -2m\vec{\Omega}\times\vec{v}_{rot}$ & so only $v_{rot}^{\perp}$ contributes $(\vec{\Omega}\times\vec{v}_{rot}^{\parallel} = 0)$.

Surface of a rotating Fluid







F_0 = contact force
$W = mg$ = weight
$F_{fictitious}$ = centrifugal force

To find the shape of the surface of a fluid on a bucket that is rotating with angular speed $|\vec{\Omega}|$, we consider in a coordinate system rotating with the bucket (so that the problem is static).

So on fluid meniscus, total force $= 0$

$$F_0\cos\phi = W = mg$$

$$F_0\sin\phi = m\Omega^2 r \quad \Rightarrow \quad \tan\phi = \frac{\Omega^2 r}{g} = \frac{dz}{dr}$$

$$\Rightarrow \int_0^z dz = \int_0^r \frac{\Omega^2 r}{g}\,dr \quad \Rightarrow \quad z = \frac{\Omega^2}{2g}r^2 \rightarrow \text{Equation of surface}$$

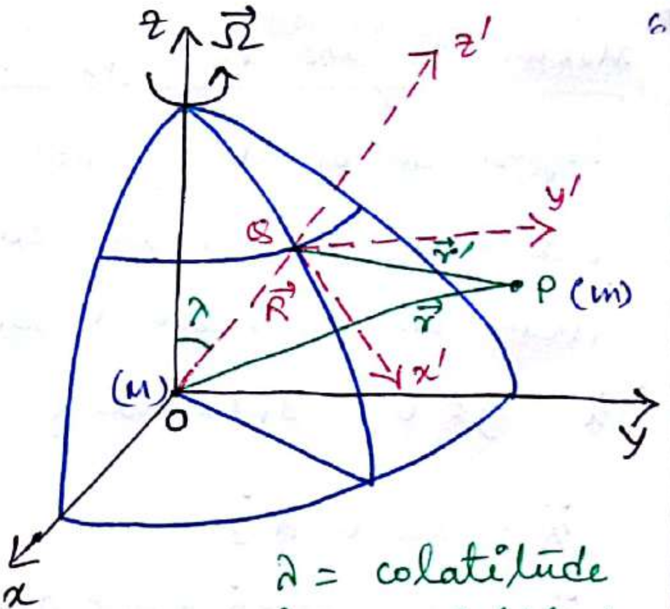
The surface is a parabola of revolution.

Equation of motion of a particle relative to an observer on Earth's surface :

Suppose Earth is spherical with center at O rotating about z-axis with angular velocity $\vec{\Omega} = \Omega\hat{k}$ & its constant, $\dot{\vec{\Omega}} = 0 = \frac{d\vec{\Omega}}{dt}$. Also the frame can be taken inertial by neglecting Earth's rotation around the Sun.

Acceleration of Q relative to O is centripetal acceleration

$$\ddot{\vec{R}} = \frac{d^2\vec{R}}{dt^2} = \vec{\Omega} \times (\vec{\Omega} \times \vec{R})$$



λ = colatitude

$90^\circ - \lambda$ = latitude

Newton's law of Gravitation,

$$\vec{F} = -\frac{GMm}{r^3}\vec{r} = m\frac{d^2\vec{r}}{dt^2}$$

Neglecting air resistance etc.

Now. $$\left.\frac{d^2\vec{r}}{dt^2}\right|_{in} = \left.\frac{d^2}{dt^2}(\vec{R} + \vec{r}')\right|_{in}$$

$$= \left.\frac{d^2\vec{R}}{dt^2}\right|_{in} + \left.\frac{d^2\vec{r}'}{dt^2}\right|_{rot} + 2\vec{\Omega} \times \left.\frac{d\vec{r}'}{dt}\right|_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}') \quad [\text{as } \dot{\vec{\Omega}} = 0]$$

$$\therefore \left.\frac{d^2\vec{r}'}{dt^2}\right|_{rot} = \underbrace{-\frac{GM}{r^3}\vec{r} - \vec{\Omega}\times(\vec{\Omega}\times\vec{R})}_{\vec{g}} - 2\vec{\Omega}\times\vec{v}_{rot} - \cancel{\vec{\Omega}\times(\vec{\Omega}\times\vec{r}')}^{\,0} \quad \text{--- (1)}$$

Near Earth's surface, contribution from $\vec{\Omega}\times(\vec{\Omega}\times\vec{r}') = 0$ & so,

$$\left.\frac{d^2\vec{r}'}{dt^2}\right|_{rot} = g - 2\vec{\Omega}\times\vec{v}_{rot}.$$ Any other external force has to be added to this equation.

Now, $$\hat{k} = (\hat{k}\cdot\hat{i}')\hat{i}' + (\hat{k}\cdot\hat{j}')\hat{j}' + (\hat{k}\cdot\hat{k}')\hat{k}'$$

$$= -\sin\lambda\,\hat{i}' + 0\,(\hat{k}\perp\hat{j}') + \cos\lambda\,\hat{k}'$$

So, $$\vec{\Omega} = \Omega\hat{k} = -\Omega\sin\lambda\,\hat{i}' + \Omega\cos\lambda\,\hat{k}'$$

$$\therefore \vec{\Omega}\times\vec{v}_{rot} = \begin{vmatrix} \hat{i}' & \hat{j}' & \hat{k}' \\ -\Omega\sin\lambda & 0 & \Omega\cos\lambda \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix}$$

$$= \hat{i}'(-\Omega\cos\lambda\,\dot{y}') + \hat{j}'(\Omega\cos\lambda\,\dot{x}' + \Omega\sin\lambda\,\dot{z}') - \hat{k}'\Omega\sin\lambda\,\dot{y}'$$

Substituting in (1) & equating coefficients, we get

$$\ddot{x}' = 2\Omega\cos\lambda\,\dot{y}', \quad \ddot{y}' = -2\Omega(\dot{x}'\cos\lambda + \dot{z}'\sin\lambda), \quad \ddot{z}' = -g + 2\Omega\dot{y}'\sin\lambda$$

Motion on the Rotating Earth

Suppose an object of mass m located at $x = y = 0$, $z = h$ & at rest is dropped to the Earth's surface. Due to the Coriolis force straight line motion is turned into circular motion. At $t = 0$, $\dot{x} = \dot{y} = 0$. Integrating the equation of motion,

$$\ddot{x} = 2\Omega\cos\lambda\,\dot{y} \qquad\qquad \dot{x} = 2\Omega\cos\lambda\, y + C_1$$

$$\ddot{y} = -2(\Omega\cos\lambda\,\dot{x} + \Omega\sin\lambda\,\dot{z}) \qquad\qquad \dot{y} = -2(\Omega\cos\lambda\, x + \Omega\sin\lambda\, z) + C_2$$

$$\ddot{z} = -g + 2\Omega\sin\lambda\,\dot{y}$$

substituting boundary condition (B.C.), $\dot{x} = 2\Omega\cos\lambda\, y$

$$\dot{y} = -2(\Omega\cos\lambda\, x + \Omega\sin\lambda\, z) + 2\Omega\sin\lambda\, h$$

$$\therefore\ \ddot{z} = -g - \cancel{4\Omega^2}^{\to 0}\sin\lambda(\cos\lambda\, x + \sin\lambda(z-h)) \simeq -g$$

[as $\Omega^2 << g$]

$$\text{or}\quad \dot{z} = -gt + C_3 \quad \&\ \text{at}\ t=0,\ \dot{z}=0 \quad \therefore\ \dot{z} = -gt$$

$$\therefore\ \ddot{y} = (-2\Omega\cos\lambda)(2\Omega\cos\lambda\, y) + (-2\Omega\sin\lambda)(-gt)$$

$$= -\cancel{4\Omega^2}^{\to 0}\cos^2\lambda\, y + 2\Omega\sin\lambda\, gt \quad \simeq 2\Omega\sin\lambda\, gt$$

$$\therefore\ \dot{y} = \Omega g\sin\lambda\, t^2 + C_4 \quad \&\ \text{at}\ t=0,\ \dot{y}=0 \quad \therefore\ C_4 = 0$$

$$\text{or}\quad \dot{y} = \Omega g\sin\lambda\, t^2 \quad \text{or}\quad y = \tfrac{1}{3}\Omega g\sin\lambda\, t^3 + C_5 \quad \&\ \text{at}\ t=0,\ y=0,\ C_5 = 0$$

$$\text{or}\quad y = \tfrac{1}{3}\Omega g\sin\lambda\, t^3$$

So after time t, object is deflected to east of the vertical.

Again, $z = -\frac{1}{2}gt^2 + C_6$ & at $t=0$, $z=h$ $\therefore\ C_6 = h$.

$\therefore\ z = h - \frac{1}{2}gt^2$. $\therefore$ By time $h - \frac{1}{2}gt^2 = 0$, $z = 0$

or $t = \sqrt{\frac{2h}{g}}$ object touches ground. The total deflection is

$$y = \frac{\Omega g}{3}\sin\lambda\left(\frac{2h}{g}\right)^{3/2} = \frac{\sqrt{8}}{3}\Omega g^{-1/2}\sin\lambda\, h^{3/2} = \sqrt{\frac{8h^3}{9g}}\,\Omega\sin\lambda.$$

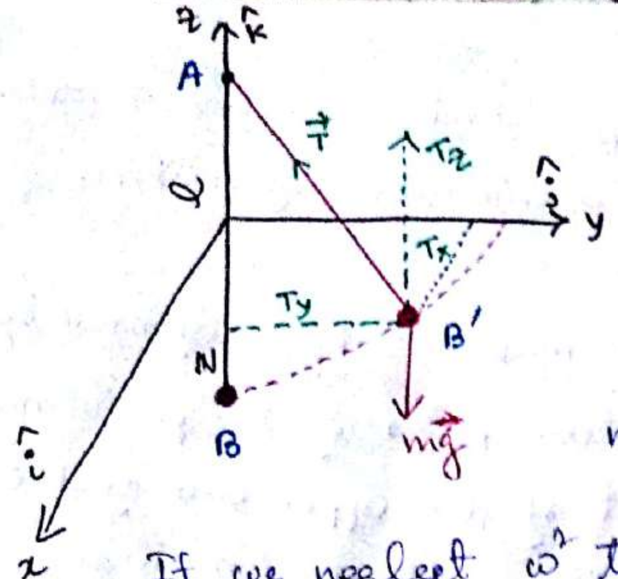
The Foucault Pendulum

It is a simple device to conveniently detect the slowest rotation of the Earth and provides a direct experimental confirmation of the existence of the Coriolis force.

Construction: It consists of a heavy mass (28 kg) suspended by a large wire (70 metre), so that the time period of the pendulum is very long (17 seconds). The attachment of the upper end of the wire allows the pendulum to swing with equal freedom in any direction, so that the period of oscillation in any plane is exactly the same. A Foucault pendulum once set in oscillation continues to oscillate for a fairly long time, in a definite vertical plain.

Working Principle: The plain of oscillation is observed to precess (rotate) around the vertical axis within a period of several hours. If the pendulum is setup at the North pole of Earth, it will oscillate as a simple pendulum in a fixed vertical plain as in an inertial frame of reference. Since the Earth rotates from west to East with an angular velocity $\vec{\omega}$, to an observer on the surface of the Earth, plane of oscillation will appear to be turning from East to west (opposite direction) with angular velocity $\vec{\omega}$. It is not necessary that the pendulum to be mounted right at the North or South pole of the Earth. An apparent rotation of the plain of oscillation of the pendulum due to rotation of the Earth will be observed in any latitude on the Earth, except at the Equator.

In the Cartesian coordinate system, suppose from point A, the Focault pendulum of length l is suspended. The tension in the string is given by,



$$\vec{T} = (\vec{T}\cdot\hat{i})\hat{i} + (\vec{T}\cdot\hat{j})\hat{j} + (\vec{T}\cdot\hat{k})\hat{k}$$
$$= T\cos\alpha\,\hat{i} + T\cos\beta\,\hat{j} + T\cos\gamma\,\hat{k}$$
$$= -T\frac{x}{l}\hat{i} - T\frac{y}{l}\hat{j} + T\frac{l-z}{l}\hat{k}$$

So the equation of motion of the bob is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{T} + m\vec{g} - 2m(\vec{\omega}\times\vec{v}) - \cancel{m\vec{\omega}\times(\vec{\omega}\times\vec{r})}^{\,0}$$

If we neglect ω^2 term (last term) for simplicity, then,

$$m\frac{d^2x}{dt^2} = -T\frac{x}{l} + 2m\omega\dot{y}\cos\lambda$$
$$m\frac{d^2y}{dt^2} = -T\frac{y}{l} - 2m\omega(\dot{x}\cos\lambda + \dot{z}\sin\lambda)$$
$$m\frac{d^2z}{dt^2} = T\left(\frac{l-z}{l}\right) - mg + 2m\omega\dot{y}\sin\lambda$$

Now if we assume that the motion of the bob takes place in the XY plain, then $z = \dot{z} = \ddot{z} = 0$.

So, $0 = T - mg + 2m\omega\dot{y}\sin\lambda$ or $T = mg - 2m\omega\dot{y}\sin\lambda$.

$$\therefore \frac{d^2x}{dt^2} = -\frac{gx}{l} + \cancel{\frac{2\omega x}{l}\dot{y}\sin\lambda}^{\,0} + 2\omega\dot{y}\cos\lambda.$$
$$\frac{d^2y}{dt^2} = -\frac{gy}{l} + \cancel{\frac{2\omega y}{l}\dot{y}\sin\lambda}^{\,0} - 2\omega\dot{x}\cos\lambda.$$

The above nonlinear differential equation can be linearized in the limit, x, y, ω are small, so that $x\dot{y}\omega$ or $y\dot{y}\omega$ etc term can be neglected.

$$\therefore \ddot{x} = -\frac{gx}{l} + 2\omega\dot{y}\cos\lambda, \quad \ddot{y} = -\frac{gy}{l} - 2\omega\dot{x}\cos\lambda.$$

Suppose that initially the bob is in the YZ plain and is given a displacement from the z-axis of magnitude A. So initial condition is at $t=0$, $x = \dot{x} = 0$, $y = A$, $\dot{y} = 0$. Conveniently we put $k^2 = \frac{g}{l}$ and $\omega\cos\lambda = \alpha$; so that equations become,

$\ddot{x} = -k^2 x + 2\alpha \dot{y}$ and $\ddot{y} = -k^2 y + 2\alpha \dot{x}$. We can solve this linear second order coupled differential equation (See Prob. 6.20, Chapter-6, M.R. Spiegel) to get.

$$x = A \cos kt \sin \alpha t = A \cos\left(\sqrt{\frac{g}{l}}\, t\right) \sin(\omega t \cos\lambda)$$

$$y = A \cos kt \cos \alpha t = A \cos\left(\sqrt{\frac{g}{l}}\, t\right) \cos(\omega t \cos\lambda)$$ or in vector form

$\vec{r} = x\hat{i} + y\hat{j} = A \cos \sqrt{\frac{g}{l}}\, t\ \hat{n}$ where, $\hat{n} = \sin(\omega t \cos\lambda)\hat{i} + \cos(\omega t \cos\lambda)\hat{j}$ is a unit vector. The time period of $\cos\left(\sqrt{\frac{g}{l}}\, t\right)$ $\left[T = 2\pi\sqrt{\frac{l}{g}}\right]$ is very small compared to the time period of $\hat{n}$ $[T' = 2\pi/\omega\cos\lambda]$, so that $\hat{n}$ is a very slowly precessing (rotating) unit vector. Thus, physically the Foucault pendulum oscillates in a plane through the z-axis which is slowly rotating about the z-axis.

At $t = 0$, $\hat{n} = \hat{j}$, $y = A$. After $t = \frac{T'}{8} = \frac{2\pi}{8\omega\cos\lambda}$, so that $\hat{n} = \sin\frac{\pi}{4}\hat{i} + \cos\frac{\pi}{4}\hat{j}$, rotation of the plane is in the clockwise direction as observed from Earth's surface in the northern hemisphere (where $\cos\lambda > 0$) and counterclockwise direction in the southern hemisphere (where $\cos\lambda < 0$). This rotation of the plane was observed by Foucault in 1851.

Q. Calculate the latitude where the plane of vibration rotates once a day. $T' = \frac{2\pi}{\omega\cos\lambda}$ & $\frac{2\pi}{\omega} = 1$ day $= 24$ hrs.

$\therefore T' = 24$ hrs. $\therefore \cos\lambda = 1 \Rightarrow \lambda = 0°$ i.e. at poles.

Fluid Motion

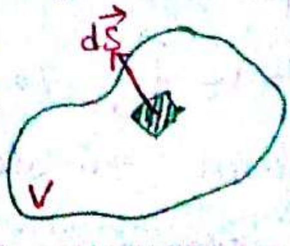
Ideal Fluids - The flow of "dry" Water

Fluid dynamics concerns with the motion of fluids (liquid & gas) in a macroscopic sense to regard as a continuous medium. Infinitely small elements of volume - fluid particle & point in a fluid means very small compared to volume of body but large compared to the molecular distance.

The Equation of Continuity: Mathematically, the state of a moving fluid is given by the fluid velocity distribution over space & time, $\vec{v} = \vec{v}(x,y,z,t)$ and of any two thermodynamic quantities, say, pressure $p(x,y,z,t)$ and density $\rho(x,y,z,t)$. So if given 3-components of velocity, pressure & density, state of the fluid is completely determined. Additionally, a conducting fluid will carry an electric current whose density $\vec{j} = \vec{j}(x,y,z,t)$. Similarly temperature or Magnetic field have similar effect.

We neglect first EM field, temperature variation & assume that density is constant or variation in pressure is very small (or the fluid is incompressible). So if the flow velocity is much less than the speed of sound wave in the fluid, density variation can be neglected. $\boxed{\rho = \text{constant}}$

Conservation of Mass: If matter flows away then there must be decrease in the amount of matter left behind. The mass of fluid flowing in unit time through a surface element $d\vec{S}$ bounding the volume is $\rho\vec{v}\cdot d\vec{S}$ & its positive if flowing out (negative otherwise), so that the total mass is $\oint_S \rho\vec{v}\cdot d\vec{S}$



Decrease in fluid mass per unit time is $-\frac{\partial}{\partial t}\int_V \rho \, dV$.

Therefore $\frac{\partial}{\partial t}\int_V \rho \, dV = -\oint_S \rho \vec{v}\cdot d\vec{S} = \int \vec{\nabla}\cdot(\rho\vec{v})\, dV$ using Green's theorem. $\therefore \int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right] dV = 0$. Since this equation must hold for any volume V, the integrand must vanish. So $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$ $\vec{j} = \rho\vec{v}$ = mass flux density.

This is the hydrodynamic equation of continuity leading to conservation of mass. For incompressible fluid ρ = constant & so $\vec{\nabla}\cdot\vec{v} = 0$. Like Magnetic field $\vec{B}$, fluid velocity has zero divergence.

Euler's equation of motion: Change of velocity due to forces, torques, so that Newton's 2nd law become,

Rate of increase of momentum of fluid particle = Sum of forces on fluid particle.

There are 2 types of forces on fluid particles,

- Surface forces $\Rightarrow$ - pressure force, viscous force, gravity force
- Body forces $\Rightarrow$ - centrifugal force, Coriolis force, EM force.

So total force acting on the volume $= -\oint_S P\, d\vec{S} = -\int_V \vec{\nabla}P\, dV$

So fluid surrounding any volume element dV exerts a force $-\vec{\nabla}P\, dV$ or $-\vec{\nabla}P$ per unit volume. There are external forces like electromagnetic, gravity. For conservative force with ϕ = potential per unit mass, $-\rho\vec{\nabla}\phi$ = force density, otherwise for non-conservative force $\vec{f}_{ext}$ has to be taken care. Due to shearing stress in a flowing fluid, there are internal force per unit volume $\vec{f}_{visc}$, so that Newton's law become,

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla}p - \rho\vec{\nabla}\phi + \cancel{f_{visc}}$$ dry water/ thin liquid (inviscid flow)

The derivative $\frac{d\vec{v}}{dt}$ denotes not the change of rate of the fluid velocity at a fixed point in space but the rate of change of the velocity of a given fluid particle as it moves about in space. So to express $\frac{d\vec{v}}{dt}$ in terms of quantities referring to fixed in space we see composition from two parts, ① change during dt in the velocity at a fixed point in space, ② velocity difference at same instant at two points $d\vec{r}$ apart.

$$\therefore \rho \frac{d\vec{v}}{dt} = \rho \frac{\partial \vec{v}}{\partial t} + \frac{\rho}{dt}\left(dx \frac{\partial \vec{v}}{\partial x} + dy \frac{\partial \vec{v}}{\partial y} + dz \frac{\partial \vec{v}}{\partial z} \right)$$

$$= \rho \frac{\partial \vec{v}}{\partial t} + \rho\left(\frac{d\vec{r}}{dt}\cdot\vec{\nabla}\right)\vec{v} = \rho\, \partial_t \vec{v} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v}$$

Note that there can be acceleration even though $\frac{\partial \vec{v}}{\partial t} = 0$ so that velocity at a given point is not changing, e.g. water flowing in a circle at constant speed is accelerating due to change in direction of the centripetal acceleration.

$$\therefore \boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi}$$ Euler's Equation

If we define $\vec{\Omega} = \vec{\nabla}\times\vec{v}$, then using the vector identity

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = (\vec{\nabla}\times\vec{v})\times\vec{v} + \frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v}) = \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2$$

$$\therefore \frac{\partial \vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2 = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi$$

$\vec{\Omega}$ is called the "vorticity" & for an irrotational flow, $|\vec{\Omega}| = 0$

Circulation of a vector field around any arbitrary closed loop in a fluid at a given instant is

$$\text{Circulation } \Gamma = \oint_c \vec{v}\cdot d\vec{l} \quad \text{(line integral)}$$

Circulation Γ $= \int_S \vec{\nabla}\times\vec{v}\cdot d\vec{S}$ (using Stoke's theorem)

$= \int_S \vec{\Omega}\cdot d\vec{S}$. So vorticity $\vec{\Omega}$ is the circulation around a unit area & perpendicular to the direction of $\vec{\Omega}$.

Conservation of Circulation:

Change in circulation around a "fluid contour" moving over space $= \frac{d}{dt}\oint_C \vec{v}\cdot\delta\vec{l} = \oint_C \frac{d\vec{v}}{dt}\cdot\delta\vec{l} + \oint_C \vec{v}\cdot\frac{d\delta\vec{l}}{dt}$

Now $\vec{v}\cdot\frac{d\delta\vec{l}}{dt} = \vec{v}\cdot\delta\frac{d\vec{l}}{dt} = \vec{v}\cdot\delta\vec{v} = \frac{1}{2}\delta(v^2)$ and then

$\oint_C \frac{1}{2}\delta(v^2) = 0$ as total differential along closed contour $= 0$.

$$\therefore \frac{d\Gamma}{dt} = \oint_C \frac{d\vec{v}}{dt}\cdot\delta\vec{l} = \oint_C\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right)\cdot\delta\vec{l}$$

$$= -\oint_C \vec{\nabla}\left(\frac{P+\phi}{\rho}\right)\cdot\delta\vec{l} \quad \text{(Using Euler's equation)}$$

$$= -\int_S \cancelto{0}{\vec{\nabla}\times\vec{\nabla}\left(\frac{P+\phi}{\rho}\right)}\cdot\delta\vec{S} = 0.$$

$\therefore \oint \vec{v}\cdot d\vec{l} =$ constant (Kelvin's theorem of conservation of circulation)

So for irrotational flow, $\frac{\partial\Omega}{\partial t} = 0$ & so $\vec{\nabla}\cdot\vec{v} = 0$ & $\vec{\nabla}\times\vec{v} = 0$

It is also called "<u>Potential flow</u>". As $\vec{\nabla}\times\vec{v} = 0$ on streamlines, steady flow past any body with a uniform incident flow at infinity must be a potential flow.

Bernoulli's theorem from Euler's equation, taking $\vec{v}\cdot$ operation

$\vec{v}\cdot\vec{\Omega}\times\vec{v} = 0$ & so

$\vec{v}\cdot\vec{\nabla}\left(\frac{P}{\rho} + \phi + \frac{1}{2}v^2\right) = 0$ for steady streamline flow $\frac{\partial v}{\partial t} = 0$.

So for a small displacement in the direction of the fluid velocity

$$\boxed{\frac{P}{\rho} + \phi + \frac{1}{2}v^2 = \text{constant}}$$ for all points along a streamline.

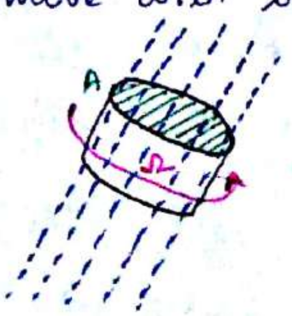
This is called Bernoulli's equation for potential flow. The constant in R.H.S. is constant along any given streamline but is different for different streamlines, while for a potential flow (irrotational), it is constant throughout the fluid.

$$\boxed{\frac{P}{\rho} + \phi + \frac{1}{2}v^2 = \text{constant}}$$ (everywhere).

Vortex lines In terms of vorticity, we have already noted the Euler's equation, $\frac{\partial \vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2 = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi$. By taking a curl, we can permanently eliminate pressure, so that for an incompressible liquid,

$$\boxed{\vec{\nabla}\cdot\vec{v} = 0, \quad \vec{\Omega} = \vec{\nabla}\times\vec{v} \quad \text{and} \quad \frac{\partial\vec{\Omega}}{\partial t} + \vec{\nabla}\times(\vec{\Omega}\times\vec{v}) = 0}$$ describes the velocity field everywhere. Also, if $\vec{\Omega} = 0$ at any time t, $\frac{\partial\vec{\Omega}}{\partial t} = 0$ so at all time $\vec{\Omega} = 0$ or the flow remains permanently irrotational. The equations to be solved are $\vec{\nabla}\cdot\vec{v} = 0$, $\vec{\nabla}\times\vec{v} = 0$

As Helmholtz proposed, imagine that in the fluid we want to draw vortex lines, rather than streamline. Vortex lines are field lines in the direction of $\vec{\Omega}$ and density proportional to the magnitude $|\vec{\Omega}|$. They are similar to magnetic induction $\vec{B}$, without any beginning or end, revolving in closed loops, and move with the fluid. Suppose at time t, a small cylinder of

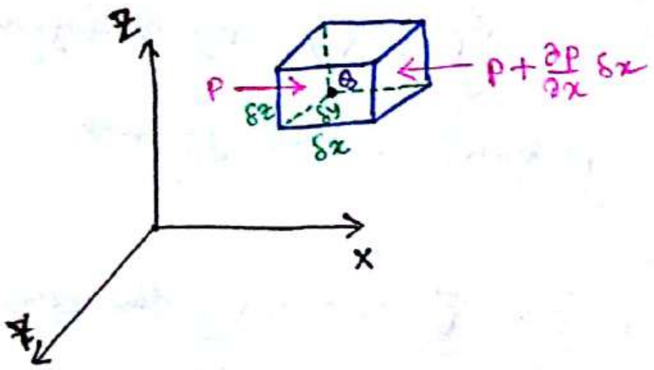


the liquid with axis parallel to vortex lines is transported at $t+dt$ to another cylinder with area A_2.

So, $\Omega_1 A_1 = \Omega_2 A_2$ as $\Omega \propto$ density because mass is same in both situation, we see $A_1 = \pi r_1^2$ and $A_2 = \pi r_2^2$ and $M_1 = M_2$ gives

$\pi M_1 r_1^2 \Omega_1 = \pi M_2 r_2^2 \Omega_2$ is $L_1 = L_2$ or in the absence of viscosity, angular momentum of an element of the fluid is invariant. This is "ideal" "dry" water case as it means if $\vec{\Omega} = 0$ then $\vec{\Omega}$ cannot be created or there will not be any vorticity.

Fluid Statics : Condition of Equilibrium of a fluid



Consider a container of fluid at rest, & within it an infinitesimal rectangular parallelopiped is taken in which at point Q, we calculate the body force. F_x, F_y, F_z are components of the body force $\vec{F}$ at $Q(x,y,z)$. Now, force due to pressure p on the elemental area $\delta y \delta z$ along x-axis is $p\,\delta y \delta z$. & force on opposite face of the parallelopiped is $-(p + \frac{\partial p}{\partial x}\delta x)\delta y \delta z$.

$$\therefore \text{ The resultant force } = p\,\delta y \delta z - (p + \frac{\partial p}{\partial x}\delta x)\delta y \delta z$$
$$= -\frac{\partial p}{\partial x}\delta x \delta y \delta z .$$

So for equilibrium under the action of the body force

$$F_x \rho\, \delta x \delta y \delta z - \frac{\partial p}{\partial x}\delta x \delta y \delta z = 0 \quad \text{or,} \quad F_x = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similarly for y and z direction, $F_y = \frac{1}{\rho}\frac{\partial p}{\partial y}$, $F_z = \frac{1}{\rho}\frac{\partial p}{\partial z}$.

$$\therefore \vec{F} = \frac{1}{\rho}\vec{\nabla}p \quad \text{and for } \rho = \text{constant}, \ \vec{\nabla}\times\vec{F} = 0.$$

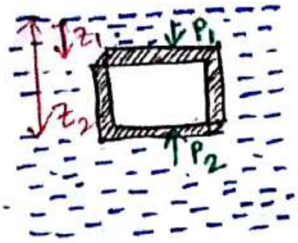
If the force is gravity then $F_z = -g$, $F_x = F_y = 0$.

$$\text{So, } -g - \frac{1}{\rho}\frac{dp}{dz} = 0 \quad \Rightarrow \quad dp = -g\rho\, dz .$$

$\therefore P = -\rho g z + C$ where at $z=0$, $P = P_0$ gives $C = P_0$

$\therefore \boxed{P = P_0 - \rho g z}$ Equation of hydrostatics for incompressible liquid

As $P - P_0$ does not depend on P_0 or pressure exerted by external forces on the fluid is transmitted equally in all directions. This is Pascal's principle.



Again, consider a body of cross-sectional area A immersed in a fluid with pressure P_1 & P_2 at upper & lower surface, then

$$P_1 = -\rho g z_1 + C_1, \quad P_2 = -\rho g z_2 + C_2$$

$$\therefore (P_1 - P_2) = \rho g (z_2 - z_1).$$

$$\therefore (P_1 - P_2)A = \text{thrust} = \rho g A (z_2 - z_1) = \text{weight of}$$

the fluid dispersed in upward direction.

This is the Archimede's principle.

This can be easily derived from Bernoulli's theorem

$\frac{v^2}{2} + \frac{P}{\rho} + gh = \text{constant}$ by substituting $v = 0$, $h = z$.

For compressible gases, Boyle's law give $P \propto \rho$.

$\therefore \frac{P}{P_0} = \frac{\rho}{\rho_0}$. So from $dP = -g\rho\, dz$ we get $dP = -g\, dz \frac{\rho_0}{P_0} P$

$$\therefore \frac{dP}{P} = -\frac{g\rho_0}{P_0} dz \qquad \therefore \int_{P_0}^{P} \frac{dP}{P} = -\frac{\rho_0 g}{P_0} \int_0^z dz$$

$\therefore \ln \frac{P}{P_0} = -\frac{\rho_0 g z}{P_0}$ [where P_0 = pressure at surface of Earth or $z=0$.]

$\therefore P = P_0 e^{-\frac{\rho_0 g}{P_0} z}$ This expression correctly shows exponential fall of pressure with distance but flawed as temperature variation is not accounted for. But from $\frac{dP}{dz} = -\rho g$, using Clausius-Clapeyron's equation

$P = n k_B T$ (k_B = Boltzmann's constant $= 1.38 \times 10^{-23}$ J/k $= \frac{R}{N}$

$= \dfrac{\text{universal gas constant}}{\text{Avogadro Number}} = \dfrac{8.314\ \text{J/mol K}}{6.023\times10^{23}}$), M = molecular weight of gas.

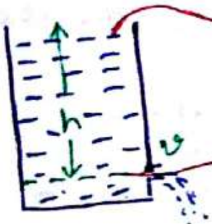
So $\dfrac{dP}{dz} = -\rho g = -\dfrac{Mg}{RT}P$ [As, $mn=\rho$, $mN = M$, $k_B = \dfrac{R}{N}$]   [As $P = nk_BT = \dfrac{\rho}{m}\dfrac{R}{N}T = \dfrac{\rho R}{M}T$]

or $\dfrac{dP}{P} = -\dfrac{Mg}{RT}dz$

or $\ln P = -\dfrac{Mg}{RT}z + \ln P_0$

or $\boxed{P = P_0 e^{-Mgz/RT}}$ This is called "law of atmosphere."

Torricelli's theorem Velocity of efflux of a liquid through an orifice is equal to the velocity attained by a body in falling freely from the surface of the liquid to the orifice.



→ Total energy = KE + PE + pressure = $0 + gh + 0 = gh$.

→ Total energy = $\frac{1}{2}v^2 + 0 + 0$.   or $\frac{1}{2}v^2 = gh$ or $v = \sqrt{2gh}$

Eulerian and Lagrangian description of conservation laws

The rate of change of a field variable $\phi(t, \bar{x})$ with respect to fixed position of space is called Eulerian derivative $\dfrac{\partial\phi}{\partial t}$ while derivative following a moving parcel is called Lagrangian derivative or substantial derivative or material derivative $\dfrac{D\phi}{Dt}$

$$\frac{D\phi}{Dt} = \underbrace{\frac{\partial\phi}{\partial t}}_{\text{local rate of change}} + \underbrace{\vec{v}\cdot\vec{\nabla}\phi}_{\text{convective rate of change}}$$

So changes in the properties of a moving fluid can be measured either on a fixed point in space while fluid particles are crossing it (Eulerian) or by following a fluid parcel along its path (Lagrangian)

Eulerian $v(t, \vec{x}(\vec{x}_0, t)) = \dfrac{\partial}{\partial t}\vec{X}(t, \vec{x}_0)$

Reynold's transport theorem As we have defined our conservation laws in Lagrangian description, Reynold's transport theorem gives the Eulerian equivalent of the integral taken over a moving material volume of a fluid.

$$\left(\frac{d\phi}{dt}\right)_{\substack{\text{material}\\ \text{volume}}} = \frac{d}{dt}\int_{V(t)} \psi \rho \, dV + \iint_{S(t)} \psi \rho \vec{v}\cdot\hat{n}\, ds \,, \quad \left[\psi = \frac{d\phi}{dm} = \text{amount of } \phi \text{ per unit mass}\right]$$

Divergence theorem

$$= \int_V \left[\frac{\partial}{\partial t}(\psi\rho) + \vec{\nabla}\cdot(\rho\vec{v}\psi)\right] dV = \int_V \left[\frac{D}{Dt}(\rho\psi) + \rho\psi\vec{\nabla}\cdot\vec{v}\right] dV$$

Notice that conservation law indicate no source or sink meaning $\cancel{\frac{d\phi}{dm}}$ ~~H(ψ)≠0~~ $\frac{dm}{dt} = 0$, meaning $\psi = \frac{d\phi}{dm} = 1$ when $\phi = m$.

$$\boxed{\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{v} = 0}$$ mass conservation law in Eulerian coordinate system.

So incompressibility ($\vec{\nabla}\cdot\vec{v} = 0$) means $\frac{D\rho}{Dt} = 0$. or ρ is not a constant but ρ does not change along a streamline.

In presence of external force $\vec{f}$ per unit volume, the non-conservative form with $\psi = \vec{v}$ is

$$\frac{D}{Dt}(\rho\vec{v}) + \rho\vec{v}\,\vec{\nabla}\cdot\vec{v} = \vec{f}$$

check!

$$\text{or} \quad \rho\frac{D\vec{v}}{Dt} + v\underbrace{\left(\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{v}\right)}_{=0 \text{ (continuity)}} = \vec{f} \quad \text{or} \quad \boxed{\rho\frac{D\vec{v}}{Dt} = \vec{f}}$$

Using Reynold's transport theorem, we find the conservative form

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot(\underbrace{\rho\vec{v}\vec{v}}_{\text{Dyadic product}}) = \vec{f} = \vec{f}_{surface} + \vec{f}_{body}$$

Now $\int_V \vec{f}_s \, dV = \int_S \vec{\vec{\sigma}}\cdot\hat{n}\, ds = \int_V \vec{\nabla}\cdot\vec{\vec{\sigma}}\, dV$ where $\vec{\vec{\sigma}}$ = Stress tensor or stress matrix.

$$\bar{\bar{\sigma}} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \sigma_{xx}+p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}+p & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}+p \end{pmatrix}$$

$= -p\bar{\bar{\mathbb{I}}} + \bar{\bar{\tau}}$ ↳ Deviatoric / viscous stress tensor.

↳ Thermodynamic pressure ($= nk_BT$). So $\vec{\nabla}\cdot\bar{\bar{\sigma}} = -\vec{\nabla}p + \vec{\nabla}\cdot\bar{\bar{\tau}}$.

$$\vec{f}_b = \underbrace{\rho\vec{g}}_{\text{gravity}} - \underbrace{2\rho\vec{\omega}\times\vec{v}}_{\text{coriolis force}} - \underbrace{\rho\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{centrifugal force}}.$$

As gravitation & centrifugal forces are dependent on position but not velocity, so they can be absorbed into a modified pressure & hence effectively ignored. Coriolis force however has to be treated explicitly.

So conservation of momentum equation becomes,

$$\boxed{\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot(\rho\vec{v}\vec{v}) = -\vec{\nabla}p + \vec{\nabla}\cdot\vec{\tau} + \vec{f}_b}$$

Stress tensor for Newtonian fluid

$$\vec{\vec{\tau}} = \underbrace{\eta}_{\substack{\text{viscosity}\\\text{coefficient}\\\text{(molecular)}}}\{\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T\} + \underbrace{\lambda}_{\substack{\text{viscosity}\\\text{coefficient}\\\text{(bulk)}}}(\vec{\nabla}\cdot\vec{v})\,\mathbb{I} \qquad \left[\lambda = -\tfrac{2}{3}\mu\right]$$

$$\text{So } \frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot(\rho\vec{v}\vec{v}) = -\vec{\nabla}p + \vec{\nabla}\cdot[\eta\{\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T\}] + \vec{\nabla}(\underbrace{\cancel{\lambda\vec{\nabla}\cdot\vec{v}}}_{\substack{\text{incompressible}\\\text{flow}}}) + \vec{f}_b$$

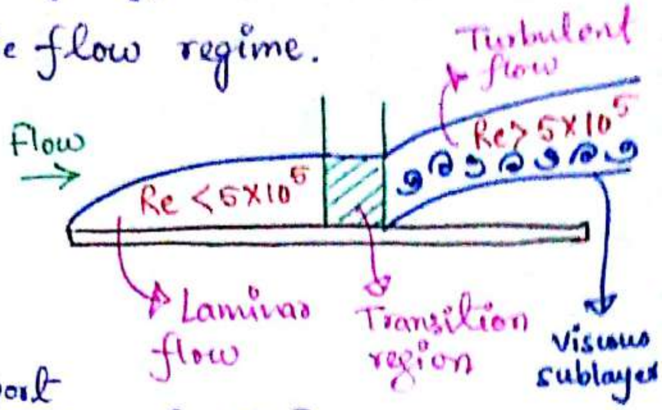
$$= -\vec{\nabla}p + \underbrace{\cancel{\eta\nabla^2\vec{v}}}_{\substack{\text{inviscid}\\\text{flow}}} + \vec{f}_b = -\vec{\nabla}p + \vec{f}_b$$

Similarly energy conservation equation can be derived.

General form

$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\text{[unsteady term]}} + \underbrace{\vec{\nabla}\cdot(\rho\vec{v}\phi)}_{\text{[convection term]}} = \underbrace{\vec{\nabla}\cdot(\Gamma\vec{\nabla}\phi)}_{\text{[diffusion term]}} + \underbrace{S}_{\text{[source term]}}$$

Like Reynolds number $Re = \frac{\rho v L}{\eta} = \frac{\text{advection (inertia)}}{\text{diffusion (viscous)}}$ reveals the boundary layer characteristic of the flow if momentum fluxes are in same direction, e.g. if diffusion is in the cross-stream direction then Re conveys the flow regime.



Similarly Schmidt number

$$Sc = \frac{\text{momentum diffusivity}}{\text{mass diffusivity}}$$

Peclet number $\quad Pe = \dfrac{\text{advective transport}}{\text{diffusive transport}} = Re \times Sc$

where Pr = prandtl number $= \dfrac{\text{hydrodynamic boundary layer}}{\text{thermal boundary layer}}$

$$= \frac{\text{convection}}{\text{conduction}}.$$

Grashof number $= \dfrac{\text{buoyancy force}}{\text{viscous force}}$

Mach number (M) $= \dfrac{\text{object speed}}{\text{speed of sound}}$ where $v_{sound} = \sqrt{\gamma\left(\frac{\partial p}{\partial \rho}\right)_T}$

$= \sqrt{\gamma R T}$ (ideal gas)

$\gamma = C_p/C_v$

$M < 0.2$ is incompressible flow.

$M < 1$ is subsonic flow.

$M = 1$ is sonic flow.

$M < 5$ is supersonic flow

$M > 5$ is hypersonic flow.

flow classification

Euler equation in Lagrangian form

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})v\right) = -\vec{\nabla}p - \rho\vec{\nabla}\varphi + \eta\nabla^2\vec{v} \quad \& \quad \vec{\Omega} = \vec{\nabla}\times\vec{v}$$

gives, $\dfrac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla}\times(\vec{\Omega}\times\vec{v}) = \dfrac{\eta}{\rho}\nabla^2\vec{\Omega}$

↳ specific viscosity $= \dfrac{1}{Re}$

If we non-dimensionalize the equation (see Feynman lectures, vol 2 section 41.3), then $\dfrac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla}\times(\vec{\Omega}\times\vec{v}) = \dfrac{1}{Re}\nabla^2\vec{\Omega}$

So if Mach number & Reynold's number are same then velocities near or above sound speed in two scenarios are equivalent.