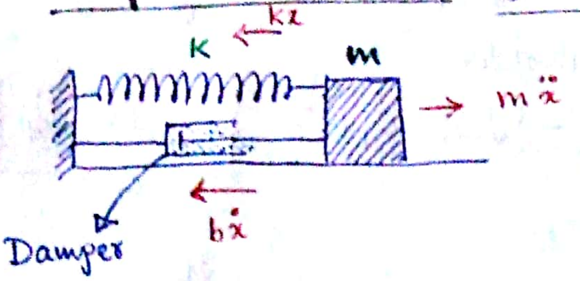
Free Damped harmonic motion

Damping of a real system is a complex phenomena involving several kind of damping force. Damping force of a body in a fluid is a function of velocity. This is called "**viscous damping**." When an oscillating body is contact with a surface, the frictional force is called "**Coulomb friction**". Also in solids, energy is partly lost due to internal friction & imperfect elasticity of the material. Experiments suggest that such resistive force is independent of frequency & proportional to amplitude. This is called "**structural damping**." The viscouse damping force may be represented as $F = -Av + \cancel{Bv^2} - \cancelto{0}{Cv^3} + \cdots$ and such approximation is "linear damping".

Damped oscillation of a system with 1 degree of freedom



inertial force $m\ddot{x}$ is balanced by elastic restoring force Kx & viscous damping force $b\dot{x}$

$$\therefore\ m\ddot{x} = -b\dot{x} - Kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \underbrace{\frac{b}{m}}_{=\gamma}\frac{dx}{dt} + \underbrace{\frac{K}{m}}_{=\omega_0^2}x = 0.$$

This is a linear homogeneous 2^{nd} order ODE.

Let the trial solution $x = Ae^{\alpha t}$, substituting we get

$$(\alpha^2 + \gamma\alpha + \omega_0^2)Ae^{\alpha t} = 0 \Rightarrow \alpha^2 + \gamma\alpha + \omega_0^2 = 0.$$

$$\therefore\ \alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2}$$

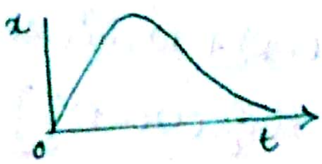
$$\therefore\ \text{Solution}\quad x = A_1\exp\left[-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t + A_2\exp\left[-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t$$

$$= e^{-\gamma t/2}\left[A_1\exp\left(\sqrt{\gamma^2/4 - \omega_0^2}\,t\right) + A_2\exp\left(-\sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,t\right)\right]$$

We can have three possibilities:

(a) Heavy damping $\gamma^2/4 > \omega_0^2$ $\quad q = \sqrt{\gamma^2/4 - \omega_0^2} > 0$.

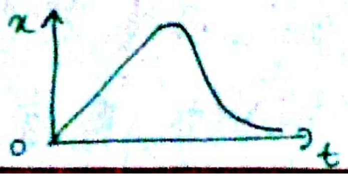
$x = e^{-\gamma t/2}(A_1e^{qt} + A_2e^{-qt})$. This means that x cannot be negative and at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & e^{qt} contributes like exponential then at $t \to \infty$, it'll damp to x (initial). If we had started at $x = 0$, after a time interval it decays back to zero $\Rightarrow$ "Dead beat Galvanometer." no oscillation



(b) Critical damping $\gamma^2/4 = \omega_0^2$: $x = (A_1 + A_2)e^{-\gamma t/2}$. The damping is slower but it has a discrepancy that at $x = 0$ at $t = 0$, $v = 0$ which is not true. Changing the trial solution, we can derive

$x \sim t\,e^{-\gamma t/2}$ mean at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & $x \propto t$ & later $t \to \infty$, $e^{-\gamma t/2}$ dominates. x is never negative $\Rightarrow$ no oscillation

"pointer type galvanometer"



(c) Weak damping $\frac{\gamma^2}{4} < \omega_0^2$ $\quad q = \sqrt{\frac{\gamma^2}{4} - \omega_0^2} =$ imaginary.

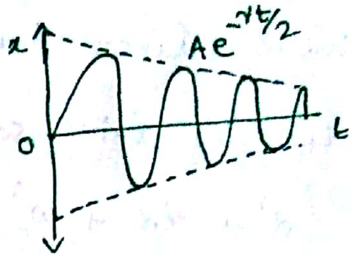
This gives oscillatory damped harmonic motion.

$$\boxed{\omega = \sqrt{\omega_0^2 - \gamma^2/4}}$$

$$x = e^{-\gamma t/2}\left[A_1 e^{i\sqrt{\omega_0^2 - \gamma^2/4}\,t} + A_2 e^{-i\sqrt{\omega_0^2 - \gamma^2/4}\,t}\right]$$

$$= e^{-\gamma t/2}\left(A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right)$$

$$= e^{-\gamma t/2}\Big[\underbrace{(A_1 + A_2)}_{A\cos\delta}\cos\omega t + \underbrace{i(A_1 - A_2)}_{A\sin\delta}\sin\omega t\Big] = A e^{-\gamma t/2}\cos(\omega t - \delta)$$



Amplitude decreases in due time

Angular frequency is less than undamped motion.

$\tau = 2/\gamma$ = mean life time of oscillation.

Energy of a weakly damped oscillator

Using $x = Ae^{-\gamma t/2}\cos(\omega t - \delta)$ we develop expression for average energy.

$$\dot{x} = -\frac{\gamma}{2} A e^{-\gamma t/2}\cos(\omega t - \delta) - Ae^{-\gamma t/2}\omega\sin(\omega t - \delta)$$

∴ Kinetic energy (instantaneous) of the vibrating body

$$\frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \gamma\omega\cos(\omega t - \delta)\sin(\omega t - \delta)\right]$$

$$\text{Potential energy} = \int_0^x F\,dx = \int_0^x Kx\,dx = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2e^{-\gamma t}\cos^2(\omega t - \delta)$$

$$= \frac{1}{2}m\omega_0^2 A^2 e^{-\gamma t}\cos^2(\omega t + \delta)$$

∴ Total energy = KE + PE =

$$\frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos^2(\omega t - \delta) + \omega^2\sin^2(\omega t - \delta) + \omega_0^2\cos^2(\omega t - \delta) + \frac{\gamma\omega}{2}\sin\{2(\omega t - \delta)\}\right]$$

For small damping, $\gamma << 2\omega_0$, then $e^{-\gamma t}$ does not change appreciably during one time period $T = \frac{2\pi}{\omega}$, then time averaged energy of the oscillator is

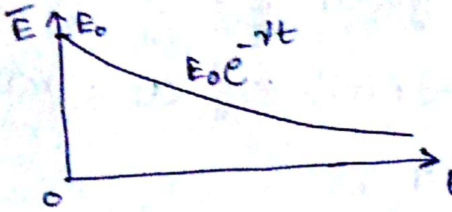
$$\langle E\rangle = \frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\langle\cos^2(\omega t - \delta)\rangle + \omega^2\langle\sin^2(\omega t - \delta)\rangle + \omega_0^2\langle\cos^2(\omega t - \delta)\rangle + \frac{\gamma\omega}{2}\underbrace{\langle\sin\{2(\omega t - \delta)\}\rangle}_{0}\right]$$

$$\text{Now } \langle\cos^2(\omega t - \delta)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\omega t - \delta)\,d(\omega t) = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}2\cos^2 x\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(1 + \cos 2x)\,dx = \frac{1}{2} = \langle\sin^2(\omega t - \delta)\rangle$$

$$\therefore \langle E \rangle = \frac{1}{2} m A^2 e^{-\gamma t}\left[\cancel{\frac{\gamma^2}{8}} + \left(\omega_0^2 - \cancel{\frac{\gamma^2}{4}}\right)\frac{1}{2} + \frac{\omega_0^2}{2}\right] = \frac{1}{2} m\omega_0^2 A^2 e^{-\gamma t}$$

$\boxed{\langle E \rangle = E_0 e^{-\gamma t}}$ where $E_0 = \frac{1}{2} m\omega_0^2 A^2$ is energy of undamped oscillator

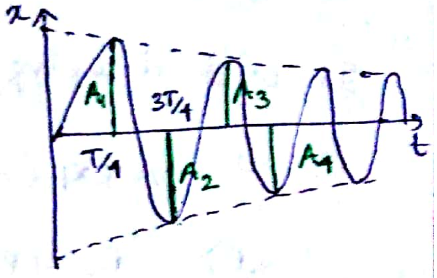


The average power dissipation in one time period

$$\langle P(t) \rangle = \frac{d}{dt}\langle E(t) \rangle = \gamma \langle E(t) \rangle.$$ due to friction.

Estimation of Damping

There are various ways of estimation of the damping of an oscillator. Let us choose initial condition at $t=0$, $x=0$, $\frac{dx}{dt} = v_0$ and $\delta = \pi/2$, $x = Ae^{-\gamma t/2}\cos(\omega t - \pi/2) = Ae^{-\gamma t/2}\sin\omega t$



Logarithmic Decrement

$$x = Ae^{-\gamma t/2}\sin\omega t = Ae^{-\gamma t/2}\sin\frac{2\pi t}{T}$$

at $t = \frac{T}{4}$, $x_1^{max} = Ae^{-\gamma T/8}\sin\frac{2\pi}{T}\frac{T}{4} = Ae^{-\gamma T/8}$

at $t = \frac{3T}{4}$, $x_2^{max} = Ae^{-3\gamma T/8}$

at $t = \frac{5T}{4}$, $x_3^{max} = Ae^{-5\gamma T/8}$ etc.

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \frac{x_3^{max}}{x_4^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\gamma T/4} = d$$ (constant)

"d" is called decrement of the motion. $\lambda = \ln d$ is the logarithmic decrement of the motion $= \ln e^{\gamma T/4} = \frac{\gamma T}{4}$

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \cdots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\lambda}$$

Multiplying, $\frac{x_1^{max}}{x_n^{max}} = e^{(n-1)\lambda}$ or $\lambda = \frac{1}{n-1}\ln\left(\frac{x_1^{max}}{x_n^{max}}\right)$

$$\lambda = \frac{2.303}{n-1}\log_{10}\left(\frac{x_1^{max}}{x_n^{max}}\right)$$

This method is used to determine the corrected last throw of a Ballistic galvanometer due to damping.

Relation between undamped throw θ_0 & first throw θ_1 is

$$\theta_1 = \theta_0 e^{-\gamma T/8} \quad \therefore \theta_0 = \theta_1 e^{\gamma T/8} = \theta_1 e^{\lambda/2} \simeq \theta_1\left(1 + \frac{\lambda}{2}\right) \text{ for } \lambda \ll 1$$

So knowing λ, we can correct θ_1 for damping.

Quality Factor (Q-Value)

Another method to express damping in an oscillatory system is to measure the rate of decay of energy. Quality factor $Q = \frac{\omega}{\gamma}$, $= \frac{\omega_0}{\gamma}\sqrt{1-\gamma^2/4\omega_0^2}$. While $\langle E\rangle = E_0 e^{-\gamma t}$, power $\langle P(t)\rangle = \frac{d}{dt}\langle E\rangle = \gamma\langle E\rangle$

So the average energy dissipated in time period T is:

$$\gamma T\langle E\rangle = \frac{2\pi\gamma}{\omega}\langle E\rangle = \frac{2\pi}{Q}\langle E\rangle = \frac{2\pi}{Q} \times \text{average energy stored.}$$

$$\therefore \boxed{Q = 2\pi \times \frac{\text{Average energy stored in one time period}}{\text{Average energy lost in one time period}}}$$

In weak damping limit $\frac{\gamma^2}{4\omega_0^2} << 1$, $Q = \frac{\omega_0}{\gamma}$. As $\gamma \to 0$, $Q \to \infty$

$\therefore$ $x = A\exp(-\frac{\omega_0 t}{2Q})\cos(\omega_0 t - \delta)$ in limit $\gamma^2/4\omega_0^2 << 1$

$\langle E\rangle = E_0 \exp(-\frac{\omega_0 t}{Q})$ and see that $\tau_1 = \frac{Q}{\omega_0}$, $\langle E\rangle = E_0 e^{-1}$

and no. of complete oscillation if is n, then $n = \frac{\omega_0}{2\pi}\tau_1 = \frac{Q}{2\pi}$

So $\langle E\rangle$ reduces to e^{-1} of $\langle E\rangle$ in $Q/2\pi$ cycles of oscillation.

Note that $\boxed{\lambda = \frac{\gamma T}{4},\ \tau = \frac{2}{\gamma}\ \&\ Q = \frac{\omega_0}{\gamma},\ \tau_1 = \frac{Q}{\omega_0} = \frac{1}{\gamma}}$.

"Moving coil Galvanometer" is the example of damped harmonic motion. Similarly, current or charge oscillation in LCR circuit, mechanical vibration of a string or tuning fork etc.

Forced Vibration

Vibrating system with damping + periodic force = forced vibration natural vibration dies out, system tunes to the frequency of force. For example, a bridge vibrates in the influence of marching soldiers. Contributions are restoring force kx, damping force $b\dot{x}$, inertial force $m\ddot{x}$ & external periodic force $F(t) = F_0\cos\omega t$.

$\therefore$ Equation of motion of the body is

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F(t)$$

or $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0^2 x = f_0\cos\omega t$, $\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{K}{m}$, $f_0 = \frac{F_0}{m}$.

linear homogeneous 2^{nd} order ODE. Solution of this we can separate out as $\dfrac{d^2x_1}{dt^2} + \nu\dfrac{dx_1}{dt} + \omega_0^2 x_1 = f_0\cos\omega t$ & $\dfrac{d^2x_2}{dt^2} + \nu\dfrac{dx_2}{dt} + \omega_0^2 x_2 = 0$ so that $x_1 + x_2$ is a solution. Now we know $x_2 = Ae^{-\gamma t/2}\cos(\omega^* t - \delta)$ with $\omega^* = \sqrt{\omega_0^2 - \gamma^2/4}$ & will die out in time. (transient state). For x_1, we can write $x_1 = B\cos(\omega t - \delta)$ where B & δ are to be determined. In this notation,

$$X = Re(Be^{i(\omega t - \delta)}).$$

$$\frac{d^2X}{dt^2} + \gamma\frac{dX}{dt} + \omega_0^2 X = f_0 e^{i\omega t} = f_0 e^{i(\omega t - \delta)} e^{i\delta}$$

or $\left[B\left[(\omega_0^2 - \omega^2) + i\omega\gamma\right] - f_0 e^{i\delta}\right] e^{i(\omega t - \delta)} = 0, \forall t.$

$$B(\omega_0^2 - \omega^2 + i\omega\gamma) - f_0 e^{i\delta} = 0 \quad \text{or} \quad Be^{-i\delta} = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

or $B\cos\delta - iB\sin\delta = \dfrac{f_0[\omega_0^2 - \omega^2 - i\omega\gamma]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$

$\therefore B\cos\delta = \dfrac{f_0(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$, $B\sin\delta = \dfrac{f_0\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$ $\therefore B = \dfrac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$

$\delta = \tan^{-1}\left(\dfrac{\omega\gamma}{\omega_0^2 - \omega^2}\right)$

$$\therefore \boxed{x_1 = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)\right)}$$

Steady state solution

Its dependent on $F_0, m, \omega, \omega_0, \gamma$ & there is a phase difference δ between force & displacement. When $D = (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2$ is minimum B is maximum amplitude. If this frequency is ω_r then $\left.\dfrac{dD}{d\omega}\right|_{\omega = \omega_r} = 0$ and $\left.\dfrac{d^2D}{d\omega^2}\right|_{\omega = \omega_r} > 0$. $\therefore -2(\omega_0^2 - \omega_r^2)2\omega_r + 2\omega_r\gamma^2 = 0$

or $\omega_r = \sqrt{\omega_0^2 - \gamma^2/2}$ and convince yourself $\dfrac{d^2D}{d\omega^2} > 0$ if $\dfrac{\gamma^2}{2} < \omega_0^2$.

Thus amplitude of forced oscillation is maximum if frequency of the driving force is nearly equal to frequency of natural oscillation.

At $\omega = \omega_r$, $B_{max} = \dfrac{f_0}{\nu(\omega_0^2 - \nu^2/4)^{1/2}}$ and $\nu << \omega_0$, $B_{max} \approx \dfrac{f_0}{\nu\omega_0}$

$= \dfrac{F_0}{m\nu\omega_0} = \dfrac{F_0}{b\omega_0}$

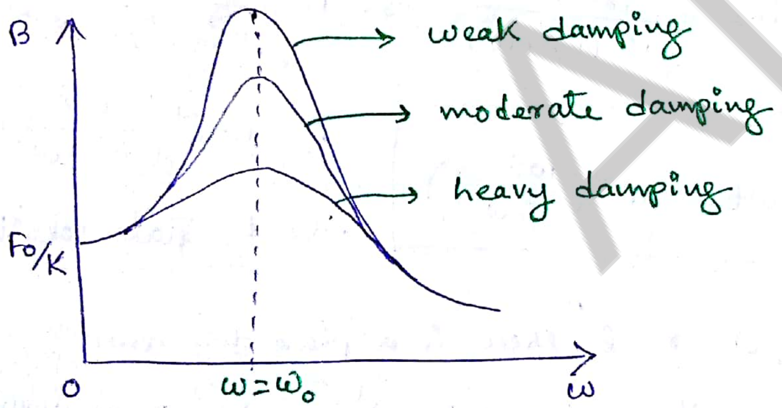
Thus in this limit $\omega_r \simeq \omega_0$ and the amplitude is controlled by "b" and the forced oscillator is "resistance controlled."

Recall $B = \dfrac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\nu^2}}$, In limit $\omega << \omega_0$, $B \simeq \dfrac{F_0/m}{\omega_0^2\sqrt{1+\frac{\omega^2}{\omega_0^2}\frac{\nu^2}{\omega_0^2}}}$

$\simeq \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{K}$

This displacement a constant force F_0 would produce. when $\omega \to 0$, $F(t) \to F_0$ or we get back $m\dfrac{d^2x}{dt^2} = -m\omega^2 x$ very small role than Kx term. $\therefore$ Response of the oscillator is controlled by the stiffness constant K & the oscillator is "stiffness controlled."

Similarly for $\omega >> \omega_0$, $B \simeq \dfrac{F_0/m}{\omega^2\sqrt{1+\frac{\nu^2}{\omega_0^2}\frac{\omega_0^2}{\omega^2}}}$ which for weak damping $\nu << \omega_0$ is $B \simeq \dfrac{F_0}{m\omega^2}$ and $m\omega^2 x$ is dominating. and the oscillator is "mass or inertia controlled."



amplitude resonance at $\omega = \omega_0$ when $\nu^2/2 < \omega_0^2$.

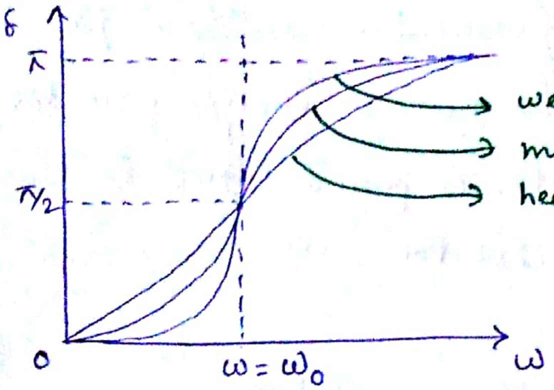
Also when $\omega << \omega_0$,

$\tan\delta = \dfrac{\omega\gamma}{\omega_0^2-\omega^2} \simeq \dfrac{\omega}{\omega_0}\dfrac{\gamma}{\omega_0}$

as $\omega \to 0$, $\delta \to 0$. Thus for low frequency of driving force, displacement is nearly in phase with driving force. If $\omega >> \omega_0$, $\tan\delta \simeq -\dfrac{\gamma}{\omega} \simeq \dfrac{\gamma}{\omega_0}\dfrac{\omega_0}{\omega}$ which for weak damping $\nu << \omega_0$ has small negative value or $\delta \simeq \pi$.

$\therefore$ If frequency of driving force >> natural frequency of free oscillations, then displacement will be out of phase with driving force. Also ~~when~~ ~~[illegible]~~ accelaration will be in phase with driving force.

But at resonance, $\omega \simeq \omega_0$ & $\tan\delta = \infty$ $\therefore$ $\delta = \pi/2$ or displacement is maximum when driving force is zero.



Displacement x_1 lags the force $F(t)$ by δ.

Velocity Resonance

$x_1 = B\cos(\omega t - \delta)$ $\therefore$ $\dot{x}_1 = -\omega B \sin(\omega t - \delta)$

or $v = v_0 \cos(\omega t - \phi)$ where $v_0 = \omega B = \dfrac{F_0/m}{\sqrt{\dfrac{(\omega_0^2 - \omega^2)^2}{\omega^2} + \nu^2}}$

$= v_0 \cos(\omega t - \delta + \pi/2)$

and $\phi = \delta - \pi/2$. $[-\sin(\omega t - \delta) = \cos(\omega t - \delta + \frac{\pi}{2})]$

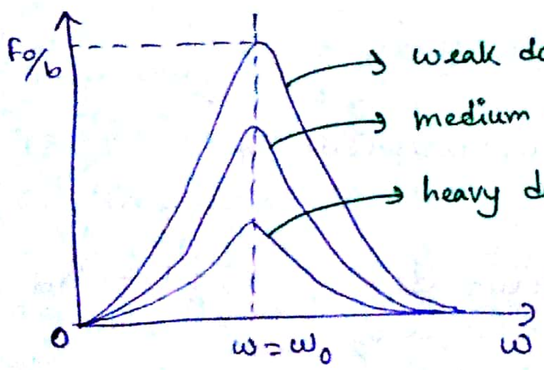
$\therefore$ Velocity leads the displacement in phase by $\pi/2$. V_0 is maximum when denominator is minimum. $\dfrac{d}{d\omega}\left[\dfrac{(\omega_0^2 - \omega^2)^2}{\omega^2} + \nu^2\right]\Bigg|_{\omega = \omega_r} = 0$

or $\omega_r = \omega_0$. So at $\omega = \omega_0$, V_0 is maximum, velocity resonance

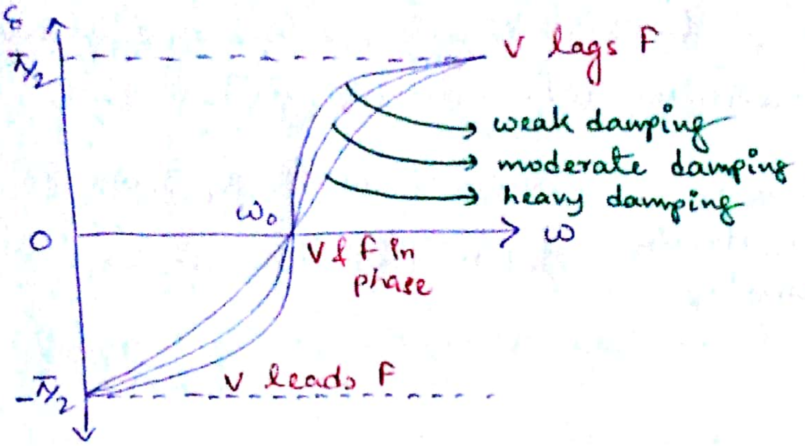
$V_0^{max} = \dfrac{F_0/m}{\nu} = \dfrac{F_0}{b}$, so as "b" increases, V_0^{max} decreases.

For $\omega >> \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega}$ and if ν is not large then $V_0 \to 0$ for $\omega \to \infty$

for $\omega << \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{m\omega^2}\dfrac{\omega^2}{\omega_0^2} \to 0$ for $\omega \to \infty$.



Phase of velocity relative to the force is $\phi = \delta - \pi/2$. For $\omega << \omega_0$, $\delta \simeq 0$, so $\phi = -\pi/2$. As ϕ is angle by which velocity lags behind the force, so here velocity leads the force by an angle $\pi/2$. For $\omega >> \omega_0$, $\delta \simeq \pi$, $\phi = \pi - \pi/2 = \pi/2$ so for very high frequencies, velocity lags the force by $\pi/2$. At resonance $\omega = \omega_0$, $\delta = \pi/2$ and $\phi = 0$ & velocity is in phase with force.



This is therefore the most favourable condition for transfer of energy from the external periodic force to the oscillator.

Power transfer from driving force to the oscillator

Energy of a damped oscillator decreases exponentially as $E(t) = E_0 e^{-\gamma t}$. In order to maintain steady state oscillation, driving force transfers energy to oscillator. Now

$$x = B\cos(\omega t - \delta) = B\cos\delta \cos\omega t + B\sin\delta \sin\omega t$$
$$= B_{el}\cos\omega t + B_{ab}\sin\omega t$$

where B_{el} = elastic amplitude $B\cos\delta = \dfrac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2+\omega^2)^2+\gamma^2\omega^2}$ [in phase with force]

B_{ab} = absorptive amplitude $B\sin\delta = \dfrac{f_0\,\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ [out of phase $\pi/2$ with force]

$v = \dot{x} = \omega(-B_{el}\sin\omega t + B_{ab}\cos\omega t)$ & thus the power by driving force $F_0\cos\omega t$ / second is the work done by the force/second

$$P(t) = F_0\cos\omega t \; v = F_0\,\omega\cos\omega t\,(-B_{el}\sin\omega t + B_{ab}\cos\omega t).$$

∴ Time averaged input power over one complete cycle is

$$P_{input} = \langle P(t)\rangle = \frac{1}{T}\int_0^T P(t)\,dt = -\frac{F_0\,\omega B_{el}}{T}\int_0^T \underbrace{\sin(\omega t)\cos(\omega t)\,dt}_{\to 0} + \frac{F_0\,\omega B_{ab}}{T}\int_0^T \underbrace{\cos^2(\omega t)\,dt}_{\to 1/2} = \frac{1}{2}F_0\,\omega B_{ab}$$

This input power supplied by driving force is not stored in oscillator but dissipated as work done in moving the system against friction. Instantaneous power dissipated through friction is

$$P(t) = bv\cdot v = b\left(\frac{dx}{dt}\right)^2 = b\omega^2(B_{ab}^2\cos^2\omega t + B_{el}^2\sin^2\omega t - 2B_{ab}B_{el}\cos\omega t\sin\omega t)$$

$\therefore$ Time averaged power $\langle P(t)\rangle = P_{dissipation} = \frac{b\omega^2}{2}(B_{ee}^2 + B_{ab}^2)$.

$$= \frac{b\omega^2 f_0^2}{2[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]} = \frac{1}{2}F_0\,\omega B_{ab}$$

$\therefore$ $P_{input} = P_{dissipate}$ (steady state).

Energy of the forced oscillator

Instantaneous KE is

$$\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(B_{ab}^2\cos^2\omega t + B_{ee}^2\sin^2\omega t - 2B_{ab}B_{ee}\cos\omega t\sin\omega t)$$

Instantaneous PE $\frac{1}{2}Kx^2 = \frac{1}{2}m\omega_0^2(B_{ab}^2\sin^2\omega t + B_{ee}^2\cos^2\omega t + 2B_{ab}B_{ee}\cos\omega t\sin\omega t)$

$\therefore$ Time averaged total energy is $E = \langle E(t)\rangle = \frac{1}{4}m(\omega^2+\omega_0^2)(B_{ab}^2+B_{ee}^2)$

$E_{resonance} = \frac{1}{2}m\omega_0^2(B_{ab}^2+B_{ee}^2)$ at $\omega \simeq \omega_0$

$\langle KE\rangle = \frac{1}{4}m\omega^2(B_{ab}^2+B_{ee}^2)$, $\langle PE\rangle = \frac{1}{4}m\omega_0^2(B_{ab}^2+B_{ee}^2)$

Maximum input power & Bandwidth

Time averaged input power $P_{input} = \frac{1}{2}F_0\,\omega B_{ab}$

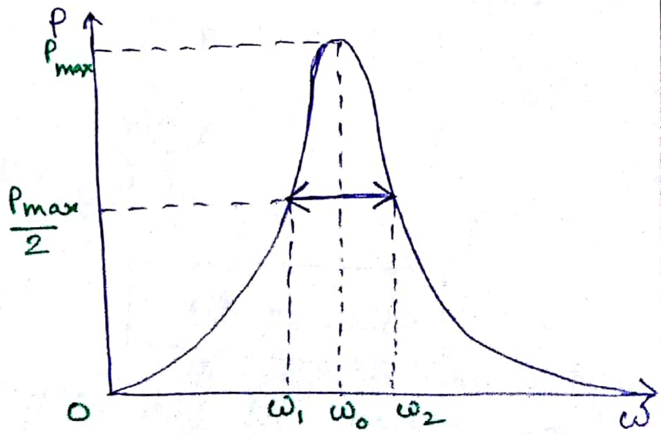
$$= \frac{F_0^2\gamma}{2m}\left[\frac{\omega^2}{(\omega_0^2-\omega)^2+\gamma^2\omega^2}\right]$$

This will be maximum for $\frac{dP}{d\omega} = 0$

& that yields $\omega = \omega_0$. Thus at resonance frequency P_{input} is maximum.

$P_{input}^{max} = \frac{F_0^2}{2m\gamma}$ $\therefore$ $P = P_{input}^{max}\,\frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

Frequency ω_1 & ω_2 at which the power drops down to 1/2 of maximum is the half power freq.



$$\frac{1}{2} = \frac{P_{input}}{P_{input}^{max}} = \frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

or $\omega^2 = \omega_0^2 \pm \gamma\omega$

$$\begin{cases}\omega_1 = -\frac{\gamma}{2} + (\omega_0^2 + \frac{\gamma^2}{4})^{1/2} \\ \omega_2 = \frac{\gamma}{2} + (\omega_0^2 + \frac{\gamma^2}{4})^{1/2}\end{cases}$$, band width $\Delta\omega = \omega_1 - \omega_2 = \gamma$.

<u>Quality Factor</u> Q is a parameter that gives the sharpness of resonance & defined a $Q = \frac{\text{resonant frequency}}{\text{band width}} = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\gamma}$

$$= 2\pi \frac{\text{Avg. energy stored in one cycle}}{\text{Avg. energy lost in one cycle}}$$

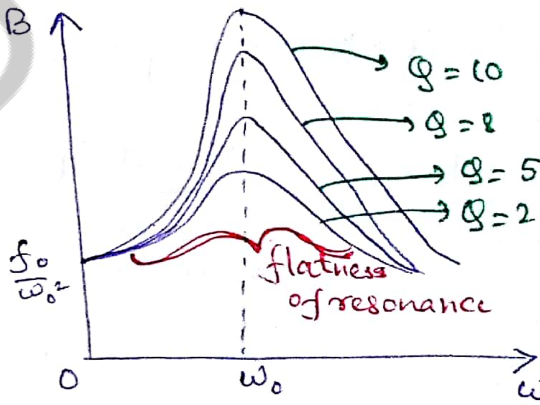
$$\therefore Q = 2\pi \frac{\langle E(t)\rangle}{P_{dissipate}\,T} = \overbrace{\left(\frac{2\pi}{T}\right)}^{=\omega} \frac{1}{\cancel{4}_{2}} m(\omega^2+\omega_0^2)\cancel{(B_{ab}^2+B_{el}^2)} \frac{\cancel{2}}{b\omega^{\cancel{2}}\cancel{(B_{ab}^2+B_{el}^2)}}$$

$$= \frac{\omega^2+\omega_0^2}{2\gamma\omega} \quad \text{and for } \omega \simeq \omega_0, \; Q^{\text{resonance}} = \frac{\omega_0}{\gamma}$$

Thus for low damping, $\gamma << \omega_0$ and Q is high that makes the resonance very ~~high~~ sharp. Thus Q measures the sharpness of resonance

Using $Q = \frac{\omega_0}{\gamma}$, the amplitude is

$$B = \frac{f_0\,Q}{\omega\omega_0\sqrt{1+Q^2\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2}}$$



Q large, B large. Q can be regarded as amplification factor. at low driving force $\omega \to 0$, $B_0 = \frac{f_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \simeq \frac{f_0}{\omega_0^2}$ and we know

$$B_{max} = \frac{f_0}{\gamma\sqrt{\omega_0^2-\gamma^2/4}}. \qquad \text{So} \quad \frac{B_{max}}{B_0} = \frac{\omega_0^2}{\gamma\sqrt{\omega_0^2-\gamma^2/4}} = \frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}$$

(for low damping) Q is very large $= Q\left(1-\frac{1}{4Q^2}\right)^{-1/2} \simeq Q\left(1+\frac{1}{8Q^2}\right)$

$$= Q.$$

$\therefore \boxed{B_{max} = Q B_0}$ The resonant amplitude is Q times the amplitude at low frequencies of the driving force.