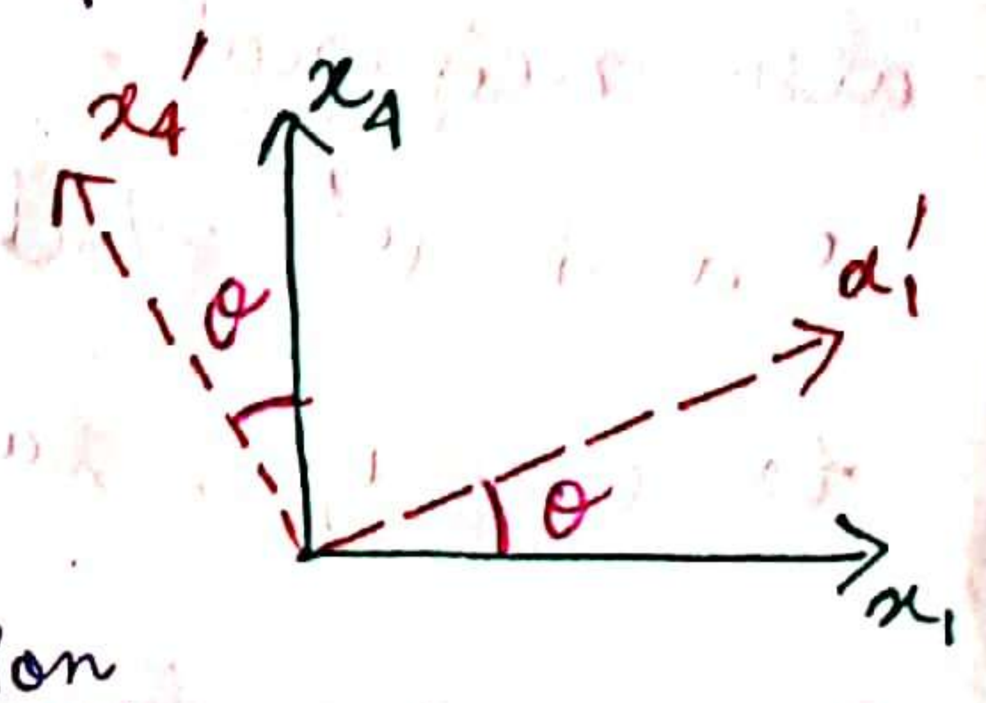
Mathematical Properties of the Space-time ; 4D Formulation of Poincaré & Minkowski

We have already seen that space-time interval $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$ remains invariant under LT. If we rewrite $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$ then $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. To note the similarity of this with rotations: (a) Orthogonal group in 2D $O(2)$ is rotation in 2D that leaves x^2+y^2 invariant, $x' = x\cos\phi - y\sin\phi$, $y' = x\sin\phi + y\cos\phi$, $0 \le \phi \le 2\pi$, then $x'^2 + y'^2 = x^2+y^2$; (b) Orthogonal group in 3D $O(3)$ that leave $x^2+y^2+z^2$ invariant. Poincaré noticed in 1906 that if we rewrite LT as

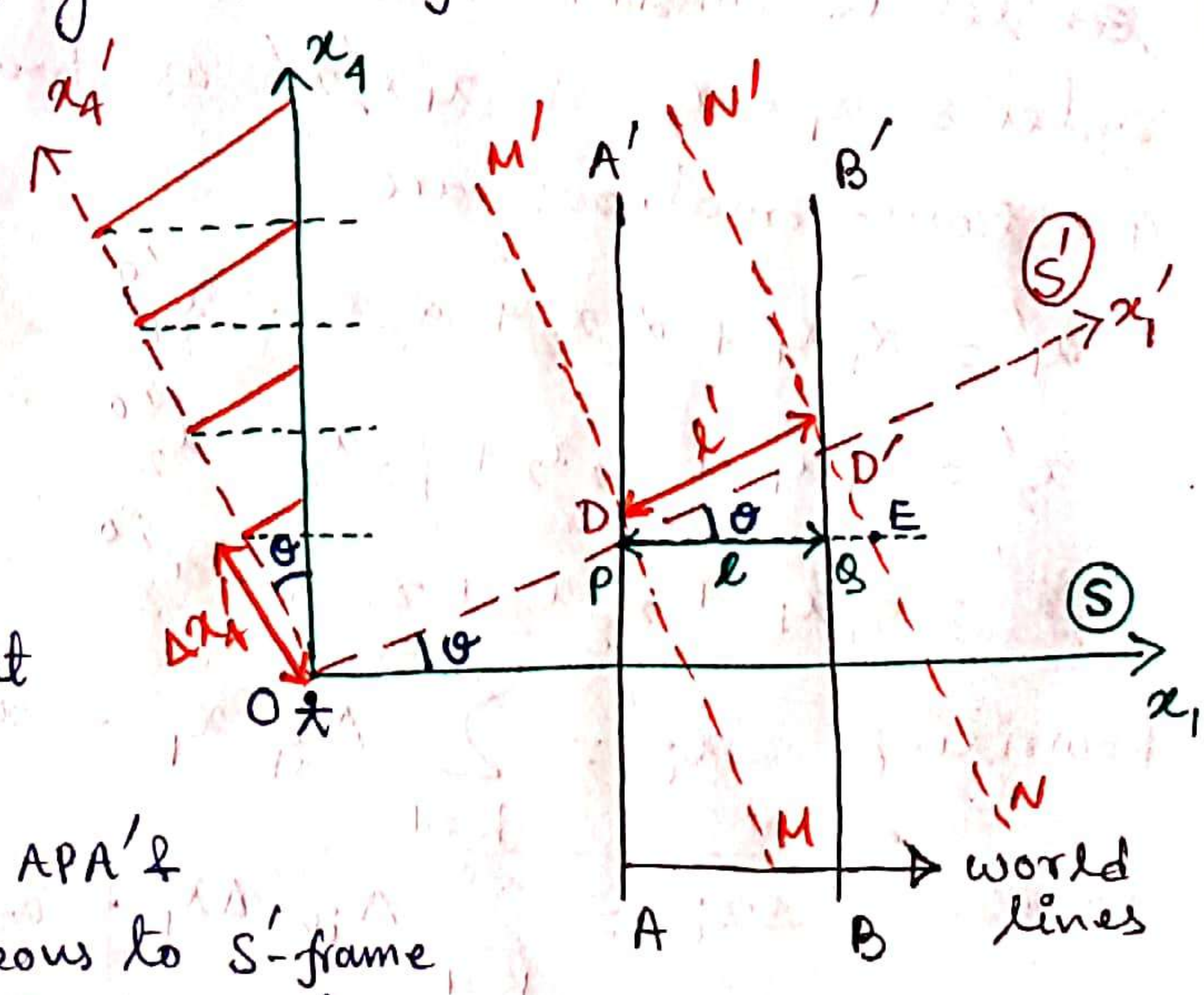


$$x_1' = \gamma(x_1 + i\beta x_4), \quad x_2' = x_2$$
$$x_4' = \gamma(x_4 - i\beta x_1), \quad x_3' = x_3$$

with $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$, then in 1+1 dimension this corresponds to rotation, with $x_1' = x_1\cos\theta + x_4\sin\theta$, $x_4' = -x_1\sin\theta + x_4\cos\theta$ so that $\cos\theta = \gamma$ and $\sin\theta = i\gamma\beta$ to yield $\tan\theta = i\beta$ ∴ $\theta = \tan^{-1}(i\beta)$

So LT is equivalent to rigid rotation of Cartesian axes in a 4D world with imaginary angle $\tan^{-1}(iv/c)$.

We can obtain Einstein's results (length contraction, time dilation, velocity addition) from Poincaré's geometrical form. Suppose that a rod of length l with end points P & Q is at rest in S-frame, so their world lines APA' & BQB' are parallel to x_4 axis.



For an observer at S' frame, world points on x_1' axis represent simultaneous events ($x_4' = 0$), so point of intersection of x_1' with APA' & BQB' (points D & D'), are simultaneous to S'-frame observer. So $l' = DD'$ & $l = PQ$ & from geometry $l' = l\sec\theta = l/\gamma$. If now rod was static in S' frame then MDM' & ND'N' are the world lines & observer-S will find $l = DE = DD'\sec\theta$

$= l'/\gamma$. So length contraction is reciprocal.

For S-frame observer at O, world line of his clock is Ox_4 & it's projection on x_4' axis will define interval for S'-frame clock observer. So $\Delta x_4' = \Delta x_4 \cos\theta = \gamma \Delta x_4$

or $ic\Delta t' = \gamma\, ic\Delta t$ or $\Delta t' = \gamma \Delta t$.

Similar argument can be given that for a stationary clock at S' frame, S-frame observer will find $\Delta t = \gamma \Delta t'$, so time dilation is also reciprocal. If we perform 2 LT one after another with velocity v and v' which is equivalent to rotations θ & θ', so that

$$\tan\theta = i\beta, \quad \tan\theta' = i\beta', \quad \text{so} \quad \tan(\theta+\theta') = \frac{\tan\theta + \tan\theta'}{1-\tan\theta\tan\theta'} = \frac{i(\beta+\beta')}{1+\beta\beta'}$$

If these transformations produces a velocity v'' corresponding to an angle θ'' then

$$\tan\theta'' = i\beta'' = \frac{i(\beta+\beta')}{1+\beta\beta'} \Rightarrow \beta'' = \frac{\beta+\beta'}{1+\beta\beta'} \Rightarrow$$

$$v'' = \frac{v+v'}{1+vv'/c^2} \quad \text{(addition law of velocity)}$$

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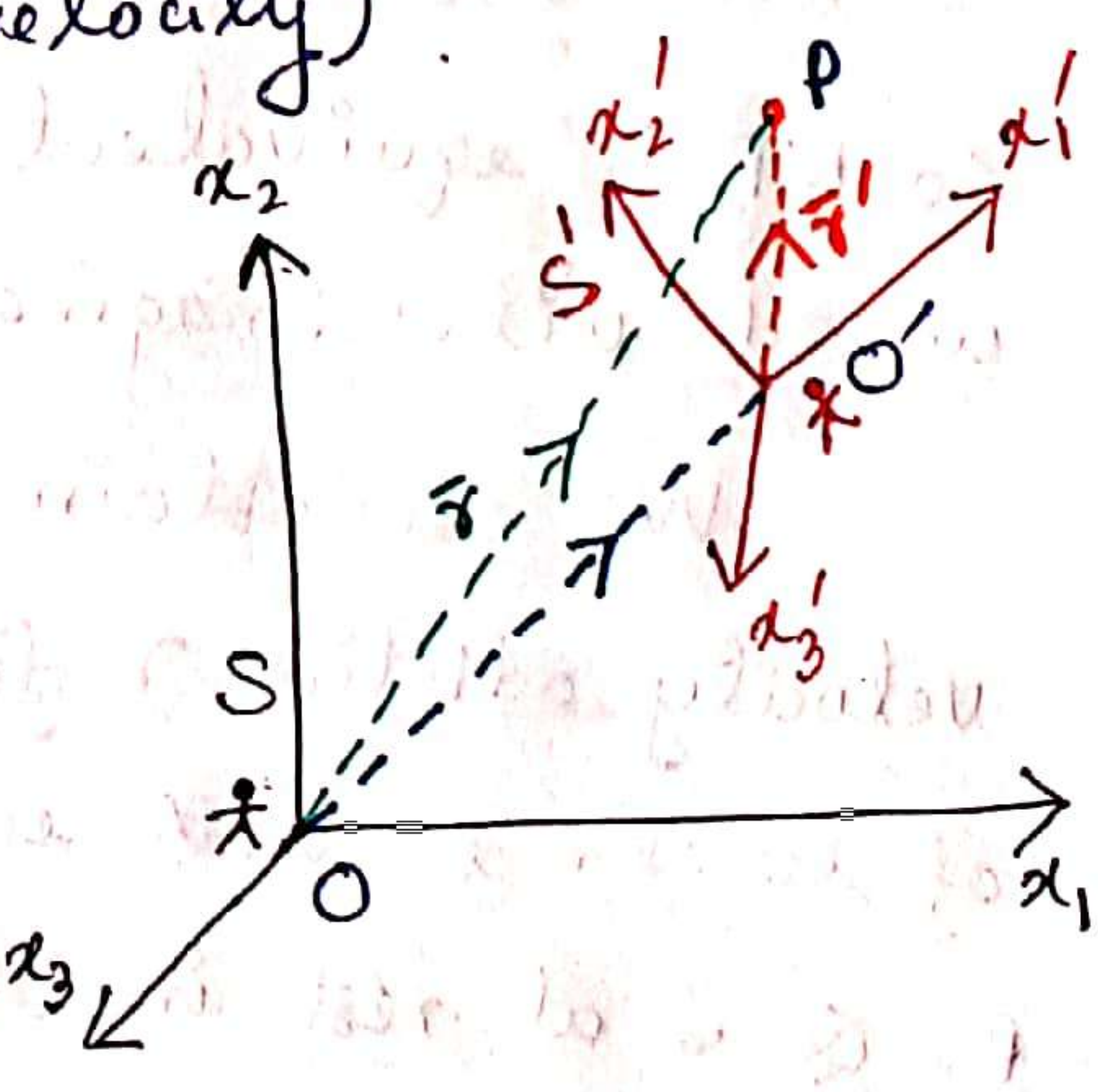
3-Vectors in O(3)



According to Poincare's formulation, let us consider 2 cartesian coordinate systems $x_1x_2x_3$ & $x_1'x_2'x_3'$ as shown.

The transformations are

$$x_1' = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 - x_1'^0$$
$$x_2' = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 - x_2'^0$$
$$x_3' = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 - x_3'^0$$

$\alpha_i, \beta_i, \gamma_i \rightarrow$ direction cosines

$x_i'^0 \rightarrow x_i'$ component of OO'.

Rewriting, $x_i' = \sum_{k=1}^{3} \Lambda_{ik} x_k - x_i'^0$, where $\Lambda_{11} = \alpha_1$, $\Lambda_{12} = \alpha_2$ etc

so that $\boxed{\Delta x_i' = \sum_{k=1}^{3} \Lambda_{ik} \Delta x_k}$ and $\boxed{\Delta x_k = \sum_{i=1} \Lambda'_{ki} \Delta x_i'}$ with $|C_{ik}| \neq 0$ (determinant)

(Orthogonal transformation) O.T.

Here $\Lambda_{ik} = \dfrac{\partial x_i'}{\partial x_k}$ & $\Lambda'_{ki} = \dfrac{\partial x_k}{\partial x_i'}$ & for OT, $\Lambda_{ik} = \Lambda'_{ki}$

Using Pythagoras theorem, we have norm of Δx_i

$$\sum_{i=1}^{3} \Delta x_i^2 = \sum_{i=1}^{3} \Delta x_i'^2 \quad \text{or} \quad \sum_{i=1}^{3} \Delta x_i \Delta x_i = \sum_{i=1}^{3} \Delta x_i'^2$$

or $\displaystyle\sum_{i=1}^{3} \sum_{k=1}^{3} \Lambda'_{ik} \Delta x'_k \sum_{l=1}^{3} \Lambda'_{il} \Delta x'_l = \sum_{i=1}^{3} \Delta x_i'^2$

or $\displaystyle\sum_{i,k,l=1}^{3} \Lambda'_{ik} \Lambda'_{il} \Delta x'_k \Delta x'_l = \sum_{i=1}^{3} \Delta x_i'^2$ can be true only if

$$\sum_{i=1}^{3} \Lambda'_{ik} \Lambda'_{il} = \delta_{kl}. \quad \text{— (1)}$$

Again using OT, $\displaystyle\Delta x_k = \sum_{i=1}^{3} \Lambda'_{ki} \Delta x'_i = \sum_{i,l=1}^{3} \Lambda'_{ki} \Lambda_{il} \Delta x_l$ can be true if

$$\sum_{i=1}^{3} \Lambda'_{ki} \Lambda_{il} = \delta_{kl} \quad \text{— (2)}$$

(1) $\times \displaystyle\sum_{l=1}^{3} \Lambda_{lm} \Rightarrow \sum_{i,l=1}^{3} \Lambda'_{ik} \Lambda'_{il} \Lambda_{lm} = \sum_{i=1}^{3} \Lambda'_{ik} \delta_{im} = \Lambda'_{mk}$

$\displaystyle\Rightarrow \sum_{i,l=1}^{3} \Lambda'_{ik} \Lambda'_{il} \Lambda_{lm} = \sum_{l=1}^{3} \delta_{kl} \Lambda_{lm} = \Lambda_{km}$

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So $\boxed{\Lambda'_{km} = \Lambda_{km}}$ for OT. (1) and (2) takes the form

$$\sum_{i=1}^{3} \Lambda_{ki} \Lambda_{li} = \delta_{kl} \quad \text{and} \quad \sum_{i=1}^{3} \Lambda_{ik} \Lambda_{il} = \delta_{kl}.$$

Taking determinant to find the transformation coefficients of OT.

$\det(AB) = \det(A)\det(B)$.

$$\det \sum_{i=1}^{3} \Lambda_{ki} \Lambda_{li} = \det \delta_{kl}$$

or $(\det \Lambda_{ki})(\det \Lambda_{li}) = \det \delta_{kl}$ or $(\det \Lambda_{ki})^2 = 1$ $(k=l)$

or $|\Lambda_{ki}| = \pm 1, e^{\pm i\theta}$.

In O(2), $x_1' = x_1 \cos\theta + x_2 \sin\theta$, $x_2' = -x_1 \sin\theta + x_2 \cos\theta$,

$$(\det \Lambda) = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1. \quad \text{(rotation)} \rightarrow \text{proper OT } O^+(3)$$

If $x_1' = x_1$, $x_2' = x_2$, $x_3' = -x_3$ then $(\det \Lambda) = -1$ (reflection)

In O(3), by solving eigenvalue equation $\Lambda u = \lambda u$ we have

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix} \quad \& \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \quad \text{so that} \quad V^{-1} \Lambda V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Vectors, Tensors, Metric & LT

The group of transformations that leave $s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$ invariant is called homogeneous Lorentz group, that contains rotation & LT. On the contrary, the group of transformations that leave invariant $s^2(x,y) = (x_0-y_0)^2 - (x_1-y_1)^2 - (x_2-y_2)^2 - (x_3-y_3)^2$ is called inhomogeneous Lorentz group (Poincaré group).

Einstein's 1st postulate dictates covariance, i.e. invariance in form under the transformations of homogeneous Lorentz group. In 4 dimensional non-Euclidean vector space, suppose a transformation between coordinates is defined as $x'^\alpha = x'^\alpha(x^0, x^1, x^2, x^3)$, $\alpha = 0,1,2,3$.

(a) Scalar (Tensor of rank 0) → interval s^2 → Lorentz scalar

(b) Vectors (Tensor of rank 1) → Contravariant vector $A^\alpha = (A^0, A^1, A^2, A^3)$
↳ Covariant vector $B_\alpha = (B_0, B_1, B_2, B_3)$

Transformation rule

$$\begin{cases} A'^\alpha = \dfrac{\partial x'^\alpha}{\partial x^\beta} A^\beta = \dfrac{\partial x'^\alpha}{\partial x^0} A^0 + \dfrac{\partial x'^\alpha}{\partial x^1} A^1 + \dfrac{\partial x'^\alpha}{\partial x^2} A^2 + \dfrac{\partial x'^\alpha}{\partial x^3} A^3 \\ B'_\alpha = \dfrac{\partial x^\beta}{\partial x'^\alpha} B_\beta = \dfrac{\partial x^0}{\partial x'^\alpha} B_0 + \dfrac{\partial x^1}{\partial x'^\alpha} B_1 + \dfrac{\partial x^2}{\partial x'^\alpha} B_2 + \dfrac{\partial x^3}{\partial x'^\alpha} B_3 \end{cases}$$

(Einstein's summation convention)

(c) 2nd rank Tensors (Tensor of rank 2) → Contravariant tensor $F'^{\alpha\beta} = \dfrac{\partial x'^\alpha}{\partial x^\gamma} \dfrac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$

→ Covariant tensor $G'_{\alpha\beta} = \dfrac{\partial x^\gamma}{\partial x'^\alpha} \dfrac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta}$

→ Mixed tensor $H'^\alpha{}_\beta = \dfrac{\partial x'^\alpha}{\partial x^\gamma} \dfrac{\partial x^\delta}{\partial x'^\beta} H^\gamma{}_\delta$

Scalar product $B \cdot A = B_\alpha A^\alpha = B^\alpha A_\alpha =$ covariant vector × contravariant vector or vice versa.

Invariance of scalar product (contraction)

$$B' \cdot A' = \underbrace{\frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\gamma} B_\beta A^\gamma}_{B_\alpha A^\alpha} = \frac{\partial x^\beta}{\partial x^\gamma} B_\beta A^\gamma = \delta^\beta{}_\gamma B_\beta A^\gamma = B \cdot A.$$

The interval (infinitesimal) $ds^2 = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}$ defines the norm of vector space of special relativity & is a special case of general differential length element $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ with $g_{\alpha\beta} = g_{\beta\alpha}$ is called the "Metric tensor." In SR, $g_{\alpha\beta}$ is diagonal with the form

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} ; \quad g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha{}^\beta = 0 \ (\alpha \neq \beta)$$
$$= 1 \ (\alpha = 0,1,2,3)$$

Covariant 4-vector x_α can be obtained from contravariant x^β by contracting with $g_{\alpha\beta}$ & inverse transformation likewise

$$x_\alpha = g_{\alpha\beta} x^\beta, \quad x^\alpha = g^{\alpha\beta} x_\beta$$

In LT, we have $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, $t' = \gamma(t - \frac{v}{c^2}x)$. If we define $A_1 = x$, $A_2 = y$, $A_3 = z$ and $ct = A_4$, then

$$A_1' = \gamma(A_1 - \tfrac{v}{c}A_4), \ A_2' = A_2, \ A_3' = A_3, \ A_4' = \gamma(A_4 - \tfrac{v}{c}A_1) \text{ and}$$
$$A^{1\prime} = \gamma(A^1 + \tfrac{v}{c}A^4), \ A^{2\prime} = A^2, \ A^{3\prime} = A^3, \ A^{4\prime} = \gamma(A^4 + \tfrac{v}{c}A^1).$$

So $A^\alpha = (A^0, \vec{A})$, $A_\alpha = (A^0, -\vec{A})$, $A^0 = A_0$, $A_1 = -A^1$, $A_2 = -A^2$, $A_3 = -A^3$.

Scalar product of 2 4-vectors $B.A = B_\alpha A^\alpha = B^0 A^0 - \vec{B}.\vec{A}$.

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right) = (\partial^0, -\vec{\partial}^i) ; \quad \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}$$
$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) = (\partial_0, \vec{\partial}_i) \quad \text{(covariant)}.$$

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4-divergence of 4-vector A is invariant

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla}.\vec{A} \quad \text{(continuity equation)}$$
$$\Box^2 = \partial_\alpha \partial^\alpha = \frac{\partial^2}{\partial x^{0\,2}} - \nabla^2 \quad \text{(operator of the wave equation)}$$

A 4-tensor of 2nd rank is a set of 16 quantities A^{ik} that transform as product of components of 2 4-vectors.

$A_{00} = A^{00}$, $A_{01} = -A^{01}$, $A_{11} = A^{11}$, ... , $A_0{}^0 = A^{00}$, $A_0{}^1 = A^{01}$, $A_1{}^0 = -A^{01}$, $A_1{}^1 = -A^{11}$

Under purely spatial transformations, 9 quantities $A^{11}, A^{12}, \ldots$ form a 3-tensor. The 3-components A^{01}, A^{02}, A^{03} & A^{10}, A^{20}, A^{30} constitute 3D vectors, A^{00} is 3D scalar.

Tensors $\delta_i{}^k$, g_{ik}, g^{ik} and ϵ^{iklm} (4th rank antisymmetric unit tensor) are special as their components are invariant in all coordinate systems.

$\epsilon^{0123} = -\epsilon_{0123} = 1$, $\epsilon^{iilm} = 0$ (as antisymmetric) & $\epsilon^{iklm}\epsilon_{iklm} = -24$ (as total 4! components). This is a "pseudotensor" as reflection cannot be transformed into rotation. If A^{ik} is an antisymmetric tensor then pseudotensor $A^{*ik} = \frac{1}{2}\epsilon^{iklm} A_{lm}$ are dual to each other. Similarly if A^i is vector then $\epsilon^{iklm} A_m$ is an antisymmetric pseudotensor of rank 3. $A^{ik} A^*_{ik}$ is a "pseudoscalar". Any completely antisymmetric tensor of rank equal to number of space dimensions is invariant under rotations of the coordinate system. Thus ϵ^{iklm} is unchanged under rotation of 4D coordinate & ϵ_{ijk} is unchanged by rotations of 3D axes.

Under a reflection (change of sign of all coordinates), components of vector changes sign are called "polar vector". Components of a vector which is a crossproduct of 2 polar vector is called "axial vector", that do not change sign under inversion. Scalar product of polar & axial vector is a "pseudoscalar", that changes sign under inversion. An axial vector is a "pseudovector", dual to antisymmetric tensor.

$$C_i = (A \times B)_i = \frac{1}{2}\epsilon_{ijk} C_{jk} \quad \text{where } C_{jk} = A_j B_k - A_k B_j$$

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Matrix representation of Proper homogeneous Lorentz Group's (PHLG) Infinitesimal generators S_i, K_i

In special relativity to stick to real coordinates (instead of complex) the metric tensor $g_{\mu\nu}$ is introduced [which is a 4x4 matrix = diag(1, −1, −1, −1)] so that the Lorentz scalar $c^2t^2 - x_i^2$ can be written as $g_{\mu\nu} x^\mu x^\nu$.

Suppose the coordinates x^0, x^1, x^2, x^3 are the components of a contravariant 4-vector which form a column vector $x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ and the covariant 4-vector

$$x_\nu = \underset{(gx)}{g_{\mu\nu} x^\mu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}.$$

The scalar product (without indices) is defined as

$$a \cdot b = a_\mu b^\mu = (a, gb) = (ga, b) = a^T g b \quad \text{where } (a,b) \equiv a^T b \text{ with}$$

$g^T = g$, $g^2 = I$ (4x4 unit matrix). What we require is a group of linear transformation of the coordinates $x' = \Lambda x$ such that norm (x, gx) is invariant, meaning

$\therefore x'^T g\, x' = x^T g x$ or, $x^T \Lambda^T g \Lambda x = x^T g x$

To satisfy this relation for all x, $\boxed{\Lambda^T g \Lambda = g}$ — ①. This is just a generalization of the 3D rotations of Cartesian axes $\Lambda^T \Lambda = \mathbb{I}$ (3x3 identity matrix) in 4D space. If we take determinant of ①, then $\det(\Lambda^T g \Lambda) = \det g\, (\det \Lambda)^2 = \det(g)$.

As $\det g = -1$, this can be true if $(\det\Lambda)^2 = 1$ or $\det\Lambda = \pm 1$.

$\therefore$ There are 2 classes of transformation: proper homogeneous Lorentz transformation (PHLG) (6 parameters) with $\det\Lambda = 1$ and improper homogeneous Lorentz transformation (PHIG) (10 parameters) with $\det\Lambda = -1$. Two counter examples of PHIG is $\Lambda = g$ (space inversion), $\det\Lambda = -1$ & $\Lambda = -\mathbb{I}$ (space & time inversion), $\det\Lambda = 1$.

Because ① is symmetric under transpose, $4^2 = 16$ parameters of Λ are not independent but $16 - (1+2+3) = 10$ linearly independent equations are obtained from ①. This means that PHLG is a 6 parameter group having 3 Euler angles (rotation) & 3 relative velocity component of $\vec{\beta} = \frac{\vec{v}}{c}$. We want to know the form (matrix) of the most general element of PHLG.

Let L is a 4x4 real traceless matrix such that $\Lambda = e^L$. This choice gives us $\det\Lambda = +1 = \det e^L = e^{\mathrm{Tr}\, L} = e^0 = +1$.

Now $\underline{\Lambda^T = e^{L^T}}$, so from ① we have $\Lambda^T g \Lambda = g$ $\;\; \times g$ (left), $\times \Lambda^{-1}$ (right)

or $g\Lambda^T g \underbrace{\Lambda\Lambda^{-1}}_{\mathbb{I}} = g g \Lambda^{-1} = \Lambda^{-1}$

or $g\Lambda^T g = \Lambda^{-1} = g e^{L^T} g = e^{gL^T g} = e^{-L}$

This means $gL^T g = -L$ or $\boxed{(gL)^T = -gL}$ — ②

So if $L = \begin{pmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix}$ then $gL = \begin{pmatrix} L_{00} & -L_{01} & -L_{02} & -L_{03} \\ L_{10} & -L_{11} & -L_{12} & -L_{13} \\ L_{20} & -L_{21} & -L_{22} & -L_{23} \\ L_{30} & -L_{31} & -L_{32} & -L_{33} \end{pmatrix}$

② $\Rightarrow \begin{pmatrix} L_{00} & L_{10} & L_{20} & L_{30} \\ -L_{01} & -L_{11} & -L_{21} & -L_{31} \\ -L_{02} & -L_{12} & -L_{22} & -L_{32} \\ -L_{03} & -L_{13} & -L_{23} & -L_{33} \end{pmatrix} = \begin{pmatrix} -L_{00} & L_{01} & L_{02} & L_{03} \\ -L_{10} & L_{11} & L_{12} & L_{13} \\ -L_{20} & L_{21} & L_{22} & L_{23} \\ -L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix}$, This can be only true if

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$L_{00} = L_{11} = L_{22} = L_{33} = 0$, $L_{01} = L_{10}$, $L_{02} = L_{20}$, $L_{03} = L_{30}$ and

$L_{ij} = -L_{ji}$ $(i \neq j)$, meaning

$$L = \left(\begin{array}{c|ccc} 0 & L_{01} & L_{02} & L_{03} \\ \hline L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{array}\right)$$

4x4 symmetric matrix (boost)

(boost)

3x3 antisymmetric spatial matrix (rotations)

$\equiv \{S_i, T_i\}$ matrices are the infinitesimal generators of PHLG. $= -\vec{\omega}\cdot\vec{S} - \vec{\xi}\cdot\vec{K}$

Rotations

$$S_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix},\quad S_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix},\quad S_3 = \begin{pmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix} \quad \&$$

$$S_1^2 = \begin{pmatrix} 0&&&0\\&0&&\\&&-1&\\0&&&-1 \end{pmatrix},\quad S_2^2 = \begin{pmatrix} 0&&&0\\&-1&&\\&&0&\\0&&&-1 \end{pmatrix},\quad S_3^2 = \begin{pmatrix} 0&&&0\\&-1&&\\&&-1&\\0&&&0 \end{pmatrix}.$$

Boosts

$$K_1 = \begin{pmatrix} 0&1&0&0\\1&&&\\0&&0&\\0&&& \end{pmatrix},\quad K_2 = \begin{pmatrix} 0&0&1&0\\0&&&\\1&&0&\\0&&& \end{pmatrix},\quad K_3 = \begin{pmatrix} 0&0&0&1\\0&&&\\0&&0&\\1&&& \end{pmatrix} \quad \&$$

$$K_1^2 = \begin{pmatrix} 1&&&0\\&1&&\\&&0&\\0&&&0 \end{pmatrix},\quad K_2^2 = \begin{pmatrix} 1&&&0\\&0&&\\&&1&\\0&&&0 \end{pmatrix},\quad K_3^2 = \begin{pmatrix} 1&&&0\\&0&&\\&&0&\\0&&&1 \end{pmatrix}.$$

If we consider a unit vector $\hat{\epsilon} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$ and $\vec{S} = \hat{i}S_1 + \hat{j}S_2 + \hat{k}S_3$, then $\hat{\epsilon}\cdot\vec{S} = \frac{1}{\sqrt{3}}(S_1 + S_2 + S_3)$

$$= \frac{1}{\sqrt{3}}\begin{pmatrix} 0&0&0&0\\0&0&-1&1\\0&1&0&-1\\0&-1&1&0 \end{pmatrix}$$

$$\therefore (\hat{\epsilon}\cdot\vec{S})^3 = \frac{1}{3\sqrt{3}}\begin{pmatrix} 0&0&0&0\\0&0&3&-3\\0&-3&0&3\\0&3&-3&0 \end{pmatrix} = -\frac{1}{\sqrt{3}}\begin{pmatrix} 0&0&0&0\\0&0&-1&1\\0&1&0&-1\\0&-1&1&0 \end{pmatrix} = -(\hat{\epsilon}\cdot\vec{S})$$

Similarly $\hat{\epsilon}' = \frac{1}{\sqrt{3}}(\hat{i}' + \hat{j}' + \hat{k}')$ and $\vec{K} = \hat{i}K_1 + \hat{j}K_2 + \hat{k}K_3$,

$(\hat{\epsilon}'\cdot\vec{K})^3 = +(\hat{\epsilon}'\cdot\vec{K})$. So any power of these matrices can be expressed as multiple of the matrix or square. Furthermore, following commutation relation hold,

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Lie algebra

$$[S_i, S_j] = \epsilon_{ijk} S_k \quad \text{(angular momentum)}$$
$$[S_i, K_j] = \epsilon_{ijk} K_k \quad \text{(K transforms as vector under rotation)}$$
$$[K_i, K_j] = -\epsilon_{ijk} S_k \quad \text{(boosts do not commute, like rotation)}$$

Physical Significance of L :

An infinitesimal LT and RLT can be defined as

$$x'^i = (g^{ij} + \epsilon^{ij}) x_j \quad \& \quad x^i = (g^{ij} + \epsilon'^{ij}) x'_j , \qquad \epsilon^{ij}, \epsilon'^{ij} \text{ are infinitesimal.}$$

Substituting one into the other we obtain

$$x'^i = (g^{ij} + \epsilon^{ij}) g_{ij} x^i = (g^{ij} + \epsilon^{ij}) g_{ij} (g^{ij} + \epsilon'^{ij}) x'_j$$

$$\text{or } x'^i = (g^{ij} + \epsilon^{ij})(\mathbb{I} + g_{ij} \epsilon'^{ij}) x'_j = (g^{ij} + \epsilon^{ij})(\mathbb{I} + g_{ij} \epsilon'^{ij}) g_{ij} x'^i$$

$$\therefore (g^{ij} + \epsilon^{ij})(g_{ij} + g_{ij} \epsilon'^{ij} g_{ij}) = \mathbb{I}.$$

$$\text{or } \mathbb{I} + \epsilon^{ij} g_{ij} + \epsilon'^{ij} g_{ij} + \cancel{O(\epsilon\epsilon')} = \mathbb{I} , \qquad \therefore \epsilon^{ij} = -\epsilon'^{ij}$$

Again, to preserve the norm, we must have $x'^T g x' = x^T g x$

$$\text{Now } x'^{iT} g^{ij} x'^i = [(g^{ij} + \epsilon^{ij}) x_j]^T g^{ij} (g^{ij} + \epsilon^{ij}) x_j$$

$$= [(g^{ij} + \epsilon^{ij}) g_{ij} x^i]^T g^{ij} (g^{ij} + \epsilon^{ij}) g_{ij} x^i \quad [\because g \,\&\, \epsilon \text{ commute}]$$

$$= (g_{ij} x^i)^T (g^{ij} + \epsilon^{ij})^T \underbrace{g^{ij} g_{ij}}_{\mathbb{I}} (g^{ij} + \epsilon^{ij}) x^i$$

$$= x^{iT} g_{ij} (g^{ij} + \epsilon^{ji})(g^{ij} + \epsilon^{ij}) x^i = x^{iT} g_{ij} x^i \text{ only if}$$

$$(g^{ij} + \epsilon^{ji})(g^{ij} + \epsilon^{ij}) = \mathbb{I} = \mathbb{I} + g^{ij}\epsilon^{ij} + g^{ij}\epsilon^{ji} + \cancel{O(\epsilon^2)}$$

$$\therefore \epsilon^{ij} = -\epsilon^{ji}$$

Again we have $\epsilon^{ij} g_{ij} + \epsilon'^{ij} g_{ij} = 0.$

$$\text{So } \epsilon'^{ij} g_{ij} = -\epsilon^{ij} g_{ij} = \epsilon^{ji} g_{ij}$$

$$\text{or } -\epsilon^{ij} g_{ij} = \epsilon^{ji} g_{ij} \quad \text{or} \quad (\epsilon g)^T = -\epsilon g.$$

Hence L matrix is the ϵ matrix that represents an infinitesimal Lorentz transformation.

$$L = \underset{\text{(rotation)}}{-\vec{\omega}\cdot\vec{S}} - \underset{\text{(boost)}}{\vec{\xi}\cdot\vec{K}} \quad \text{and} \quad \Lambda = e^{L} = e^{-(\vec{\omega}\cdot\vec{S} - \vec{\xi}\cdot\vec{K})}$$

For a boost in arbitrary direction $\Lambda = e^{-\vec{\xi}\cdot\vec{K}} = e^{-\hat{\beta}\cdot\vec{K}\tanh^{-1}\beta}$

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where the boost vector $\vec{\xi} = \hat{\beta} \tanh^{-1}\beta$ with $\beta = \frac{v}{c}$ and $\hat{\beta}$ is a unit vector in the direction of the inertial frames.

In the hyperbolic form $\gamma = \cosh\xi$, $\gamma\beta = \sinh\xi$ so that $\beta = \tanh\xi$

$$\Lambda = e^{-\xi(\hat{\beta}\cdot\vec{K})} = 1 - (\hat{\beta}\cdot\vec{K})\xi + (\hat{\beta}\cdot\vec{K})^2\frac{\xi^2}{2!} - (\hat{\beta}\cdot\vec{K})^3\frac{\xi^3}{3!} + (\hat{\beta}\cdot\vec{K})^4\frac{\xi^4}{4!} - \dots$$

$$= 1 - (\hat{\beta}\cdot\vec{K})\left(\xi + \frac{\xi^3}{3!} + \frac{\xi^5}{5!} + \cdots\right) + (\hat{\beta}\cdot\vec{K})^2\left(1 + \frac{\xi^2}{2!} + \frac{\xi^4}{4!} + \cdots\right) - (\hat{\beta}\cdot\vec{K})^2$$

$$= 1 - (\hat{\beta}\cdot\vec{K})\sinh\xi + (\hat{\beta}\cdot\vec{K})^2(\cosh\xi - 1)$$

Now $\hat{\beta} = \frac{\beta_1}{\beta}\hat{i} + \frac{\beta_2}{\beta}\hat{j} + \frac{\beta_2}{\beta}\hat{k}$, $\vec{K} = K_1\hat{i} + K_2\hat{j} + K_3\hat{k}$, so that

$$\hat{\beta}\cdot\vec{K} = \frac{\beta_1}{\beta}K_1 + \frac{\beta_2}{\beta}K_2 + \frac{\beta_3}{\beta}K_3 = \begin{pmatrix} 0 & \beta_1/\beta & \beta_2/\beta & \beta_3/\beta \\ \beta_1/\beta & & & \\ \beta_2/\beta & & 0 & \\ \beta_3/\beta & & & \end{pmatrix} \text{ and}$$

$$(\hat{\beta}\cdot\vec{K})^2 = \begin{pmatrix} 0 & \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \\ \frac{\beta_1}{\beta} & 0 & 0 & 0 \\ \frac{\beta_2}{\beta} & 0 & 0 & 0 \\ \frac{\beta_3}{\beta} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \\ \frac{\beta_1}{\beta} & 0 & 0 & 0 \\ \frac{\beta_2}{\beta} & 0 & 0 & 0 \\ \frac{\beta_3}{\beta} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \underbrace{\frac{\beta_1^2+\beta_2^2+\beta_3^2}{\beta^2}}_{=1} & 0 & 0 & 0 \\ 0 & \frac{\beta_1^2}{\beta^2} & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_1\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_2^2}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_3}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} & \frac{\beta_3^2}{\beta^2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta_1^2}{\beta^2} & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_1\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_2^2}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_3}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} & \frac{\beta_3^2}{\beta^2} \end{pmatrix}$$

$$\therefore \Lambda = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} - \gamma\beta\begin{pmatrix} 0 & \frac{\beta_1}{\beta} & \frac{\beta_2}{\beta} & \frac{\beta_3}{\beta} \\ \frac{\beta_1}{\beta} & & & \\ \frac{\beta_2}{\beta} & & 0 & \\ \frac{\beta_3}{\beta} & & & \end{pmatrix} + (\gamma-1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta_1^2/\beta^2 & \frac{\beta_1\beta_2}{\beta^2} & \frac{\beta_1\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_2}{\beta^2} & \beta_2^2/\beta^2 & \frac{\beta_2\beta_3}{\beta^2} \\ 0 & \frac{\beta_1\beta_3}{\beta^2} & \frac{\beta_2\beta_3}{\beta^2} & \frac{\beta_3^2}{\beta^2} \end{pmatrix}$$

$$\text{or } \boxed{\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\frac{(\gamma-1)\beta_1^2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1+\frac{(\gamma-1)\beta_2^2}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1+\frac{(\gamma-1)\beta_3^2}{\beta^2} \end{pmatrix}}$$

To remind,

$X' = \Lambda X$, where

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}.$$

The same matrix obtained without 4-vector notation in Generalized 3D LT with $X = \begin{pmatrix} \vec{x} \\ t \end{pmatrix}$.

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