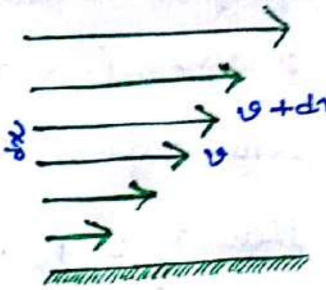
VISCOSITY



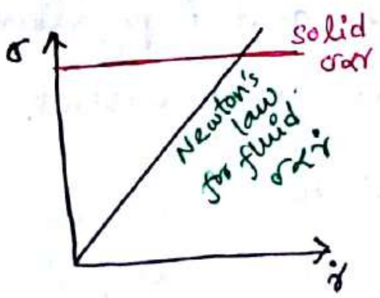
The central property that distinguishes a fluid from a solid is that, a fluid cannot sustain a "shear stress" for any period of time, meaning if a shear is applied to a fluid, it will move under the shear. Thicker liquids like paint, honey, cornstarch solution move less easily than fluids like water, glycerol or air. The measure of the ease with which a fluid yields is its viscosity.

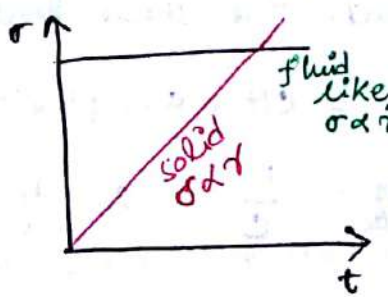
When a liquid is at rest, we do not observe any rigidity or shape elasticity in it but when the liquid is in orderly motion (not turbulent, but streamline), there comes into play a tangential stress between any two layers of the fluid, that are moving relative to each other. Difference in velocities between these two layers gives rise to internal friction, as a result of which the faster layer tends to accelerate the slower one & vice versa.

Newton found that for a fluid moving in parallel layers, the shearing stress at any point where the velocity gradient $\frac{\partial v_x}{\partial y}$ perpendicular to the direction of motion, the frictional force F_x is proportional to area of fluid layer & $\frac{\partial v_x}{\partial y}$.

$$\therefore \quad F_x \propto A \qquad \qquad \therefore \quad F_x = \eta A \frac{\partial v_x}{\partial y}$$

$$\propto \frac{\partial v_x}{\partial y} \qquad \text{or} \quad \frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y} \qquad \text{or} \quad \underbrace{\sigma}_{\text{stress}} = \underbrace{\eta}_{\text{coefficient of viscosity}} \underbrace{\dot{\gamma}}_{\text{strain rate}}$$





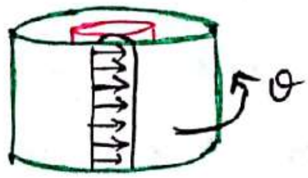
This is Newton's law of viscosity and fluid exhibiting such phenomena is called Newtonian fluid.

Compare with Hooke's law of elastic solid where stress $\propto$ strain ($\sigma = Y\gamma$, Y = Young's modulus) while for fluid stress $\propto$ strainrate ($\sigma = \eta\dot{\gamma}$, η = Coefficient of viscosity). If $A = 1$, $\frac{\partial v_y}{\partial x} = 1$ then $\eta = F_x$. So, coefficient of viscosity of a fluid is the tangential stress per unit velocity gradient. C.G.S. unit of η is "Poise". 1 Poise η means that a force of 1 dyne is required to maintain a velocity difference of 1 cm/sec between two layers of 1 square cm area which are 1 cm apart from each other.

Dimension of η: $\eta = \dfrac{F/A}{\frac{\partial v_y}{\partial x}}$, $\therefore [\eta] = \dfrac{[MLT^{-2}]/[L]^2}{[LT^{-1}]/[L]}$

$= [ML^{-1}T^{-1}]$

Fugitive elasticity:



Other than sliding planes geometry, Couette flow can be generated by sandwiching a liquid between two concentric cylinders with inner (outer) one stationary. From consideration of shape elasticity, tangential stress $\sigma = n\theta$ where n = modulus of rigidity and θ is the angle of shear.

In the limit $\lim_{\theta \to 0}$, $\sigma = n\tan\theta = n\frac{dy}{dx}$

From Newton's law of viscosity, $\sigma = \eta\frac{dv}{dx} = \eta\frac{d}{dx}\left(\frac{dy}{dt}\right)$

$= \eta\frac{d}{dt}\left(\frac{dy}{dx}\right) = \eta\frac{d\theta}{dt}$.

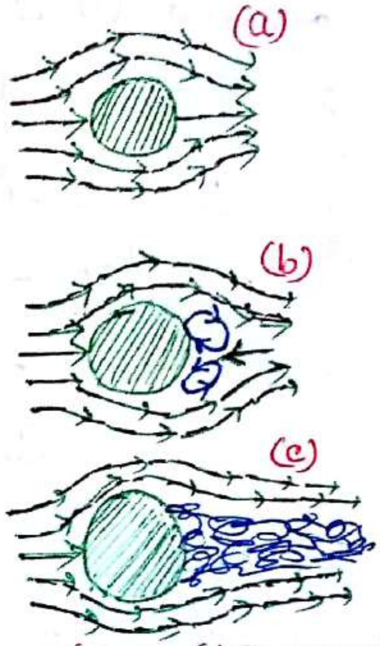
$\frac{d\theta}{dt}$ represents the rate at which the shear brakes and is proportional to the angle of shear. $\therefore \frac{d\theta}{dt} = c\theta$, c = proportionality constant.

$\therefore \sigma = \eta c\theta = n\theta$ or $\eta = \frac{n}{c}$ $\therefore \eta \propto n$

$\frac{1}{c}$ is the relaxation time of the medium fluid that measures the time taken by the shear to disappear, if force is taken off.

So the appearance of viscous force during streamline motion of a liquid is due to the existence of intermittent shear elasticity (fugitive elasticity). Maxwell stated that viscosity in a fluid is due to the existence of fugitive elasticity in it.

Streamline & Turbulent motion - Critical velocity & Reynold's number :



Viscous flow around a circular cylinder

If fluid flow is such that magnitude & direction of velocity at any point is always same then its called a steady streamline flow. As in panel (a), no two streamlines can cross each other & the tangent to a line at any point gives the flow direction. In turbulent flow as panel (b) & (c) velocity magnitude & direction changes in irregular manner in terms of eddies, vortices, zigzag motion.

A difference of pressure is maintained between ends for flow of liquid through a tube (Poiseuille's flow). The layer of liquid in contact with the wall of the tube is at rest ("no-slip" condition where both normal & tangential component of velocity is zero). Velocity of the layer increases towards the axis of the tube. The streamline flow is maintained when the velocity is below a certain limit known as "Critical velocity."

When critical velocity is reached, indifferent parts of the liquid no layer travel in a straight line along the tube & when velocity is further increased, streamline motion is completely lost (turbulent).

Reynolds using dimensional analysis showed that in Poiseuille flow, critical velocity v is related to the fluid density ρ, radius of tube r & coefficient of viscosity η.

Suppose, $v = Re\, \eta^x \rho^y r^z$ where Re = Reynold's number whose value is 1000 for narrow tubes. In general, for a liquid of high viscosity even for high velocity streamline motion is observed while for high density & wide bore of tube makes the motion turbulent.

Substituting dimensions, we have

Re is dimensionless number

$$[LT^{-1}] = [ML^{-1}T^{-1}]^x [ML^{-3}]^y [L]^z$$
$$= [M^{x+y}][L^{-x-3y+z}][T^{-x}]$$

Equating the powers of [M], [L] and [T], we have

$$\left.\begin{aligned} x + y &= 0 \\ -x - 3y + z &= 1 \\ -x &= -1 \end{aligned}\right\} \quad x = 1,\ y = -1,\ z = -1$$

$$\therefore\ v = Re\, \eta \rho^{-1} r^{-1}$$

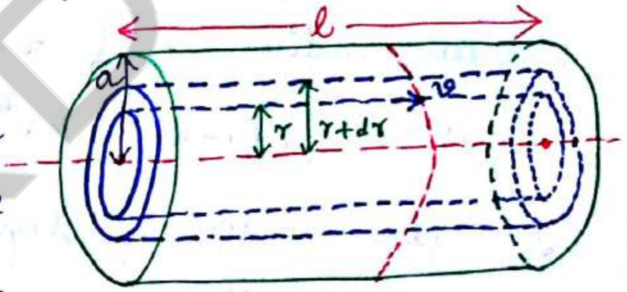
or $\boxed{Re = \dfrac{\rho v r}{\eta}}$

While discussing on motion of dry water, we will see that "vorticity" $\vec{\Omega} = \vec{\nabla} \times \vec{v}$ follows a simple kinetic equation

$$\frac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = \frac{1}{Re} \nabla^2 \vec{\Omega}.$$ This means that if we solve the flow problem for v_1 for a certain cylinder with radius r_1 & then ask about flow for a different radius r_2 for a different fluid with velocity v_2, the Reynold's number will be same means flows will appear same.

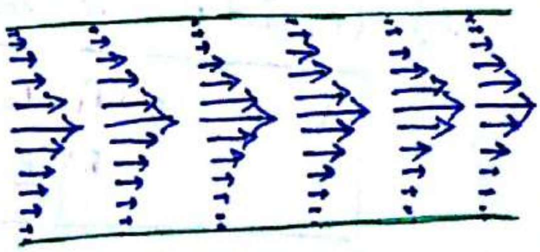
$\therefore$ $Re = \frac{\rho_1 v_1 r_1}{\eta_1} = \frac{\rho_2 v_2 r_2}{\eta_2}$. So we can determine the flow of air past an airplane wing without building an airplane to try but instead make a model with velocity to yield same Re. We can only apply so provided we are dealing with "incompressible" liquid & not with "compressible" gas. Otherwise speed of sound in terms of "Mach number" has to be taken into account. $Ma = \frac{\text{speed of sound in fluid}}{\text{speed of sound in air}}$. So for velocities near the speed of sound or above, the flows are the same in two situations if both "Ma" & "Re" are same for both situations.

Poiseuille's equation for flow of liquid through a horizontal narrow tube:



Consider a horizontal streamline motion of a liquid through a narrow tube. The lines are parallel to the axis of tube (no radial flow). Pressure varies along the length of tube & due to no-slip, velocity of the liquid gradually decreases radially from the axis towards the wall of the tube.

When steady state flow is attained, let v be the velocity at a distance r from the axis of tube & velocity gradient is $\frac{dv}{dr}$. So the tangential stress is $\eta \frac{dv}{dr}$. This force acts over the unit area of surface of cylinder at r in a direction opposite to the pressure gradient. So the total resisting force over the surface of the liquid cylinder is $2\pi r l \, \eta \frac{dv}{dr}$.



If P is the pressure difference between the ends of the tube, then the active force is $P\pi r^2$ (As $P = \frac{F}{A} = \frac{F}{\pi r^2}$). This force tends to accelerate the liquid in cylinder and therefore in steady state, this accelerating force is balanced by the viscous retarding force,

$$P\pi r^2 = -\eta \frac{dv}{dr} 2\pi r l \quad \left(\because \frac{dv}{dr} < 0 \text{ as } v \text{ decrease with increasing } r\right)$$

$$\text{or} \quad r\,dr = -\frac{2\eta l}{P} dv$$

Integrating with the Boundary condition (B.C.) $v = 0$ at $r = a$,

$$\int_r^a r\,dr = -\frac{2\eta l}{P}\int_v^0 dv \Rightarrow \boxed{v = \frac{P}{4\eta l}(a^2 - r^2)}$$

This is an equation for parabola. Now if dV is the volume of liquid that flows through the cylindrical shell per unit time between radius r & $r+dr$ then

$$dV = [\pi(r+dr)^2 - \pi r^2]v = 2\pi r\,dr\,v = \frac{\pi P}{2\eta l}(a^2 - r^2) r\,dr$$

$\therefore$ Total volume of liquid passing through the tube per unit time is

$$V = \frac{\pi P}{2\eta l}\int_0^a (a^2 - r^2) r\,dr = \frac{\pi P}{2\eta l}\left(a^2\int_0^a r\,dr - \int_0^a r^3 dr\right)$$

$$= \frac{\pi P}{2\eta l}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi P a^4}{8\eta l}$$

If P_1 and P_2 are the pressure at two ends of the tube then

$$\boxed{V = \frac{\pi (P_1 - P_2) a^4}{8\eta l}}$$

This is known as Poiseuille's equation

__Correction to Poiseuille's formula:__

Poiseuille's equation $V = \frac{\pi P a^4}{8\eta l}$ is approximately true because two important factors are not taken into account. (a) pressure difference P is utilized partly in communicating Kinetic energy to the liquid (b) acceleration of the liquid along the axis of the tube is neglected.

A finite acceleration at inlet of the tube becomes zero only after traversing a finite distance, so $1.64a$ is added to the length l. To find the kinetic energy correction, let's consider that p is the effective pressure difference that overcomes viscosity.

The workdone against viscous force per unit time is pV & kinetic energy per unit time is $= \int_0^a \frac{1}{2}\underbrace{\rho\, 2\pi r dr\, v}_{\text{mass}} . v^2$

$$= \int_0^a \rho\pi v^3 r dr = \int_0^a \rho\pi\left(\frac{p}{4\eta l}\right)^3 (a^2-r^2)^3 r dr = \rho\pi\left(\frac{p}{4\eta l}\right)^3 \int_0^a (a^2-r^2)^3 r dr$$

$$= \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8}.$$ So total loss of energy $= pV + \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8}$

and this is equal to PV

$$\therefore \quad pV + \rho\pi\left(\frac{p}{4\eta l}\right)^3 \frac{a^8}{8} = PV$$

$$\text{or} \quad pV + \frac{\rho}{\pi^2 a^4}\left(\frac{\pi p a^4}{8\eta l}\right)^3 = PV$$

$$\text{or} \quad pV + \frac{\rho V^3}{\pi^2 a^4} = PV \qquad \left(\because V = \frac{\pi p a^4}{8\eta l}\right)$$

$$\text{or} \quad p = P - \frac{\rho V^2}{\pi^2 a^4}.$$

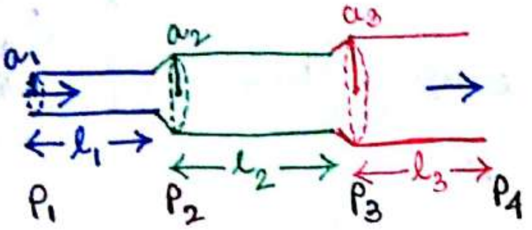
Experimentally it was found by Hagenbach, Couette & Wilberforce that correct form is $p = P - \frac{k\rho V^2}{\pi^2 a^4}$ with $k \approx 1$, differing for different scenario.

Corrected Poiseuille's formula is

$$\boxed{V = \frac{\pi a^4}{8\eta(l+1.64a)}\left(P - \frac{k\rho V^2}{\pi^2 a^4}\right)}$$

Flow of liquid through capillaries in series & parallel

Poiseuille's formula $V = \frac{P}{8\eta l/\pi a^4} = \frac{P}{R}$ can be compared with ohm's law for flow of electric current through a resistance $i = \frac{E}{R}$. So the rate of flow of the liquid V corresponds to current i, pressure difference P to the potential difference E and $\frac{8\eta l}{\pi a^4}$ to the resistance R.



Consider a series connection of three capillaries with radius a_1, a_2, a_3 & length l_1, l_2, l_3. Let P_1 & P_4 are the pressure at extreme ends & P_2, P_3 are pressure at the junctions. As there is no accumulation of the liquid at the junction, so V must be equal through all capillaries, just like current is same through any resistance connected in series.

$$\text{So,} \quad V = \frac{\pi(P_1 - P_2)a_1^4}{8\eta l_1} = \frac{\pi(P_2 - P_3)a_2^4}{8\eta l_2} = \frac{\pi(P_3 - P_4)a_3^4}{8\eta l_3}$$

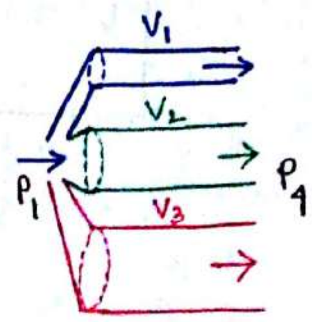
$$\therefore \left.\begin{aligned} P_1 - P_2 &= \frac{8\eta l_1 V}{\pi a_1^4} \\ P_2 - P_3 &= \frac{8\eta l_2 V}{\pi a_2^4} \\ P_3 - P_4 &= \frac{8\eta l_3 V}{\pi a_3^4} \end{aligned}\right\}$$

Summing, $P_1 - P_4 = \frac{8\eta V}{\pi}\left[\frac{l_1}{a_1^4} + \frac{l_2}{a_2^4} + \frac{l_3}{a_3^4}\right]$

$$\text{or } V = \frac{\pi P}{8\eta}\left[\frac{l_1}{a_1^4} + \frac{l_2}{a_2^4} + \frac{l_3}{a_3^4}\right]^{-1} \quad \text{—(1)}$$

where $P = P_1 - P_4$ is pressure difference across composite slab.

If we maintain $P = P_1 - P_4$ across ends of three capillaries connected in parallel, then volume of liquid flowing per unit time through them is



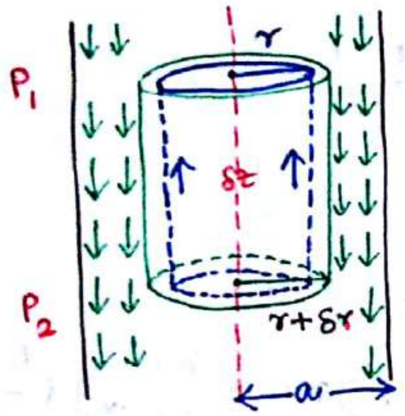
$$V = V_1 + V_2 + V_3 = \frac{\pi P a_1^4}{8\eta l_1} + \frac{\pi P a_2^4}{8\eta l_2} + \frac{\pi P a_3^4}{8\eta l_3}$$

$$V = \frac{\pi P}{8\eta}\left(\frac{a_1^4}{l_1} + \frac{a_2^4}{l_2} + \frac{a_3^4}{l_3}\right) = P\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \quad \text{--- (2)}$$

Comparing (1) and (2) we recover the effective resistance for an equivalent series connected viscous flow as $R = R_1 + R_2 + R_3$ while for parallel connected, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ where

$$R_1 = \frac{8\eta l_1}{\pi a_1^4}, \quad R_2 = \frac{8\eta l_2}{\pi a_2^4}, \quad R_3 = \frac{8\eta l_3}{\pi a_3^4}.$$

Downward flow of a liquid through a vertical narrow tube:



Consider a vertical narrow tube through which a liquid flows steadily. Consider a cylindrical shell with radius r & $r + \delta r$ and length δz. So the viscous force on the inner shell wall in vertical upward direction is $2\pi r \delta z \eta \frac{dv}{dr}$. Viscous force on the outer wall in vertical downward direction is $2\pi r \delta z \eta \frac{dv}{dr} + \frac{d}{dr}\left(2\pi r \delta z \eta \frac{dv}{dr}\right)\delta r$

Force due to pressure p on the upper annular flat surface of the element of the liquid cylinder in the vertical downward direction is $2\pi r \delta r p$ & similarly force due to pressure on the lower annular flat surface of the liquid cylinder in the upward direction is $2\pi r \delta r \left(p + \frac{dp}{dz}\delta z\right)$

At steady state, liquid acceleration = 0, resultant downward force is zero.

$$\therefore \cancel{-2\pi r \delta z \eta \frac{dv}{dr}} + \cancel{2\pi r \delta z \eta \frac{dv}{dr}} + \frac{d}{dr}\left(2\pi r \delta z \eta \frac{dv}{dr}\right)\delta r + \underbrace{2\pi r \delta r \delta z \rho g}_{\Downarrow}$$

$$+ \cancel{2\pi r \delta r p} - 2\pi r \delta r\left(\cancel{p} + \frac{dp}{dz}\delta z\right) = 0$$

[Vertical downward force due to weight of cylindrical shell]

$$\text{or} \quad 2\pi \delta z \eta \frac{d}{dr}\left(r\frac{dv}{dr}\right) + 2\pi r \delta r \delta z \rho g - 2\pi r \delta r \frac{dp}{dz}\delta z = 0$$

or, $\eta \frac{d}{dr}\left(r\frac{dv}{dr}\right) + r\rho g = r\frac{dp}{dz}$ $[\because \eta \neq \eta(r)]$

or $\frac{dp}{dz} = \frac{\eta}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \rho g.$ or $-\frac{P_1 - P_2}{l} = \frac{\eta}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \rho g$

or $\left(\rho g + \frac{P_1 - P_2}{l}\right) r = -\eta \frac{d}{dr}\left(r\frac{dv}{dr}\right)$

Integrating with B.C. $r = a$, $v = 0$

$$\left(\rho g + \frac{P_1 - P_2}{l}\right)\frac{r^2}{2} + A = -\eta r \frac{dv}{dr}$$

Integrating once again, $\left(\rho g + \frac{P_1 - P_2}{l}\right)\frac{r^2}{4} + A \ln r + B = -\eta v$

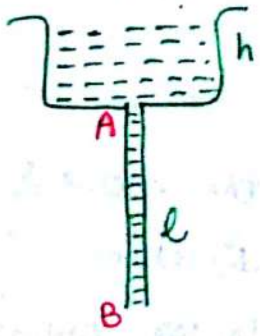
at $r = 0$, $v \neq \infty$ (remember $\ln 0 = -\infty$), A must be zero. &

substituting $r = a$, $v = 0$. $B = -\left(\rho g + \frac{P_1 - P_2}{l}\right)\frac{a^2}{4}$

$$\therefore \eta v = \left(\rho g + \frac{P_1 - P_2}{l}\right)\left(\frac{a^2 - r^2}{4}\right)$$

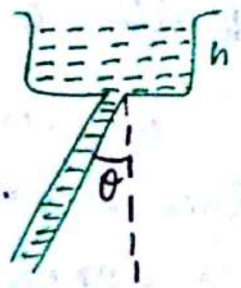
So the volume of liquid flowing per unit time is

$$V = \int_0^a 2\pi r\, dr\, v = \frac{\pi}{2\eta}\left(\rho g + \frac{P_1 - P_2}{l}\right)\int_0^a (a^2 - r^2) r\, dr = \frac{\pi a^4}{8\eta}\left(\rho g + \frac{P_1 - P_2}{l}\right)$$



In liquid viscometer, a narrow tube is connected to a liquid container. $P_1 = \pi + \rho g h$ is the pressure at inlet A and $P_2 = \pi$ is the pressure at outlet B where π is the barometric pressure.

$$\therefore V = \frac{\pi a^4}{8\eta}\left(\rho g + \frac{\rho g h}{l}\right) = \boxed{\frac{\pi a^4}{8\eta l}\rho g (l + h)}$$

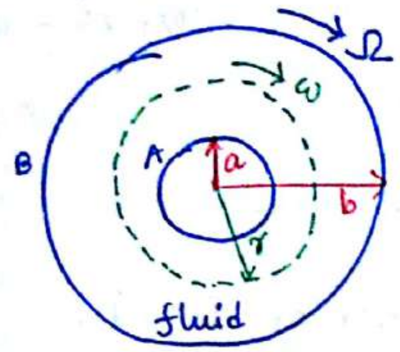


For inclined liquid viscometer, acceleration due to gravity g has $g\cos\theta$ component along vertical direction.

$$\text{So, } V = \frac{\pi a^4}{8\eta}\left(\rho g \cos\theta + \frac{P_1 - P_2}{l}\right) = \frac{\pi a^4}{8\eta}\left(\rho g \cos\theta + \frac{\rho g h}{l}\right)$$

$$= \boxed{\frac{\pi a^4}{8\eta l}\rho g (l\cos\theta + h)}$$

Torque on a cylinder immersed in a rotating fluid



A viscous liquid is filled within two coaxial cylinders A & B with cylinder B rotating about common axis with constant angular velocity Ω. The torque on A due to rotation B is required. The innermost layer of fluid at A is zero velocity (no-slip), a, b, l are the radii of A & B cylinder and length. Suppose the fluid at a distance r rotates with angular velocity ω. Its linear velocity is $r\omega$ and velocity gradient

is $\frac{d}{dr}(r\omega) = r\frac{d\omega}{dr} + \cancel{\omega}^{\to 0}$ ↳ no viscosity effect.

∴ Viscous force on the side of fluid cylinder is

$F = \eta\, 2\pi r l\, r\frac{d\omega}{dr}$ & then the viscous torque $\Gamma = F\cdot r$
$= 2\pi\eta l r^3 \frac{d\omega}{dr}$

As the fluid rotates in steady state, torque on inner cylinder is in clockwise direction

$\Gamma.$ ∴ $\Gamma\frac{dr}{r^3} = 2\pi\eta l\, d\omega$

Integrating, $\Gamma\int_a^b \frac{dr}{r^3} = 2\pi\eta l \int_0^\Omega d\omega$

or $\frac{\Gamma}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 2\pi\eta l\Omega$ or $\boxed{\Gamma = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2}}$

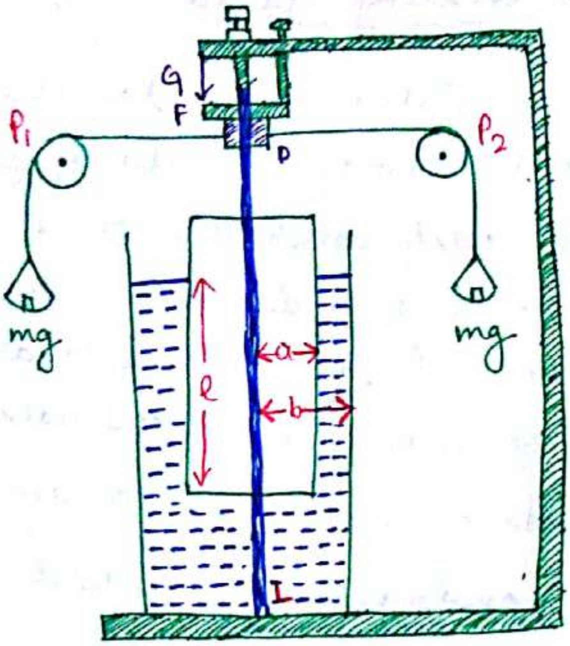
\# Suppose now A rotates with Ω & Γ is the clockwise torque

$\Gamma = -2\pi\eta l r^3\frac{d\omega}{dr}$ ($\Gamma < 0$ as $\frac{d\omega}{dr} < 0$ as ω decreases with increasing r)

∴ $-\Gamma\int_a^b \frac{dr}{r^3} = 2\pi\eta l\int_\Omega^0 d\omega$ or $\frac{\Gamma}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = -2\pi\eta l\Omega$

or $\boxed{\Gamma = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2}}$

Searle's viscometer uses this technique for measurement of η of highly viscous liquids. Two weights rotate the inner



Searle's viscometer

cylinder by ball-bearing pulleys while the outer cylinder is fixed with the fluid in between. We know

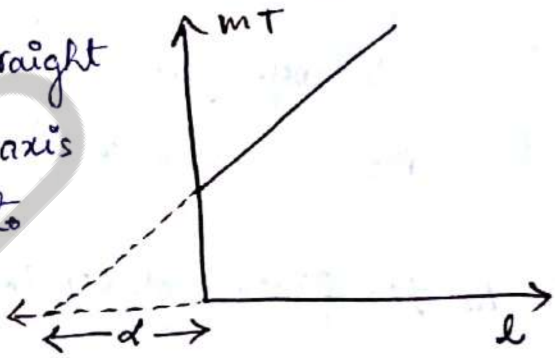
$$\Gamma = \frac{4\pi\eta l \Omega a^2 b^2}{b^2 - a^2} = mgd$$

where d = diameter of drum D.

So $\eta = \frac{gd(b^2-a^2)}{4\pi\Omega a^2 b^2} \frac{m}{l}$ and as

$T = \frac{2\pi}{\Omega}$ is the time period of rotation

$$\eta = \frac{gd(b^2-a^2)}{8\pi^2 a^2 b^2} \frac{mT}{l}.$$ For a given liquid, $\frac{mT}{l}$ = constant



Though it should be a origin-passing straight line, but experimentally found to cut Y axis so that the above expression is modified to accomodate

$$\eta = \frac{gd(b^2-a^2)}{8\pi^2 a^2 b^2} \frac{mT}{l+\alpha}$$

Viscosity of high viscous liquids

Stoke's law : Viscous resisting force on a small sphere falling through a liquid of infinite extent is $F = 6\pi\eta a v$ where v = terminal velocity of the sphere, a = radius.

viscous retarding force = effective gravitational force.

$$\frac{4}{3}\pi a^3(\rho - \sigma)g = 6\pi\eta a v, \quad \begin{cases} \rho = \text{density of the sphere} \\ \sigma = \text{density of the liquid.} \end{cases}$$

$$\text{or } \eta = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{v}$$

In practice in cylindrical vessel due to confinement, boundary effect due to wall & bottom of cylinder is corrected to yield

$$\eta = \frac{2}{9}\frac{a^2(\rho-\sigma)g}{v(1+2.4\frac{a}{R})(1+3.3\frac{a}{h})}$$, R = radius of cylinder, h = height of liquid.

Using dimensional analysis $F = 6\pi\eta a v$ can be deduced as

$F = K a^x \eta^y v^z$, K = dimensionless number.

$\therefore [MLT^{-2}] = [L]^x [ML^{-1}T^{-1}]^y [LT^{-1}]^z = M^y L^{x-y+z} T^{-y-z}$

Equating the powers of M, L, T, $y=1$, $x-y+z=1$, $-y-z=-2$

$\therefore x=y=z=1$ $\therefore F = K\eta a v$

By solving $\eta\nabla^2\vec{v} - \vec{\nabla}p = 0$ for a sphere in a liquid with incompressibility constaint $\vec{\nabla}\cdot\vec{v} = 0$, Stokes calculated $K = 6\pi$.

$\therefore F = 6\pi\eta a v$.

Observe that from $v = \frac{2}{9}\frac{ga^2(\rho-\sigma)}{\eta}$, $v \propto a^2$. For raindrop of radius $a = 10^{-3}$ cm falling through air with $\eta = 1.8\times10^{-4}$ prise terminal velocity $v = \frac{2\times981\times10^{-6}}{9\times1.8\times10^{-4}} = 1.2$ cm/sec.

$\rho = 1$, $\sigma\to0$ for air.

That is why raindrops fall with slow rate. $v \neq f(p)$ as $\eta \neq \eta(p)$. Notice also that its independent of mass.

Bigger raindrops fall rapidly through air as Stoke's law does not hold if $a > 0.01$ cm because of turbulence, where $F \propto \sqrt{v}$.

If $\sigma > \rho$ then $v < 0$ meaning if density of fluid > density of body, then body moves through the fluid in the upward direction. This is the reason, why air bubbles in water or in any other liquid rise up.

Equation of motion of a body falling through a viscous medium

Viscous force $\propto$ velocity (instantaneous), but now we have downward force mg due to gravity for sedimentation & no thermal (Brownian) force.

$$m\frac{d^2x}{dt^2} = mg - f = mg - \gamma\frac{dx}{dt}$$

or $\frac{d}{dt}\left(\frac{dx}{dt}\right) = g - \frac{\gamma}{m}\frac{dx}{dt}$ or $\frac{dv}{dt} = g - \frac{\gamma}{m}v$

or $\frac{dv}{g - \gamma/m\, v} = dt$ or $-\frac{m}{\gamma}\frac{d(g - \gamma/m\, v)}{g - \gamma/m\, v} = dt$

or $\int \frac{d(g - \gamma/m\, v)}{g - \gamma/m\, v} = -\frac{\gamma}{m}\int dt + c$

or $\ln(g - \frac{\gamma}{m}v) = -\frac{\gamma}{m}t + c$. Now substitute the boundary condition, $t=0$, $v=0$ $\therefore c = \ln g$.

or $\ln\left(1 - \frac{\gamma v}{mg}\right) = -\gamma t/m$ or $v = \frac{mg}{\gamma}(1 - e^{-\frac{\gamma}{m}t})$.

$\simeq \frac{mg}{\gamma}$ (if $\frac{\gamma t}{m} \gg$)

So the maximum (terminal) velocity is $v = mg/\gamma$

Again, $\frac{dx}{dt} = \frac{mg}{\gamma}(1 - e^{-\gamma/m\, t})$

Integrating, $x = \frac{mg}{\gamma}(t + \frac{m}{\gamma}e^{-\gamma t/m}) + c'$. Again substitute the B.C. at $t=0$, $x=0$.

$$\therefore c' = -\frac{m^2 g}{\gamma^2}$$

$$\therefore x = \frac{mg}{\gamma}(t + \frac{m}{\gamma}e^{-\gamma t/m}) - \frac{m^2 g}{\gamma^2}$$

$$= \frac{mgt}{\gamma} + ge^{-\gamma t/m} - \frac{m^2 g}{\gamma^2}$$

Flow of Gas through a narrow tube

Unlike incompressible liquids (density is independent of pressure), gas is compressible (density $\propto$ pressure). So for a liquid, volume flowing through any cross-section in a given time is constant while for a gas, mass flowing through a crosssection in a given time is constant.

$\therefore \rho \propto P$ & ρV = constant or PV = constant (Boyle's law)

Let us consider an elemental length dx with pressure difference dP within the tube, & is small compared to the tube so that density variation within dx <<<< so that we can still write Poiseuille's equation for flowing liquid

$$V = -\frac{\pi a^4}{8\eta}\frac{dP}{dx}$$. It's $-$ive because $\frac{dP}{dx} < 0$ so that $V > 0$

If P_1 & P_2 are pressure of gas at inlet & outlet end with V_1 volume entering per unit time, then

$$P_1 V_1 = PV = -P\frac{\pi a^4}{8\eta}\frac{dP}{dx}$$

$$\text{or} \int_0^l P_1 V_1\,dx = -\frac{\pi a^4}{8\eta}\int_{P_1}^{P_2} P\,dP = -\frac{\pi a^4}{8\eta}\frac{P_2^2 - P_1^2}{2}$$

$$\text{or} \quad P_1 V_1 l = \frac{\pi a^4 (P_1^2 - P_2^2)}{16\eta} \quad \text{or} \quad P_1 V_1 = \frac{\pi a^4 (P_1^2 - P_2^2)}{16\eta l}$$

$$\text{and} \quad \eta = \frac{\pi a^4 (P_1^2 - P_2^2)}{16 P_1 V_1 l}.$$

Here we have assumed "no-slip" or no relative motion between tube wall & adjacent gas layers, which breaks down at low pressure. The corrected form is

$$\eta = \frac{\pi a^4 (P_1^2 - P_2^2)}{16 P_1 V_1 l}\left(1 + \frac{4\lambda}{a}\right) - \frac{\rho V_1}{8\pi l}\left(K + \ln P_1/P_2\right)$$

λ = slipping coefficient (constant for gas) & K depends on apparatus.

Dependence of Viscosity on pressure & temperature

Liquids : η of liquids increases rapidly with pressure. However for water, glycerol, η decreases with pressure. η decreases rapidly with increase in temperature. For pure liquids $\eta = \frac{A}{(1+BT)^n}$ with A, B, n depending on nature of liquid. For liquid mixture there is no one recipe.

Gases : Using kinetic theory of gases & experiment, established is

(i) At high pressures, η increases with pressure increment.

(ii) At moderate pressures, η is independent of pressure.

(iii) At low pressures, $\eta \propto P$.

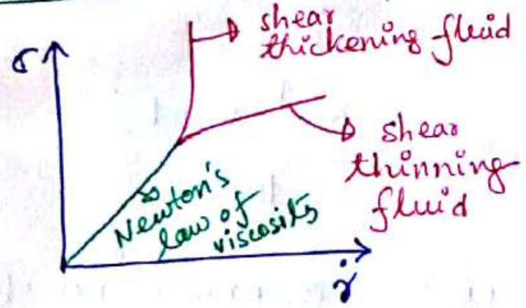
(i) At high temperature, η rapidly increases with temperature & for Mercury $\eta = \alpha T^{1.6}$ (Kinetic theory $\eta = KT^{0.5}$).

(ii) At moderate temperature, $\eta = \eta_0 \frac{\alpha T^{0.5}}{1+\frac{S}{T}}$, η_0 is at $0^\circ C$

α, S = constants

(iii) At low temperature, no one formula agrees well.

Non-Newtonian liquids



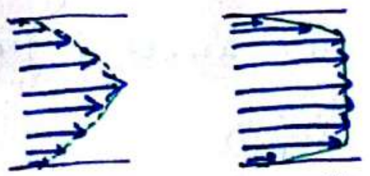
$\sigma \not\propto \dot{\gamma}$ thixotrophic fluid

sol $\leftrightarrow$ gel transition, both property of solid & fluid $\rightarrow$ Glass!!

Examples : shear thinning $\rightarrow$ "colloid"

e.g. paint, milk, blood.

Shear thickening : cornstarch solvent, molten chocolate.



Newtonian non-Newtonian

Displacement of fluid particles are usually non-affine even if Couette flow is established with linear flow profile.