ELASTICITY

Elastic Properties of Matter

When an external force acts on a body, relative displacement of its various parts takes place. By exerting a restoring force, particles tend to come back to their original position.

$$\text{Stress} = \frac{\text{Restoring force}}{\text{cross-sectional area}} = \frac{F}{A}.$$

Strain is defined as the ratio of change of length, volume or shape to the original length, volume or shape.

Young's Modulus: $Y = \dfrac{\text{applied load per unit cross-section}}{\text{increase in length per unit length}}$

$$= \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{l/L} = \frac{FL}{Al}$$

Unit (CGS) : stress = dynes/cm^2, (SI) = Newton/m^2.
strain = no unit (pure number).

$\therefore$ Y = dynes/cm^2 (CGS) or N/m^2 (SI). Dimension of Y is

$$[Y] = \left[\frac{MLT^{-2}L^{-2}}{L \cdot L^{-1}}\right] = [ML^{-1}T^{-2}].$$

Bulk Modulus: (volume elasticity)

$$K = \frac{\text{compressive or tensile force per unit area}}{\text{decrease or increase in volume per unit volume}}$$

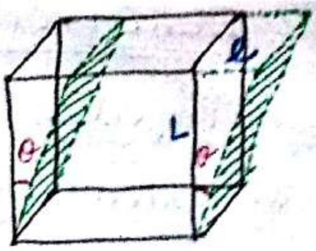
$$= \frac{\text{compressional or dilational pressure}}{\text{volume strain}} = -\frac{dp}{\frac{dV}{V}}.$$

$\therefore$ K = dynes/cm^2 or N/m^2 as, $[K] = \left[\dfrac{MLT^{-2}L^{-2}}{L^3L^{-3}}\right] = [ML^{-1}T^{-2}]$

Negative sign means increase in applied pressure causes decrease in volume.

Rigidity Modulus: $n = \dfrac{\text{tangential stress}}{\text{angle of shear or shearing strain}}$.

Consider a solid cube, whose lower face is fixed and a tangential force F is applied over the upper face, so that its displaced to a new position. As each horizontal layer of the cube is displaced with displacement proportional to its distance from the fixed lower plane,



shearing strain $= \frac{l}{L} = \tan\theta \sim \theta$ if $\lim_{\theta \to 0^\circ}$ (usually $< 4^\circ$)

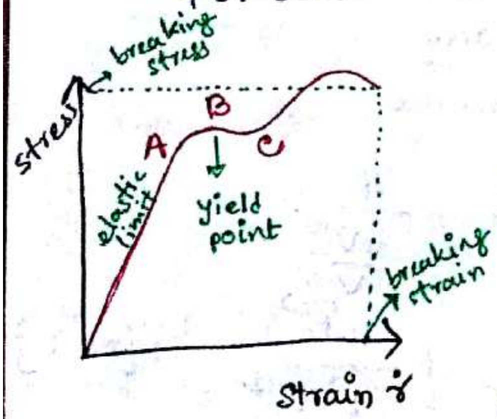
$\therefore\ n = \frac{F/A}{\theta} = \frac{F}{A\theta}$. $[n] = [ML^{-1}T^{-2}]$ & unit is dynes/cm^2 or N/m^2 as θ is a pure number.

Poisson's Ratio: When a wire is stretched, its length increases but its diameter decreases. When an elongation is produced by a longitudinal stress in a certain direction, a contraction results in the lateral dimensions of the body under strain.

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{d/D}{l/L} = \frac{dL}{Dl} \quad \text{where}$$

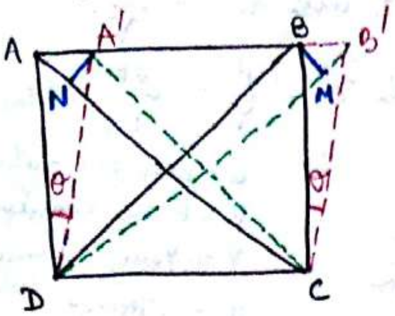
$D =$ diameter of the wire, $d =$ decrease in diameter.
$L =$ length of the wire, $l =$ increase in length.

Poisson's ratio is a dimensionless number.



Stress ∝ strain is called the elastic limit or Hooke's law (point A), so that if stress is removed, an elastic body regains its original shape. After point A, the curve is bent towards a maximum point B (permanent set). After point B, elongation is faster than AB. So after yield point B, elongation increases rapidly with rapid contraction of the area of cross-section of the wire until the breaking stress is reached where snapping occurs. It's called fracture.

Shear $\equiv$ Elongation (Extension) strain $\perp$ Compression strain
θ $\qquad$ $\theta/2$ $\qquad$ $\theta/2$



Suppose in ABCD cube of AB = BC = CD = DA = L the base CD is fixed and after applying a tangential force F the distorted cube is A'B'CD with AA' = BB' = l & $\angle ADA' = \theta$.

Now DB = DM = $\sqrt{2}L$. As θ (angle of shear) is very small, So $\Delta ANA'$ & $\Delta BMB'$ are isocles right angle triangle with

$$\angle A'AN = \angle BB'M = 45^\circ$$

$$\therefore B'M = BB'\cos 45^\circ = \frac{l}{\sqrt{2}}.$$

So Elongation (Eextension) strain along DB diagonal is

$$\frac{B'M}{DB} = \frac{l}{\sqrt{2}} \times \frac{1}{L\sqrt{2}} = \frac{l}{2L} = \frac{\theta}{2} \quad \text{as } \frac{l}{L} = \tan\theta \approx \theta.$$

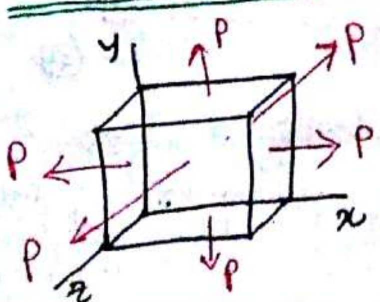
Similarly, compression strain along diagonal AC = $\frac{AN}{AC}$

$$= \frac{AA'\cos 45^\circ}{L\sqrt{2}} = \frac{l}{\sqrt{2}} \times \frac{1}{L\sqrt{2}} = \frac{l}{2L} = \frac{\theta}{2}.$$

So shear θ is equivalent to an extension and a compression strain at right angle to each other with each of value $\theta/2$.

Look for a proof that a shearing stress is equivalent to a linear tensile stress and an equal compression stress mutually at right angles.

Relation between Y, K, n, σ for a homogeneous isotropic medium



Suppose a cube in a strained medium is subjected to uniform tensile stress P over each face. So linear strain along x axis due to tensile stress along x axis is $\frac{P}{Y}$. Linear strain along x axis due to tensile stress along Y-axis is $-\frac{\sigma P}{Y}$. Also, linear strain along X-axis due to tensile stress along Z-axis is $-\frac{\sigma P}{Y}$.

So the resultant linear strain along x axis $\gamma = \frac{P}{Y} - \frac{2\sigma P}{Y}$

Similarly so for Y and Z axis.

σ = poisson's ratio
K = Bulk modulus
Y = Young's modulus
γ = strain

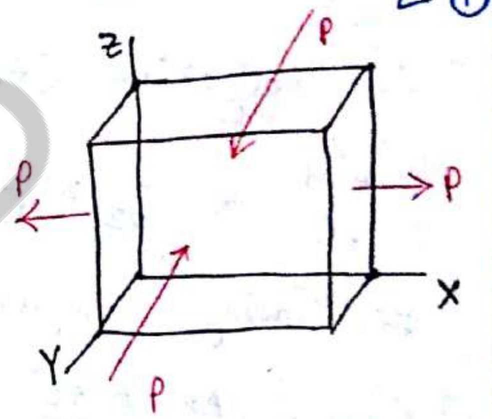
If $\frac{\delta V}{V}$ is the volume strain then $\frac{\delta V}{V} = \frac{P}{K}$

for spherically isotropic system,

$$\frac{\delta V}{V} = \frac{\frac{4}{3}\pi\left[r(1+\gamma)\, r(1+\gamma)\, r(1+\gamma)\right] - \frac{4}{3}\pi r^3}{\frac{4}{3}\pi r^3} = \cancel{1} + 3\gamma + \cancel{O(\gamma^2)+\cdots} \xrightarrow{} 0 - \cancel{1}$$

$= 3\gamma$. (Cubical expansion is 3 times linear expansion)

$\therefore \frac{\delta V}{V} = 3\gamma = \frac{3P}{Y}(1-2\sigma) = \frac{P}{K}$ $\qquad \therefore \boxed{Y = 3K(1-2\sigma)}$ ⟶ ①



Now suppose that the cube is subjected to a tensile stress along x axis & an equal compressional stress along Y axis.

So linear strain along x axis due to tensile stress along x axis is $\frac{P}{Y}$. Also, linear strain along x axis due to compressional stress along Y axis is $\frac{\sigma P}{Y}$. So the resultant linear strain along x axis is

$\gamma_x = \frac{P}{Y} + \frac{\sigma P}{Y}$. Resultant linear strain along Y axis $\gamma_y = -\frac{P}{Y} - \frac{\sigma P}{Y}$

and resultant linear strain along z axis $\gamma_z = -\cancel{\frac{\sigma P}{Y}} + \cancel{\frac{\sigma P}{Y}} = 0$

We Know $\theta = \frac{P}{n}$ and $\frac{\theta}{2} = \frac{P}{Y}(1+\sigma)$

θ = angle of shear
n = modulus of rigidity

as $\theta = \frac{\gamma_x}{2}$. So $\frac{P}{2n} = \frac{P}{Y}(1+\sigma)$ or $\boxed{Y = 2n(1+\sigma)}$ ⟶ ②

From ① and ②, we have $\frac{Y}{K} = 3 - 6\sigma$ and $\frac{Y}{n} = 2 + 2\sigma$

$\therefore Y\left(\frac{1}{K} + \frac{1}{n}\right) = 5 - 4\sigma = 5 - 4\left(\frac{Y-2n}{2n}\right)$ or $\boxed{Y = \frac{9nK}{n+3K}}$

Again from ① and ②, $\frac{Y}{3K} = 1 - 2\sigma$ and $\frac{Y}{2n} = 1 + \sigma$

$$\therefore \sigma = Y \frac{3K - 2n}{18Kn} = \frac{9Kn}{n + 3K} \frac{3K - 2n}{18Kn} = \frac{3K - 2n}{6K + 2n}$$

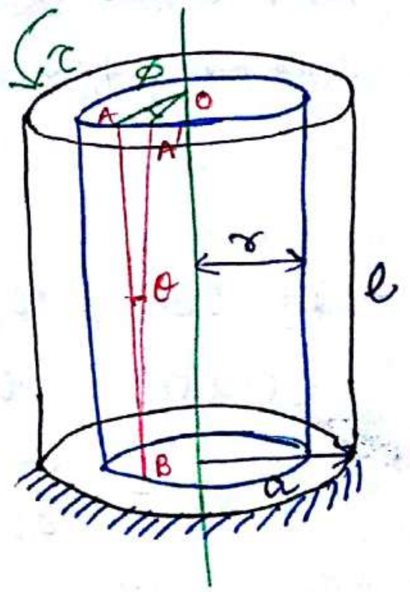
or $\boxed{\sigma = \frac{3K - 2n}{6K + 2n}}$ or $\underline{3K(1 - 2\sigma) = 2n(1 + \sigma), \quad n, K > 0.}$

When $\sigma > \frac{1}{2}$, $3K(1-2\sigma) < 0$ but $2n(1+\sigma) > 0$.

when $\sigma < -1$, $3K(1-2\sigma) > 0$ but $2n(1+\sigma) < 0$

$\left.\right\}$ $3K(1-2\sigma) \neq 2n(1+\sigma)$ violation

$$\therefore \boxed{-1 < \sigma < \frac{1}{2}}$$

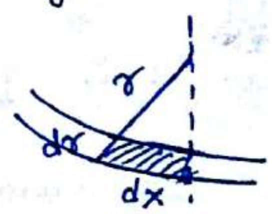


The Torsion cylinder and shear Waves

Consider a cylinder of length l and radius a with clamped (fixed) lower end and a torque is applied at upper end, because of that cylinder is twisted through an angle. If an elemental point A is displaced to A′, then $l\theta = AA' = r\phi$

where ϕ = angle of twist and r is radius of elemental cylinder.

$\therefore r\phi = l\theta$ or $\theta = r\phi/l$. Due to elasticity, there will be a restoring torque. To calculate this, consider the shell of thickness dr, length dx at r distance apart. So tangential stress $= \frac{F}{dr dx}$ and



$$\text{rigidity modulus } n = \frac{\text{tangential stress}}{\text{angle of shear}} = \frac{F/dr dx}{\theta} = \frac{F/dr dx}{r\phi/l}$$

$$= \frac{Fl}{r\phi\, dr dx}. \qquad \therefore F = \frac{n\phi}{l} r dr dx.$$

Moment of this force about cylinder axis $= F.r = \frac{n\phi}{l} r^2 dr dx$

So restoring torque over entire surface $\delta\tau = \frac{n\phi}{l} r^2 dr \sum dx$

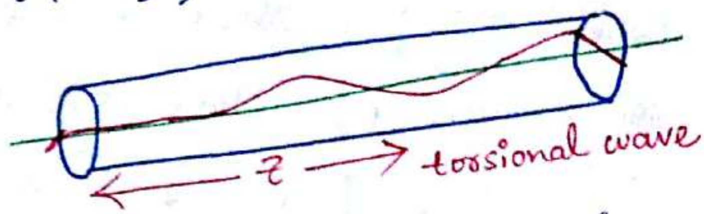
$[\Sigma dx = 2\pi r]$ $= \frac{2\pi n\phi}{l} r^3 dr$

∴ Total restoring torque for the entire cylindrical bar

$$\tau = \frac{2\pi n\phi}{l}\int_0^a r^3 dr = \frac{\pi n a^4}{2l}\phi \quad , \quad \frac{\tau}{\phi} = \text{torsional rigidity}$$

For hollow cylinder with inner & outer radii r_1 & r_2

$$\tau = \frac{2\pi n\phi}{l}\int_{r_2}^{r_1} r^3 dr = \frac{n\pi\phi}{2l}(r_1^4 - r_2^4)$$



For a static torsion, torque is same everywhere & proportional to ϕ/l. If its nonuniform then

$$\tau(z) = \frac{n\pi a^4}{2}\left(\frac{\partial\phi}{\partial z}\right)$$

↳ local torsional strain

If $\tau(z)$ & $\tau(z+dz)$ are the torque at two ends then

$$\tau(z+dz) = \tau(z) + \frac{\partial\tau}{\partial z}\Delta z \quad \text{and so,} \quad \Delta\tau = \tau(z+dz) - \tau(z)$$

$$\text{or } \Delta\tau = \frac{\partial\tau}{\partial z}\Delta z = \frac{n\pi a^4}{2}\frac{\partial^2\phi}{\partial z^2}\Delta z \quad \text{— (1)}$$

The effect of this incremental torque is to yield angular acceleration of the slice of mass $(\pi a^2 \Delta z)\rho$ or moment of inertia $\frac{1}{2}ma^2 = \frac{\pi}{2}\rho a^4 \Delta z$, so that the restoring couple is

$$I\alpha = I\frac{d^2\phi}{dt^2} = \frac{\pi\rho a^4}{2}\Delta z\frac{\partial^2\phi}{\partial t^2} \quad \text{— (2)}$$

So, from (1) & (2)

$$\frac{n\pi a^4}{2}\frac{\partial^2\phi}{\partial z^2}\Delta z = \frac{\pi\rho a^4}{2}\frac{\partial^2\phi}{\partial t^2}\Delta z$$

$$\text{or } \boxed{\frac{\partial^2\phi}{\partial t^2} = \frac{n}{\rho}\frac{\partial^2\phi}{\partial z^2}}$$

Wave equation in 1 dimension with $\boxed{v_{shear} = \sqrt{\frac{n}{\rho}}}$ = shear speed

So denser the rod for same stiffness the slower the waves & independent of radius of rod. Torsional waves are special example of shear waves, which are those in which the strains do not change the material volume, but in torsional waves, shear stresses are distributed on a circle and move with same speed.

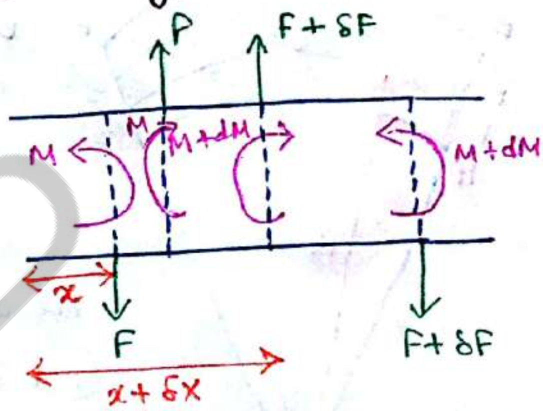
Inside a solid material there are compressional / longitudinal waves as well as on the surface Rayleigh or Love waves. In them, strains are neither purely longitudinal or transverse. Longitudinal waves travel faster than shear waves

$$\boxed{v_{long} = \sqrt{\frac{1-\sigma}{(1+\sigma)(1-2\sigma)}\frac{Y}{\rho}}}$$

and $Y > n$ as $Y = 2n(1+\sigma)$, so $\boxed{v_{long} > v_{shear}}$

Y & σ can be measured by measuring ρ and v_{long}, v_{shear}, e.g. in earthquake, distance between quakes can be measured like this.

Bending of Beam : <u>General method for determining deflection due to bending:</u>

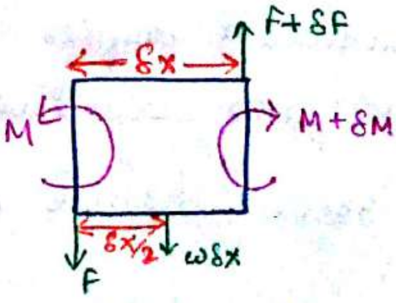


Let ω be the weight per unit length of the beam. Let us consider an element δx, the left edge of which is at a distance x from the origin and the right edge at $x+\delta x$ from the origin.

During downward displacement of the element, shearing stress at left face is F & at right face is $F+\delta F$. The corresponding internal resisting moment at the left face due to the left hand portion of the beam is M in the counterclockwise direction and at the right hand portion of the beam is $M+\delta M$ in the clockwise direction.

Considering equilibrium of the δx element due to force balance

$$F+\delta F = F + \omega\,\delta x \quad \text{or} \quad \boxed{\frac{\delta F}{\delta x} = \omega}$$



In the limit $\delta x \to 0$, $\frac{dF}{dx} = \omega$.

Considering the moment equation,

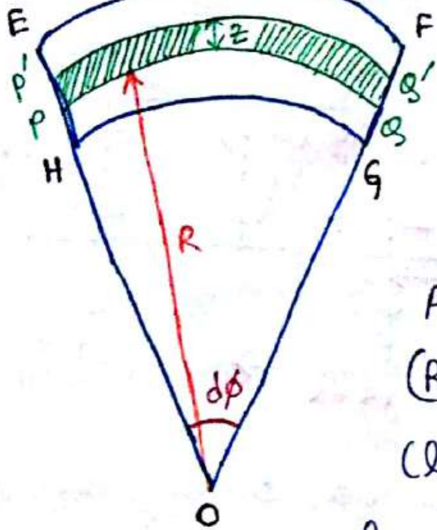
$$\cancel{M} + \delta M + \omega\cancel{\delta x}\cdot\frac{\delta x}{2} = \cancel{M} + (F+\cancel{\delta F})\delta x$$

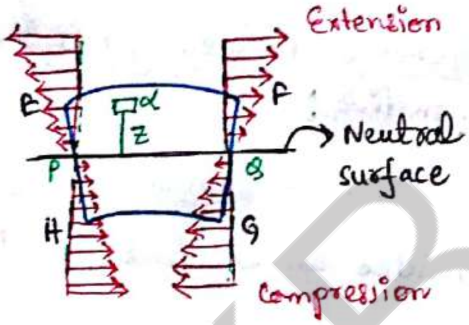
$$\lim_{\delta x\to 0},\ \delta M = F\,\delta x \quad \text{or} \quad \boxed{\frac{dM}{dx} = F}$$

We will use this two equations to determine the depression of a loaded beam.

Internal Bending Moment

A body whose length is much greater than its cross-sectional area is called a beam. When such a beam is bent by an applied torque, tensile forces act on some layers of the beam and compressional forces act on other layer, as a result of which, filaments of the beam nearest the outside curve of the bent beam are extended and the filaments nearest the inside curve get compressed. In between them, there is a surface (called Neutral surface) on which the filaments remain unaltered.





Let us consider a small portion of bent beam EFGH with length PQ and breadth EH. O is the center of curvature.

At a distance z from PQ, length of filament is $(R+z)d\phi$ & hence increase in length of filament (linear extension) $= (R+z)d\phi - Rd\phi = z\,d\phi$.

The linear extensional strain $= \dfrac{z\,d\phi}{R\,d\phi} = \dfrac{z}{R}$. If α is the cross-section of filament then longitudinal force f to resist elongation $f = \dfrac{\alpha Y z}{R}$, below neutral surface this is the force to resist compression. If two filaments are equal distance from the neutral axis then they form a couple. For infinitely many equidistant couples comes into play the internal bending moment with same magnitude as external bending moment.

Total internal moment $= \sum\limits_{\substack{\text{all} \\ \text{filaments}}} fz = \dfrac{Y}{R} \sum\limits_{\substack{\text{all} \\ \text{filaments}}} \alpha z^2$. $\sum \alpha z^2$ is called the geometrical moment of inertia, having same (equivalence) as moment of inertia with $m \Leftrightarrow \alpha$. So $\sum \alpha z^2 = Ak^2$ where

A is the total area EFGH and K is the radius of gyration of the surface about the neutral axis. $YI = YAK^2$ is called the Flexural rigidity. Radius of curvature $\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$ and if bending is small, $\frac{dy}{dx} < 1$ & so

$\frac{1}{R} = \frac{d^2y}{dx^2}$. So internal bending moment $= YI\frac{d^2y}{dx^2}$

for rectangular beam, $A = ab$ and $K^2 = \frac{b^2}{12}$, a = breadth, b = thickness.

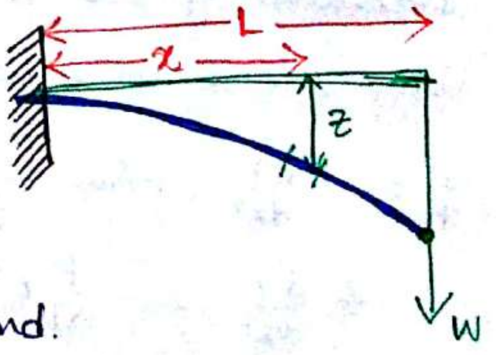
$\therefore$ Bending moment $= \frac{Yab^3}{12R}$.

for circular beam, $A = \pi r^2$, $K^2 = \frac{r^2}{4}$, Bending moment $= \frac{Y\pi r^4}{4R}$.

$$\underbrace{M}_{\text{bending moment}} = \underbrace{\frac{Y}{R}}_{\text{curvature}} \underbrace{\int z^2 dA}_{\text{moment of inertia}}$$

. So as the stiffness of the beam is proportional to Y and moment of inertia I, to make the stiffest possible beam with a given amount of steel, the mass has to be as distant from neutral surface gives larger I, but also curvature won't be much due to buckling/twisting. So structural beams are made in the form of I and H.

Cantilever



A cantilever is a uniform beam supported in such a way that both position and slope are fixed in one end (cement wall) and a concentrated force W acts on free end.

What is the shape of the beam z(x)?

If beam is long in comparison to cross section, $\frac{1}{R} = \frac{d^2z/dx^2}{\sqrt{\left[1+\left(\frac{dz}{dx}\right)^2\right]^3}} \simeq \frac{d^2z}{dx^2}$

Bending moment M is equal to the torque about the neutral axis of any cross section then $M(x) = W(L-x)$

So from moment equation, $W(L-x) = \frac{YI}{R} = YI\frac{d^2z}{dx^2}$

$\therefore \dfrac{d^2z}{dx^2} = \dfrac{W}{YI}(L-x)$, Integrating, $\cancel{\dfrac{dz}{}}$

$YI\dfrac{dz}{dx} = W\left(Lx - \dfrac{x^2}{2}\right) + C_1$. Now at $x=0$, $\dfrac{dz}{dx}=0$, so, $C_1 = 0$.

Integrating once again, $YIz = W\left(\dfrac{Lx^2}{2} - \dfrac{x^3}{6}\right) + C_2$

Again at $x=0$, $z=0$. $\therefore C_2 = 0$. $\therefore \boxed{z = \dfrac{W}{YI}\left(\dfrac{Lx^2}{2} - \dfrac{x^3}{6}\right)}$

Shape of the beam

Displacement of the end is $x = L$.

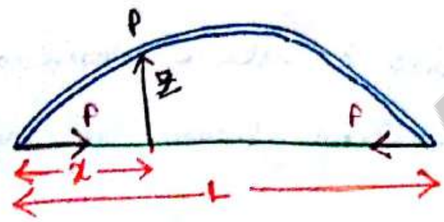
$z = \dfrac{W}{YI}\dfrac{L^3}{3}$. Substituting I for rectangular or circular beam, exact expression can be obtained. But $z \propto L^3$.

But crosssection do charge and for incompressible material, $\sigma = 0.5$

$\therefore V = \pi r^2 l$ = constant $\therefore dV = 0 = 2r\,dr\,L + r^2 dL$

$\therefore dL/L = -2\,dr/r$ or $\sigma = -\dfrac{dr/r}{dL/L} = \dfrac{1}{2} = 0.5$. (rubber)

Buckling



Consider a straight rod in bent shape by two opposite forces that push the two ends of the rod. What is the shape of the rod and magnitude of force? → "Euler force"

Deflection of rod is $z(x)$. So bending moment $\mathcal{M}$ at $P = Fz$.

Using the beam equation, $\dfrac{YI}{R} = Fz$ and for small deflection

$\dfrac{1}{R} = -\dfrac{d^2z}{dx^2}$ (minus sign because curvature is downward).

$\therefore \dfrac{d^2z}{dx^2} = -\dfrac{F}{YI}z$ — ① which is SHM equation of sine wave. So for small deflection, wavelength λ of sine wave $= 2 \times L$.

$z = A\sin\dfrac{\pi x}{L}$, $\dfrac{d^2z}{dx^2} = -\dfrac{A\pi^2}{L^2}\sin\dfrac{\pi x}{L} = -\dfrac{\pi^2}{L^2}z$. — ②

① & ② $\therefore \dfrac{F}{YI} = \dfrac{\pi^2}{L^2}$ $\therefore$ $\boxed{F = YI\dfrac{\pi^2}{L^2}}$. So for small bendings the force is independent of the bending displacement z. Below this Euler force, there will be no bending at all & for above this force there will be large amount of bending + buckling. If loading on 2nd floor of a building exceeds Euler force, the building will collapse.