Energy of a particle in SHM

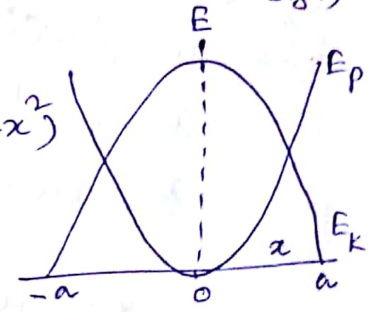
Work is done on particle to displace → restoring force. So P.E. in spring stored & motion is K.E. Total energy constant

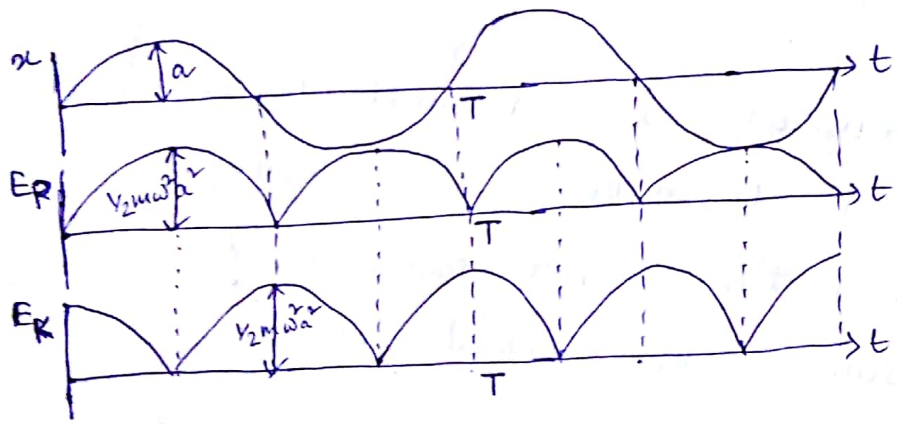
P.E. $F = mf = -m\omega^2 x$ $\therefore$ $dW = F\,dx = m\omega^2 x\,dx$ (against so no −ive sign)

$\therefore E_p = \int_0^x m\omega^2 x\,dx = \frac{1}{2} m\omega^2 x^2.$

K.E. $v = \omega\sqrt{a^2-x^2}$, $E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2-x^2)$

$E_{Tot} = E_K + E_p = \frac{1}{2}m\omega^2 a^2 = \text{constant}.$





Examples of SHM

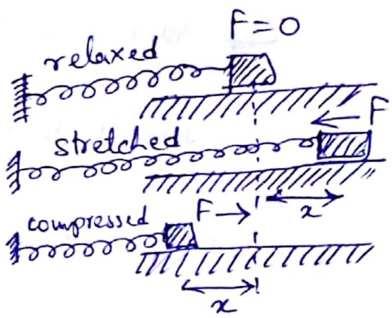
Horizontal oscillations

$F = -kx = m\ddot{x}$

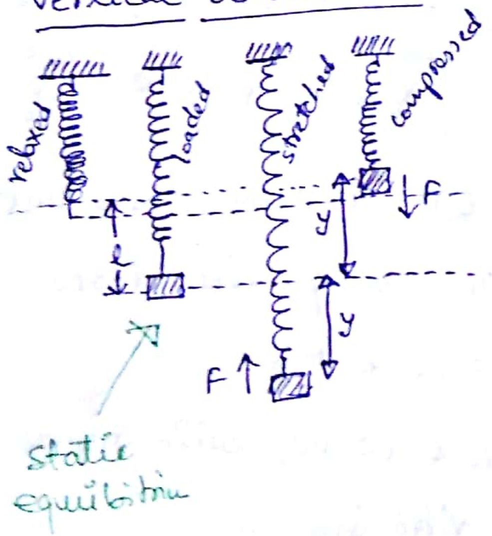
$\ddot{x} + \omega^2 x = 0$ $\omega = \sqrt{\frac{K}{m}}$

$x = A\cos(\omega t + \phi)$, $T = 2\pi\sqrt{\frac{m}{K}}$

initial condⁿ. material.



Vertical oscillations



Tension on spring $F_0 = Kl$

Force on mass $= mg$.

Static eq. $mg = Kl$.

stretched Tension on spring $= K(l+y)$

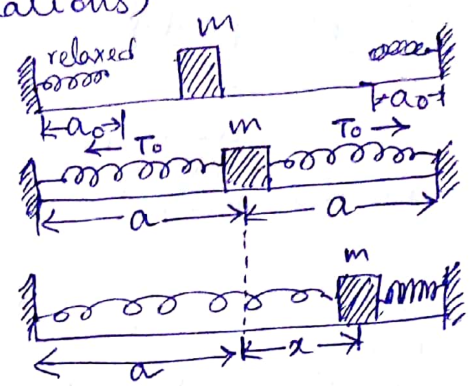
$mg - F = K(l+y) = Kl + Ky$
$= \cancel{mg} + Ky$

$F = -Ky.$

compressed $mg + F = K(l-y) = \cancel{mg} - Ky$

$F = -Ky.$

Two spring system (Longitudinal oscillations)



horizontal frictionless surface, rigid wall, massless spring, relaxed length a_0.

After connection, static equilibrium

$$T_0 = K(a - a_0)$$

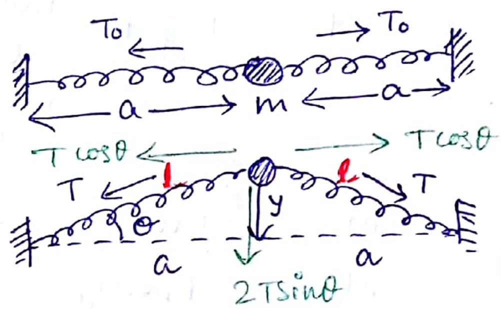
x = displacement to right. restoring force by left spring $-K(a+x-a_0)$

force on right spring $K(a-x-a_0)$

$$\therefore \quad F_x = K(a-x-a_0) - K(a+x-a_0) = -2Kx$$

$$m\ddot{x} = -2Kx \quad \text{or} \quad \ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{long} = 2\pi\sqrt{\frac{m}{2K}}}$$

Two spring system (Transverse oscillations)



$$T_0 = K(a - a_0)$$

$$T = K(l - a_0)$$

$$F_y = -2T\sin\theta = -2T\frac{y}{l}$$

$l = \sqrt{y^2 + a^2}$

$$\text{or} \quad m\ddot{y} + \frac{2T}{l}y = 0 \quad \text{or} \quad \ddot{y} + \omega^2 y = 0$$

$$\omega^2 = \frac{2T}{ml} = \frac{2K(l - a_0)}{ml}, \quad \text{but} \quad l = f(y).$$

So $\ddot{y} + \frac{2K}{m}\left(1 - \frac{a_0}{f(y)}\right) y = 0$ is not a SHM.

ⓐ Slinky approximation $a \gg a_0$ or $\frac{a_0}{a} \ll 1$.

$$\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{l}\right) = \frac{2K}{m}\left(1 - \frac{a_0}{a}\frac{a}{l}\right) \quad \text{as } l > a.$$

$$= \frac{2K}{m}. \quad \text{Then SHM.} \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K}}}$$

"large" harmonic oscillations

Spring

ⓑ small oscillation approximation $a \not\gg a_0$ but $y \ll a$ or l.

$$\therefore \quad l = \sqrt{y^2 + a^2} = a\sqrt{\frac{y^2}{a^2} + 1} \sim a$$

Then also $\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{a}\right)$ or $\boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K\left(1 - \frac{a_0}{a}\right)}}}$

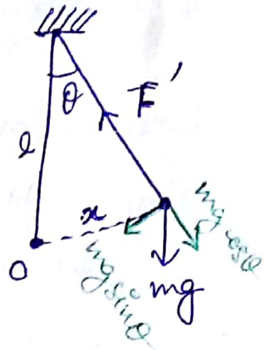
SHM.

Rubber

$$\therefore \quad T_{long} = \sqrt{1 - \frac{a_0}{a}}\; T_{trans}.$$

So longitudinal is faster than transverse.

Simple pendulum



$F' = mg\cos\theta$
(tension in string)

$[\lim_{\theta\to 0}]$

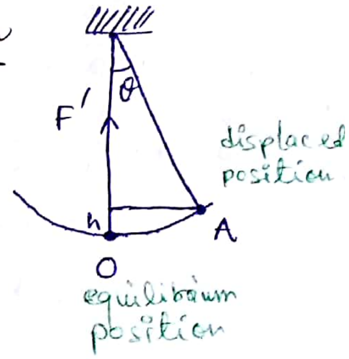
F (restoring force) $= -mg\sin\theta$
$= -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \simeq -mg\theta$

$x = l\theta$

or, $m\ddot{x} = -mg\frac{x}{l}$ or $\ddot{x} + \frac{g}{l}x = 0.$

$\therefore \omega = \sqrt{g/l}, \quad T = 2\pi\sqrt{\frac{l}{g}}.$ (mass independent)

String tension when pendulum at mean position



$F' = mg + \frac{mv^2}{l}$
(centrifugal force)

at A, Energy $= KE + PE = 0 + mgh = mgh$
at O, Energy $= KE + PE = \frac{1}{2}mv^2 + 0 = \frac{1}{2}mv^2$

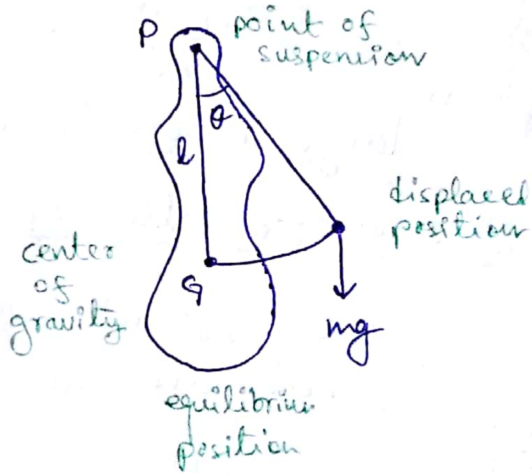
Conservation of energy $\Rightarrow \frac{1}{2}mv^2 = mgh$ or $v^2 = 2gh.$

or $v^2 = 2g(l - l\cos\theta) = 2gl(1-\cos\theta) = 2gl \times 2\sin^2\frac{\theta}{2}$

$\simeq 4gl\left(\frac{\theta}{2}\right)^2 = gl\theta^2.$

$\therefore F' = mg + \frac{m}{\cancel{l}}g\cancel{l}\theta^2 = mg(1+\theta^2).$

Compound Pendulum



arbitrary shaped rigid body oscillating about a horizontal axis passing through it.

restoring force $\leftrightarrow$ reactive couple or torque

moment of restoring force
$= -mgl\sin\theta$

angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$, moment of inertia $= I$.

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

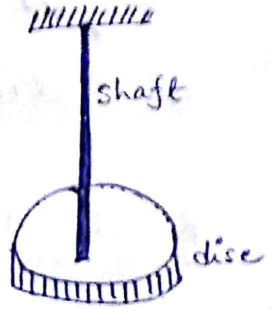
or $$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\sin\theta \simeq -\frac{mgl}{I}\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

If we consider moment of inertia about a parallel axis through G,
K = radius of gyration then using parallel axis theorem,

$$I = mK^2 + ml^2 \quad \therefore T = 2\pi\sqrt{\frac{K^2/l + l}{g}} = 2\pi\sqrt{\frac{L'}{g}}.$$

equivalent length of simple pendulum $= \frac{K^2}{l} + l$.



Torsional Pendulum

twist of shaft → torsional oscillations

torsional couple $= -\tau\theta$

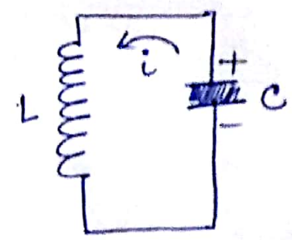
couple due to accelaration $= I\frac{d^2\theta}{dt^2}$

$$I\frac{d^2\theta}{dt^2} = -\tau\theta \quad , \quad T = 2\pi\sqrt{\frac{I}{\tau}}$$

From classical mechanics course, $\tau = \frac{\pi\eta d^4}{32 l} = \frac{\pi\eta r^4}{2l}$

d = shaft diameter, η = modulus of rigidity,
$= 2r$.

Electrical oscillator



Capacitor is charged ⇒ electrostatic energy in dielectric media. It discharges through the inductor electrostatic energy ⇔ magnetic energy. (no dissipation of heat)

voltage across inductor $= -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$ q = charge

voltage across capacitor $= \frac{q}{C}$.

No e.m.f. circuit, $\frac{q}{C} = -L\frac{d^2q}{dt^2}$ or $\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$

$\omega^2 = \frac{1}{LC}$, $q = q_0\sin(\omega t + \phi)$. charge on capacitor varies harmonically.

$$i = \frac{dq}{dt} = \omega q_0 \cos(\omega t + \phi)$$

$$V = \frac{q}{C} = \frac{q_0}{C} \sin(\omega t + \phi)$$

Total energy = magnetic energy + electric energy

$$= \int iV\,dt + \frac{1}{2}CV^2 = \int i L \frac{di}{dt}\,dt + \frac{1}{2}CV^2$$

$$= \int L i\,di + \frac{1}{2}CV^2 = \frac{1}{2}Li^2 + \frac{1}{2}CV^2 = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}CV^2$$

In mechanical oscillation, Total energy $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

equivalence

$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

In electrical oscillation, Total energy $= \frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2$