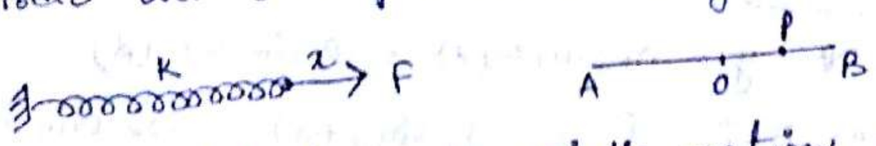
SHM

Motion : Translation, rotation, vibration/oscillation

periodic motion $f(t) = f(t+T)$ e.g. $\sin \frac{2\pi t}{T}$, $\cos \frac{2\pi t}{T}$

if periodic over same path → oscillatory motion

elasticity & inertia



SHM is a special type of periodic motion where restoring force on particle is proportional to displacement & directed to the mean position.

oscillation between point A & B, mean position O. at time t, particle is at P & displacement is x. F – restoring force

$F \propto -x$ or $F = -kx$ or $ma = -kx$ "small oscillation approximation"

$\therefore \quad a = -\frac{k}{m}x = -\omega^2 x$

characteristics (1) linear motion → to-n-fro in straight line.
(2) $F \propto -x$

linear harmonic motion ↔ angular harmonic motion.
(pendulum) (torsional pendulum)
$F \propto -x$ $\tau \propto -\theta$

complete oscillation : one point to same point. (time period)

amplitude : maximum displacement on both sides

frequency : no. of oscillations in 1 second.

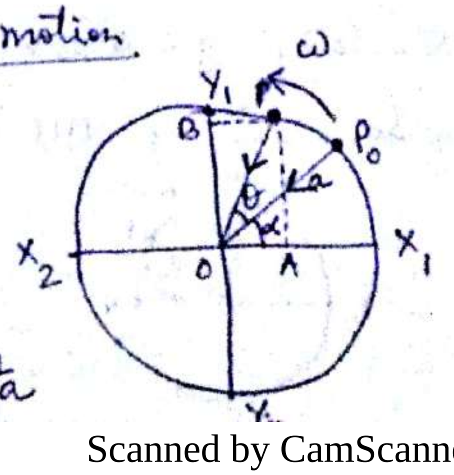
phase : displacement, velocity, accelaration & direction of motion. After 1 oscillation, phase is same.

$t=0$, initial phase.

Relation between SHM & uniform circular motion.

$\theta = \omega t$
$s = a\theta$

$OA = x$, $OB = y$.
$= OP\cos(\theta+\alpha) = a\cos(\theta+\alpha)$
$= a\cos(\omega t+\alpha)$



speed $v = \omega a$, centripetal acc $f_r = \frac{v^2}{a} = \omega^2 a$

Accelaration of A is component of f_r along X_1OX_2

$$f_A = -f_r \cos(\omega t+\alpha) = -\omega^2 a\cos(\omega t+\alpha) = -\omega^2 x$$

$\therefore$ $\underline{f_A \propto -x.}$

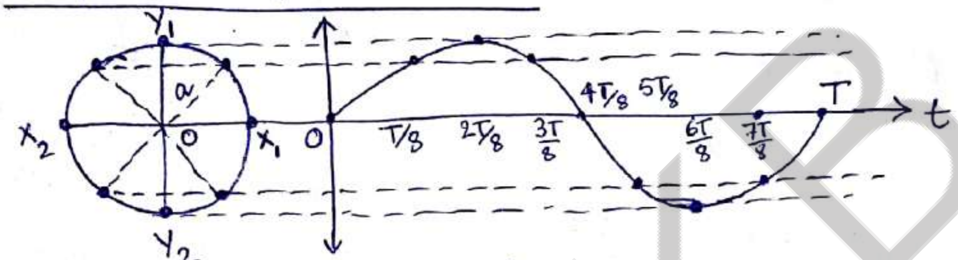
Similarly, $OB = y = OP\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

Accelaration of B is $f_B = -f_r\sin(\theta+\alpha) = -\omega^2 a\sin(\omega t+\alpha) = -\omega^2 y$

$\therefore$ $\underline{f_B \propto -y.}$

$\therefore$ SHM is defined as the projection of uniform circular motion along diameter of circle.

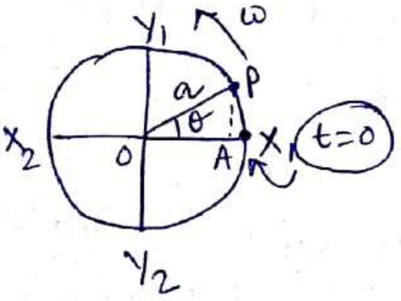
__Graphical representation__

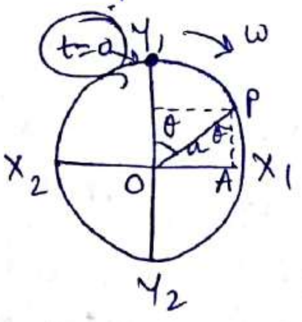


Time period = T.

$$y = a\sin\frac{2\pi}{T}t$$
(SHM along y-axis)

__Displacement__ In SHM, displacement at time t is the distance of the particle from the mean position.





$OA = OP\cos\theta$

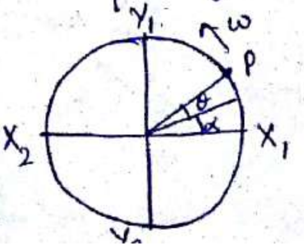
$x = a\cos\omega t$

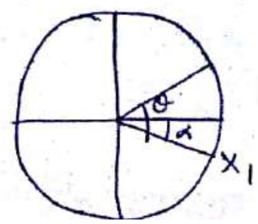
$OA = OP\cos(\pi/2 - \theta)$

$x = a\sin\theta = a\sin\omega t$

Similarly, $y = a\cos\omega t$ & $y = a\sin\omega t$.

So, eq^n of SHM can be derived from any instant t.



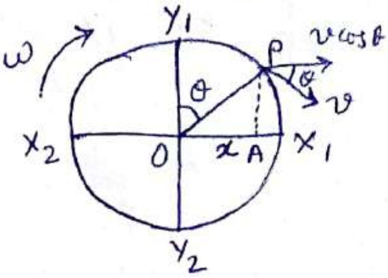


$x = a\cos(\theta+\alpha) = a\cos(\omega t+\alpha)$

Similarly, $x = a\sin(\theta+\alpha) = a\sin(\omega t+\alpha)$.

if initial position is X_1 (2^{nd} pic) then $x = a\cos(\omega t-\alpha)$

or $x = a\sin(\omega t-\alpha)$



<u>Velocity & accelaration</u>

velocity of SHM is component of the particle's velocity along x-axis at time t.

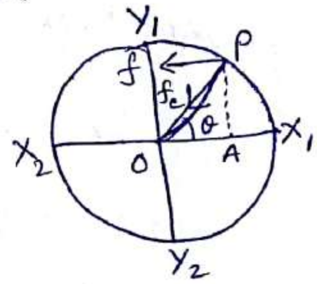
$V = a\omega$, V parallel to OA $= V\cos\theta$

$= a\omega\cos\theta = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$x = a\sin\theta$

$\therefore \boxed{v = \omega\sqrt{a^2-x^2}}$

v_{max} is at $x=0$, $v_{max} = a\omega$. & $x=a$, $v_{min}=0$.



Same with accelaration ⟹ SHM is the projection along X-axis is component of accelaration along X-axis. $f_c = -\omega^2 a$ & component around X_1X_2 is $-\omega^2 a\cos\theta = -\omega^2 a\cos\omega t = -\omega^2 x$.

$\therefore f = -\omega^2 x$.

$f_{max} = -\omega^2 a$ when $x = \pm a$, $f_{max} = \pm\omega^2 a$.

$f_{min} = 0$ when $x = 0$.

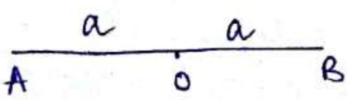
<u>Calculus:</u> $x = a\sin\omega t$, $V = \dot{x} = a\omega\cos\omega t = a\omega\sqrt{1-\frac{x^2}{a^2}}$

$= \omega\sqrt{a^2-x^2}$.

$f = \ddot{x} = -a\omega^2\sin\omega t = -\omega^2 x$.

$\omega^2 = f/x$ (neglect)

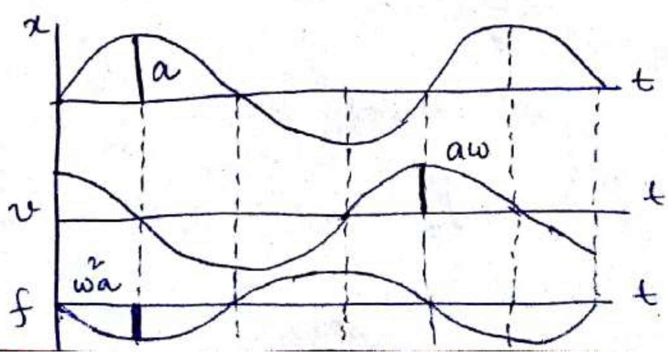
Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{f}}$



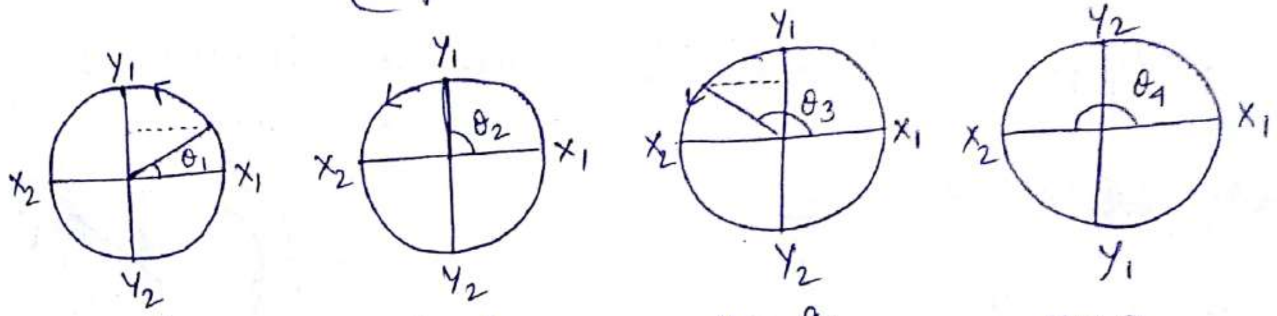
$x = a\sin\omega t = a\sin\frac{2\pi}{T}t$

$v = a\omega\cos\omega t = a\omega\cos\frac{2\pi}{T}t$

$f = -a\omega^2\sin\omega t = -a\omega^2\sin\frac{2\pi}{T}t$



Phase you see, a & ω (angular velocity) are constant.
(amplitude) $\theta = \omega t$ is changing = phase.



$y = a/2$, $\theta_1 = 30^\circ$, v upwards

$y_1 = a$, $\theta_2 = 90^\circ$, $V = 0$

$y = a/2$, $\theta_3 = 150^\circ$, V = downwards

$y = 0$, $\theta_4 = 180^\circ$, v = downwards.

phase difference 2 particles. $\phi = \theta_1 - \theta_2 = 0$ (in phase)
$= 180^\circ$ (out of phase)

Differential form & solution Homogeneous, 2nd order, ODE with constant coefficients

$F = -kx$ or $m\ddot{x} = -kx$ or $\ddot{x} + \omega^2 x = 0$, $\omega = \sqrt{\frac{k}{m}}$.

Solution: Multiply by $2\dot{x}$, $2\dot{x}\ddot{x} + 2\omega^2 x\dot{x} = 0$

Integrating $\dot{x}^2 = -\omega^2 x^2 + c$

when displacement is maximum, $x = a$, $\dot{x} = 0 \Rightarrow c = \omega^2 a^2$

$\therefore \quad v = \dot{x} = \pm\omega\sqrt{a^2 - x^2}$

or $\pm \dfrac{dx}{\sqrt{a^2 - x^2}} = \omega dt$, Integrating $\sin^{-1}\dfrac{x}{a} = \omega t + \phi$

or $x = a\sin(\omega t + \phi)$

See, $x = a\cos(\omega t + \phi)$ also satisfy $\ddot{x} + \omega^2 x = 0$.

$x = a\sin(\omega t + \phi) = a\sin\omega t\cos\phi + a\cos\omega t\sin\phi$
$= A\sin\omega t + B\cos\omega t$.

In operator form, $\dfrac{d^2x}{dt^2} = D^2 x$, $\dfrac{dx}{dt} = Dx$

$D^2 x + \omega^2 x = 0$ or $D^2 = -\omega^2$ or $D = \pm i\omega$

$\therefore$ General solution $x = Ae^{i\omega t} + Be^{-i\omega t}$

For real value of x, $A = B^*$ $\quad A = a+ib, \quad B = a-ib$

you can also have $x = ae^{i(\omega t+\phi)}$

sinusoidal or cosinusoidal.

CW 1. Oscillatory motion of a particle is represented by $x = ae^{i\omega t}$. Establish the motion is SHM. Similarly if $x = a\cos\omega t + b\sin\omega t$ then SHM.

$$x = ae^{i\omega t}, \quad \dot{x} = ai\omega e^{i\omega t}, \quad \ddot{x} = -a\omega^2 e^{i\omega t} = -\omega^2 x \quad (SHM)$$

$$x = a\cos\omega t + b\sin\omega t, \quad \dot{x} = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$\ddot{x} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t = -\omega^2 x \quad (SHM).$$

2. Which periodic motion is not oscillatory?

→ earth around sun or moon around earth.

3. Dimension of force constant of vibrating spring.

$$F = -Kx \qquad [K] = \frac{[Force]}{[displacement]} = \frac{[Newton]}{[Metre]}$$

also called "stiffness".

$$= \frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$$

HW 1. In SHM, displacement is $x = a\sin(\omega t+\phi)$. at $t=0$, $x = x_0$ with velocity v_0, show that $a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$ & $\tan\phi = \frac{\omega x_0}{v_0}$.

2. Particle is vibrated at frequency 5 Hz in SHM. Show that when displacement exceeds 10^{-2} metre, the particle loses contact with the vibrator. Given $g = 9.8\ m/s^2$

3. In SHM, a particle has speed 80 cm/s & 60 cm/s with displacement 3 cm & 4 cm. Calculate amplitude of vibration

Energy of a particle in SHM

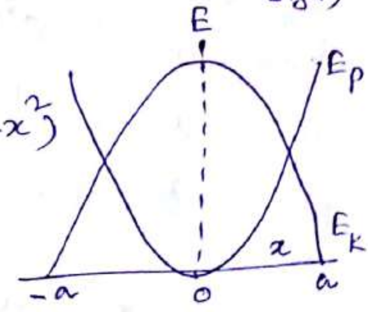
Work is done on particle to displace → restoring force. So P.E. in spring stored & motion is K.E. Total energy constant

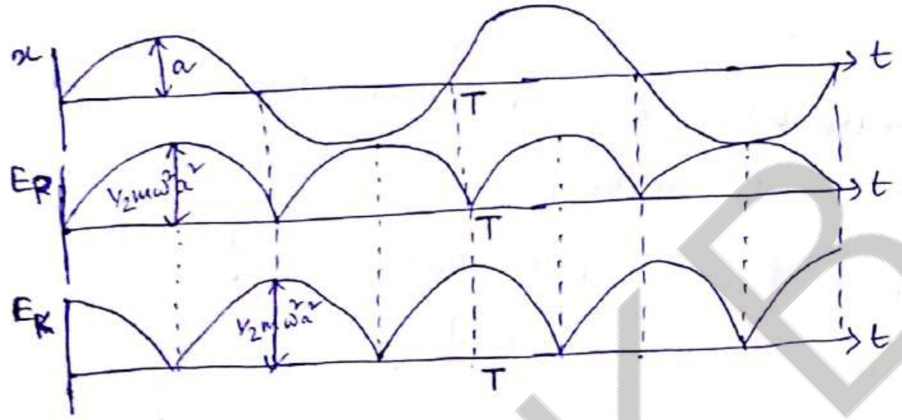
P.E. $F = mf = -m\omega^2 x \quad \therefore \; dW = F\,dx = m\omega^2 x\,dx$ (against so no −ive sign)

$$\therefore \; E_p = \int_0^x m\omega^2 x\,dx = \frac{1}{2} m\omega^2 x^2.$$

K.E. $v = \omega\sqrt{a^2 - x^2}, \quad E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(a^2 - x^2)$

$$E_{Tot} = E_K + E_p = \frac{1}{2}m\omega^2 a^2 = \text{constant}.$$





Examples of SHM

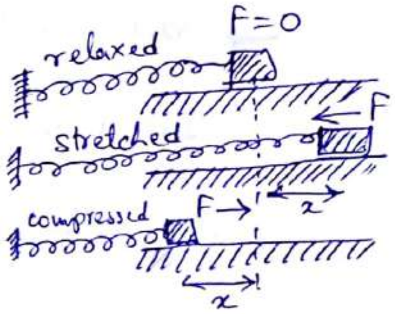
Horizontal oscillations

$F = -kx = m\ddot{x}$

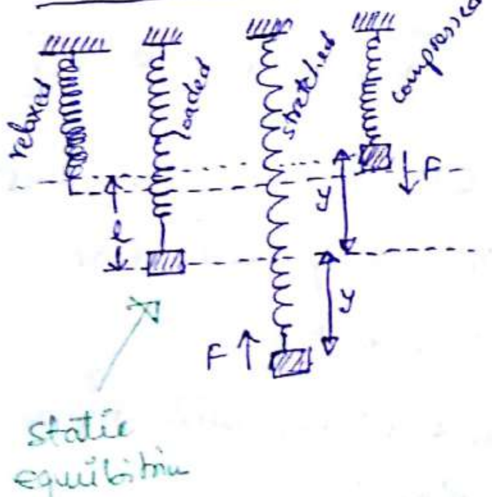
$\ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{K}{m}}$

$x = A\cos(\omega t + \phi), \quad T = 2\pi\sqrt{\frac{m}{K}}$

initial cond^n. material.



Vertical oscillations



Tension on spring $F_0 = Kl$

Force on mass $= mg$.

Static eq. $mg = Kl$.

stretched tension on spring $= K(l+y)$

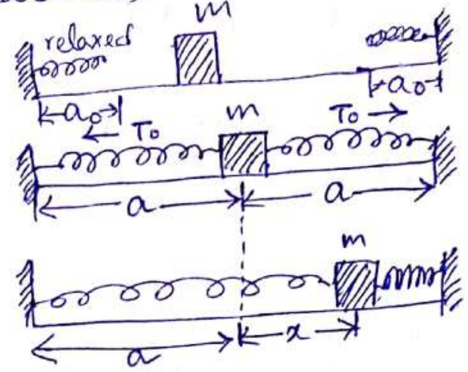
$$mg - F = K(l+y) = Kl + Ky = \cancel{mg} + Ky$$

$$F = -Ky.$$

compressed $mg + F = K(l-y) = \cancel{mg} - Ky$

$$F = -Ky.$$

Two spring system (Longitudinal oscillations)



horizontal frictionless surface, rigid wall, massless spring, relaxed length a_0.

After connection, static equilibrium

$$T_0 = K(a-a_0)$$

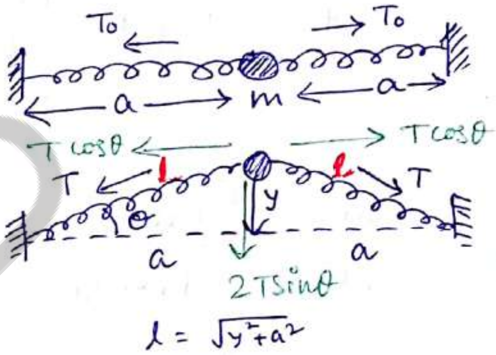
x = displacement to right. restoring force by left spring $-K(a+x-a_0)$

force on right spring $K(a-x-a_0)$

$$\therefore \; F_x = K(a-x-a_0) - K(a+x-a_0) = -2Kx$$

$$m\ddot{x} = -2Kx \quad \text{or} \quad \ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{long} = 2\pi\sqrt{\frac{m}{2K}}}$$

Two spring system (Transverse oscillations)



$$T_0 = K(a-a_0)$$

$$T = K(l-a_0)$$

$$F_y = -2T\sin\theta = -2T\frac{y}{l}$$

$$\text{or} \quad m\ddot{y} + \frac{2T}{l}y = 0 \quad \text{or} \quad \ddot{y} + \omega^2 y = 0$$

$$\omega^2 = \frac{2T}{ml} = \frac{2K(l-a_0)}{ml}, \quad \text{but } l = f(y).$$

So $\ddot{y} + \frac{2K}{m}\left(1 - \frac{a_0}{f(y)}\right)y = 0$ is not a SHM.

(a) Slinky approximation $a \gg a_0$ or $\frac{a_0}{a} \ll 1$.

$$\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{l}\right) = \frac{2K}{m}\left(1 - \frac{a_0}{a}\frac{a}{l}\right) \quad \text{as } l > a.$$

$$= \frac{2K}{m}. \quad \text{Then SHM.} \qquad \omega = \sqrt{\frac{2K}{m}}, \quad \boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K}}}$$

Spring

"large" harmonic oscillations

(b) small oscillation approximation $a \not\gg a_0$ but $y \ll a$ or l.

$$\therefore \; l = \sqrt{y^2+a^2} = a\sqrt{\frac{y^2}{a^2}+1} \sim a$$

Then also $\omega^2 = \frac{2K}{m}\left(1 - \frac{a_0}{a}\right)$ or $\boxed{T_{trans} = 2\pi\sqrt{\frac{m}{2K\left(1-\frac{a_0}{a}\right)}}}$

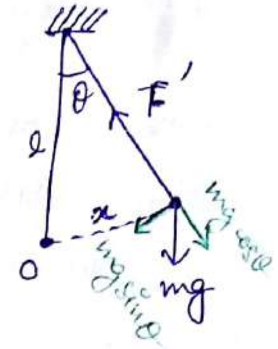
SHM.

Rubber

$$\therefore \; T_{long} = \sqrt{1 - \frac{a_0}{a}}\; T_{trans}.$$

So longitudinal is faster than transverse.

Simple pendulum



$F' = mg\cos\theta$
(tension in string)

$\left[\lim_{\theta\to 0}\right]$

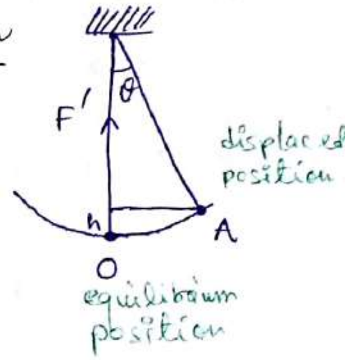
$F = -mg\sin\theta$
(restoring force) $= -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \simeq -mg\theta$

$x = l\theta$

or, $m\ddot{x} = -mg\frac{x}{l}$ or $\ddot{x} + \frac{g}{l}x = 0$.

$\therefore\ \omega = \sqrt{g/l}$, $T = 2\pi\sqrt{\frac{l}{g}}$. (mass independent)

String tension when pendulum at mean position



$F' = mg + \frac{mv^2}{l}$
(centrifugal force)

at A, Energy $= KE + PE = 0 + mgh = mgh$

at O, Energy $= KE + PE = \frac{1}{2}mv^2 + 0 = \frac{1}{2}mv^2$

Conservation of energy $\Rightarrow \frac{1}{2}mv^2 = mgh$ or $v^2 = 2gh$.

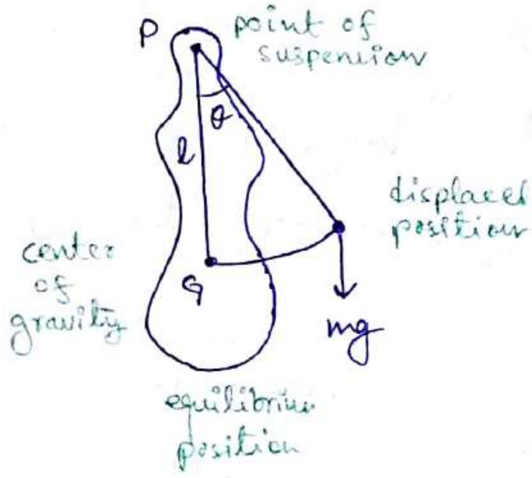
or $v^2 = 2g(l - l\cos\theta) = 2gl(1-\cos\theta) = 2gl \times 2\sin^2\frac{\theta}{2}$

$\simeq 4gl\left(\frac{\theta}{2}\right)^2 = gl\theta^2$.

$\therefore\ F' = mg + \frac{m}{l}gl\theta^2 = mg(1+\theta^2)$.

Compound Pendulum

arbitrary shaped rigid body oscillating about a horizontal axis passing through it.



restoring force $\leftrightarrow$ reactive couple or torque

moment of restoring force
$= -mgl\sin\theta$

angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$, moment of inertia $= I$.

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$\text{or } \frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\sin\theta \simeq -\frac{mgl}{I}\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

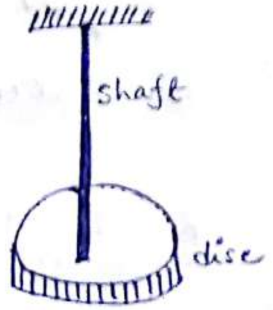
$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

If we consider moment of inertia about a parallel axis through G,
K = radius of gyration then using parallel axis theorem,

$$I = mK^2 + ml^2 \quad \therefore T = 2\pi\sqrt{\frac{K^2/l + l}{g}} = 2\pi\sqrt{\frac{l'}{g}}.$$

equivalent length of simple pendulum $= \frac{K^2}{l} + l$.

Torsional Pendulum



twist of shaft → torsional oscillations

torsional couple $= -\tau\theta$

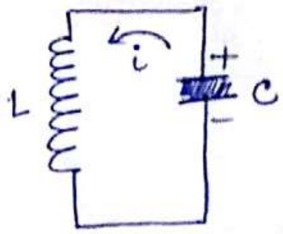
couple due to accelaration $= I\frac{d^2\theta}{dt^2}$

$$I\frac{d^2\theta}{dt^2} = -\tau\theta \quad , \quad T = 2\pi\sqrt{\frac{I}{\tau}}$$

From classical mechanics course, $\tau = \frac{\pi\eta d^4}{32\,l} = \frac{\pi\eta r^4}{2l}$

d = shaft diameter, η = modulus of rigidity,
$= 2r$.

Electrical oscillator



Capacitor is charged ⇒ electrostatic energy in dielectric media. It discharges through the inductor electrostatic energy ⇔ magnetic energy. (no dissipation of heat)

voltage across inductor $= -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$ $\quad q$ = charge

voltage across capacitor $= \frac{q}{C}$.

No e.m.f. circuit, $\frac{q}{C} = -L\frac{d^2q}{dt^2}$ or $\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$

$\omega^2 = \frac{1}{LC}$, $q = q_0\sin(\omega t + \phi)$. charge on capacitor varies harmonically.

$$i = \frac{dq}{dt} = \omega q_0 \cos(\omega t + \phi)$$

$$V = \frac{q}{C} = \frac{q_0}{C} \sin(\omega t + \phi)$$

Total energy = magnetic energy + electric energy

$$= \int iV\,dt + \frac{1}{2}CV^2 = \int i\,L\frac{di}{dt}\,dt + \frac{1}{2}CV^2$$

$$= \int Li\,di + \frac{1}{2}CV^2 = \frac{1}{2}Li^2 + \frac{1}{2}CV^2 = \frac{1}{2}L\dot{q}^2 + \frac{1}{2}CV^2$$

In mechanical oscillation, Total energy $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2$

equivalence

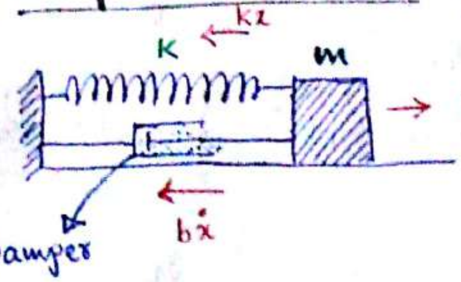
$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

In electrical oscillation, Total energy $= \frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2$

Free Damped harmonic motion

Damping of a real system is a complex phenomena involving several kind of damping force. Damping force of a body in a fluid is a function of velocity. This is called "**viscous damping.**" When an oscillating body is contact with a surface, the frictional force is called "**Coulomb friction**". Also in solids, energy is partly lost due to internal friction & imperfect elasticity of the material. Experiments suggest that such resistive force is independent of frequency & proportional to amplitude. This is called "**structural damping**." The viscouse damping force may be represented as $F = -Av + \cancel{Bv^2} - \cancel{Cv^3} + \cdots$ and such approximation is "linear damping".

Damped oscillation of a system with 1 degree of freedom



inertial force $m\ddot{x}$ is balanced by elastic restoring force kx & viscous damping force $b\dot{x}$

$$\therefore\ m\ddot{x} = -b\dot{x} - kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \underbrace{\frac{b}{m}}_{=\gamma}\frac{dx}{dt} + \underbrace{\frac{k}{m}}_{=\omega_0^2}x = 0.$$

This is a linear homogeneous 2nd order ODE.

Let the trial solution $x = Ae^{\alpha t}$, substituting we get

$$(\alpha^2 + \gamma\alpha + \omega_0^2)Ae^{\alpha t} = 0 \Rightarrow \alpha^2 + \gamma\alpha + \omega_0^2 = 0.$$

$$\therefore\ \alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\gamma^2/4 - \omega_0^2}$$

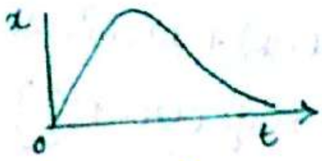
$$\therefore\ \text{Solution } x = A_1\exp\left[-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t + A_2\exp\left[-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right]t$$

$$= e^{-\gamma t/2}\left[A_1\exp\left(\sqrt{\gamma^2/4 - \omega_0^2}\,t\right) + A_2\exp\left(-\sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,t\right)\right]$$

We can have three possibilities:

(a) Heavy damping $\gamma^2/4 > \omega_0^2$ $\quad q = \sqrt{\gamma^2/4 - \omega_0^2} > 0$.

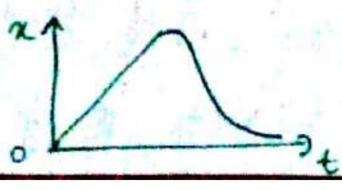
$x = e^{-\gamma t/2}(A_1e^{qt} + A_2e^{-qt})$. This means that x cannot be negative and at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & e^{qt} contributes like exponential then at $t \to \infty$, it'll damp to x (initial). If we had started at $x = 0$, after a time interval it decays back to zero $\Rightarrow$ "Dead beat no oscillation Galvanometer."



(b) Critical damping $\gamma^2/4 = \omega_0^2$: $x = (A_1 + A_2)e^{-\gamma t/2}$. The damping is slower but it has a discrepancy that at $x = 0$ at $t = 0$, $v = 0$ which is not true. Changing the trial solution, we can derive

$x \sim t e^{-\gamma t/2}$ mean at $t \approx 0$, $e^{-\gamma t/2} \approx 1$ & $x \propto t$ & later $t \to \infty$, $e^{-\gamma t/2}$ dominates. x is never negative $\Rightarrow$ no oscillati

"pointer_type galvanometer"



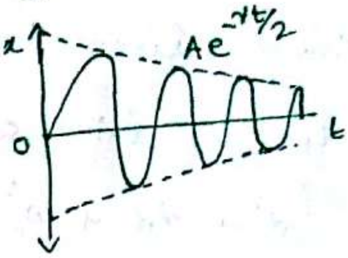
(c) Weak damping $\gamma^2/4 < \omega_0^2$ $\qquad q = \sqrt{\gamma^2/4 - \omega_0^2}$ = imaginary.

This gives oscillatory damped harmonic motion

$$x = e^{-\gamma t/2}\left[A_1 e^{i\sqrt{\omega_0^2-\gamma^2/4}\,t} + A_2 e^{-i\sqrt{\omega_0^2-\gamma^2/4}\,t}\right] \qquad \boxed{\omega = \sqrt{\omega_0^2 - \gamma^2/4}}$$

$$= e^{-\gamma t/2}\left(A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right)$$

$$= e^{-\gamma t/2}\left[\underbrace{(A_1+A_2)}_{A\cos\delta}\cos\omega t + \underbrace{i(A_1-A_2)}_{A\sin\delta}\sin\omega t\right] = Ae^{-\gamma t/2}\cos(\omega t-\delta)$$



Amplitude decreases in due time

Angular frequency is less than undamped motion.

$\tau = 2/\gamma$ = mean life time of oscillation.

Energy of a weakly damped oscillator

Using $x = Ae^{-\gamma t/2}\cos(\omega t-\delta)$ we develop expression for average energy.

$$\dot{x} = -\frac{\gamma}{2}Ae^{-\gamma t/2}\cos(\omega t-\delta) - Ae^{-\gamma t/2}\omega\sin(\omega t-\delta)$$

∴ Kinetic energy (instantaneous) of the vibrating body

$$\frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\left[\frac{\gamma^2}{4}\cos^2(\omega t-\delta) + \omega^2\sin^2(\omega t-\delta) + \gamma\omega\cos(\omega t-\delta)\sin(\omega t-\delta)\right]$$

$$\text{Potential energy} = \int_0^x F\,dx = \int_0^x Kx\,dx = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2e^{-\gamma t}\cos^2(\omega t-\delta)$$
$$= \frac{1}{2}m\omega_0^2A^2e^{-\gamma t}\cos^2(\omega t+\delta)$$

∴ Total energy = KE + PE =

$$\frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos^2(\omega t-\delta) + \omega^2\sin^2(\omega t-\delta) + \omega_0^2\cos^2(\omega t-\delta) + \frac{\gamma\omega}{2}\sin\{2(\omega t-\delta)\}\right]$$

For small damping, $\gamma << 2\omega_0$, then $e^{-\gamma t}$ does not change appreciably during one time period $T = \frac{2\pi}{\omega}$, then time averaged energy of the oscillator is

$$\langle E\rangle = \frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\langle\cos^2(\omega t-\delta)\rangle + \omega^2\langle\sin^2(\omega t-\delta)\rangle + \omega_0^2\langle\cos^2(\omega t-\delta)\rangle + \frac{\gamma\omega}{2}\langle\sin\{2(\omega t-\delta)\}\rangle\right]$$

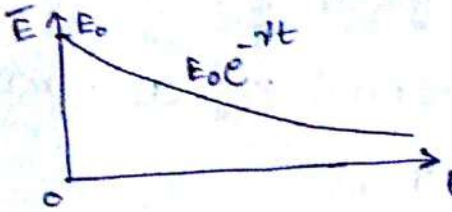
with $\langle\sin\{2(\omega t-\delta)\}\rangle \to 0$.

$$\text{Now } \langle\cos^2(\omega t-\delta)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\omega t-\delta)\,d(\omega t) = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}2\cos^2x\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(1+\cos 2x)\,dx = \frac{1}{2} = \langle\sin^2(\omega t-\delta)\rangle$$

$$\therefore \langle E\rangle = \frac{1}{2}mA^2e^{-\gamma t}\left[\cancel{\frac{\gamma^2}{8}} + \left(\omega_0^2 - \cancel{\frac{\gamma^2}{4}}\right)\frac{1}{2} + \frac{\omega_0^2}{2}\right] = \frac{1}{2}m\omega_0^2A^2e^{-\gamma t}$$

$\boxed{\langle E\rangle = E_0\, e^{-\gamma t}}$ where $E_0 = \frac{1}{2}m\omega_0^2 A^2$ is energy of undamped oscillator

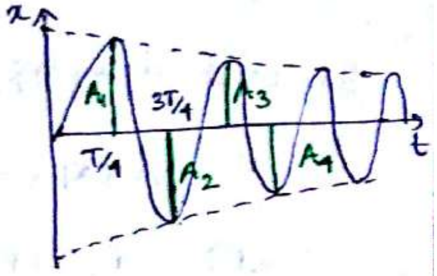


The average power dissipation in one time period

$$\langle P(t)\rangle = \frac{d}{dt}\langle E(t)\rangle = \gamma\langle E(t)\rangle.$$ due to friction

Estimation of Damping

There are various ways of estimation of the damping of an oscillator. Let us choose initial condition at $t=0$, $x=0$, $\frac{dx}{dt}=v_0$ and $\delta = \pi/2$, $x = Ae^{-\gamma t/2}\cos(\omega t - \pi/2) = Ae^{-\gamma t/2}\sin\omega t$



Logarithmic Decrement

$$x = Ae^{-\gamma t/2}\sin\omega t = Ae^{-\gamma t/2}\sin\frac{2\pi t}{T}$$

at $t = \frac{T}{4}$, $x_1^{max} = Ae^{-\gamma T/8}\sin\frac{2\pi}{T}\frac{T}{4} = Ae^{-\gamma T/8}$

at $t = \frac{3T}{4}$, $x_2^{max} = Ae^{-3\gamma T/8}$

at $t = \frac{5T}{4}$, $x_3^{max} = Ae^{-5\gamma T/8}$ etc.

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \frac{x_3^{max}}{x_4^{max}} = \dots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\gamma T/4} = d$$ (constant)

"d" is called decrement of the motion. $\lambda = \ln d$ is the logarithmic decrement of the motion $= \ln e^{\gamma T/4} = \frac{\gamma T}{4}$

$$\therefore \frac{x_1^{max}}{x_2^{max}} = \frac{x_2^{max}}{x_3^{max}} = \dots = \frac{x_{n-1}^{max}}{x_n^{max}} = e^{\lambda}$$

Multiplying, $\frac{x_1^{max}}{x_n^{max}} = e^{(n-1)\lambda}$ or $\lambda = \frac{1}{n-1}\ln\left(\frac{x_1^{max}}{x_n^{max}}\right)$

$$\lambda = \frac{2.303}{n-1}\log_{10}\left(\frac{x_1^{max}}{x_n^{max}}\right)$$

This method is used to determine the corrected last throw of a Ballistic galvanometer due to damping. Relation between undamped throw θ_0 & first throw θ_1 is

$\theta_1 = \theta_0 e^{-\gamma T/8}$ $\therefore \theta_0 = \theta_1 e^{\gamma T/8} = \theta_1 e^{\lambda/2} \simeq \theta_1(1+\frac{\lambda}{2})$ for $\lambda \ll 1$

So knowing λ, we can correct θ_1 for damping.

Quality Factor (Q-Value)

Another method to express damping in an oscillatory system is to measure the rate of decay of energy. Quality factor $Q = \frac{\omega}{\gamma} = \frac{\omega_0}{\gamma}\sqrt{1-\gamma^2/4\omega_0^2}$. While $\langle E\rangle = E_0 e^{-\gamma t}$, power $\langle P(t)\rangle = \frac{d}{dt}\langle E\rangle = \gamma\langle E\rangle$

So the average energy dissipated in time period T is

$$\gamma T\langle E\rangle = \frac{2\pi\gamma}{\omega}\langle E\rangle = \frac{2\pi}{Q}\langle E\rangle = \frac{2\pi}{Q} \times \text{average energy stored.}$$

$$\therefore \boxed{Q = 2\pi \times \frac{\text{Average energy stored in one time period}}{\text{Average energy lost in one time period}}}$$

In weak damping limit $\frac{\gamma^2}{4\omega_0^2} << 1$, $Q = \frac{\omega_0}{\gamma}$. As $\gamma \to 0$, $Q \to \infty$

$\therefore$ $x = A\exp\left(-\frac{\omega_0 t}{2Q}\right)\cos(\omega_0 t - \delta)$ in limit $\gamma^2/4\omega_0^2 << 1$

$\langle E\rangle = E_0 \exp\left(-\frac{\omega_0 t}{Q}\right)$ and see that $\tau_1 = \frac{Q}{\omega_0}$, $\langle E\rangle = E_0 e^{-1}$

and no. of complete oscillation if is n, then $n = \frac{\omega_0}{2\pi}\tau_1 = \frac{Q}{2\pi}$

So $\langle E\rangle$ reduces to e^{-1} of $\langle E\rangle$ in $Q/2\pi$ cycles of oscillation.

Note that $\boxed{\lambda = \frac{\gamma T}{4},\ \tau = \frac{2}{\gamma}\ \&\ Q = \frac{\omega_0}{\gamma},\ \tau_1 = \frac{Q}{\omega_0} = \frac{1}{\gamma}}$

"Moving coil Galvanometer" is the example of damped harmonic motion. Similarly, current or charge oscillation in LCR circuit, mechanical vibration of a string or tuning fork etc.

Forced Vibration

Vibrating system with damping + periodic force = forced vibration

Vibrating system tunes to the frequency of natural vibration dies out, system tunes to the frequency of force. For example, a bridge vibrates in the influence of marching soldiers. Contributions are restoring force kx, damping force $b\dot{x}$, inertial force $m\ddot{x}$ & external periodic force $F(t) = F_0\cos\omega t$.

$\therefore$ Equation of motion of the body is

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F(t)$$

or $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f_0\cos\omega t$, $\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{k}{m}$, $f_0 = \frac{F_0}{m}$.

linear homogeneous 2^{nd} order ODE. Solution of this we can separate out as $\frac{d^2x_1}{dt^2} + \gamma\frac{dx_1}{dt} + \omega_0^2 x_1 = f_0\cos\omega t$ & $\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \omega_0^2 x_2 = 0$ so that $x_1 + x_2$ is a solution. Now we know $x_2 = Ae^{-\gamma t/2}\cos(\omega^* t - \delta)$ with $\omega^* = \sqrt{\omega_0^2 - \gamma^2/4}$ & will die out in time. (transient state). For x_1, we can write $x_1 = B\cos(\omega t - \delta)$ where B & δ are to be determined

$X = Re(Be^{i(\omega t - \delta)})$. In this notation,

$$\frac{d^2X}{dt^2} + \gamma\frac{dX}{dt} + \omega_0^2 X = f_0 e^{i\omega t} = f_0 e^{i(\omega t - \delta)} e^{i\delta}$$

$$\text{or } \left[B\left[(\omega_0^2 - \omega^2) + i\omega\gamma\right] - f_0 e^{i\delta}\right] e^{i(\omega t - \delta)} = 0, \forall t.$$

$$B(\omega_0^2 - \omega^2 + i\omega\gamma) - f_0 e^{i\delta} = 0 \quad \text{or} \quad Be^{-i\delta} = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

$$\text{or } B\cos\delta - iB\sin\delta = \frac{f_0\left[\omega_0^2 - \omega^2 - i\omega\gamma\right]}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\therefore B\cos\delta = \frac{f_0(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}, \quad B\sin\delta = \frac{f_0\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \quad \therefore B = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)$$

$$\therefore \boxed{x_1 = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right)\right)}$$

Steady state solution

It's dependent on $F_0, m, \omega, \omega_0, \gamma$ & there is a phase difference δ between force & displacement. When $D = (\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2$ = minimum B is maximum amplitude. If this frequency is ω_r then $\left.\frac{dD}{d\omega}\right|_{\omega = \omega_r} = 0$ and $\left.\frac{d^2D}{d\omega^2}\right|_{\omega = \omega_r} > 0$. $\therefore -2(\omega_0^2 - \omega_r^2)2\omega_r + 2\omega_r\gamma^2 = 0$

or $\omega_r = \sqrt{\omega_0^2 - \gamma^2/2}$ and convince yourself $\frac{d^2D}{d\omega^2} > 0$ if $\frac{\gamma^2}{2} < \omega_0^2$.

Thus amplitude of forced oscillation is maximum if frequency of the driving force is nearly equal to frequency of natural oscillation

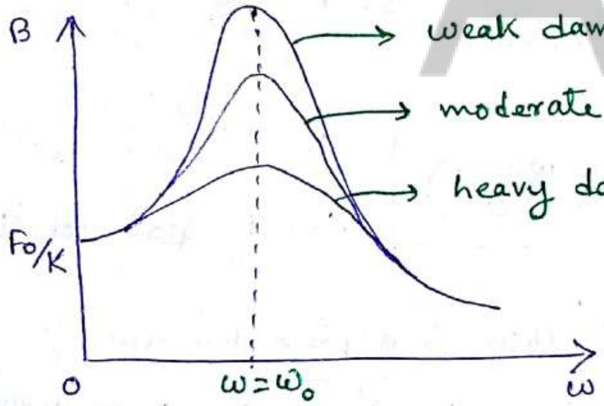
At $\omega = \omega_r$, $B_{max} = \frac{f_0}{\gamma(\omega_0^2 - \gamma^2/4)^{1/2}}$ and $\gamma << \omega_0$, $B_{max} \approx \frac{f_0}{\gamma\omega_0} = \frac{F_0}{m\gamma\omega_0} = \frac{F_0}{b\omega_0}$

Thus in this limit $\omega_r \simeq \omega_0$ and the amplitude is controlled by "b" and the forced oscillator is **"resistance controlled."**

Recall $B = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}}$, In limit $\omega << \omega_0$, $B \approx \frac{F_0/m}{\omega_0^2\sqrt{1+\frac{\omega^2}{\omega_0^2}\frac{\gamma^2}{\omega_0^2}}} \simeq \frac{F_0}{m\omega_0^2} = \frac{F_0}{K}$

This displacement a constant force F_0 would produce. when $\omega \to 0$, $F(t) \to F_0$ or we get back $m\frac{d^2x}{dt^2} = -m\omega^2 x$ very small role than Kx term. ∴ Response of the oscillator is controlled by the stiffness constant K & the oscillator is **"stiffness controlled."**

Similarly for $\omega >> \omega_0$, $B \simeq \frac{F_0/m}{\omega^2\sqrt{1+\frac{\gamma^2}{\omega_0^2}\frac{\omega_0^2}{\omega^2}}}$ which for weak damping $\gamma << \omega_0$ is $B \simeq \frac{F_0}{m\omega^2}$ and $m\omega^2 x$ is dominating and the oscillator is **"mass or inertia controlled."**



amplitude resonance at $\omega = \omega_0$ when $\gamma^2/2 < \omega_0^2$.

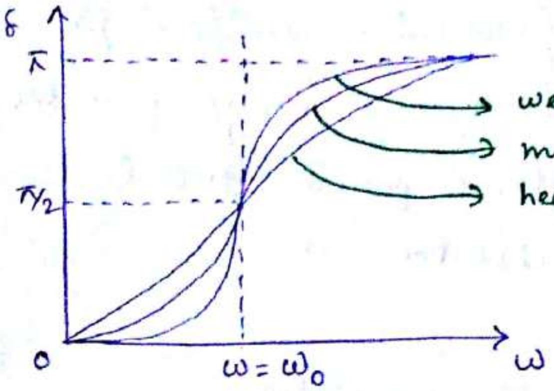
Also when $\omega << \omega_0$,

$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \simeq \frac{\omega}{\omega_0}\frac{\gamma}{\omega_0}$

as $\omega \to 0$, $\delta \to 0$. Thus for low frequency of driving force, displacement is nearly in phase with driving force. If $\omega >> \omega_0$, $\tan\delta \simeq -\frac{\gamma}{\omega} \simeq \frac{\gamma}{\omega_0}\frac{\omega_0}{\omega}$ which for weak damping $\gamma << \omega_0$ has small negative value or $\delta \simeq \pi$.

∴ If frequency of driving force >> natural frequency of free oscillations, then displacement will be out of phase with driving force. Also ~~when~~ ~~[illegible]~~ accelaration will be in phase with driving force.

But at resonance, $\omega \simeq \omega_0$ & $\tan\delta = \infty$ $\therefore$ $\delta = \pi/2$ or displacement is maximum when driving force is zero.



Displacement x_1 lags the force $F(t)$ by δ.

Velocity Resonance $\quad x_1 = B\cos(\omega t - \delta)$ $\therefore$ $\dot{x}_1 = -\omega B \sin(\omega t - \delta)$

or $v = v_0 \cos(\omega t - \phi)$ where $v_0 = \omega B = \dfrac{F_0/m}{\sqrt{\dfrac{(\omega_0^2 - \omega^2)^2}{\omega^2} + \gamma^2}}$

$= v_0 \cos(\omega t - \delta + \pi/2)$

and $\phi = \delta - \pi/2$. $[-\sin(\omega t - \delta) = \cos(\omega t - \delta + \pi/2)]$

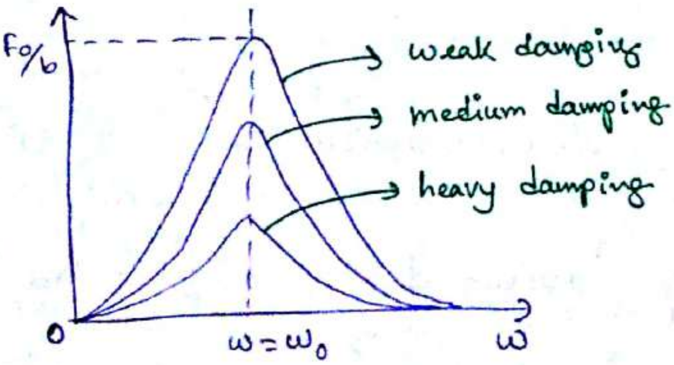
$\therefore$ Velocity leads the displacement in phase by $\pi/2$. V_0 is maximum when denominator is minimum. $\dfrac{d}{d\omega}\left[\dfrac{(\omega_0^2 - \omega^2)^2}{\omega^2} + \gamma^2\right]\Bigg|_{\omega = \omega_r} = 0$

or $\omega_r = \omega_0$. So at $\omega = \omega_0$, V_0 is maximum, velocity resonance

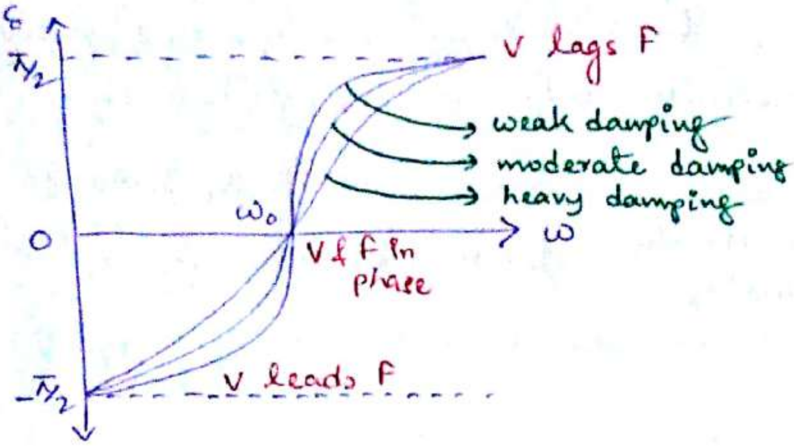
$V_0^{max} = \dfrac{F_0/m}{\gamma} = \dfrac{F_0}{b}$, so as "b" increases, V_0^{max} decreases.

for $\omega >> \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega^2}$ and if γ is not large then $V_0 \to 0$ for $\omega \to \infty$

for $\omega << \omega_0$, $V_0 \simeq \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{m\omega^2}\dfrac{\omega^2}{\omega_0^2} \to 0$ for $\omega \to \infty$.



Phase of velocity relative to the force is $\phi = \delta - \pi/2$. For $\omega << \omega_0$, $\delta \simeq 0$, so $\phi = -\pi/2$. As ϕ is angle by which velocity lags behind the force, so here velocity leads the force by an angle $\pi/2$. For $\omega >> \omega_0$, $\delta \simeq \pi$, $\phi = \pi - \pi/2 = \pi/2$ so for very high frequencies, velocity lags the force by $\pi/2$. At resonance $\omega = \omega_0$, $\delta = \pi/2$ and $\phi = 0$ & velocity is in phase with force.



This is therefore the most favourable condition for transfer of energy from the external periodic force to the oscillator.

Power transfer from driving force to the oscillator

Energy of a damped oscillator decreases exponentially as $E(t)=E_0 e^{-\gamma t}$. In order to maintain steady state oscillation, driving force transfers energy to oscillator. Now

$$x = B\cos(\omega t-\delta) = B\cos\delta\cos\omega t + B\sin\delta\sin\omega t$$
$$= B_{el}\cos\omega t + B_{ab}\sin\omega t$$

where B_{el} = elastic amplitude $B\cos\delta = \dfrac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2+\omega^2)^2+\gamma^2\omega^2}$ [in phase with force]

B_{ab} = absorptive amplitude $B\sin\delta = \dfrac{f_0\,\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ [out of phase $\pi/2$ with force]

$v = \dot{x} = \omega(-B_{el}\sin\omega t + B_{ab}\cos\omega t)$ & thus the power by driving force $F_0\cos\omega t$ / second is the work done by the force/second

$$P(t) = F_0\cos\omega t\; v = F_0\omega\cos\omega t\,(-B_{el}\sin\omega t + B_{ab}\cos\omega t).$$

$\therefore$ Time averaged input power over one complete cycle is

$$P_{input} = \langle P(t)\rangle = \frac{1}{T}\int_0^T P(t)\,dt = -F_0\omega\frac{B_{el}}{T}\int_0^T \underbrace{\sin(\omega t)\cos(\omega t)}_{0}\,dt +$$
$$F_0\omega\frac{B_{ab}}{T}\int_0^T \underbrace{\cos^2(\omega t)\,dt}_{\to 1/2} = \frac{1}{2}F_0\omega B_{ab}.$$

This input power supplied by driving force is not stored in oscillator but dissipated as work done in moving the system against friction. Instantaneous power dissipated through friction is

$$P(t) = bv\cdot v = b\left(\frac{dx}{dt}\right)^2 = b\omega^2(B_{ab}^2\cos^2\omega t + B_{el}^2\sin^2\omega t - 2B_{ab}B_{el}\cos\omega t\sin\omega t)$$

$\therefore$ Time averaged power $\langle P(t)\rangle = P_{dissipation} = \frac{b\omega^2}{2}(B_{el}^2 + B_{ab}^2)$.

$$= \frac{b\omega^2 f_0^2}{2[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2]} = \frac{1}{2}F_0\omega B_{ab}$$

$\therefore$ $P_{input} = P_{dissipate}$ (steady state).

<u>Energy of the forced oscillator</u> Instantaneous KE is

$$\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(B_{ab}^2\cos^2\omega t + B_{el}^2\sin^2\omega t - 2B_{ab}B_{el}\cos\omega t\sin\omega t)$$

Instantaneous PE $\frac{1}{2}Kx^2 = \frac{1}{2}m\omega_0^2(B_{ab}^2\sin^2\omega t + B_{el}^2\cos^2\omega t + 2B_{ab}B_{el}\cos\omega t\sin\omega t)$

$\therefore$ Time averaged total energy is $E = \langle E(t)\rangle = \frac{1}{4}m(\omega^2+\omega_0^2)(B_{ab}^2+B_{el}^2)$

$E_{resonance} = \frac{1}{2}m\omega_0^2(B_{ab}^2+B_{el}^2)$ at $\omega \simeq \omega_0$

$\langle KE\rangle = \frac{1}{4}m\omega^2(B_{ab}^2+B_{el}^2)$, $\langle PE\rangle = \frac{1}{4}m\omega_0^2(B_{ab}^2+B_{el}^2)$

<u>Maximum input power & Bandwidth</u>

Time averaged input power $P_{input} = \frac{1}{2}F_0\omega B_{ab}$

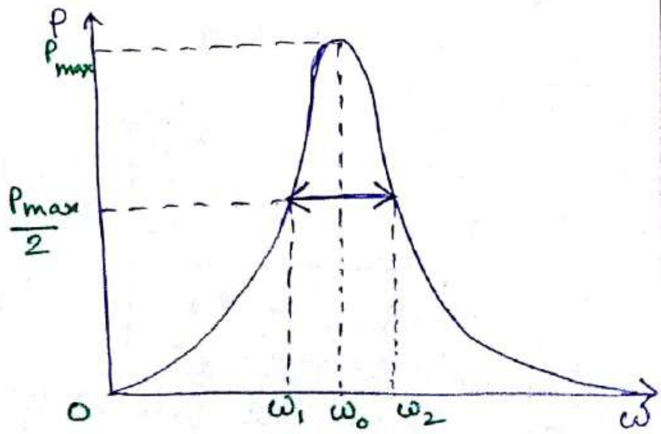
$$= \frac{F_0^2\gamma}{2m}\left[\frac{\omega^2}{(\omega_0^2-\omega)^2+\gamma^2\omega^2}\right]$$

This will be maximum for $\frac{dP}{d\omega} = 0$

& that yields $\omega = \omega_0$. Thus at resonance frequency P_{input} is maximum.

$P_{input}^{max} = \frac{F_0^2}{2m\gamma}$ $\therefore$ $P = P_{input}^{max}\frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

Frequency ω_1 & ω_2 at which the power drops down to 1/2 of maximum is the half power freq.



$$\frac{1}{2} = \frac{P_{input}}{P_{input}^{max}} = \frac{\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

or $\omega^2 = \omega_0^2 \pm \gamma\omega$

$$\begin{cases}\omega_1 = -\frac{\gamma}{2} + (\omega_0^2+\frac{\gamma^2}{4})^{1/2} \\ \omega_2 = \frac{\gamma}{2} + (\omega_0^2+\frac{\gamma^2}{4})^{1/2}\end{cases}$$

, band width $\Delta\omega = \omega_1 - \omega_2 = \gamma$.

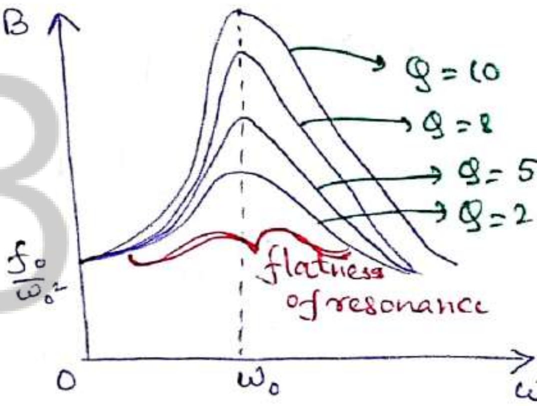
<u>Quality Factor</u> Q is a parameter that gives the sharpness of resonance & defined as $Q = \dfrac{\text{resonant frequency}}{\text{band width}} = \dfrac{\omega_0}{\Delta\omega} = \dfrac{\omega_0}{\nu}$

$$= 2\pi \frac{\text{Avg. energy stored in one cycle}}{\text{Avg. energy lost in one cycle}}$$

$$\therefore Q = 2\pi \frac{\langle E(t)\rangle}{P_{dissipate}\, T} = \underbrace{\left(\frac{2\pi}{T}\right)}_{=\omega} \frac{1}{\cancel{4}_{2}} m(\omega^2+\omega_0^2)\cancel{(B_{ab}^2+B_{el}^2)} \frac{\cancel{2}}{b\omega^{\cancel{2}}\cancel{(B_{ab}^2+B_{el}^2)}}$$

$$= \frac{\omega^2+\omega_0^2}{2\nu\omega} \quad \text{and for } \omega \approx \omega_0,\ Q^{\text{resonance}} = \frac{\omega_0}{\gamma}$$

Thus for low damping, $\nu \ll \omega_0$ and Q is high. that makes the resonance very ~~high~~. sharp. Thus Q measures the sharpness of resonance



Using $Q = \frac{\omega_0}{\nu}$, the amplitude is

$$B = \frac{f_0 Q}{\omega\omega_0\sqrt{1+Q^2\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2}}$$

Q large, B large. Q can be regarded as amplification factor. at low driving force $\omega \to 0$, $B_0 = \dfrac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \simeq \dfrac{f_0}{\omega_0^2}$ and we know

$$B_{max} = \frac{f_0}{\gamma\sqrt{\omega_0^2-\gamma^2/4}}. \qquad \text{So} \quad \frac{B_{max}}{B_0} = \frac{\omega_0^2}{\gamma\sqrt{\omega_0^2-\nu^2/4}} = \frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}$$

(for low damping) Q is very large $= Q\left(1-\frac{1}{4Q^2}\right)^{-1/2} \simeq Q\left(1+\frac{1}{8Q^2}\right)$

$$= Q.$$

$\therefore \boxed{B_{max} = QB_0}$ The resonant amplitude is Q times the amplitude at low frequencies of the driving force.

ELASTICITY

Elastic Properties of Matter

When an external force acts on a body, relative displacement of its various parts takes place. By exerting a restoring force, particles tend to come back to their original position.

$$\text{Stress} = \frac{\text{Restoring force}}{\text{cross-sectional area}} = \frac{F}{A}.$$

Strain is defined as the ratio of change of length, volume or shape to the original length, volume or shape.

Young's Modulus: $Y = \dfrac{\text{applied load per unit cross-section}}{\text{increase in length per unit length}}$

$$= \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{l/L} = \frac{FL}{Al}$$

Unit (CGS) : stress = dynes/cm^2, (SI) = Newton/m^2.
strain = no unit (pure number).

$\therefore$ Y = dynes/cm^2 (CGS) or N/m^2 (SI). Dimension of Y is

$$[Y] = \left[\frac{MLT^{-2}L^{-2}}{L\cdot L^{-1}}\right] = [ML^{-1}T^{-2}].$$

Bulk Modulus: (volume elasticity)

$$K = \frac{\text{compressive or tensile force per unit area}}{\text{decrease or increase in volume per unit volume}}$$

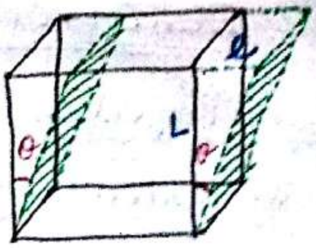
$$= \frac{\text{compressional or dilational pressure}}{\text{volume strain}} = -\frac{dp}{\frac{dV}{V}}.$$

$\therefore$ K = dynes/cm^2 or N/m^2 as, $[K] = \left[\dfrac{MLT^{-2}L^{-2}}{L^3L^{-3}}\right] = [ML^{-1}T^{-2}]$

Negative sign means increase in applied pressure causes decrease in volume.

Rigidity Modulus: $n = \dfrac{\text{tangential stress}}{\text{angle of shear or shearing strain}}$.

Consider a solid cube, whose lower face is fixed and a tangential force F is applied over the upper face, so that its displaced to a new position. As each horizontal layer of the cube is displaced with displacement proportional to its distance from the fixed lower plane,



$$\text{shearing strain} = \frac{l}{L} = \tan\theta \sim \theta \quad \text{if } \lim_{\theta\to 0} \text{ (usually } < 4^\circ\text{)}$$

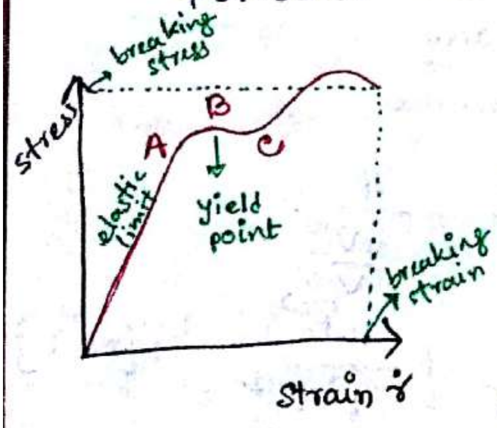
$$\therefore n = \frac{F/A}{\theta} = \frac{F}{A\theta}. \qquad [n] = [ML^{-1}T^{-2}]$$ & unit is dynes/cm^2 or N/m^2 as θ is a pure number.

Poisson's Ratio: When a wire is stretched, its length increases but its diameter decreases. When an elongation is produced by a longitudinal stress in a certain direction, a contraction results in the lateral dimensions of the body under strain.

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{d/D}{l/L} = \frac{dL}{Dl} \quad \text{where}$$

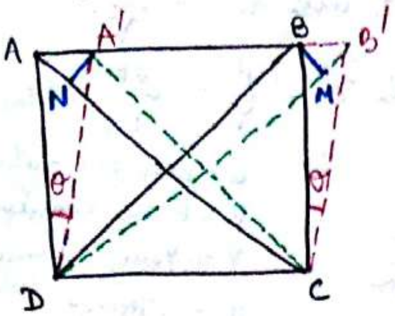
D = diameter of the wire, d = decrease in diameter.
L = length of the wire, l = increase in length.

Poisson's ratio is a dimensionless number.



Stress ∝ strain is called the elastic limit or Hooke's law (point A), so that if stress is removed, an elastic body regains its original shape. After point A, the curve is bend towards a maximum point B (permanent set). After point B, elongation is faster than AB. So after yield point B, elongation increases rapidly with rapid contraction of the area of cross-section of the wire until the breaking stress is reached where snapping occurs. It's called fracture.

Shear ≡ Elongation (Extension) strain ⊥ Compression strain
θ $\qquad$ $\theta/2$ $\qquad$ $\theta/2$



Suppose in ABCD cube of AB = BC = CD = DA = L the base CD is fixed and after applying a tangential force F the distorted cube is A′B′CD with AA′ = BB′ = l & $\angle ADA' = \theta$.

Now DB = DM = $\sqrt{2}L$. As θ (angle of shear) is very small, So $\Delta ANA'$ & $\Delta BMB'$ are isocles right angle triangle with

$$\angle A'AN = \angle BB'M = 45^\circ$$

$$\therefore B'M = BB'\cos 45^\circ = \frac{l}{\sqrt{2}}.$$

So Elongation (Eextension) strain along DB diagonal is

$$\frac{B'M}{DB} = \frac{l}{\sqrt{2}} \times \frac{1}{L\sqrt{2}} = \frac{l}{2L} = \frac{\theta}{2} \quad \text{as} \quad \frac{l}{L} = \tan\theta \approx \theta.$$

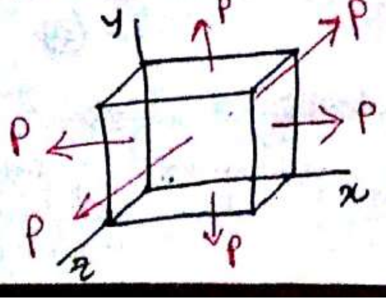
Similarly, compression strain along diagonal AC = $\frac{AN}{AC}$

$$= \frac{AA'\cos 45^\circ}{L\sqrt{2}} = \frac{l}{\sqrt{2}} \times \frac{1}{L\sqrt{2}} = \frac{l}{2L} = \frac{\theta}{2}.$$

So shear θ is equivalent to an extension and a compression strain at right angle to each other with each of value $\theta/2$.

Look for a proof that a shearing stress is equivalent to a linear tensile stress and an equal compression stress mutually at right angles.

Relation between Y, K, n, σ for a homogeneous isotropic medium



Suppose a cube in a strained medium is subjected to uniform tensile stress P over each face. So linear strain along x axis due to tensile stress along X axis is $\frac{P}{Y}$. Linear strain along x axis due to tensile stress along Y-axis is $-\frac{\sigma P}{Y}$. Also, linear strain along X-axis due to tensile stress along Z-axis is $-\frac{\sigma P}{Y}$.

So the resultant linear strain along x axis $\gamma = \frac{P}{Y} - \frac{2\sigma P}{Y}$

Similarly so for Y and Z axis.

σ = Poisson's ratio
K = Bulk modulus
Y = Young's modulus
γ = strain

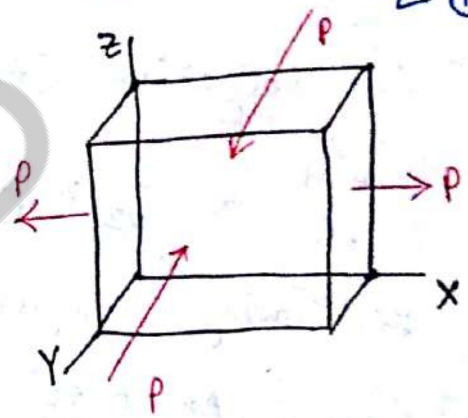
If $\frac{\delta V}{V}$ is the volume strain then $\frac{\delta V}{V} = \frac{P}{K}$

For spherically isotropic system,

$$\frac{\delta V}{V} = \frac{\frac{4}{3}\pi\left[r(1+\gamma)\,r(1+\gamma)\,r(1+\gamma)\right] - \frac{4}{3}\pi r^3}{\frac{4}{3}\pi r^3} = \cancel{1} + 3\gamma + \cancel{O(\gamma^2)}^{\,0} + \dots - \cancel{1}$$

$= 3\gamma$. (Cubical expansion is 3 times linear expansion)

$\therefore \frac{\delta V}{V} = 3\gamma = \frac{3P}{Y}(1-2\sigma) = \frac{P}{K}$ $\therefore$ $\boxed{Y = 3K(1-2\sigma)}$ —— ①



Now suppose that the cube is subjected to a tensile stress along x axis & an equal compressional stress along Y axis.

So linear strain along x axis due to tensile stress along x axis is $\frac{P}{Y}$. Also, linear strain along x axis due to compressional stress along Y axis is $\frac{\sigma P}{Y}$. So the resultant linear strain along x axis is

$\gamma_x = \frac{P}{Y} + \frac{\sigma P}{Y}$. Resultant linear strain along Y axis $\gamma_y = -\frac{P}{Y} - \frac{\sigma P}{Y}$

and resultant linear strain along z axis $\gamma_z = -\cancel{\frac{\sigma P}{Y}} + \cancel{\frac{\sigma P}{Y}} = 0$

We know $\theta = \frac{P}{n}$ and $\frac{\theta}{2} = \frac{P}{Y}(1+\sigma)$

θ = angle of shear
n = modulus of rigidity

as $\theta = \frac{\gamma_x}{2}$. So $\frac{P}{2n} = \frac{P}{Y}(1+\sigma)$ or $\boxed{Y = 2n(1+\sigma)}$ —— ②

From ① and ②, we have $\frac{Y}{K} = 3 - 6\sigma$ and $\frac{Y}{n} = 2 + 2\sigma$

$\therefore Y\left(\frac{1}{K} + \frac{1}{n}\right) = 5 - 4\sigma = 5 - 4\left(\frac{Y-2n}{2n}\right)$ or $\boxed{Y = \frac{9nK}{n+3K}}$

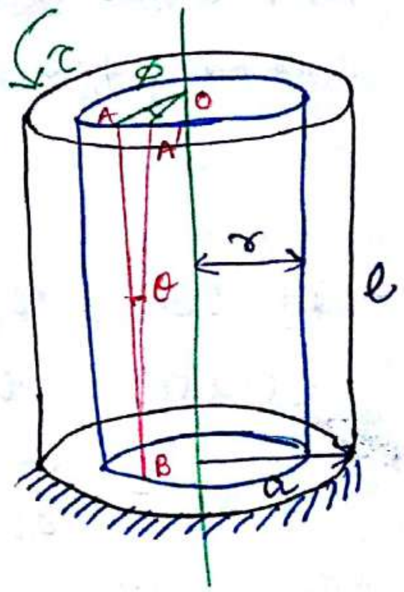
Again from ① and ②, $\frac{Y}{3K} = 1 - 2\sigma$ and $\frac{Y}{2n} = 1 + \sigma$

$$\therefore \sigma = Y \frac{3K - 2n}{18Kn} = \frac{9Kn}{n + 3K} \frac{3K - 2n}{18Kn} = \frac{3K - 2n}{6K + 2n}$$

or $\boxed{\sigma = \frac{3K - 2n}{6K + 2n}}$ or $3K(1 - 2\sigma) = 2n(1 + \sigma)$, $n, K > 0$.

When $\sigma > \frac{1}{2}$, $3K(1-2\sigma) < 0$ but $2n(1+\sigma) > 0$.
when $\sigma < -1$, $3K(1-2\sigma) > 0$ but $2n(1+\sigma) < 0$ $\Big\}$ $3K(1-2\sigma) \neq 2n(1+\sigma)$ violation

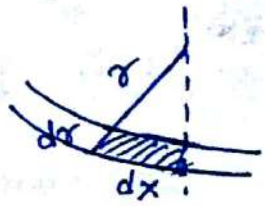
$$\therefore \boxed{-1 < \sigma < \frac{1}{2}.}$$



The Torsion cylinder and shear Waves

Consider a cylinder of length l and radius a with clamped (fixed) lower end and a torque is applied at upper end, because of that cylinder is twisted through an angle. If an elemental point A is displaced to A', then $l\theta = AA' = r\phi$ where ϕ = angle of twist and r is radius of elemental cylinder.

$\therefore r\phi = l\theta$ or $\theta = r\phi/l$. Due to elasticity, there will be a restoring torque. To calculate this, consider the shell of thickness dr, length dx at r distance apart. So tangential stress $= \frac{F}{dr\,dx}$ and



$$\text{rigidity modulus } n = \frac{\text{tangential stress}}{\text{angle of shear}} = \frac{F/dr\,dx}{\theta} = \frac{F/dr\,dx}{r\phi/l}$$

$$= \frac{Fl}{r\phi\,dr\,dx}. \qquad \therefore F = \frac{n\phi}{l} r\,dr\,dx.$$

Moment of this force about cylinder axis $= F.r = \frac{n\phi}{l} r^2 dr\,dx$

So restoring torque over entire surface $\delta\tau = \frac{n\phi}{l} r^2 dr \sum dx$

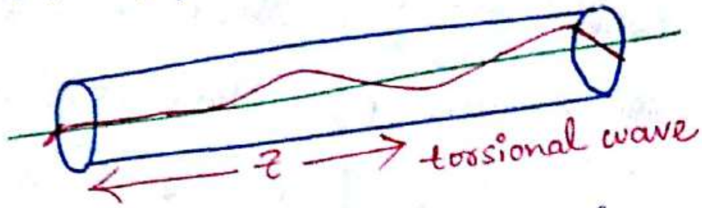
$[\Sigma dx = 2\pi r]$ $= \frac{2\pi n\phi}{l} r^3 dr$

∴ Total restoring torque for the entire cylindrical bar

$$\tau = \frac{2\pi n\phi}{l}\int_0^a r^3 dr = \frac{\pi n a^4}{2l}\phi \ , \quad \frac{\tau}{\phi} = \text{torsional rigidity}$$

For hollow cylinder with inner & outer radii r_1 & r_2

$$\tau = \frac{2\pi n\phi}{l}\int_{r_2}^{r_1} r^3 dr = \frac{n\pi\phi}{2l}(r_1^4 - r_2^4)$$



For a static torsion, torque is same everywhere & proportional to ϕ/l. If its nonuniform then

$$\tau(z) = \frac{n\pi a^4}{2}\left(\frac{\partial\phi}{\partial z}\right)$$

↳ local torsional strain

If $\tau(z)$ & $\tau(z+dz)$ are the torque at two ends then

$$\tau(z+dz) = \tau(z) + \frac{\partial\tau}{\partial z}\Delta z \quad \text{and so,} \quad \Delta\tau = \tau(z+dz) - \tau(z)$$

$$\text{or } \Delta\tau = \frac{\partial\tau}{\partial z}\Delta z = \frac{n\pi a^4}{2}\frac{\partial^2\phi}{\partial z^2}\Delta z \quad \text{— (1)}$$

The effect of this incremental torque is to yield angular acceleration of the slice of mass $(\pi a^2 \Delta z)\rho$ or moment of inertia $\frac{1}{2}ma^2 = \frac{\pi}{2}\rho a^4 \Delta z$, so that the restoring couple is

$$I\alpha = I\frac{d^2\phi}{dt^2} = \frac{\pi\rho a^4}{2}\Delta z\frac{\partial^2\phi}{\partial t^2}. \quad \text{— (2)}$$

So, from (1) & (2)

$$\frac{n\pi a^4}{2}\frac{\partial^2\phi}{\partial z^2}\Delta z = \frac{\pi\rho a^4}{2}\frac{\partial^2\phi}{\partial t^2}\Delta z$$

$$\text{or } \boxed{\frac{\partial^2\phi}{\partial t^2} = \frac{n}{\rho}\frac{\partial^2\phi}{\partial z^2}}$$

Wave equation in 1 dimension with $\boxed{v_{shear} = \sqrt{\frac{n}{\rho}}}$ = shear speed

So denser the rod for same stiffness the slower the waves & independent of radius of rod. Torsional waves are special example of shear waves, which are those in which the strains do not change the material volume, but in torsional waves, shear stresses are distributed on a circle and move with same speed.

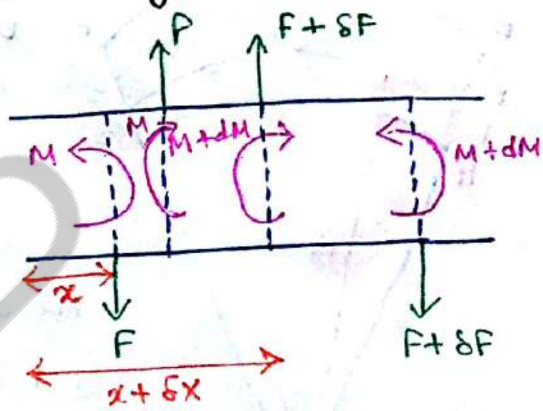
Inside a solid material there are compressional / longitudinal waves as well as on the surface Rayleigh or Love waves. In them, strains are neither purely longitudinal or transverse. Longitudinal waves travel faster than shear waves

$$\boxed{v_{long} = \sqrt{\frac{1-\sigma}{(1+\sigma)(1-2\sigma)}\frac{Y}{\rho}}}$$

and $Y > n$ as $Y = 2n(1+\sigma)$, so $\boxed{v_{long} > v_{shear}}$

Y & σ can be measured by measuring ρ and v_{long}, v_{shear}, e.g. in earthquake, distance between quakes can be measured like this.

Bending of Beam: General method for determining deflection due to bending:

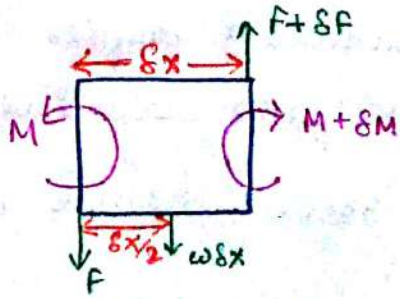


Let ω be the weight per unit length of the beam. Let us consider an element δx, the left edge of which is at a distance x from the origin and the right edge at $x + \delta x$ from the origin.

During downward displacement of the element, shearing stress at left face is F & at right face is $F + \delta F$. The corresponding internal resisting moment at the left face due to the left hand portion of the beam is M in the counterclockwise direction and at the right hand portion of the beam is $M + \delta M$ in the clockwise direction.

Considering equilibrium of the δx element due to force balance

$$F + \delta F = F + \omega\,\delta x \quad \text{or} \quad \boxed{\frac{\delta F}{\delta x} = \omega}$$



In the limit $\delta x \to 0$, $\frac{dF}{dx} = \omega$.

Considering the moment equation,

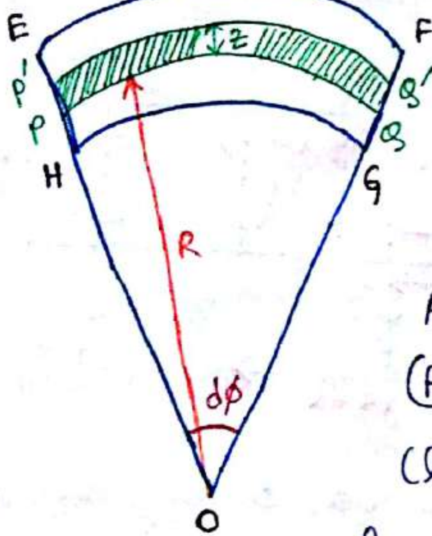
$$\cancel{M} + \delta M + \omega\,\cancel{\delta x}\cdot\frac{\delta x}{2} = \cancel{M} + (F + \cancel{\delta F})\,\delta x$$

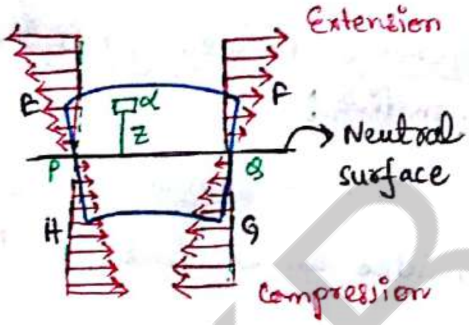
$$\lim_{\delta x \to 0}, \quad \delta M = F\,\delta x \quad \text{or} \quad \boxed{\frac{dM}{dx} = F}$$

We will use this two equations to determine the depression of a loaded beam.

Internal Bending Moment

A body whose length is much greater than its cross-sectional area is called a beam. When such a beam is bent by an applied torque, tensile forces act on some layers of the beam and compressional forces act on other layer, as a result of which, filaments of the beam nearest the outside curve of the bent beam are extended and the filaments nearest the inside curve get compressed. In between them, there is a surface (called Neutral surface) on which the filaments remain unaltered.





Let us consider a small portion of bent beam EFGH with length PQ and breadth EH. O is the center of curvature.

At a distance z from PQ, length of filament is $(R+z)d\phi$ & hence increase in length of filament (linear extension) $= (R+z)d\phi - Rd\phi = z\,d\phi$.

The linear extensional strain $= \frac{z\,d\phi}{R\,d\phi} = \frac{z}{R}$. If α is the cross-section of filament then longitudinal force f to resist elongation $f = \frac{\alpha Y z}{R}$, below neutral surface this is the force to resist compression. If two filaments are equal distance from the neutral axis then they form a couple. For infinitely many equidistant couples comes into play the internal bending moment with same magnitude as external bending moment.

Total internal moment $= \sum\limits_{\substack{\text{all}\\\text{filaments}}} fz = \frac{Y}{R}\sum\limits_{\substack{\text{all}\\\text{filaments}}} \alpha z^2$. $\sum \alpha z^2$ is called the geometrical moment of inertia, having same (equivalence) as moment of inertia with $m \Leftrightarrow \alpha$. So $\sum \alpha z^2 = Ak^2$ where

A is the total area EFGH and K is the radius of gyration of the surface about the neutral axis. $YI = YAK^2$ is called the flexural rigidity. Radius of curvature $\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$ and if bending is small, $\frac{dy}{dx} < 1$ & so

$\frac{1}{R} = \frac{d^2y}{dx^2}$. So internal bending moment $= YI\frac{d^2y}{dx^2}$

for rectangular beam, $A = ab$ and $K^2 = \frac{b^2}{12}$, $a =$ breadth, $b =$ thickness.

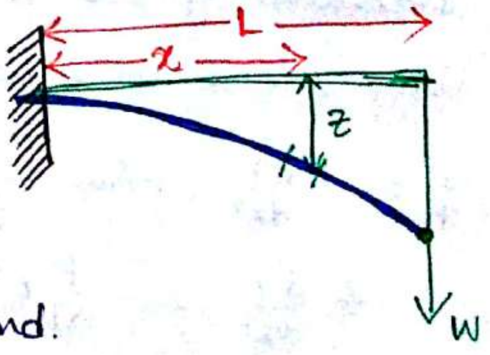
$\therefore$ Bending moment $= \frac{Yab^3}{12R}$.

for circular beam, $A = \pi r^2$, $K^2 = \frac{r^2}{4}$, Bending moment $= \frac{Y\pi r^4}{4R}$.

$$\underbrace{M}_{\text{bending moment}} = \underbrace{\frac{Y}{R}}_{\text{curvature}}\underbrace{\int z^2 dA}_{\text{moment of inertia}}$$

So as the stiffness of the beam is proportional to Y and moment of inertia I, to make the stiffest possible beam with a given amount of steel, the mass has to be as distant from neutral surface gives larger I, but also curvature won't be much due to buckling/twisting. So structural beams are made in the form of I and H.

Cantilever



A cantilever is a uniform beam supported in such a way that both position and slope are fixed in one end (cement wall) and a concentrated force W acts on free end.

What is the shape of the beam $z(x)$?

If beam is long in comparison to cross section, $\frac{1}{R} = \frac{d^2z/dx^2}{\sqrt{\left[1+\left(\frac{dz}{dx}\right)^2\right]^3}} \simeq \frac{d^2z}{dx^2}$

Bending moment M is equal to the torque about the neutral axis of any cross section then $M(x) = W(L-x)$

So from moment equation, $W(L-x) = \frac{YI}{R} = YI\frac{d^2z}{dx^2}$

$\therefore \frac{d^2z}{dx^2} = \frac{W}{YI}(L-x)$, Integrating, $\cancel{\frac{dz}{}}$

$YI\frac{dz}{dx} = W\left(Lx - \frac{x^2}{2}\right) + C_1$. Now at $x=0$, $\frac{dz}{dx} = 0$, so, $C_1 = 0$.

Integrating once again, $YI z = W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$

Again at $x=0$, $z=0$. $\therefore C_2 = 0$. $\therefore \boxed{z = \frac{W}{YI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)}$

Shape of the beam

Displacement of the end is $x = L$.

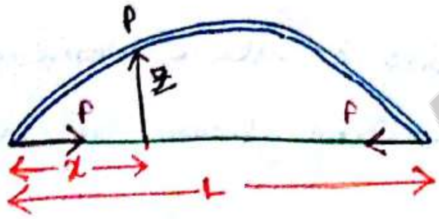
$z = \frac{W}{YI}\frac{L^3}{3}$. Substituting I for rectangular or circular beam, exact expression can be obtained. But $z \propto L^3$.

But crosssection do charge and for incompressible material, $\sigma = 0.5$

$\therefore V = \pi r^2 l$ = constant $\therefore dV = 0 = 2r\,dr\,L + r^2 dL$

$\therefore dL/L = -2\,dr/r$ or $\sigma = -\frac{dr/r}{dL/L} = \frac{1}{2} = 0.5$. (rubber)

Buckling



Consider a straight rod in bent shape by two opposite forces that push the two ends of the rod. What is the shape of the rod and magnitude of force? $\rightarrow$ "Euler force"

Deflection of rod is $z(x)$. So bending moment M at $P = Fz$.

Using the beam equation, $\frac{YI}{R} = Fz$ and for small deflection

$\frac{1}{R} = -\frac{d^2z}{dx^2}$ (minus sign because curvature is downward).

$\therefore \frac{d^2z}{dx^2} = -\frac{F}{YI}z$ — ① which is SHM equation of sine wave. So for small deflection, wavelength λ of sine wave $= 2\times L$.

$z = A\sin\frac{\pi x}{L}$, $\frac{d^2z}{dx^2} = -\frac{A\pi^2}{L^2}\sin\frac{\pi x}{L} = -\frac{\pi^2}{L^2}z$. — ②

① & ② $\therefore \frac{F}{YI} = \frac{\pi^2}{L^2}$ or $\boxed{F = YI\frac{\pi^2}{L^2}}$. So for small bendings the force is independent of the bending displacement z. Below this Euler force, there will be no bending at all & for above this force there will be large amount of bending + buckling. If loading on 2^{nd} floor of a building exceeds Euler force, the building will collapse. 5