Optimal Matrix Multiplication

Optimal Matrix Multiplication

- Given a sequence of matrices, find the most efficient way to multiply these matrices together.

- matrix multiplication is associative.

- no matter how we parenthesize the product, the result will be the same.

- the order in which we parenthesize the product affects the number of simple arithmetic operations needed to compute the product, or the efficiency

Optimal Matrix Multiplication

- Given a sequence of matrices, find the most efficient way to multiply these matrices together.
  - A, B, C are 3 matrices
  - A is a 10 × 30 matrix, B is a 30 × 5 matrix, and C is a 5 × 60 matrix
  - ABC
- matrix multiplication is associative.
  - (AB)C or
  - A(BC)
- no matter how we parenthesize the product, the result will be the same.

Optimal Matrix Multiplication

$$A = \begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{matrix} \qquad B = \begin{matrix} 7 & 8 & 1 \\ 5 & 6 & 2 \end{matrix}$$

$$AB = \begin{matrix} 1*7+2*5 & 1*8+2*6 & 1*1+2*2 \\ 3*7+4*5 & 3*8+4*6 & 3*1+4*2 \\ 5*7+6*5 & 5*8+6*6 & 5*1+6*2 \end{matrix}$$

- 18 Multiplications
- 3*2*3 Multiplications

- A (3 x 2) B (2 x 3) AB=(3 x 3)
- 3*2*3 Multiplications

Optimal Matrix Multiplication

A is a 10 × 30 matrix, B is a 30 × 5 matrix, and C is a 5 × 60 matrix

AB = 10*30*5 multiplications & order will be 10 x 5

- (AB)C = (10*30*5) + (10*5*60) = 1500 + 3000 = 4500 multiplications

BC= 30*5*60 multiplications & order will be 30 x 60

- A(BC) = (30*5*60) + (10*30*60) = 9000 + 18000 = 27000 multiplications

Optimal Matrix Multiplication

Find the optimal matrix multiplication of

A1(10 x100) A2(100 x 5) A3(5 x 50) A4(50 x 1)

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

- $A_1 (A_2 (A_3 A_4))$ - 1750 multiplications
- $(A_1 A_2) (A_3 A_4)$ - 5300 multiplications
- $((A_1 A_2) A_3) A_4$ - 8000 multiplications
- $(A_1 (A_2 A_3)) A_4$ - 75,500 multiplications
- $A_1 ((A_2 A_3) A_4)$ - 31,000 multiplications

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

- Using Dynamic programming

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

Multiplications →

0			
	0		
		0	
			0

Ordering/Pairing →

0			
	0		
		0	
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

- $A_{12} = A_1 A_2$
  $= 10*100*5 = 5000$
- $A_{23} = A_2 A_3$
  $= 100*5*50 = 25000$
- $A_{34} = A_3 A_4$
  $= 5*50*1 = 250$

0			
	0		
		0	
			0

0			
	0		
		0	
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

- $A_{12} = A_1 A_2$
  $= 10*100*5 = 5000$
- $A_{23} = A_2 A_3$
  $= 100*5*50 = 25000$
- $A_{34} = A_3 A_4$
  $= 5*50*1 = 250$

0	5000		
	0	25000	
		0	250
			0

0	A_1A_2 10 x 5		
	0	A_2A_3 100 x 50	
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{13} = A_1A_2A_3$

- $A_{13} = (A_1A_2)A_3$
  =
- $A_{13} = A_1(A_2A_3)$
  =

0	5000		
	0	25000	
		0	250
			0

0	A_1A_2 10 x 5		
	0	A_2A_3 100 x 50	
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{13} = A_1A_2A_3$

- $A_{13} = (A_1A_2)A_3$
  $= 5000+10*5*50 = 7500$
- $A_{13} = A_1(A_2A_3)$
  $= 10*100*50 + 25000 = 75000$

0	5000		
	0	25000	
		0	250
			0

0	A_1A_2 10 x 5		
	0	A_2A_3 100 x 50	
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{13} = A_1A_2A_3$

- $A_{13} = (A_1A_2)A_3$
  $= 5000+10*5*50 = 7500$
- $A_{13} = A_1(A_2A_3)$
  $= 10*100*50 + 25000 = 75000$

0	5000	7500	
	0	25000	
		0	250
			0

0	A_1A_2 10 x 5	$(A_1A_2)A_3$ 10 x 50	
	0	A_2A_3 100 x 50	
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$m[i,j]=m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$

- $A_{13} = (A_1A_2)A_3$

$m[1,3]= m[1,2] +m[3,3]+p_0p_2p_3$

$=5000+0+10*5*50 = 7500$

0	5000	7500	
	0	25000	
		0	250
			0

0	A_1A_2 10 x 5	$(A_1A_2)A_3$ 10 x 50	
	0	A_2A_3 100 x 50	
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{24} = A_2A_3A_4$

- $A_{24} = (A_2A_3)A_4$
  $= 25000 + 100*50*1 = 30000$
- $A_{24} = A_2(A_3A_4)$
  $= 100*5*1 + 250 = 750$

0	5000	7500	
	0	25000	750
		0	250
			0

0	A_1A_2 10 x 5	$(A_1A_2)A_3$ 10 x 50	
	0	A_2A_3 100 x 50	$A_2(A_3A_4)$ 100 x 1
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{14} = A_1A_2A_3A_4$

- $A_{14} = A_1(A_2A_3A_4)$
  =
- $A_{14} = (A_1A_2A_3)A_4$
  =
- $A_{14} = (A_1A_2)(A_3A_4)$
  =

0	5000	7500	
	0	25000	750
		0	250
			0

0	A_1A_2 10 x 5	$(A_1A_2)A_3$ 10 x 50	
	0	A_2A_3 100 x 50	$A_2(A_3A_4)$ 100 x 1
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{14} = A_1A_2A_3A_4$

- $A_{14} = A_1(A_2A_3A_4)$
  $= 10*100*1 + 750 = 1750$
- $A_{14} = (A_1A_2A_3)A_4$
  $= 7500 + 10*50*1 = 8000$
- $A_{14} = (A_1A_2)(A_3A_4)$
  $= 5000 + 250 + 10*5*1 = 5300$

0	5000	7500	
	0	25000	750
		0	250
			0

0	A_1A_2 10 x 5	$(A_1A_2)A_3$ 10 x 50	
	0	A_2A_3 100 x 50	$A_2(A_3A_4)$ 100 x 1
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{14} = A_1A_2A_3A_4$

- $A_{14} = A_1(A_2A_3A_4)$
  $= 10*100*1 + 750 = 1750$
- $A_{14} = (A_1A_2A_3)A_4$
  $= 7500 + 10*50*1 = 8000$
- $A_{14} = (A_1A_2)(A_3A_4)$
  $= 5000 + 250 + 10*5*1 = 5300$

0	5000	7500	1750
	0	25000	750
		0	250
			0

0	A_1A_2 10 x 5	$(A_1A_2)A_3$ 10 x 50	$A_1(A_2(A_3A_4))$ 10 x 1
	0	A_2A_3 100 x 50	$A_2(A_3A_4)$ 100 x 1
		0	A_3A_4 5 x 1
			0

Optimal Matrix Multiplication

$A_1(10 \times 100)$ $A_2(100 \times 5)$ $A_3(5 \times 50)$ $A_4(50 \times 1)$

$A_{14} = A_1A_2A_3A_4$

Answer = $A_1(A_2(A_3A_4))$

- 1750 multiplications

0	5000	7500	1750
	0	25000	750
		0	250
			0

0	A_1A_2 10 x 5	$(A_1A_2)A_3$ 10 x 50	$A_1(A_2(A_3A_4))$ 10 x 1
	0	A_2A_3 100 x 50	$A_2(A_3A_4)$ 100 x 1
		0	A_3A_4 5 x 1
			0

Thank you