Computer Vision 3D (67542): Exercise 4

By: Jeremie Sasson, ID: 336319470

Amit Halbreich, ID: 208917393

1. Figure that describes the scene:

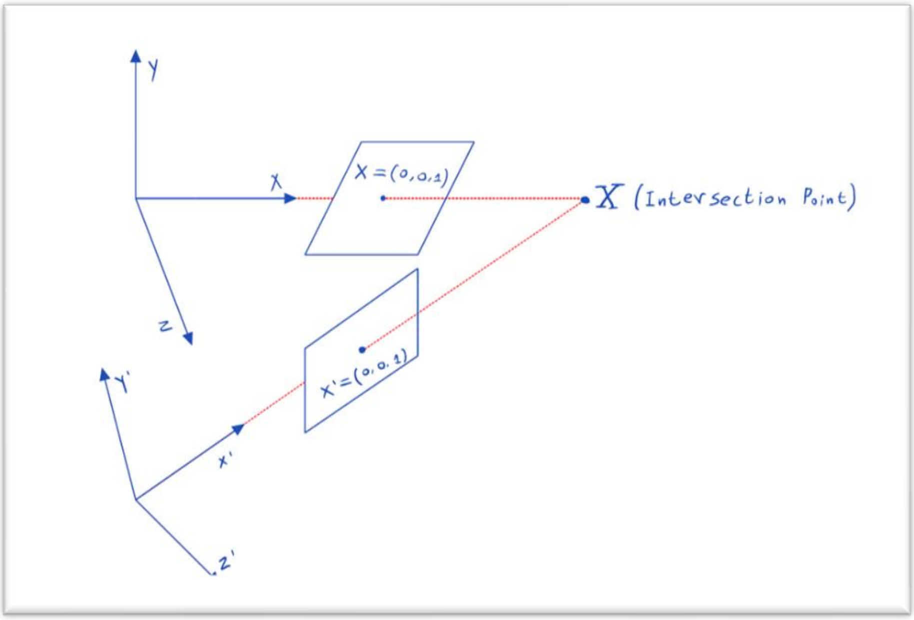


Figure 1: Scene Sketch

We know that for every corresponding points x_i and $x_i{}'$ the following equation holds:

$$x_i^T \cdot F \cdot x_i{}' = 0$$

For two cameras that fixate on a point in space such that their principal axes intersect at the exact point, for normalized image coordinates, if we know that the coordinates origin coincides with the principal point, we have:

$$x_i^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \quad , x_i' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

By multiplying F with x_i' first, we get the 3^{rd} column of F as the multiplication result:

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_{13} \\ f_{23} \\ f_{33} \end{pmatrix} = 0$$

$$f_{13} \cdot 0 + f_{32} \cdot 0 + f_{33} \cdot 1 = 0$$

$$\boxed{f_{33} = 0}$$

We get what we wanted to prove in this question, directly from the equation above.

2. Denote an object O and its reflection in a mirror O'.

We will show that that a camera that views the object O and its reflection O' in plane mirror is equivalent to 2 views of the object situation from 2 different cameras.

Denote a n arbitrary coordinate on the object by x. The same image on the mirror will appear on the coordinate R_f where R_f is the reflection transformation matrix from the object to the mirror.

Denote by P the camera's calibration matrix. Therefore, we have the coordinates $Px, PR_f x$ which represents the coordinates in the obtained image that matches $x, R_f x$ accordingly.

Let's develop the equation of the transformation matrices P, R_f such that they will represent canonical cameras and show that there exists a fundamental matrix F that relates between the 2 matrices.

The reflection transformation matrix R_f could be written as:

$$R_f = \underbrace{\begin{bmatrix} R_{11} & R_{12} & R_{13} & t_1 \\ R_{21} & R_{22} & R_{23} & t_2 \\ R_{31} & R_{32} & R_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{H_e} \cdot \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} R_{11} & R_{12} & R_{13} & t_1 \\ R_{21} & R_{22} & R_{23} & t_2 \\ R_{31} & R_{32} & R_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}}_{H_e^{-1}}$$

Where H_e represents "Euaclides" matrix composed from Rotation Matrix R and translation vector $\vec{t}$.

The matrix P can be expressed as:

$$(1) \quad P = K[I \mid 0]$$

Therefore we have:

$$(2) \quad PR_f = K[I \mid 0] \cdot H_e A H_e^{-1} = K[RAR^T \mid -RAR^T + t] = K[RAR^T \mid RBR^T t]$$

Where $B = I - A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Now we wish to find the matrix that links between the 2 matrices P and PR_f, according to page 254 in the course book we have:

Result 9.9. *The fundamental matrix corresponding to a pair of camera matrices* $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$ *and* $\mathtt{P}' = [\mathtt{M} \mid \mathbf{m}]$ *is equal to* $[\mathbf{m}]_\times \mathtt{M}$.

In our case $M = RAR^T$ and $m' = KRBR^T t$, thus we get the relation for the fundamental matrix F:

$$F = [KRBR^T t]_\times \, [KRAR^T K^{-1}]$$

Here is the python code that calculates the fundamental matrix F:

```
import sympy as sp

# Symbolic variables
k11, k12, k13 = sp.symbols('k11 k12 k13')
k21, k22, k23 = sp.symbols('k21 k22 k23')
k31, k32, k33 = sp.symbols('k31 k32 k33')

r11, r12, r13 = sp.symbols('r11 r12 r13')
r21, r22, r23 = sp.symbols('r21 r22 r23')
r31, r32, r33 = sp.symbols('r31 r32 r33')

t1, t2, t3 = sp.symbols('t1 t2 t3')

# K matrix
K = sp.Matrix([[k11, k12, k13], [k21, k22, k23], [k31, k32, k33]])

# R matrix
R = sp.Matrix([[r11, r12, r13], [r21, r22, r23], [r31, r32, r33]])

# t vector
t = sp.Matrix([[t1], [t2], [t3]])

# Calculate F
B = sp.Matrix([[2, 0, 0], [0, 0, 0], [0, 0, 0]])
A = sp.Matrix([[-1, 0, 0], [0, 1, 0], [0, 0, 1]])

R_T = R.T
inter_term = K * R * B * R_T * t
right_term = K * R * A * R_T * K.inv()
left_term = sp.Matrix([[0, right_term[2], -right_term[1]], [-
right_term[2], 0, right_term[0]], [right_term[1], -right_term[0],
0]])
F = left_term * right_term
print(F)
```

We got the result:

```
[[0, -k13*(k31*t1 + k32*t2 + k33*t3), k12*(k31*t1 + k32*t2 +
k33*t3)],
 [k13*(k31*t1 + k32*t2 + k33*t3), 0, -k11*(k31*t1 + k32*t2 +
k33*t3)],
[-k12*(k31*t1 + k32*t2 + k33*t3), k11*(k31*t1 + k32*t2 + k33*t3), 0]]
```

$$f_{12} = -f_{21}$$

$$f_{13} = -f_{31}$$

$$f_{23} = -f_{32}$$

As we can see, the matrix is skew-symmetric because we got the **skew-symmetric property** for each pair of elements in the matrix F we have:

$$\boldsymbol{f_{ij} = -f_{ji}}$$

In other words, F satisfies the equation $x^T F x = 0$. Let's show that equation holds:

$$(x_1 \quad x_2 \quad x_3) \cdot \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [-ax_2 - bx_3 \quad ax_1 - cx_3 \quad bx_1 + cx_2] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Where $F = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

Which simplifies to:

$$= -ax_2x_1 - bx_3x_1 + ax_1x_2 - cx_3x_2 + bx_1x_3 + cx_2x_3 = 0$$

These properties holds:

Pure Translation: Our case is similar to pure translation. Both matrices are **Skew-Symmetric** as well as they are **Auto-Epipolar**. In both cases, the matrices have two degrees of freedom.

Pure Planar Motion: The case is not similar to the pure planar motion case. While in our matrix there are two degrees of freedom, in the pure planar motion case, there are 5.

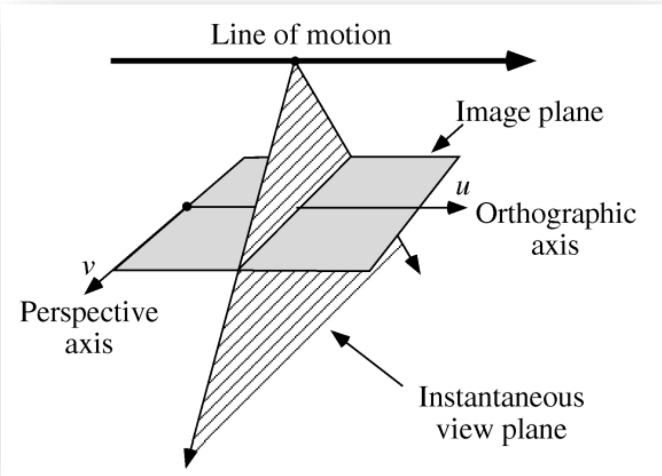
3. If the vanishing line of a plane contains the epipole, it means that the vanishing point in the baseline's direction lies on this vanishing line. Parallel planes intersect in a common line in the plane at infinity (Π_∞), and the image of this line is the vanishing line of the plane. Therefore, if the vanishing point in the baseline's direction lies on the vanishing line of a plane, it implies that the baseline must lie in a plane parallel to the given plane.

4. This question was a bit confusing, we have to approaches to solve it –

- 1. What we understood regarding the article and section 9.2.2 in the study book.
- 2. Solution that came up after discussing the question with Mike in class.

Solution 1 – according to the book and the article:

We will mention several facts about the geometry of the fundamental matrix F in the pushbroom experiment. The model is described in the image below:



In this model, we assume that camera motion is planar and perpendicular to the image plane, and it is constant. It is observed that the form of the matrix F is:

$$F = \begin{bmatrix} 0 & 0 & f_{13} & f_{14} \\ 0 & 0 & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix}$$

Furthermore, as mentioned in the article, it is known that the "epipolar line" is actually a hyperbola in the case of the pushbroom.

To demonstrate this, let's perform the calculation. It is observed that a point (u, v) in the image can be represented as (u, uv, v). The corresponding point (u', v') can be represented as $(u', u'v', v')$. Therefore, based on the definition of the fundamental matrix, we have:

$$(u', u'v', v', 1)^T F(u, uv, v, 1) = 0$$

$$\rightarrow \quad (u', u'v', v', 1)^T \begin{bmatrix} 0 & 0 & f_{13} & f_{14} \\ 0 & 0 & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} (u, uv, v, 1) = 0$$

which simplifies to:

$$u'(vf_{13} + f_{14}) + u'v'(vf_{23} + f_{24}) + v'(uf_{31} + uvf_{32} + vf_{33} + f_{34}) + f_{31} + f_{32} + f_{33} + f_{34}) = 0$$

Now, after rearranging the terms, we can observe that this is equivalent to the expression:

$$au' + bu'v' + cv' + d = 0$$

which represents the equation of a hyperbola as required.

Additionally, in Section 9.2.2, it is shown that there is another representation for the fundamental matrix in the general epipolar geometry given two matrices representing the cameras P and P'. The projection of a ray x by the camera P is defined as:

$$PX = x$$

In other words, the solution family takes the form:

$$X\lambda = P^+x + \lambda C$$

where C is the zero vector of P.

The ray is defined by the scalar λ, and specifically, we can identify two points on the ray: P^+x and C. Therefore, the epipolar line (in our case, a hyperbola) connects these two points:

$$l' = (P'C)_\times (P'P^+x)$$

The point $P'C$ represents the epipole of the second camera, and thus, we can derive the equation for the fundamental matrix:

$$l' = [e']_\times (P'P^+)x = Fx \leftrightarrow F = [e']_\times (P'P^+)$$

We have seen that we have found two different ways to express the fundamental matrix in the pushbroom. These forms allow us to infer information about the cameras and the epipole on the matrix F and vice versa.

Solution 2 - According to Mike's explanation after discussion in class:

We know that Linear Pushbrooms cameras are projective in one coordinate and orthographic on the other coordinate.

Therefore, we have the following equation for an arbitrary 3D world coordinate $A = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for projective transformation P:

$$P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ \frac{y}{z} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\rightarrow \quad \begin{cases} x = u \\ y = vz \end{cases}$$

The operator P is non-linear. Denote 2 linear pushbrooms cameras by C and C'.

Let's assume, without loss of generality that camera C is located in the origin $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with no rotation at all. Let's define the other camera's location and orientation C' to be with addition of translation vector $\vec{t}$ and rotation matrix R.

If a coordinate $B = \begin{bmatrix} u \\ v \end{bmatrix}$ in the second camera's C' image, then the epipolar line in the first camera's image can be written as:

$$P(RP^{-1}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) + \vec{t}\,)$$

$$\rightarrow \quad \boxed{\left\{\boldsymbol{P}\left(\boldsymbol{R}\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v\lambda} \\ \boldsymbol{\lambda} \end{bmatrix} + \vec{\boldsymbol{t}}\right), \qquad \forall \boldsymbol{\lambda} \in \mathbb{R}\right.}$$