Asymptotic Properties of Powers of Nonnegative Matrices, with Applications

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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ABSTRACT

Let A be a finite square (reducible) nonnegative matrix. The main theorem of the paper gives first order approximations of $f_n(A)$, for certain sequences f_n of analytic functions. In particular, the theorem holds when $f_n(A) = A^n$. The theorem is applied to study the local behavior of large powers of a nonnegative matrix, to study the limiting output vectors of a nonnegative multiplicative process, and to characterize the nonnegative eigenvectors of a nonnegative matrix. Finally, an application to absorbing Markov chains is given.

1. INTRODUCTION

In the study of analytic functions (in particular powers) of a square nonnegative matrix A, we can without loss of generality assume that the rows and columns of A are ordered so that A is given in the so-called Frobenius normal form (1.1), where the A_{jj} are either nonzero irreducible nonnegative matrices or A_{jj} is a 1×1 matrix with the single entry 0 (which can be viewed as a degenerate irreducible nonnegative matrix):

$$A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \\ A_{s1} & A_{s2} & A_{s3} & \cdots & A_{ss} \end{bmatrix}. \tag{1.1}$$

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Of course, $s=1$ corresponds to the case when A itself is *irreducible*. Otherwise, A is called *reducible*.

Throughout the paper we shall think of A as *fixed*, given by (1.1). The *entries* of A are denoted A_{xy}, while the blocks defined by (1.1) are denoted A_{jk}. Thus when reading the paper one should have in mind that j, k are *block* indices, while x, y are *entry* indices.

As A is considered fixed, we may without confusion for any matrix B of the same dimension as A use B_{jk} to denote the *block* of B corresponding to the same indices as A_{jk} defined by (1.1). This will be done throughout the paper.

The basic theorem of the paper is Theorem 3.1, giving first order approximations of certain sequences $f_n(A)$ of analytic functions of A (in particular, the result applies to powers A^n). Separate approximations are given for each of the blocks

$$[f_n(A)]_{jk}, \qquad j=k, k+1, \dots,$$

in terms of the dominating eigenvalues and the corresponding eigenprojections of the *diagonal* blocks A_{jj}. The theorem thus generalizes the main result of Mandl [11], who considered only *aperiodic* A and required nondegenerate diagonal blocks A_{jj}. The aperiodic case was also considered by Lindqvist [9].

Studies in the same direction as ours have been performed by Rothblum [13, 14] and Friedland and Schneider [6]. However, they do not give as explicit results as we do in this paper.

We prove our Theorem 3.1 by first stating and proving a more general result, Theorem 2.1, concerning matrices of the form (1.1), but not necessarily nonnegative. Roughly, this theorem shows how one may obtain the dominant spectral properties of the blocks A_{jk} from the corresponding spectral properties of the diagonal blocks A_{jj}.

Theorem 2.1 is proved by using simple spectral properties of finite dimensional linear operators, given for example in Dunford and Schwarz [5]. As will be seen, this spectral approach is much more powerful than the more analytical approaches of [11] and [6]. Roughly speaking, while analytical proofs give results for eigenvalues of dominant absolute value only, our Theorem 2.1 give corresponding results for *any* eigenvalue of A. A similar approach was used in the recent paper [10] by the present author. In addition, we utilize a graph theoretic description of the problem, essentially corresponding to the one used by Friedland and Schneider [6].

Sections 4–6 are devoted to applications of Theorem 3.1.

In Section 4 we study the local behavior of the powers A^n at single entries (x, y), as well as at blocks (j, k). In particular we study the local periods of A introduced by Friedland and Schneider in [6]. However, [6] in effect studies merely the periods of A at the blocks (j, k). In the present paper we improve their notion of local period to define periods at single entries as well. Moreover, we give a theorem throwing additional light on local periods at the block level.

Section 5 contains an application of Theorem 3.1 to multiplicative processes, i.e. processes in which a vectorial input p in n time units is transformed into the output vector pA^n $(n = 1, 2, \dots, n)$. The purpose of the section is to study the asymptotic behavior of pA^n as n tends to infinity. In particular we obtain a sufficient condition for a class of input vectors p to have the same (normalized) limiting output vector.

In Section 6 we characterize, using results of Section 5, the nonnegative eigenvectors of a nonnegative matrix A.

Square nonnegative matrices and functions of them occur in a variety of problems from fields such as numerical analysis, mathematical economics, and dynamic programming (see e.g. Seneta [16]). An application of results related to ours is given in the final section of [6]. Further examples, concerning multiplictive processes, are mentioned by Rothblum and Tan [15]. The main motivation for the present work has been to study the asymptotic properties of reducible Markov chains. An outline of such applications is given in Section 7.

2. A BASIC RESULT

Let A be an $q \times q$ complex matrix (q finite). Let $\sigma(A)$ denote the spectrum of A. For $\lambda \in \sigma(A)$, let

$$X_n^\lambda = \{ z \in C^q : (A - \lambda I)^n z = 0 \}, \qquad n = 1, 2, \dots .$$

(Here C^q is the q-dimensional complex space.) The index of λ, $\nu(\lambda, A)$, is defined as the smallest n for which $X_n^\lambda = X_{n+1}^\lambda$. [When no confusion can arise, we shall often write $\nu(\lambda)$ instead of $\nu(\lambda, A)$.] $X^\lambda \equiv X_{\nu(\lambda)}^\lambda$ is called the *algebraic eigenspace* of λ. Note that C^q is the direct sum of the spaces X^λ [5].

Let $f(z)$ be a complex function analytic in some open set containing $\sigma(A)$. Then $f(A)$ can be defined, and by Theorem 8, Chapter VII in [5],

$$f(A) = \sum_{\lambda \in \sigma(A)} \sum_{\alpha = 0}^{\nu(\lambda) - 1} \frac{f^{(\alpha)}(\lambda)}{\alpha!} (A - \lambda I)^\alpha Z^\lambda, \tag{2.1}$$

where Z^λ, $\lambda \in \sigma(A)$ (called the *eigenprojection* of A at λ) is a nonzero complex polynomial in A satisfying

$$(Z^\lambda)^2 = Z^\lambda, \qquad Z^\lambda Z^{\lambda'} = 0, \qquad \sum_{\lambda \in \sigma(A)} Z^\lambda = I,$$

$$(A - \lambda I)^{\nu(\lambda)-1} Z^\lambda \neq 0, \qquad (A - \lambda I)^{\nu(\lambda)} Z^\lambda = 0 \tag{2.2}$$

whenever $\lambda, \lambda' \in \sigma(A)$, $\lambda \neq \lambda'$. Here $f^{(j)}$ is the jth derivative of f, and $B^0 = I$ (identity matrix) for any square matrix B. It can be shown that

$$X^\lambda = \{ Z^\lambda z : z \in C^q \}.$$

We shall for simplicity let $C^{\alpha,\lambda} = (A - \lambda I)^{\alpha-1} Z^\lambda$ for $\lambda \in \sigma(A)$, $\alpha = 1, 2, \dots$ [so $C^{\alpha,\lambda} = 0$ for $\alpha > \nu(\lambda)$]. Whenever necessary, we will write $Z^\lambda(A)$ or $C^{\alpha,\lambda}(A)$ to emphasize the underlying matrix.

Suppose now A is partitioned into blocks as in (1.1), and let $Z^\lambda, C^{\alpha,\lambda}$ be partitioned in the same way.

The graph-theoretic description below turns out to be convenient. It is essentially similar to the one used in [6]. For $s \geqslant j \geqslant k \geqslant 1$ we define a path from j to k to be a sequence $h = (k_1, \dots, k_p)$ where $p \geqslant 1$, $s \geqslant j = k_1 > k_2 > \cdots > k_p = k \geqslant 1$, and $A_{k_l k_{l+1}} \neq 0$ for $l = 1, \dots, p-1$. [If $j = k$, then $h = (j)$ is the only possibility.] The *support* of h is the set $\operatorname{supp} h = \{k_1, \dots, k_p\}$. The set of paths h from j to k (which may be empty) is denoted H_{jk}. We let

$$\operatorname{supp} H_{jk} = \bigcup_{h \in H_{jk}} \operatorname{supp} h,$$

$$\sigma_{jk} = \bigcup_{l \in \operatorname{supp} H_{jk}} \sigma(A_{ll}).$$

For $\lambda \in \sigma(A)$, $h \in H_{jk}$, let $\alpha(\lambda, h) = \sum_{l=1}^{p} \nu(\lambda, A_{k_l k_l})$, where by convention we put $\nu(\lambda, A_{jj}) = 0$ if $\lambda \notin \sigma(A_{jj})$. Let now $\alpha^*(\lambda; j, k) = \max\{\alpha(\lambda, h) : h \in H_{jk})\}$. We call $\alpha^*(\lambda; j, k)$ the λ-distance from j to k. Let $H_{jk}^*(\lambda) = \{h \in H_{jk} : \alpha(\lambda, h) = \alpha^*(\lambda; j, k)\}$. The paths in $H_{jk}^*(\lambda)$ are called λ-*maximal paths*.

The main result of this section is the following:

THEOREM 2.1. *Let A be as in* (1.1) *and let $\lambda \in \sigma(A)$. Let $s \geqslant j \geqslant k \geqslant 1$ be given, and put $\alpha^* = \alpha^*(\lambda; j, k)$, $\tau_j = \nu(\lambda, A_{jj})$. If $\alpha^* \geqslant 1$, then*

$$C_{jk}^{\alpha^*,\lambda} = \sum_{h \in H_{jk}^*(\lambda)} R_{k_1} A_{k_1 k_2} R_{k_2} \cdots A_{k_{p-1} k_p} R_{k_p},$$

where $h = (k_1, \dots, k_p)$ and

$$R_l = \begin{cases} C^{\tau_l,\lambda}(A_{ll}) & \text{if } \lambda \in \sigma(A_{ll}), \\ (\lambda I - A_{ll})^{-1} & \text{if } \lambda \notin \sigma(A_{ll}). \end{cases}$$

Moreover, $C_{jk}^{r,\lambda} = 0$ for all $r > \alpha^*$. If $\alpha^* = 0$, then $C_{jk}^{\alpha,\lambda} = 0$ for all $\alpha = 1, 2, \dots$.

In order to prove the theorem we need to introduce some new notation and some lemmas.

A matrix B partitioned as in (1.1) is called a *path matrix* if $B_{jk} = 0$ for $j - k \geqslant 2$ and $B_{j+1,j} \neq 0$ for $j = 1, 2, \dots, s-1$ (the diagonal blocks B_{jj} may be 0 or $\neq 0$).

Let A be as in (1.1), and let $H = (k_1, \dots, k_p) \in H_{jk}$. Then we shall let $A(h)$ denote a path matrix with

$$B_{jj} = A_{k_j k_j}, \quad B_{j+1,j} = A_{k_{j+1}, k_j}, \qquad j = 1, 2, \dots, p.$$

(This explains the term "path matrix," which is adopted from [6].)

From the definition of matrix multiplication it follows that

$$A_{jk}^n = \sum_{h \in H_{jk}} \left[A(h)^n \right]_{p,1} \tag{2.3}$$

where the sum is over $h = (k_1, \dots, k_p)$ (which explains the p above). Thus, essentially, in a study of the A_{jk}^n one need only consider the path matrices (this fact was also used in [11] and [6]). Our lemma below shows that such a decomposition holds for any matrix $C^{\alpha,\lambda}$ in the spectral decomposition (2.1). Thus in the proof of Theorem 2.1 we can essentially restrict attention to path matrices A.

LEMMA 2.2. *Let A be as in (2.3), and let $\lambda \in \sigma(A)$. Then for $s \geq j > k \geq 1$ we have*

$$C_{jk}^{\alpha,\lambda} = \sum_{h \in H_{jk}} \left[C^{\alpha,\lambda}(A(h))\right]_{p1},$$

where the sum is over all $h = (k_1, \ldots, k_p) \in H_{jk}$.

Proof. By (2.1),

$$A_{jk}^n = \sum_{\lambda \in \sigma(A)} \sum_{\alpha=0}^{\nu(\lambda)-1} \binom{n}{\alpha} \lambda^{n-\alpha} C_{jk}^{\alpha+1,\lambda}.$$

On the other hand,

$$\begin{aligned} A_{jk}^n &= \sum_{h \in H_{jk}} \left[A(h)^n\right]_{p1} \\ &= \sum_{h \in H_{jk}} \sum_{\lambda \in \sigma(A)} \sum_{\alpha=0}^{\nu(\lambda)-1} \binom{n}{\alpha} \lambda^{n-\alpha} \left[C^{\alpha+1,\lambda}(A(h))\right]_{p1}. \end{aligned}$$

The lemma is obtained by equating coefficients of the functions $\binom{n}{\alpha}\lambda^{n-\alpha}$, which are linearly independent as functions of n. ∎

The following lemma is well known. For a proof, see e.g. [2].

LEMMA 2.3. *Let*

$$T = \begin{bmatrix} U & 0 \\ K & V \end{bmatrix},$$

and assume that $\nu(\lambda, U) = \tau \geq 0$, $\nu(\lambda, V) = \gamma \geq 0$. Then $\max\{\tau, \gamma\} \leq \nu(\lambda, T) \leq \tau + \gamma$.

LEMMA 2.4. *Let T be block-partitioned as*

$$T = \begin{bmatrix} U & 0 \\ K & V \end{bmatrix},$$

and let $\nu(\lambda, U) = \tau \geq 0$.

(i) *If* $\nu(\lambda, V) = 0$, *then* $\nu(\lambda, T) = \tau$ *and*

$$C^{\tau,\lambda}(T) = \begin{bmatrix} C^{\tau,\lambda}(U) & 0 \\ (\lambda I - V)^{-1} K C^{\tau,\lambda}(U) & 0 \end{bmatrix}$$

(ii) *If* $\nu(\lambda, V) = \gamma \geqslant 1$, *then*

$$\textit{if} \quad \tau \geqslant 1, \qquad C^{\tau+\gamma,\lambda}(T) = \begin{bmatrix} 0 & 0 \\ C^{\gamma,\lambda}(V) K C^{\tau,\lambda}(U) & 0 \end{bmatrix}$$

and

$$\textit{if} \quad \tau = 0, \qquad C^{\gamma,\lambda}(T) = \begin{bmatrix} 0 & 0 \\ C^{\gamma,\lambda}(V) K (\lambda I - U)^{-1} & C^{\gamma,\lambda}(V) \end{bmatrix}.$$

If $\tau \geqslant 1$, *then* $\nu(\lambda, T) < \tau + \gamma$ *or* $= \tau + \gamma$ *according as* $C^{\tau+\gamma,\lambda}(T) = 0$ *or* $\neq 0$. *If* $\tau = 0$, *then* $\nu(\lambda, T) = \gamma$.

Proof. (i): We can assume here that $\tau \geqslant 1$. By Lemma 2.3 we have $\nu(\lambda, T) = \tau$ and thus

$$(T - \lambda I) C^{\tau,\lambda}(T) = 0. \tag{2.4}$$

Moreover, for some K^*,

$$C^{\tau,\lambda}(T) = \begin{bmatrix} C^{\tau,\lambda}(U) & 0 \\ K^* & 0 \end{bmatrix},$$

so (2.4) implies $K C^{\tau,\lambda}(U) + (V - \lambda I) K^* = 0$, giving

$$K^* = (\lambda I - V)^{-1} K C^{\tau,\lambda}(U).$$

(ii): Suppose first $\tau \geqslant 1$. From Lemma 2.3 it follows that $\nu(\lambda, T) \leqslant \tau + \gamma$. Clearly, for some K^*,

$$C^{\tau+\gamma,\lambda}(T) = \begin{bmatrix} 0 & 0 \\ K^* & 0 \end{bmatrix}.$$

Using $C^{\tau+\gamma,\lambda}(T) = Z^{\lambda}(T)C^{\tau+\gamma,\lambda}(T)$, we get

$$K^* = Z^{\lambda}(V)K^*. \tag{2.5}$$

Writing

$$C^{\tau,\lambda}(T) = \begin{bmatrix} C^{\tau,\lambda}(U) & 0 \\ K^{**} & R \end{bmatrix}$$

and using $C^{\tau+\gamma,\lambda}(T) = (T-\lambda I)^{\gamma}C^{\tau,\lambda}(T)$, we get

$$K^* = \tilde{K}^{\gamma}C^{\tau,\lambda}(U) + (V-\lambda I)^{\gamma}K^{**}, \tag{2.6}$$

where $\tilde{K}^{\gamma}$ is the block corresponding to K in $(T-\lambda I)^{\gamma}$. Multiplication from left by $Z^{\lambda}(V)$ and using (2.5) in (2.6) yields

$$K^* = Z^{\lambda}(V)\tilde{K}^{\gamma}C^{\tau,\lambda}(U). \tag{2.7}$$

Now from $(T-\lambda I)^{\gamma} = (T-\lambda I)^{\gamma-1}(T-\lambda I)$ we get

$$\tilde{K}^{\gamma} = \tilde{K}^{\gamma-1}(U-\lambda I) + (V-\lambda I)^{\gamma-1}K.$$

Substituting this into (2.7), we get

$$K^* = C^{\gamma,\lambda}(V)KC^{\tau,\lambda}(U).$$

The case $\tau = 0$ is similar to (i). ■

LEMMA 2.5. *Let A be a path matrix where the diagonal blocks A_{jj} are all nonsingular. Then for $j > k$ we have*

$$(A^{-1})_{jk} = (-1)^{j-k}A_{jj}^{-1}A_{j,j-1}A_{j-1,j-1}^{-1}\cdots A_{k+1,k}A_{kk}^{-1}.$$

Proof. The proof is straightforward and is omitted. ■

LEMMA 2.6. *Let A be a path matrix on the form* (1.1). *Let $\lambda \in \sigma(A)$, and put $\tau_j = \nu(\lambda, A_{jj})$. Then*:

(i) $\nu(\lambda, A) \leqslant \sum_{j=1}^{n} \nu(\lambda, A_{jj}) \equiv \tau$.
(ii) *We have*

$$C_{s1}^{\tau,\lambda} \equiv D_s = R_s A_{s,s-1} R_{s-1} A_{s-1,s-2} \cdots A_{21} R_1,$$

where

$$R_j = \begin{cases} C^{\tau_j \lambda}(A_{jj}) & \text{if } \lambda \in \sigma(A_{jj}), \\ (\lambda I - A_{jj})^{-1} & \text{if } \lambda \notin \sigma(A_{jj}). \end{cases}$$

Proof. (i) follows easily from Lemma 2.3. To prove (ii), let $m \geqslant 1$ be the smallest index with $\lambda \in \sigma(A_{mm})$. It is clear that $m \leqslant s$. We prove first that (ii) holds for $s = m$. If $A_{(j)}$ denotes the matrix consisting of the upper left j times j blocks in A, then we can write

$$A_{(m)} = \left[\begin{array}{cccc|c} & & A_{(m-1)} & & 0 \\ \hline 0 & \cdots & 0 & A_{m,m-1} & A_{m,m} \end{array}\right],$$

so Lemma 2.4 yields

$$D_m = C^{\tau,\lambda}(A_{mm})\begin{bmatrix} 0 & \cdots & 0 & A_{m,m-1} \end{bmatrix}\left[(\lambda I - A_{(m-1)})^{-1}\right]_1,$$

where $[U]_1$ means the first column of blocks of the matrix U. Now we apply Lemma 2.5 to the path matrix $(\lambda I - A_{(m-1)})^{-1}$. Note that the powers of -1 in Lemma 2.5 disappear because the blocks of $(\lambda I - A_{(m-1)})^{-1}$ below the diagonal have a minus sign in front of them. Thus (ii) follows. That (ii) also holds for $s = m+1,\ m+2, \ldots$ follows easily by induction using Lemma 2.4. ∎

Proof of Theorem 2.1. Lemma 2.2 permits us to consider separately the path matrices $A(h)$ for each $h \in H_{jk}$. Pick therefore $h \in H_{jk}$. Then Lemma 2.6 gives an expression for $[C^{\alpha,\lambda}(A(h))]_{p1}$ when $\alpha = \sum_{l=1}^{p} \nu(\lambda, A_{k_l k_l})$. Moreover, this expression is of the form announced in the theorem. Also, Lemma

2.6 gives us $[C^{\alpha+1,\lambda}(A(h))]_{p1}=0$. But then, noting the definition of $\alpha^*(\lambda;\, j,k)$, Theorem 2.1 is seen to hold. ■

3. APPLICATION TO REDUCIBLE NONNEGATIVE MATRICES. THE MAIN THEOREM

We shall say that a matrix A is *nonnegative* if $A_{xy} \geqslant 0$ for all x, y. A will be called *strictly positive*, written $A \gg 0$, if $A_{xy} > 0$ for all x, y.

It is well known by the Perron-Frobenius theory of nonnegative matrices (see e.g. Seneta [16]) that any square nonnegative matrix $A \neq 0$ has a nonnegative eigenvalue μ with $|\lambda| \leqslant \mu$ for all $\lambda \in \sigma(A)$. The value μ will be called the *Perron-Frobenius value* (PF value) of A.

Let in the following A be a nonnegative square matrix of the Frobenius normal form (1.1). Let μ_j denote the PF value of A_{jj} [where $\mu_j \equiv 0$ if $A_{jj}=(0)$]. For $A_{jj} \neq (0)$ there exists an integer $d_j \geqslant 1$ (the period of A_{jj}) such that the d_j complex roots of the equation $z^{d_j} - \mu_j^{d_j} = 0$ are all simple eigenvalues (i.e. of algebraic multiplicity 1) of A_{jj} and every other eigenvalue $\lambda \in \sigma(A_{jj})$ satisfies $|\lambda| < \mu_j$. Moreover, to each $\lambda \in \sigma(A_{jj})$ with $|\lambda| = \mu_j$ correspond unique (up to constant multiples) and strictly nonzero (i.e. every entry is $\neq 0$) right and left eigenvectors. We shall let u_j^λ (${}^t v_j^\lambda$) denote the right (left) eigenvectors, scaled so that ${}^t v_j^\lambda u_j^\lambda = 1$ (where t denotes transpose). For $\lambda = \mu_j$, the eigenvectors are strictly positive. Now we have

$$Z^\lambda(A_{jj}) = u_j^\lambda\, {}^t v_j^\lambda.$$

In the degenerate case $A_{jj}=(0)$ we shall put $d_j = 1$, $\mu_j = 0$, $Z^0(A_{jj}) = (1)$. By this convention it is seen that, in particular, (2.1) and (2.2) hold. Thus Theorem 2.1 also covers this degenerate case.

Consider now a block (j,k). Let $\mu_{jk} = \max\{\mu_l : l \in \operatorname{supp} H_{jk}\}$ and let $\tau_{jk} = \alpha^*(\mu_{jk};\, j,k)$. In words, μ_{jk} is the largest PF value occurring among the matrices A_{ll} located at some path from j to k. Also, τ_{jk} is the μ_{jk}-distance from j to k. Note that $\nu(\mu_{jk}, A_{ll}) \leqslant 1$ for all $l \in \operatorname{supp} H_{jk}$, so τ_{jk} is simply the largest possible number of matrices with PF value μ_{jk} on paths from j to k.

Clearly, for all $\lambda \in \sigma_{jk}$ we have $|\lambda| \leqslant \mu_{jk}$. Suppose now that $\lambda \in \sigma_{jk}$ satisfies $|\lambda| = \mu_{jk}$, $\lambda \neq \mu_{jk}$. Then for some $d \geqslant 2$, $\lambda^d = \mu_{jk}^d$. Moreover, it is clear that $\alpha^*(\lambda;\, j,k) \leqslant \alpha^*(\mu_{jk}, j,k) \equiv \tau_{jk}$, since $\lambda \in \sigma(A_{ll}) \Rightarrow \mu_{jk} \in \sigma(A_{ll})$ (see also [12]). Let now $\sigma_{jk}^* = \{\lambda \in \sigma_{jk} : |\lambda| = \mu_{jk},\ \alpha^*(\lambda;\, j,k) = \tau_{jk}\}$, and for $\lambda \in \sigma_{jk}^*$ let d_λ denote the smallest d for which $\lambda^d = \mu_{jk}^d$. Then (Rothblum

[12]) $\lambda \in \sigma_{jk}^*$ if and only if there exists $h \in H_{jk}^*(\mu_{jk})$ such that d_λ divides the greatest common divisor of the integers.

$$\{d_l : l \in \operatorname{supp} h,\ \mu_l = \mu_{jk}\}.$$

In the degenerate case where $\mu_{jk} = 0$ it is clear that $\sigma_{jk}^* = \sigma_{jk} = \{0\}$.

The following theorem is obtained by applying Theorem 2.1 to the A considered in this section.

THEOREM 3.1. *Let A be a nonnegative matrix written in the Frobenius form (1.1). Let $\{f_n\}$ be a sequence of complex functions analytic in an open set Γ containing $\sigma(A)$. Suppose $\{f_n\}$ has the property that for $z_1, z_2 \in \Gamma$ we have $f_n^{(\alpha)}(z_1) = o(f_n^{(\beta)}(z_2))$ as $n \to \infty$ for all $\alpha, \beta = 0, 1, 2, \ldots$ if $|z_1| < |z_2|$ and for all $\alpha < \beta$ if $|z_1| = |z_2|$. Let $s \geq j \geq k \geq 1$ be given, with $H_{jk} \neq \varnothing$. Put, for simplicity, $\mu = \mu_{jk}$, $\tau = \tau_{jk}$. Then as $n \to \infty$*

$$[f_n(A)]_{jk} = \sum_{\lambda \in \sigma_{jk}^*} \frac{f_n^{(\tau-1)}(\lambda)}{(\tau-1)!} C_{jk}^{\tau,\lambda} + o\big(f_n^{(\tau-1)}(\mu)\big), \tag{3.1}$$

where

$$\begin{aligned} C_{jk}^{\tau,\lambda} &= \sum_{h \in H_{jk}^*(\lambda)} R_{k_1} A_{k_1 k_2} R_{k_2} \cdots A_{k_{p-1} k_p} R_{k_p}, \\ h &= (k_1, \ldots, k_p), \\ R_j &= \begin{cases} Z^\lambda(A_{jj}) & \text{if } \lambda \in \sigma(A_{jj}), \\ (\lambda I - A_{jj})^{-1} & \text{if } \lambda \notin \sigma(A_{jj}). \end{cases} \end{aligned} \tag{3.2}$$

Moreover,

$$C_{jk}^{\tau,\mu} \gg 0.$$

REMARK. The case $\mu_{jk} = 0$ is not excluded in the theorem. Recall that if $A_{jj} = (0)$ we have $Z^0(A_{jj}) = (1)$.

Proof. The result follows from Theorem 2.1, except for the fact that $C_{jk}^{\tau,\mu} \gg 0$. This is seen as follows. We know that $Z^\mu(A_{jj}) \gg 0$ when $\mu \in$

$\sigma(A_{jj})$. Now if $\mu \notin \sigma(A_{jj})$ we have $\mu_j < \mu$ and so, as is well known, $(\mu I - A_{jj})^{-1} = \mu^{-1}\sum_{n=0}^{\infty} A_{jj}^n$. That each entry of this matrix is strictly positive is a consequence of the irreducibility of A_{jj}. Finally, that $C_{jk}^{\tau,\mu} \gg 0$ follows from the given expression (3.2) on using that if X, Y, Z are matrices such that $X \gg 0$, $Y \gneq 0$, $Z \gg 0$, then the product $XYZ \gg 0$. ∎

As shown by an example in Rothblum [12], we may have $C_{jk}^{\tau,\lambda} = 0$ if $\lambda \in \sigma_{jk}^* - \{\mu_{jk}\}$. Thus the sum in (3.1) may in some cases be taken over a strict subset of σ_{jk}^*. Moreover, as we shall see, it may happen that some but not all entries of some $C_{jk}^{\tau,\lambda}$ are 0. In this case the index set of the sum in (3.1) can be reduced for some but not all entries corresponding to the block (j, k). The expression (3.2) for $C_{jk}^{\tau,\lambda}$ turns out to be useful in such a study. As a direct consequence of Theorem 3.1 we get:

COROLLARY 3.2. *Let $\lambda \in \sigma_{jk}^*$. Then*

(i) *if $\lambda \in \sigma(A_{jj})$, the columns of $C_{jk}^{\tau,\lambda}$ are multiples of u_j^λ, and thus either each entry is equal to 0 or each entry is nonzero;*

(ii) *if $\lambda \in \sigma(A_{kk})$, the rows of $C_{jk}^{\tau,\lambda}$ are multiples of ${}^t v_k^\lambda$, and thus either each entry is equal to 0 or each entry is nonzero;*

(iii) *if $\lambda \in \sigma(A_{jj}) \cap \sigma(A_{kk})$, then either $C_{jk}^{\tau,\lambda} \equiv 0$ or* all *entries of $C_{jk}^{\tau,\lambda}$ are nonzero;*

(iv) *if $\lambda \notin \sigma(A_{jj}) \cup \sigma(A_{kk})$ and $C_{jk}^{\tau,\lambda} \neq 0$, then $C_{jk}^{\tau,\lambda}$ can contain one or more zero columns and at the same time one or more zero rows.*

Proof. Statements (i), (ii), and (iii) follow from Theorem 3.1 and the fact that the u_j^λ and ${}^t v_j^\lambda$ are strictly nonzero. Statement (iv) follows from this example: Let

$$A = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 3 & 2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 6 & 2 & 1 & 0 \end{array}\right].$$

Here $\mu_{32} = \mu_{21} = \mu_{31} = 2$, $\tau_{32} = \tau_{21} = \tau_{31} = 1$,

$$\begin{aligned} C_{31}^{1,-2} &= (-2I - A_{33})^{-1} A_{32} Z^{-2}(A_{22}) A_{21} (-2I - A_{11})^{-1} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \end{aligned}$$

whereas

$$C_{21}^{1,-2} = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \qquad C_{32}^{1,-2} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix},$$

which shows that (iv) holds (by the form of $C_{31}^{1,-2}$) and shows as well that both zero and nonzero columns and rows may occur in the cases covered by (i) and (ii), respectively. ■

The next theorem and its corollary are needed in Section 5. As will be seen, the results are extensions of Theorem 3.1.

THEOREM 3.3. *Let the situation be as in Theorem* 3.1, *and define* $B = A^d$, *where* $d = \operatorname{lcm}\{d_j : 1 \leqslant j \leqslant s\}$. *Then*

$$[f_n(B)]_{jk} = \frac{f_n^{(\tau-1)}(\mu^d)}{(\tau-1)!} D_{jk}^{\tau,\mu} + o\left(f_n^{(\tau-1)}(\mu^d)\right), \tag{3.3}$$

where

$$\begin{aligned} D_{jk}^{\tau,\mu} &= \sum_{h \in H_{jk}^{*}(\mu)} R_{k_1} A^d_{k_1,k_2} R_{k_2} \cdots A^d_{k_{p-1},k_p} R_{k_p}, \\ h &= (k_1, \ldots, k_p), \\ R_j &= \begin{cases} Z^{\mu^d}\left(A^d_{jj}\right) & \text{if } \mu \in \sigma(A_{jj}), \\ \left(\mu^d I - A^d_{jj}\right)^{-1} & \text{if } \mu \notin \sigma(A_{jj}). \end{cases} \end{aligned} \tag{3.4}$$

Moreover, $D_{jk}^{\tau,\mu}$ *can be block partitioned into a* $d_j \times d_k$ *block matrix of the form*

$$\begin{bmatrix} U_{11} & \cdots & U_{1d_k} \\ \vdots & & \vdots \\ U_{d_j 1} & \cdots & U_{d_j d_k} \end{bmatrix} \tag{3.5}$$

with $U_{pq} = 0$ *or* $U_{pq} \gg 0$ *for each* $1 \leqslant p \leqslant d_j$, $1 \leqslant q \leqslant d_k$. *Moreover, each row (column) of blocks contains at least one strictly positive block matrix.*

Proof. The expressions (3.3) and (3.4) follow directly from Theorem 2.1, since A^d may also be written on the form (1.1), with the same blocks as A. Note that the only assumption made in Theorem 2.1 is that the matrix is written in the form (1.1). To prove the last part of the theorem, consider first the diagonal blocks A^d_{mm} of A^d. Since d_m divides d, the matrix A^d_{mm} can be written in block form with irreducible blocks of period 1 on the diagonal and zero blocks off the main diagonal. There are d_m blocks, corresponding to the d_m cyclic classes determined by A_{mm} (see Section 1.3 in Seneta [16]). Thus it is seen, generalizing the argument used to prove the last part of Theorem 3.1, that each R_m can be written in block form with strictly positive blocks on the main diagonal and zero blocks elsewhere. In proving the stated property of the blocks U_{pq}, we may assume that A is a path matrix. This is so because of the expression (3.4), which is a sum over paths. Again by (3.4), we are done if we can show that each product

$$R_m A^d_{m,m-1} R_{m-1} \tag{3.6}$$

has the property of having at least one strictly positive block in each row and column. Thus in fact it suffices to study the case $s=2$, $j=2$, $k=1$. Suppose therefore

$$A = \begin{bmatrix} U & 0 \\ K & V \end{bmatrix}$$

with U, V irreducible, periodic with periods dividing d and $K \neq 0$. Now

$$K^d = \sum_{i=0}^{d-1} V^i K U^{d-i-1},$$

and it may be shown that since $K \neq 0$, then with K^d written in block form as in (3.5), with blocks named K^d_{pq}, each row (column) of K^d will contain a nonzero block. This follows from the irreducibility of U, V, writing them in block form with blocks determined by the corresponding cyclic classes (again see Seneta [16]). The desired property of (3.6) finally follows by using the already described positivity property of the R_j and the fact that $A^d_{m,m-1}$ has the described property of K^d. ■

COROLLARY 3.4. *In the same situation as in Theorem* 3.3,

$$\sum_{\lambda \in \sigma_{jk}^*} \lambda^{-\tau+1} C_{jk}^{\tau,\lambda} = \left(\frac{d}{\mu^d}\right)^{\tau-1} D_{jk}^{\tau,\mu}.$$

Proof. Compute $(A^{dn})_{jk}$ first by using $f_n(t) = t^{dn}$ in Theorem 3.1, then by using $f_n(t) = t^n$ in Theorem 3.3, and compare the coefficient of $n^{\tau-1}\mu^{dn}$. (Note that $\lambda^d = \mu^d$ for all $\lambda \in \sigma_{jk}^*$ by the definition of d in Theorem 3.3.) ■

4. LOCAL BEHAVIOR OF POWERS OF A NONNEGATIVE MATRIX. LOCAL PERIODS

Let A be a nonnegative $q \times q$ matrix in the Frobenius form (1.1). In this section we apply Theorem 3.1 in order to study the local behavior of A^n for large n. In particular we shall consider *local periods* of A. In the following x, y will denote elements in $S \equiv \{1,2,\dots,q\}$, whereas j, k will denote block numbers as before. We shall say that (x, y) belongs to block (j, k) if the entry A_{xy} is in block A_{jk} of A.

For given block indices j, k and integer $d \geqslant 1$, call $\lambda_1, \lambda_2 \in \sigma_{jk}^*$ d-equivalent if $\lambda_1^d = \lambda_2^d$.

THEOREM 4.1. *Let $x, y \in S$, where (x, y) belongs to the block (j, k) with $\mu = \mu_{jk}$, $\tau = \tau_{jk}$. Then for given $d \geqslant 1$ and $0 \leqslant r \leqslant d-1$, the following are equivalent*:

(i) $\lim_{n\to\infty}[(dn)^{\tau-1}\mu^{dn+r-\tau+1}]^{-1}A_{xy}^{dn+r}$ *exists.*

(ii) *For each d-equivalence class R of σ_{jk}^*, we have either $\lambda^d = \mu^d$ for all $\lambda \in R$ or*

$$\sum_{\lambda \in R}\left(\frac{\lambda}{\mu}\right)^{r-\tau+1} C_{xy}^{\tau,\lambda} = 0. \tag{4.1}$$

Moreover, if (i) *holds, then the limit equals*

$$[(\tau-1)!]^{-1}\sum_{\lambda \in L}\left(\frac{\lambda}{\mu}\right)^{r-\tau+1} C_{xy}^{\tau,\lambda}, \tag{4.2}$$

where $L = \{\lambda \in \sigma_{jk}^* : \lambda^d = \mu^d\}$.

In the proof we need the following lemma, taken from [6, Lemma 3.3].

LEMMA 4.2 [6]. *Let λ_α, z_α, $\alpha = 1,\dots,r$, be complex numbers, where the λ_α are pairwise distinct and $|\lambda_\alpha| = 1$. If $\lim_{n\to\infty}(\sum_{\alpha=1}^r \lambda_\alpha^n z_\alpha)$ exists, then $z_\alpha = 0$ when $\lambda_\alpha \neq 1$.*

Proof of Theorem 4.1. Let

$$f_n(t) = t^{dn+r}.$$

Then using Theorem 3.1 it is obtained that

$$\left[(dn)^{\tau-1}\mu^{dn+r-\tau+1}\right]^{-1} A_{xy}^{dn+r}$$

$$= \sum_{\lambda \in \sigma_{jk}^*} \frac{1}{(\tau-1)!}\left(\frac{\lambda}{\mu}\right)^{r-\tau+1}\left(\frac{\lambda}{\mu}\right)^{dn} C_{xy}^{\tau,\lambda} + o(1).$$

Now by Lemma 4.2 this limit exists if and only if the condition of the theorem holds. Also, Lemma 4.2 shows that the limit, when it exists, necessarily equals (4.2). ■

COROLLARY 4.3. *Let (x, y) and (j, k) be as in Theorem 4.1. Then for given $d \geqslant 1$, the following are equivalent:*

(i) $\lim_{n\to\infty}[(dn)^{\tau-1}\mu^{dn+r-\tau+1}]^{-1}A_{xy}^{dn+r}$ *exists for* all $r = 0, 1, \ldots, d-1$.
(ii) *For each $\lambda \in \sigma_{jk}^*$ we have either $\lambda^d = \mu^d$ or $C_{xy}^{\tau,\lambda} = 0$.*

Moreover, if the limits (i) *exist, then they equal* (4.2), *which is nonzero for at least one $r = -, -, \ldots, d - -$.*

Proof. The "if" part follows directly from Theorem 4.1. To prove "only if," note first that each d-equivalence class of σ_{jk}^* contains at most d different λ's. To see this, note that λ_1, λ_2 are d-equivalent iff

$$(\lambda_1/\lambda_2)^d = 1. \tag{4.3}$$

Now as the equation $z^d = 1$ has exactly d roots, for a given λ_1 there are at most d possible λ_2 for which (4.3) holds. But then as (4.1) is to hold for $r = 0, 1, \ldots, d-1$, it follows by linear independence (using the Vandermonde determinant) that $C_{xy}^{\tau,\lambda} = 0$ for all $\lambda \in R$ when R is an equivalence class of λ's for which $\lambda^d \neq \mu^d$. This proves the first part of the corollary.

Finally, suppose the condition of the corollary holds and the limit (4.2) equals 0 for each $r = 0, 1, \ldots, d-1$. Then as there are at most d different λ with $\lambda^d = \mu^d$, it follows as above that the $C_{xy}^{\tau,\lambda} = 0$ for all $\lambda \in \sigma_{jk}^*$. But this contradicts the fact that $C_{xy}^{\tau,\lambda} \gg 0$. Hence at most one limit (4.2) must be nonzero. ∎

Theorem 4.1 and Corollary 4.3 motivate the definition of *local period* that will be suggested below. However, before giving the definition we show by an example that the conditions of Theorem 4.1 are strictly weaker than those of Corollary 4.3.

EXAMPLE 1. Let

$$A = \left[\begin{array}{c|cccccc} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

with entries numbered $x_1, x_2, \ldots, x_6, x_7$. Then straightforward computations show that

$$\lim_{n\to\infty} A^{6n}_{x_2,x_1} = \lim_{n\to\infty} A^{6n+3}_{x_2,x_1} = \tfrac{4}{7},$$

$$\lim_{n\to\infty} A^{6n+1}_{x_2,x_1} = \tfrac{2}{7},$$

$$\lim_{n\to\infty} A^{6n+4}_{x_2,x_1} = \tfrac{16}{7},$$

$$\lim_{n\to\infty} A^{6n+2}_{x_2,x_1} = \lim_{n\to\infty} A^{6n+5}_{x_2,x_1} = \tfrac{8}{7}.$$

Thus $\lim_{n\to\infty} A^{3n}_{x_2,x_1}$ exists (and equals $\frac{4}{7}$), and $\lim_{n\to\infty} A^{3n+2}_{x_2,x_1}$ exists (and equals $\frac{8}{7}$), while $\lim_{n\to\infty} A^{3n+1}_{x_2,x_1}$ does *not* exist.

DEFINITION. For given $x, y \in S$ belonging to the block (j,k) of A, we call A (d,r)*-periodic at* (x,y) if condition (ii) [and hence (i)] of Theorem 4.1 holds.

The *period* $d(x,y)$ *of* A *at* (x,y) is the least integer $d \geqslant 1$ for which condition (ii) [and hence (i)] of Corollary 4.3 holds.

Note that $d(x, y)$ equals $\operatorname{lcm}\{d_\lambda : \lambda \in \sigma_{jk}^*,\ C_{xy}^{\tau,\lambda} \neq 0\}$. Note also that Theorem 4.1 and Corollary 4.3 hold with (x, y) replaced by block indices (j, k). Thus definitions similar to the ones above can be given for the behavior of the block matrices A^n_{jk}. In particular we have:

DEFINITION. The *period* $d(j, k)$ *of* A *at the block* (j, k) is the least integer $d \geq 1$ such that for each $\lambda \in \sigma_{jk}^*$ we have either $\lambda^d = \mu^d$ or $C_{jk}^{\tau,\lambda} = 0$.

As for $d(x, y)$, it is seen that $d(j, k)$ equals $\operatorname{lcm}\{d_\lambda : \lambda \in \sigma_{jk}^*,\ C_{jk}^{\tau,\lambda} \neq 0\}$. Thus we can also see that $d(j, k)$ is the least common multiple of the $d(x, y)$ with (x, y) belonging to the block (j, k).

From Corollary 3.2 we get the following result, giving additional information on the periods $d(x, y)$ for (x, y) within particular blocks (j, k).

THEOREM 4.4. *Let (j, k) be a fixed block. For (x, y) belonging to (j, k) we have*:

(i) *If $\mu_{jk} \in \sigma(A_{jj})$, then $d(x, y)$ does not depend on x for fixed y.*

(ii) *If $\mu_{jk} \in \sigma(A_{kk})$, then $d(x, y)$ does not depend on y for fixed x.*

(iii) *If $\mu_{jk} \in \sigma(A_{kk}) \cap \sigma(A_{jj})$, then $d(x, y)$ is constant within the block (j, k), with value $d(j, k)$.*

Proof. We use the fact that $d(x, y) = \operatorname{lcm}\{d_\lambda : \lambda \in \sigma_{jk}^*,\ C_{xy}^{\tau,\lambda} \neq 0\}$. In case (i) we must have $\sigma_{jk}^* \subseteq \sigma(A_{jj})$, by the definition of σ_{jk}^*. But then Corollary 3.2(i) implies that the set $\{\lambda \in \sigma_{jk}^* : C_{xy}^{\tau,\lambda} \neq 0\}$ does not depend on x for fixed y. This proves (i). The proofs of (ii) and (iii) are similar. ∎

We remark that the local period defined by Friedland and Schneider [6] in our notation is the least common multiple of the set $\{d_\lambda : \lambda \in \sigma_{jk}^*\}$, which may be larger than our local period. (See example below.) Moreover, [6] merely defines the local period at a pair (x, y) to be their local period at the associated block (j, k).

EXAMPLE 2. Let

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right].$$

Then $\lim_{n\to\infty} A_{21}^n = (\frac{1}{2}\ \frac{1}{2})$, so $d(2,1)=1$ with our definition of local period, while the local period defined in [6] here equals 2.

Note also that Friedland and Schneider [6] define local periods in terms of convergence of matrix sequences of the type

$$B^{a,d,n} \equiv A^n(I + a^{-1}A + a^{-2}A^2 + \cdots + a^{-d+1}A^{d-1})$$

for $a>0$, d integer $\geqslant 1$, and n integer $\geqslant 0$, with $n\to\infty$. As we have seen, our local periods were defined in terms of the convergence as $n\to\infty$ of sequences A^{dn+r}.

The following theorem shows that the two approaches in fact are equivalent.

THEOREM 4.5. *The following are equivalent when $x,y\in S$ and (x,y) belongs to the block (j,k) with $\mu=\mu_{jk}$, $\tau=\tau_{jk}$:*

(i) $\lim_{n\to\infty}(n^{\tau-1}\mu^{n-\tau+1})^{-1}B_{xy}^{\mu,d,n}$ *exists.*
(ii) *Condition* (ii) *of Corollary* 4.3 *holds.*

Moreover, if the limit (i) *exists, then it equals*

$$\frac{d}{(\tau-1)!}C_{xy}^{\tau,\mu}.$$

Proof. Put

$$f_n(t) = t^n(1+\mu^{-1}t+\cdots+\mu^{-d+1}t^{d-1}). \tag{4.4}$$

Then for $k\geqslant 1$,

$$f_n^{(k)}(t) = n^k t^{-k} f_n(t) + o(n^k t^n).$$

Thus by Theorem 3.1 we get

$$B_{xy}^{\mu,d,n} = [f(A)]_{xy} = \sum_{\lambda\in\sigma_{jk}^*}\frac{n^{\tau-1}\lambda^{-\tau+1}f_n(\lambda)}{(\tau-1)!}C_{xy}^{\tau,\lambda} + o(n^{\tau-1}\mu^n),$$

so that

$$[n^{\tau-1}\mu^{n-\tau+1}]^{-1}B_{xy}^{\mu,d,n}$$

$$= \sum_{\lambda\in\sigma_{jk}^*} \frac{1}{(\tau-1)!}\left(\frac{\lambda}{\mu}\right)^{n-\tau+1}\left(1+\frac{\lambda}{\mu}+\cdots+\left(\frac{\lambda}{\mu}\right)^{d+1}\right)C_{xy}^{\tau,\lambda}+o(1), \quad (4.5)$$

the limit of which exists by Lemma 4.2 iff for each $\lambda \in \sigma_{jk}^*$, $\lambda \neq \mu$, either $C_{xy}^{\tau,\lambda} = 0$ or $1+(\lambda/\mu)+\cdots+(\lambda/\mu)^{d-1} = 0$. The last condition is equivalent to $\lambda^d = \mu^d$, and so the equivalence (i) $\Leftrightarrow$ (ii) is proved. Moreover, it follows from Lemma 4.2 that if the limit (4.2) exists, then the limit is given by the term on the right hand side of (4.5) corresponding to $\lambda = \mu$. This is $[d/(\tau-1)!]C_{xy}^{\tau,\lambda}$. ■

5. APPLICATION TO MULTIPLICATIVE PROCESSES: LIMITING OUTPUT VECTORS

Let A be a nonnegative matrix as before. We consider in the present section a multiplicative process in which a vectorial input p (a row vector of same dimension as A) is transformed in n time units to the output vector pA^n ($n = 1,2,\dots$). We will use the same *block* notation for p as we do for A,

$$p = (p_1, p_2, \dots, p_s).$$

Thus we can write

$$pA^n \equiv p^n \equiv (p_1^n, p_2^n, \dots, p_s^n),$$

where

$$p_k^n = \sum_{j=k}^{s} p_j A_{jk}^n, \qquad k = 1,\dots,s.$$

In the following we study the properties of p^n as $n \to \infty$. It will be tacitly assumed, throughout the section, that $p \neq 0$. Let

$$\mu(p) = \max\{\mu_{jk} : p_j \neq 0,\ H_{jk} \neq \varnothing\}$$

and

$$\tau(p) = \max\{\tau_{jk} : p_j \neq 0,\ H_{jk} \neq \varnothing,\ \mu_{jk} = \mu(p)\}.$$

To each p we assign a set

$$K(p) \equiv \{(j,k) : p_j \neq 0,\ \mu_{jk} = \mu(p),\ \tau_{jk} = \tau(p)\},$$

and we let

$$\overline{K}(p) = \{k : (j,k) \in K(p) \text{ for some } j\}$$

$$K_k(p) = \{j : (j,k) \in K(p)\} (= \varnothing \text{ if } k \notin \overline{K}(p))$$

$$S_k(p) = \bigcup_{j \in K_k(p)} \sigma_{jk}^*.$$

The following lemma will be used later in the section. The proof is easy and is omitted. If $\mu > 0$, then we shall say that a (block index) j, $1 \leqslant j \leqslant s$, is *$\mu$-final* if $\mu_j = \mu$ and $\mu_k < \mu$ whenever $H_{jk} \neq \varnothing$. If K is a set of μ-final indices, then we set

$$D(K) = \{k : H_{jk} \neq \varnothing \text{ for some } j \in K,\ j \neq k\},$$

i.e., $D(K)$ is the set of blocks that can be reached from K, not including states in K.

LEMMA 5.1. *Suppose $\mu(p) > 0$. Then*

$$\overline{K}(p) = \overline{\overline{K}}(p) \cup D\big(\overline{\overline{K}}(p)\big),$$

where

$$\overline{\overline{K}}(p) = \{k : (j,k) \in K(p) \text{ for some } j,\ k \text{ is } \mu(p)\text{-final}\}.$$

The next lemma is similar to Theorem 4.1.

LEMMA 5.2. *Suppose $\mu(p) > 0$, and put $\mu = \mu(p)$, $\tau = \tau(p)$.*
If $k \in \bar{K}(p)$, then for given $d \geqslant 1$ and $0 \leqslant r \leqslant d-1$, the following two conditions are equivalent:

(i) $\lim_{n \to \infty} [(dn)^{\tau-1}\mu^{dn+r-\tau+1}]^{-1} p_k^{dn+r}$ *exists.*

(ii) *For each d-equivalence class R (see definition in Section 3) in $S_k(p)$ we have either $\lambda^d = \mu^d$ for all $\lambda \in R$ or*

$$\sum_{j \in K_k(p)} \sum_{\lambda \in \sigma_{jk}^* \cap R} \left(\frac{\lambda}{\mu}\right)^{r-\tau+1} p_j C_{jk}^{\tau,\lambda} = 0. \tag{5.1}$$

Moreover, if (i) *holds, then the limit equals*

$$[(\tau-1)!]^{-1} \sum_{j \in K_k(p)} \sum_{\lambda \in \sigma_{jk}^* \cap L_k} \left(\frac{\lambda}{\mu}\right)^{r-\tau+1} p_j C_{jk}^{\tau,\lambda}, \tag{5.2}$$

where $L_k = \{\lambda \in S_k(p): \lambda^d = \mu^d\}$.
If $k \notin \bar{K}(p)$, then the limit (i) *exists and equals* 0 *for all d and r.*

Proof. By Theorem 3.1 we can write

$$p_k^{dn+r} = \sum_{j \in K_k(p)} \sum_{\lambda \in \sigma_{jk}^*} \frac{(dn)^{\tau-1}}{(\tau-1)!} \lambda^{dn+r-\tau+1} p_j C_{jk}^{\tau,\lambda} + o(n^{\tau-1}\mu^{dn}). \tag{5.3}$$

The result of the lemma then follows in the same way as Theorem 4.1. ■

THEOREM 5.3. *Suppose $\mu(p) > 0$, and put $\mu = \mu(p)$, $\tau = \tau(p)$. Then for any given integer $d \geqslant 1$ the following three conditions are equivalent:*

(i) $\lim_{n \to \infty} [(dn)^{\tau-1}\mu^{dn+r_0-\tau+1}]^{-1} p^{dn+r_0}$ *exists for some r_0 with $0 \leqslant r_0 \leqslant d-1$.*

(ii) $\lim_{n \to \infty} [(dn)^{\tau-1}\mu^{dn+r-\tau+1}]^{-1} p^{dn+r}$ *exists for all $r = 0, 1, \ldots, d-1$.*

(iii) *For each $k \in \bar{K}(p)$ and each $\lambda \in S_k(p)$ with $\lambda^d \neq \mu^d$ we have*

$$\sum_{j \in K_k(p)} p_j C_{jk}^{\tau,\lambda} = 0.$$

Moreover, if the limit in (i) *exists and is nonzero for some r_0, then the limits in* (ii) *are all nonzero.*

REMARK. Let $d = \text{lcm}\{d_{jk} : (j,k) \in K(p)\}$. Then $\lambda^d = \mu^d$ for all $\lambda \in S_k(p)$, for all $k \in \bar{K}(p)$. Thus there always exist d for which condition (iii) above holds.

Proof. That (i) implies (ii) follows from

$$p^{dn+r} = \begin{cases} p^{dn+r_0}A^{r-r_0} & \text{if} \quad r_0 \leqslant r \leqslant d-1, \\ p^{(d-1)n+r_0}A^{d+r-r_0} & \text{if} \quad 0 \leqslant r < r_0. \end{cases} \tag{5.4}$$

Thus (i) $\Leftrightarrow$ (ii). Furthermore these two conditions are equivalent to the condition that (ii) of Lemma 5.2 holds for all $k \in \bar{K}(p)$ and $0 \leqslant r \leqslant d-1$. By a linear independence argument (using the Vandermonde determinant) as in Corollary 4.3, but now applied to (5.1), we conclude that this is equivalent to condition (iii) of the present theorem. The last part of the theorem follows from (5.4). ■

Let p be a given input vector with $\mu = \mu(p)$ and $\tau = \tau(p)$. In order to simplify expressions, in the following we shall put

$$k_{n,d} = \left[(dn)^{\tau-1}\mu^{dn-\tau+1}\right]^{-1}.$$

THEOREM 5.4. *Let the situation be as in Theorem* 5.3.

(i) *The limit*

$$\bar{p} = \lim_{n\to\infty} k_{n,d}p^{dn} \tag{5.5}$$

is independent of d, provided d satisfies Theorem 5.3(iii). *Moreover, $\bar{p}$ is given by*

$$\bar{p}_k = \left[(\tau-1)!\right]^{-1} \sum_{j \in K_k(p)} \sum_{\lambda \in \sigma_{jk}^* \cap L_k} (\lambda/\mu)^{-\tau+1} p_j C_{jk}^{\tau,\lambda}$$

for $k \in \bar{K}(p)$, where $L_k = \{\lambda \in S_k(p) : \lambda^d = \mu^d\}$ for any d satisfying Theorem 5.3(iii), *and*

$$\bar{p}_k = 0 \qquad \textit{for} \quad k \notin \bar{K}(p).$$

(ii) *Suppose $\bar{p} \neq 0$. Then if the limit*

$$\lim_{n \to \infty} \left[(d'n)^{a-1} \beta^{d'n-a+1} \right]^{-1} p^{d'n}$$

exists and is nonzero for some a, β, d', we have $a = \tau(p)$, $\beta = \mu(p)$, and d' satisfies Theorem 5.3(iii).

Proof. The result (i) follows from Lemma 5.2, (5.2), and Theorem 5.3(iii). To prove (ii), let d satisfy Theorem 5.3(iii) and write

$$\left[(d'dn)^{a-1} \beta^{d'dn-a+1} \right]^{-1} p^{d'dn} = (dd'n)^{\tau - a} (\mu/\beta)^{dd'n - \tau + a} k_{d'n,d} p^{dd'n}.$$

By assumption, the limit as $n \to \infty$ exists and is nonzero. Moreover, the limit of $k_{d'nd} p^{dd'n}$ exists and equals $\bar{p} \neq 0$. But this implies that

$$\lim_{n \to \infty} (dd'n)^{\tau - a} (\mu/\beta)^{dd'n - \tau + a}$$

exists and is nonzero, which can only happen if $a = \tau$ and $\beta = \mu$. This completes the proof. ∎

The following example shows that the assumption $\bar{p} \neq 0$ is necessary in Theorem 5.4(ii). Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad p = (1, -1).$$

Then $pA = 2p$, so $p^n = pA^n = 2^n p$ for $n = 1, 2, \ldots$. Thus $\lim_{n \to \infty} 2^{-n} p^n$ exists (and is nonzero). However, $\mu(p) = 4$ by our definition of $\mu(p)$, so the conclusion of Theorem 5.4(ii) does not hold.

As we shall see in Theorem 5.5, we have $\bar{p} \neq 0$ whenever $p \geqslant 0$. With the above example in mind, this indicates that the approach of the present section is best suited for *nonnegative* multiplicative processes, i.e. processes with nonnegative input vectors p.

Definition. For an input vector p with $\mu(p) > 0$ we define the period $d(p)$ as the least $d \geqslant 1$ for which Theorem 5.3(iii) holds.

THEOREM 5.5. *Suppose* $\mu(p) > 0$.

(i) *If* $p \geqslant 0$, *then* $\bar{p}_k \neq 0$ *for all* $k \in \overline{K}(p)$.
(ii) *If* $p \geqslant 0$ *and* $d(p) = 1$, *then* $\bar{p}_k \gg 0$ *for all* $k \in \overline{K}(p)$.

Proof. By Corollary 3.4 we have, for any $k \in \overline{K}(p)$,

$$\bar{p}_k = [(\tau - 1)!]^{-1} \frac{d}{\mu^{(d-1)(\tau-1)}} \sum_{j \in K_k(p)} p_j D_{jk}^{\tau, \mu^d},$$

which is nonzero, since each $D_{jk}^{\tau, \mu^d} \geqslant 0$ and each row of D_{jk}^{τ, μ^d} contains at least one positive entry by Theorem 3.3. This proves (i). To prove (ii), note that in this case by Theorem 5.4(i) we may write

$$\bar{p}_k = [(\tau - 1)!]^{-1} \sum_{j \in K_k(p)} p_j C_{jk}^{\tau, \mu}, \tag{5.6}$$

where each $C_{jk}^{\tau, \mu} \gg 0$ by Theorem 3.1. ■

THEOREM 5.6. *Let* p *be a given input vector, with* $\mu \equiv \mu(p) > 0$, $d \equiv d(p)$. *Then*

(i) $K(\bar{p}) \subseteq \{(j, k) : j \in \overline{\overline{K}}(p),\ H_{jk} \neq \varnothing\}$, *with equality if* $p \geqslant 0$,
(ii) $\mu(\bar{p}) = \mu$, $\tau(\bar{p}) = 1$,
(iii) $\bar{p}A^d = \mu^d \bar{p}$,
(iv) $\underline{\bar{p}^{dn}} = \mu^{dn}\bar{p}$ *for* $n = 1, 2, \ldots$,
(v) $\overline{(\bar{p})} = \bar{p}$,
(vi) $d(\bar{p}) \leqslant d$.

Proof. (i) and (ii) follow from Theorem 5.5 and the definitions of $K(\cdot)$, $\mu(\cdot)$, $\tau(\cdot)$ by noting that $\bar{p}_k \neq 0$ only if $k \in \overline{K}(p)$ (if and only if, provided $p \geqslant 0$), and using Lemma 5.1. Statements (iii) and (iv) are easy consequences of the definition (5.5) of $\bar{p}$. Finally, (v) and (vi) follow from (iv) and Theorem 5.4(ii). ■

The above theorem implies that if $d(p) = 1$, then $\bar{p}$ is a (left) eigenvector of A with corresponding eigenvalue $\mu(p)$. Moreover, it follows that $d(\bar{p}) = 1$ and $\tau(\bar{p}) = 1$.

THEOREM 5.7. *For a given input vector* p *with* $\mu(p) > 0$, *the period* $d(p)$ *is the* gcd *of the* d *for which Theorem* 5.3(iii) *holds.*

Proof. We need to show that if $d_1, \ldots, d_l$ satisfy Theorem 5.3(iii), then with $d = \gcd\{d_1, \ldots, d_l\}$ we have convergence of $k_{n,d} p^{dn}$. We shall prove this by showing that any subsequence $\{n'\}$ of $\{n\}$ contains a further subsequence $\{n''\}$ for which $k_{n'',d} p^{dn''}$ converges to $\bar{p}$. Now for all large n we can write

$$dn = \sum_{j=1}^{l} m_{j,n} d_j$$

with all $m_{j,n} \geqslant 0$, which follows from Lemma A.3 in Seneta [16].

Let now $\{n'\}$ be a subsequence of $\{n\}$. Then for *some* j there must be a subsequence $\{n''\}$ of $\{n\}$ with $m_{j,n''} \to \infty$. Without loss of generality, assume this is $j = 0$, and write for $j = 1, \ldots, l$

$$m_{j,n} = q_{j,n} d_0 + r_{jn}, \qquad \text{where} \quad 0 \leqslant r_{j,n} < d_0.$$

Now for a subsequence $\{\bar{n}\}$ of $\{n''\}$ we will have $r_{j,\bar{n}} \equiv r_j$, independent of $\bar{n}$, for $j = 1, \ldots, l$. Thus

$$\begin{aligned} d\bar{n} &= \left(m_{0,\bar{n}} + \sum_{j=1}^{l} q_{j,\bar{n}} d_j \right) d_0 + \sum_{j=1}^{l} r_j d_j \\ &\equiv m_{\bar{n}} d_0 + r \end{aligned}$$

with $m_{\bar{n}} \to \infty$ as $\bar{n} \to \infty$. Now we have

$$\begin{aligned} k_{\bar{n},d} p^{d\bar{n}} &= k_{\bar{n},d} p^{m_{\bar{n}} d_0 + r} \\ &= k_{\bar{n},d} k_{m_{\bar{n}},d_0}^{-1} k_{m_{\bar{n}},d_0} p^{m_{\bar{n}} d_0} A^r. \end{aligned} \tag{5.7}$$

Here

$$\lim_{n \to \infty} k_{\bar{n},d} k_{m_{\bar{n}},d_0}^{-1} = \mu^{-r},$$

since $m_{\bar{n}} d_0 = d\bar{n} - r$, where r is a fixed constant as $\bar{n} \to \infty$. Moreover,

$$\lim_{\bar{n} \to \infty} k_{m_{\bar{n}},d_0} p^{m_{\bar{n}} - d_0} = \bar{p},$$

so the limit in (5.7) exists and equals

$$\mu^{-r}\bar{p}A^r = \mu^{-r}\bar{p}A^{\sum_{j=1}^{l} r_j d_j} = \bar{p},$$

where the last equality follows from Theorem 5.6(iv), applied to each d_j, $j = 1, \dots, l$. ∎

It is well known that if A is itself irreducible and aperiodic, then for any input vector $p \neq 0$ the limit $\bar{p}$ is proportional to the unique (up to scalar multiplication) left eigenvector ${}^t v$ of A at the PF value of A.

The next result can be viewed as a generalization of this result to reducible matrices A. The special case of our result obtained by putting $j_0 = s$, assuming $d_j = 1$ and $A_{jj} \neq (0)$ for all $1 \leqslant j \leqslant s$, was given by Mandl [11].

THEOREM 5.8. *Let A be as before, and suppose $\mathscr{P}$ is a class of input vectors such that for some $1 \leqslant j_0 \leqslant s$ with $\mu_{j_0} > 0$*

(i) *$\mu(p) = \mu_{j_0} \equiv \mu$ for all $p \in \mathscr{P}$;*
(ii) *$d(p) = 1$ for all $p \in \mathscr{P}$;*
(iii) *for any given $p \in \mathscr{P}$, $(j, k) \in K(p)$, $h \in H_{jk}^*(\mu)$ we have $j_0 \in \operatorname{supp} h$.*

Then there exists a vector $\pi \geqslant 0$, $\pi \neq 0$ such that $\bar{p}$ is proportional to π for all $p \in \mathscr{P}$.

REMARK. Note that $\tau(p)$ may well depend on $p \in \mathscr{P}$.

Proof. Let $p \in \mathscr{P}$ and let $(j, k) \in K(p)$. Assumption (iii) states that every path in $H_{jk}^*(\mu)$ passes through j_0 and hence $H_{jk}^*(\mu)$ consists of exactly the h in H_{jk} consisting of an $h_1 \in H_{jj_0}^*(\mu)$ immediately followed by an $h_2 \in H_{j_0, k}^*(\mu)$. Thus also $\bar{K} \equiv \bar{K}(p)$ is independent of p, and $K_k(p)$ is independent of k when $p \in \mathscr{P}$. Moreover, by Theorem 3.1 we must have

$$C_{jk} = C_{jj_0} C_{j_0 k}, \tag{5.8}$$

where we have suppressed the upper indices on the C's (which are anyway clear) and where we have used that $[Z^\mu(A_{j_0 j_0})]^2 = Z^\mu(A_{j_0 j_0})$. But (5.8) can now be written

$$C_{jk} = C_{jj_0} u_{j_0} {}^t v_{j_0} C_{j_0 k} = \mu_j^* \, {}^t v_k^*,$$

where $u_j^* = C_{jj_0}u_{j_0}$, ${}^t v_k^* = {}^t v_{j_0} C_{j_0 k}$, so by (5.6) for $k \in \overline{K}(p) \equiv \overline{K}$ we have [with $\tau = \tau(p)$]

$$\begin{aligned}\bar{p}_k &= [(\tau-1)!]^{-1} \sum_{j \in K_k(p)} p_j C_{jk}\\ &= [(\tau-1)!]^{-1} \sum_{j \in K_k(p)} p_j u_j^* \, {}^t v_k^*\\ &= c(p)\, {}^t v_k^*,\end{aligned}$$

where

$$c(p) = [(\tau-1)!]^{-1}\Bigg(\sum_{j \in K_k(p)} p_j u_j^*\Bigg)$$

is a number not depending on k, as $K_k(p)$ is independent of k. Since $\bar{p}_k = 0$ for $k \in \overline{K}$ it follows that

$$\bar{p} = c(p)\pi;$$

π is the vector given by

$$\pi_k = \begin{cases} {}^t v_k^* & \text{for} \quad k \in \overline{K},\\ 0 & \text{for} \quad k \notin \overline{K}.\end{cases}$$

Note that ${}^t v_k^* \gg 0$, which follows from its definition. ∎

We close the section by considering input vectors p with $\mu(p) = 0$. This is the case if $p_j \neq 0$ implies $\mu_j = 0$ and $\mu_k = 0$ for all k with $H_{jk} \neq \varnothing$. Note here that if $\mu_j = 0$, then A_{jj} is the 1×1 matrix (0), so that "block j" is in fact "entry x" for some x. The following result follows easily:

THEOREM 5.9. *If $\mu(p) = 0$, then $p^n = 0$ if and only if $n \geqslant \tau(p)$.*

6. NONNEGATIVE EIGENVECTORS OF NONNEGATIVE MATRICES

In this section we shall use results from the previous section to give a characterization of nonnegative eigenvectors of the matrix A.

First note that if $\pi \geqslant 0$ is an eigenvector of A, then the associated eigenvalue is necessarily nonnegative.

THEOREM 6.1. *Suppose A is a nonnegative matrix given as in* (1.1) *in the Frobenius normal form, and suppose that $\pi \geqslant 0$ is a left eigenvector for the eigenvalue $\mu > 0$. Then*:

(i) $\mu = \mu_j$ *for some* $1 \leqslant j \leqslant s$.

(ii) *There is a set $K \subseteq \{k : k \text{ is } \mu\text{-final}\}$ such that $\pi_k \gg 0$ whenever $k \in K \cup D(K)$ and $\pi_k = 0$ otherwise.*

(iii) $\pi_k = c_k v_k^\mu$ *for $k \in K$, where $c_k > 0$, and*

$$\pi_k = \sum_{j \in K_k} \pi_j C_{jk}^{1,\mu} \qquad \textit{for} \quad k \in D(K),$$

where $K_k = \{ j \in K : H_{jk} \neq \varnothing \}$.

Conversely, any π satisfying (i)–(iii) (*with* $\mu > 0$) *is a left eigenvector of A at μ.*

Proof. Suppose $\pi A = \mu \pi$. Then $\pi^n = \mu^n \pi$ for $n = 1, 2, \ldots$, so

$$\lim_{n \to \infty} \mu^{-n} \pi^n$$

exists and equals π. As $\pi \neq 0$, we have $\bar{\pi} \neq 0$ by Theorem 5.5(i), and hence Theorem 5.4(ii) implies that $\mu(\pi) = \mu$ [which proves (i)], $\tau(\pi) = 1$, $d(\pi) = 1$, and in fact $\bar{\pi} = \pi$. Thus Theorem 5.5(i) implies that $\pi_k \gg 0$ for all $k \in K(\pi)$ and $\pi_k = 0$ otherwise. Condition (ii) of the present theorem is now a direct consequence of Lemma 5.1. Finally, (iii) follows from (5.6), since

$$K_k(\pi) = \begin{cases} \{k\} & \text{for} \quad k \in K \\ \{ j \in K : H_{jk} \neq \varnothing \} & \text{for} \quad k \in D(K) \end{cases} \tag{6.1}$$

and $\pi_k C_{kk}^{1,\mu} = \pi_k Z^\mu (A_{kk}) = c_k v_k^\mu$ for some $c_k > 0$.

Conversely, suppose π satisfies (i)–(iii). Then $\mu(\pi) = \mu$, $\tau(\pi) = 1$, $\overline{K}(\pi) = K \cup D(K)$, $K_k(\pi)$ is given by (6.1), and

$$S_k(\pi) = \begin{cases} \sigma_{kk}^* & \text{for} \quad k \in K, \\ \displaystyle\bigcup_{j \in K : H_{jk} \neq \varnothing} \sigma_{jk}^* & \text{for} \quad k \in D(K). \end{cases}$$

Now for $j \in K$, $H_{jk} \neq \varnothing$, for any $\lambda \in \sigma_{jk}^*$, $\lambda \neq \mu$, we have by the expression for $C_{jk}^{1,\lambda}$ given by Theorem 3.1,

$$\pi_j C_{jk}^{1,\lambda} = c_j v_j^{\mu} Z^{\lambda}(A_{jj}) M = 0$$

(where M is implicitly given by Theorem 3.1), since $Z^{\mu}(A_{jj})Z^{\lambda}(A_{jj}) = 0$ by (2.2) and the rows of $Z^{\mu}(A_{jj})$ are proportional to v_j^{μ}. Thus condition (iii) of Theorem 5.3 is satisfied with $d = 1$, so by the same theorem,

$$\lim_{n \to \infty} \mu^{-n} \pi^{n} = \bar{\pi}$$

exists, and $\bar{\pi} = \pi$ by (5.6). Thus π is an eigenvector of A at μ by Theorem 5.6(iii). ∎

To illustrate the use of the theorem we give two corollaries, the results of which were proved earlier by Cooper [3]. Moreover, it should be noted that Corollary 6.3 below is essentially a restatement of Theorem 13.6 in Gantmacher [7].

Corollary 6.2. *Let $\mu = \mu_j$ for some $1 \leqslant j \leqslant s$. Then the dimension of the linear space spanned by the nonnegative left eigenvectors at the eigenvalue μ equals the number of block indices k for which k is μ-final. A basis is given by the vectors obtained in Theorem 6.1 by choosing K as all possible one point sets.*

Proof. By Theorem 6.1 we can freely choose exactly the c_k for which k is μ-final. ∎

Corollary 6.3. *Let μ be the PF value of A, and let $F = \{k : k$ is μ-final$\}$. Then the dimension of the space spanned by the strictly positive left eigenvectors of A at μ is 0, unless we have $F \cup D(F) = \{1, 2, \ldots, s\}$, in which case the dimension equals the number of elements in F.*

We shall finally show that Theorem 6.1 still holds for $\mu = 0$. We must, however, modify our definition of a μ-final block index to say that j, $1 \leqslant j \leqslant s$, is 0-final if $\mu_j = 0$ and $H_{jk} = \varnothing$ for all $k < j$. Note also that we have $K(\pi) = \{j : \pi_j \neq 0,\ j$ is 0-final$\}$ if $\mu(\pi) = 0$.

Theorem 6.4. *Suppose A is a nonnegative matrix given as in (1.1) in Frobenius normal form, and suppose that $\pi \geqslant 0$, $\pi \neq 0$ satisfies $\pi A = 0$.*

Then:

(i) $\mu_j = 0$ *for some* $1 \leqslant j \leqslant s$.

(ii) *There is a set* $K \subseteq \{k : k$ *is* 0*-final*$\}$ *such that* $\pi_k > 0$ *if and only if* $k \in K$.

Conversely, if (i) *holds, then any* π *satisfying* (ii) *is a left eigenvector of* A *at* 0.

Proof. (i) follows from $\pi A = 0$, which implies that A is singular. Moreover, we have $\pi A^n = 0$ for all $n \geqslant 1$. Thus by Theorem 5.5(i) we cannot have $\mu(\pi) > 0$, as this would lead to a (normalized) limit $\bar{\pi} \neq 0$, and so $\mu(\pi) = 0$. Theorem 5.9 implies that $\tau(\pi) = 1$. Thus condition (ii) of the present theorem must hold. Conversely, if (i) holds and π satisfies (ii), then of course $\pi A = 0$. ■

7. APPLICATION TO ABSORBING MARKOV CHAINS. LIMITING CONDITIONAL DISTRIBUTION

A Markov chain is a mathematical model for describing a system that can be in a certain set of states and jumps at unit time intervals from one state to another. The Markov chain is called *absorbing* if, informally, there are one or more states from which further jumps are impossible (a precise definition is given below).

Absorbing Markov chains are frequently used as models in medicine, biology, quality control, reliability studies, etc. The absorbing state(s) may then represent the occurrence of certain events like death, recovery, failure of a machine, etc.

Of particular interest in such problems is the time to absorption from a given initial state or a given probability distribution for the initial state. Various interesting exit times with applications in e.g. reliability analysis are considered e.g. in Keilson [8]. Brook and Evans [1] use absorbing Markov chain models to compute the probability distribution of CUSUM run lengths with applications in quality control.

Let S denote the set of *non*absorbing states. The initial state of the Markov chain will always be assumed to belong to S. It is well known that if S is finite, then the probability of staying in S forever is 0. In cases where the time to absorption is very long, however, also the behavior of the process within S is of importance. This leads to the study of the so-called "quasistationary distributions" or "limiting conditional distributions," which roughly describe the limiting behavior of the Markov chain conditional on the event that absorption has not taken place; see Darroch and Seneta [4].

Let $A = P = (P_{xy} : x, y \in S)$ be the matrix of transition probabilities P_{xy} where P_{xy} is the probability of jumping to y given that the previous state is x. The nth power P^n $(n \geqslant 1)$ is the matrix of n step probabilities P^n_{xy}, which is the probability of the event that the process is in state y at time n, given that it is in state x at time 0. In case of an absorbing Markov chain, the matrix P is clearly *substochastic*, i.e. the row sums are all less than or equal to 1, and at least one of them is strictly less than one (corresponding to a positive probability of absorption). More formally we shall say that P (or rather the Markov chain with transition matrix P) is *absorbing* if $\lim P^n = 0$. Equivalently, P is absorbing if $|\lambda| < 1$ for all eigenvalues λ of P.

In the literature, most attention has been given to the case when P is primitive, i.e. P^n is strictly positive (in each entry) for sufficiently large n. The more general case when P is reducible and aperiodic was considered by Mandl [11] (a summary is found in [4]), who studied the asymptotic properties of P^n as $n \to \infty$. He also obtained results on the "limiting conditional behavior" of the chain.

Let now π be the probability vector on S defining the initial condition of the chain at time 0. Then $\pi^n \equiv \pi A^n$ is the vector whose xth entry is the probability that the chain is in state x at time n. As A is absorbing, we have $\lim_{n \to \infty} \pi^n = 0$. Theorem 3.1 with $f_n(z) = z^n$ gives us asymptotic expressions for π^n as $n \to \infty$ which may be of interest in applications. If 1 denotes a column of 1's, then $\pi A^n 1 = \pi^n 1$ is the probability that absorption has not taken place at time n. Theorem 3.1 thus can be used to approximate the tail probabilities of the distribution of the random variable T, the time of absorption. Brook and Evans [1], for the case when A is irreducible and aperiodic, also computed the sth factorial moment of T, i.e. the expected value of the random variable $T(T-1)(T-2)\cdots(T-s+1)$, to be

$$s!A^{s-1}(I-A)^{-s}1. \tag{7.1}$$

Since $f(z) = z^{s-1}(1-z)^{-s}$ satisfies the condition of Theorem 3.1, this theorem can also be used to approximate (7.1) for large s.

We shall now show how the results of Section 5 can be used to obtain information on the limiting conditional behavior of the chain. The distribution of the state at time n, given that absorption has not taken place, is

$$(\pi A^n 1)^{-1} \pi A^n \tag{7.2}$$

provided $\pi A^n 1 > 0$. Clearly (7.2) defines a probability distribution on S. If A

is irreducible and aperiodic, then (Darroch and Seneta [4]) the distribution (7.2) converges as $n \to \infty$ to the (unique) normalized left eigenvector of A at the PF value of A. Limit results for (7.2) in the general case are given by the theorems in Section 5 of the present paper. In particular Theorem 5.4 gives an expression for the limit when it exists.

Mandl [11] considered the special case when A is reducible and each A_{jj} is aperiodic and nonzero. He proved in this case that the limit of (7.2) always exists and obtained necessary and sufficient conditions for the limit to be positive. Also, he gave conditions insuring that the limit is independent of π, when π is restricted to some specified class of distributions. Our Theorem 5.8, as already mentioned, is a generalization of this last mentioned result of Mandl. In fact our result covers periodic cases as well as more general classes of initial distributions than those considered by Mandl.

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