A Short Proof of the Truemper-Tseng Theorem on Max-Flow Min-Cut Matroids

Robert E. Bixby*
Department of Mathematical Sciences
Rice University
Houston, Texas 77251

and

Arvind Rajan†
AT&T Bell Laboratories

Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

1. INTRODUCTION

Seymour has characterized the matroids satisfying the integral max-flow min-cut property with respect to a fixed element [3]. Truemper and Tseng [9] subsequently proved a decomposition theorem for this class, similar in spirit to Wagner's characterization of the graphs containing no K_5 minor [11] and Seymour's characterization of the regular (totally unimodular) matroids [4]. The purpose of this paper is to give a short, self-contained[1] exposition of the Truemper-Tseng result.

2. MAX-FLOW MIN-CUT MATROIDS

Throughout this paper M denotes a matroid on a finite set E. Fix $l \in E$, and let A be the $\{0,1\}$ matrix with columns indexed on elements $e \in E - l$ (braces being omitted, since $\{l\}$ is a singleton) and rows indexed on circuits

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[1]A basic knowledge of matroid theory is assumed, at the level of Chapter 3 in [1].

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$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Fano matroid F_7 Dual Fano matroid F_7^*

FIG. 1. Binary representations of the Fano matroids.

C of M containing l, such that the (C, e) entry is 1 iff $e \in C$. In the case when M is graphic, the rows of A correspond to paths joining the end vertices of l.

Let $\mathscr{C}^*$ be the family of cocircuits of M containing l. We say that M is *l-MFMC*, that is, has the (*integral*) *max-flow min-cut property* with respect to l, if for every choice of nonnegative integral vector w defined on E,

$$\min_{C^* \in \mathscr{C}^*} w(C^* - l) = \max\{ y^T 1 : y^T A \leqslant w^T,\ y \geqslant 0 \text{ and integral} \},$$

where $w(C^* - l) = \sum_{e \in C^* - l} w(e)$ and 1 is a vector of all 1s. Seymour [3] proved that a connected matroid is l-MFMC if and only if it is binary and has no F_7^* minor (see Figure 1) containing the fixed element l. Denote this class by $\mathscr{M}$.

The proof of Seymour's theorem is difficult. However, a natural strengthening of the hypotheses of this theorem yields a theorem that has an easy, or at least easier, proof. This strengthening is obtained by further restricting the given matroid M, insisting that it satisfy the max-flow min-cut property for *every* choice of l, not simply one fixed choice. Designating such a matroid as *MFMC*, the expected theorem then holds: M is MFMC iff it is binary and contains no F_7^* minor. This result can be proved by a straightforward calculation, using the fact that regular matroids are known to be MFMC, after first proving a structural result. This structural result, which is a consequence of Seymour's "splitter" theory (see (7.6) of [4]), states that every connected binary matroid with no F_7^* minor can be built up using 2-sums[2] from copies of F_7 and regular matroids. Thus, the only 3-connected, binary, nonregular MFMC matroid is the Fano matroid F_7. The Truemper-Tseng theorem provides a similar result for the much more difficult class $\mathscr{M}$.

[2]Sums are not discussed in this paper. See [4] or [8].

We give a short proof of the Truemper-Tseng theorem. It should be noted, however, that neither this proof nor the original version in [9] includes a proof of Seymour's theorem. Both simply characterize the class $\mathscr{M}$. The difficulty results from the presence of the special element l and the fact that the Truemper-Tseng theorem involves 3-sums, rather than 2-sums. An alternative proof *can* be found in [8], which gives a polynomial-time algorithm to recognize membership in the class $\mathscr{M}$ and to solve the maximum-flow problem over this class.

3. PARTIAL REPRESENTATIONS

Partial representations were introduced in [5]. Most of the material in this section can be found in that paper.

Let X be a base of M. The *partial representation* B (or B^M) of the matroid M with respect to X is the $\{0,1\}$ matrix with rows indexed on elements $x \in X$ and columns indexed on elements $y \in Y = E - X$ such that the (x, y) entry is 1 iff $x \in C(X, y)$, where $C(X, y)$ is the unique circuit contained in $X \cup y$. We abbreviate the term partial representation by PR. PRs for F_7 and F_7^* are obtained from Figure 1 by deleting the identity columns. Indeed, the PRs that result in this case are actual representations, since, with the identity columns included and linear independence interpreted over the binary field, the independent sets of each matroid correspond exactly to the linearly independent subsets of columns in each matrix. PRs provide a generalization when no such representation over a field is available.

Note that if $x \in X$ and $y \in Y$, then deleting row x of B yields a PR, $B - x$, for the contraction minor M/x, and deleting column y of B yields a PR, $B - y$, for the deletion minor $M \backslash y$. Contractions and deletions of this type are said to be *visible* in B. For a fixed PR B, this definition yields a notion of *visible minor*: $N = M/X_0 \backslash Y_0$ is visible in B if $X_0 \subseteq X$ and $Y_0 \subseteq Y$. Note that the submatrix $B - \{X_0 \cup Y_0\}$ is a PR of N. Indeed, submatrices of B are in one-to-one correspondence with PRs of visible minors of M (with respect to B).

Pivoting on a nonzero element at position (x, y) in B means replacing the base X by the base $(X - x) \cup y$ and replacing B by the PR corresponding to this new base. In the case that B is an actual representation of M over some field, this pivot can be carried out in the usual linear-algebraic way, taking the missing identity matrix appropriately into account.

Crucial to several arguments in this paper are convenient PR interpretations of the notions "span" and "parallel." These interpretations stem from an appropriate definition of rank for PRs. Let $B(X', Y')$ denote a nonempty square submatrix of B with row index set X' and column index set Y'. We say that $B(X', Y')$ is *nonsingular* if $Y' \cup (X - X')$ is a base of M. Note that $|Y' \cup (X - X')| = |Y'| + |X| - |X'| = |X|$, since $B(X', Y')$ is square, so this set does have the proper cardinality to be a base. Note also that nonzero 1×1 submatrices are nonsingular, since for $x \in C(X, y)$, $(X - x) \cup y$ is a base. For arbitrary $X' \subseteq X$, $Y' \subseteq Y$ define the *matrix rank* of the submatrix $B(X', Y')$ by

$$\mathrm{rk}(B(X', Y')) = r(Y' \cup (X - X')) - |X - X'|,$$

where r is the usual matroid rank function for M.

LEMMA 1. *Let $X' \subseteq X$, $Y' \subseteq Y$, and suppose $B(X', Y')$ is nonzero. Then all maximal nonsingular submatrices of $B(X', Y')$ have $\mathrm{rk}(B(X', Y'))$ rows.*

Since 1×1 nonzero submatrices are nonsingular, it follows that nonzero submatrices have nonzero rank.

COROLLARY. *If the rank of a submatrix of a PR is reduced by deleting some of its columns (rows), then there is some single column (row) that can be reincluded to increase the rank.*

Define two columns (rows) of a submatrix of B to be *parallel* if they are each nonzero and together form a rank-one submatrix. Clearly, distinct $y_1, y_2 \in Y$ are parallel in M (form a circuit) iff columns y_1 and y_2 are parallel, and distinct $x_1, x_2 \in X$ are in series in M (form a cocircuit) iff rows x_1 and x_2 are parallel. A column (row) subvector is *spanned* by a submatrix if it is on the same row (column) set and appending it to the submatrix does not increase the rank of that submatrix. Note that two nonzero columns (rows) of a submatrix are parallel iff each one spans the other.

The *dual* of the matroid M on E is the matroid M^* on E with bases the complements of the bases of M. It is elementary to prove that where B is a PR of M, B^T is a PR of M^*.

LEMMA 2. *If B is a PR for M and A is a submatrix of B, then $\mathrm{rk}_B(A) = \mathrm{rk}_{B^T}(A^T)$, where rk_{B^T} is interpreted over M^*.*

This lemma is applied at the start of the proof of Lemma 7. The final result of this section implies that when computing matrix rank relative to a visible minor, we need not bother to specify which visible minor.

LEMMA 3. *If B is a PR of M, B' is a submatrix of B, and B'' is a submatrix of B', then*

$$\mathrm{rk}_B(B'') = \mathrm{rk}_{B'}(B''),$$

where rk_B is interpreted relative to M and $\mathrm{rk}_{B'}$ is interpreted relative to the visible minor corresponding to B'.

4. INDUCED SEPARATIONS

The key idea in the proof of the Truemper-Tseng result is the notion of an induced separation.

4.1. *Definitions*

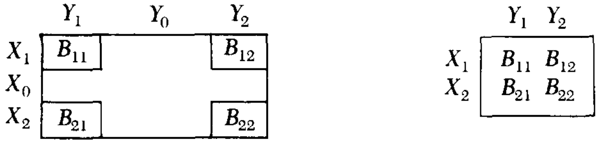
A *k-separation* of the matroid M, for $k \geqslant 1$, is a partition (T_1, T_2) of E such that $r(T_1) + r(T_2) \leqslant r(E) + k - 1$ and $|T_1| \geqslant k \leqslant |T_2|$; the k-separation is *exact* if $r(T_1) + r(T_2) = r(E) + k - 1$. M is *2-connected*, usually called simply *connected*, if it has no 1-separation, and *3-connected* if it has no 1- or 2-separation. Suppose N is a minor of M. We say that a k-separation (S_1, S_2) of N *induces* the k-separation (T_1, T_2) of M if $S_i \subseteq T_i$ $(i = 1,2)$.

We need interpretations of k-separations and induced k-separations in terms of PRs. To this end, suppose that $N = M/X_0 \backslash Y_0$ gives N as a visible minor of M with respect to a PR B of M determined by a base X. Let (S_1, S_2) be a k-separation of N, and denote $X_i = S_i \cap X$ and $Y_i = S_i \cap Y$ $(i = 1,2)$, where $Y = E - X$. Define $B_{ij} = B(X_i, Y_j)$. This situation is depicted in Figure 2.

Using the above notation, an equivalent definition of k-separation in terms of PRs is obtained from the computation

$$\begin{aligned}\mathrm{rk}(B_{21}) + \mathrm{rk}(B_{12}) &= r\big((\overline{X} - X_2) \cup Y_1\big) - |\overline{X} - X_2| \\ &\quad + r\big((\overline{X} - X_1) \cup Y_2\big) - |\overline{X} - X_1| \\ &= r(S_1) + r(S_2) - r(\overline{E}) \\ &\leqslant k - 1, \end{aligned} \tag{1}$$

where $\overline{X} = X_1 \cup X_2$ and $\overline{E} = E(N)$. Thus, a bipartition of the ground set of a matroid, in this case the matroid N, is a k-separation iff the blocks of the



PR for M PR for N

FIG. 2.

bipartition are sufficiently large and, in any PR of N, the corresponding "off-diagonal" submatrices determined by this bipartition have total matrix rank at most $k-1$.

Now, returning to the PR B of M in Figure 2, we see that finding an induced k-separation amounts to splitting B vertically somewhere inside Y_0 and horizontally somewhere inside X_0, so that the resulting submatrices of B that contain, respectively, B_{12} and B_{21} have matrix ranks not totaling more than $k-1$. In the important special case that (S_1, S_2) is exact, the containing submatrix must in each case have matrix rank equal to the matrix rank of B_{12} or B_{21}, whichever it contains.

4.2. *Two Lemmas on Induced Separations*

The proof of the next result is a simple exercise using matrix rank.

LEMMA 4. *If the k-separation (S_1, S_2) of N does not induce a k-separation of M, then B must have a submatrix of one of the following two types*:

$$\underset{\textit{Type } \mathrm{c}}{\boxed{\begin{matrix} B_{11} & e & B_{12} \\ B_{21} & f & B_{22} \end{matrix}}} \quad \textit{or} \quad \underset{\textit{Type } \mathrm{r}}{\boxed{\begin{matrix} B_{11} & B_{12} \\ e^T & f^T \\ B_{21} & B_{22} \end{matrix}}},$$

where in type c *the column vector e is not spanned by B_{12}, and in type* r *the row vector e^T is not spanned by B_{21}.*

Proof. If

$$\mathrm{rk}\left(\begin{bmatrix} B_{10} & B_{12} \end{bmatrix}\right) = \mathrm{rk}(B_{12}) \quad \text{and} \quad \mathrm{rk}\left(\begin{bmatrix} B_{01} \\ B_{21} \end{bmatrix}\right) = \mathrm{rk}(B_{21}),$$

then clearly M has an induced k-separation (T_1, T_2) where $T_1 = S_1$ and $T_2 = S_2 \cup X_0 \cup Y_0$. On the other hand if, say, $\mathrm{rk}([B_{10}\ \ B_{12}]) > \mathrm{rk}(B_{12})$, then type c occurs by the corollary to Lemma 1, and similarly for type r. ■

The proof of the following result uses a standard submodularity argument.

LEMMA 5. *If the k-separation (S_1, S_2) of the minor N is exact and does not induce a k-separation of M, but does induce a k-separation in any proper visible minor of M containing N, then B cannot have submatrices of types both* r *and* c.

Proof. If the lemma fails, then B has a submatrix of the following form, where e_r^T is not spanned by B_{21} and e_c is not spanned by B_{12}:

$$\begin{array}{cc} & y \\ x & \begin{array}{|ccc|} \hline B_{11} & e_c & B_{12} \\ e_r^T & & f_r^T \\ B_{21} & f_c & B_{22} \\ \hline \end{array} \end{array}$$

The assumed minimality of M implies that M/x has a k-separation (T_1', T_2') induced by (S_1, S_2). Thus

$$r'(T_1') + r'(T_2') - r'(E - x) \leqslant k - 1, \tag{2}$$

where r' is the rank function of M/x; moreover, since e_c is not spanned by B_{12}, we conclude, using (1) and Lemma 3, that $y \in T_1'$. Rewriting (2) in terms of r, we get

$$r(T_1' \cup x) - 1 + r(T_2' \cup x) - 1 - r(E) + 1 \leqslant k - 1. \tag{3}$$

Similarly, $M \backslash y$ has an induced k-separation (T_1'', T_2''), so that

$$r(T_1'') + r(T_2'') - r(E - y) \leqslant k - 1, \tag{4}$$

and e_r^T not spanned by B_{21} implies $x \in T_1''$. Adding (3) to (4) and applying submodularity, we have

$$\begin{aligned} 2(k-1) \geqslant\ & r(T_1' \cup T_1'') + r((T_1' \cap T_1'') \cup x) \\ & + r(T_2' \cup T_2'' \cup x) + r(T_2' \cap T_2'') - 2r(E) - 1. \end{aligned} \tag{5}$$

Now consider the partition $(T_1' \cap T_1'', T_2' \cup T_2'')$ of $E - \{x, y\}$. Let r'' be the rank function of $M/x\backslash y$. Since $S_1 \subseteq T_1' \cap T_1''$ and $S_2 \subseteq T_2' \cup T_2''$, and (S_1, S_2) is an exact k-separation of a minor of $M/x\backslash y$, (1) and Lemma 3 imply

$$r''(T_1' \cap T_1'') + r''(T_2' \cup T_2'') - r''(E - \{x, y\}) \geqslant k - 1.$$

Rewriting this inequality in terms of r yields

$$r((T_1' \cap T_1'') \cup x) + r(T_2' \cup T_2'' \cup x) - r(E) - 1 \geqslant k - 1. \tag{6}$$

Adding (5) and (6) and canceling terms yields

$$r(T_1' \cup T_1'') + r(T_2' \cap T_2'') - r(E) \leqslant k - 1.$$

But $(T_1' \cup T_1'', T_2' \cap T_2'')$ is a partition of E with $S_1 \subseteq T_1' \cup T_2''$ and $S_2 \subseteq T_2' \cap T_2''$, so we have found an induced k-separation of M, a contradiction. ■

4.3. Series-Parallel (S-P) *Reduction*

Let (M, N, S_1, S_2) be such that N is a minor of M, N has no loops or coloops, and (S_1, S_2) is a k-separation of N. For convenience assume that $N = M/X_0\backslash Y_0$ specifies N as a visible minor relative to a PR B for M. This causes no loss of generality, being equivalent to assuming that X_0 is independent and Y_0 is coindependent; this form for X_0 and Y_0 can always be arranged. Let $y' \in Y_0$ and $y \in S_1$ be such that y' and y are parallel in $M/X_0\backslash(Y_0 - y')$, noting that since N contains no loops, this condition forces $y' \in Y_0$, independent of the particular choice of Y_0. A *parallel reduction* of the 4-tuple (M, N, S_1, S_2) is the 4-tuple $(M\backslash y, N', (S_1 - y) \cup y', S_2)$ where $N' = (M\backslash y)/X_0\backslash(Y_0 - y')$. Clearly $N' \cong N$, using the bijection that maps y' to y and fixes all other elements. Note also that $((S_1 - y) \cup y', S_2)$ is a k-separation of N'. Let $x' \in X_0$ and $x \in S_1$ be such that x' and x are in series in $M/(X_0 - x')\backslash Y_0$. Then a *series reduction* of (M, N, S_1, S_2) is the 4-tuple $(M/x, N', (S_1 - x') \cup x, S_2)$, where $N' = (M/x)/(X_0 - x')\backslash Y_0$. As in the case of a parallel reduction, $N' \cong N$ and $((S_1 - x) \cup x', S_2)$ is a k-separation of N'. An *s-p reduction* is a reduction of either type, and an *s-p minor* is a 4-tuple obtained by a sequence of s-p reductions.

It is important to understand what these operations look like in PRs. Consider, for example, a parallel reduction. As in the definition, $y' \in Y_0$ and $y \in S_1$, so using the notation of Section 4.1, $y \in Y_1 \cup X_1$. If $y \in Y_1$, the condition that y' and y are parallel in $\overline{M} = M/X_0\backslash(Y_0 - y')$ says exactly that

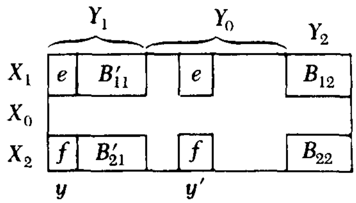


FIG. 3.

the parts of columns y' and y in rows $X_1 \cup X_2$ are parallel in B. This situation is pictured in Figure 3, where the appearance of common vectors e and f in columns y' and y reflects the parallelism condition. The indicated parallel reduction is obtained by deleting column y from Y_1 and replacing it by y'. If $y \in X_1$, the condition that y and y' are parallel in $\overline{M}$ says that when restricted to rows $X_1 \cup X_2$, column y' is an identity column with its 1 in row y; pivoting on the (y, y') entry exchanges y and y' in the basis X, and column y can then be deleted to obtain the indicated parallel reduction. Note that even though the above operations do change B, they do not affect the submatrix corresponding to N, either its nonzero pattern or the ranks of any of its submatrices. Note also that, given a starting PR, we can, in a natural way, associate a sequence of PRs with a sequence of s-p reductions, even though the reductions themselves need not *a priori* be tied to a particular PR. This observation makes it much easier to carry out PR based arguments for s-p reductions.

A *coloop* of a matroid is an element contained in every base. In what follows we need a characterization of when an element is a coloop of a specific deletion minor.

LEMMA 6.

(a) *$y \in Y_1$ is a coloop of $N \setminus S_2$ iff the portion of column y in B_{21} is not spanned by the remaining columns of B_{21}, and*

(b) *$x \in X_1$ is a coloop of $N \setminus S_2$ iff the portion of row x in B_{11} is spanned by the rows of B_{21}.*

Proof. To prove (a), note that y is not spanned by the remaining columns of B_{21} iff $\mathrm{rk}(B'_{21}) < \mathrm{rk}(B_{21})$, where B'_{21} is B_{21} with column y deleted. Applying the definition of rk then yields $r(S_1 - y) - |X_1| < r(S_1) - |X_1|$, which says exactly that y is in every base of $N \setminus S_2$, that is, that y is a coloop of $N \setminus S_2$. Similarly, x is spanned by the rows of B_{21} iff $r(S_1) - |X_1| = r(S_1 - x) - |X_1 - x|$, that is, iff $r(S_1) > r(S_1 - x)$. ■

The crucial tool in the proof of the Truemper-Tseng theorem is the following result, a special case of results in [7] (see Lemma 10.11).

LEMMA 7. *Assume that* (S_1, S_2) *is an exact k-separation of* N *and does not induce a k-separation of* M, *but that for any s-p reduction* (M', N', S_1', S_2) *of* (M, N, S_1, S_2), *the exact k-separation* (S_1', S_2) *of* N' *induces a k-separation of* M'. *Then the following conclusions hold*:

(a) *If* B *has a type-*r *submatrix, then*:

  (i) *If* f^T *is parallel to some row of* B_{12} *spanned by the remainder of* B_{12}, *then* $[e^T \ f^T]$ *is not parallel to any row of* $[B_{11} \ B_{12}]$.

  (ii) *If* e^T *has a* 1 *in some column and the part of this column in* B_{11} *is not spanned by* B_{12}, *then this is not the only* 1 *in* $[e^T \ f^T]$.

(b) *If* B *has a type-*c *submatrix, then*:

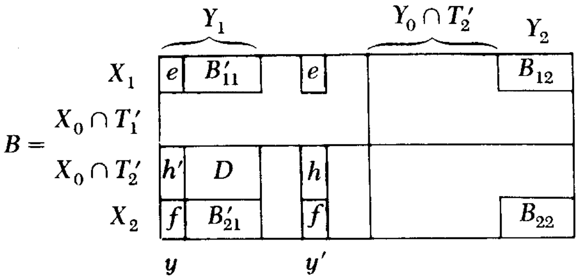
  (i) *If* f *is parallel to some column of* B_{21} *spanned by the remainder of* B_{21}, *then* $\begin{bmatrix} e \\ f \end{bmatrix}$ *is not parallel to any column of* $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$.

  (ii) *If* e *has a* 1 *in some row and the part of this row in* B_{11} *is not spanned by* B_{21}, *then this is not the only* 1 *in* $\begin{bmatrix} e \\ f \end{bmatrix}$.

Proof. It suffices to prove (b). (a) then follows by duality, using Lemma 2. In fact, it suffices to prove only (b)(i), for suppose (b)(ii) occurs. Let x be the row in which e has its 1, and let y' be the column containing e. By Lemma 6(b), x is not a coloop of $N \setminus S_2$; moreover, x is parallel to y' in $M/X_0 \setminus (Y_0 - y')$. Pick any element y of Y_1 such that the (x, y) entry of B is 1 (such an entry exists, since the part of row x in B_{11} is not spanned by B_{21}). Pivot on this entry. This leaves S_1 and S_2 unchanged. It simply exchanges y for x in the base, so that now $x \in Y_1$. But x is still not a coloop of $N \setminus S_2$, and x and y' are still parallel in $M/X_0 \setminus (Y_0 - y')$. Hence, after this pivot B has the form (b)(i) [using Lemma 6(a)]. Moreover, since (M, N, S_1, S_2) is the same, the required minimality property still holds. Thus, if (b)(i) is excluded, then so is (b)(ii).

Now we prove (b)(i). The idea is to perform a single-element s-p reduction, find the guaranteed induced k-separation for the reduced M, and then show that this k-separation induces a k-separation in M, a contradiction. We are assuming that B has a type-c submatrix. Let $y' \in Y_0$ be the column of B containing $\begin{bmatrix} e \\ f \end{bmatrix}$, and suppose, contrary to (b)(i), that there is an element $y \in Y_1$ parallel to y' in $M/X_0 \setminus (Y_0 - y')$. Then $(M \setminus y, N', (S_1 - y) \cup y', S_2)$ is a parallel reduction of (M, N, S_1, S_2), where $N' = (M \setminus y)/X_0 \setminus (Y_0 - y')$, and so the exact k-separation $((S_1 - y) \cup y', S_2)$ of N' induces a k-separation

(T_1', T_2') of $M \backslash y$. This situation may be pictured as follows:



where $[h' \;\; D]$ is a submatrix of $B_{01} = B(X_0, Y_1)$. When column y is deleted, the crossing horizontal and vertical lines above delineate the k-separation (T_1', T_2'). Thus, the upper right quadrant contains and has the same rank as B_{12}, and the lower left quadrant, which we denote B_{21}'', contains and has the same rank as B_{21}'.

If

$$\begin{bmatrix} D \\ B_{21}' \end{bmatrix} \quad \text{spans} \quad \begin{bmatrix} h' \\ f \end{bmatrix},$$

then so does B_{21}'', which implies $(T_1' \cup y, T_2')$ is a k-separation of M induced by N, a contradiction. Hence,

$$\mathrm{rk}\left(\begin{bmatrix} h' & D \\ f & B_{21}' \end{bmatrix}\right) > \mathrm{rk}\left(\begin{bmatrix} D \\ B_{21}' \end{bmatrix}\right) \geqslant \mathrm{rk}(B_{21}') = \mathrm{rk}(B_{21}),$$

where the last equality follows because we assumed B_{21}' spans f. We conclude that $[h' \;\; D]$, and hence B_{01}, has a row not spanned by B_{21}, and this row together with e contradicts Lemma 5. ■

5. MAIN THEOREM

Given two matroids N_1 and N_2 both containing a distinguished element l, an *l-isomorphism* of N_1 onto N_2 is an isomorphism that maps l onto itself.

Truemper-Tseng Theorem *Let M be a 3-connected, binary, nonregular matroid with distinguished element l. Assume $M \neq F_7$ and that M has no F_7^* minor containing l. Then M has a minor N containing l and with a PR as*

$$\begin{array}{c|cccc|} & i & j & k & l \\ \hline a & 1 & 1 & 1 & 0 \\ b & 1 & 1 & 0 & 1 \\ c & 1 & 0 & 1 & 1 \\ d & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

FIG. 4. Labeled PR for N from Truemper-Tseng theorem.

shown in Figure 4; *moreover*, M *has a* 3*-separation induced by one of the following* 3*-separations of any such* N: $(\{a, i, j, k\}, \{b, c, d, l\})$, $(\{a, b, c, i\}, \{d, j, k, l\})$, $(\{b, c, j, k\}, \{a, d, i, l\})$.

Proof. It is readily verified that the given 3-separations of N are in fact 3-separations, and that they are the only *big* 3-separations (S_1, S_2) of N, that is, the only 3-separations such that $|S_1| \geq 4 \leq |S_2|$.

Since M is binary but not regular, it contains either an F_7 or an F_7^* minor by Tutte's characterization of regular matroids [10]. If it contains no F_7^* minor, then it follows from the 3-connectedness of M and Seymour's splitter theory (see (7.6) of [4]) that $M = F_7$. Hence M does have an F_7^* minor. By assumption, l is not an element of this F_7^* minor, and so it follows, by another result of Seymour [2] and the connectedness of M, that M has a minor N such that N is connected, $l \in E(N)$, and either $N \backslash l$ or $N/l = F_7^*$. Note that this implies that N is in fact 3-connected, for otherwise l is in series with or parallel to some element of F_7^* and can be exchanged with this element to yield an F_7^* minor containing l, a contradiction.

Clearly, a PR for N can be obtained from one for F_7^* by addition of a single row if $N/l = F_7^*$ or by addition of a single column if $N \backslash l = F_7^*$. But adding *any* row violates 3-connectedness, so it must be that $N \backslash l = F_7^*$. Considering the ways that a single column e can be added, there are eight possibilities that do not produce parallel columns. Among these $e^T = [0 \ 1 \ 1 \ 1]$ produces an F_7^* minor containing l. The remaining seven are l-isomorphic to the one in Figure 4, $e^T = [0 \ 1 \ 1 \ 0]$. Fix a PR B of M such that the PR of N in Figure 4 is visible.

Suppose that none of the big 3-separations of N induces a 3-separation of M. Consider the particular 3-separation $(S_1, S_2) = (\{a, i, j, k\}, \{b, c, d, l\})$ of N. Since none of the big 3-separations of N induces a 3-separation of M, neither does this one. Let $\tilde{M}$ be a minimal visible minor of M with respect to B and with the properties that N is a visible minor of $\tilde{M}$, and $\tilde{M}$ has no 3-separation induced by (S_1, S_2). Now let (M', N', S_1', S_2) be an s-p minor of $(\tilde{M}, N, S_1, S_2)$ such that the 3-separation (S_1', S_2) of N' does not induce a

3-separation of M', but the corresponding 3-separation of any s-p reduction of (M', N', S_1', S_2) does. Clearly such an (M', N', S_1', S_2) exists. Let B' be the PR of M derived from B by the sequence of operations used to derive N'—see the discussion in the second paragraph of Section 4.3. Following the notation of Section 4.1, this situation may be viewed as below:

$$B' = \begin{bmatrix} B_{11}' & & B_{12}' \\ & & \\ B_{21}' & & B_{22}' \end{bmatrix} = \begin{array}{c} \\ a' \\ X_0' \\ b \\ c \\ d \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} i' & j' & k' & Y_0' & l \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & & 0 \\ & & & & \\ 1 & 1 & 0 & & 1 \\ 1 & 0 & 1 & & 1 \\ 0 & 1 & 1 & & 0 \end{array}\right] \end{array},$$

where a', i', j', and k' are the images of a, i, j, and k, respectively, under the l-isomorphism $N' \cong N$ generated by the s-p reductions that produced (M', N', S_1', S_2).

We first prove that *some* 3-separation of N' does induce a 3-separation of M. This proof uses the machinery set up in Lemmas 4–7. Then we show, by carefully using the properties of s-p reductions, that there is a corresponding inducing 3-separation in N. This last part of the proof depends on the special form of N and the particular elements in the inducing 3-separation that we find in N'. This fact explains the care taken surrounding the derivation of (7) below.

Given the minimality assumption on M' and N', we may apply Lemmas 4 and 7. Lemma 4 guarantees that a type-r row or a type-c column is present, and the two lemmas together restrict the form that this row or column can take. The possible type-c columns $\begin{bmatrix} e \\ f \end{bmatrix}$ have $e = [1]$ and $f^T = [1\ 1\ 1]$, $[1\ 0\ 0]$, $[0\ 1\ 0]$, or $[0\ 0\ 1]$. $e = [1]$ because e is not spanned by B_{12}'. No or two nonzeros in f^T are ruled out by Lemma 7(b)(ii) and (b)(i), respectively. Similarly, the possible type-r rows $[e^T\ f^T]$ have $e^T = [1\ 0\ 0]$, $[0\ 1\ 0]$, $[0\ 0\ 1]$, or $[1\ 1\ 1]$ and $f = [1]$.

It can be checked that the last of these row possibilities yields a visible F_7^* containing l, and that each of the first three yield an F_7^* containing l after just one pivot. We conclude that only the four column cases can occur, and these are all l-isomorphic. Consider the particular column case

$$\begin{bmatrix} e \\ f \end{bmatrix} = [1 \quad 1 \quad 1 \quad 1]^T.$$

Call the one-element extension of N' obtained by appending this column N'', and denote the element corresponding to this column by m.

Examining the 3-separations of N listed in the theorem, we see that the 3-separation $(\bar{S}_1', \bar{S}_2) = (\{b, c, j', k'\}, \{a', d, i', l\})$ of N' induces a 3-separation $(\bar{S}_1', \bar{S}_2 \cup m)$ of N''. After rearranging and performing a pivot on the (c, i') entry we obtain

$$B^{N''} = \begin{array}{c} \\ b \\ a' \\ i' \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} c & j' & k' & l & m \end{array} \\ \boxed{\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}} \end{array}. \tag{7}$$

Carrying out a similar calculation for each of the other three column possibilities,

$$\begin{bmatrix} e \\ f \end{bmatrix} = [1 \quad 1 \quad 0 \quad 0]^T, [1 \quad 0 \quad 1 \quad 0]^T, \text{and } [1 \quad 0 \quad 0 \quad 1]^T,$$

we obtain exactly the same matrix except that the column labels (k', j') are replaced by (i', a'), (a', i'), and (k', j'), respectively, and the row labels (a', i') are replaced by (k', j'), (j', k'), and (i', a'), respectively.

Now this N'' is a minor of M, and if the exhibited 3-separation does not induce a 3-separation of M, then we can take an s-p minimal example generated from N'' and M. Applying Lemmas 4 and 7, the possible new rows that can be added are [1 1 1 0 1], [1 0 0 0 1], [0 1 0 0 1], and [0 0 1 0 1]. In each case the zero in the fourth position follows from the previous claim that an F_7^* using l results in all the row cases in going from N' to N''.

Let's consider what happens if we add one of these rows to $B^{N''}$. Suppose we add [1 0 0 0 1]. Then a pivot on the (i', j') entry produces F_7^* using l. The remaining cases reduce to this one as follows: For [0 1 0 0 1] pivot on the (b, j') entry, for [0 0 1 0 1] pivot on the (b, k') entry, and for [1 1 1 0 1] pivot on the (b, c) entry. Thus, all row cases are excluded.

Now consider the column case. The transposes of the columns that can be added are [1 1 1 1], [1 1 0 0], [1 0 1 0], and [1 0 0 1]. These are all l-isomorphic—[1 1 1 1] may be obtained from each of the final three possibilities, by a single pivot on the entries (b, j'), (b, k'), and (b, c), respectively. We may therefore suppose that the column $[1 \ 1 \ 1 \ 1]^T$ has been added. But then a pivot on the (i', m) entry produces an F_7^* minor containing l. Hence, all column cases are excluded.

We have proved that the 3-separation $(\bar{S}_1', \bar{S}_2 \cup m)$ of N'' induces a 3-separation, say (T_1, T_2), of M. Clearly, this implies that the 3-separation $(\bar{S}_1', \bar{S}_2)$ of N' also induces (T_1, T_2). N' on the other hand was produced by a sequence of s-p reductions starting with N. To finish the proof of the theorem we find a k-separation $(\bar{S}_1, \bar{S}_2)$ of N that induces this same (T_1, T_2). The $(\bar{S}_1, \bar{S}_2)$ we take is the one identified with $(\bar{S}_1', \bar{S}_2)$ under the l-isomorphism between N and N' generated by these s-p reductions. We prove that $(\bar{S}_1, \bar{S}_2)$ has the desired property by showing that for each pair of elements exchanged in each s-p reduction used to obtain N', either both are in T_1 or both are in T_2. Let us assume that exactly one such s-p reduction is involved. The general case will then follow by induction.

Thus, we assume N' was obtained from N by an s-p reduction involving two elements z and z' where $z' \in E(M) - E(N)$ and $z \in S_1 = \{a, i, j, k\}$ (the exact form of S_1 is needed here!). Denote by B'' the PR of M obtained by performing on B' the (c, i') pivot used to obtain $B^{N''}$. Let X'' denote the corresponding base with $Y'' = E - X''$, and let X_0'' and Y_0'' be such that $B^{N''} = B'' - (X_0'' \cup Y_0'')$. There are now two cases to consider: when the reduction is a parallel reduction and when it is a series reduction. Consider the parallel case. Clearly $z \in Y_0''$ (otherwise contracting z to form N' from M leaves z' a loop). If $z' = j'$ or k', then column z of B'' must have a 1 in row b, and so $z, z' \in T_1$; otherwise, the inclusion of z in T_2 would imply that $\mathrm{rk}(B''(X'' \cap T_1, Y'' \cap T_2)) > \mathrm{rk}(B''(X'' \cap \bar{S}_1', Y'' \cap \bar{S}_2)) = 0$. On the other hand, if $z' = a'$ or i', then z must have exactly one 1 among the rows a', i', and d (namely, in row a' or i', respectively) and so $z, z' \in T_2$. Now consider the series case. Then we have $z \in X_0''$. If $z' = j'$ or k', row z has exactly one 1 in columns c, j', or k', and so $z, z' \in T_1$. Similarly, if $z' = a'$ or i', then row z must have a 1 in column l (which row d does not), and so $z, z' \in T_2$. This completes the proof in the case where the added column, m, has all 1s. The reamining three choices for column m are handled by the identical argument, given our observations following (7) about the way in which each choice alters the labeling of $B^{N''}$.

This completes the proof if just one s-p reduction was used to obtain N' from N. But this actually proves the result in general, by induction on the number of reductions, since for any single such s-p reduction only the identities of a', i', j', and k' can change and not their entries in the submatrix corresponding to the current copy of "N." ∎

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