On the Distribution of the Maximum Eigenvalue of Graphs

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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ABSTRACT

Given a graph G, let $\lambda(G)$ denote the largest eigenvalue of the adjacency matrix of G. We prove that for any $\lambda \geqslant \sqrt{2+\sqrt{5}}$ $(=2.058+)$ there exists a sequence of graphs $G_1, G_2, \dots$ such that $\lim_{k\to\infty}\lambda(G_k)=\lambda$, thus answering a question posed by Hoffman.

In [2] Hoffman defined **A** to be the set of all symmetric matrices (of all finite orders), every entry of which is a nonnegative integer. For $A \in \mathbf{A}$ let $\lambda(A)$ be the largest eigenvalue (spectral radius) of A. Let $\mathbf{R} = \{\lambda \mid \lambda = \lambda(A)$ for some $A \in \mathbf{A}\}$. Hoffman posed the problem of finding the limit points of **R**, and in [2] found all limit points of $\mathbf{R} \leqslant \sqrt{2+\sqrt{5}}$. Here we complete the solution of the problem posed by Hoffman by showing that every point in $[\sqrt{2+\sqrt{5}}, \infty)$ is a limit point of **R**. In fact we show this holds when **A** is restricted to be the set of adjacency matrices of finite graphs.

Our result is the following theorem.

THEOREM. *For any* $\lambda \geqslant \sqrt{2+\sqrt{5}}$ *there exists a sequence of graphs* $G_1, G_2, \dots$ *such that* $\lim_{k\to\infty}\lambda(G_k)=\lambda$.

Proof. Let

$$\phi_1 = \frac{\lambda - \sqrt{\lambda^2-4}}{2}, \qquad \phi_2 = \frac{\lambda + \sqrt{\lambda^2-4}}{2}$$

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be the roots of $x^2 - \lambda x + 1 = 0$. Define sequences $a_0, a_1, \ldots; n_0, n_1, \ldots$ recursively as follows: $a_0 = 1$, $a_1 = \lambda$, $n_0 = 0$,

$$n_k = \max\left\{ j \in \mathbf{Z} \,\middle|\, \lambda - \frac{a_{k-1}}{a_k} - \frac{j}{\lambda} \geqslant \phi_1 \right\}, \qquad k \geqslant 1, \tag{1}$$

$$a_{k+1} = \left[\lambda - \left(\frac{n_k}{\lambda}\right)\right] a_k - a_{k-1}, \qquad k \geqslant 1. \tag{2}$$

Let G_k be a tree consisting of a path $P_0 - P_1 - \cdots P_k$ with n_j leaves $Q_{j1}, \ldots, Q_{jn_j}$ connected to P_j, $j = 1, \ldots, k$ (see Figure 1). We claim $\lim_{k \to \infty} \lambda(G_k) = \lambda$.

We must first show that the $n_k \geqslant 0$ so that the G_k are well defined. Define $r_k = a_k / a_{k-1}$, $k = 1, \ldots$. Then

$$n_k = \max\left\{ j \in \mathbf{Z} \,\middle|\, \lambda - \frac{1}{r_k} - \frac{j}{\lambda} \geqslant \phi_1 \right\}, \qquad k \geqslant 1, \tag{3}$$

$$r_{k+1} = \lambda - \frac{n_k}{\lambda} - \frac{1}{r_k}, \qquad k \geqslant 1. \tag{4}$$

Note that $r_1 = \lambda > \phi_1$, and (3) and (4) imply $r_{k+1} \geqslant \phi_1$. Hence $r_k \geqslant \phi_1$ for any $k \geqslant 1$. Hence $\lambda - 1/r_k - \phi_1 \geqslant \lambda - 1/\phi_1 - \phi_1 = 0$. Therefore j may be assumed to be $\geqslant 0$ so $n_k \geqslant 0$.

Next note that G_k is a subgraph of G_{k+1}, so the $\lambda(G_k)$ form an increasing sequence. We now assert that $\lambda(G_k) < \lambda$ for all k. To see this consider the sequences a_j and r_j as functions of λ (while holding the sequence n_j fixed). We claim that λ' is an eigenvalue of G_k iff $a_{k+1}(\lambda') = 0$. For let $b_j = a_j / \lambda'$, $j = 1, \ldots, k$. Suppose v is an eigenvector of G_k. We may assume $v(P_0) = 1$. Then we may determine $v(P_j) = a_j$ and $v(Q_{ji}) = b_j$, $i = 1, \ldots, n_j$, $j = 1, \ldots, k$, recursively. For v to be an eigenvector we must have $\lambda' v(P_k) = v(P_{k-1}) + n_k b_k$, or $\lambda' a_k = a_{k-1} + n_k a_k / \lambda'$, or $(\lambda' -$

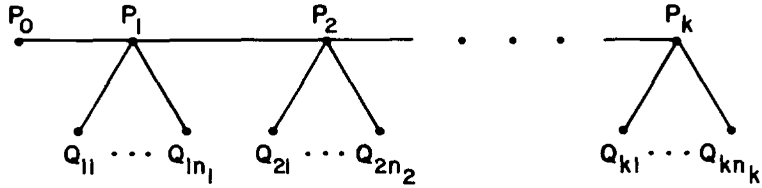

FIG. 1.

$n_k/\lambda')a_k - a_{k-1} = 0$, or $a_{k+1} = 0$ as claimed. Suppose $\lambda' > \lambda$. Then clearly $r_1(\lambda') > r_1(\lambda) \geqslant \phi_1 > 0$. Also $r_j(\lambda') > r_j(\lambda) \geqslant \phi_1 > 0$ together with (4) implies $r_{j+1}(\lambda') > r_{j+1}(\lambda)$, and we have already seen that $r_{j+1}(\lambda) \geqslant \phi_1 > 0$. Hence by induction we obtain $r_j(\lambda') > r_j(\lambda) \geqslant \phi_1 > 0$ for all $j = 1, \ldots$. Hence for $\lambda' > \lambda$, $a_{k+1}(\lambda') = r_{k+1}(\lambda')r_k(\lambda') \cdots r_1(\lambda') > r_{k+1}(\lambda)r_k(\lambda) \cdots r_1(\lambda) \geqslant \phi_1^{k+1} > 0$. Hence the largest root of $a_{k+1}(\lambda')$ is $< \lambda$ which means $\lambda(G_k) < \lambda$, as asserted.

We have shown so far that $\lambda(G_1) < \lambda(G_2) < \cdots < \lambda(G_k) < \cdots < \lambda$. To complete the proof it will suffice to show that for any $\varepsilon > 0$ we can find a k so that $\lambda(G_k) > \lambda - \varepsilon$. We need the following theorem. Let M be a symmetric matrix, let $\lambda(M)$ be the largest eigenvalue of M, and let v be any nonzero vector. Then $\lambda(M) \geqslant v^TMv/v^Tv$. For a proof see for example [1]. We apply this theorem as follows. Let M be the adjacency matrix of G_k. Let a vector v be defined on G_k as follows: $v(P_j) = a_j$, $j = 0, \ldots, k$; $v(Q_{ji}) = b_j = a_j/\lambda$, $i = 1, \ldots, n_j$, $j = 1, \ldots, k$. Then we have $v^TMv = \lambda v^Tv - a_k a_{k+1}$. Hence $\lambda(G_k) \geqslant \lambda - a_k a_{k+1}/v^Tv$. Now $v^Tv \geqslant a_1^2 + \cdots + a_k^2$. Suppose $a_j \leqslant B$ for all j. Then $\lim_{k\to\infty} a_k a_{k+1}/(a_1^2 + \cdots + a_k^2) = 0$ [for either $\lim_{k\to\infty} a_k = 0$ or $\lim_{k\to\infty}(a_1^2 + \cdots + a_k^2) = \infty$]. Hence it remains to show that the a_k are bounded. From (3) and (4) we know that r_{k+1} lies in the interval $[\phi_1, \phi_1 + 1/\lambda)$. Hence if $\phi_1 + 1/\lambda \leqslant 1$ (i.e. $\lambda \geqslant 2.325-$), then $r_{k+1} < 1$ for $k > 1$, which means that $a_1 > a_2 > \cdots$ and the theorem follows at once. For $\lambda \in [2.058+, 2.325-]$ the argument is more complicated.

Suppose $\lambda \in (\theta_1, \theta_2)$, where $\theta_1 = \sqrt{2+\sqrt{5}} = 2.058+$ and θ_2 is the root of $\phi_1 + 1/\lambda = 1$ (or $\lambda^3 - 3\lambda^2 + 2\lambda - 1 = 0$; $\theta_2 = 2.325-$). Then some of the r_k may be greater than 1, but any run of r_j's greater than 1 is of bounded length and is followed immediately by a run of r_j's less than 1 such that the product of the r_j's in both runs is less than 1. Suppose $1 \leqslant r_k < \phi_1 + 1/\lambda$ and $n_k > 0$. Then we claim $r_{k+1}r_k < 1$. For

$$r_{k+1} = \lambda - \frac{n_k}{\lambda} - \frac{1}{r_k}$$

$$\Rightarrow \quad r_k r_{k+1} \leqslant \left(\lambda - \frac{1}{\lambda}\right) r_k - 1 < \left(\lambda - \frac{1}{\lambda}\right)\left(\phi_1 + \frac{1}{\lambda}\right) - 1.$$

Now a simple but tedious calculation, which we omit, shows $\lambda \geqslant \sqrt{2+\sqrt{5}} \Rightarrow (\lambda - 1/\lambda)(\phi_1 + 1/\lambda) - 1 \leqslant 1$, so we have $r_k r_{k+1} < 1$, as claimed. Note since $r_k > 1$, this implies $r_{k+1} < 1$. Hence if $r_j, r_{j+1}, \ldots, r_{j+i}, r_{j+i+1}$ is a run of r_j's > 1, we must have $n_j = n_{j+1} = \cdots = n_{j+i} = 0$. Now consider the map $f(r) = \lambda - 1/r$. This map has fixed points ϕ_1 and ϕ_2 and is increasing on (ϕ_1, ϕ_2). Note $\lambda > \theta_1$ implies $\phi_1 + 1/\lambda < \phi_2$ (by another calculation which we omit). Hence if $1 \leqslant r_k < \phi_1 + 1/\lambda$, some iterate of f on r_k will exceed $\phi_1 + 1/\lambda$. If $f^{j+1}(r_k)$ is the first iterate $\geqslant \phi_1 + 1/\lambda$, then it is clear that $r_{k+i} = f^i(r_k)$, $i = 0, \dots, j$; $n_{k+i} = 0$, $i = 0, \dots, j-1$; $n_{k+j} > 0$. Hence $r_{k+j+1} < 1$ by the preceding claim. Hence runs of r_j's > 1 must be finite (and in fact can be bounded by the number of iterates of f on 1 which it takes to exceed $\phi_1 + 1/\lambda$; this number is clearly finite, as the iterates are increasing to ϕ_2 and we have $\phi_1 + 1/\lambda < \phi_2$). Suppose $r_j, \dots, r_k$ is such a run (i.e. $r_{j-1} \leqslant 1 < r_j < r_{j+1} < \cdots < r_k$; $n_j = \cdots = n_{k-1} = 0$; $n_k > 0$, so $r_{k+1} < 1$). We have shown $r_k r_{k+1} < 1$. We claim $r_{k-1} r_{k+2} < 1$. For note that $r_k = \lambda - 1/r_{k-1}$, $r_{k+2} \leqslant \lambda - 1/r_{k+1}$. Therefore $r_{k-1} = 1/(\lambda - r_k)$, so

$$r_{k-1} r_{k+2} \leqslant \frac{1}{\lambda - r_k}\left(\lambda - \frac{1}{r_{k+1}}\right) = \frac{\lambda r_{k+1} - 1}{\lambda r_{k+1} - r_k r_{k+1}} < 1,$$

since $r_k r_{k+1} < 1$. Similarly, $r_{k-1} r_{k+2} < 1 \Rightarrow r_{k-2} r_{k+3} < 1$, etc. Now $1 < r_j < \cdots < r_k$, so $r_{k-l} r_{k+1+l} < 1$ implies $r_{k+1+l} < 1$, $0 \leqslant l \leqslant k - j$. Also, $r_j r_{j+1} \cdots r_k r_{k+1} \cdots r_{k+1+k-j} < 1$, which establishes our claim that any run of r_j's > 1 is balanced by a following run of r_j's < 1. Thus the a_j remain bounded (in fact, slightly more careful estimates would show $a_j < C(1-\varepsilon)^j$ for some $\varepsilon > 0$.) This completes the proof of the theorem for $\lambda > \sqrt{2+\sqrt{5}}$. When $\lambda = \sqrt{2+\sqrt{5}}$ this construction fails, but the theorem still holds, since for any $\varepsilon > 0$ we may find a graph G by the above construction such that $\lambda < \lambda(G) < \lambda + \varepsilon$. Letting $\varepsilon \to 0$, we obtain a sequence of such graphs, so that $\lim_{k \to \infty} \lambda(G_k) = \lambda$ as desired. ∎

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