Subspaces with Well-Scaled Frames

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Robert E. Bixby

ABSTRACT

In this paper, classes of subspaces are introduced that generalize the regular (unimodular) subspaces of rational coordinate space. These spaces are shown to be of interest in the study of linear and integer programming. Fundamental properties of standard representative matrices for classes of subspaces are established. Certain subspaces are shown to be projectively equivalent to regular subspaces. Well-known examples of totally unimodular matrices arising from graphs and intervals of the real line are generalized in this context. A generalization of Heller's theorem, bounding the number of distinct columns of a rank-r, totally unimodular matrix, is established in this setting.

1. INTRODUCTION AND NOTATION

In this paper, a generalization of regular (unimodular) subspaces of rational coordinate space is presented. Implicit is a generalization of totally unimodular matrices. A rational matrix is *totally unimodular* if every square nonsingular submatrix has determinant ± 1. The *regular* subspaces are precisely those subspaces that can be expressed as the row or null spaces of totally unimodular matrices [40]. Such matrices include the constraint matrices for the node-arc formulation of the classical single commodity network flow problem. A further class of examples is given by the $\{0,1\}$-valued matrices having consecutive ones in each column [24]. Not only do totally unimodular matrices arise in practical contexts, their structure implies that

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the associated linear programming problems have integer-valued optima (whenever the data is integer-valued and an optimal solution exists) [24]. Moreover, such optima may be found quickly via combinatorial methods [7]. Totally unimodular matrices also arise in the study of abstract dependence, or matroid theory. Totally unimodular matrices correspond in a natural way to regular matroids [40]. Matroids in this class are determined by abstract dependence relations that can be achieved as linear dependence relations over any field.

A motivation for studying generalizations of the class of totally unimodular matrices is that linear and integer-linear programming problems having totally unimodular constraint matrices are easier to solve than arbitrary problems of similar dimensions. From an algorithmic viewpoint, a useful generalization of unimodularity retains the combinatorial properties of these matrices that make the associated programming problems easier to solve than unstructured problems of similar size. The generalization should be broad enough so that the associated class of programming problems includes significant problems that have been heretofore attacked generically. Using these two criteria, the results in this paper offer a fair amount of evidence in support of the generalization of unimodularity presented herein. However, the most important algorithmic issues are left unsettled.

Another reason for the study of generalizations of unimodularity is to find natural extensions of results concerning regular matroids to more general classes of representable matroids. This viewpoint is taken in [32].

Before giving an overview of the paper, the following remark is in order. The approach taken is to generalize the null (or row) spaces of totally unimodular matrices (namely, the regular subspaces), rather than the matrices themselves, since much of the practical and theoretical significance of the totally unimodular matrices lies in the structure of the associated spaces.

Section 2 contains definitions of some generalizations of regular subspaces. Also in this section, the behavior of subspaces under the usual monadic subspace operations (deletion, contraction, and orthogonalization) is established. In Section 3, it is shown how the behavior of a linear-programming augmentation scheme is related to the structure of the null space of its constraint matrix. In Section 4, it is shown that local search heuristics for integer programming problems may have favorable performance when the null space of the constraint matrix is well scaled. The relationship between restrictions on the subdeterminant values of a matrix and the structure of its row (or null) space is examined in Section 5. The results of this section, together with the discussion of orthogonalization in Section 2, serve to demonstrate that the generalizations presented in the present paper provide a richer structure than one would achieve by simply restricting matrix subdeterminants to some set of values. This is in sharp contrast to the equivalent manners in which regular subspaces can be defined. In Section 6, results of

Section 5 are used to show how the structure of the row space of a linear-programming constraint matrix yields basic solutions having well-behaved denominators when the right-hand side vector is integer-valued. Furthermore, the structure of the null space yields integer-valued basic solutions when the right-hand side is well behaved. The extent to which such linear programming properties can be used to characterize the structure of a subspace is also investigated.

The problem of finding a basis of a full-row-rank matrix M having minimum determinant magnitude is studied in Section 7. This problem is fundamental in the study of integer-valued basic solutions to the system $Mx = b$. Local search methods for finding a "minimum basis" are studied. In particular, it is shown how the structure of the row space of M can sometimes be exploited in the solution of this fundamental problem. In Section 8, it is shown that subspaces in a very restricted class of rational subspaces can be converted, by coordinate scaling, to regular subspaces. Algorithmic consequences, as they relate to Sections 3 and 7, are examined. Section 9 contains examples of "2-regular" subspaces that generalize well-known examples of regular subspaces. Linear programming problems having constraint matrices related to these subspaces can be solved rapidly by combinatorial methods after reformulating the programs. In Section 10, a generalization of Heller's theorem, bounding the number of distinct columns of a rank-r totally unimodular matrix, is established. This result, coupled with results of Section 3, has interesting algorithmic consequences in linear programming.

The notation used herein is standard. Possible exceptions to this are as follows. The symbol $\vec{0}$ (respectively, $\vec{1}$) denotes a vector with all entries equal to 0 (1). In a matrix, a single 0 is often used to denote a block of zeros. The field of rationals (respectively, reals) is denoted by $\mathbf{Q}$ ($\mathbf{R}$). The symbol $\mathbf{Z}$ (respectively, $\mathbf{Z}_{\oplus}$) denotes the integers (nonnegative integers). For a field $\mathbf{F}$, and matrix A over $\mathbf{F}$, $\text{n.s.}_{\mathbf{F}}(A)$ [respectively, $\text{r.s.}_{\mathbf{F}}(A)$] denotes the null space [row space] of A; the field is omitted when it is clear from context. For a matrix A, $A_{i\cdot}$ (respectively, $A_{\cdot j}, a_{ij}$) denotes row i (column j, the entry in row i and column j) of A. For a positive integer r, I_r denotes an identity matrix of order r. For a set S of positive integers, l.c.m.(S) denotes the least common multiple of the elements in S. For a real vector x, $\|x\|_\infty$ (respectively, $\|x\|_1$) denotes the usual ∞-norm (1-norm) of x. For a singleton e, the set $\{e\}$ is often abbreviated by e. The symbol $\dot{\cup}$ denotes disjoint set union.

2. DEFINITIONS AND FUNDAMENTALS

Let $\mathbf{F}$ be an arbitrary field. For any finite, nonempty set E, $\mathbf{F}^E$ denotes the vector space of $|E|$-tuples over $\mathbf{F}$ having coordinates indexed by the elements of E. For any $x \in \mathbf{F}^E$, the *support* of x, denoted $\underline{x}$, is the set of coordinates for which x is nonzero, i.e. $\underline{x} = \{ j \in E : x_j \neq 0 \}$. Let V be a vector subspace of $\mathbf{F}^E$. A vector $x \in V$ is an *elementary vector* of V if $x \neq \vec{0}$ and x has minimal support in $V \setminus \vec{0}$, i.e.,

$$x \in V \setminus \vec{0} \quad \text{and} \quad \nexists y \in V \setminus \vec{0} \text{ with } \underline{y} \subsetneq \underline{x}. \tag{2.1}$$

Tutte [40], Camion [10], Fulkerson [18], and Rockafellar [35] provide what appear to be the first explicit treatments of elementary vectors and their relation to matroid theory. The *frame* of V, denoted $\mathscr{F}(V)$, is simply the set of elementary vectors of V.

Let $\mathbf{F}$ be an ordered field. A vector $y \in \mathbf{F}^E$ *conforms to* $x \in \mathbf{F}^E$ if for each coordinate where y is nonzero, x is also nonzero and has the same sign as y, i.e.,

$$x_j y_j > 0 \qquad \forall j \in \underline{y}. \tag{2.2}$$

For V a subspace of $\mathbf{F}^E$, every nonzero vector $x \in V$ can be expressed as the sum of at most $\dim(V)$ vectors from $\mathscr{F}(V)$ with each summand conforming to x.

A new family of classes of subspaces will now be defined. Let V be a subspace of $\mathbf{F}^E$, where $\mathbf{F}$ is an arbitrary field, and let $S \subseteq \mathbf{F} \setminus 0$. The space V is *S-regular* (over $\mathbf{F}$) if every elementary vector of V can be scaled, by a nonzero of $\mathbf{F}$, so that all of its nonzero entries are in S, i.e.,

$$\forall x \in \mathscr{F}(V), \qquad \exists \lambda \in \mathbf{F} \setminus 0 \text{ such that } \lambda x_j \in S \ \forall j \in \underline{x}. \tag{2.3}$$

This generalizes the notion of a regular space, for which $\mathbf{F} = \mathbf{Q}$ and $S = \{-1, +1\}$. Another new family of classes of subspaces will be defined which has finer structure than the family of S-regular classes. Let V be a subspace of $\mathbf{F}^E$, where $\mathbf{F}$ is now an ordered field. Let T be a finite subset of positive elements from $\mathbf{F}$, and Ω be a nonnegative integer. The space V is *T-adic of order* Ω (over $\mathbf{F}$) if V is S-regular for the choice of

$$S = \left\{ \pm \prod_{t \in T} t^{p_t} : p_t \in \mathbf{Z}_{\oplus}\ (t \in T),\ \sum_{t \in T} p_t \leqslant \Omega \right\}. \tag{2.4}$$

A T-adic space of order 1 is $\pm(T \cup \{1\})$-regular. In the case that $\mathbf{F} \supseteq \mathbf{Q}$ and $T = \{k\}$, for a positive integer k, *k-adic* will be written instead of the more formal $\{k\}$-adic. *Dyadic* replaces the term 2-adic (following Zaslavsky[1] [42]). In the case that $\mathbf{F} \supseteq \mathbf{Q}$ and V is $\{1,2,\dots,k\}$-adic of order 1, V is also $\{\pm 1, \pm 2,\dots, \pm k\}$-regular, and is referred to as *k-regular*. A dyadic space of order 1 is 2-regular. A 1-adic space is always 1-regular, and is called *regular* [40] or *unimodular* [24].

Properties of a subspace are most easily established by representing them in some canonical manner. In the remainder of this section, representative matrices will be reviewed, and then the behavior of the subspaces defined above, under the usual monadic subspace operations, will be established.

Let V be an r-dimensional subspace of $\mathbf{F}^E$, where $\mathbf{F}$ is an arbitrary field. The space V can be *represented* as the row space of a matrix M, with entries from $\mathbf{F}$ and columns indexed by the elements of E. The *representative matrix* M is a *standard representative matrix* if it has r rows and it contains an $r \times r$ identity matrix as a submatrix. Any $r \times r$ nonsingular submatrix B of M is called a *basis of* M. In linear-algebraic terms, the columns of B form a basis for the column space of M. The definition here is more restrictive than the usual one, since the columns of B are required to be chosen from those of M. It is easy to see that for any choice of basis B of M, the rows of $B^{-1}M$ are in $\mathscr{F}(\text{r.s.}(M))$. Moreover, any $x \in \mathscr{F}(\text{r.s.}(M))$ can be obtained (up to nonzero scalar multiplication) as a row of $B^{-1}M$ for some basis B of M. Choosing a row i and column j of a full-row-rank matrix M, with m_{ij} nonzero, and performing the steps which follow is called a *pivot* in row i and column j (or, when unambiguous, a pivot on m_{ij}):

$$\begin{aligned}
&\text{(i) Replace each row } k \neq i \text{ with } M_{k\cdot} - \frac{m_{kj}}{m_{ij}} M_{i\cdot}.\\
&\text{(ii) Replace row } i \text{ with } \frac{1}{m_{ij}} M_{i\cdot}.
\end{aligned} \tag{2.5}$$

These row operations correspond to a step of Gauss–Jordan elimination for the related system of equations $Mx = b$. For any choice of basis B of M, M can be transformed into $B^{-1}M$ in at most r pivot steps.

Let V be a subspace of $\mathbf{F}^E$, where $\mathbf{F}$ is an arbitrary field. The *orthogonal space* of V, denoted $V^{\perp}$, is the set of vectors in $\mathbf{F}^E$ having zero inner product with every vector in V, i.e., $V^{\perp} = \{x \in \mathbf{F}^E : \langle x, y \rangle = 0 \ \forall y \in V\}$. If the characteristic of $\mathbf{F}$ is 0, then $V^{\perp}$ is the usual *orthogonal complement*. If V is the row space of a standard representative matrix $M = [I \mid A]$, then $V^{\perp} = \text{n.s.}(M) = \text{r.s.}([-A^{\mathrm{T}} \mid I])$.

[1] Zaslavsky has recently undertaken the study of $\{t\}$-adic subspaces over $\mathbf{R}$, where $t \in \mathbf{R}\backslash 0$.

Let V be a subspace of $\mathbf{F}^E$, where $\mathbf{F}$ is an arbitrary field. For any $F \subsetneq E$, there are two natural and useful maps that take V onto a subspace of $\mathbf{F}^{E \setminus F}$. The subspace V/F (read V *contract* F) is obtained by projecting V onto the subspace $\{x \in \mathbf{F}^E : x_e = 0\ \forall e \in F\}$, and then omitting the F coordinates. The subspace $V \setminus F$ (read V *delete* F) is obtained by intersecting V with the subspace $\{x \in \mathbf{F}^E : x_e = 0\ \forall e \in F\}$, and then omitting the F coordinates. For disjoint F and H, with $F \cup H \subsetneq E$, the following useful facts are immediate.

$$(V \setminus F) \setminus H = (V \setminus H) \setminus F, \tag{2.6a}$$

$$(V/F)/H = (V/H)/F, \tag{2.6b}$$

$$(V/F) \setminus H = (V \setminus H)/F, \tag{2.6c}$$

$$(V/F)^{\perp} = V^{\perp} \setminus F. \tag{2.6d}$$

Any subspace of V obtained by a sequence of deletions and contractions is called a *minor* of V.

Suppose that M is a standard representative matrix having V as its row space. For any $e \in E$, it is easy to obtain standard representative matrices for V/e and $V \setminus e$ from M. A standard representative matrix for V/e is obtained in the following manner. If the column of M indexed by e appears in every basis of M, then follow the procedure for $V \setminus e$. Otherwise, choose any basis B that does not contain the column indexed by e, form the matrix $B^{-1}M$, and remove the column indexed by e; this can be done in at most one pivot. A standard representative matrix for $V \setminus e$ is obtained in the following manner. If the column indexed by e is a $\vec{0}$ column, then follow the procedure for V/e. Otherwise, choose any basis B that contains the column indexed by e, form the matrix $B^{-1}M$, remove row l where the column of M indexed by e is the lth column of B, and then remove the column indexed by e; this also can be accomplished with at most one pivot. It is easy to see, via these constructions, that if V is S-regular, then so are minors of V.

A subspace V is *c-coregular* if $V^{\perp}$ is c-regular. The *regularity* of V is the smallest k for which V is k-regular. If there is no such k, the regularity is defined to be ∞. The *coregularity* of V is the regularity of $V^{\perp}$.

LEMMA 2.1. *The maximum coregularity of a k-regular space is* l.c.m.$(\{1,2,\ldots,k\})$.

Proof. Let V be k-regular. Given an arbitrary element of $\mathscr{F}(V^{\perp})$, a standard representative matrix $M = [I \mid A]$ for V can be chosen so that a nonzero scalar multiple of the element, say y, is a row of $M^{\perp} = [-A^{\mathrm{T}} \mid I]$. It is assumed, without loss of generality, that all entries of y have absolute value no greater than one; if this were not the case, a pivot on an entry of y with greatest magnitude would achieve this. Every entry of y is of the form $\pm\sigma/\tau$, with $\sigma \in \{0, 1, \ldots, \tau\}$ and $\tau \in \{1, 2, \ldots, k\}$ (since y has an entry of 1). The vector $y' = \mathrm{l.c.m.}(\{1, 2, \ldots, k\})y$ is an integer-valued vector and has no entry with magnitude exceeding $\mathrm{l.c.m.}(\{1, 2, \ldots, k\})$. Hence $\mathrm{l.c.m.}(\{1, 2, \ldots, k\})$ is an upper bound on the coregularity of a k-regular space V. The following example shows that this bound is sharp:

$$\left[\begin{array}{ccccc|c} 1 & & & & & 1 \\ & 1 & & 0 & & 1/2 \\ & & 1 & & & 1/3 \\ & 0 & & \ddots & & \vdots \\ & & & & 1 & 1/k \end{array}\right]. \qquad \blacksquare \quad (2.7)$$

This phenomenon extends to T-adic spaces of order Ω. The orthogonal space of a T-adic space of order Ω is always T-adic, but its order may be as large as $|T|\Omega$. For example, choose $T = \{t_1, t_2, \ldots, t_{|T|}\}$, where T consists of the first $|T|$ positive primes in $\mathbf{Z}$; the following matrix has a (rational) row space that is T-adic of order Ω, but a null space that is T-adic of order $|T|\Omega$:

$$\left[\begin{array}{ccccc|c} 1 & & & & & t_1^{-\Omega} \\ & 1 & & 0 & & t_2^{-\Omega} \\ & & 1 & & & t_3^{-\Omega} \\ & 0 & & \ddots & & \vdots \\ & & & & 1 & t_{|T|}^{-\Omega} \end{array}\right]. \qquad (2.8)$$

This example shows that the order of a T-adic space cannot be completely characterized by properties of standard representative matrices that are preserved under the "orthogonalization map" $[I \mid A] \to [-A^{\mathrm{T}} \mid I]$. One such property is the set of magnitudes of subdeterminants of the standard representative matrix. This is explored further in Section 5.

Perhaps the first algorithmic question which comes to mind after associating subspaces with matrices is whether or not the associated recognition problems are solvable in polynomial time. That is, for fixed S, is there a polynomial-time algorithm for testing whether or not the null space of a matrix is S-regular? By establishing a very deep structure theorem for regular subspaces, Seymour has solved the associated recognition problem [37]. The possibility of solving the recognition problem for 2-regular subspaces is remote. Coupling results from Section 8 of the present paper with Seymour's work leads to a solution of the recognition problem for k-adic subspaces when $k > 2$.

3. BLAND'S AUGMENTATION SCHEME

Consider the following canonical form of a linear programming problem over **Q**:

$$\begin{aligned} \max \quad & dx \\ \text{s.t.} \quad & Ax = b \\ \text{and} \quad & l \leqslant x \leqslant u. \end{aligned} \tag{3.1}$$

The matrix A is rational with column indices E. The vectors d and b are **Q**-valued, while the elements of l and u are from $\mathbf{Q} \cup \{\pm\infty\}$. The program (3.1) can be solved in an amount of time bounded by a polynomial in the length of its standard data encoding by a variety of algorithms (see [29], [33], and [26]). These algorithms are all noncombinatorial. Dantzig's simplex method [12] is a combinatorial algorithm for solving (3.1). Convergence of certain versions of the simplex method [5, 39] can be proven combinatorially. Convergence of any of the polynomial algorithms for (3.1) has not been verified using purely combinatorial arguments. For various highly restrictive classes of data for (3.1), there are algorithms that are both combinatorial and polynomial. In the case that n.s.(A) is regular, Bland and Edmonds [7] have supplied such an algorithm. Their procedure uses an augmentation scheme of Bland [4]. These techniques properly generalize the algorithms of Edmonds and Karp [17] for the maximum-flow and minimum-cost flow problems, when A is the node-arc incidence matrix of a directed graph. Below, the major ideas of Bland's augmentation scheme are reviewed. Highlighted are the features that are retained, and the major difficulty that arises, when n.s.(A) is k-regular with $k > 1$.

Let $V = \text{n.s.}(A)$. Suppose that $\dim(V) = r$. For the present, V will be considered to be a general rational subspace. It is supposed that x^0 is an initial feasible solution to (3.1). At step i of the algorithm, x^i is known and is a feasible solution. A vector $y^i \in \mathbf{Q}^E$ is called *unit augmenting with respect to* x^i if the following four properties hold:

$$y^i \in V, \tag{3.2a}$$

$$y^i_j \leqslant 0 \quad \text{if } x^i_j = u_j \qquad (j \in E), \tag{3.2b}$$

$$y^i_j \geqslant 0 \quad \text{if } x^i_j = l_j \qquad (j \in E), \tag{3.2c}$$

$$dy^i = 1. \tag{3.2d}$$

It is not difficult to show that feasible x^i is optimal if and only if there fails to exist a unit augmenting elementary y^i with respect to x^i. In the event that x^i is nonoptimal and y^i is an appropriate augmentation, x^{i+1} is set to $x^i + \lambda^i y^i$, where λ^i is chosen to be positive and as large as possible, i.e.,

$$\lambda^i = \max\big(\{\lambda : l \leqslant x^i + \lambda y^i \leqslant u\}\big). \tag{3.3}$$

The objective value of x^{i+1} is $dx^{i+1} = dx^i + \lambda^i$, an improvement. If the maximum in (3.3) does not exist, then (3.1) is unbounded. There is enough flexibility in this algorithm so that, if the elements of l and u are from $\mathbf{R} \cup \{\pm\infty\}$, it might approach a nonoptimal solution or never discover unboundedness.

Let the *length* of a unit augmenting y be $\|y\|_1$. The following result indicates that this augmentation scheme may be refined to have desirable properties in certain instances.

PROPOSITION 3.1 (Bland [4]). *If a shortest unit elementary augmentation is taken at each step, then the number of augmentations required is less than* $\nu|E|$, *where* $\nu = |\{\|y\|_1 : y \in \mathscr{F}(V),\ dy = 1\}|$.

A sketch of the proof of Proposition 3.1 provides some intuition. One first shows that on successive augmentations, the length of a shortest augmentation never decreases. By demonstrating that after no more than $|E|$ augmentations the length must increase, the result follows. Proposition 3.1 suggests the examination of pairs of V and d for which ν is small.

PROPOSITION 3.2. *If V is k-regular, and d is integer-valued, then*

$$\nu \leqslant \begin{cases} \frac{1}{2}\|d\|_\infty km(km+1) & \text{for} \quad m \leqslant \dfrac{\|d\|_1}{\|d\|_\infty}, \\ \|d\|_1 k^2 m + \frac{1}{2}\|d\|_1 k\left(1 - k\dfrac{\|d\|_1}{\|d\|_\infty}\right) & \text{for} \quad m \geqslant \dfrac{\|d\|_1}{\|d\|_\infty}, \end{cases}$$

where $m \equiv |E| - r + 1$.

Proof. Any $y \in \mathscr{F}(V)$ satisfies $|y| \leqslant |E| - r + 1$. If $dy = 1$, then y can be scaled by a positive λ so that $y' = \lambda y$ is $\{0, \pm 1, \pm 2, \dots, \pm k\}$-valued. Clearly, $1 \leqslant \|y'\|_1 \leqslant k(|E| - r + 1)$. The objective value dy' of y' satisfies $1 \leqslant dy' \leqslant \min(\{\|d\|_\infty \|y'\|_1, k\|d\|_1\})$. Since $\|y\|_1 = \|y'\|_1 / dy'$, the number of possible values for $\|y\|_1$ is at most

$$\sum_{i=1}^{k(|E|-r+1)} \min\{\|d\|_\infty i, k\|d\|_1\}.$$

Evaluating this sum gives the required bound. ■

The bound given by Proposition 3.2 is not sharp, but it is sufficient to suggest potentially reasonable algorithmic performance for classes of problems where k and $\|d\|_\infty$ grow no faster than polynomials in the length of the data encoding. If d is a standard unit vector, then $\nu \leqslant k^2 m$. In any case, ν is $O(\|d\|_1 k^2 m)$ asymptotically in m.

The problem of finding a shortest augmentation (by a combinatorial polynomial-time algorithm) has been solved by Bland and Edmonds [7] when V is regular and d is a standard unit vector. Their procedure makes use of Seymour's decomposition theorem for regular subspaces [37]. If d is an arbitrary integer-valued vector, and V is regular, Bland and Edmonds use a successive approximation scheme to solve (3.1), generalizing the Edmonds-Karp approach to the minimum-cost flow problem. Their procedure involves approximately $\log_2(\|d\|_\infty)$ major iterations, each of which consists of the solution of no more than $|E|$ problems with standard unit-vector objective functions. For $k > 1$, the problem of finding a shortest augmentation is open; a special 2-regular subclass, which has a simple solution, is discussed in Section 9. The problem is also solved for k-adic subspaces, when $k > 2$, in Section 8.

4. INTEGER-PROGRAMMING SEARCH REGIONS

Consider the canonical integer programming problem

$$\begin{aligned} \max \quad & dx \\ \text{s.t.} \quad & Ax = b, \\ & x \geqslant \vec{0} \text{ and integer-valued.} \end{aligned} \tag{4.1}$$

The program (4.1) is, in general, NP-hard; thus, one often considers heuristics for its solution. Suppose that $V = \text{n.s.}(A)$ is k-regular and $\dim(V) = r$.

PROPOSITION 4.1. *If* (4.1) *is feasible, and x^* is any feasible solution to the linear-programming relaxation of* (4.1), *then there exists a feasible solution x' to* (4.1) *with $\|x' - x^*\|_\infty \leqslant kr$.*

In particular, there is a feasible solution "near" any optimal solution to the linear-programming relaxation. It also follows that if x^* is optimal to the linear-programming relaxation of (4.1), and x' is an optimal solution to (4.1), then $dx' \geqslant dx^* - kr\|d\|_1$. In some instances, this lower bound on dx' may justify local search for an integer-valued feasible solution in the neighborhood of a linear-programming optimum. It also may be of use in branch-and-bound schemes to solve (4.1).

PROPOSITION 4.2. *If x^* is a feasible solution to* (4.1), *and there exists a feasible solution with greater objective value, then there exists a feasible solution x' to* (4.1) *with $\|x' - x^*\|_\infty \leqslant kr$ and $dx' > dx^*$.*

Proposition 4.2 suggests local improvement procedure for solving (4.1).

The proof of these results uses a well-known technique (see Bland [6] and Cook et al. [11]) which, for a general integer program with integer-valued constraint matrix A, produces a bound of Dr, where D is the magnitude of the largest subdeterminant of A. If n.s.(A) has regularity k, and A is a standard representative matrix with n columns, D may be as large as k^{n-r} and is always at least k (see Section 5). The structure of n.s.(A), not a more sophisticated proof technique, yields this improvement. It is worth remarking that proximity results of this type are not possible over $\mathbf{R}$.

Proof of Proposition 4.2. Let x^* be a feasible, but nonoptimal, solution to (4.1). Let x' be a closest feasible solution to x^* having greater objective value than that of x^*. Let $y' = x' - x^*$, and suppose that $\|y'\|_\infty > kr$. Since $\dim(V) = r$, $y' \in V$ can be expressed as a conformal sum of at most r vectors in $\mathscr{F}(V)$. Let L be the set of summands having ∞-norm greater than k. Observe that L is nonempty, since $\|y'\|_\infty > kr$. A positive scalar λ_y can be chosen, for each summand y, so that $\lambda_y y$ is $\{0, \pm 1, \dots, \pm k\}$-valued. For each $y \in L$, λ_y is less than 1. If $dy > 0$ for some $y \in L$, then $x^* + \lambda_y y$ has objective value greater than x^*, is feasible, and is closer to x^* than x' is. If $dy \leqslant 0$ for all $y \in L$, then $x^* + y' - \Sigma_{y \in L} \lambda_y y$ has objective value no less than that of x', is feasible, and is closer to x^* than x' is. The last fact follows because if $|y'_j| > kr$, then $|y_j| > 0$ for some $y \in L$. In either case, the choice of x' is contradicted, thus proving the proposition. ■

The proof of Proposition 4.1 is similar.

5. SUBDETERMINANTS OF STANDARD REPRESENTATIVE MATRICES

Totally unimodular matrices have an intimate connection with regular subspaces. Every regular subspace can be expressed as the row space of a totally unimodular matrix (in fact, a totally unimodular standard representative matrix). Conversely, the row space of any totally unimodular matrix is regular. In this section, subdeterminants of standard representative matrices of more general classes of subspaces will be explored. It will be shown how the equivalence between regular subspaces and totally unimodular matrices is not mimicked, in a straightforward manner, by more general classes of subspaces and matrices. Nonetheless, the results established will be exploited in the balance of the paper.

Let T be a finite subset of positive elements from an ordered field $\mathbf{F}$, and let Ω be a nonnegative integer. For each positive integer r, define

$$\delta(T, r, \Omega) = \left\{ \pm \prod_{t \in T} t^{p_t} : \sum_{\substack{t \in T \\ p_t > 0}} p_t \leqslant r\Omega,\ \sum_{\substack{t \in T \\ p_t < 0}} (-p_t) \leqslant r\Omega,\ p_t \in \mathbf{Z}\ (t \in T) \right\}. \tag{5.1}$$

Proposition 5.1. *The row space of an $m \times n$ standard representative matrix M is T-adic of order Ω only if every square nonsingular submatrix of order r $(1 \leqslant r \leqslant m)$ has its determinant in $\delta(T, r, \Omega)$.*

Proof. Let M be an $m \times n$ standard representative matrix, and suppose $V = \text{r.s.}(M) \subseteq \mathbf{F}^E$ is T-adic of order Ω. Suppose that R is an $r \times r$ nonsingular submatrix of M. Let V' be the r-dimensional subspace obtained from V by contracting all elements of E corresponding to nonbasic columns of M that do not include columns of R, and deleting all elements of E corresponding to basic columns of M that do not have their single 1 in a row of R, i.e.,

$$V' = \text{r.s.}(M'), \qquad \text{where} \quad R = \left[R' \;\middle|\; \frac{0}{I} \right] \text{ and } M' = [R' \mid I]. \tag{5.2}$$

The matrix M' has r rows; thus $R^{-1}M'$ can be pivoted to (from M') in at most r pivots. The magnitude of $\det(R)$ is the magnitude of the product of the associated pivot elements. Each of these elements is in $\delta(T, 1, \Omega)$, since every row of M' has an entry of 1. Hence, the result follows. ∎

COROLLARY 5.2. *There is a constant $C_{|T|}$, depending only on $|T|$, such that every $m \times n$ standard representative matrix with T-adic row space of order Ω has no more than $C_{|T|}(r\Omega)^{|T|}$ order-r $(1 \leqslant r \leqslant m)$ subdeterminant values.*

Proposition 5.1 has a partial converse that is fairly weak but useful.

PROPOSITION 5.3. *If all bases of an $m \times n$ full-row-rank representative matrix M have determinants in*

$$\left\{ \pm \prod_{t \in T} t^{p_t} : \sum_{\substack{t \in T \\ p_t > 0}} p_t \leqslant \Omega,\ \sum_{\substack{t \in T \\ p_t < 0}} (-p_t) \leqslant \eta,\ p_t \in \mathbf{Z}\ (t \in T) \right\},$$

then r.s.(M) (*and* n.s.(M)) *are T-adic of order $\eta + \Omega$. If M is integer-valued and T is a set of positive integers, then the spaces are T-adic of order Ω.*

Proof. The result is clear by Cramer's rule. ∎

It is of no use to restrict order-r nonzero subdeterminants $(1 \leqslant r \leqslant m)$ in order to obtain a stronger result than Proposition 5.3. Such restrictions will only yield a stronger result if they imply a stronger hypothesis on the determinants of bases. The pair of examples that follow indicate that more stringent bounds (or auxiliary conditions) must be placed on subdeterminants in order to characterize spaces of fixed order (such as k-regular spaces, $k \geqslant 2$):

$$[\, I_m \mid 2\cdot I_m \mid 3\cdot I_m \mid \cdots \mid k\cdot I_m \,], \tag{5.3}$$

$$\left[\begin{array}{ccccc|ccccc|ccccc|c|ccccc}
1 & & & & & 1 & 1 & & & & 2 & 1 & & & & & k & 1 & & & \\
 & 1 & & \mathbf{0} & & & 1 & 1 & & \mathbf{0} & & 2 & 1 & & \mathbf{0} & & & k & 1 & & \mathbf{0} \\
 & & 1 & & & & & 1 & 1 & & & & 2 & 1 & & \cdots & & & k & 1 & \\
 & & & \ddots & & & & & \ddots & \ddots & & & & \ddots & \ddots & & & & & \ddots & \ddots \\
 & \mathbf{0} & & & 1 & & \mathbf{0} & & & 1 & 1 & \mathbf{0} & & 2 & 1 & & \mathbf{0} & & & k & 1 \\
 & & & & & 1 & & & & & 1 & & & & 2 & & & & & & k
\end{array}\right]. \tag{5.4}$$

The matrices of (5.3) and (5.4) have the determinants of all of their square nonsingular submatrices of order r in $\{\pm\prod_{i=1}^{r}\tau_i : \tau_i \in \{1,2,\ldots,k\}\ (1 \leqslant i \leqslant r)\}$ $(1 \leqslant r \leqslant m)$. Moreover, for each r $(1 \leqslant r \leqslant m)$ and choices of $\tau_i \in \{1,2,\ldots,k\}$ $(1 \leqslant i \leqslant r)$, there is a submatrix of order r from each of these two matrices with determinant $\prod_{i=1}^{r}\tau_i$. The source of distress is that the row space of (5.3) has regularity k, while that of (5.4) has regularity k^m. This phenomenon is not surprising in light of Lemma 2.1 and the comments that follow it.

6. BASIC SOLUTIONS OF LINEAR PROGRAMS

Regularity assumptions involving the constraint matrix of a linear program determine certain scaling characteristics of the associated basic solutions. Consider the canonical linear program

$$\begin{aligned} \max \quad & dx \\ \text{s.t.} \quad & Mx = b \\ \text{and} \quad & x \geqslant \vec{0}. \end{aligned} \tag{6.1}$$

The matrix M is an $m \times n$ standard representative matrix, and all vectors are appropriately dimensioned. A *basic solution* of (6.1) is one that arises by choosing a basis B of M, and for each column $B_{.i}$ arising from column $M_{.\beta_i}$ of M, assigning x_{β_i} the value $(B^{-1})_{i.}b$. All other coordinates of x are set to zero. If a linear program has an optimal solution, then it has a basic optimal solution. The following proposition, which is easily verified by appealing to Cramer's rule, describes basic solutions to (6.1) under certain regularity hypotheses.

PROPOSITION 6.1. *Let T be a set of positive integers. If* r.s.$_{\mathbf{Q}}(M)$ *is T-adic of order Ω, and* n.s.$_{\mathbf{Q}}(M)$ *is T-adic of order γ, then for each basis B of M, there exist $\Delta_i \in \{\prod_{t \in T} t^{p_t}: p_t \in \mathbf{Z}_{\oplus}\ (t \in T),\ \sum_{t \in T} p_t \leqslant \Omega\}$ $(1 \leqslant i \leqslant m)$ and $\theta_i \in \{\prod_{t \in T} t^{p_t}: p_t \in \mathbf{Z}_{\oplus}\ (t \in T),\ \sum_{t \in T} p_t \leqslant \gamma\}$ $(1 \leqslant i \leqslant m)$ such that for all $z \in \mathbf{Z}^m$, the basic solution x determined by B and $b = z$ has $\Delta_i x_{\beta_i} \in \mathbf{Z}$ (for all basic indices β_i), and the basic solution $\bar{x}$ determined by B and $b =$* diag$(\theta_i)z$ *has $\bar{x}_{\beta_i} \in \mathbf{Z}$ (for all basic indices β_i).*

The proposition asserts that basic solutions satisfy a scaling property for arbitrary integer-valued b, and that basic solutions are integer-valued when b satisfies a scaling property. This has implications for "packing" (and "covering" programs). Consider, for example, the program

$$\begin{aligned} \max \quad & dx \\ \text{s.t.} \quad & Ax \leqslant b, \\ & x \geqslant \vec{0} \text{ and integer-valued,} \end{aligned} \tag{6.2}$$

where A is an $m \times n$ nonnegative matrix, and b is integer-valued. Suppose that (x^*, s^*) is an optimal basic solution to the program obtained from (6.2) by relaxing the integrality restriction and introducing the slack vector s. It is easy to see that the solution (x', s'), obtained from (x^*, s^*) by rounding down the components of x^* to integers, and adjusting the slack vector appropriately, satisfies $dx' \geqslant dx^* - m\|d\|_\infty$. This bound can be improved upon under regularity hypotheses. For example, if r.s.$([A \mid I])$ is 2-regular, then $dx' \geqslant dx^* - m\|d\|_\infty/2$. Similar bounds can be obtained for other regularity hypotheses.

A partial converse of Proposition 6.1 holds when T consists of a single positive prime integer k.

PROPOSITION 6.2. *Let k be a positive prime. Let M, an $m \times n$ full-row-rank representative matrix, have the property that there exist $q_i \in \mathbf{Z}_{\oplus}$ $(1 \leqslant i \leqslant m)$, satisfying $\sum_{i=1}^m q_i \leqslant \eta$, such that* diag$(k^{q_i})M$ *is integer-valued. If for* *each basis B of M there exist $p_i \in \mathbf{Z}_{\oplus}$ $(1 \leqslant i \leqslant m)$, with $\sum_{i=1}^{m} p_i \leqslant \Omega$, such that the basic solution x, determined by B and b, satisfies $k^{p_i} x_{\beta_i} \in \mathbf{Z}$ (for all basic indices β_i) for all integer-valued b, then* r.s.(M) *(and* n.s.(M)) *are k-adic of order $\eta + \Omega$.*

If M is integer-valued, then η may be taken to be 0, and the spaces will be k-adic of order Ω.

Proof. Let B be an arbitrary basis of M. The existence of $\{p_i : 1 \leqslant i \leqslant m\}$ implies that $\operatorname{diag}(k^{p_i}) B^{-1}$ is integer-valued. This follows by considering the choices of $b = e^l$ $(1 \leqslant l \leqslant m)$, where e^l denotes the lth standard unit vector. Now, $\operatorname{diag}(k^{q_i}) B$ is integer-valued; hence

$$\det(\operatorname{diag}(k^{q_i}) B) = (k^{\sum_{i=1}^{m} q_i}) \det(B)$$

is an integer. This quantity is also equal to

$$\frac{k^{\sum_{i=1}^{m} q_i + \sum_{i=1}^{m} p_i}}{\det(\operatorname{diag}(k^{p_i}) B^{-1})}.$$

This latter quantity has an integer denominator. Since it is equal to the previous quantity, it must also be an integer, and hence be of the form $\pm k^{\tau}$, with $0 \leqslant \tau \leqslant \sum_{i=1}^{m} q_i + \sum_{i=1}^{m} p_i$ (since k is assumed to be prime). Now

$$|\det(B)| = k^{\tau - \sum_{i=1}^{m} q_i};$$

hence $|\det(B)| = k^{\sigma}$ for some integer σ satisfying $-\eta \leqslant -\sum_{i=1}^{m} q_i \leqslant \sigma \leqslant \sum_{i=1}^{m} p_i \leqslant \Omega$. By Proposition 5.3, r.s.(M) [and n.s.(M)] are k-adic of order $\eta + \Omega$. ■

Proposition 6.2 generalizes Hoffman and Kruskal's characterization of totally unimodular matrices [24]. Their result is obtained by setting $\eta = \Omega = 0$, and letting k be an arbitrary prime.

7. MINIMUM BASES

Let M be a full-row-rank representative matrix over an ordered field $\mathbf{F}$. A basis B of M is a *minimum basis* if it has minimum magnitude determinant among all bases of M. The following simple lemma provides a useful canonical form for a representative matrix of r.s.(M). The lemma will be used in establishing Proposition 10.1.

LEMMA 7.1. *For any matrix M, of full-row-rank, over an ordered field* $\mathbf{F}$, *there exists a basis B of M such that all nonzero entries of $B^{-1}M$ have magnitude at least* 1.

Proof. It is sufficient to choose such a B that is a minimum basis of M. The result then follows by Cramer's rule. ∎

A further use of a minimum basis is in the study of integer-valued basic solutions to systems of equations. Let M be a full-row-rank integer-valued matrix. If B is a basis of M and $|\det(\mathrm{B})| = 1$, then the basic solution determined by B is integer-valued for all integer-valued vectors b. On the other hand, if $|\det(\mathrm{B})| > 1$, then B^{-1} must have a fractional entry, say $(B^{-1})_{ij}$. In this case $(B^{-1})_{i.}b$ is fractional for the choice of b equal to the jth standard unit vector. It is noteworthy that the system $Mx = b$ may have integer-valued solutions for all integer-valued vectors b, but M need not have a basis with determinant value ± 1. For example, let

$$M = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}. \tag{7.1}$$

The vector

$$x = (b_2, -b_1, b_1, b_1 - b_2)^{\mathrm{T}} \tag{7.2}$$

is an integer-valued solution to the system $Mx = b$ for all integer-valued vectors $b = (b_1, b_2)^{\mathrm{T}}$, but M has no basis having determinant ± 1.

No good algorithm seems to be known for finding a minimum basis of an arbitrary matrix, of full-row-rank, over an ordered field. For restricted classes of matrices M, the problem may be easier; consider the problem when M is restricted to be totally unimodular! In fact, the problem is solvable in polynomial time for a fairly broad class of matrices. Consider the following "local search" heuristic.

1-OPT.

(0) Given an $m \times n$ matrix M, of full row rank, over ordered field $\mathbf{F}$, choose an arbitrary basis B_0 of M. Let $M^{(0)} \leftarrow B_0^{-1}M$, $l \leftarrow 0$, and $\beta \leftarrow |\det(\mathrm{B}_0)|$.

(1) If all nonzero entries of $M^{(l)}$ have absolute value at least 1, then go to (2); otherwise, choose any row i and column j with $0 < |m_{ij}^{(l)}| < 1$ and pivot on $m_{ij}^{(l)}$. Let $\beta \leftarrow \beta |m_{ij}^{(l)}|$. Let $M^{(l+1)}$ be the resulting standard representative matrix. Let $l \leftarrow l+1$. Go to (1).

(2) Stop. Output basis B is the matrix solution to $BM^{(l)} = M$, and $|\det(B)| = \beta$.

1-OPT must terminate in a finite number of pivots since at iteration l, $\beta = |\det(B^{(l)})|$ (where $B^{(l)}$ denotes the matrix solution to $B^{(l)}M^{(l)} = M$). Since β decreases at every pivot, a basis is never repeated; since there are only a finite number of bases, 1-OPT must terminate.

1-OPT has some flexibility in that any pivot element with magnitude less than 1 can be selected in step (1). Consider the following refinement of 1-OPT.

Choose a pivot element that has the least nonzero magnitude in its row. (7.3)

PROPOSITION 7.2. *Let M be a full-row-rank matrix with k-adic row space (of arbitrary order) for some positive integer k. The procedure* 1-OPT, *with refinement* (7.3), *produces a minimum basis in no more pivots than the number of rows of M.*

Proof. Suppose a pivot occurs on $m_{ij}^{(l)}$ according to (7.3). Row i of $M^{(l+1)}$ will be integer-valued, since the entry of least nonzero absolute value in a k-adic elementary vector must evenly divide every entry of that vector. Every row of $M^{(l)}$ that is integer-valued will be integer-valued in $M^{(l+1)}$, since a row, say t, of $M^{(l+1)}$ is of the form $M_{t\cdot}^{(l+1)} = M_{t\cdot}^{(l)} - (m_{tj}^{l}/m_{ij}^{(l)})M_{i\cdot}^{(l)}$; the value $m_{ij}^{(l)}$ divides every entry of $M_{i\cdot}^{(l)}$ evenly, so if $M_{t\cdot}^{(l)}$ is integer-valued then so is $M_{t\cdot}^{(l+1)}$. Thus, 1-OPT with (7.3) must terminate in no more pivots than the number of rows of M. The basis B must be a minimum basis, since $B^{-1}M$ is integer-valued. All bases of $B^{-1}M$ have determinants of magnitude at least 1. The identity basis, I, has determinant of magnitude 1, and is thus a minimum basis for $B^{-1}M$. Now merely observe that B_1 and B_2 are bases of M with $|\det(B_1)| \leqslant |\det(B_2)|$ if and only if $B^{-1}B_1$ and $B^{-1}B_2$ are bases of $B^{-1}M$ with $|\det(B^{-1}B_1)| = |\det(B^{-1})|\,|\det(B_1)| \leqslant |\det(B^{-1})|\,|\det(B_2)| = |\det(B^{-1}B_2)|$. ∎

Consider the application of 1-OPT to the full-row-rank representative matrices of the class of T-adic spaces with $|T| \leqslant \tau$. Corollary 5.2 implies that

1-OPT will terminate within $O((r\Omega)^{\tau})$ pivots for any choice of pivot rule obeying the constraints of the algorithm; as usual, T is a finite set of positive elements from an ordered field $\mathbf{F}$, Ω the order of the T-adic space, and r its dimension. This bound can be improved upon for fixed T. Let $t_1 = \max(T)/\min(T)$, and let $t_2 = \max(\{\alpha : \alpha \in \delta(T, 1, \Omega),\ \alpha < 1\})$. Each pivot of 1-OPT reduces the magnitude of the determinant by at least a factor of t_2. The greatest possible reduction is by a factor of $t_1^{-r\Omega}$. The number of pivots in 1-OPT is, then, bounded above by the least integer s such that $t_2^s \leqslant t_1^{-r\Omega}$. Hence, $s = \lceil r\Omega\{-\log(t_1)/\log(t_2)\}\rceil = O(r\Omega)$. The constant in this bound depends on the relative magnitudes of the elements in T. This bound is arbitrarily large if only $|T|$ is fixed, whereas the $O((r\Omega)^{|T|})$ bound has its constant depending only on $|T|$ and not on the actual elements of T.

1-OPT may produce a nonminimum basis in the T-adic setting. The following example, of a matrix with a 3-regular row space, demonstrates this:

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \tag{7.4}$$

Indeed, in the general T-adic setting, 1-OPT may produce a basis that is only "locally minimum" in the sense that no adjacent basis has a determinant of smaller magnitude, but there are bases with smaller-magnitude determinant. It is worth noting that Lemma 7.1 only asserts the existence of a basis that is locally minimum. For a matrix with k-adic row space, a local minimizer is a global minimizer.

A natural extension of 1-OPT is p-OPT. Let M be an $m \times n$ full-row-rank representative matrix. In p-OPT, if there is a $t \times t$ $(1 \leqslant t \leqslant \min(p, m))$ nonsingular submatrix G of $M^{(l)}$ with $|\det(G)| < 1$, then G is extended to a basis H of $M^{(l)}$ by the addition of identity columns from $M^{(l)}$, and then $M^{(l+1)} \leftarrow H^{-1}M^{(l)}$. The magnitude of the determinant, $\beta^{(l+1)}$, of the matrix solution $B^{(l+1)}$ to $B^{(l+1)}M^{(l+1)} = M$ is $|\det(H)|\beta^{(l)} = |\det(G)|\beta^{(l)} < \beta^{(l)}$ (where $\beta^{(l)}$ denotes the magnitude of the determinant of the matrix solution $B^{(l)}$ to $B^{(l)}M^{(l)} = M$). For $p = 1$, this procedure is simply 1-OPT. It generalizes the local search idea in that p-OPT produces a basis for which there is no basis with smaller magnitude determinant differing in at most p columns. The work per iteration of p-OPT is the computation of no more than $\binom{m}{p} \times \binom{n}{p}$ subdeterminants, each requiring no more than $O(p^3)$ basic arithmetic operations, which, for any constant p, is a polynomial amount of computation. Moreover, the number of iterations, when the row space of the matrix is T-adic of order Ω, is also $O((r\Omega)^{|T|})$ for fixed $|T|$ and $O(r\Omega)$ for fixed T.

It would be interesting to find nice classes of matrices (of full-row-rank) for which 1-OPT may fail to produce a minimum basis, but for some constant

p, p-OPT always does. An obvious candidate is the 3-regular class—the simplest rational T-adic class that is not k-adic for any k. The following family of $m \times 2m$ ($m \geqslant 2$) examples, however, have row spaces of regularity 3, and have all $p \times p$ ($1 \leqslant p < m$) non-singular submatrices having determinants of magnitude either 1 or $\frac{3}{2}$, but each has a basis (the first m columns) with a determinant of $\frac{1}{2}$.

$$\left[\begin{array}{cccccc|ccccc} \frac{3}{2} & 1 & & & & & 1 & & & & \\ & 1 & 1 & & 0 & & & 1 & & 0 & \\ & & 1 & 1 & & & & & 1 & & \\ & & & \ddots & \ddots & & & & & \ddots & \\ & 0 & & & 1 & 1 & & 0 & & & 1 & \\ (-1)^m & & & & & 1 & & & & & 1 \end{array}\right]. \tag{7.5}$$

8. PROJECTIVE EQUIVALENCE

Let V_1 and V_2 be subspaces of $\mathbf{F}^E$, where $\mathbf{F}$ is an arbitrary field. The subspace V_1 is *projectively equivalent* to V_2 if there exist $\lambda_j \in \mathbf{F} \setminus 0$ ($j \in E$) such that $V_2 = \{y \in \mathbf{F}^E : x \in V_1,$ where $x_j = (1/\lambda_j)y_j\}$. Projective equivalence is clearly an equivalence relation. Let A^1 and A^2 be full-row-rank representative matrices over $\mathbf{F}$ with common column indices E. It is easy to see that n.s.$_{\mathbf{F}}(A^1)$ and n.s.$_{\mathbf{F}}(A^2)$ are projectively equivalent if and only if there exist nonsingular L and nonsingular diagonal D such that $A^2 = LA^1D$. The determination of an appropriate L and D is not difficult. Without loss of generality, assume that $A^l = [B_l \mid N_l]$, where B_l is a basis of A^l ($l = 1,2$). Let $M^l = B_l^{-1}N_l$ ($l = 1,2$). Apply the following well-known (see [3]) scaling technique to find nonsingular diagonal matrices D_1 and D_2 so that $D_1M^1D_2 = M^2$. Let $L = B_2D_1B_1^{-1}$ and $D = \mathrm{diag}(D_1^{-1}, D_2)$, so that $LA^1D = A^2$.

Let R (respectively, K) be the common set of row (column) indices of M^1 and M^2. Define a bipartite graph $G = (R \cup K, \mathscr{E})$ where

$$\mathscr{E} = \left\{(i,j) : i \in R,\ j \in K,\ m_{ij}^l \neq 0\ (l = 1,2)\right\}. \tag{8.1}$$

Observe that $m_{ij}^1 \neq 0$ if and only if $m_{ij}^2 \neq 0$, whenever n.s.(A^1) is projectively equivalent to n.s.(A^2). Now apply the following iterative procedure:

$$
\begin{array}{ll}
(0) & U \leftarrow R \cup K,\ L \leftarrow \varnothing, \text{ and } S \leftarrow \varnothing. \\
(1) & \text{If } U = \varnothing, \text{ then stop.} \\
 & \text{If } L = \varnothing, \text{ then choose any } x \in U, \\
 & \text{and set } \mu_x = 1,\ U \leftarrow U \setminus x, \text{ and } L \leftarrow \{x\}. \\
(2) & \text{Choose any } x \in L.\ L \leftarrow L \setminus x, \text{ and } S \leftarrow S \cup \{x\}. \\
 & \text{For each } y \in U, \text{ such that either } e = (x, y) \in \mathscr{E} \text{ or} \\
 & e = (y, x) \in \mathscr{E}, \text{ set } \mu_y = m_e^2/(m_e^1 \mu_x),\ U \leftarrow U \setminus y, \\
 & \text{and } L \leftarrow L \cup \{y\}. \\
 & \text{Go to step (1).}
\end{array}
\tag{8.2}
$$

The diagonal matrices, D_1 and D_2, are then determined by the $\{\mu_i\}$ $(i \in R \cup K)$ as follows:

$$
\begin{aligned}
d_{ii}^1 &= \mu_i \quad \text{for} \quad i \in R, \\
d_{ii}^2 &= \mu_i \quad \text{for} \quad i \in K.
\end{aligned}
\tag{8.3}
$$

Projective equivalence can play an important role in linear programming. Consider the program

$$
\begin{aligned}
\max \quad & dx \\
\text{s.t.} \quad & Ax = b \\
& \text{and} \quad l \leqslant x \leqslant u,
\end{aligned}
\tag{8.4}
$$

where A is $m \times n$, and all of the data are rational. Suppose that L is an $m \times m$ nonsingular matrix and D is an $n \times n$ diagonal matrix with $d_{ii} > 0$ $(i = 1, 2, \ldots, n)$. By letting $d' = dD$, $A' = LAD$, $b' = Lb$, $l' = D^{-1}l$, $u' = D^{-1}u$, and $x' = D^{-1}x$, the program (8.5) can be solved instead:

$$
\begin{aligned}
\max \quad & d'x' \\
\text{s.t.} \quad & A'x' = b' \\
& \text{and} \quad l' \leqslant x' \leqslant u'.
\end{aligned}
\tag{8.5}
$$

Linear programming algorithms that exploit the structure of the null space of the constraint matrix may benefit from this transformation if n.s.(A') is more manageable than n.s.(A).

PROPOSITION 8.1. *If a subspace V is k-adic, for some integer $k > 2$, then V is projectively equivalent to a unimodular subspace.*

This sharpens a result of Zaslavsky[2] [42]. The proof of Proposition 8.1 is most easily accomplished by the application of results from matorid theory, and appears in [32]. A sketch of the proof is given below.

Proof. If rational subspace V is not projectively equivalent to a unimodular subspace, then V has a subspace minor isomorphic to $\text{r.s.}_{\mathbf{Q}}(U_2^4)$, where

$$U_2^4 = \begin{bmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & a_3 & a_4 \end{bmatrix}. \tag{8.6}$$

The values a_1, a_2, a_3, and a_4 are nonzero and $a_1 a_4 - a_3 a_2 \neq 0$. This follows from results of Tutte [40] and of Brylawski and Lucas [8]. Since $\text{r.s.}_{\mathbf{Q}}(U_2^4)$ is also k-adic, U_2^4 is of the form

$$U_2^4 = \begin{bmatrix} 1 & 0 & \pm k^{p_1} & \pm k^{p_2} \\ 0 & 1 & \pm k^{p_3} & \pm k^{p_4} \end{bmatrix}, \tag{8.7}$$

with $(\pm k^{p_1+p_4}) - (\pm k^{p_2+p_3}) = \pm k^{p_5}$, for some integer p_5, by Proposition 5.1. This is impossible for an integer $k > 2$. ■

If A^1 is an $m \times n$ matrix having k-adic row space of order Ω, with $k > 2$, it is easy to find nonsingular L and positive diagonal D so that $A^2 = LA^1 D$ is totally unimodular. For simplicity, assume that A^1 has full-row-rank, and that $A^1 = [B_1 | N_1]$, where B_1 is a basis of A^1. Let $M^1 = B_1^{-1} N_1$, and let $M^2 = \text{sgn}(M^1)$ (i.e., $m_{ij}^2 = 0$ if $m_{ij}^1 = 0$, and $m_{ij}^2 = m_{ij}^1 / |m_{ij}^1|$ if $m_{ij}^1 \neq 0$). Proposition 8.1 implies that M^1 can have its rows and columns scaled by nonzeros so that it becomes totally unimodular. A result of Camion [9] states that any two totally unimodular matrices with the same support are scaling-equivalent. Therefore, the scaling procedure (8.2) can be applied to M^1 and M^2. It is not difficult to see that the matrix D thus determined has diagonal elements of the form k^τ, $-h\Omega \leqslant \tau \leqslant h\Omega$, where $h \leqslant 2\min(\{m, n-m\}) - 1$. The value h is merely the height (measured in edges) of the search tree generated by (8.2). In passing from the linear program (8.4) to (8.5), the objective $d' = dD$ is formed. With a standard binary data encoding, the number of bits needed to encode d' is polynomially bounded in the length of the encoding

[2]Zaslavsky has shown that a matroid represented by a $\{t\}$-adic subspace ($t \in \mathbf{R} \setminus 0$) can also be represented by a dyadic subspace.

of the original linear program. In particular, if d is integer-valued, and d' is scaled to be integer-valued, the number of bits used to encode d' is $O(n[\log(\max_{1 \leqslant j \leqslant n} d_j) + h \log k])$. The appearance of $h \log k$ in the bound is not disturbing, since it simply reflects the standard binary storage requirements of the matrix M^1. Hence, the Bland-Edmonds algorithm applied to (8.5) can be considered to be a polynomially bounded, combinatorial algorithm for the original program. The original program, however, may admit a more parsimonious encoding than the standard binary encoding. A nonzero, $\pm k^{\pm\tau}$ $(0 \leqslant \tau \leqslant \Omega)$, of M^1, is most concisely encoded in $2 + \log \tau$ bits (provided k is stored, once, with $\log k$ bits). This does not affect the scaling procedure (8.2), except to make it slightly faster; additions and subtractions replace multiplications and divisions, since the computations can be done on the exponents of k. A similar encoding of the resulting d' now only requires $O(n[\log(\max_{1 \leqslant j \leqslant n} d_j) + \log h] + \log k)$ bits. It is not apparent how the Bland-Edmonds algorithm can be made polynomially bounded with respect to this data-encoding scheme. Even the "strongly polynomial" scheme of Tardos [38] is not polynomial here, since arithmetic must be done on the elements of d' and it is not apparent how to retain this compact representation. It should also be noted that with this parsimonious data encoding, the noncombinatorial algorithms of Khachiyan [29], Karmarkar [26], and Levin and Yamnitsky [33] are not polynomially bounded either.

Another interesting consequence of Proposition 8.1 is an alternative method of finding minimum bases of full-row-rank representative matrices with k-adic null space (Section 7) when $k > 2$. Without loss of generality, let $A = [I \mid M]$ be a standard representative matrix with k-adic null space $(k > 2)$. Let D_1 and D_2 be diagonal matrices, with positive diagonal elements, so that $D_1 M D_2$ is totally unimodular. Therefore, $D_1 A D$ is totally unimodular, where $D = \operatorname{diag}(D_1^{-1}, D_2)$. The determinant of any basis of A, with column index set β, has magnitude $\det(D_1^{-1})\prod_{i \in \beta} 1/d_{ii}$. Hence, a basis that maximizes $\sum_{i \in \beta} \log_k(d_{ii})$ is a minimum basis of A. Such a basis can be found by applying Edmond's "greedy" algorithm [15].

9. EXAMPLES

A *mixed graph* is an ordered quadruple $G = (\mathscr{W}, \mathscr{E}_+, \mathscr{E}_-, \mathscr{A})$ where $\mathscr{W}$, the *vertex set*, is a finite set; $\mathscr{E}_+$ and $\mathscr{E}_-$, the *positive* and *negative edges*, respectively, are multisets of unordered pairs from $\mathscr{W}$; and $\mathscr{A}$, the *arc set*, is a multiset of ordered pairs from $\mathscr{W}$. An edge (but not an arc) may have both of its elements identical, in which case it is called a (positive or negative) *loop*. An arc (but not an edge) may have one (respectively, both) of its elements void, in which case it is called a *half-arc* (*zero-arc*). In the event that $\mathscr{E}_-$ and $\mathscr{A}$ are both empty, G is called an *undirected graph*; in this event, $\mathscr{E}_+$ is called the *edge set* and G is denoted $G=(\mathscr{W},\mathscr{E}_+)$. If $\mathscr{E}_+$ and $\mathscr{E}_-$ are both empty, G is called a *directed graph*; in this event, $\mathscr{W}$ is called the *node set*, and G is denoted $G=(\mathscr{W},\mathscr{A})$.

The *incidence matrix* M of $G(\mathscr{W},\mathscr{E}_+,\mathscr{E}_-,\mathscr{A})$ has a row for each vertex and a column for each edge and arc. For each positive (respectively, negative) loop $e=(w,w)$, $m_{w,e}=+2$ (-2). For all other positive (respectively, negative) edges $e=(w_1,w_2)$, $m_{w_1,e}=m_{w_2,e}=+1$ (-1). For each arc $a=(w_1,w_2)$, $m_{w_1,a}=-m_{w_2,a}=+1$. All unspecified entries of M are zero.

The term "mixed graph" has appeared, at least informally, elsewhere (see [30]). Mixed graphs, and their incidence matrices, have been studied previously under the names *bidirected graphs* [14] and *signed graphs* [20, 41]

The rational null space of a mixed (respectively, directed) graph incidence matrix is called *mixed graphic* (*graphic*), and the rational row space is called *mixed cographic* (*cographic*). All of these properties of a subspace are preserved under the minor operations of deletion and contraction (see [41]). Mixed graphic (and mixed cographic) spaces are dyadic, as is implicit in Balinski [1] and as can easily be deduced from Zaslavsky [41]. That they are actually 2-regular can be demonstrated by using a basis characterization of Zaslavsky [41], and then noting the structure of *adjacent bases*, bases differing in just one column. The entirety of these arguments is presented below.

Proposition 9.1. *Mixed graphic (and mixed cographic) spaces are 2-regular.*

Proof. Let G be a mixed graph with vertex set $\mathscr{W}$. Without loss of generality, assume that G has a single copy of every possible edge and arc for a mixed graph on $\mathscr{W}$. Let M be the incidence matrix of G. It will be shown that $V=\text{r.s.}_{\mathbb{Q}}(M)$ is 2-regular, and hence so are $V^{\perp}$ and all minors of V and $V^{\perp}$ (which allows the choice of G above). First, it will be demonstrated that V is dyadic. The matrix M is a standard representitive matrix. Suppose that B is a basis of M. A square submatrix, B^0, is a *block* of B if there exist permutation matrices Π^1 and Π^2 such that

$$\Pi^1 B \Pi^2 = \left[\begin{array}{c|c} B^0 & 0 \\ \hline 0 & * \end{array}\right]. \tag{9.1}$$

The submatrix B^0 is *indecomposable* if it has no nontrivial blocks. It will be shown that $|\det(B)|$ is a power of 2. In doing so, there is no loss of generality in assuming that B is indecomposable. If B has any columns or rows with a single nonzero, the determinant of B can be expanded along them and a multiplicative factor of a power of 2 will be accumulated for the absolute value of the determinant of B. This process may be continued until a matrix, say B', remains which has exactly two nonzeros (all necessarily of unit magnitude) in every column and row. Consider the representation of the determinant of the reduced matrix B', of order say p, as the sum of $p!$ terms each of which has as absolute value the product of p entries from B'. The terms arise by considering the signed product of all $p!$ choices of p entries from B', exactly one entry from each row and column. Only two of these $p!$ terms can be nonzero (hence ± 1), since for either choice of the two nonzeros in row 1 of B', the choice of the remaining nonzeros is uniquely determined to obtain a nonzero product. Therefore $\det(B') = (\pm 1) + (\pm 1) = \pm 2$, since $\det(B') \neq 0$. Therefore the absolute value of the determinant of B is a power of 2, and hence V is dyadic by Proposition 5.3.

The determinant of B is $\pm 2^{T+C}$, where T is the number of entries in B of value ± 2, and C is the number of square indecomposable submatrices of B with exactly two nonzeros in every row and column. Consider an adjacent basis $\hat{B}$. In passing from B to $\hat{B}$, a column of B is deleted, which causes $T + C$ to decrease by 0 or 1, and then a column is appended, which causes $T + C$ to increase by 0 or 1. Hence, $|\det(\hat{B})| / |\det(B)| = \frac{1}{2}$, 1, or 2. Therefore V is 2-regular. ∎

An important subclass of the loop-free mixed graph incidence matrices are those that satisfy (9.2) below:

> The rows can be partitioned into two sets T_1 and T_2 (one of which may be empty), so that for each column in which there are two nonzeros, these entries appear in the same part of the partition if and only if they have opposite signs. (9.2)

A well-known result [22, 23] states that a loop-free mixed-graph incidence matrix is totally unimodular if and only if it satisfies (9.2). A somewhat larger class of mixed-graph incidence matrices have regular null spaces, as the well-known [2] example below indicates:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}. \tag{9.3}$$

There are two familiar optimization problems associated with the incidence matrix M of a loop-free, zero-arc-free mixed graph:

$$\begin{aligned} \max \quad & dx \\ \text{s.t.} \quad & g \leqslant Mx \leqslant b, \\ & l \leqslant x \leqslant u \text{ and integer-valued.} \end{aligned} \tag{9.4}$$

and

$$\begin{aligned} \max \quad & fy \\ \text{s.t.} \quad & l \leqslant M^{\mathsf{T}} y \leqslant u, \\ & g \leqslant y \leqslant b \text{ and integer-valued.} \end{aligned} \tag{9.5}$$

The programming problem (9.4) has been well solved [13, 14, 16], and is called the bidirected flow problem. It has as special cases the single-commodity network flow problem and the degree-constrained subgraph problem. The program (9.5) is NP-hard in general. It remains NP-hard if $f = \vec{1}$, $l = \vec{0}$, $u = \vec{1}$, $b = \vec{1}$, $g = \vec{0}$, and M is the incidence matrix of an undirected graph [27]. In this instance, (9.5) is the maximum-cardinality vertex-packing problem. It is polynomially solvable, in the general case, if M satisfies (9.2). If the integrality restriction is removed from (9.5), the "relaxed" problem can be solved by a polynomial-time combinatorial algorithm by reducing the problem to another (relaxed) instance of (9.5) for which the incidence matrix satisfies (9.2). Suppose that M has row set R and column set E. Construct a matrix M' with row set $R' = R_1 \cup R_2$ and column set $E' = E_1 \cup E_2$, where $R_l = \{r_l : r \in R\}$ $(l = 1,2)$ and $E_l = \{e_l : e \in E\}$ $(l = 1,2)$. The nonzero entries of M' are as follows. For each column e of M with a 1 (respectively, -1) in both of rows i and j, M' has a column, e_1, with a 1 (-1) in both of rows i_1 and j_2, and another column, e_2, with a 1 (-1) in both of rows i_2 and j_1. For each column e of M with a 1 in row i and a -1 in row j, M' has a column, e_1, with a 1 in row i_1 and a -1 in row j_1, and another column, e_2, with a 1 in row i_2 and a -1 in row j_2. For each column e of M with a single nonzero $\varepsilon = \pm 1$ in row i, M' has a column, e_1, with ε in row i_1, and another column, e_2, with ε in row i_2. The matrix M' is totally unimodular, as can be seen by the partition of the rows $T_1 = R_1$ and $T_2 = R_2$. An equivalent linear programming problem, over the constraint matrix M', has objective f' defined by $f'_{i_1} = f'_{i_2} = f_i/2$ $(\forall i \in R)$, constraint bounds l' and u' defined by $l'_{e_1} = l'_{e_2} = l_e$

and $u'_{e_1} = u'_{e_2} = u_e$ ($\forall e \in R$), and variable bounds g' and b' defined by $g'_{i_1} = g'_{i_2} = g_i$ and $b'_{i_1} = b'_{i_2} = b_i$ ($\forall i \in R$). The solution $y_i = (y'_{i_1} + y'_{i_2})/2$ $(i \in R)$ is feasible for the original problem if y'_{i_1}, y'_{i_2} $(i \in R)$ is feasible for the derived problem. Moreover, the objective values are identical. Also, if y_i $(i \in R)$ is feasible for the original problem, then $y'_{i_1} = y'_{i_2} = y_i$ $(i \in R)$ is feasible for the derived problem; again, the objective values are identical.

Two new classes of matrices will now be introduced which have 2-regular null (and row) spaces. Let $\mathscr{I}$ and $\mathscr{J}$ be finite sets of intervals of the real line, and let $\mathscr{P}$ be a finite set of points on the line. Define a matrix A, with column set $\mathscr{I} \cup \mathscr{J}$ and row set $\mathscr{P}$, as follows:

$$a_{pI} = \begin{cases} 1 & \text{if} \quad p \in I \quad (I \in \mathscr{I}), \\ 0 & \text{if} \quad p \notin I \quad (1 \in \mathscr{I}), \\ -1 & \text{if} \quad p \in L(I) \quad (I \in \mathscr{J}), \\ 1 & \text{if} \quad p \in R(I) \quad (I \in \mathscr{J}), \\ 0 & \text{if} \quad p \in I \quad (I \in \mathscr{J}) \end{cases} \tag{9.6}$$

for $p \in \mathscr{P}$, $I \in \mathscr{I} \cup \mathscr{J}$. Here $L(I)$ denotes $\{\alpha : \alpha < \beta \ \ \forall \beta \in I\}$; similarly $R(I) = \{\alpha : \alpha > \beta \ \ \forall \beta \in I\}$. The matrix A is called a *skew interval matrix*. If $\mathscr{J} = \varnothing$, then A is called an *interval matrix* [19]. Another new class of matrices are those with the *skew consecutive-ones property*. A matrix has this property if its rows can be permuted so that each column is of the form

$$\begin{aligned} &\pm(0,0,\ldots,0,1,1,\ldots,1,0,0,\ldots,0)^{\mathrm{T}}, \\ \text{or} \quad &\pm(-1,-1,\ldots,-1,0,0,\ldots,0,1,1,\ldots,1)^{\mathrm{T}}, \end{aligned} \tag{9.7}$$

where any of the continuous substrings may have length zero. Every skew interval matrix has the skew consecutive-ones property, as can be seen by ordering the rows of the matrix so as to correspond to the natural ordering of the associated points on the line. A nonnegative matrix with the skew consecutive-ones property is said to have the *consecutive-ones property* [24]. Matrices with the the consecutive-ones property are known to be totally unimodular, and to have graphic null spaces [24].

PROPOSITION 9.2. *A matrix A has the skew consecutive-ones property only if* $\text{n.s.}_{\mathrm{Q}}(A)$ *is mixed graphic.*

Proof. Suppose that A has r rows. Let M be the incidence matrix of a complete mixed graph on r vertices. Consider the choice of basis

$$B = \begin{bmatrix} 1 & & & & & & 1 \\ -1 & 1 & & & & \mathbf{0} & \\ & -1 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & \mathbf{0} & & -1 & & 1 & \\ & & & & & -1 & 1 \end{bmatrix}. \tag{9.8}$$

For $M_{\cdot j} = \pm(0,0,\ldots,0,1,0,\ldots,0,\ -1,0,\ldots,0)^{\mathrm{T}}$,

$$B^{-1}M_{\cdot j} = \pm(0,0,\ldots,0,1,1,\ldots,1,0,0,\ldots,0)^{\mathrm{T}}.$$

For $M_{\cdot j} = \pm(0,0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0)^{\mathrm{T}}$,

$$B^{-1}M_{\cdot j} = \pm(-1,\ -1,\ldots,\ -1,0,0,\ldots,0,1,1,\ldots,1)^{\mathrm{T}}.$$

For $M_{\cdot j} = \pm(0,0,\ldots,0,2,0,\ldots,0)^{\mathrm{T}}$,

$$B^{-1}M_{\cdot j} = \pm(-1,\ -1,\ldots,\ -1,1,1,\ldots,1)^{\mathrm{T}}.$$

For $M_{\cdot j} = \vec{0}$,

$$B^{-1}M_{\cdot j} = \vec{0}.$$

The matrix $B^{-1}M$ has all possible columns of A (and, in addition, columns of the form $\pm(-\frac{1}{2},\ -\frac{1}{2},\ldots,\ -\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})^{\mathrm{T}}$). Therefore, n.s.$(A)$ is a minor of n.s.(M), and thus is mixed graphic. ■

An example of a skew consecutive-ones matrix A for which n.s.(A) is regular but not graphic is the well-known matrix [2] below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{9.9}$$

This matrix has precisely the same null space as the matrix of (9.3). A simple example of a matrix with the skew consecutive-ones property having null space of regularity 2 is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}. \tag{9.10}$$

Another area in which the study of regularity may be useful is in the consideration of 2-commodity flow problems. Let $G = (N, \mathscr{A})$ be a directed graph with the property that if $a = (v_1, v_2) \in \mathscr{A}$ then $-a = (v_2, v_1) \notin \mathscr{A}$. Let A be the node-arc incidence matrix of G. The following linear program is under consideration:

$$\begin{aligned} \max \quad & c^1x^1 + c^2x^2 \\ \text{s.t.} \quad & Ax^1 = \vec{0}, \\ & Ax^2 = \vec{0}, \\ & -u \leqslant Ix^1 + Ix^2 \leqslant u, \\ & -u \leqslant Ix^1 - Ix^2 \leqslant u. \end{aligned} \tag{9.11}$$

This linear program is equivalent to the problem of finding a flow of two commodities through a network to maximize a linear profit function on the arc flows of the two commodities. There are symmetric (about 0) lower and upper bounds on the total flow in each arc.

The program (9.11) can be reformulated in a simple way by defining variables y^1 and y^2 implicitly as follows.

$$\begin{aligned} x^1 &= y^1 + y^2 - u, \\ x^2 &= y^1 - y^2. \end{aligned} \tag{9.12}$$

The program (9.11) can now be recast as

$$\begin{aligned} \max \quad & (c^1 + c^2)y^1 + (c^1 - c^2)y^2 - c^1u \\ \text{s.t.} \quad & Ay^1 = \tfrac{1}{2}Au, \\ & Ay^2 = \tfrac{1}{2}Au, \end{aligned} \tag{9.13}$$

$$0 \leqslant y^1 \leqslant u, \quad \text{and} \quad 0 \leqslant y^2 \leqslant u.$$

The formulation (9.13) (similar to the one in Sakarovitch [36]) has a totally unimodular constraint matrix, and thus the "$\frac{1}{2}$-integrality" theorems of Hu [25] are immediate. Rebman [34] has shown that subdeterminants of the constraint matrix of (9.11) (and even more restrictive ones) can be very ill behaved; Fibonacci and Lucas numbers arise, making the regularity difficult to discern. Moreover, (9.13) suggests the use of network-flow algorithms, while (9.11) suggests the use of slower general linear-programming algorithms, or special-purpose multicommodity flow methods [28].

10. A GENERALIZATION OF A THEOREM OF HELLER

In this section, an upper bound will be established for the number of distinct columns of representative matrices of k-regular subspaces. The bound is a polynomial, of degree $2k$, in the rank of the matrices. An exact polynomial is not determined; only the order is established. The order of the polynomial is sharp for $k = 1$ and coincides with the order of the exact sharp bound produced by Heller [21].

PROPOSITION 10.1. *For each positive integer k, there is a constant C_k such that all matrices with k-regular row space of dimension r have at most $C_k r^{2k}$ distinct columns.*

Before proceeding to the proof of Proposition 10.1, an algorithmic consequence will be outlined. Recall the linear-programming augmentation scheme of Section 3. Let $\{\bar{A}_{\cdot 1}, \dots, \bar{A}_{\cdot p}\}$ be the set of distinct columns of the constraint matrix A. It is useful to allow multiple copies of columns in the constraint matrix so that piecewise linear, separable, concave objective functions can be modeled. For each l $(1 \leqslant l \leqslant p)$, let $N(l)$ denote the number of columns of A that are identical to $\bar{A}_{\cdot l}$, and let $\{j_1, \dots, j_{N(l)}\}$ be the set of column indices in A of the copies of $\bar{A}_{\cdot l}$. Without loss of generality, assume that the objective coefficients satisfy $d_{j_1} \geqslant d_{j_2} \geqslant \cdots \geqslant d_{j_{N(l)}}$ $(1 \leqslant l \leqslant p)$. Provided an initial feasible solution of the program (3.1) is known, it is easy to arrange for one of the form

$$x_{j_t} = u_{j_t} \quad \text{for } t < T(l), \quad \text{and} \quad x_{j_t} = l_{j_t} \quad \text{for } t > T(l), \tag{10.1}$$

where the index $T(l)$ satisfies $1 \leqslant T(l) \leqslant N(l)$ $(1 \leqslant l \leqslant p)$. A shortest augmentation y, with respect to a solution of the form (10.1), must have $\underline{y} \cap \{j_1, \dots, j_{N(l)}\} \in \{\varnothing, \{j_{T(l)}\}, \{j_{T(l)+1}\}, \{j_{T(l)-1}\}\}$. Furthermore, under the augmentation scheme of Section 3, each iterate will satisfy (10.1). These observations have an interesting consequence. If the null space of the constraint matrix A is k-regular, then Proposition 10.1 (together with Lemma 2.1) implies that the search for a shortest augmentation can always be confined to the null space of a matrix having the number of its columns bounded by a polynomial in the number of rows of A—regardless of the number of columns of A.

Proof of Proposition 10.1. Attention may be restricted to standard representative matrices, since if M is an arbitrary representative matrix, there always exists a nonsingular matrix E and a permutation matrix Π such that

$$EM\Pi = \left[\begin{array}{c|c} I & A \\ \hline 0 & 0 \end{array}\right] \tag{10.2}$$

is a standard representative matrix with, possibly, some zero rows. Since E is nonsingular, the number of distinct columns of M and $[I \mid A]$ are identical $[\text{r.s.}([I \mid A]) = \text{r.s.}(M\Pi)]$.

Suppose, then, that $M = [I \mid A]$ is a standard representative matrix of rank r. Further, suppose r.s.(M) is k-regular and has coregularity c. The proof will be by induction on r. Since an asymptotic bound is sought, it need only be observed that for $r = 1$, the base case of the induction, the number of distinct columns of M is finite (k is fixed). Suppose that the result holds for all standard representative matrices with k-regular row space of dimension $r - 1$. Consider the standard representative matrix M', of rank $r - 1$, obtained from M by performing the deletion operation (see Section 2) on the coordinate corresponding to the first identity column of M:

$$\begin{aligned} M &= [I_r \mid A] \\ &= \left[\begin{array}{c|c|c} 1 & \vec{0}^{T} & * \\ \hline \vec{0} & I_{r-1} & A' \end{array}\right], \\ M' &= [I_{r-1} \mid A']. \end{aligned} \tag{10.3}$$

The row space of M' is k-regular and c-coregular. The matrix M' has no more than $C_k(r-1)^{2k}$ distinct columns by the inductive hypothesis. It will be shown that M has no more than $C_k r^{2k}$ distinct columns by showing that the number of distinct columns of M cannot exceed those of M' by more than $D_k r^{2k-1}$, for some constant D_k depending only on k. The result will follow, since C_k can be chosen large enough to satisfy $C_k(r-1)^{2k} + D_k r^{2k-1} \leqslant C_k r^{2k}$ (for all positive integers r).

A column $\binom{a}{x}$ of M is called an *original* of column x of M'. A column x of M' may have multiple originals in M. It will be shown that there is a choice of a maximal linearly independent set $\mathscr{I}$ of columns, amongst those columns of M' with multiple originals, so that the remaining nonnull columns with multiple originals are minimal linear combinations of no more than $2k-1$ columns, each from $\mathscr{I}$. This will suffice, since any nonnull column y of M' has a unique representation (up to scalar multiplication) as a linear combination of some minimal set of columns, from the basic set $\mathscr{I}$, of the form

$$\varepsilon y = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_t x_t. \tag{10.4}$$

The coefficients can be chosen to be in $\{\pm 1, \pm 2, \ldots, \pm c\}$, since the null space of M' is c-regular and the coefficients correspond to the entries in a vector from $\mathscr{F}(\text{n.s.}(M'))$. If t can be restricted to be no greater than $2k-1$, then there are at most

$$\sum_{j=1}^{2k-1} \binom{r-1}{j} 2^j c^{j+1}$$

such linear combinations. Each such column of M' may have as many as $2k+1$ originals in M, as may the column $\vec{0}$ if it appears in M'. Thus, it will be shown that the number of distinct columns of M in excess of the number of such columns of M' is no greater than

$$2k + 2k\left[\sum_{j=1}^{2k-1} \binom{r-1}{j} 2^j c^{j+1}\right] \leqslant 2k + 2k\left[c^{2k+1} 2^{2k} \sum_{j=1}^{2k-1} \binom{r-1}{j}\right], \tag{10.5}$$

which is bounded above by $D_k r^{2k-1}$ for appropriately large D_k (recall that $c \leqslant \text{l.c.m.}\{1, 2, \ldots, k\}$ from Section 2).

It remains, then, to show that t of (10.4) can be taken to be no greater than $2k-1$. Let M^* be the submatrix of M' composed of those nonnull columns with multiple originals in M. Choose $\mathscr{I}$, a basis from M^*, with the property that the unique minimal linear dependence relation

$$y_j = \varepsilon_{j_1} x_{j_1} + \varepsilon_{j_2} x_{j_2} + \cdots + \varepsilon_{j_n} x_{j_n} \qquad (x_{j_i} \in \mathscr{I}, \quad 1 \leqslant i \leqslant n) \tag{10.6}$$

has $|\varepsilon_{j_i}| \geqslant 1$ $(1 \leqslant i \leqslant n)$, for all columns y_j of M^*. Such a basic set $\mathscr{I}$ can be chosen by Lemma 7.1. Suppose that there is a column y of M^* that is minimally dependent on at least $2k$ columns of $\mathscr{I}$, as follows:

$$y = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_{2k} x_{2k} + L(x_{2k+1}, \ldots, x_t). \tag{10.7}$$

Let $\hat{x}_i$ and $\bar{x}_i$ be two of the originals of x_i $(i = 1, 2, \ldots, t)$. Similarly, let $\hat{y}$ and $\bar{y}$ be two of the originals of y. Observe now that $\hat{x}_i$ and $\bar{x}_i$ differ from $\binom{0}{x_i}$ by multiples of $b_1 = \binom{1}{\vec{0}}$ $(i = 1, 2, \ldots, t)$. Similarly, $\hat{y}$ and $\bar{y}$ differ from $\binom{0}{y}$ by multiples of b_1. Hence, the following relationships are also minimal linear dependencies:

$$\begin{aligned} \hat{y} &= \hat{\kappa} b_1 + \varepsilon_1 \hat{x}_1 + \varepsilon_2 \hat{x}_2 + \cdots + \varepsilon_{2k} \hat{x}_{2k} + L(\hat{x}_{2k+1}, \ldots, \hat{x}_t), \\ \bar{y} &= \bar{\kappa} b_1 + \varepsilon_1 \hat{x}_1 + \varepsilon_2 \hat{x}_2 + \cdots + \varepsilon_{2k} \hat{x}_{2k} + L(\hat{x}_{2k+1}, \ldots, \hat{x}_t), \end{aligned} \tag{10.8}$$

where $\hat{\kappa}$ and $\bar{\kappa}$ are suitable values. Since $\hat{y} \neq \bar{y}$, we have $\hat{\kappa} \neq \bar{\kappa}$; this fact will be used shortly. Denote by σ_i the value such that $\bar{x}_i = \sigma_i b_1 + \hat{x}_i$ $(i = 1, 2, \ldots, t)$. It is not true, in general, that the values $\{\sigma_i\}$ are at least 1 in absolute value as the $\{\varepsilon_i\}$ are. Let B be a basis of M containing $\{b_1, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2k}\}$. Consider the matrix $B^{-1}M$; it has the following as a submatrix:

$$B^{-1}M = \begin{array}{c} \\ b_1 \\ \hat{x} \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & \hat{y} & \bar{y} & \hat{x}^{\mathrm{T}} & \bar{x}^{\mathrm{T}} \end{array} \\ \begin{pmatrix} 1 & \hat{\kappa} & \bar{\kappa} & \vec{0}^{\mathrm{T}} & \sigma^{\mathrm{T}} \\ \vec{0} & \varepsilon & \varepsilon & I_{2k} & I_{2k} \end{pmatrix} \end{array}, \tag{10.9}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2k})^{\mathrm{T}}$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{2k})^{\mathrm{T}}$, $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2k})^{\mathrm{T}}$, and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2k})^{\mathrm{T}}$. Pivoting in the $\bar{x}_i$ column and the $\hat{x}_i$ row changes the entry in the b_1 row and $\hat{y}$ (respectively, $\bar{y}$) column from $\hat{\kappa}$ $(\bar{\kappa})$ to $\hat{\kappa} - \sigma_i \varepsilon_i$ $(\bar{\kappa} - \sigma_i \varepsilon_i)$. Define τ, P, and N as follows:

$$\begin{aligned} \tau &= \min(\{|\sigma_i| : 1 \leqslant i \leqslant 2k\}) > 0, \\ P &= \{i : \sigma_i \varepsilon_i \geqslant \tau,\ 1 \leqslant i \leqslant 2k\}, \\ N &= \{i : \sigma_i \varepsilon_i \leqslant -\tau,\ 1 \leqslant i \leqslant 2k\}. \end{aligned} \tag{10.10}$$

Pivoting successively in the $\bar{x}_i$ column and $\hat{x}_i$ row, for all $i \in P$, causes the entries in row b_1 and columns $\hat{y}$ and $\bar{y}$ to decrease by at least $|P|\tau$. Completing these pivots for all $i \in N$ causes these entries to increase by at least $|N|\tau$. Since $|P|+|N| = 2k$ ($P \cup N$ is a partition of $\{1,2,\dots,2k\}$) and $\hat{\kappa} \neq \bar{\kappa}$, one of the following four values exceeds $k\tau$:

$$\left|\hat{\kappa} - \sum_{i \in P} \sigma_i \varepsilon_i\right|, \quad \left|\hat{\kappa} - \sum_{i \in N} \sigma_i \varepsilon_i\right|, \quad \left|\bar{\kappa} - \sum_{i \in P} \sigma_i \varepsilon_i\right|, \quad \text{and} \quad \left|\bar{\kappa} - \sum_{i \in N} \sigma_i \varepsilon_i\right|. \tag{10.11}$$

Upon completion of either all P or all N pivots, an entry in the b_1 row and either the $\hat{y}$ or $\bar{y}$ column will exceed $k\tau$ in absolute value, and the entry in either the $\bar{x}_l$ or the $\hat{x}_l$ column of this row will have magnitude τ, where $l = \operatorname{argmin}(\{|\sigma_i| : 1 \leqslant i \leqslant 2k\})$. This will represent a contradiction to the assumption that r.s.(M) is k-regular. ■

For $k = 1$, Heller [21] obtained an exact bound of $r(r+1)+1$; this bound is attained, as he pointed out, by the node-arc incidence matrix of a complete directed graph on $r+1$ nodes (with a $\vec{0}$ column). The order of the polynomial obtained here (for $k = 1$) is the same as in Heller's bound. In fact, if a little more care is taken in (10.5), and many other details are omitted, the proof presented here reduces to Heller's (for $k = 1$), thus providing his exact sharp bound.

The bound provided by Proposition 10.1 can be used to establish lower bounds on the regularity of classes of subspaces. For example, it is easy to establish that the null spaces of arc-chain incidence matrices of source-sink paths in graphs on v vertices have regularities bounded by no constant, as v increases.

This paper is based on part of the author's doctoral dissertation [31], which had the benefit of being supervised by Robert Bland. Bland suggested the possibility of generalizing regular subspaces in the spirit of Proposition 3.1. Further announcements of the results in [31] can be found in [32].

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