Sufficient Matrices and the Linear Complementarity Problem

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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ABSTRACT

We pose and answer two questions about solutions of the linear complementarity problem (LCP). The first question is concerned with the conditions on a matrix M which guarantee that for every vector q, the solutions of the LCP (q, M) are identical to the Karush-Kuhn-Tucker points of the natural quadratic program associated with (q, M). In answering this question we introduce the class of "row sufficient" matrices. The transpose of such a matrix is what we call "column sufficient." The latter matrices turn out to furnish the answer to our second question, which asks for the conditions on M under which the solution set of (q, M) is convex for every q. In addition to these two main results, we discuss the connections of these two new matrix classes with other well-known matrix classes in linear complementarity theory.

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1. INTRODUCTION

The present investigation belongs to a long-standing tradition in the study of the linear complementarity problem (q, M):

$$q + Mz \geqslant 0,$$
$$z \geqslant 0,$$
$$z^{\mathrm{T}}(q + Mz) = 0.$$

A *feasible* linear complementarity problem (q, M) is one for which

$$F(q, M) = \{ z : q + Mz \geqslant 0,\ z \geqslant 0 \} \neq \varnothing .$$

Naturally, $F(q, M)$ is called the *feasible set* of (q, M). The problem (q, M) is solvable if

$$S(q, M) = \{ z : z \in F(q, M),\ z^{\mathrm{T}}(q + Mz) = 0 \} \neq \varnothing .$$

In this paper, we introduce a class of matrices (called "row sufficient" matrices) which characterizes a certain property of the linear complementarity problem. We say that a matrix is "column sufficient" if its transpose is row sufficient, thereby obtaining a second class of matrices. We demonstrate that the class of column sufficient matrices characterizes another interesting property of the same problem. In addition to these two main results, we discuss the connections of these two new matrix classes with other well-known matrix classes in linear complementarity theory.

Matrix classes have always played a prominent role in the theory of the linear complementarity problem (LCP). Indeed, even before the LCP had been given a name, Samelson, Thrall, and Wesler [23] showed that for fixed $M \in R^{n \times n}$, the LCP (q, M) has a unique solution for every $q \in R^n$ if and only if M belongs to the class **P** of matrices with positive principal minors. This theorem is one of several wherein an interesting property of the linear complementarity problem actually characterizes a class of matrices. Of course, by virtue of the equivalence, the characterization goes both ways.

Our point of departure for this study is an analytic (as opposed to constructive) proof technique that has been used by several authors for demonstrating the existence of a solution to a feasible linear complementarity problem. This technique does not work in all cases, but it does for linear complementarity problems involving positive semidefinite matrices [3] among others [5]. This raises a question: *What is the scope of the technique, i.e., the class of all matrices for which it works?* Our characterization result (Theorem 4) shows that this class consists precisely of the row sufficient matrices.

An extension of the existence result [3] is due to Adler and Gale [1], who show that a feasible LCP (q, M) with a positive semidefinite matrix M has a nonempty convex polyhedral solution set. This raises a second question: *What kinds of linear complementarity problems have convex solution sets?* We give more than one characterization in answer to this question. Interestingly enough, one such characterization is intimately related to the answer to the first question. (See Theorem 6.)

2. THE ASSOCIATED QUADRATIC PROGRAM

There is a close and much-noted connection between the LCP (q, M) and the quadratic program

$$\begin{aligned} \text{minimize} \quad & \phi(z) = q^{\mathrm{T}}z + z^{\mathrm{T}}Mz \\ \text{subject to} \quad & q + Mz \geqslant 0 \\ & z \geqslant 0. \end{aligned} \tag{1}$$

The feasible set of (1) is precisely $F(q, M)$. In order to compensate for the fact that M may be asymmetric, it is customary to use the equivalent representation of the objective function:

$$\phi(z) = q^{\mathrm{T}}z + \tfrac{1}{2}z^{\mathrm{T}}(M + M^{\mathrm{T}})z. \tag{2}$$

It is clear that for any $M \in R^{n \times n}$ and $q \in R^n$, any solution of (q, M) is an optimal solution of (1). However, the converse of this observation is not valid unless the optimal objective function value of (1) happens to be 0. Part of this paper will characterize certain instances when any Karush-Kuhn-Tucker point of (1) will be a solution of the LCP (q, M).

The following lemma is extracted from the proofs of several earlier existence theorems. (Cf. [3], [4], [5].) It is included here because of its strong motivating influence.

LEMMA 1. *Let (q, M) be a feasible LCP. Then, the quadratic program (1) has an optimal solution, z^*. Moreover, there exists a vector u^* of* *multipliers satisfying the conditions*

$$q+(M+M^{\mathrm{T}})z^{*}-M^{\mathrm{T}}u^{*}\geqslant 0, \tag{3}$$

$$(z^{*})^{\mathrm{T}}\left[q+(M+M^{\mathrm{T}})z^{*}-M^{\mathrm{T}}u^{*}\right]=0, \tag{4}$$

$$u^{*}\geqslant 0, \tag{5}$$

$$(u^{*})^{\mathrm{T}}(q+Mz^{*})=0. \tag{6}$$

Finally, the vectors z^* and u^* satisfy

$$(z^{*}-u^{*})_{i}\left[M^{\mathrm{T}}(z^{*}-u^{*})\right]_{i}\leqslant 0 \qquad \textit{for all} \quad i=1,\dots,n. \tag{7}$$

Proof. Since (q, M) is feasible, so is the quadratic program (1). As the objective function of the quadratic program is bounded below on the feasible region, it follows from the Frank-Wolfe theorem [12] that there exists an optimal solution to (1). Such an optimal solution z^*, together with a suitable vector u^* of multipliers, will satisfy the Karush-Kuhn-Tucker conditions (3)–(6). (See [17].) To prove (7), we examine the inner product (4) at the componentwise level and deduce that for all $i = 1, \dots, n$

$$z^{*}_{i}\left[M^{\mathrm{T}}(z^{*}-u^{*})\right]_{i}\leqslant 0, \tag{8}$$

using the fact that z^* is feasible. Similarly, multiplying the ith component in (3) by u_i^* and then invoking the complementarity condition $u_i^*(q + Mz^*)_i = 0$, which is implied by (5), (6), and the feasibility of z^*, we obtain

$$-u_{i}^{*}\left[M^{\mathrm{T}}(z^{*}-u^{*})\right]_{i}\leqslant 0. \tag{9}$$

Now, (7) follows by adding (8) and (9). ■

REMARK. The conditions (3)–(9) are satisfied by any *local* minimum for (1) and corresponding vector of Lagrange multipliers. It is not necessary to assume that z^* is a *global* minimum for (1). As we shall see below, the stronger conclusion of global optimality may also be inferred under some circumstances.

3. SIGN REVERSING

From the standpoint of this paper, the most important conclusion of Lemma 1 is embodied in the system of inequalities (7), which say that M^T "reverses the sign" of $z^* - u^*$. Credit for this terminology is due to Gale and Nikaido [13], who define this concept as follows.

Definition. The matrix $M \in R^{n \times n}$ *reverses the sign* of the vector $x \in R^n$ if $x_i(Mx)_i \leqslant 0$ for all $i = 1, \dots, n$.

It helps the discussion somewhat to consider the set, $\operatorname{rev} M$, of all vectors whose sign is reversed by a given matrix $M \in R^{n \times n}$:

$$\operatorname{rev} M = \{x : x_i(Mx)_i \leqslant 0,\ i = 1, \dots, n\}.$$

Note that for all $M \in R^{n \times n}$, $\operatorname{rev} M$ contains the zero vector. In general, $\operatorname{rev} M$ is a cone (not necessarily convex) containing $\ker M$, the kernel (nullspace) of M. The conditions under which $\operatorname{rev} M$ and $\ker M$ are equal can be characterized through the notion of column adequacy (defined below). A special result of this kind can be found in the 1962 paper [10] by Fiedler and Pták, who gave several equivalent conditions for a matrix M to belong to $\mathbf{P}$. One pair of them [namely, 1° and 2° in their Theorem (3,3)] amounts to saying that

$$M \in \mathbf{P} \quad \text{if and only if} \quad \operatorname{rev} M = \{0\}.$$

Several discussions in this paper concern submatrices and subvectors. Since such objects are described in terms of rows and/or columns of specific matrices and vectors, this requires the use of notation pertaining to *index sets*. Relative to the positive integer n, an index set α is either a subset of $\{1, \dots, n\}$ of the form $\{i_1, \dots, i_k\}$ where $1 \leqslant i_1 < \cdots < i_k \leqslant n$ or else the empty set. If $\alpha \subset \{1, \dots, n\}$ is an index set, then $\bar{\alpha}$ is the (complementary) index set obtained by taking the elements of the set $\{1, \dots, n\} \setminus \alpha$ in their natural order. For a given matrix M and index set α, the matrix $M_{\alpha\cdot}(M_{\cdot\alpha})$ denotes the submatrix of M consisting of the rows (columns) indexed by α. When $M \in R^{n \times n}$, its principal submatrices can be expressed as $M_{\alpha\alpha}$ where $\alpha \subset \{1, \dots, n\}$.

DEFINITION. Let $M \in R^{n\times n}$ be a matrix with nonnegative principal minors.[1] Then M is

(i) *row adequate* if for all $\alpha \subset \{1,\dots,n\}$

$$\det M_{\alpha\alpha} = 0 \quad \Leftrightarrow \quad \text{the rows of } M_{\alpha\cdot} \text{ are linearly dependent,}$$

(ii) *column adequate* if for all $\alpha \subset \{1,\dots,n\}$

$$\det M_{\alpha\alpha} = 0 \quad \Leftrightarrow \quad \text{the columns of } M_{\cdot\alpha} \text{ are linearly dependent,}$$

(iii) *adequate* if it is row and column adequate.

The class of adequate matrices was introduced by Ingleton [14] to capture some of the properties of positive semidefinite matrices vis-à-vis the LCP. In [15], Ingleton identified the three matrix classes described in the preceding definition. The concept of row adequacy also appeared in the work of Eaves [8, 9]. It is obvious that every **P**-matrix is adequate. Ingleton [15, 1.2.3] noted the converse: every *nonsingular* adequate matrix belongs to **P**. The proof of this given in [5, Lemma 3] can be modified to show the following very slight generalization.

LEMMA 2. *Let $M \in R^{n\times n}$ be nonsingular. Then M is row (column) adequate if and only if $M \in$* **P**.

For ease of reference and as motivation for our new results, we state the following known theorem in detail.

THEOREM 3. *Let $M \in R^{n\times n}$. Then*

(a) *M is row adequate if and only if*

$$x_i(M^{\mathrm{T}}x)_i \leqslant 0 \quad \textit{for all } i=1,\dots,n \quad \Rightarrow \quad M^{\mathrm{T}}x = 0.$$

(b) *M is column adequate if and only if*

$$x_i(Mx)_i \leqslant 0 \quad \textit{for all } i=1,\dots,n \quad \Rightarrow \quad Mx = 0.$$

Proof. Part (a) was proved by Eaves [8; 9, Lemma (2), p. 622]. Part (b) follows from part (a). ■

[1] The class of real square matrices whose principal minors are all nonnegative is denoted $\mathbf{P}_0$.

There is a more elegant way to state Theorem 3. The matrix $M \in R^{n\times n}$ is

(a) row adequate if and only if $\operatorname{rev} M^{\mathrm{T}} = \ker M^{\mathrm{T}}$,
(b) column adequate if and only if $\operatorname{rev} M = \ker M$.

These characteristics of (row and column) adequacy inspire the more general notion of (row and column) sufficiency. Before giving the definition, we define a special mapping.

DEFINITION. For $M \in R^{n\times n}$, let $h_M: R^n \to R^n$ be the mapping $x \mapsto x * Mx$, the Hadamard product of x and Mx. Thus, for every $x \in R^n$, $h_M(x)$ is the n-vector $(x_i(Mx)_i)$.

Note that for any $M \in R^{n\times n}$,

$$\operatorname{rev} M = \{x : h_M(x) \leqslant 0\}.$$

To facilitate the language, we let $\operatorname{rev} h_M = \operatorname{rev} M$ and then put

$$\ker h_M = \{x : h_M(x) = 0\}.$$

The relation $\ker h_M \subset \operatorname{rev} h_M$ is automatic. When the reverse inclusion holds for M and/or M^{T}, we get the notions of sufficiency.

DEFINITION. The matrix $M \in R^{n\times n}$ is

(i) *row sufficient* if

$$x_i(M^{\mathrm{T}}x)_i \leqslant 0 \quad \text{for all } i \quad \Rightarrow \quad x_i(M^{\mathrm{T}}x)_i = 0 \text{ for all } i, \tag{10}$$

(ii) *column sufficient* if

$$x_i(Mx)_i \leqslant 0 \quad \text{for all } i \quad \Rightarrow \quad x_i(Mx)_i = 0 \text{ for all } i, \tag{11}$$

(iii) *sufficient* if it is row and column sufficient.

Alternatively, we can say that $M \in R^{n \times n}$ is

(a) *row sufficient* if $\operatorname{rev} h_{M^{\mathrm{T}}} = \ker h_{M^{\mathrm{T}}}$,
(b) *column sufficient* if $\operatorname{rev} h_M = \ker h_M$.

Row sufficient matrices and column sufficient matrices must belong to the class $\mathbf{P}_0$. This follows from the Fiedler-Pták [11] characterization of $\mathbf{P}_0$, which says that the matrix $M \in R^{n \times n} \cap \mathbf{P}_0$ if and only if for every nonzero vector $x \in R^n$ there exists an index $k \in \{1, \ldots, n\}$ such that $x_k \neq 0$ and $x_k (Mx)_k \geqslant 0$.

We note the following facts.

(i) Positive semidefinite matrices are sufficient, but not necessarily adequate.

(ii) Row (column) adequate matrices are row (column) sufficient.

(iii) There are row (column) sufficient matrices which are neither positive semidefinite nor row (column) adequate. [The direct sum of an indefinite **P**-matrix and a nonadequate positive semidefinite matrix is a sufficient matrix of this sort.]

(iv) Every principal submatrix of a row (column) sufficient matrix is row (column) sufficient.

4. THE ROLE OF ROW SUFFICIENCY

In Section 2, we saw (Lemma 1) that when the quadratic program (1) associated with the LCP (q, M) is feasible, there must exist a Karush-Kuhn-Tucker pair (z^*, u^*) such that M^{T} reverses the sign of $z^* - u^*$. [To say that (z^*, u^*) is a Karush-Kuhn-Tucker pair for (1) means z^* is feasible for (1) and the vectors z^* and u^* satisfy (3)–(6).] We now link this fact with the property of row sufficiency of the matrix M.

THEOREM 4. *Let $M \in R^{n \times n}$. The following two statements are equivalent*:

(a) *M is row sufficient.*

(b) *For each vector $q \in R^n$, if (z^*, u^*) is a Karush-Kuhn-Tucker pair of the quadratic program* (1), *then z^* solves the LCP (q, M).*

Proof. (a) $\Rightarrow$ (b): Effectively, it was shown in Lemma 1 that $z^* - u^* \in \operatorname{rev} h_{M^{\mathrm{T}}}$. When M is row sufficient, $z^* - u^* \in \ker h_{M^{\mathrm{T}}}$. Thus, for all

$i = 1, \ldots, n$,

$$z_i^* \left[M^{\mathrm{T}}(z^* - u^*) \right]_i = u_i^* \left[M^{\mathrm{T}}(z^* - u^*) \right]_i.$$

Combining this with (4), (8), and (9), we deduce that

$$z_i^*(q + Mz^*)_i = 0 \qquad \text{for all} \quad i = 1, \ldots, n.$$

It now follows that z^* solves (q, M).

(b) $\Rightarrow$ (a): Suppose M is *not* row sufficient. Then there exists a vector $x \in \operatorname{rev} h_{M^{\mathrm{T}}}$ such that $x_j(M^{\mathrm{T}}x)_j < 0$ for some j. Without loss of generality, we may assume $x_j > 0$. We now use the vector x to develop a contradiction to (b). To this end, let $z^* = x^+$, $u^* = x^-$, and $q = -Mz^* + (M^{\mathrm{T}}x)^-$. It is then easy to show that (z^*, u^*) is a Karush-Kuhn-Tucker pair for the quadratic program (1) defined by the given M and the constructed q. We now obtain a contradiction. By construction, $z_j^* > 0$ and $(q + Mz^*)_j > 0$, yet according to (b), the vector z^* solves (q, M). ■

Remarks. This theorem has some important implications.

(1) Row sufficient matrices belong to the class $\mathbf{Q}_0$ consisting of all real square matrices M for which the feasibility of (q, M) implies its solvability. In fact, we now observe that since row sufficient matrices belong to $\mathbf{P}_0 \cap \mathbf{Q}_0$, they can be "processed" by Lemke's algorithm [18]. See Aganagić and Cottle [2, p. 230].

(2) The row sufficiency property precisely delimits the class of matrices for which the analytic existence proof technique via Karush-Kuhn-Tucker pairs of the quadratic program (1) will work. Thus, we have now answered the first question asked in Section 1.

(3) The preceding remark and the existence of other matrix classes belonging to $\mathbf{Q}_0$ lead one to speculate about the possibility of finding other analytic methods. The copositive-plus matrices introduced by Lemke [18] (see also Cottle and Dantzig [6]) exemplify a familiar class of matrices that need not be row sufficient.

5. THE ROLE OF COLUMN SUFFICIENCY

In this section we address our question on the convexity of the solution set $S(q, M)$ of an LCP (q, M). We begin with a preliminary theorem which is reminiscent of the Adler-Gale [1] characterization of the solution set of

(q, M) when M is positive semidefinite. Much more general results were obtained by Jansen [16], who studied the structure of the solution set of an arbitrary LCP (q, M). Jansen showed that $S(q, M)$ is the finite union of polyhedral sets.

THEOREM 5. *Let $M \in R^{n \times n}$ and $q \in R^n$ be given. The following statements are equivalent:*

(a) *The solution set of (q, M) is polyhedral.*
(b) *The solution set of (q, M) is convex.*
(c) *The equation*

$$(z^1)^{\mathrm{T}}(q + Mz^2) = (z^2)^{\mathrm{T}}(q + Mz^1) = 0 \tag{12}$$

holds for any two solutions z^1 and z^2 of (q, M).

(d) *There exist complementary index sets α and $\bar{\alpha}$ contained in $\{1, \dots, n\}$ such that every solution to (q, M) satisfies*

$$\begin{aligned} q_\alpha + M_{\alpha\alpha} z_\alpha &= 0, \\ q_{\bar{\alpha}} + M_{\bar{\alpha}\alpha} z_\alpha &\geqslant 0, \\ z_\alpha &\geqslant 0, \\ z_{\bar{\alpha}} &= 0. \end{aligned} \tag{13}$$

Proof. (a) $\Rightarrow$ (b): This is obvious, as all polyhedral sets are convex.

(b) $\Rightarrow$ (c): Let z^1 and z^2 be any two solutions of (q, M). By the convexity assumption, the vector $z = \tau z^1 + (1 - \tau) z^2$ is also a solution for any $\tau \in (0, 1)$. By letting $w^i = q + Mz^i$ $(i = 1, 2)$, we have

$$0 = \left[\tau w^1 + (1 - \tau) w^2\right]^{\mathrm{T}} \left[\tau z^1 + (1 - \tau) z^2\right] = \tau(1 - \tau)\left[(w^1)^{\mathrm{T}} z^2 + (w^2)^{\mathrm{T}} z^1\right],$$

from which (12) follows.

(c) $\Rightarrow$ (d): The desired index set is

$$\alpha = \left\{ i : z_i > 0 \text{ for some } z \in S(q, M) \right\}.$$

It follows that $i \in \bar{\alpha}$ if and only if $z_i = 0$ for all $z \in S(q, M)$. Let $\bar{z} \in S(q, M)$ be arbitrary. It suffices to show that $\bar{w}_\alpha = q_\alpha + M_{\alpha\alpha}\bar{z}_\alpha = 0$. Now choose an arbitrary $i \in \alpha$. Then $z_i > 0$ for some $z \in S(q, M)$. By (12), with $z^1 = z$ and $z^2 = \bar{z}$, we deduce that $\bar{w}_i = 0$. Thus $\bar{w}_\alpha = 0$, and (d) follows.

(d) $\Rightarrow$ (a): Let Z denote the solution set of (13). Then Z is clearly polyhedral, and $Z = S(q, M)$. ∎

Theorem 5 gives necessary and sufficient conditions for the convexity of $S(q, M)$, but only for the individual vector q. In the theorem below, we remove this restriction, thereby obtaining a universal result in terms of M alone. The proof of the theorem is reminiscent of Murty's proof in [21] of the Samelson-Thrall-Wesler characterization of the class **P**.

THEOREM 6. *Let $M \in R^{n \times n}$. The following two statements are equivalent*:

(a) *For each vector $q \in R^n$, the LCP (q, M) has a (possibly empty) convex solution set.*

(b) *M is column sufficient.*

Proof. (a) $\Rightarrow$ (b): Suppose M is not column sufficient. Then there exists a vector x such that $x_i(Mx)_i \leqslant 0$ for all $i = 1, \dots, n$ and $x_j(Mx)_j < 0$ for some j. Now let $z^1 = x^+$ and $z^2 = x^-$. For brevity, let $u^+ = (Mx)^+$ and $u^- = (Mx)^-$. Define the vector $q = u^+ - Mx^+$. Notice that $q = u^- - Mx^-$ also. It is not difficult to verify that z^1 and z^2 are both solutions of (q, M) with the so-defined q. Nevertheless, we have either $z_j^1(q + Mz^2)_j > 0$ or $z_j^2(q + Mz^1)_j > 0$, depending on whether $x_j > 0$ or $x_j < 0$. This contradicts the convexity of the solution set of (q, M).

(b) $\Rightarrow$ (a): Let $q \in R^n$ be given. We may assume (q, M) has at least two solutions, for otherwise there is nothing to prove. Let z^1 and z^2 be two solutions of (q, M), and define $w^k = q + Mz^k$ for $k = 1, 2$. Then, for each $i = 1, \dots, n$, we have

$$0 \geqslant (z^1 - z^2)_i(w^1 - w^2)_i = (z^1 - z^2)_i\left[M(z^1 - z^2)\right]_i. \tag{14}$$

By the column sufficiency of M, the terms on the right-hand side of (14) must equal zero, and hence equality must hold throughout. In particular, $z_i^1 w_i^2 = z_i^2 w_i^1 = 0$ for $i = 1, \dots, n$. Since this is just a version of (12), the convexity follows. ∎

A characterization of sufficient matrices can now be obtained by combining Theorems 4 and 6.

COROLLARY 7. *Let $M \in R^{n \times n}$. The following two statements are equivalent*:

(a) *For each $q \in R^n$, the set of Karush-Kuhn-Tucker points of the quadratic program* (1) *is convex and equal to* $S(q, M)$.

(b) *M is sufficient.*

Another way to look at the universal convexity question is through the geometry of the complementary cones relative to M. Complementary cones were first introduced by Samelson, Thrall, and Wesler [23] and later studied in the LCP context by Murty [21]. To carry out this part of the analysis, we recall two important definitions.

DEFINITION. Let $A \in R^{n \times m}$. Then the positive cone spanned by A is the set

$$\operatorname{pos}(A) = \{ y : y = Ax,\ x \geqslant 0 \}.$$

For any matrix A, pos(A) is a convex (finitely generated) cone. It is well known[2] that the *relative interior* of pos(A) is precisely the set of all strictly positive linear combinations of the columns of A. That is,

$$\operatorname{ri}(\operatorname{pos}(A)) = \{ y : y = Ax,\ x > 0 \}. \tag{15}$$

DEFINITION. Let $M \in R^{n \times n}$ be given. For each $\alpha \subset \{1, \dots, n\}$ let $C_M(\alpha)$ denote the matrix specified by

$$(C_M(\alpha))_{\cdot i} = \begin{cases} -M_{\cdot i} & \text{if} \quad i \in \alpha, \\ I_{\cdot i} & \text{otherwise.} \end{cases}$$

Sets of the form pos$(C_M(\alpha))$ are called *complementary cones* (relative to M). (For discussions of complementary cones relative to a single, fixed matrix, M, we simplify the notation by dropping the subscript M.)

The connection between complementary cones and the solvability of an LCP is well known: An LCP (q, M) has a solution if and only if $q \in \operatorname{pos} C(\alpha)$ for some α. In general, two complementary cones $C(\alpha)$ and $C(\beta)$ may be equal for distinct index sets α and β. Part of the following result asserts that this cannot be the case if $S(q, M)$ is convex for every q.

[2] See for example Rockafellar [22, Theorem 6.9].

THEOREM 8. *Let $M \in R^{n \times n}$. The following two statements are equivalent*:

(a) *For each vector $q \in R^n$, the LCP (q, M) has a (possibly empty) convex solution set.*

(b) *The distinctly generated complementary cones (relative to M) have disjoint relative interiors; that is,*

$$\operatorname{ri}(\operatorname{pos} C(\alpha)) \cap \operatorname{ri}(\operatorname{pos} C(\beta)) = \varnothing \qquad \text{for all} \quad \alpha, \beta \subset \{1, \dots, n\}, \quad \alpha \neq \beta.$$

Proof. (a) $\Rightarrow$ (b): Suppose there exist distinct index sets α and β such that

$$\operatorname{ri}(\operatorname{pos} C(\alpha)) \cap \operatorname{ri}(\operatorname{pos} C(\beta)) \neq \varnothing .$$

Let $U = C(\alpha)$ and $V = C(\beta)$. We may assume that there is an index k such that $U_{\cdot k} = I_{\cdot k}$ and $V_{\cdot k} = -M_{\cdot k}$. Now let $q \in \operatorname{ri}(\operatorname{pos} C(\alpha)) \cap \operatorname{ri}(\operatorname{pos} C(\beta))$ be arbitrary. Then [as in (15) above] there exist positive scalars λ_i and μ_i $(i = 1, \dots, n)$ such that

$$q = \sum_{i=1}^{n} \lambda_i U_{\cdot i} = \sum_{i=1}^{n} \mu_i V_{\cdot i}.$$

Define z^1, w^1, z^2, and w^2 by the rules

$$z_i^1 = \begin{cases} \lambda_i, & i \in \alpha, \\ 0, & i \in \bar{\alpha}, \end{cases} \qquad w_i^1 = \begin{cases} 0, & i \in \alpha, \\ \lambda_i, & i \in \bar{\alpha}, \end{cases}$$

$$z_i^2 = \begin{cases} \mu_i, & i \in \beta, \\ 0, & i \in \bar{\beta}, \end{cases} \qquad w_i^2 = \begin{cases} 0, & i \in \beta, \\ \mu_i, & i \in \bar{\beta}. \end{cases}$$

Thus, $z^1, z^2 \in S(q, M)$. But $k \in \bar{\alpha} \cap \beta$. Hence

$$(z^1)^{\mathrm{T}} w^2 \geqslant z_k^1 w_k^2 = \lambda_k \mu_k > 0,$$

which contradicts Theorem 5.

(b) $\Rightarrow$ (a): Again, the proof is by contradiction. Suppose there exists a vector $q \in R^n$ such that $S(q, M)$ is not polyhedral. Then there exist two solutions of (q, M) for which (12) is violated. The idea is now to use this information to construct a vector $\tilde{q}$ that belongs to the relative interiors of two distinctly generated complementary cones.

From the above, it follows that

$$q = \sum_{i=1}^{n} \lambda_i U_{\cdot i} = \sum_{i=1}^{n} \mu_i V_{\cdot i},$$

where for some index k, we have $U_{\cdot k} = I_{\cdot k}$, $V_{\cdot k} = -M_{\cdot k}$, $\lambda_k > 0$, and $\mu_k > 0$. Let $\gamma = \{i : U_{\cdot i} = V_{\cdot i}\}$. Then we can write

$$\begin{aligned} q &= \sum_{i \in \bar{\gamma}} \lambda_i U_{\cdot i} + \sum_{i \in \gamma} \lambda_i U_{\cdot i} \\ &= \sum_{i \in \bar{\gamma}} \mu_i V_{\cdot i} + \sum_{i \in \gamma} \mu_i U_{\cdot i}. \end{aligned}$$

Next define the following index sets:

$$\begin{aligned} \delta_1 &= \{ i \in \bar{\gamma} : \lambda_i > 0,\ \mu_i = 0 \}, \\ \delta_2 &= \{ i \in \bar{\gamma} : \lambda_i = 0,\ \mu_i > 0 \}, \\ \delta_3 &= \{ i \in \bar{\gamma} : \lambda_i = 0,\ \mu_i = 0 \}, \\ \delta_4 &= \{ i \in \bar{\gamma} : \lambda_i > 0,\ \mu_i > 0 \}. \end{aligned}$$

By its definition, $k \in \delta_4$, which is therefore nonempty. Now, using the two representations of q, define

$$\begin{aligned} \tilde{q} &= q + \sum_{i \in \delta_1 \cup \delta_3} U_{\cdot i} + \sum_{i \in \delta_2} V_{\cdot i} + \sum_{i \in \gamma} U_{\cdot i} \\ &= \sum_{i \in \delta_1} (\lambda_i + 1) U_{\cdot i} + \sum_{i \in \delta_2} V_{\cdot i} + \sum_{i \in \delta_3} U_{\cdot i} + \sum_{i \in \delta_4} \lambda_i U_{\cdot i} + \sum_{i \in \gamma} (\lambda_i + 1) U_{\cdot i} \\ &= \sum_{i \in \delta_1} U_{\cdot i} + \sum_{i \in \delta_2} (\mu_i + 1) V_{\cdot i} + \sum_{i \in \delta_3} U_{\cdot i} + \sum_{i \in \delta_4} \mu_i V_{\cdot i} + \sum_{i \in \gamma} (\mu_i + 1) U_{\cdot i}. \end{aligned}$$

An examination of the index sets and the coefficients in these equations reveals that $\tilde{q}$ has been represented as a vector belonging to the relative interiors of two distinctly generated complementary cones, namely $C(\alpha)$ and $C(\beta)$ where $k \in \bar{\alpha} \cap \beta$, thereby making $\alpha \neq \beta$. ■

6. A SPECIAL CASE

Generally speaking, the implication (11) through which column sufficiency is defined involves *nonlinear* inequalities and *all* vectors $x \in R^n$. In this section, we consider linear complementarity problems (q, M) in which the matrix M belongs to $\mathbf{Z}$, the class of real square matrices with nonpositive off-diagonal elements. Much has been written about this class of matrices in the LCP context and otherwise. See for instance [7] and the references therein. The main result of this section, Theorem 9, shows that if M is a $\mathbf{Z}$-matrix, then the implication (11) can be restricted to the nonnegative vectors $x \in R^n$. In turn, this restricted implication is equivalent to a *finite* set of *linear* inequality systems. The upshot of the theorem is that if $M \in \mathbf{Z}$, then the column sufficiency property of M and the universal convexity of solution sets of the associated linear complementarity problems can be tested by a finite procedure.[3]

THEOREM 9. *Let M be a $\mathbf{Z}$-matrix of order n. The following four statements are equivalent*:

(a) *For all $q \in R^n$, if (q, M) is feasible, then $S(q, M)$ is polyhedral.*
(b) *For all $\alpha \subset \{1, \dots, n\}$ the system*

$$0 \neq M_{\alpha\alpha} x_\alpha \leqslant 0, \qquad x_\alpha > 0 \tag{16}$$

has no solution.
(c) *For all $x \geqslant 0$, $x \in \operatorname{rev} h_M$ implies $x \in \ker h_M$.*
(d) *The matrix M is column sufficient.*

Proof. (a) $\Rightarrow$ (b): Suppose (b) is false, i.e., that for some α, (16) has a solution $\tilde{x}_\alpha$. We shall now construct a solvable LCP (q, M) for which $S(q, M)$ is not polyhedral. Indeed, define a nonnegative vector q so that $q_\alpha = -M_{\alpha\alpha}\tilde{x}_\alpha$ and $q_{\bar{\alpha}} > -M_{\bar{\alpha}\alpha}\tilde{x}_\alpha$. Now define $\tilde{z}$ so that $\tilde{z}_\alpha = \tilde{x}_\alpha$ and $\tilde{z}_{\bar{\alpha}} = 0$.

[3]The transpose of a Z-matrix is, of course, a Z-matrix. Hence, in this case, row sufficiency is also finitely testable.

Then $\tilde{z} \in S(q, M)$. Since $q \geqslant 0$, it follows that $0 \in S(q, M)$. But $\tilde{z}^{\mathrm{T}}(q + M0) > 0$, for $\tilde{z}_\alpha > 0$ and $0 \neq q_\alpha \geqslant 0$. We now have a contradiction, for by Theorem 5, $S(q, M)$ is not polyhedral.

(b) $\Rightarrow$ (c): If the nonnegative vector $\tilde{x}$ belongs to $\operatorname{rev} h_M$ and not to $\ker h_M$, then $\tilde{x}$ must be nonzero. It follows that $\alpha = \{i : \tilde{x}_i > 0\}$ is nonempty and $\tilde{x}_\alpha$ is a solution of (16), in contradiction to the hypothesis.

(c) $\Rightarrow$ (d): Assume $\tilde{x} \in \operatorname{rev} h_M$. Using the **Z**-property of M, we can show that

$$\tilde{x}_i(M\tilde{x})_i \geqslant |\tilde{x}_i|(M|\tilde{x}|)_i \qquad \text{for all} \quad i = 1, \ldots, n,$$

where $|\tilde{x}|$ denotes the vector whose ith component is $|\tilde{x}_i|$. Thus, $|\tilde{x}|$ belongs to $\operatorname{rev} h_M$, and since $|\tilde{x}| \geqslant 0$, it follows from (c) that $|\tilde{x}| \in \ker h_M$. Accordingly, we have

$$0 \geqslant \tilde{x}_i(M\tilde{x})_i \geqslant |\tilde{x}_i|(M|\tilde{x}|)_i = 0 \qquad \text{for all} \quad i = 1, \ldots, n.$$

Hence $\tilde{x} \in \ker h_M$, as required.

(d) $\Rightarrow$ (a): This is immediate from Theorems 5 and 6. ∎

EXAMPLE. We have observed that row and column sufficient matrices belong to $\mathbf{P}_0$. We might then inquire whether the converse holds for **Z**-matrices. Does $M \in \mathbf{Z} \cap \mathbf{P}_0$ imply that M is sufficient? (Since $M \in \mathbf{Z} \cap \mathbf{P}_0$ if and only if $M^{\mathrm{T}} \in \mathbf{Z} \cap \mathbf{P}_0$, there is no point in asking about row or column sufficiency separately.) The answer is no, for the **Z**-matrix

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

has nonnegative principal minors, and the vector $x^{\mathrm{T}} = (1,2)$ belongs to $\operatorname{rev} h_M \setminus \ker h_M$. The same vector with $\alpha = \{1,2\}$ gives a solution to (16), showing that condition (b) of Theorem 9 is not satisfied. For results on linear complementarity problems with $M \in \mathbf{Z} \cap \mathbf{P}_0$, see Mohan [19, 20].

Linear complementarity problems with **Z**-matrices are known to have two particularly interesting features. First, they belong to the class $\mathbf{Q}_0$. Thus, whenever the polyhedral set $F(q, M)$ is nonempty, then so is $S(q, M)$. Second, when $F(q, M)$ is nonempty, $S(q, M)$ contains a vector z^* such that $z^* \leqslant z$ for every $z \in F(q, M)$. The vector z^* is called the *least element* of $F(q, M)$. It is clear that such a vector must be unique. Thus, a feasible linear complementarity problem with a **Z**-matrix has a unique least-element solution.

As a concluding result, we show that the least-element solution provides an interesting structural property of the solution set of a linear complementarity problem (q, M) in which M is a Z-matrix.

THEOREM 10. *Let $M \in \mathbf{Z}$, and assume that $F(q, M) \neq \varnothing$. Let $w^* = q + Mz^*$, where z^* is the least-element solution of (q, M). Then*

$$z \in \mathrm{S}(q, M) \quad \Leftrightarrow \quad z - z^* \in \mathrm{S}(w^*, M).$$

Proof. Let $z \in \mathrm{S}(q, M)$. Then $\xi = z - z^* \geqslant 0$. Since $z \in \mathrm{S}(q, M)$, we have

$$w^* + M\xi = q + Mz^* + M\xi = q + Mz \geqslant 0.$$

Moreover,

$$0 \leqslant \xi^{\mathrm{T}}(w^* + M\xi) = (z - z^*)^{\mathrm{T}}(q + Mz) = -(z^*)^{\mathrm{T}}(q + Mz) \leqslant 0.$$

This implies $\xi^{\mathrm{T}}(w^* + M\xi) = 0$ and hence that $\xi \in \mathrm{S}(w^*, M)$.

Conversely, let $\xi \in \mathrm{S}(w^*, M)$. Then $z = z^* + \xi \geqslant 0$, and

$$q + Mz = q + Mz^* + M\xi = w^* + M\xi \geqslant 0.$$

It follows from the definitions that

$$z^{\mathrm{T}}(q + Mz) = (z^* + \xi)^{\mathrm{T}}(w^* + M\xi) = (z^*)^{\mathrm{T}}(w^* + M\xi). \tag{17}$$

Now if $(w^* + M\xi)_i > 0$, then $\xi_i = 0$ because $\xi \in \mathrm{S}(w^*, M)$. This implies $(M\xi)_i \leqslant 0$, since $M \in \mathbf{Z}$ and $\xi \geqslant 0$. Combining these observations, we note that $w_i^* > 0$ and hence $z_i^* = 0$. Accordingly, we deduce that $(z^*)^{\mathrm{T}}(w^* + M\xi) = 0$, which by Equation (17) implies that $z \in \mathrm{S}(w^*, M)$. ∎

REMARK. Theorem 10 says that $\mathrm{S}(q, M) = \{z^*\} + \mathrm{S}(w^*, M)$. This is noteworthy for two reasons. First, $\mathrm{S}(w^*, M)$ is the solution set of a problem based upon an LCP in which the constant column is a *nonnegative* vector. Second, the solution set $\mathrm{S}(q, M)$ is a *translate* (by z^*) of another solution set, $\mathrm{S}(w^*, M)$. Hence $\mathrm{S}(q, M)$ is polyhedral if and only if $\mathrm{S}(w^*, M)$ is polyhedral.

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