The Limit Points of Eigenvalues of Graphs*

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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ABSTRACT

If $r \geqslant \tau^{1/2} + \tau^{-1/2}$ (τ is the golden mean), then there exists a sequence of graphs whose kth largest eigenvalues converge to r. If $r \leqslant -(\tau^{1/2} + \tau^{-1/2})$, then there exists a sequence of graphs whose kth smallest eigenvalues converge to r.

The study of the limit points of the eigenvalues of graphs was initiated by Alan Hoffman [5]. There he described all of the limit points of the maximum eigenvalues of graphs that are less than $\tau^{1/2} + \tau^{-1/2}$. It turns out that the limit points for all graphs are just those of the maximum eigenvalues of trees. In [6] he showed that the limit points of the least eigenvalues of graphs that are greater than -2 are actually found by just considering trees. These limit points were also described in the paper by Dragoš Cvetković and Michael Doob [4] in their investigation of generalized line graphs. More recently, the work of Hoffman [5] was extended by James Shearer [7] to show that every real number $r \geqslant \tau^{1/2} + \tau^{-1/2}$ is the limit point of the largest eigenvalues of graphs. Here τ, as usual, is the golden mean.

In [5] Hoffman remarked, "On least eigenvalues, I can find all limit points $\geqslant -2$ of least eigenvalues of graphs (and these are algebraic integers), but I know nothing about the range < -2. And I know nothing at all about limit points for eigenvalues other than the greatest and least." It seems appropriate in a volume dedicated to Alan Hoffman to consider these limit

*This research has been supported by Grant A7547 of NSERC of Canada.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

points. We shall show that any $r \geqslant \tau^{1/2} + \tau^{-1/2}$ is the limit point of the kth largest eigenvalue of a graph, with a similar result for the kth smallest eigenvalue.

We shall use the Hoffman notation: for a graph G, we denote by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ the eigenvalues of the adjacency matrix of G in descending order, while $\lambda^1(G) \leqslant \lambda^2(G) \leqslant \cdots \leqslant \lambda^n(G)$ are the eigenvalues in ascending order. In addition, we denote by $T(n_1,\dots,n_k)$ the tree formed by taking the path $v_1, v_2,\dots, v_k$ and attaching $n_j \geqslant 0$ vertices of degree one to the vertex v_j. $\mathscr{T} = \{T(n_1,\dots,n_k) \mid k = 1,2,\dots\}$. These graphs are sometimes known as *caterpillars*. Shearer uses families of caterpillars which converge to certain limit points:

THEOREM 1 [7]. *If $r \geqslant \tau^{1/2} + \tau^{-1/2}$, then there exists a sequence of nonnegative integers $\{n_i \mid i = 1,2,\dots\}$ such that*

$$\lim_{t \to \infty} \lambda_1(T(n_1,\dots,n_t)) = r.$$

We wish to extend Shearer's results from the case of the largest eigenvalue to that of the eigenvalues that are either the kth largest or kth smallest.

THEOREM 2. *For any $k \geqslant 1$ and $r \geqslant \tau^{1/2} + \tau^{-1/2}$, there exists a sequence of nonnegative integers $\{n_i \mid i = 1,2,\dots\}$ such that*

$$\lim_{t \to \infty} \lambda_k(T(n_1,\dots,n_t)) = r.$$

THEOREM 3. *For any $k \geqslant 1$ and $r \leqslant -(\tau^{1/2} + \tau^{-1/2})$, there exists a sequence of nonnegative integers $\{n_i \mid i = 1,2,\dots\}$ such that*

$$\lim_{t \to \infty} \lambda^k(T(n_1,\dots,n_t)) = r.$$

In fact Theorem 3 will follow from Theorem 2, since our construction will only use trees in $\mathscr{T}$, and, since trees are bipartite, they have a symmetric spectrum (see [3], for example).

PROPOSITION 4. *Suppose G is a graph with n vertices and eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. H is formed by taking t copies of G, taking one additional vertex v, and arbitrarily adding edges joining v to the copies of G. Then λ_i is an eigenvalue of H with multiplicity (at least) $t-1$, and the remaining $n+1$ eigenvalues of H interlace the eigenvalues of G.*

Proof. Let $A(H)$ be the adjacency matrix; consider the principal submatrix formed by deleting the row and column corresponding to the vertex v.

The eigenvalues of this matrix consists of each λ_i with multiplicity t, and the eigenvalues of H must interlace these eigenvalues. ■

This simple result, surprisingly enough, is actually sufficiently strong to produce our first result.

COROLLARY 5. *If* $\lambda_1(G)=\lambda$, *and* H *is connected, then* $\lambda=\lambda_2(H)=\lambda_3(H)=\cdots=\lambda_t(H)$.

Since H is connected and G is a proper subgraph of H, it follows that $\lambda_1(G)<\lambda_1(H)$.

COROLLARY 6. *If t is given, G_i is a sequence of graphs, and H_i is constructed from G_i as in Proposition 4, then*

$$\lim_{i\to\infty}\lambda_1(G_i)=r \quad\Rightarrow\quad \lim_{i\to\infty}\lambda_j(H_i)=r$$

for $j=2,\dots,t$.

Note that if G is a tree, then H may also be made a tree by appropriate choice of edges joining the new vertex. However, even if G is a tree in $\mathscr{T}$, H cannot be in $\mathscr{T}$ unless $t=1$ or $t=2$. Nonetheless we can extend our result to show that Corollary 6 is still true if we restrict our attention to caterpillars.

If $T_1=T(m_1,\dots,m_k)$ and $T_2=T(n_1,\dots,n_l)$, define $T_1\circ T_2$ to be the graph $T(m_1,\dots,m_k,0,n_1,\dots,n_l)$. It is clear that $T\circ T$ is a special case of the construction used for Proposition 4. Let T^k denote the product of k factors of T using the (obviously associative) $\circ$ product. Clearly T^k is in $\mathscr{T}$.

LEMMA 7. *For any* $k\geqslant 1$,

$$\lambda_k(T^k)=\lambda_1(T).$$

The proof of Lemma 7 is an easy inductive argument using a slight variation of Proposition 4.

PROPOSITION 8. *Suppose* $T_i\in\mathscr{T}$ *for* $i=1,2,\dots$ *and* $\lim_{i\to\infty}\lambda_1(T_i)=r$. *Then*

$$\lim_{i\to\infty}\lambda_k\left(T_i^k\right)=r.$$

Hence the T_i^k, $i = 1, 2, \dots$, form a sequence of graphs whose kth largest eigenvalues have r as a limit point.

There is more than can be said about the limit points of least eigenvalues. If T is a tree and v is a univalent vertex of T, then $\hat{A}(T, v) = K^T K - 2I$, where K is the vertex-incidence matrix of T with the row corresponding to v deleted.

THEOREM 9. *The real number r is a limit point of least eigenvalues of graphs if*

(i) $-2 \leqslant r \leqslant 0$ *and* $r = \lambda^1(\hat{A}(T, v))$,

(ii) $\tau^{1/2} + \tau^{-1/2} \leqslant r \leqslant 2$, *and r is one of an infinite number of points of the form* $-(\beta^{1/2} + \beta^{-1/2})$ *(β is a particular type of algebraic integer),*

(iii) $r \leqslant \tau^{1/2} + \tau^{-1/2}$.

Proof. The first conclusion is a result of A. J. Hoffman [6]; the second follows from the consideration of graphs G satisfying $2 < \lambda_1(G) \leqslant \tau^{1/2} + \tau^{-1/2}$. Since they are all bipartite [1, 3], any limit point in that range gives a limit point for least eigenvalues too. The limit points for $\lambda_1(G)$ have been determined in [3]. The final conclusion follows from Theorem 3. ■

It is possible that the conditions given in parts (i) and (ii) of Theorem 9 are necessary as well as sufficient.

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Received 26 January 1988; final manuscript accepted 20 September 1988