Homotopy and Crossings of Systems of Curves on a Surface

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Robert E. Bixby

ABSTRACT

Let $C_1,\dots,C_k$ and $C_1',\dots,C_k'$ be closed curves on a compact orientable surface S. We characterize (in terms of counting crossings) when there exists a permutation π of $\{1,\dots,k\}$ such that, for each $i=1,\dots,k$, $C'_{\pi(i)}$ is freely homotopic to C_i or C_i^{-1}. The characterization is equivalent to the nonsingularity of a certain infinite symmetric matrix.

We prove the following theorem:

THEOREM. *Let $C_1,\dots,C_k$ and $C_1',\dots,C_{k'}'$ be primitive closed curves on a compact orientable surface S. Then the following are equivalent:*

(i) *$k=k'$, and there exists a permutation π of $\{1,\dots,k\}$ such that for each $i=1,\dots,k$*

$$C'_{\pi(i)} \sim C_i \quad or \quad C'_{\pi(i)} \sim C_i^{-1};$$

(ii) *for each closed curve D on S,*

$$\sum_{i=1}^{k} \operatorname{mincr}(C_i, D) = \sum_{i=1}^{k'} \operatorname{mincr}(C_i', D). \tag{1}$$

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Here we use the following terminology and notation. A *closed curve C on S* is a continuous function $C: S_1 \to S$, where S_1 is the unit circle $\{z \in \mathbb{C} \,|\, |z| = 1\}$. Two closed curves C and C' are called *(freely) homotopic*, and we write $C \sim C'$, if there exists a continuous function Φ: $[0,1] \times S_1 \to S$ such that $\Phi(0, z) = C(z)$ and $\Phi(1, z) = C'(z)$ for all $z \in S_1$. A closed curve C is *primitive* if there do not exist a closed curve D and an integer $n \geqslant 2$ such that $C \sim D^n$. For a closed curve D and integer n, D^n is the closed curve defined by $D^n(z) := D(z^n)$ for $z \in S_1$.

Finally, for closed curves C and D:

$$\begin{aligned} \operatorname{cr}(C, D) &:= \left|\{(y, z) \in S_1 \times S_1 \,|\, C(y) = D(z)\}\right| \qquad \text{if} \quad C \neq D, \\ \operatorname{cr}(C, C) &:= \left|\{(y, z) \in S_1 \times S_1 \,|\, C(y) = C(z),\ y \neq z\}\right|, \end{aligned} \tag{2}$$

$$\operatorname{mincr}(C, D) := \min\left\{\operatorname{cr}(\tilde{C}, \tilde{D}) \,\middle|\, \tilde{C} \sim C,\ \tilde{D} \sim D\right\}.$$

Proof. The implication (i) $\Rightarrow$ (ii) being trivial [as $\operatorname{mincr}(C^{-1}, D) = \operatorname{mincr}(C, D)$], we show (ii) $\Rightarrow$ (i). By symmetry we may assume

$$\sum_{i=1}^{k'} \sum_{j=1}^{k'} \operatorname{mincr}(C_i', C_j') \leqslant \sum_{i=1}^{k} \sum_{j=1}^{k} \operatorname{mincr}(C_i, C_j). \tag{3}$$

By a result of Baer [1] there exist $\tilde{C}_1 \sim C_1', \dots, \tilde{C}_{k'} \sim C_{k'}'$ such that

$$\operatorname{cr}\left(\tilde{C}_i, \tilde{C}_j\right) = \operatorname{mincr}\left(C_i', C_j'\right) \qquad \text{for} \quad i, j = 1, \dots, k'. \tag{4}$$

We may assume that in fact $\tilde{C}_i = C_i'$ for $i = 1, \dots, k'$, that $C_i' \neq C_j'$ if $i \neq j$, and that each point of S is passed at most twice by the C_i' (so no two crossings of the C_i' coincide).

Let $G = (V, E)$ be the graph made up by the curves C_i'. So G is a graph embedded on S. Each point of S passed twice by the C_i' is a vertex, of degree 4, of G. Moreover, we take as vertices some of the points of S passed exactly once by the C_i', in such a way that G will be a graph without loops or parallel edges. So each vertex of G has degree 2 or 4.

Now by (1), for each closed curve $D: S_1 \to S \setminus V$,

$$\sum_{i=1}^{k} \operatorname{mincr}(C_i, D) = \sum_{i=1}^{k'} \operatorname{mincr}(C_i', D) \leqslant \sum_{i=1}^{k'} \operatorname{cr}(C_i', D) = \operatorname{cr}(G, D), \tag{5}$$

where $\mathrm{cr}(G, D) := |\{z \in S_1 \mid D(z) \in G\}|$. Hence, by the "homotopic circulation theorem" in [2], there exist cycles $C_{11}, \dots, C_{1t_1}, \dots, C_{k1}, \dots, C_{kt_k}$ in G and rationals $\lambda_{11}, \dots, \lambda_{1t_1}, \dots, \lambda_{k1}, \dots, \lambda_{kt_k} > 0$ such that

$$C_{ij} \sim C_i \qquad (i = 1, \dots, k;\ j = 1, \dots, t_i), \tag{6a}$$

$$\sum_{j=1}^{t_i} \lambda_{ij} = 1 \qquad (i = 1, \dots, k), \tag{6b}$$

$$\sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{ij} \chi^{C_{ij}}(e) \leqslant 1 \qquad (e \in E). \tag{6c}$$

Here a *cycle* in G is a sequence

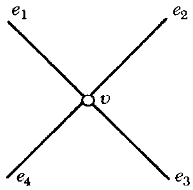
$$C = (v_0, e_1, v_1, e_2, v_2, \dots, v_{d-1}, e_d, v_d), \tag{7}$$

where $v_0, \dots, v_d$ are vertices of G, $v_0 = v_d$, and e_i is the edge connecting v_{i-1} and v_i $(i = 1, \dots, d)$. ($v_1, \dots, v_d, e_1, \dots, e_d$ need not be distinct.) With each cycle in G we can associate in the obvious way a closed curve on S—unique up to homotopy. For any cycle (7) and any edge e of G,

$$\chi^C(e) := \text{number of } i \in \{1, \dots, d\} \text{ with } e_i = e. \tag{8}$$

We call a cycle (7) *nonreturning* if $e_i \neq e_{i-1}$ for $i = 1, \dots, d$, and $e_1 \neq e_d$. Clearly, we may assume the C_{ij} to be nonreturning.

Consider now any vertex of G of degree 4, and denote the edges incident with v by e_1, e_2, e_3, e_4 in cyclic order:



(9)

We call edges e_1 and e_3 *opposite in* v, and similarly, we call e_2 and e_4 *opposite in* v.

The remainder of this proof consists of showing:

$$\begin{array}{ll}\text{(i) for each edge } e\text{, equality holds in (6c);} & \\ \text{(ii) for each cycle } C_{ij} \text{ and each vertex } v \text{ of degree 4,} & \\ \quad\text{each time when } C_{ij} \text{ passes } v\text{, it goes from one edge} & \\ \quad\text{to the edge opposite in } v. & \end{array} \tag{10}$$

Having shown this, it follows that each C_{ij} belongs to $\{C_1',(C_1')^{-1},\ldots,C_{k'}',(C_{k'}')^{-1}\}$, and hence we have (i) in our theorem.

In order to prove (10), we first show two lemmas. For each vertex v of degree 4, we fix one choice e_1, e_2, e_3, e_4 as in (9). For any cycle C in G, any vertex of degree 4, in G, and any $i, j \in \{1,2,3,4\}$, let

$$\alpha_{ij}^{v}(C) := \text{number of times } C \text{ passes } v \text{ by going from } e_i \text{ to } e_j \text{ or from } e_j \text{ to } e_i. \tag{11}$$

LEMMA A. *For any pair of nonreturning cycles C, D in G,*

$$\begin{aligned}\operatorname{mincr}(C,D) \leqslant & \sum_{v \in W} \{\beta_{13}\gamma_{24} + \beta_{24}\gamma_{13} + \tfrac{1}{2}[(\beta_{12}+\beta_{34})(\gamma_{13}+\gamma_{14}+\gamma_{23}+\gamma_{24}) \\ & + (\beta_{13}+\beta_{24})(\gamma_{12}+\gamma_{14}+\gamma_{23}+\gamma_{34}) + (\beta_{14}+\beta_{23})(\gamma_{12}+\gamma_{13}+\gamma_{24}+\gamma_{34})]\},\end{aligned} \tag{12}$$

where $\beta_{ij} := \alpha_{ij}^{v}(C)$ and $\gamma_{ij} := \alpha_{ij}^{v}(D)$, and $W := \{v \in V \mid v$ has degree $4\}$.

[Note that the term in (12) with factor $\frac{1}{2}$ contains all products $\beta_{gh}\gamma_{ij}$ with $g \neq h$, $i \neq j$, and $|\{g,h\} \cap \{i,j\}| = 1$.]

Proof of Lemma A. We can represent C and D as

$$\begin{aligned} C &= (v_0, f_1, v_1, f_2, v_2, \ldots, v_{s-1}, f_s, v_s), \\ D &= (w_0, g_1, w_1, g_2, w_2, \ldots, w_{t-1}, g_t, w_t), \end{aligned} \tag{13}$$

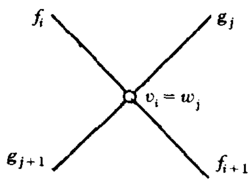
where $v_0, \ldots, v_s$ and $w_0, \ldots, w_t$ are vertices of G with $v_s = v_0$ and $w_t = w_0$, f_i is an edge of G connecting v_{i-1} and v_i ($i = 1, \ldots, s$), and g_i is an edge of

G connecting w_{i-1} and w_i $(i = 1,\dots,t)$, so that $f_i \neq f_{i-1}$ for $i = 1,\dots,s$ and $g_i \neq g_{i-1}$ for $i = 1,\dots,t$ (taking indices of v and f mod s, and indices of w and g mod t).

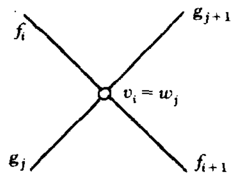
Let λ be the number of pairs $(i, j) \in \{1,\dots,s\} \times \{1,\dots,t\}$ such that

$$v_i = w_j \in W,\ f_i \text{ and } f_{i+1} \text{ are opposite in } v_i, \text{ and } g_j \text{ and } g_{j+1} \text{ are opposite in } w_j, \text{ while } \{f_i, f_{i+1}\} \neq \{g_j, g_{j+1}\}. \tag{14}$$

So (14) corresponds to



or



(15)

Let μ be the number of pairs $(i, j) \in \{1,\dots,s\} \times \{1,\dots,t\}$ such that

$$v_i = w_j \in W, \qquad f_{i+1} = g_{j+1}, \quad \text{and} \quad f_i \neq g_j. \tag{16}$$

So μ is also equal to the number of pairs $(i, j) \in \{1,\dots,s\} \times \{1,\dots,t\}$ such that

$$v_i = w_j \in W, \qquad f_i = g_j, \quad \text{and} \quad f_{i+1} \neq g_{j+1}. \tag{17}$$

Similarly, let ν be the number of pairs $(i, j) \in \{1,\dots,s\} \times \{1,\dots,t\}$ such that

$$v_i = w_j \in W, \qquad f_{i+1} = g_j, \quad \text{and} \quad f_i \neq g_{j+1}. \tag{18}$$

Again, ν is also equal to the number of pairs $(i, j) \in \{1,\dots,s\} \times \{1,\dots,t\}$ such that

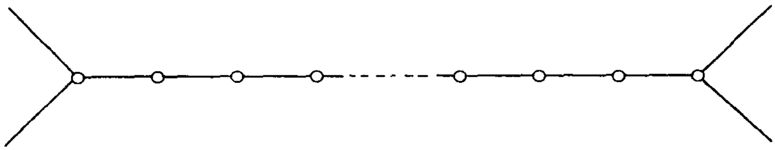
$$v_i = w_j \in W, \qquad f_i = g_{j+1}, \quad \text{and} \quad f_{i+1} \neq g_j. \tag{19}$$

Note that the right-hand side of (12) is equal to $\lambda + \mu + \nu$. To see that $\operatorname{mincr}(C, D) \leqslant \lambda + \mu + \nu$, note that μ is equal to the number of pairs $(i, j) \in$

$\{1,\ldots,s\}\times\{1,\ldots,t\}$ such that there exists a number $b\geqslant 1$ with

$$f_i \neq g_j, \qquad v_i = w_j, \qquad f_{i+1} = g_{j+1}, \qquad v_{i+1} = w_{j+1},$$
$$f_{i+2} = g_{j+2},\ldots, \qquad v_{i+b} = w_{j+b}, \qquad f_{i+b+1} \neq g_{j+b+1}, \tag{20}$$

which corresponds to pictures of type



(21)

Similarly, ν is equal to the number of pairs $(i,j)\in\{1,\ldots,s\}\times\{1,\ldots,t\}$ such that there exists a number $b\geqslant 1$ with

$$f_i \neq g_{j+1}, \qquad v_i = w_j, \qquad f_{i+1} = g_j, \qquad v_{i+1} = w_{j-1},$$
$$f_{i+2} = g_{j-1},\ldots, \qquad v_{i+b} = w_{j-b}, \qquad f_{i+b+1} \neq g_{j-b}. \tag{22}$$

Again, this corresponds to a picture of type (21).

Since each of the intersections of type (21) can be replaced by parts that have one crossing or none at all, we obtain $\operatorname{mincr}(C,D)\leqslant \lambda+\mu+\nu$. ∎

Next we study the pattern of the C_{ij} at one fixed vertex v of degree 4. Again, let the neighborhood of v be as in (9), and denote for $g,h\in\{1,2,3,4\}$

$$\alpha_{gh}^v := \alpha_{gh} := \sum_{i=1}^{k}\sum_{j=1}^{t_i}\lambda_{ij}\alpha_{gh}^v(C_{ij}). \tag{23}$$

Then:

Lemma B. *For each fixed vertex v of degree 4,*

$$2\alpha_{13}\alpha_{24}+\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{14}+\alpha_{12}\alpha_{23}+\alpha_{12}\alpha_{24}+\alpha_{13}\alpha_{14}+\alpha_{13}\alpha_{23}+\alpha_{13}\alpha_{34}$$
$$+\alpha_{14}\alpha_{24}+\alpha_{14}\alpha_{34}+\alpha_{23}\alpha_{24}+\alpha_{23}\alpha_{34}+\alpha_{24}\alpha_{34}\leqslant 2, \tag{24}$$

with equality only if $\alpha_{13}=\alpha_{24}=1$ *and* $\alpha_{12}=\alpha_{23}=\alpha_{34}=\alpha_{14}=0$.

Proof of Lemma B. The left-hand side of (24) is not larger than the first expression in the following series of inequalities (as the latter is obtained by adding $2\alpha_{12}\alpha_{34}+2\alpha_{14}\alpha_{23}$):

$$\begin{aligned}
&2\alpha_{13}\alpha_{24}+2\alpha_{12}\alpha_{34}+2\alpha_{14}\alpha_{23}+\alpha_{12}\alpha_{13}+\alpha_{12}\alpha_{14}+\alpha_{12}\alpha_{23}+\alpha_{12}\alpha_{24}+\alpha_{13}\alpha_{14}\\
&\qquad+\alpha_{13}\alpha_{23}+\alpha_{13}\alpha_{34}+\alpha_{14}\alpha_{24}+\alpha_{14}\alpha_{34}+\alpha_{23}\alpha_{24}+\alpha_{23}\alpha_{34}+\alpha_{24}\alpha_{34}\\
&=\tfrac{1}{2}\big[\alpha_{12}(\alpha_{13}+\alpha_{14}+\alpha_{23}+\alpha_{24}+2\alpha_{34})+\alpha_{13}(\alpha_{12}+\alpha_{14}+\alpha_{23}+\alpha_{34}+2\alpha_{24})\\
&\qquad+\alpha_{14}(\alpha_{12}+\alpha_{13}+\alpha_{24}+\alpha_{34}+2\alpha_{23})+\alpha_{23}(\alpha_{12}+\alpha_{24}+\alpha_{13}+\alpha_{34}+2\alpha_{14})\\
&\qquad+\alpha_{24}(\alpha_{12}+\alpha_{23}+\alpha_{14}+\alpha_{34}+2\alpha_{13})+\alpha_{34}(\alpha_{13}+\alpha_{23}+\alpha_{14}+\alpha_{24}+2\alpha_{12})\big]\\
&=\tfrac{1}{2}\big[\alpha_{12}(\delta_3+\delta_4)+\alpha_{13}(\delta_2+\delta_4)+\alpha_{14}(\delta_2+\delta_3)+\alpha_{23}(\delta_1+\delta_4)\\
&\qquad+\alpha_{24}(\delta_1+\delta_3)+\alpha_{34}(\delta_1+\delta_2)\big], \qquad (25)
\end{aligned}$$

where, for $g\in\{1,2,3,4\}$,

$$\delta_g := \sum_{i=1}^{k}\sum_{j=1}^{t_i}\lambda_{ij}\chi^{C_{ij}}(e_g). \qquad (26)$$

So by (6c), $\delta_g\leqslant 1$ for each $g\in\{1,2,3,4\}$. Moreover,

$$\begin{aligned}
\delta_1&=\alpha_{12}+\alpha_{13}+\alpha_{14}, &\qquad \delta_2&=\alpha_{12}+\alpha_{23}+\alpha_{24},\\
\delta_3&=\alpha_{13}+\alpha_{23}+\alpha_{34}, &\qquad \delta_4&=\alpha_{14}+\alpha_{24}+\alpha_{34}.
\end{aligned} \qquad (27)$$

Hence the last expression in (25) is not larger than the first expression in

$$\alpha_{12}+\alpha_{13}+\alpha_{14}+\alpha_{23}+\alpha_{24}+\alpha_{34}=\tfrac{1}{2}(\delta_1+\delta_2+\delta_3+\delta_4)\leqslant 2. \qquad (28)$$

This proves the inequality (24). In order to have equality we should have

$$\alpha_{12}\alpha_{34}=0, \qquad \alpha_{14}\alpha_{23}=0, \qquad \delta_1=\delta_2=\delta_3=\delta_4=1. \qquad (29)$$

Now (27) and (29) imply

$$\begin{aligned} \alpha_{12} &= \tfrac{1}{2}(\delta_1+\delta_2-\delta_3-\delta_4)+\alpha_{34}=\alpha_{34},\\ \alpha_{14} &= \tfrac{1}{2}(\delta_1+\delta_4-\delta_2-\delta_3)+\alpha_{23}=\alpha_{23}. \end{aligned} \tag{30}$$

Hence $\alpha_{12}=\alpha_{34}=\alpha_{14}=\alpha_{23}=0$ and $\alpha_{13}=\alpha_{24}=1$. ∎

From Lemmas A and B we derive

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{i'=1}^{k}\operatorname{mincr}(C_i,C_{i'})\\
&=\sum_{i=1}^{k}\sum_{j=1}^{t_i}\sum_{i'=1}^{k}\sum_{j'=1}^{t_{i'}}\lambda_{ij}\lambda_{i'j'}\operatorname{mincr}(C_{ij},C_{i'j'})\\
&\leqslant\sum_{i=1}^{k}\sum_{j=1}^{t_i}\sum_{i'=1}^{k}\sum_{j'=1}^{t_{i'}}\lambda_{ij}\lambda_{i'j'}\sum_{v\in W}\Big(\alpha_{13}^v(C_{ij})\alpha_{24}^v(C_{i'j'})+\alpha_{24}^v(C_{ij})\alpha_{13}^v(C_{i'j'})\\
&\quad+\tfrac{1}{2}\Big\{\big[\alpha_{12}^v(C_{ij})+\alpha_{34}^v(C_{ij})\big]\big[\alpha_{13}^v(C_{i'j'})+\alpha_{14}^v(C_{i'j'})+\alpha_{23}^v(C_{i'j'})+\alpha_{24}^v(C_{i'j'})\big]\\
&\qquad+\big[\alpha_{13}^v(C_{ij})+\alpha_{24}^v(C_{ij})\big]\big[\alpha_{12}^v(C_{i'j'})+\alpha_{14}^v(C_{i'j'})+\alpha_{23}^v(C_{i'j'})+\alpha_{34}^v(C_{i'j'})\big]\\
&\qquad+\big[\alpha_{14}^v(C_{ij})+\alpha_{23}^v(C_{ij})\big]\big[\alpha_{12}^v(C_{i'j'})+\alpha_{13}^v(C_{i'j'})+\alpha_{24}^v(C_{i'j'})+\alpha_{34}^v(C_{i'j'})\big]\Big\}\Big)\\
&=\sum_{v\in W}\Big\{\alpha_{13}^v\alpha_{24}^v+\alpha_{24}^v\alpha_{13}^v+\tfrac{1}{2}\big[(\alpha_{12}^v+\alpha_{34}^v)(\alpha_{13}^v+\alpha_{14}^v+\alpha_{23}^v+\alpha_{24}^v)\\
&\qquad+(\alpha_{13}^v+\alpha_{24}^v)(\alpha_{12}^v+\alpha_{14}^v+\alpha_{23}^v+\alpha_{34}^v)+(\alpha_{14}^v+\alpha_{23}^v)(\alpha_{12}^v+\alpha_{13}^v+\alpha_{24}^v+\alpha_{34}^v)\big]\Big\}\\
&=\sum_{v\in W}\big(2\alpha_{13}^v\alpha_{24}^v+\alpha_{12}^v\alpha_{13}^v+\alpha_{12}^v\alpha_{14}^v+\alpha_{12}^v\alpha_{23}^v+\alpha_{12}^v\alpha_{24}^v+\alpha_{13}^v\alpha_{14}^v+\alpha_{13}^v\alpha_{23}^v+\alpha_{13}^v\alpha_{34}^v\\
&\qquad+\alpha_{14}^v\alpha_{24}^v+\alpha_{14}^v\alpha_{34}^v+\alpha_{23}^v\alpha_{24}^v+\alpha_{23}^v\alpha_{24}^v+\alpha_{23}^v\alpha_{34}^v+\alpha_{24}^v\alpha_{34}^v\big)\\
&\leqslant 2|W|=\sum_{i=1}^{k'}\sum_{i'=1}^{k'}\operatorname{mincr}(C_i',C_{i'}').
\end{aligned}\tag{31}$$

By our assumption (3), we should have equality throughout in (31). Hence by Lemma B, for each vertex v in W,

$$\alpha_{13}^v = \alpha_{24}^v = 1, \qquad \alpha_{12}^v = \alpha_{23}^v = \alpha_{34}^v = \alpha_{14}^v = 0. \tag{32}$$

So (10) holds, and hence we have (i) in our theorem. □□

Our theorem above can be formulated equivalently as the nonsingularity of a certain infinite symmetric matrix. Let $\mathscr{C}$ be the family of equivalence classes of closed curves on S, with respect to the equivalence relation $\sim$. For $\Gamma, \Delta \in \mathscr{C}$ we define

$$\operatorname{mincr}(\Gamma, \Delta) := \operatorname{mincr}(C, D) \tag{33}$$

for (arbitrary) $C \in \Gamma$ and $D \in \Delta$. So mincr is considered also as a function from $\mathscr{C} \times \mathscr{C}$ to $\mathbb{Z}_+$ (= set of nonnegative integers). We can represent this function as an infinite symmetric matrix M, with both rows and columns indexed by $\mathscr{C}$.

The rows of M are not linearly independent. First of all, the row corresponding to the trivial class $\langle 0 \rangle$ is all-zero (where 0 is a homotopically trivial closed curve, and where $\langle \cdots \rangle$ denotes the equivalence class containing $\cdots$). Moreover, the rows corresponding to $\langle C \rangle$ and to $\langle C^{-1} \rangle$ are the same, since $\operatorname{mincr}(C^{-1}, D) = \operatorname{mincr}(C, D)$ for each closed curve D. More generally, it is shown in [2] that for each pair of closed curves C, D on S and each $n \in \mathbb{Z}$,

$$\operatorname{mincr}(C^n, D) = |n| \operatorname{mincr}(C, D). \tag{34}$$

So the row corresponding to $\langle C^n \rangle$ is a multiple of the row corresponding to $\langle C \rangle$.

Now the theorem above actually says that (34) generate *all* linear dependencies of rows of M. To explain this, we mention the following result of [2]:

> for each homotopically nontrivial closed curve C on S there exists a primitive closed curve D on S and an integer $n \geqslant 1$ such that $C \sim D^n$. The integer n and closed curve D are unique (up to homotopy). (35)

Let $\mathscr{C}_p \subseteq \{\langle C\rangle \mid C$ a primitive closed curve$\}$, so that for each primitive closed curve C exactly one of $\langle C\rangle$ and $\langle C^{-1}\rangle$ belongs to $\mathscr{C}_p$, which we denote by $[C]$. Let M' be the $\mathscr{C}_p \times \mathscr{C}_p$ submatrix of M. Then the following is equivalent to our theorem above:

EQUIVALENT FORM OF THE THEOREM. *The matrix M' is nonsingular, i.e., the rows of M' are linearly independent.*

Proof of Equivalence.

I. We first derive the equivalent form from the theorem. Suppose M' has linearly dependent rows. That is, there are distinct $\Gamma_1, \ldots, \Gamma_t \in \mathscr{C}_p$ and $\lambda_1, \ldots, \lambda_t \in \mathbb{R} \setminus \{0\}$ (with $t \geqslant 1$) such that for each $\Delta \in \mathscr{C}_p$

$$\sum_{i=1}^{t} \lambda_i \operatorname{mincr}(\Gamma_i, \Delta) = 0. \tag{36}$$

Since M' is an integer matrix, we may assume that the λ_i are rational, and hence integer. By repeating each Γ_i $|\lambda_i|$ times, we obtain $\Gamma_1', \ldots, \Gamma_{t'}'$ and $\Gamma_1'', \ldots, \Gamma_{t''}''$ in $\mathscr{C}_p$ (with $t' + t'' \geqslant 1$), so that for each $\Delta \in \mathscr{C}_p$

$$\sum_{i=1}^{t'} \operatorname{mincr}(\Gamma_i', \Delta) = \sum_{i=1}^{t''} \operatorname{mincr}(\Gamma_i'', \Delta), \tag{37}$$

and so that $\{\Gamma_1', \ldots, \Gamma_{t'}'\} \cap \{\Gamma_1'', \ldots, \Gamma_{t''}''\} = \varnothing$. Now (34) and (35) imply that (37) in fact holds for every $\Delta \in \mathscr{C}$. But then our theorem gives that $t' = t''$, and there exists a permutation π of $\{1, \ldots, t'\}$ such that $\Gamma_i' = \Gamma_{\pi(i)}''$ for each $i = 1, \ldots, t'$. This is a contradiction.

II. To see the reverse implication, note that condition (ii) of the Theorem implies that for each $\Delta \in \mathscr{C}_p$

$$\sum_{i=1}^{k} \operatorname{mincr}([C_i], \Delta) = \sum_{i=1}^{k'} \operatorname{mincr}([C_i'], \Delta). \tag{38}$$

Since by the equivalent form of the theorem the rows of M' are linearly independent, we must have $k = k'$ and $[C_i] = [C'_{\pi(i)}]$ for each $i = 1, \ldots, k$, for some permutation π of $\{1, \ldots, k\}$. ∎

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