On Cycle Cones and Polyhedra

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Robert E. Bixby

ABSTRACT

Given an undirected graph G and a cost associated with each edge, the *weighted girth problem* is to find a simple cycle of G having minimum total cost. We consider several variants of the weighted girth problem, some of which are NP-hard and some of which are solvable in polynomial time. We also consider the polyhedra associated with each of these problems. Two of these polyhedra are the *cycle cone* of G, which is the cone generated by the incidence vectors of cycles of G, and the *cycle polytope* of G, which is the convex hull of the incidence vectors of cycles of G. First we give a short proof of Seymour's characterization of the cycle cone of G. Next we give a polyhedral composition result for the cycle polytope of G. In particular, we prove that if G decomposes via a 3-edge cut into graphs G_1 and G_2, say, then defining linear systems for the cycle polytopes of G_1 and G_2 can be combined in a certain way to obtain a defining linear system for the cycle polytope of G. We also describe a

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polynomial decomposition-based algorithm for the weighted girth problem on Halin graphs, and we give a complete linear description for the cycle polytope of G, in the case G is a Halin graph.

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1. INTRODUCTION

Given an undirected graph $G=(V,E)$ and a vector $c=(c_e: e\in E)$ of edge costs, the *weighted girth problem* is to find a simple cycle in G for which the sum of the costs is minimized. This problem is NP-hard in general, because an instance of the traveling salesman problem can be reduced to it by subtracting a large positive constant from the cost of each edge. On the other hand, the weighted girth problem is polynomially solvable when certain restrictions are placed on the cost vector.

For example, suppose $c_e \geqslant 0$ for all $e\in E$. Then the minimum cost cycle containing edge $e=uv$ can be found by deleting e and then finding a minimum cost path joining u to v in the resulting graph. This can be done by replacing each edge with a pair of oppositely directed arcs each having the cost of the original edge, and then applying a shortest path algorithm. (See Figure 1.) The minimum cost cycle can thus be found by performing $|E|$ shortest path calculations.

Now suppose there are negative cost edges, but the sum of costs is nonnegative for every cycle in G. The previous reduction no longer works because of negative cost directed cycles of length 2 in the constructed digraph. However, the problem can still be solved by solving $|V|$ minimum cost 2-factor problems, as follows. For each node v, find a minimum cost cycle containing v, by adding a zero cost loop to each node $w\in V\backslash\{v\}$, and then find a minimum cost 2-factor of the resulting graph. (See Figure 2.)

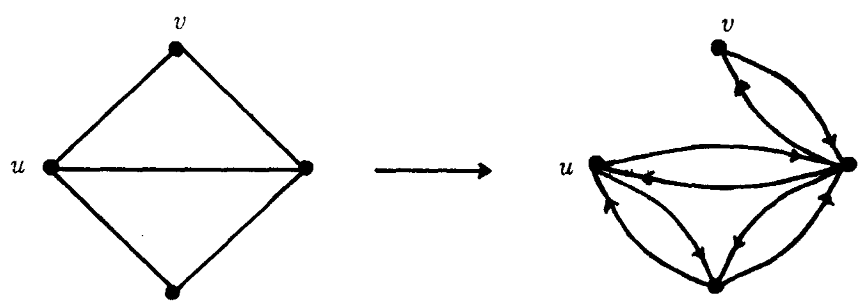


FIG. 1.

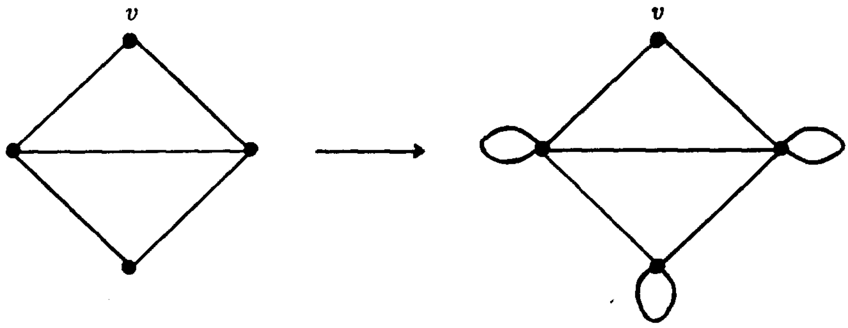


FIG. 2.

This procedure can be applied whether or not negative cycles exist. When it is applied at node v, the solution consists of a cycle containing v, together with some other cycles (and loops). If all these other cycles have nonnegative (and hence zero) cost, then a minimum cost cycle through v has been found. If the solution contains cycles of negative cost, then the cycle through v may not be of minimum cost, but a negative cycle has been found. (To simply check for a negative cycle, it suffices to add a loop at every node and solve a single minimum cost 2-factor problem. A minimum cost 2-factor can be found by using a direct algorithm of Edmonds and Johnson (see [4]) or by transforming the problem to a minimum cost perfect matching problem. These transformations are described in Lawler [10].)

The polyhedra associated with the various optimization problems described above are the main subjects of this paper. The *cycle polytope*, $P(G)$, of a graph $G=(V,E)$ is the convex hull of the set $\{\chi(C): C \in \mathscr{C}\}$ of incidence vectors of the edge sets of the simple cycles of G. The *cycle cone* $C(G)$ is the cone generated by $\{\chi(C): C \in \mathscr{C}\}$. We are interested in obtaining linear systems whose solution sets are these polyhedra. Since the weighted girth problem is equivalent to minimizing cx for $x \in P(G)$, and is NP-hard, it is unlikely that we can obtain a tractable, complete linear system defining $P(G)$ (cf. Karp and Papadimitriou [9]). On the other hand, optimizing over $C(G)$ is equivalent to finding a negative cost cycle, in which case the problem is unbounded, or determining that none exist, in which case the zero vector is optimal. As we have seen, this problem can be solved polynomially, and indeed, Seymour [13] gave a linear system that defines $C(G)$.

The *sum* of polyhedra P and Q, denoted $P+Q$, is the set of all z such that $z=x+y$ where $x \in P$ and $y \in Q$. Solving the weighted girth problem for nonnegative cost vectors is equivalent to minimizing cx for $x \in P(G)+\mathbb{R}_+^E$, where $\mathbb{R}_+^E$ denotes the nonnegative orthant indexed on E. This sum is called the *dominant* of $P(G)$. Solving the weighted girth problem in the absence of negative cost cycles is equivalent to minimizing cx for $x \in P(G) + C(G)$. Since both these cases are polynomially solvable, it is not unreasonable to hope for tractable linear systems defining these two polyhedra; however, none are known at present.

This paper is structured as follows. In Section 2, we consider a relaxation of $C(G)$ and show how a characterization of its extreme rays results in a very short proof of Seymour's theorem. In Section 3, the polyhedra $L(G)$ and $U(G)$ are defined via parameters μ_x and λ_x associated with a vector $x \in C(G)$. Next we present polyhedral composition results for $L(G)$ and $U(G)$. In particular, we prove that if G is a graph that decomposes via a 3-edge cut into graphs G_1 and G_2, say, then defining linear systems for $L(G_1)$ and $L(G_2)$ can be combined to obtain a defining linear system for $L(G)$; a similar result holds for $U(G)$. In Section 4 we turn to Halin graphs. First we describe a polynomial algorithm for the weighted girth problem on Halin graphs. Next we give linear systems for $C(G)$ on Halin graphs and wheels. In Section 5 we address the polyhedral question associated with the weighted girth problem; that is, we find a linear description for $P(G)$. Finally, in Section 6 we show how to derive a linear description for TSP(G), the convex hull of incidence vectors of hamiltonian cycles, from that for $P(G)$.

Throughout this paper we are dealing with undirected graphs. Given a graph $G = (V, E)$, V denotes the node set of G, and E the edge set of G. For a node $v \in V$, $\delta(v)$ is the set of edges incident with v, and given a set $X \subseteq V$, $\delta(X)$ is the set of edges with exactly one end in X. Given a partition $\{X, Y\}$ of V, $\delta(X, Y)$ is the set of edges with one end in X and one end in Y. The collection $\mathscr{K}$ of *cuts* of G consists of those sets $\delta(X, Y)$ where $\{X, Y\}$ is a partition of V and $X \neq \varnothing \neq Y$. Given a cut $K = \delta(X, Y)$, the sets X and Y are called the *shores* of K. The term *cycle* always means *simple cycle*, and $\mathscr{C}$ is used to denote the collection of cycles of G.

We assume familiarity with basic polyhedral theory. For an introductory survey in this topic, see [11]. For a complete reference, see [12]:

2. THE CYCLE CONE

Given a graph $G = (V, E)$, let $\mathscr{C}$ be the collection of edge sets of cycles of G and let $\mathscr{K}$ be the collection of edge sets of cuts of G. The *cycle cone*, $C(G)$, is the cone generated by the set $\{\chi(C) : C \in \mathscr{C}\}$ of incidence vectors of cycles. In [13] Seymour showed that $C(G)$ is the solution set of the following linear system:

$$\begin{aligned} x(K\setminus\{e\}) - x_e &\geqslant 0 \qquad \text{for all} \quad K \in \mathscr{K} \ \text{ and all } \ e \in K, \\ x_e &\geqslant 0 \qquad \text{for all} \quad e \in E. \end{aligned} \tag{2.1}$$

Alternative proofs have been given by Aráoz et al. [1] and by Hoffman and Lee [7]. Here we provide a short, elementary proof by first characterizing the extreme rays of the cone, $\tilde{C}(G)$, defined by the following subsystem of (2.1):

$$\begin{aligned} x(\delta(v)\setminus\{e\}) - x_e &\geqslant 0 \qquad \text{for} \quad v \in V \text{ and } e \in \delta(v), \\ x_e &\geqslant 0 \qquad \text{for} \quad e \in E. \end{aligned} \tag{2.2}$$

Lemma 2.1. *The cone $\tilde{C}(G)$ is generated by the set $\{\chi(C): C \in \mathscr{C}\}$, together with the set of vectors of the form $\chi(C_1) + \chi(C_2) + 2\chi(P)$, where C_1 and C_2 are node-disjoint cycles and P is a path joining C_1 and C_2 having no internal nodes in C_1 or C_2.*

Proof. Let $\tilde{x}$ be an extreme ray solution to (2.2). Then $\tilde{x}$ is the unique (up to scalar multiplication) nonzero solution to a system obtained from (2.2) by setting some of the inequalities to equations. Let N be the set of equations of the form $x_e = 0$ satisfied by $\tilde{x}$, and among the other equations satisfied by $\tilde{x}$, take I to be a subset such that $I \cup N$ is maximally linearly independent. Clearly $|I \cup N| = |E| - 1$, and where $\tilde{E} = \{e: \tilde{x}_e > 0\}$, $|I| = |\tilde{E}| - 1$. Let G be the subgraph induced by E. By uniqueness of $\tilde{x}$, G is connected. Moreover, by the feasibility of $\tilde{x}$, every node of $\tilde{G}$ has degree at least 2, implying $\tilde{G}$ has at most $|\tilde{E}|$ nodes.

Each equation in I is associated with some node v of $\tilde{G}$; we next establish that each node of G is associated with at most one such equation in I. Suppose for some node v and edges $e, f \in \delta(v)$, the equations

$$x(\delta(v)\setminus\{e\}) - x_e = 0$$

and

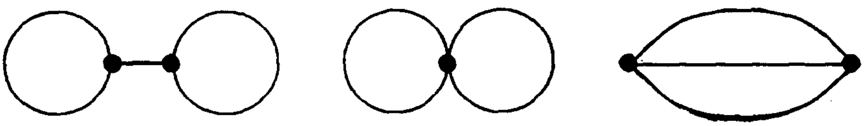
$$x(\delta(v)\setminus\{f\}) - x_f = 0$$

are in I. Adding the two gives that $x_j = 0$ for all $j \in \delta(v)\setminus\{e, f\}$, implying that N together with one of the two equations spans the other equation, a contradiction. Thus, the equations in I correspond to $|\tilde{E}|-1$ distinct nodes of G.

Therefore $\tilde{G}$ has either $|\tilde{E}|$ or $|\tilde{E}|-1$ nodes.

Suppose first that $\tilde{G}$ has $|\tilde{E}|$ nodes. Then $\tilde{G}$ is a cycle, and $\tilde{x}_e = \tilde{x}_f$ for all $e, f \in \tilde{E}$, as desired.

Suppose now that $\tilde{G}$ has $|\tilde{E}|-1$ nodes. Then $\tilde{G}$ is a subdivision of one of the graphs below:



(a) (b) (c)

(That is, $\tilde{G}$ may be obtained from one of these graphs by replacing some of the edges by paths.)

Now every node of $\tilde{G}$ corresponds to some equation in I, implying that if v is incident to exactly two edges, e and f, then $\tilde{x}_e = \tilde{x}_f$. It is straightforward to check that if $\tilde{G}$ is of type (b) or (c), then the incidence vector of some cycle properly contained in G satisfies all the equations in $I \cup N$, contradicting uniqueness of $\tilde{x}$. If G is of type (a), then letting C_1 and C_2 be the two cycles and P be the path between them, we have that the vector $\chi(C_1)+\chi(C_2)+2\chi(P)$ satisfies all the equations in $I \cup N$, which, by uniqueness of $\tilde{x}$, completes the proof. ■

Now we can prove Seymour's theorem.

THEOREM 2.2 (Seymour [13]). *The cycle cone $C(G)$ is the solution set of the system* (2.1).

Proof. Since the incidence vector of every cycle is a solution to (2.1), the inequalities (2.1) are valid for $C(G)$. It suffices to show that every vector on an extreme ray of the cone given by (2.1) lies in $C(G)$.

We proceed by induction on $|V|$. Let $\tilde{x}$ be an extreme ray solution to (2.1). Suppose first that $\tilde{x}$ is also an extreme ray solution to (2.2). Then by Lemma 2.1, $\tilde{x}$ is either a multiple of the incidence vector of a cycle, and we are finished, or $\tilde{x}$ is a multiple of a vector of the form $\chi(C_1)+\chi(C_2)+2\chi(P)$, in which case $\tilde{x}$ violates an inequality of (2.1) corresponding to the cut $\delta(X)$, where X is the node set of C_1, a contradiction.

Assume $\tilde{x}$ is not an extreme ray solution to (2.2). Then there is some cut K that is not the star of a node, and an edge $e \in K$, such that

$$\tilde{x}(K\setminus\{e\}) - \tilde{x}_e = 0.$$

Let V_1 and V_2 be the shores of K, and let $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ be the graphs obtained from G by contracting the subgraphs induced by V_2 and V_1, respectively, to a single node; let w denote this node in both graphs. For $i=1,2$, let $\tilde{x}^i$ be the restriction of $\tilde{x}$ to E_i. Since every cut of G_i is also a cut of G, it follows by induction that $\tilde{x}^i \in C(G_i)$ $(i=1,2)$. That is, $\tilde{x}^i$ can be expressed as a positive linear combination of incidence vectors of a set $\mathscr{C}_i$ of cycles of G_i $(i=1,2)$. Let $\mathscr{C}_i'$ be the set of members of $\mathscr{C}_i$ that meet K $(i=1,2)$. Since $\tilde{x}^i(K\setminus\{e\}) - \tilde{x}^i_e = \tilde{x}(K\setminus\{e\}) - \tilde{x}_e = 0$, each cycle of $\mathscr{C}_i'$ must contain e and exactly one other edge of K, for $i=1,2$. Therefore the cycles of $\mathscr{C}_1'$ and $\mathscr{C}_2'$ can be combined to form a set $\mathscr{C}'$ of cycles of G, each containing e and one other edge of K, and $\tilde{x}$ is a positive linear combination of the incidence vectors of the members of $\mathscr{C}',(\mathscr{C}_1\setminus\mathscr{C}_1'),(\mathscr{C}_2\setminus\mathscr{C}_2')$. Hence $\tilde{x} \in C(G)$ as required. ■

3. $L(G)$ AND $U(G)$

Let $G=(V,E)$ be a graph, with cycle polytope $P(G)$ and cycle cone $C(G)$. Given a vector $x \in C(G)$, there are, in general, many ways that x may be expressed as a nonnegative linear combination of incidence vectors of cycles. Since each such expression can be viewed as a vector γ in $\mathbb{R}^{\mathscr{C}}$, we define $Q_x \subseteq \mathbb{R}^{\mathscr{C}}$ to be the collection of those vectors, that is, Q_x is the polyhedron given by the system

$$\sum_{C \in \mathscr{C}} \gamma_C \chi(C) = x,$$

$$\gamma_C \geqslant 0 \qquad \text{for all} \quad C \in \mathscr{C}.$$

Since Q_x is a bounded polyhedron, we can define the numbers λ_x and μ_x to be

$$\lambda_x = \min\{1\cdot\gamma : \gamma \in Q_x\},$$

$$\mu_x = \max\{1\cdot\gamma : \gamma \in Q_x\}.$$

Then $P(G)$ can be expressed in terms of λ and μ:

$$P(G) = \{x \in C(G) : \lambda_x \leqslant 1 \text{ and } \mu_x \geqslant 1\}.$$

In addition, we can define the polyhedra

$$L(G) = \{x \in C(G) : \lambda_x \leqslant 1\} \quad \text{and} \quad U(G) = \{x \in C(G) : \mu_x \geqslant 1\},$$

and then we have that $P(G) = L(G) \cap U(G)$ and $C(G) = L(G) \cup U(G)$. Clearly $U(G) = P(G) + C(G)$.

By linear programming duality, we have that μ_x is the optimal value of the following linear programming problem:

$$\begin{array}{ll} \min & \pi x \\ \text{subject to} & \pi(C) \geqslant 1 \quad \text{for all } C \in \mathscr{C}, \\ & \pi \text{ unrestricted.} \end{array}$$

As we saw in Section 1, the separation problem over the polyhedron

$$U^*(G) = \{\pi \in \mathbb{R}^E : \pi(C) \geqslant 1 \text{ for all } C \in \mathscr{C}\}$$

can be solved in polynomial time as follows: We consider π to be a vector of edge costs. If there is a negative cost cycle, then the constraint corresponding to that cycle is violated. Otherwise, the weighted girth problem is polynomially solvable, and its optimal solution provides a violated constraint, if one exists. Since the separation and optimization problems are polynomially equivalent (see [5]), μ_x can be computed in polynomial time. However, an open problem is to find a defining system for $U(G)$. This is equivalent to finding a generating set for $U^*(G)$, i.e., sets H and $K \subseteq \mathbb{R}^E$ such that $U^*(G)$ is equal to the convex hull of H plus the cone generated by K. This can be seen as follows.

First we show that $H \subseteq U^*(G)$ includes at least one member of each minimal nonempty face of $U^*(G)$ if and only if $U(G) = \{x \in C(G) : ax \geqslant 1$ for all $a \in H\}$. To see the necessity, first note that for all $a \in H$, the inequality $ax \geqslant 1$ is clearly valid for $U(G)$. Let $\hat{x} \in C(G)$ be such that $a\hat{x} \geqslant 1$ for all $a \in H$. Then $a\hat{x} \geqslant 1$ for all $a \in U^*(G)$, implying $\mu_{\hat{x}} \geqslant 1$, and therefore $\hat{x} \in U(G)$, as required.

To see sufficiency, let F be a minimal nonempty face of $U^*(G)$. Then there is some $\hat{x} \in C(G)$ such that F is exactly the set of members π of

$U^*(G)$ satisfying $\pi\hat{x} = \mu_{\hat{x}}$. Clearly, $\hat{x}$ can be chosen (by scaling) so that $\mu_{\hat{x}} = 1$; it follows that $\hat{x}$ is on some facet of $U(G)$. That is, $a\hat{x} = 1$ for some $a \in H$. Clearly, $a \in U^*(G)$, from which it follows that $a \in F$, as required.

Therefore, since a linear system for $C(G)$ is given by the system (2.1), the problem of finding a defining system for $U(G)$ reduces to finding a set H of representatives of the minimal faces of $U^*(G)$. Indeed, that is the open problem in finding a set H, K of generators for $U^*(G)$. The set K must contain a set of generators of the cone

$$\left\{ \pi \in \mathbb{R}^E : \pi(C) \geqslant 0 \text{ for all } C \in \mathscr{C} \right\}.$$

Equivalently, the cone

$$\{ x \in \mathbb{R}^E : ax \geqslant 0 \text{ for all } a \in K \}$$

must be equal to the cone generated by the incidence vectors of the cycles of G. Therefore, by Theorem 2.2, we can obtain a suitable set K from the coefficients of the constraints of (2.1).

A similar analysis holds for $L(G)$. In this case we are interested in the polyhedron

$$L^*(G) = \left\{ \pi \in \mathbb{R}^E : \pi(C) \leqslant 1 \text{ for all } C \in \mathscr{C} \right\}.$$

This has the property that given a generating set H', K' for $L^*(G)$, where $L^*(G)$ is the convex hull of H' plus the cone generated by K', the polytope $L(G)$ is given by

$$L(G) = \{ x \in C(G) : ax \leqslant 1 \text{ for all } a \in H' \}.$$

We remark that in the general sense of Johnson [8], $U(G)$ and $U^*(G)$ form an example of a blocking pair of polyhedra, and $L(G)$ and $L^*(G)$ form an example of an antiblocking pair of polyhedra.

It is easy to see that $L(G)$ is the convex hull of the incidence vectors of the cycles of G, together with the zero vector. Since it is NP-hard, in general, to optimize over $L(G)$, as well as to separate over $L^*(G)$, it is unlikely that a set H' can be found, in general. However, we show in the next section that the weighted girth problem is polynomially solvable for the special class of Halin graphs, for arbitrary objective functions. In Section 5 we give defining linear systems for $L(G)$ and $U(G)$ for this class.

We conclude this section with a theorem that facilitates computing μ_x and λ_x for graphs with nontrivial 3-edge cuts, followed by composition results for $L(G)$ and $U(G)$.

A cut is *nontrivial* if both its shores have at least two nodes. For a cut $K = \delta(V_1, V_2)$, we let $G_1 = (V_1 \cup \{w\}, E_1)$ be the graph obtained from G by shrinking the subgraph induced by V_2 to a single pseudonode w, and let $G_2 = (V_2 \cup \{w\}, E_2)$ be defined analogously. Given $x \in \mathbb{R}^E$, let x^i be the restriction of x to the edges in E_i, for $i = 1,2$.

THEOREM 3.1. *Let $K = \delta(V_1, V_2)$ be a nontrivial 3-edge cut in $G = (V, E)$, and let $x \in \mathbb{R}^E$. Then $x \in C(G)$ if and only if $x^i \in C(G_i)$, $i = 1,2$. Moreover, if $x \in C(G)$, then*

$$\lambda_x = \lambda_{x^1} + \lambda_{x^2} - \tfrac{1}{2}x(K), \tag{a}$$

and

$$\mu_x = \mu_{x^1} + \mu_{x^2} - \tfrac{1}{2}x(K). \tag{b}$$

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the sets of cycles of G_1 and G_2, respectively, that contain no edges of K, and let $\mathscr{C}_3$ be the set of cycles of G that intersect K, and hence contain exactly two edges of K. Then $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ forms a partition of $\mathscr{C}$, the set of cycles of G. Let $\mathscr{C}_3^1$ and $\mathscr{C}_3^2$ be the sets of cycles of G_1 and G_2, respectively, that contain edges of K. For any $C \in \mathscr{C}_3$, let C^1 and C^2 denote the induced cycles of G_1 and G_2, respectively.

Let $x \in C(G)$. Then there is a vector $\gamma \in \mathbb{R}^{\mathscr{C}}$ such that

$$x = \sum_{C \in \mathscr{C}_1} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_2} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_3} \gamma_C \chi(C),$$

implying

$$x^1 = \sum_{C \in \mathscr{C}_1} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_3} \gamma_C \chi(C^1)$$

and

$$x^2 = \sum_{C \in \mathscr{C}_2} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_3} \gamma_C \chi(C^2).$$

Thus, $x^i \in C(G_i)$ for $i = 1,2$. Also, since every cycle in $\mathscr{C}_3$ contains exactly two edges of K, we have $\sum_{C \in \mathscr{C}_3} \gamma_C = \frac{1}{2}x(K)$. Combining these, we obtain

$$\lambda_x \geqslant \lambda_{x^1} + \lambda_{x^2} - \tfrac{1}{2}x(K),$$

and

$$\mu_x \leqslant \mu_{x^1} + \mu_{x^2} - \tfrac{1}{2}x(K).$$

Now let $x \in \mathbb{R}^E$, and assume that $x^i \in C(G_i)$ for $i = 1,2$. Let γ be such that

$$x^1 = \sum_{C \in \mathscr{C}_1} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_3^1} \gamma_C \chi(C),$$

$$x^2 = \sum_{C \in \mathscr{C}_2} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_3^2} \gamma_C \chi(C).$$

We use an argument from [2] which shows that the cycles of $\mathscr{C}_3^1$ and $\mathscr{C}_3^2$ can be suitably combined. For each unordered pair $\{j, k\}$ of edges of K, let $\mathscr{C}_3^1(j,k)$ and $\mathscr{C}_3^2(j,k)$ be the cycles of $\mathscr{C}_3^1$ and $\mathscr{C}_3^2$, respectively, that contain these two edges. This partitions each of $\mathscr{C}_3^1$ and $\mathscr{C}_3^2$ into three sets, and if $K = \{j, k, l\}$, then we have the relationships

$$\sum_{C \in \mathscr{C}_3^1(j,k)} \gamma_C + \sum_{C \in \mathscr{C}_3^1(j,l)} \gamma_C = x_j^1 = x_j^2,$$

$$\sum_{C \in \mathscr{C}_3^1(j,l)} \gamma_C + \sum_{C \in \mathscr{C}_3^1(k,l)} \gamma_C = x_l^1 = x_l^2,$$

$$\sum_{C \in \mathscr{C}_3^1(k,l)} \gamma_C + \sum_{C \in \mathscr{C}_3^1(j,k)} \gamma_C = x_k^1 = x_k^2.$$

Analogous relations hold for $\mathscr{C}_3^2$. Solving these three equations, we obtain

$$\sum_{C \in \mathscr{C}_3^1(j,k)} \gamma_C = \tfrac{1}{2}\left(x_j^1 + x_k^1 - x_l^1\right),$$

$$\sum_{C \in \mathscr{C}_3^1(k,l)} \gamma_C = \tfrac{1}{2}\left(x_k^1 + x_l^1 - x_j^1\right),$$

$$\sum_{C \in \mathscr{C}_3^1(j,l)} \gamma_C = \tfrac{1}{2}\left(x_j^1 + x_l^1 - x_k^1\right),$$

and analogous relations for $\mathscr{C}_3^2$. From these it follows that we can combine pairs of cycles from $\mathscr{C}_3^1(j,k)$ and $\mathscr{C}_3^2(j,k)$ for each pair $j,k \in K$ and define suitable γ'_C for $C \in \mathscr{C}_3$ to obtain

$$x = \sum_{C \in \mathscr{C}_1} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_2} \gamma_C \chi(C) + \sum_{C \in \mathscr{C}_3} \gamma'_C \chi(C),$$

which implies $x \in C(G)$. Note that

$$\sum_{C \in \mathscr{C}_3} \gamma'_C = \frac{1}{2}\left(\sum_{C \in \mathscr{C}_3^1} \gamma_C + \sum_{C \in \mathscr{C}_3^2} \gamma_C \right) = \tfrac{1}{2}x(K).$$

This implies

$$\lambda_x \leqslant \lambda_{x^1} + \lambda_{x^2} - \tfrac{1}{2}x(K)$$

and

$$\mu_x \geqslant \mu_{x^1} + \mu_{x^2} - \tfrac{1}{2}x(K),$$

completing the proof. ∎

Now suppose that $U(G_i)$, $i = 1,2$ are known. Equivalently, we know H_i, sets of representatives from the minimal nonempty faces of $U^*(G_i)$, $i = 1,2$. Then the following corollary provides a linear system sufficient to define $U(G)$, and hence H, a set of representatives from the minimal nonempty faces of $U^*(G)$.

Given a set $A \subseteq E$ and a vector $a \in \mathbb{R}^A$, let $\tilde{a} \in \mathbb{R}^E$ be defined as follows:

$$\tilde{a}_e = \begin{cases} a_e & \text{if } e \in A, \\ 0 & \text{otherwise}. \end{cases}$$

COROLLARY 3.2. *Let $K = (V_1, V_2)$ be a 3-edge cut in $G = (V, E)$, and assume for $i = 1,2$ we have*

$$U(G_i) = \{x \in C(G_i) : ax \geqslant 1 \textit{ for all } a \in H_i\}.$$

Then

$$U(G) = \{x \in C(G) : hx \geqslant 1 \textit{ for all } h \in H\},$$

where

$$H = \left\{ \tilde{a} + \tilde{b} - \tfrac{1}{2}\chi(K) : a \in H_1 \text{ and } b \in H_2 \right\}.$$

Proof. It suffices to show that $H \subseteq U^*(G)$ and that H contains a member of each minimal nonempty face of $U^*(G)$.

Let $h \in H$, and let $C \in \mathscr{C}$ be arbitrary. If $C \cap K = \varnothing$, then $h(C) \geqslant 1$ follows from the fact that C must be a cycle of either G_1 or G_2. If $C \cap K \neq \varnothing$, then $|C \cap K| = 2$, and $C \cap E_i$ is a cycle for both $i = 1$ and $i = 2$, from which it follows that $h(C) \geqslant 1$. Thus, $H \subseteq U^*(G)$.

Now let F be a minimal nonempty face of $U^*(G)$. There exists a vector $x \in C(G)$ such that $F = \{\pi \in U^*(G) : \pi x = \mu_x\}$. For $i = 1,2$, there is a minimal nonempty face F_i such that $\pi x^i = \mu_{x^i}$, for all $\pi \in F_i$. Let $a^i \in H_i \cap F_i$, $i = 1,2$, and let $h = \tilde{a}^1 + \tilde{a}^2 - \frac{1}{2}\chi(K)$. Now $h \in H$, and

$$\begin{aligned} hx &= a^1x^1 + a^2x^2 - \tfrac{1}{2}x(K) \\ &= \mu_{x^1} + \mu_{x^2} - \tfrac{1}{2}x(K) \\ &= \mu_x, \end{aligned}$$

by Theorem 3.1, which implies $h \in F$, as desired. ∎

Similarly, linear systems for $L(G_i)$, $i = 1,2$, can be combined to give a linear system for $L(G)$:

COROLLARY 3.3. *Let $K = (V_1, V_2)$ be a 3-edge cut in $G = (V, E)$, and assume for $i = 1,2$ we have*

$$L(G_i) = \{ x \in C(G_i) : ax \leqslant 1 \text{ for all } a \in H_i' \}.$$

Then

$$L(G) = \{ x \in C(G) : hx \leqslant 1 \text{ for all } h \in H' \},$$

where

$$H' = \left\{ \tilde{a} + \tilde{b} - \tfrac{1}{2}\chi(K) : a \in H_1' \text{ and } b \in H_2' \right\}.$$

4. HALIN GRAPHS

A *Halin graph* $G = (V, T \cup C)$ is a planar graph consisting of a tree T that has no degree-two nodes, together with a simple cycle, C, whose nodes are the degree-one nodes of T. (See Figure 3.) Halin [6] introduced these graphs as an example of a class of planar minimally 3-connected graphs, where a graph is minimally 3-connected if it is 3-connected and the deletion of any edge yields a 2-separable graph.

Halin graphs have the property that every edge $e \in T$ is in a unique 3-edge cut that contains 2 edges of C; we denote this cut by K_e.

Given a positive integer n, the *wheel* W_n is a graph consisting of a cycle with n nodes, together with one additional node, called the *center*, which is adjacent to each of the other n nodes. The edges incident to the center are called *spokes*, and the remaining edges and nodes are called *rim edges* and *rim nodes*, respectively. Wheels are exactly those Halin graphs $G = (V, T \cup C)$ for which T is a star.

If a Halin graph $G = (V, T \cup C)$ is not a wheel, then for any nonpendant edge e of T, K_e is a nontrivial cut; that is, $K_e = \delta(V_1, V_2)$, where $|V_1| \geqslant 2 \leqslant |V_2|$. Let $G_1(e) = (V_1, T_1 \cup C_1)$ be the graph obtained from G by contracting the edges with both ends in V_2 to a special node called w; define $G_2(e)$ analogously. Then $G_1(e)$ and $G_2(e)$ are both Halin graphs. Continuing this procedure on $G_1(e)$ and $G_2(e)$, every Halin graph decomposes into a collection of wheels. In fact, it is always possible to choose $e \in T$ so that one of $G_1(e)$, $G_2(e)$ is a wheel. It suffices to choose e so that e is the only nonpendant edge incident to some node of T. Since e is a pendant edge in both T_1 and T_2, the total number of nonpendant edges in $G_1(e)$ and $G_2(e)$ is one less than that of G. Thus, a Halin graph $G = (V, T \cup C)$ decomposes into p wheels, where p is the number of nonpendant edges of T; that is, $p = |V| - |C| - 1$.

Next we describe a polynomial algorithm for the weighted girth problem on Halin graphs. For each rim node v, we define a cost $c_v = \infty$. As the

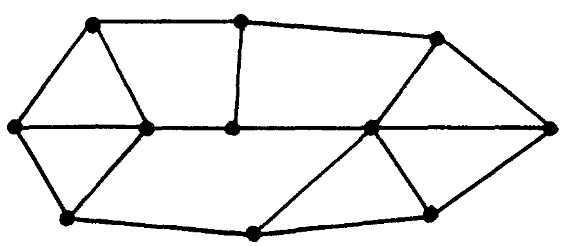


FIG. 3.

shrinking process proceeds, we will define these values for the new rim nodes created in such a way that they always equal the minimum cost of a cycle completely contained in the subgraph shrunk to form the node.

A wheel W_n has only

$$1+2\binom{|C|}{2}$$

simple cycles, so the weighted girth problem can be solved polynomially for wheels, by enumeration. Next we describe the basic reduction. Assume $G=(V,T\cup C)$ is a Halin graph, and e is a nonpendant edge chosen so that $G_1(e)$ is a wheel. Let $K_e=\{e,f,g\}$, and let $c_{e,f}$, $c_{e,g}$, $c_{f,g}$, and $c_{\varnothing}$ denote the costs of optimal cycles in $G_1(e)$ that meet $\{e,f,g\}$ in $\{e,f\}$, $\{e,g\}$, $\{f,g\}$, and $\varnothing$, respectively. Again, since $G_1(e)$ is a wheel, these values can be computed polynomially (e.g., by enumeration).

Let α, β, γ be defined by the following system:

$$\alpha+\beta=c_{e,f},$$

$$\alpha+\gamma=c_{e,g},$$

$$\beta+\gamma=c_{f,g}.$$

This system has the unique solution

$$\alpha=\tfrac{1}{2}(c_{e,f}+c_{e,g}-c_{f,g}),$$

$$\beta=\tfrac{1}{2}(c_{e,f}-c_{e,g}+c_{f,g}),$$

$$\gamma=\tfrac{1}{2}(-c_{e,f}+c_{e,g}+c_{f,g}).$$

Define the costs of the edges e, f, g in G_2 to be α, β, and γ respectively. We define c_w for the new rim node on $G_2(e)$ formed by shrinking $G_1(e)$ by

$$c_w=\min(\{c_{\varnothing}\}\cup\{c_v: v\in V(G_1(e))\}.$$

Now suppose that C is a minimum cost cycle in $G_2(e)$. If $c(C)\geqslant c_w$, then the minimum cost cycle in G is obtained from a cycle in $G_1(e)$, which provided the bound on c_w. Suppose $c(C)<c_w$. If C contains no edges of $\delta(w)$, then it is the optimum solution. Otherwise, C contains two edges of

$\delta(w)$ and can be extended through $G_1(e)$ by using a cycle of $G_1(e)$ which gave the bound $c_{j,k}$ corresponding to those edges.

Now we state the algorithm.

ALGORITHM

Input. A Halin graph $G = (V, T \cup C)$, with costs $(c_c : e \in T \cup C)$.

Output. A minimum cost simple cycle.

Step 1. If G is a wheel, then find a minimum cost cycle. If the cost of this cycle is less than c_v for every rim node v, then return this cycle. If not, return v, the rim node for which c_v minimum. (Note that in this case v is a pseudonode formed by shrinking.) Otherwise, choose e to be a nonpendant edge of T such that $G_1(e)$ is a wheel. Let $K_e = \{e, f, g\}$.

Step 2. By enumeration, compute $c_{e,f}$, $c_{e,g}$, $c_{f,g}$, and $c_\varnothing$, the costs of optimal cycles in $G_1(e)$ that meet $\{e, f, g\}$ in $\{e, f\}$, $\{e, g\}$, $\{f, g\}$, and $\varnothing$, respectively.

Step 3. Modify $G_2(e)$ as described in the basic reduction, and recursively apply the algorithm to $G_2(e)$. If the rim node w is returned, then return a cycle in the subgraph shrunk to form w of cost c_w. If a cycle is returned that contains w, then extend it to a cycle of G by including the appropriate edges of G_2. Stop.

Since a wheel $G = (V, T \cup C)$ has exactly $1 + 2\binom{|C|}{2}$ cycles, the values $c_{e,f}$, $c_{e,g}$, and $c_{f,g}$ can be computed by considering the cycles in order of increasing size; the values $c_{e,f}$, $c_{e,g}$, and $c_{f,g}$ can be computed for a wheel in time $O(|C|^2)$. From this it follows that the total running time is also $O(|C|^2)$. [In fact, Gunther Rote observed that the values $c_{e,f}$, $c_{e,g}$, and $c_{f,g}$ can be computed in time $O(|C|)$, implying an overall time $O(|C|)$ for Halin graphs.]

Since the weighted girth problem is polynomially solvable on Halin graphs, it is reasonable to hope for a linear description of the cycle polytope $P(G)$. The remainder of this paper solves this problem. We begin by showing that the cycle cone $C(G)$ has a simpler description for Halin graphs than that of general graphs. The only cut inequalities needed are those that correspond to the 3-edge cuts K_e.

THEOREM 4.1. *If $G = (V, T \cup C)$ is a Halin graph, then the cycle cone $C(G)$ is given by the following system*:

$$x(\delta(v) \setminus \{j\}) - x_j \geqslant 0 \qquad \text{for all} \quad v \in V \text{ and } j \in \delta(v),$$

$$x(K_e \setminus \{j\}) - x_j \geqslant 0 \qquad \text{for all} \quad e \in T \text{ and } j \in K_e.$$

Proof. This can be proved in exactly the same way as Theorem 2.2, observing only that if H is a subgraph of G consisting of two disjoint cycles joined by a path, then there is some 3-edge cut K_e such that $|K_e \cap H| = 1$.

Alternatively, if we use Theorem 2.2, then it suffices to show that the system (2.1) reduces to the above system in the case of Halin graphs. To see this, let K be an arbitrary cut of G and let $e \in K$. Clearly, we can assume that the shores of K induce connected subgraphs of G. First suppose that K contains no edge of C. Then K is the set of pendant edges of some subtree T' of T. Let w be the degree-one node of T' that is incident to edge e. For every node v of T', let $j(v)$ be the edge of T' that is on the path in T' from v to w. Now summing the inequalities

$$x(\delta(v)\backslash\{j(v)\}) - x_{j(v)} \geqslant 0$$

over all the internal nodes of T', we obtain the inequality

$$x(K\backslash\{e\}) - x_e \geqslant 0.$$

Next assume that K contains edges of C. Since the shores of K induce connected subgraphs of G, K must contain exactly two edges of C, and K is a dual path of G; that is, K corresponds to a path in the dual graph of G. Traversing this dual path, beginning with an edge of C produces a sequence $e_1 f_1 \cdots f_{k-1} e_k f_k \cdots f_{n-1} e_n$, where $\{e_1, \ldots, e_n\} = K$, edge $e = e_k$, and $f_1, \ldots, f_{n-1}$ is the sequence of faces of G encountered. For $i = 1, \ldots, n-1$, let $j(i)$ be the edge of C that is on face f_i. Now summing the inequalities

$$x(K_{e_i}\backslash\{j(i)\}) - x_{j(i)} \geqslant 0$$

for $i = 2, \ldots, k-1$, together with the inequalities

$$x(K_{e_i}\backslash\{j(i-1)\}) - x_{j(i-1)} \geqslant 0$$

for $i = k+1, \ldots, n-1$, together with the single inequality

$$x(K_{e_k}\backslash\{e_k\}) - x_{e_k} \geqslant 0,$$

we obtain the inequality

$$x(K\backslash\{e\}) - x_e \geqslant 0,$$

as desired.

Finally, to see that the nonnegativity constraints are implied by the 3-edge-cut constraints, observe that $x_j \geqslant 0$ is obtained by adding together two of the cut constraints corresponding to a 3-edge cut containing edge j. ■

COROLLARY 4.2. *If $G=(V, E)$ is a wheel, then the cycle cone $C(G)$ is given by the following system*:

$$x(\delta(v)\setminus\{j\}) - x_j \geqslant 0 \qquad \text{for all} \quad v \in V \text{ and } j \in \delta(v).$$

5. THE CYCLE POLYTOPE FOR HALIN GRAPHS

As described in Section 3, given complete linear systems sufficient to define the polyhedra $L(G)$ and $U(G)$, a linear system for $P(G)$ is simply the union of the two. For general graphs, one is unlikely to find a system for $L(G)$, but $U(G)$ may be tractable. In the case of Halin graphs, we can give both. We do this by establishing min-max results for λ_x and μ_x, from which the systems for $L(G)$ and $U(G)$ follow immediately. These min-max results are first proved for wheels, and then, using the 3-edge decomposition, for general Halin graphs. We begin with λ_x.

LEMMA 5.1. *If $G=(V, E)$ is a wheel and $x \in C(G)$, then*

$$\lambda_x = \tfrac{1}{2}\max\{x(\delta(v)) : v \in V\}.$$

Proof. Assume $x = \sum_{C \in \mathscr{C}} \gamma_C \chi(C)$, where $\gamma_C \geqslant 0$ for all $C \in \mathscr{C}$, and $\sum_{C \in \mathscr{C}} \gamma_C = \lambda_x$. Let $\mathscr{C}' = \{C \in \mathscr{C} : \gamma_C > 0\}$. Since for every node v we have $\sum_{C \in \mathscr{C}} \gamma_C \geqslant \frac{1}{2}x(\delta(v))$, it suffices to show that equality holds for some v. That is, it suffices to show that for some $v \in V$, every cycle $C \in \mathscr{C}'$ contains v.

Let C_r be the rim cycle. If $\gamma_{C_r} = 0$, then every cycle $C \in \mathscr{C}'$ contains the center node c, as desired. Assume $\gamma_{C_r} > 0$. If there is some rim node contained in every $C \in \mathscr{C}'$, then we are finished, so assume there is no such rim node.

Suppose there are cycles $C_1, C_2 \in \mathscr{C}'$ that have no rim node in common. Let v_1 and w_1 (v_2 and w_2) be the first and last rim nodes of C_1 (C_2) encountered, traversing C_r in a clockwise direction. Let C_3 (C_4) be the cycle containing all of v_1, w_1, v_2, w_2 that contains edges w_1c and v_2c (v_1c and w_2c). Note that

$$\chi(C_1) + \chi(C_2) + \chi(C_r) = \chi(C_3) + \chi(C_4).$$

Now let $\theta = \min\{\gamma_{C_1}, \gamma_{C_2}, \gamma_{C_r}\}$. Then $\theta > 0$. Put

$$\gamma'_C = \begin{cases} \gamma_C - \theta & \text{if} \quad C \in \{C_1, C_2, C_r\}, \\ \gamma_C + \theta & \text{if} \quad C \in \{C_3, C_4\}, \\ \gamma_C & \text{otherwise}. \end{cases}$$

Then we have that $x = \sum_{C \in \mathscr{C}} \gamma'_C \chi(C)$ and $\sum_{C \in \mathscr{C}} \gamma'_C = \sum_{C \in \mathscr{C}} \gamma_C - \theta$, a contradiction.

Thus, every pair of cycles in $\mathscr{C}'$ shares a rim node. If there is no rim node common to all cycles in $\mathscr{C}'$, then there must be three cycles $C_1, C_2, C_3 \in \mathscr{C}'$ such that there is no rim node common to all three. It follows that $C_r \subseteq C_1 \cup C_2 \cup C_3$. Let $\{V_1, V_{1,2}, V_2, V_{2,3}, V_3, V_{3,1}\}$ be the partition of the rim nodes, where V_i are those nodes contained only in C_i, $i = 1,2,3$, and $V_{i,j}$ are those nodes common to C_i and C_j, $i, j \in \{1,2,3\}$. Let $C_4(C_5, C_6)$ be the cycle that contains all the rim nodes except those in V_1 (V_2, V_3, respectively). Note that

$$\chi(C_1) + \chi(C_2) + \chi(C_3) + \chi(C_r) = \chi(C_4) + \chi(C_5) + \chi(C_6).$$

Let $\theta = \min\{\gamma_{C_1}, \gamma_{C_2}, \gamma_{C_3}, \gamma_{C_r}\}$, and put

$$\gamma'_C = \begin{cases} \gamma_C - \theta & \text{if} \quad C \in \{C_1, C_2, C_3, C_r\}, \\ \gamma_C + \theta & \text{if} \quad C \in \{C_4, C_5, C_6\}, \\ \gamma_C & \text{otherwise}. \end{cases}$$

Again we have $x = \sum_{C \in \mathscr{C}} \gamma'_C \chi(C)$ and $\sum_{C \in \mathscr{C}} \gamma'_C = \sum_{C \in \mathscr{C}} \gamma_C - \theta$, a contradiction. ∎

THEOREM 5.2. *If $G = (V, E)$ is a wheel, then $L(G)$ is given by the following system*:

$$x(\delta(v) \setminus \{j\}) - x_j \geq 0 \qquad \textit{for all} \quad v \in V \textit{ and } j \in \delta(V),$$

$$x(\delta(v)) \leq 2 \qquad \textit{for all} \quad v \in V.$$

Proof. The theorem follows immediately from Corollary 4.2, Lemma 5.1, and the definition of $L(G)$. ∎

Since we know λ_x for wheels, λ_x for a Halin graph can be computed by using the 3-edge decomposition and applying Theorem 3.1. Moreover, we can actually give an explicit min-max theorem for λ_x for Halin graphs as follows.

Given a Halin graph $G=(V, T \cup C)$, let $\tilde{T} \subseteq T$ be the set of pendant edges of T. Let $R \subseteq V$, $P \subseteq T$, and $M \subseteq T \setminus \tilde{T}$. Then (R, P, M) is called a *lower block set* if R contains no degree-one nodes of T, $P \cap M = \varnothing$, and each component of $T \setminus M$ contains exactly one member of $P \cup R$.

THEOREM 5.3. *If $G=(V, T \cup C)$ is a Halin graph and $x \in C(G)$, then*

$$\lambda_x = \tfrac{1}{2} \max\left\{ \sum_{v \in R} x(\delta(v)) + \sum_{e \in P} x(K_e) - \sum_{e \in M} x(K_e) : (R, P, M) \text{ is a lower block set.} \right\}$$

Proof. For convenience, let

$$f(x, R, P, M) = \frac{1}{2}\left(\sum_{v \in R} x(\delta(v)) + \sum_{e \in P} x(K_e) - \sum_{e \in M} x(K_e) \right).$$

We proceed by induction on $|V|$. Suppose G is a wheel. Then for any lower block set (R, P, M), we have that $M = \varnothing$ and $|R| + |P| = 1$. Now $f(x, R, P, M)$ reduces to $\frac{1}{2}x(\delta(v))$ if $\{v\} = R$, or $\frac{1}{2}x(K_e)$ if $\{e\} = P$, and the result follows from Lemma 5.1.

Now assume G is not a wheel and therefore T has a nonpendant edge e. Let $G_i(e) = (V_i, T_i \cup C_i)$, $i = 1,2$, be the graphs obtained by decomposing G along K_e, and let x^i be the restriction of x to $T_i \cup C_i$, $i = 1,2$.

A lower block set (R, P, M) may be decomposed into lower block sets (R_i, P_i, M_i) of $G_i(e)$, $i = 1,2$, as follows:

$$R_i = R \cap V_i;$$

$$P_i = \begin{cases} P \cap T_i & \text{if } e \in M, \text{ or if } e \notin M \text{ and the element of } P \cup R \text{ in} \\ & \text{the component of } T \setminus M \text{ that contains } e \text{ is in } T_i, \\ (P \cap T_i) \cup \{e\} & \text{if } e \notin M \text{ and the element of } P \cup R \text{ in the compo-} \\ & \text{nent of } T \setminus M \text{ that contains } e \text{ is not in } T_i; \end{cases}$$

$$M_i = (M \cap T_i) \setminus \{e\}.$$

Conversely, lower block sets (R_i, P_i, M_i) of $G_i(e)$, $i = 1,2$, may be composed to form a lower block set (R, P, M) of G as follows:

$$R = R_1 \cup R_2;$$

$$P = \begin{cases} P_1 \cup P_2 & \text{if} \quad e \in P_1 \cap P_2, \\ (P_1 \cup P_2) \setminus \{e\} & \text{if} \quad e \notin P_1 \cap P_2; \end{cases}$$

$$M = \begin{cases} M_1 \cup M_2 & \text{if} \quad e \in P_1 \cup P_2, \\ M_1 \cup M_2 \cup \{e\} & \text{if} \quad e \notin P_1 \cup P_2. \end{cases}$$

It is straightforward to check that this decomposition followed by this composition yields the original lower block set, and conversely. Moreover, we have that

$$f(x, R, P, M) = f(x^1, R_1, P_1, M_1) + f(x^2, R_2, P_2, M_2) - \tfrac{1}{2}x(K_e).$$

Now let (R, P, M) be an arbitrary lower block set of G. We first show that $\lambda_x \geqslant f(x, R, P, M)$. Using the decomposition of (R, P, M), we have that

$$\begin{aligned} f(x, R, P, M) &= f(x^1, R_1, P_1, M_1) + f(x^2, R_2, P_2, M_2) - \tfrac{1}{2}x(K_e) \\ &\leqslant \lambda_x^1 + \lambda_x^2 - \tfrac{1}{2}x(K_e) \\ &= \lambda_x, \end{aligned}$$

where the inequality follows by induction and the last equality follows from Theorem 3.1.

We can construct a lower block set (R, P, M) for which equality holds by using induction to construct such lower block sets of G_1 and G_2, and then composing them as described above. ■

COROLLARY 5.4. *If $G = (V, T \cup C)$ is a Halin graph, then $L(G)$ is given by the following system:*

$$\begin{aligned} x(\delta(v) \setminus \{j\}) - x_j &\geqslant 0 && \textit{for all} \quad v \in V \textit{ and } j \in \delta(v), \\ x(K_e \setminus \{j\}) - x_j &\geqslant 0 && \textit{for all} \quad e \in T \textit{ and } j \in K_e, \\ \sum_{v \in R} x(\delta(v)) + \sum_{e \in P} x(K_e) & && \\ - \sum_{e \in M} x(K_e) &\leqslant 2 && \textit{for all lower block sets } (R, P, M). \end{aligned}$$

Proof. The result follows immediately from Theorem 4.1, Theorem 5.3, and the definition of $L(G)$. ∎

Now we establish the defining system for $U(G)$. The development is analogous to that for $L(G)$. We begin by giving the min-max result for μ_x for wheels.

LEMMA 5.5. *If $G=(V,E)$ is a wheel with center node c and $x \in C(G)$, then*

$$\mu_x = \tfrac{1}{2}\left[x(\delta(c)) + \min\{x_j + x_k - x_l\}\right],$$

where the minimum is taken over edge triples $\{j,k,l\}$ where j and k are rim edges incident to the same rim node v and l is the spoke edge incident to v.

Proof. Assume $x = \sum_{C \in \mathscr{C}} \gamma_C \chi(C)$, where $\gamma_C \geq 0$ for all $C \in \mathscr{C}$. Since every cycle except the rim cycle, C_r, contains node c, we have that $\sum_{C \in \mathscr{C}} \gamma_C = \frac{1}{2}x(\delta(c)) + \gamma_r$. Thus, $\mu_x = \frac{1}{2}x(\delta(c)) + t^*$, where $t^* = \max\{t \geq 0 : x - t\chi(C_r) \in C(G)\}$.

By Corollary 4.2, we have that

$$t^* = \tfrac{1}{2}\min\left\{x(\delta(v)\setminus\{j\}) - x_j : v \text{ is a rim node and } j \text{ is the spoke edge incident to } v\right\}. \qquad ∎$$

THEOREM 5.6. *If $G=(V,E)$ is a wheel, then $U(G)$ is given by the following system:*

$$x(\delta(v)\setminus\{j\}) - x_j \geq 0 \qquad \textit{for all} \quad v \in V \textit{ and } j \in \delta(v),$$

$$x(\delta(c)) + x(\delta(v)\setminus\{j\}) - x_j \geq 2 \qquad \textit{for all} \quad v \in (V\setminus\{c\}) \textit{ and } j \textit{ the spoke incident to } v.$$

Proof. The theorem follows immediately from Corollary 4.2, Lemma 5.5, and the definition of $U(G)$. ∎

Given a Halin graph $G=(V, T \cup C)$, again let $\tilde{T} \subseteq T$ be the set of pendant edges of T. Let $P \subseteq T$ and $M \subseteq T \setminus \tilde{T}$. Then (P, M) is called an *upper block set if* $P \cap M = \varnothing$ and each component of $T \setminus M$ contains exactly one member of P.

THEOREM 5.7. *If $G=(V, T\cup C)$ is a Halin graph and $x\in C(G)$, then*

$$\mu_x=\tfrac{1}{2}x(\tilde{T})+\tfrac{1}{2}\min\Big\{\sum_{e\in P}\big(x(K_e\setminus\{e\})-x_e\big)-\sum_{e\in M}\big(x(K_e\setminus\{e\})-x_e\big):$$

$$(P, M) \text{ is an upper block set}\Big\}.$$

Proof. For convenience, let

$$g(x,P,M)=\frac{1}{2}\Big(x(\tilde{T})+\sum_{e\in P}\big[x(K_e\setminus\{e\})-x_e\big]-\sum_{e\in M}\big[x(K_e\setminus\{e\})-x_e\big]\Big).$$

We proceed by induction on $|V|$. Suppose G is a wheel. Then for any upper block set (P, M), we have that $M=\varnothing$ and $|P|=1$, and $g(x,P,M)$ reduces to $\frac{1}{2}x(\delta(c))+x(K_e\setminus\{e\})-x_e$, where $\{e\}=P$, and the result follows from Lemma 5.5.

Now assume G is not a wheel and therefore T has a nonpendant edge e. Let $G_i(e)=(V_i, T_i\cup C_i)$, $i=1,2$, be the graphs obtained by decomposing G along K_e, and let x^i be the restriction of x to $T_i\cup C_i$, $i=1,2$.

An upper block set (P, M) may be decomposed into upper block sets (P_i, M_i) of $G_i(e)$, $i=1,2$, as follows:

$$P_i=\begin{cases} P\cap T_i & \text{if } e\in M, \text{ or if } e\notin M \text{ and the element of } P\cup R \text{ in} \\ & \text{the component of } T\setminus M \text{ that contains } e \text{ is in } T_i, \\ (P\cap T_i)\cup\{e\} & \text{if } e\notin M, \text{ and the element of } P\cup R \text{ in the compo-} \\ & \text{nent of } T\setminus M \text{ that contains } e \text{ is not in } T_i; \end{cases}$$

$$M_i=(M\cap T_i)\setminus\{e\}.$$

Conversely, upper block sets (P_i, M_i) of $G_i(e)$, $i=1,2$, may be composed to form an upper block set (P, M) of G as follows:

$$P=\begin{cases} P_1\cup P_2 & \text{if} \quad e\in P_1\cap P_2, \\ (P_1\cup P_2)\setminus\{e\} & \text{if} \quad e\notin P_1\cap P_2; \end{cases}$$

$$M=\begin{cases} M_1\cup M_2 & \text{if} \quad e\in P_1\cap P_2, \\ M_1\cup M_2\cup\{e\} & \text{if} \quad e\notin P_1\cap P_2. \end{cases}$$

It is straightforward to check that this decomposition followed by this composition yields the original upper block set, and conversely. Moreover, we have that

$$g(x, P, M) = g(x^1, P_1, M_1) + g(x^2, P_2, M_2) - x_e + \tfrac{1}{2}[x(K_e\setminus\{e\}) - x_e]$$

$$= g(x^1, P_1, M_1) + g(x^2, P_2, M_2) - \tfrac{1}{2}x(K_e)$$

Now let (P, M) be an arbitrary upper block set of G. We first show that $\mu_x \leqslant g(x, P, M)$. Using the decomposition of (P, M), we get that

$$\begin{aligned} g(x, P, M) &= g(x^1, P_1, M_1) + g(x^2, P_2, M_2) - \tfrac{1}{2}x(K_e). \\ &\geqslant \mu_x^1 + \mu_x^2 - \tfrac{1}{2}x(K_e) \\ &= \mu_x, \end{aligned}$$

where the inequality follows by induction and the last equality follows from Theorem 3.1.

We can construct an upper block set (P, M) for which equality holds by using induction to construct such upper block sets of G_1 and G_2 and then composing them as described above. ∎

COROLLARY 5.8. *If $G = (V, T \cup C)$ is a Halin graph, then $U(G)$ is given by the following system*:

$$x(\delta(v)\setminus\{j\}) - x_j \geqslant 0 \qquad \textit{for all} \quad v \in V \textit{ and } j \in \delta(v),$$

$$x(K_e\setminus\{j\}) - x_j \geqslant 0 \qquad \textit{for all} \quad e \in T \textit{ and } j \in K_e,$$

$$x(\tilde{T}) + \sum_{e \in P} [x(K_e\setminus\{e\}) - x_e] - \sum_{e \in M} [x(K_e\setminus\{e\}) - x_e] \geqslant 2 \qquad \textit{for all upper block sets } (P, M).$$

Proof. The result follows immediately from Theorem 4.1, Theorem 5.7, and the definition of $U(G)$. ∎

Finally, the cycle polytope $P(G)$ for Halin graphs is obtained by combining the systems for $L(G)$ and $U(G)$. From the way they are presented here it may appear that the systems for $L(G)$ and $U(G)$ have coefficients only $+1$, -1, and 0. However, that is not the case. Indeed, the coefficients can be integers with absolute value as large as $O(|T|)$.

6. THE TSP POLYTOPE

Given a graph $G=(V, E)$, the traveling salesman polytope, TSP(G), is the convex hull of hamiltonian cycles of G. In [3], Cornuéjols et al. gave a defining linear system for TSP(G) in the case where G is a Halin graph.

THEOREM 6.1 [3]. *If $G=(V, T\cup C)$ is a Halin graph, then* TSP(G) *is given by the following system*:

$$\begin{aligned} x_j &\leq 1 && \textit{for all} \quad j\in C, \\ x(\delta(v)) &= 2 && \textit{for all} \quad v\in V, \\ x(K_e) &= 2 && \textit{for all} \quad e\in T. \end{aligned} \tag{6.1}$$

Proof. We show how TSP(G) can be derived from $L(G)$. First note that the inequalities

$$x(\delta(v))\leq 2 \qquad \text{for all} \quad v\in V$$

are valid for $L(G)$, since $(\{v\},\varnothing,\varnothing)$ is a lower block set if $v\notin C$, and $(\varnothing,\{e\},\varnothing)$ is a lower block set if $v\in C$ and e is the pendant edge of T incident with v. Thus, including the equations

$$x(\delta(v))=2 \qquad \text{for all} \quad v\in V,$$

we obtain TSP(G), which is a face of $L(G)$. Thus we have that TSP(G) for

Halin graphs is given by the following system:

$$
\begin{aligned}
x(\delta(v)\setminus\{j\}) - x_j &\geqslant 0 \qquad \text{for all} \quad v \in V \text{ and } j \in \delta(v),\\
x(K_e\setminus\{j\}) - x_j &\geqslant 0 \qquad \text{for all} \quad e \in T \text{ and } j \in K_e,\\
\sum_{v \in R} x(\delta(v)) + \sum_{e \in P} x(K_e) & \\
- \sum_{e \in M} x(K_e) &\leqslant 2 \qquad \text{for all lower block sets } (R, P, M),\\
x(\delta(v)) &= 2 \qquad \text{for all} \quad v \in V.
\end{aligned}
\tag{6.2}
$$

To derive Theorem 6.1 from the system (6.2) above, first note that the constraints of the system (6.1) are all valid for TSP(G) for Halin graphs. Thus, it suffices to show that the inequalities of the system (6.2) are all implied by the system (6.1). First we will show that the system (6.3) below implies all the inequalities of the system (6.2):

$$
\begin{aligned}
x_j &\leqslant 1 \qquad \text{for all} \quad j \in T \cup C,\\
x(\delta(v)) &= 2 \qquad \text{for all} \quad v \in V,\\
x(K_e) &= 2 \qquad \text{for all} \quad e \in T.
\end{aligned}
\tag{6.3}
$$

Then we will show that the inequalities $x_j \leqslant 1$ for $j \in T$ are redundant.

The system (6.2) has three types of inequalities. For convenience, these are referred to as *degree constraints*, *cut constraints*, and *lower block set constraints*.

It is trivial to see that the degree constraints and the cut constraints are implied by the system (6.3).

Let (R, P, M) be a lower block set. Then summing the degree and cut equations of (6.3) corresponding to the sets R, P, and subtracting those corresponding to M, we get

$$\sum_{v \in R} x(\delta(v)) + \sum_{e \in P} x(K_e) - \sum_{e \in M} x(K_e) = 2|R| + 2|P| - 2|M| = 2.$$

Thus, the lower block set constraints are implied.

Finally we show that the inequalities $x_e \leqslant 1$ for $e \in T$ are redundant in (6.3). Let $e \in T$ be arbitrary, and let $K_e = \{e, f, g\}$. Let v be an end of e

that is incident to at least two other edges in T, and among the edges incident to v, let j be on the face containing e and f and let k be on the face containing e and g. Then $K_j = \{j, f, l\}$ and $K_k = \{k, g, m\}$, for some l, $m \in C$. Subtracting the upper bound constraints for l and m from the cut constraints for K_j and K_k, we get

$$x_j + x_f \geqslant 1$$

and

$$x_k + x_g \geqslant 1,$$

and adding, we get

$$x_j + x_k + x_f + x_g \geqslant 2.$$

For $h \in T$, the nonnegativity constraint $x_h \geqslant 0$ is obtained by subtracting the upper bound constraints for the edges in $K_h \cap C$ from the cut constraint for K_h. Thus, we can subtract the nonnegativity constraints of the edges in $\delta(v)\backslash\{e, j, k\}$ to obtain

$$x_j + x_k + x_e \leqslant 2.$$

Now, adding this inequality to the cut equation for K_e and subtracting the above inequality, we get the desired inequality $x_e \leqslant 1$. ∎

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