Interesting Pointsets in Generalized Quadrangles and Partial Geometries

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

Ovoids, m-ovoids, k-arcs, and hemisystems of generalized quadrangles and partial geometries are studied. In several of the proofs the same matrix technique is used, which was developed in a slightly less general form by P. J. Cameron. As an appendix we show that this elementary technique can be used to prove the Krein inequalities for strongly regular graphs, without the need of Hadamard multiplication and the theory of minimal idempotents.

1. INTRODUCTION

Let $H(3, q^2)$ be the generalized quadrangle (GQ) arising from the nonsingular Hermitian variety H of $PG(3, q^2)$. If V is a lineset of $H(3, q^2)$ such that each point of $H(3, q^2)$ is incident with a constant number m of lines of V, then it was shown by B. Segre that $m = q+1$ or $m = \frac{1}{2}(q+1)$ [8]. If $m = q+1$, then V consists of all lines of $H(3, q^2)$; if $m = \frac{1}{2}(q+1)$, then V contains half of the lines of $H(3, q^2)$ and is called a hemisystem of $H(3, q^2)$. The only known example of a hemisystem of $H(3, q^2)$ occurs in the case $q = 3$. A short proof of Segre's result was given by J. A. Thas [12]. In the course of it he showed that a hemisystem V defines a strongly regular graph (SRG) with the lines of V as vertices, two lines of V being adjacent iff they are concurrent. In [3] P. J. Cameron proves that in any GQ with parameters

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$s = q^2$ and $t = q$, the graph associated with a hemisystem is strongly regular (Thas's proof only applies to the classical GQ). Equivalently, the points of the GQ together with the lines of the hemisystem form a dual partial quadrangle, in the sense of P. J. Cameron [2].

Here we will consider the dual problem. Let $S = (P, B, I)$ be a GQ of order (s, t), and let V be a pointset of S such that each line of B contains a constant number m of points of V. What can be said about m and V? If $m = 1$, then V is called an ovoid of S. Let V be an ovoid of S. Then it was shown by E. E. Shult [7, 10] that $s = 1$ or $t \leqslant s^2 - s$.

More generally, we will consider the analogous problem for partial geometries (PG).

In [7] k-arcs of GQ were studied. A k-arc of a GQ is a set of k points no two of which are collinear. If s and t are the parameters of the GQ, then $k \leqslant st + 1$ and a k-arc with $k = st + 1$ is the same as an ovoid. Here we will also study k-arcs of PG.

2. A MATRIX TECHNIQUE

The proof of Cameron's theorem on hemisystems depends on a lemma about partitioned semidefinite symmetric matrices. In his paper he suggests looking at other applications of this lemma. In our paper several applications will be given. We now prove a slight variation of Cameron's lemma.

THEOREM 1. *Let the real positive semidefinite symmetric matrix A be partitioned as*

$$A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$$

with respect to a principal submatrix B ($\neq A$) with $\operatorname{rank} B = \operatorname{rank} A$. *Suppose that B and C have constant row sums b and c respectively. If c' is the average of the row sums of C^T, and d the average of the row sums of D, then $bd = cc'$ holds. Moreover, if A has nonzero constant row sums, then $c = d$ and $c' = b$; if moreover C is square, then $b = c = c' = d$.*

Proof. Let $\operatorname{rank} A = \operatorname{rank} B = r$, and suppose that A is an $n \times n$ matrix and B is an $n' \times n'$ matrix. Since A is real positive semidefinite, it can be written as $A = E^T E$ with $\operatorname{rank} E = r$. Let $E = [X \;\; Y]$ with X a matrix with n' columns. Then $B = X^T X$, $C = X^T Y$, and $D = Y^T Y$. We have $r = \operatorname{rank} B =$

$\operatorname{rank}(X^TX) \leqslant \operatorname{rank} X \leqslant \operatorname{rank}[X \ \ Y] = r$. Hence $\operatorname{rank} X = \operatorname{rank}[X \ \ Y]$. Consequently we have $Y = XU$, with U some $n' \times (n - n')$ matrix. Thus we have

$$B = X^TX, \qquad C = X^TXU, \quad \text{and} \quad D = U^TX^TXU.$$

Let J be the matrix $[1 \ 1 \ \cdots \ 1]^T$ with n' rows, and let J' be the matrix $[1 \ 1 \ \cdots \ 1]^T$ with $n - n'$ rows. Then

$$X^TXJ = bJ \quad \text{and} \quad X^TXUJ' = cJ.$$

Hence

$$C^TJ = U^TX^TXJ = bU^TJ \quad \text{and} \quad DJ' = U^TX^TXUJ' = cU^TJ.$$

Consequently

$$cC^TJ = bDJ'. \tag{1}$$

Adding the elements of the matrices in (1), we obtain $(n - n')cc' = (n - n')bd$. Hence $cc' = bd$.

Suppose that A has nonzero constant row sum. Then $b + c = c' + d \neq 0$. Elimination of c' yields $(b + c)(d - c) = 0$. Consequently $d = c$. Hence also $b = c'$. Suppose now that moreover C is square. Adding the elements of C, we obtain $n'c' = n'c$. Hence $c = c'$, and so $b = d = c = c'$. ∎

In the original lemma of P. J. Cameron, also the matrices C^T and D have constant row sum.

3. m-OVOIDS OF GENERALIZED QUADRANGLES

Let $S = (P, B, I)$ be a GQ of order (s, t). An m-ovoid V of S is a subset of P such that each line in B contains exactly m points of V. We always assume that $1 \leqslant m \leqslant s$. If V is an m-ovoid of S, then $P - V$ clearly is an $(s + 1 - m)$-ovoid of S. A 1-ovoid of S is called an ovoid of S. Many results on ovoids can be found in [7]. If $m = (s + 1)/2$, then the m-ovoid is also called a hemisystem.

LEMMA 1. *If V is an m-ovoid of the GQ $S = (P, B, I)$ of order (s, t), then $|V| = m(st + 1)$.*

Proof. Count in different ways the number of pairs (x, L), with $x \in V$, $L \in B$, and xIL. We obtain $|V|(t+1) = (t+1)(st+1)m$. Hence $|V| = m(st+1)$. ∎

For $s = 1$ the GQ has exactly two ovoids. For $t = 1$, it is possible to construct an m-ovoid for any m with $1 \leqslant m \leqslant s$. From now on we always assume that $s > 1$ and $t > 1$.

Let G be the point graph of the GQ S, and let U be a pointset of S. Then the graph induced by G on U will be denoted by $G(U)$.

THEOREM 2. *Let $S = (P, B, I)$ be a GQ of order (s, t), with $t \geqslant s$, and suppose that S contains an m-ovoid V with $m \leqslant s(t+1)/(s+t)$. Then*

$$\begin{aligned} &t^2(s^2m^2 - 3sm^2 + m^2 - s^3m + 2s^2m + 2sm - m - s^2) \\ &\quad + t(-s^4m^2 + 3s^3m^2 - 5s^2m^2 + sm^2 + s^5m \\ &\qquad - 2s^4m + 2s^3m + 4s^2m - sm - 2s^3) \\ &\quad - s^3m^2 + s^4m + s^3m - s^4 \geqslant 0. \end{aligned} \tag{2}$$

If equality holds in (2), *then the graph $G(P - V)$ is strongly regular. If equality holds in* (2) *and if moreover $m \geqslant (s^2 + t)/(s+t)$, then also the graph $G(V)$ is strongly regular.*

Proof. Let M be the adjacency matrix of the complement of the point graph of the GQ. The matrix M has eigenvalues $s^2t, t, -s$ with respective multiplicities $1, s^2(st+1)/(s+t), st(s+1)(t+1)/(s+t)$. We may assume that

$$M = \begin{bmatrix} B' & C \\ C^T & D' \end{bmatrix},$$

where the rows and columns of B' correspond to the points of $P - V$. So B' is the adjacency matrix of the complement of $G(P - V)$. The matrix $M + sI = A$ is positive semidefinite with rank equal to $1 + s^2(st+1)/(s+t)$.

Row sum of $B' + sI = B$: $t(s^2 - sm + m) + s = b$.
Row sum of $D' + sI = D$: $t(sm - m + 1) + s = d$.
Row sum of C: $t(s-1)m = c$.
Row sum of C^T: $t(s-1)(s-m+1) = c'$.

Since $c = d$ or $c' = b$ implies $s + t = 0$, we have $c \neq d$ and $b \neq c'$. Then by Theorem 1 rank $A \neq$ rank B. Consequently $\operatorname{rank}(B' + sI) \leqslant s^2(st + 1)/(s + t)$. It follows that $-s$ is an eigenvalue of B' with multiplicity at least $(s - m + 1)(st + 1) - s^2(st + 1)/(s + t) = (st + 1)(st - sm - tm + s + t)/(s + t)$. Since $m \leqslant s(t + 1)/(s + t)$, we have $(st + 1)(st - sm - tm + s + t)/(s + t) > 0$. The matrix B' has constant row sum, so it has the simple eigenvalue $t(s^2 - sm + m)$. The eigenvalues of B' different from $-s$ and $t(s^2 - sm + m)$ are denoted by $\lambda_0, \lambda_1, \ldots,$ and their multiplicities by $m_0, m_1, \ldots$. We have

$$0 \leqslant \sum m_i \leqslant (s + 1)(st + 1) - m(st + 1) - 1 - \frac{(st + 1)(st - sm - tm + s + t)}{s + t}$$

$$= \frac{(s - 1)(s^2t + st + s + t)}{s + t}. \tag{3}$$

Further

$$\sum m_i\lambda_i + t(s^2 - sm + m) - \frac{s(st + 1)(st - sm - tm + s + t)}{s + t} \geqslant \operatorname{tr} B' = 0,$$

i.e.,

$$\sum m_i\lambda_i \geqslant \frac{s(st + 1)(st - sm - tm + s + t)}{s + t} - t(s^2 - sm + m). \tag{4}$$

Suppose the right side of (4) is strictly negative. Then

$$m > \frac{s(s^2t^2 + st + s + t)}{s^3t + s^2t^2 - s^2t + s^2 + 2st - st^2 + t^2}. \tag{5}$$

Since

$$m \leqslant \frac{s(t + 1)}{s + t},$$

we have

$$\frac{s(t + 1)}{s + t} > \frac{s(s^2t^2 + st + s + t)}{s^3t + s^2t^2 - s^2t + s^2 + 2st - st^2 + t^2}.$$

Hence

$$st(s-1)(s^2-t^2) > 0,$$

contradicting $t \geqslant s$. Consequently

$$\sum m_i \lambda_i \geqslant \frac{s(st+1)(st-sm-tm+s+t)}{s+t} - t(s^2-sm+m) \geqslant 0. \quad (6)$$

Also

$$\sum m_i \lambda_i^2 + t^2(s^2-sm+m)^2 + \frac{s^2(st+1)(st-sm-tm+s+t)}{s+t}$$
$$\leqslant \operatorname{tr} B'^2 = (s-m+1)(st+1)t(s^2-sm+m),$$

i.e.,

$$0 \leqslant \sum m_i \lambda_i^2 \leqslant t(s^2-sm+m)(st-tm+s-m+1)$$
$$- \frac{s^2(st+1)(st-sm-tm+s+t)}{s+t}. \quad (7)$$

Let F be the family consisting of m_0 times λ_0, m_1 times $\lambda_1, \ldots$. Up to a strictly positive factor the variance L of F is equal to

$$\left(\sum m_i\right)\left(\sum m_i \lambda_i^2\right) - \left(\sum m_i \lambda_i\right)^2.$$

Since $L \geqslant 0$, by (3), (6), and (7) we have

$$\frac{(s-1)(s^2t+st+s+t)}{s+t}\left(t(s^2-sm+m)(st-tm+s-m+1)\right.$$
$$\left. - \frac{s^2(st+1)(st-sm-tm+s+t)}{s+t}\right)$$
$$-\left(\frac{s(st+1)(st-sm-tm+s+t)}{s+t} - t(s^2-sm+m)\right)^2 \geqslant 0. \quad (8)$$

This inequality is equivalent to

$$(s+t)(st+1)\big[t^2(s^2m^2-3sm^2+m^2-s^3m+2s^2m+2sm-m-s^2)$$
$$+t(-s^4m^2+3s^3m^2-5s^2m^2+sm^2+s^5m$$
$$-2s^4m+2s^3m+4s^2m-sm-2s^3)$$
$$-s^3m^2+s^4m+s^3m-s^4\big]\geqslant 0. \tag{9}$$

This proves the first part of the theorem.

Suppose now that equality holds in (9). Then

$$\left(\sum m_i\right)\left(\sum m_i\lambda_i^2\right)-\left(\sum m_i\lambda_i\right)^2=0.$$

Since the variance of F is equal to zero, F consists of m_0 times λ_0. Hence the matrix B' has at most three distinct eigenvalues. It follows that the complement of $G(P-V)$, and consequently also $G(P-V)$, is strongly regular.

Finally, assume that equality holds in (9) and that moreover $m\geqslant(s^2+t)/(s+t)$. Then $s+1-m\leqslant s(t+1)/(s+t)$. Replacing m by $s+1-m$ in the left side of (9), we obtain exactly the same expression. Hence interchanging the roles of V and $P-V$, by the previous section we see that also $G(V)$ is strongly regular. ∎

REMARK 1. If the graph $G(P-V)$ is strongly regular, it has parameters

$$v=|P|-|V|=(s+1-m)(st+1),$$
$$k=(t+1)(s-m),$$
$$\lambda=s-1-m,$$
$$\mu=\frac{k(k-1-\lambda)}{v-k-1}=\frac{(t+1)(s-m)^2}{s^2-ms+m}.$$

If the graph $G(V)$ is strongly regular, the parameters are

$$v' = m(st+1),$$

$$k' = (t+1)(m-1),$$

$$\lambda' = m-2,$$

$$\mu' = \frac{(t+1)(m-1)^2}{ms-m+1}.$$

COROLLARY 1. *If the GQ $S=(P,B,I)$ of order (s,t) contains an ovoid O, then $t \leqslant s^2-s$. If O is an ovoid of the GQ $S=(P,B,I)$ of order (s,s^2-s), then the graph $G(P-O)$ is strongly regular with parameters $v=s(s^3-s^2+1)$, $k=(s^2-s+1)(s-1)$, $\lambda=s-2$, $\mu=(s-1)^2$.*

Proof. Suppose that the GQ S with $t > s^2-s$ contains an ovoid O. Here $t \geqslant s$ and $m=1 \leqslant s(t+1)/(s+t)$. The inequality (2) becomes $st(s-1)^2(s^2-s-t) \geqslant 0$. Hence $t \leqslant s^2-s$, a contradiction. Suppose now that $t=s^2-s$ and that S contains an ovoid O. By Theorem 2 the graph $G(P-O)$ is strongly regular. The parameters of this graph are given in Remark 1. ■

Only one GQ of order (s,s^2-s) is known, namely the unique GQ with $s=t=2$ [7]. This GQ has an ovoid. The corresponding strongly regular graph has parameters $v=10$, $k=3$, $\lambda=0$, $\mu=1$. Such a graph is the well-known Petersen graph.

COROLLARY 2. *If the GQ $S=(P,B,I)$ of order (s,s^2) contains an m-ovoid V, then $m=(s+1)/2$. If V is an $(s+1)/2$-ovoid of the GQ $S=(P,B,I)$ of order (s,s^2), then the graphs $G(V)$ and $G(P-V)$ are strongly regular with parameters $v=(s+1)(s^3+1)/2$, $k=(s^2+1)(s-1)/2$, $\lambda=(s-3)/2$, $\mu=(s-1)^2/2$.*

Proof. We have $t \geqslant s$. By Corollary 1 we may assume that $m \leqslant s-1$. Hence $m \leqslant s(t+1)/(s+t)$. The inequality (2) becomes $-s^4(2m-s-1)^2 \geqslant 0$. Hence $m=(s+1)/2$. Now let V be a $[(s+1)/2]$-ovoid of S. Since $s+1-m=m$, both graphs $G(V)$ and $G(P-V)$ are strongly regular. The parameters are given in Remark 1. ■

Only for $s=3$ is an $[(s+1)/2]$-ovoid of a GQ S of order (s,s^2) known. Here the parameters of the strongly regular graphs are $v=v'=56$, $k=k'=$

10, $\lambda = \lambda' = 0$, $\mu = \mu' = 2$. Such a graph is the well-known Gewirtz graph [4].

REMARK 2. Putting $t = s^2 - s$ in (2), we obtain $s^5(m-1)(s-m) \geqslant 0$. This gives no restriction on m. Moreover, equality only occurs in the case of an ovoid or the complement of an ovoid.

REMARK 3. Putting $m = (s+1)/2$ in (2), we obtain $(s^3t - t + s^2 - s) \times (s^2 - t)(s-1) \geqslant 0$. Since $s^3t - t + s^2 - s > 0$ and $s^2 \geqslant t$, this gives no restriction on s and t. Moreover, equality occurs only in the case $t = s^2$. In [3] P. J. Cameron shows that for $m = (s+1)/2$ the graph $G(P-V)$ [or $G(V)$] is strongly regular iff $s^2 = t$.

THEOREM 3. *Let V be an m-ovoid of the GQ $S = (P, B, I)$ of order (s, t). If the graph $G(V)$ is strongly regular, then one of the following cases occurs*:

(i) $m = (s+1)/2$ *and* $t = s^2$.
(ii) $m < (s+1)/2$; *if* $m = 1$ *then* $t \leqslant s^2 - s$; *if* $m > 1$ *then* $t \leqslant s^2 - 2s$.
(iii) $m > (s+1)/2$ *and* $t \in \{s^2 - s, s^2 - s - 1\}$; *if* $t = s^2 - s - 1$ *then* $m \neq s$.

The graphs $G(V)$ and $G(P-V)$ are both strongly regular iff one of the following occurs:

(a) $m = (s+1)/2$ *and* $t = s^2$,
(b) $m \in \{1, s\}$ *and* $t = s^2 - s$.

Proof. Suppose that V is an m-ovoid of the GQ $S = (P, B, I)$ of order (s, t), and assume that the graph $G(V)$ is strongly regular. By Remark 1 we have $\mu' = (1+t)(m-1)^2/(ms - m + 1)$.

Fix a point $x \in P - V$. The set of all points of V collinear with x is denoted by W. We have $|W| = m(t+1)$. Let W' be the set of all points of $P - V$ not collinear with x. We have $|W'| = (s+1)(st+1) - m(st+1) - 1 - (t+1)(s-m) = t(s^2 - sm + m)$. The points of W' are denoted by $x_1, x_2, \ldots$. Further, let s_i be the number of points of W which are collinear with x_i.

Count in different ways the number of ordered pairs (x_i, y), with $x_i \in W'$, $y \in W$, and x, y collinear. We obtain

$$\sum s_i = m(t+1)t(s+1-m). \tag{10}$$

Now we count in different ways the ordered triples (x_i, y, y') with $x_i \in W'$, $y \in W$, $y' \in W$, $y \neq y'$, the points x, y collinear, and the points

x, y' collinear. We obtain

$$\sum s_i(s_i - 1) = m(t+1)mt(t-\mu') = m^2t(t+1)\left(t - \frac{(t+1)(m-1)^2}{ms-m+1}\right). \tag{11}$$

From (10) and (11) follows

$$\sum s_i^2 = mt(t+1)\left(s+1-m+mt - \frac{m(t+1)(m-1)^2}{ms-m+1}\right). \tag{12}$$

Since

$$|W'|\left(\sum s_i^2\right) - \left(\sum s_i\right)^2 \geqslant 0, \tag{13}$$

we have

$$t(s^2 - sm + m)mt(t+1)\left(s+1-m+mt - \frac{m(t+1)(m-1)^2}{ms-m+1}\right)$$
$$- m^2(t+1)^2t^2(s+1-m)^2 \geqslant 0. \tag{14}$$

Equivalently

$$(s-2m+1)\left[-mt(s-m+1) + s^3m\right.$$
$$\left. + s^2(-m^2-m+1) + s(2m^2-2m)\right] \geqslant 0, \tag{15a}$$

i.e.,

$$(s-2m+1)\left(-t+s^2-2s+\frac{s^2}{m(s+1-m)}\right) \geqslant 0. \tag{15b}$$

First assume that $m = (s+1)/2$. Then by Remark 3 we have $t = s^2$ and also the graph $G(P-V)$ is strongly regular.

Next, suppose that $m > (s+1)/2$. Then $t \geqslant s^2 - 2s + s^2/[m(s+1-m)]$. By Corollary 2 we have $t < s^2$. Hence $t \leqslant s^2 - s$ by 1.2.5 of [7]. So $s^2 - 2s < t \leqslant s^2 - s$. Let $t = s^2 - s - d$, with $s > d \geqslant 0$. Since $s + t \mid st(s+1)(t+1)$

[7], we have $s^2 - d \mid s(s^2 - s - d)(s+1)(s^2 - s - d + 1)$. So $s^2 - d \mid s^2(s+1)(s-1)$. Hence $s^2 - d \mid d(d-1)$. Let $d \notin \{0,1\}$. Then $s^2 - d \leqslant d(d-1)$, hence $s^2 \leqslant d^2$, so $s \leqslant d$, a contradiction. Consequently $d \in \{0,1\}$ and $t \in \{s^2 - s, s^2 - s - 1\}$. If $t = s^2 - s - 1$, then from $t \geqslant s^2 - 2s + s^2/[m(s+1-m)]$ it follows that $m \neq s$.

Finally, let $m < (s+1)/2$. Then $t \leqslant s^2 - 2s + s^2/[m(s+1-m)]$. If $m = 1$ then $t \leqslant s^2 - s$. Next, let $m > 1$. Then $t \leqslant s^2 - s - d$ with $0 < d < s$. By the previous paragraph $t = s^2 - s - 1$ or $t \leqslant s^2 - 2s$. By $m < (s+1)/2$ and $t \leqslant s^2 - 2s + s^2/[m(s+1-m)]$ the case $t = s^2 - s - 1$ cannot occur. Hence $t \leqslant s^2 - 2s$.

Assume now that the graphs $G(V)$ and $G(P-V)$ are both strongly regular. Then s_i is a constant for each choice of x, so we have equality in (13) for each choice of x, so we have equality in (15b), i.e., $m = (s+1)/2$ or $t = s^2 - 2s + s^2/[m(s+1-m)]$.

If $m = (s+1)/2$, then by Remark 3 we have $t = s^2$.

Now let $t = s^2 - 2s + s^2/[m(s+1-m)]$ and $m \neq (s+1)/2$. By a previous part of the proof $t \in \{s^2 - s, s^2 - s - 1\}$. If $t = s^2 - s$, then $m \in \{1, s\}$. If $t = s^2 - s - 1$, then $s - 1 = s^2/[m(s+1-m)]$, so $s - 1 \mid s^2$, i.e., $s = 2$ and $t = 1$, a contradiction.

Conversely, if we have one of the cases (a) or (b) in the statement of the theorem, then by Remark 3 and Corollary 1 the graphs $G(V)$ and $G(P-V)$ are strongly regular. ∎

COROLLARY 3. *Let V be an m-ovoid of the GQ $S = (P, B, I)$ of order (s,t). If $(s^2 + t)/(s+t) \leqslant m \leqslant s(t+1)/(s+t)$ and if equality holds in* (2), *then $m = (s+1)/2$ and $t = s^2$.*

Proof. Since $(s^2 + t)/(s+t) \leqslant s(t+1)/(s+t)$, we have $t \geqslant s$. By Theorem 2 the graphs $G(V)$ and $G(P-V)$ are strongly regular. By Theorem 3 we have $m = (s+1)/2$ and $t = s^2$, or $m \in \{1, s\}$ and $t = s^2 - s$. Since $m \geqslant (s^2 + t)/(s+t)$, we have $m \neq 1$; since $m \leqslant s(t+1)/(s+t)$, we have $m \neq s$. ∎

4. m-OVOIDS OF PARTIAL GEOMETRIES

A partial geometry (PG) is an incidence structure $S = (P, B, I)$ of points and lines satisfying the following axioms:

(i) each point is incident with $t+1$ $(t \geqslant 1)$ lines, and two distinct points are incident with at most one common line;

(ii) each line is incident with $s+1$ $(s \geqslant 1)$ points, and two distinct lines are incident with at most one common point;

(iii) for any point x and any line L not incident with x, there are exactly α $(\alpha > 0)$ points which are incident with L and collinear with x.

If $|P| = v$ and $|B| = b$, then $v = (s+1)(1+st/\alpha)$ and $b = (t+1)(1+st/\alpha)$ [11].

If $\alpha = 1$, then the PG is a GQ; if $\alpha = t$, then the PG is a net of order $s+1$ and degree $t+1$; if $\alpha = s+1$, then the PG is a 2-$(v, s+1, 1)$ design.

Let $S = (P, B, I)$ be a PG with parameters s, t, α. An m-ovoid V of S is a subset of P such that each line in B contains exactly m points of V. We always assume that $1 \leqslant m \leqslant s$. If V is an m-ovoid of S, then $P - V$ clearly is an $(s+1-m)$-ovoid of S. A 1-ovoid of S is called an ovoid of S. If $m = (s+1)/2$, then the m-ovoid is also called a hemisystem.

Lemma 2. *If V is an m-ovoid of the PG $S = (P, B, I)$ with parameters s, t, α, then $|V| = m(1+st/\alpha)$.*

Proof. Count in different ways the number of pairs (x, L) with $x \in V$, $L \in B$, and xIL. We obtain $|V|(t+1) = (t+1)(1+st/\alpha)m$. Hence $|V| = m(1+st/\alpha)$. ■

Let G be the point graph of the PG S, and let U be a pointset of S. Then the graph induced by G on U will be denoted by $G(U)$.

Theorem 4. *If the PG $S = (P, B, I)$ with parameters s, t, α contains an ovoid O, then $s = \alpha$ or $t \leqslant (s-1)(s+1-\alpha)$. In particular, if the PG $S = (P, B, I)$ with parameters $s, t, \alpha = s$ or $s, t = (s-1)(s+1-\alpha)$, α contains an ovoid O, then the graph $G(P-O)$ is strongly regular.*

Proof. Let M be the adjacency matrix of the complement of the point graph of the PG. The matrix M has eigenvalues $st(s+1-\alpha)/\alpha, t, \alpha - s - 1$ with respective multiplicities $1, s(s+1-\alpha)(st+\alpha)/[\alpha(s+t+1-\alpha)]$, $st(s+1)(t+1)/[\alpha(s+t+1-\alpha)]$. We may assume that

$$M = \begin{bmatrix} B' & C \\ C^T & D' \end{bmatrix},$$

where the rows and columns of B' correspond to the points of $P - O$. So B' is the adjacency matrix of the complement of $G(P-O)$. The matrix $M +$

$(s+1-\alpha)I=A$ is positive semidefinite with rank equal to $1+s(s+1-\alpha)\times(st+\alpha)/[\alpha(s+t+1-\alpha)]$.

Row sum of $B'+(s+1-\alpha)I=B$: $(t/\alpha)(s^2-s\alpha+\alpha)+(s+1-\alpha)=b$.
Row sum of $D'+(s+1-\alpha)I=D$: $st/\alpha+s+1-\alpha=d$.
Row sum of C: $st/\alpha-t=c$.
Row sum of C^T: $(st/\alpha)(s-\alpha)=c'$.

Since $c=d$ or $c'=b$ implies $s+t+1-\alpha=0$, we have $c\neq d$ and $b\neq c'$. Then by Theorem 1, rank $A\neq$ rank B. Consequently rank$[B'+(s+1-\alpha)I]\leqslant s(s+1-\alpha)(st+\alpha)/[\alpha(s+t+1-\alpha)]$. It follows that $-s-1+\alpha$ is an eigenvalue of B' with multiplicity at least

$$s\left(\frac{st}{\alpha}+1\right)-\frac{s(s+1-\alpha)(st+\alpha)}{\alpha(s+t+1-\alpha)}=\alpha\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}.$$

The matrix B' has constant row sum, so it has simple eigenvalue $t(s^2-s\alpha+\alpha)/\alpha$. The eigenvalues of B' different from $-s-1+\alpha$ and $t(s^2-s\alpha+\alpha)/\alpha$ are denoted by $\lambda_0,\lambda_1,\dots,$ and their multiplicities by $m_0,m_1,\dots$. We have

$$\begin{aligned}0\leqslant\sum m_i&\leqslant s\left(\frac{st}{\alpha}+1\right)-1-s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}\\&=s\left(\frac{st}{\alpha}+1\right)\frac{s+1-\alpha}{s+t+1-\alpha}-1.\end{aligned}\tag{16}$$

Further

$$\sum m_i\lambda_i+\frac{t}{\alpha}(s^2-s\alpha+\alpha)-(s+1-\alpha)s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}\geqslant \operatorname{tr}B'=0,$$

i.e.,

$$\sum m_i\lambda_i\geqslant(s+1-\alpha)s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}-\frac{t}{\alpha}(s^2-s\alpha+\alpha).\tag{17}$$

Since a 2-$(v, s+1, 1)$ design has no ovoid, we have $s \leqslant \alpha$. Hence

$$(s+1-\alpha)s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}-\frac{t}{\alpha}(s^2-s\alpha+\alpha)$$

$$\geqslant (s+1-\alpha)s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}-\frac{t}{\alpha}(s^2-s\alpha+s)$$

$$=(s+1-\alpha)\frac{st((s-1)(t-1)+2(\alpha-1))}{\alpha(s+t+1-\alpha)} \geqslant 0. \tag{18}$$

Also

$$\sum m_i\lambda_i^2+\frac{t^2}{\alpha^2}(s^2-s\alpha+\alpha)^2+(s+1-\alpha)^2 s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}$$

$$\leqslant \operatorname{tr} B'^2=\frac{t}{\alpha}(s^2-s\alpha+\alpha)s\left(\frac{st}{\alpha}+1\right),$$

i.e.,

$$0 \leqslant \sum m_i\lambda_i^2 \leqslant \frac{st}{\alpha}\left(\frac{st}{\alpha}+1\right)(s^2-s\alpha+\alpha)-\frac{t^2}{\alpha^2}(s^2-s\alpha+\alpha)^2$$

$$-(s+1-\alpha)^2 s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha}. \tag{19}$$

Let F be the family consisting of m_0 times λ_0, m_1 times $\lambda_1, \dots$. Up to a strictly positive factor the variance L of F is equal to

$$\left(\sum m_i\right)\left(\sum m_i\lambda_i^2\right)-\left(\sum m_i\lambda_i\right)^2.$$

Since $L \geqslant 0$, by (16), (17), (18), (19) and after some calculations we finally obtain

$$st\left(\frac{st}{\alpha}+1\right)(s-\alpha)^2(-t+(s-1)(s+1-\alpha)) \geqslant 0. \tag{20}$$

Hence $s=\alpha$ or $t \leqslant (s-1)(s+1-\alpha)$, which proves the first part of the theorem.

Suppose now that $s=\alpha$ or $t=(s-1)(s+1-\alpha)$. Then equality holds in (20); hence

$$\left(\sum m_i\right)\left(\sum m_i\lambda_i^2\right)-\left(\sum m_i\lambda_i\right)^2=0.$$

Since the variance of F is equal to zero, F consists of m_0 times λ_0. Hence the matrix B' has at most three distinct eigenvalues. It follows that the complement of $G(P-O)$, and consequently also $G(P-O)$, is strongly regular. ∎

REMARK 4. In fact Theorem 4 is Cvetcovic's bound on strongly regular graphs [6] applied to an ovoid of a partial geometry.

COROLLARY 4. *Let* $S=(P,B,I)$ *be a PG with* $(\alpha,s)\neq(1,1)$ *for which the Krein equality* [4] $(s+1-2\alpha)t=(s-1)(s+1-\alpha)^2$ *is satisfied. Then* S *has no ovoid.*

Proof. Suppose that the Krein equality $(s+1-2\alpha)t=(s-1)(s+1-\alpha)^2$ is satisfied and that S has an ovoid. If $s=\alpha$, then necessarily $\alpha=s=1$, a contradiction. So by the previous theorem we have $t\leqslant(s-1)(s+1-\alpha)$. Hence $(s-1)(s+1-\alpha)^2=(s+1-2\alpha)t\leqslant(s+1-2\alpha)(s-1)(s+1-\alpha)$, so $s+1-\alpha\leqslant s+1-2\alpha$, again a contradiction. ∎

The Case $\alpha=s$

Consider a PG $S=(P,B,I)$ with $s=\alpha$, i.e., consider a dual net. This PG has exactly $s+1$ ovoids, and these are the $s+1$ classes of "parallel" points. For each ovoid O the graph $G(P-O)$ is strongly regular with parameters $v^*=|P-O|=s(t+1)$, $k^*=(t+1)(s-1)$, $\lambda^*=(s-2)(t+1)$, $\mu^*=(t+1)(s-1)$. Clearly this graph is a complete multipartite graph.

The case $t=(s-1)(s+1-\alpha)$

Consider a PG $S=(P,B,I)$ with parameters $s, t=(s-1)(s+1-\alpha), \alpha$, and suppose that S has an ovoid O. By the previous theorem the graph $G(P-O)$ is strongly regular. Now we calculate the parameters $v^*, k^*, \lambda^*, \mu^*$ of this graph. Clearly $v^*=s(st/\alpha+1)=s(s(s-1)(s+1-\alpha)/\alpha+1)$ and $k^*=(t+1)(s-1)=(s^2-\alpha s+\alpha)(s-1)$. From the proof of Theorem 4 we have

$$\lambda_0=\frac{\sum m_i\lambda_i}{\sum m_i},$$

with

$$\sum m_i\lambda_i = (s+1-\alpha)s\left(\frac{st}{\alpha}+1\right)\frac{t}{s+t+1-\alpha} - \frac{t}{\alpha}(s^2 - s\alpha + \alpha)$$

and

$$\sum m_i = s\left(\frac{st}{\alpha}+1\right)\frac{s+1-\alpha}{s+t+1-\alpha} - 1.$$

Since $s+t+1-\alpha = s(s+1-\alpha)$, we have

$$\sum m_i\lambda_i = \frac{st}{\alpha}(t-s+\alpha) \quad \text{and} \quad \sum m_i = \frac{st}{\alpha}.$$

Hence $\lambda_0 = t - s + \alpha$. Consequently, besides its valency, the graph $G(P-O)$ has eigenvalues $s-\alpha = r$ and $s-t-\alpha-1 = l$. Since $\mu^* = k^* + rl$ and $\lambda^* = \mu^* + r + l$, we have

$$\mu^* = \alpha(t-s+\alpha) = \alpha(s^2 - s\alpha - s + 2\alpha - 1)$$

and

$$\lambda^* = (s-2)(\alpha s + 2\alpha - s - \alpha^2).$$

The only known PGs with $t = (s-1)(s+1-\alpha)$ and $s \neq \alpha$ have parameters $s = 2^m$, $t = 2(2^m - 1)$, $\alpha = 2^m - 1$ with $m \geqslant 1$ [1]. They are constructed as follows. Consider a dual hyperoval of PG$(2,q)$, $q = 2^{m+1}$ and $m \geqslant 1$, i.e., consider a set of $q+2$ lines $L_1, L_2, \ldots, L_{q+2}$ of PG$(2,q)$ no three of which are concurrent. The points of the PG are the points of $L_1 \cup L_2 \cup \cdots \cup L_{q+2}$; the lines of the PG are the lines of PG$(2,q)$ different from $L_1, L_2, \ldots, L_{q+2}$; the incidence is that of PG$(2,q)$. Such a PG has parameters $s = q/2 = 2^m$, $t = q-2 = 2(2^m-1)$, $\alpha = q/2 - 1 = 2^m - 1$. The ovoids of this PG are the lines $L_1, L_2, \ldots, L_{q+2}$. If O is an ovoid of the PG, then the graph $G(P-O)$ has parameters $v^* = 2^m(2^{m+1}+1)$, $k^* = (2^{m+1}-1)(2^m-1)$, $\mu^* = (2^m-1)(2^{m+1}-3)$, $\lambda^* = (2^m-2)(2^{m+1}-3)$. Hence $G(P-O)$ is the complement of the triangular graph $T(2^{m+1}+1)$ [9].

Quasipartial Geometries

Let $S = (P, B, I)$ be a PG with $s = \alpha \neq 1$ or $t = (s-1)(s+1-\alpha)$, and let O be an ovoid of S. Then the incidence structure $S' = (P - O, B, I')$, with I'

induced by I, satisfies the following properties:

(i′) Each point is incident with $t+1$ $(t \geqslant 1)$ lines, and two distinct lines are incident with at most one common point;

(ii′) each line is incident with $s = s'+1$ $(s' \geqslant 1)$ points, and two distinct points are incident with at most one common line;

(iii′) if x is a point of $P-O$ and L is a line in B which is not incident with x, then there are $\alpha - 1$ or α $(\alpha > 0)$ points of $P-O$ which are incident with L and collinear with x;

(iv′) the point graph of S' is strongly regular.

An incidence structure satisfying properties (i′)–(iv′) will be called a quasipartial geometry with parameters s', t, α, λ, and μ, where λ (μ) is the number of points of S' collinear with two given collinear (noncollinear) points of S'. The notion of quasipartial geometry clearly is a generalization of the notion of partial geometry. The quasipartial geometries with $\alpha = 1$ are just the partial quadrangles introduced by P. J. Cameron [2].

THEOREM 5. *If $S = (P, B, I)$ is a PG with parameters s, t, α which satisfies the Krein equality $t(s+1-2\alpha) = (s-1)(s+1-\alpha)^2$, then for any hemisystem V the graphs $G(V)$ and $G(P-V)$ are strongly regular.*

Proof. In the trivial case $s = \alpha = 1$ the theorem clearly is satisfied. For $\alpha = 1$ and $s > 1$, the theorem is satisfied by Corollary 2. From now on we assume that $\alpha \geqslant 2$.

The method of proof is the one we used to prove Theorem 4. Here we obtain consecutively the following. The rank of the matrix $B' + (s+1-\alpha)I$ is less than or equal to $s(s+1-\alpha)(st+\alpha)/[\alpha(s+t+1-\alpha)]$. It follows that $-s-1+\alpha$ is an eigenvalue of B' with multiplicity at least

$$\left(\frac{st}{\alpha}+1\right)\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}\right).$$

Since $t(s+1-2\alpha) = (s-1)(s+1-\alpha)^2$, we have $t > s-1$, so

$$\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha} = \frac{s(t-s+\alpha)+(t+1-\alpha)}{2(s+t+1-\alpha)} > 0. \qquad (21)$$

Next,

$$0 \leqslant \sum m_i \leqslant \frac{s(s+1-\alpha)}{s+t+1-\alpha}\left(\frac{st}{\alpha}+1\right)-1, \qquad (22)$$

and

$$\sum m_i\lambda_i \geqslant (s+1-\alpha)\left(\frac{st}{\alpha}+1\right)\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}\right)$$

$$-\frac{t}{2\alpha}(s^2+s-\alpha s+\alpha). \tag{23}$$

Since $t(s+1-2\alpha)=(s-1)(s+1-\alpha)^2$, we have $t>s$ and $s>\alpha$, so

$$(s+1-\alpha)\left(\frac{st}{\alpha}+1\right)\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}\right)-\frac{t}{2\alpha}(s^2+s-\alpha s+\alpha)$$

$$>(s+1-\alpha)\left(\frac{st}{\alpha}+1\right)\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}\right)$$

$$-\frac{t}{2\alpha}(s^2+s-\alpha s+s^2+s-\alpha s)$$

$$=(s+1-\alpha)\left(\frac{st}{\alpha}+1\right)\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}\right)-\frac{st}{\alpha}(s+1-\alpha)$$

$$>(s+1-\alpha)\frac{st}{\alpha}\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}\right)-\frac{st}{\alpha}(s+1-\alpha) \qquad \text{[by (21)]}$$

$$=(s+1-\alpha)\frac{st}{\alpha}\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}-1\right)$$

$$=(s+1-\alpha)\frac{st}{\alpha}\left(\frac{t(s-1)-s^2-2s+s\alpha+\alpha-1}{2(s+t+1-\alpha)}\right)$$

$$\geqslant(s+1-\alpha)\frac{st}{\alpha}\left(\frac{(s+1)(s-1)-s^2-2s+s\alpha+\alpha-1}{2(s+t+1-\alpha)}\right)$$

$$=\frac{(s+1-\alpha)st(s+1)(\alpha-2)}{2\alpha(s+t+1-\alpha)}\geqslant 0.$$

Consequently the right side of (23) is strictly positive. Further,

$$0 \leqslant \sum m_i \lambda_i^2 \leqslant \frac{t(s+1)(s^2+s-\alpha s+\alpha)}{4\alpha}\left(\frac{st}{\alpha}+1\right) - \frac{t^2}{4\alpha^2}(s^2+s-\alpha s+\alpha)^2$$

$$-(s+1-\alpha)^2\left(\frac{st}{\alpha}+1\right)\left(\frac{s+1}{2}-\frac{s(s+1-\alpha)}{s+t+1-\alpha}\right). \tag{24}$$

Hence

$$\begin{aligned} 0 &\leqslant \left(\sum m_i\right)\left(\sum m_i \lambda_i^2\right) - \left(\sum m_i \lambda_i\right)^2 \\ &\leqslant \frac{1}{4\alpha^2(s+t+1-\alpha)^2}(\alpha+st)(\alpha-s-t-1) \\ &\quad \times\left[t(\alpha s^2+\alpha-s^3-s^2)+\alpha^2 s-\alpha^2-\alpha s^2+\alpha\right] \\ &\quad \times\left[-t(s+1-2\alpha)+(s-1)(s+1-\alpha)^2\right]. \end{aligned} \tag{25}$$

Since $t(s+1-2\alpha)=(s-1)(s+1-\alpha)^2$, we have

$$\left(\sum m_i\right)\left(\sum m_i \lambda_i^2\right) - \left(\sum m_i \lambda_i\right)^2 = 0,$$

and so the graph $G(P-V)$ is strongly regular. The roles of V and $P-V$ can be interchanged; hence also $G(V)$ is strongly regular. ∎

REMARK 5. As in the ovoidal case with $t=(s-1)(s+1-\alpha)$, one can calculate the parameters of the strongly regular graphs $G(P-V)$ and $G(V)$.

REMARK 6. Let us consider again (25), but this time without restriction on the parameters s, t, α. Since $t(s+1-2\alpha) \leqslant (s-1)(s+1-\alpha)^2$ for any PG and $t(\alpha s^2+\alpha-s^3-s^2)+\alpha^2 s-\alpha^2-\alpha s^2+\alpha = t[\alpha-s^2(s+1-\alpha)] - \alpha(s-1)(s+1-\alpha) \leqslant 0$ for $\alpha \neq s+1$, the right side of (25) is always positive for $\alpha \neq s+1$. Moreover, for $\alpha \neq s+1$ and $t(s+1-2\alpha) \neq (s-1)(s+1-\alpha)^2$ the right side of (25) is always strictly positive. For $\alpha = s+1$ the right side of (23) is strictly negative. Hence, in applying this method to a hemisystem of a PG with parameters s, t, α there arises no necessary condition on the parameters. Moreover we have equality in (25) iff the Krein equality $t(s+1-2\alpha)=(s-1)(s+1-\alpha)^2$ is satisfied.

5. k-ARCS OF PARTIAL GEOMETRIES

A k-arc K of a PG $S=(P,B,I)$ is a set of k points no two of which are collinear. If the PG S has parameters s,t,α, then counting the lines of S which are incident with an element of K, we obtain $(t+1)k \leqslant (t+1)(st/\alpha+1)$. Hence $k \leqslant st/\alpha+1$. Clearly the ovoids of S are exactly the $(st/\alpha+1)$-arcs of S. The k-arc K is called complete if it is not contained in a $(k+1)$-arc. Ovoids evidently are complete.

A section on k-arcs of GQs is contained in [7].

THEOREM 6. *If K is a complete k-arc of the PG $S=(P,B,I)$ with parameters s,t,α, then*

$$k=\frac{st}{\alpha}+1 \quad or \quad k \leqslant \frac{st}{\alpha}-\frac{t-(\alpha-1)(s+1)}{\alpha s}.$$

Proof. Let K be a complete k-arc of the PG $S=(P,B,I)$, and assume that $k<st/\alpha+1$. Then $k=st/\alpha-\rho$ with $\rho>-1$.

Let W be the set of all lines having no point in common with K. We have

$$w=|W|=(t+1)\left(\frac{st}{\alpha}+1\right)-(t+1)\left(\frac{st}{\alpha}-\rho\right)=(t+1)(\rho+1).$$

Choose a line L of W, an let $x_0,x_1,\dots,x_s$ be the points incident with L. The number of lines of W (of $B-W$) which are incident with x_i, but distinct from L, is denoted by t_i (by s_i). We have $s_i+t_i=t$. Counting in distinct ways the pairs (y,x_i) with $y\in K$ and y,x_i collinear, we obtain

$$\sum s_i=\left(\frac{st}{\alpha}-\rho\right)\alpha.$$

Hence

$$\sum t_i = \sum (t - s_i) = (s+1)t - \left(\frac{st}{\alpha} - \rho\right)\alpha = t + \rho\alpha. \tag{26}$$

If M is a line of W which is incident with x_i, then the number of lines of $W - \{M\}$ which are concurrent with M at points different from x_i is equal to $t + \rho\alpha - t_i$. Let γ_i be the number of lines of W which are concurrent with at least one line of W through x_i but which are not incident with x_i. Counting in different ways the ordered pairs (M, N), with $M, N \in W$, $x_i I M$, $x_i \not{I} N$, and M, N concurrent, we obtain

$$(t_i + 1)(t + \rho\alpha - t_i) \leqslant \gamma_i \alpha.$$

Hence

$$\gamma_i \geqslant \frac{(t_i + 1)(t + \rho\alpha - t_i)}{\alpha}. \tag{27}$$

Clearly

$$w = (t+1)(\rho+1) \geqslant \gamma_i + t_i + 1 \geqslant \frac{(t_i + 1)(t + \rho\alpha - t_i)}{\alpha} + t_i + 1. \tag{28}$$

Consequently

$$\alpha(t+1)(\rho+1) \geqslant (t_i + 1)(t + \rho\alpha - t_i) + \alpha(t_i + 1),$$

i.e.,

$$(t_i - t)(t_i - \rho\alpha - \alpha + 1) \geqslant 0.$$

Since K is complete, $K \cup \{x_i\}$ is not a $(k+1)$-arc, so we have $t_i < t$. It follows that

$$t_i \leqslant \rho\alpha + \alpha - 1. \tag{29}$$

Hence by (26) and (29)

$$t + \rho\alpha = \sum_{i=0}^{s} t_i \leqslant (s+1)(\rho\alpha + \alpha - 1), \tag{30}$$

i.e.,

$$\rho \geqslant \frac{t - (\alpha - 1)(s+1)}{\alpha s}$$

i.e.,

$$k \leqslant \frac{st}{\alpha} - \frac{t - (\alpha-1)(s+1)}{\alpha s}. \qquad \blacksquare$$

COROLLARY 5 [7]. *If K is a complete k-arc of the GQ S of order (s, t), then $k = st + 1$ or $k \leqslant st - t/s$.*

Proof. Straightforward. $\blacksquare$

THEOREM 7. *Let K be a k-arc of the PG $S = (P, B, I)$ with parameters s, t, α. If $k > st/\alpha - [t - (\alpha-1)(s+1)]/\alpha s$, then there is a unique ovoid O which contains K.*

Proof. For $k = st/\alpha + 1$ the theorem is trivial, so let us assume that

$$\frac{st}{\alpha} + 1 > k > \frac{st}{\alpha} - \frac{t - (\alpha-1)(s+1)}{\alpha s}.$$

By Theorem 6 the k-arc K is contained in a $(k+1)$-arc K'. If K' is not an ovoid, then again by Theorem 6 the $(k+1)$-arc K' is contained in a $(k+2)$-arc K'', etc. Finally, K is completed to an ovoid O. Let O and O' be distinct ovoids containing K, and let $x \in O' - O$. Then each of the $t+1$ lines incident with x contains a point of $O - K$. Hence $|O - K| = st/\alpha + 1 - k \geqslant t+1$. So

$$\frac{st}{\alpha} + 1 - \frac{st}{\alpha} + \frac{t-(\alpha-1)(s+1)}{s\alpha} \geqslant t+1,$$

i.e., $t(s\alpha - 1) + (s+1)(\alpha-1) \leqslant 0$, i.e., $\alpha = s = 1$. But in the case $\alpha = s = 1$ the k-arc K clearly is contained in a unique ovoid, a contradiction. We conclude that K is contained in a unique ovoid. ∎

We remark that in the case $s = \alpha$ each k-arc, with $k \neq 0$, is contained in a unique ovoid.

COROLLARY 6. *If the PG $S = (P, B, I)$ with parameters s, t, α, where $s \neq \alpha$, contains a k-arc K with*

$$k > \frac{st}{\alpha} - \frac{t-(\alpha-1)(s+1)}{\alpha s},$$

then $t \leqslant (s-1)(s+1-\alpha)$.

Proof. Straightforward from Theorems 7 and 4. ∎

THEOREM 8. *If K is a complete k-arc, with*

$$k = \frac{st}{\alpha} - \frac{t-(\alpha-1)(s+1)}{\alpha s},$$

of the PG $S = (P, B, I)$ with parameters s, t, α, then the lines of S having no point in common with K together with the points incident with these lines form a partial subgeometry of S with parameters $s' = s$, $t' = (t - \alpha + 1)/s$, $\alpha' = \alpha$.

Proof. Let K be a complete k-arc with $k = st/\alpha - [t-(\alpha-1)\times(s+1)]/\alpha s$. From (29) and (30) in the proof of Theorem 6 it follows that $t_i = (t-\alpha+1)/s$ for all $i = 0, 1, \dots, s$. Then from (28) it follows that $w = \gamma_i + (t-\alpha+1)/s+1$ for all $i = 0,1,\dots, s$, i.e., each line of W is concurrent with at least one of the $(t-\alpha+1)/s+1$ lines of W through x_i. Since we have also equality in (27), each line of W which is not incident with x_i is concurrent with exactly α lines of W through x_i. Now it is clear that the lines of W together with the points on these lines form a partial subgeometry [5] of S with parameters $s' = s$, $t' = (t-\alpha+1)/s$, $\alpha' = \alpha$. ■

REMARK 7. Let K be a complete k-arc, $k = st/\alpha - [t-(\alpha-1)\times(s+1)]/\alpha s$, of the PG S, and let x be a point which is not contained in the corresponding partial subgeometry S′. Then the points of S′ which are collinear with x form an α-ovoid of S′. Further, it is easy to show that any line of S, but not in S′, contains exactly one point of S′.

EXAMPLE. Consider the GQ $W(2)$ with 15 points and 15 lines [7]. Let L be a line of PG(3, 2) which does not belong to $W(2)$. Then L is a complete 3-arc of $W(2)$. The lines of $W(2)$ having no point in common with L together with the points of these lines form a subquadrangle of order $(s, 1)$.

6. APPENDIX: THE KREIN INEQUALITIES

The matrix technique of Theorem 1 was used in Theorems 2, 4, and 5. In each of these applications the row sums of the matrices B, D, C, C^T were constants. So the slightly less general version of Theorem 1, due to P. J. Cameron [3], was sufficient to prove these theorems. Now we shall apply Theorem 1 in a case where the row sums of D and C^T are not constant. Moreover, by this technique we shall obtain the Krein inequalities for strongly regular graphs, and also an interpretation of the equalities, in an elementary way, without using Hadamard multiplication and the theory of the minimal idempotents of the Bose-Mesner algebra [4].

Let G be a noncomplete strongly regular graph (SRG) with parameters v, k, λ, μ, with $0 < \mu < k$. The vertex set of G is denoted by P. If $x \in P$, then let $\Gamma(x) = \{y \in P \mid y \sim x\}$ and $\Delta(x) = \{y \in P - \{x\} \mid y \nsim x\}$. The eigenvalues of the adjacency matrix of G are k, r, l, with $r > 0$ and $l < 0$. We have $r + l = \lambda - \mu$ and $rl = \mu - k$ [4].

THEOREM 9. *The strongly regular graph G satisfies the so-called Krein inequalities*

$$(l+1)(k+l+2rl) \leqslant (k+l)(r+1)^2 \tag{31}$$

and

$$(r+1)(k+r+2rl) \leqslant (k+r)(l+1)^2. \tag{32}$$

If we have equality in (31) [*in* (32)], *then for any vertex x of G the graph induced on $\Delta(x)$ [$\Gamma(x)$] is also strongly regular.*

Proof. Let M be the adjacency matrix of the complement of G. The matrix M has eigenvalues $v-k-1$, $-r-1$, $-l-1$ with respective multiplicities 1, $f=-k(l+1)(k-l)/[(k+rl)(r-l)]$, $g=k(r+1)(k-r)/[(k+rl)(r-l)]$. We assume that

$$M=\begin{bmatrix} B' & C \\ C^T & D' \end{bmatrix},$$

where the rows and columns of B' correspond to the vertices of $\Delta(x)$, for a given vertex x. The matrix $M+(r+1)I=A$ is positive semidefinite with rank equal to $v-f$.

Row sum of $B'+(r+1)I=B$: $v-2k-2+\mu+r+1=(l+1)(r^2l-k)/(k+rl)=b$.

Row sum of C: $k-\mu+1=-rl+1=c$.

Average of the row sums of $D'+(r+1)I=D$: $k(k-\lambda-1)/(k+1)+r+1=-(r+1)(l+1)k/(k+1)+r+1=(r+1)(1-lk)/(k+1)=d$.

Average of the row sums of C^T: $[v-k-1+k(v-2k+\lambda)]/(k+1)=k(r+1)(l+1)(lr-1)/[(k+1)(k+rl)]=c'$.

Since $c=d$ or $c'=b$ implies $(l+1)(k-r)=0$, i.e., $\mu=0$ or G complete, we have $c\neq d$ and $c'\neq b$. Then by Theorem 1, rank $A\neq$ rank B. Consequently rank$[B'+(r+1)I]\leqslant v-f-1$. It follows that $-r-1$ is an eigenvalue of B' with multiplicity at least $v-k-1-v+f+1=f-k$.

Assume that $f-k>0$, i.e., $-k(l+1)(k-l)/[(k+rl)(r-l)]>k$, i.e., $\mu<l(l+1)$. The matrix B' has constant row sum, so it has the simple eigenvalue $v-2k-2+\mu=v-k+rl-2$. The eigenvalues of B' different from $-r-1$ and $v-k+rl-2$ are denoted by $\lambda_0,\lambda_1,\ldots$, and their multiplicities by $m_0,m_1,\ldots$. We have

$$0\leqslant \sum m_i\leqslant v-k-1-1-f+k=v-f-2=\alpha \tag{33}$$

and

$$\begin{aligned}\sum m_i\lambda_i &\geqslant -(v-k+rl-2)+(r+1)(f-k)+\operatorname{tr}B'\\ &=-(v-k+rl-2)+(r+1)(f-k)=\beta.\end{aligned} \tag{34}$$

Since

$$\operatorname{tr}B'^2=(v-k-1)(v-k+rl-2),$$

we have

$$\begin{aligned}0\leqslant \sum m_i\lambda_i^2 &\leqslant -(v-k+rl-2)^2-(r+1)^2(f-k)+\operatorname{tr}B'^2\\ &=(v-k+rl-2)(-rl+1)-(r+1)^2(f-k)=\gamma.\end{aligned} \tag{35}$$

Let F be the family consisting of m_0 times λ_0, m_1 times $\lambda_1,\ldots$. Up to a strictly positive factor, the variance L of F is equal to

$$\left(\sum m_i\right)\left(\sum m_i\lambda_i^2\right)-\left(\sum m_i\lambda_i\right)^2.$$

If $\beta\geqslant 0$, then by $L\geqslant 0$, (33), (34), and (35), we have $\alpha\gamma-\beta^2\geqslant 0$. Now let $\beta<0$. We have $\alpha+\beta=r(f-k-l)>0$ and $\beta+\gamma=r[-l(v-k+rl-2)-(r+1)(f-k)]>-r\beta>0$. Hence again $\alpha\gamma-\beta^2>0$. After some calculations we obtain

$$\begin{aligned}(k+rl)(r-l)(\alpha\gamma-\beta^2)=kr(l+1)(r+1)\big[&(l+1)(k+l+2rl)\\ &-(k+l)(r+1)^2\big].\end{aligned}$$

Since $k + rl = \mu > 0$, $r - l > 0$, $kr > 0$, $l + 1 < 0$, $r + 1 > 0$, it follows that

$$(l+1)(k+l+2rl) \leqslant (k+l)(r+1)^2. \tag{36}$$

Next, assume that $f - k \leqslant 0$, i.e., $\mu \geqslant l(l+1)$. Now we show that $(l+1)(k+l+2rl) \geqslant (k+l)(r+1)^2$ implies that G is the pentagon, and in such a case equality holds. If G is a conference graph [4], then $k + 2l + 1 = 2\mu - \sqrt{4\mu + 1} > 0$ for $\mu \neq 1$. For the pentagon, i.e., the conference graph with $\mu = 1$, we have $\mu = l(l+1)$ and $(l+1)(k+l+2rl) = (k+l)(r+1)^2$. If G is not a conference graph, then $r, l \in \mathbb{Z}$ [9]. For $r \geqslant 2$, we have $k + 2l \geqslant k + rl = \mu > 0$; for $r = 1$, w have $\lambda = 1 + 2l + k \geqslant 0$. Consequently, if G is not the pentagon, then $(-k-l)/(l+1) \geqslant 1$. In such a case $(l+1)(k+l+2rl) \geqslant (k+l)(r+1)^2$ implies $-k-l-2rl \geqslant (r+1)^2$, so $-\mu \geqslant (r+1)(r+1+l)$, so $r + 1 + l < 0$, so $\mu < l(l+1) + rl$, so $\mu < l(l+1)$, a contradiction.

The complement $\bar{G}$ of G has parameters $\bar{v} = v$, $\bar{k} = v - k - 1$, $\bar{\lambda} = v - 2k + \mu - 2$, and $\bar{\mu} = v - 2k + \lambda$. The eigenvalues corresponding with $\bar{G}$ are $\bar{k} = v - k - 1$, $\bar{l} = -r - 1$ and $\bar{r} = -l - 1$. Since $(\bar{l}+1)(\bar{k}+\bar{l}+2\bar{r}\bar{l}) \leqslant (\bar{k}+\bar{l})(\bar{r}+1)^2$, it follows that

$$(r+1)(k+r+2rl) \leqslant (k+r)(l+1)^2. \tag{37}$$

Next, assume that we have equality in at least one of (31), (32), e.g. $(l+1)(k+l+2rl) = (k+l)(r+1)^2$. Further, let x be any vertex of G. If G is the pentagon, then clearly the graphs induced on $\Gamma(x)$ and $\Delta(x)$ are strongly regular. So assume that G is not the pentagon. By one of the preceding paragraphs we have $\mu < l(l+1)$. With the notation of the second paragraph we have $\alpha\gamma - \beta^2 = 0$ and $\beta \geqslant 0$, so $(\Sigma m_i)(\Sigma m_i \lambda_i^2) - (\Sigma m_i \lambda_i)^2 = 0$. Since the variance of F is equal to zero, F consists of m_0 times λ_0. Hence the matrix B' has at most three distinct eigenvalues. There follows that the graph induced on $\Delta(x)$ is strongly regular. ■

The case $(l+1)(k+l+2rl) = (k+l)(r+1)^2$

Suppose that the strongly regular graph G satisfies $(l+1)(k+l+2rl) = (k+l)(r+1)^2$, and let x be any vertex of G. Now we calculate the parameters $v_1, k_1, \lambda_1, \mu_1$ of the SRG G_1 induced on $\Delta(x)$. We exclude the trivial case of the pentagon.

We have $v_1 = v - k - 1 = -k(r+1)(l+1)/(k+rl)$ and $k_1 = k - \mu = -rl$. From the proof of Theorem 9 we have $\lambda_0 = \Sigma m_i \lambda_i / \Sigma m_i$. Hence

$$\lambda_0 = \frac{-v - rl + 2 - rk + (r+1)f}{v - f - 2}.$$

Since $(k+rl)(l-r)=(r+1)(kr-l^2)$, we have

$$\lambda_0=-1+r\frac{f-k-l}{v-f-2}$$

$$=-1+r\frac{\dfrac{k(1+r)(-k+l^2+l-lr)}{(k+rl)(r-l)}-l}{\dfrac{k(1+r)(k-r)+(k+rl)(l-r)}{(k+rl)(r-l)}}$$

$$=-1+r\frac{k(1+r)(-k+l^2+l-lr)-l(r+1)(l^2-kr)}{k(1+r)(k-r)+(r+1)(kr-l^2)}$$

$$=-1+r\frac{l^2-k}{k+l}=-1+\frac{r^2-l}{2}.$$

Consequently, besides its valency the graph $\Delta(x)$ has eigenvalues $r_1=r$ and $l_1=(l-r^2)/2$. Hence $\mu_1=k_1+r_1l_1=-r(l+r^2)/2$ and $\lambda_1=\mu_1+r_1+l_1=(1-r)(l+r^2+2r)/2$.

REMARK 8. In [4] it is shown that if we have equality in at least one of (31), (32), then the graphs induced on $\Gamma(x)$ and $\Delta(x)$ are both strongly regular. Suppose e.g. that we have equality in (31), and let $v_2, k_2, \lambda_2, \mu_2$ be the parameters of the SRG G_2 induced on $\Gamma(x)$. If $\Gamma(x)$ is not a coclique of G, then we have $v_2=k$, $k_2=\lambda=r+l+k+rl$, $\mu_2=k_2+r_2l_2$, $\lambda_2=r_2+l_2+k_2+r_2l_2$, with $r_2=r$ and $l_2=\frac{1}{2}(r^2+2r+l)$ [4].

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