# Uniform Refinement of Curves

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Dedicated to Alan J. Hoffman with friendship and esteem on the occasion of his 65th birthday.

Submitted by Hans Schneider

ABSTRACT

We propose and analyze a class of algorithms for the generation of curves and surfaces. These algorithms encompass some well-known methods of subdivision for Bernstein-Bézier curves (de Casteljau's algorithm) and $B$-spline curves (Lane and Riesenfeld's algorithm). Several results concerning properties of the limiting curves as well as related questions are discussed.

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## 0. INTRODUCTION

Subdivision algorithms provide important techniques for the fast generation of curves and surfaces. A continual smoothing (usually by a corner cutting procedure) of a given control polygon will lead in the limit to a desired visually smooth object. Many subdivision strategies have been proposed, for instance, by Chaikin [4], Catmull and Clark [3], Doo [5], Lane and Riesenfeld [9], and de Rham [11, 12]. However, it is often not clear what kind of curves and surfaces are produced by these procedures (Catmull and Clark [3], Doo [5]), though some thought has been given to this issue (Doo and Sabin [6], Riesenfeld [13]). Sometimes each iteration of a smoothing method can be viewed as a representation of the limiting curve or surface relative to a refined basis. For such *refinement algorithms* an analysis of the convergence properties of the algorithms is more likely to be possible.

In this paper we consider uniform subdivision, i.e. smoothing methods that use at each iteration the same smoothing strategy. This investigation is inspired by algorithms for curve generation presented in [2, 3, 9, 12]. We unify their essential structure within a family of smoothing methods, which allows for a systematic study of the limiting curves. We show that they are *uniformly refinable*, that is, scaled versions of the curves can be represented in terms of the original curve, a property clearly valid for curves modeled by polynomial components. Finally, we demonstrate the intimate relationship between smoothness of the limiting curve and the existence of polynomial components. These results were announced in [10].

## 1. UNIFORM REFINEMENT

Let $\{\psi_1,\dots,\psi_n\}$ be a set of real-valued functions. We say that they are uniformly refinable over the interval $[0,1]$ with scale $p\in\mathbb{N}$ if there exist $p$ $(n\times n)$ matrices $B_i$ $(i=0,\dots,p-1)$ such that

$$\psi(x)=B_i\psi(px-i),\qquad x\in\left[\frac{i}{p},\frac{i+1}{p}\right],\tag{1.1}$$

where $\psi=[\psi_1,\dots,\psi_n]^T$. We will call $\psi$ a *fundamental curve in* $\mathbb{R}^n$ *associated with the matrices* $B_i$, $i=0,1,\dots,p-1$. Whenever the components of a curve $\psi$ are polynomials and span $\pi_j$, $j \in \mathbb{Z}$, then $\{\psi_1, \dots, \psi_n\}$ is certainly uniformly refinable.

The refinement of a basis has its geometric counterpart in the refinement of a *control polygon*. Thus, let us form a curve in $\mathbb{R}^d$ by some linear combination of $\psi_1, \dots, \psi_n$, say $\mathbf{c}^T\psi$, where $\mathbf{c} \in (\mathbb{R}^d)^n$. The vector $\mathbf{c}$ will be called the control polygon, and its components in $\mathbb{R}^d$ are referred to as *control points*. Visually, the control points are connected by line segments to form the control polygon. The control polygon then represents an initial representation of the shape of the desired smooth limiting curve $\mathbf{c}^T\psi$.

From (1.1) we have

$$\mathbf{c}^T\psi(x) = \mathbf{c}^T B_i \psi(px - i), \tag{1.2}$$

which means that

$$\mathbf{c}^T\psi\left[\frac{i}{p}, \frac{i+1}{p}\right] = \mathbf{c}^T B_i \psi[0,1] = \left(B_i^T\mathbf{c}\right)^T \psi[0,1], \tag{1.3}$$

or, in other words, the control polygon $B_i^T\mathbf{c}$ gives a representation of the curve segment $\mathbf{c}^T\psi(x)$, $x \in [i/p, (i+1)/p]$, $i = 0, 1, \dots, p-1$. Therefore, the new control polygons $B_0^T\mathbf{c}, \dots, B_{p-1}^T\mathbf{c}$ give a refined representation of the original curve determined by the vector $\mathbf{c}$.

These remarks indicate that a *subdivision method* (refining a control polygon) may be described either through how a fundamental curve $\psi$ can be refined [through the matrices $B_i$ and formula (1.1)] or through how the control polygon is refined [by means of the matrices $B_i^T$ and formula (1.3)]. This suggests two approaches to an analysis of uniform refinement. One may consider the sequence of control polygons generated by iterated refinement and ask whether they converge to a curve and what the nature of the limiting curve will be. This approach is motivated by algorithms of de Casteljau [2], Chaikin [4], Lane and Riesenfeld [9, 13], and de Rham [11, 12]. In these cases, it is known that the sequence of control polygons converges. Recent convergence properties of this type are given in de Boor [1] and Goldman and De Rose [8]. The second approach is to investigate whether there is a solution to the refinement equations (1.1) and to analyze its properties directly. We will study both approaches in this paper. Although they are interrelated, the second one proves to be more useful to us. As we shall see, requiring that refined control polygons converge to a continuous curve restricts the form of the matrices $B_i$. Nevertheless, the refinement of a control polygon is a frequently used tool for curve generation, and when it converges, it provides a computational means for solving the refinement equation.

## 2. THE SUBDIVISION ALGORITHM AND ITS CONVERGENCE

We consider a refinement algorithm given by $p$ $(n \times n)$ matrices $B_0, \dots, B_{p-1}$. For our convenience we will call the transposed matrices $A_0, \dots, A_{p-1}$ respectively, that is, $A_i = B_i^T$, $i = 0, 1, \dots, p-1$. Let $\psi : [0,1] \to \mathbb{R}^n$ be the corresponding fundamental curve, and assume that any sequence of refined control polygons converges to the curve it describes. This means that for any control polygon $\mathbf{c} \in (\mathbb{R}^d)^n$ the curve of interest $\mathbf{c}^T\psi$ is generated by the equation

$$\mathbf{c}^T\psi(x) = \mathbf{f}^T \lim_{l \to \infty} A_{x_l} \cdots A_{x_1}\mathbf{c}, \qquad \mathbf{f} \in \mathbb{R}^n \text{ with } \mathbf{f}^T\mathbf{e} = 1, \quad \mathbf{e} := [1, \dots, 1]^T, \tag{2.1}$$

where $(x_1, x_2, \dots) \in \mathbb{Z}_p^\infty$ is a $p$-adic expansion of $x \in [0,1]$, i.e.

$$x = \sum_{i=1}^{\infty} x_i p^{-i}. \tag{2.2}$$

Given some $n \times n$ matrices $A_0, \dots, A_{p-1}$, a main theme of this paper is to investigate whether the limits in (2.1) exist. Requiring convergence, in particular, means that all the limits $\lim_{l \to \infty} A_i^l$ $(i = 0, \dots, p-1)$ exist and must be matrices with equal rows. From this we obtain the fact that the matrices $A_0, \dots, A_{p-1}$ have only eigenvalues of moduli less than or equal to one, and one is the only eigenvalue of modulus one, which also must be simple.

For most subdivision algorithms used in curve and surface design, the refined control polygon lies within the convex hull of the original control polygon. To insure this property in the case at hand means that the matrices $A_0, \dots, A_{p-1}$ must be *stochastic* (i.e. have nonnegative entries with row sums one). Assuming that our matrices are in fact stochastic, we will give a necessary and sufficient condition on $A_0, \dots, A_{p-1}$ which insure that the limit in (2.1) exists.

THEOREM 2.1. *Let $A_0, \dots, A_{p-1}$ be $p$ $(n \times n)$ stochastic matrices. The sequence $A_{x_l} \cdots A_{x_1}\mathbf{c}$ converges to a multiple of the vector $\mathbf{e} = [1, \dots, 1]^T$ for all $p$-adic expansions $(x_1, x_2, \dots) \in \mathbb{Z}_p^\infty$ and for all control polygons $\mathbf{c} \in \mathbb{R}^n$ if and only if there exists an integer $k \leqslant 2^{n^2}$ such that the product $A_{y_1} \cdots A_{y_k}$ possesses a positive column for all $(y_1, \dots, y_k) \in \mathbb{Z}_p^k$.*

*Proof.* First we will prove that the condition stated guarantees convergence. To this end, we consider a control polygon $\mathbf{c} \in \mathbb{R}^n$ and a $p$-adic expansion $(x_1, x_2, \ldots) \in \mathbb{Z}_p^\infty$. For ease of exposition we introduce the notation

$$\mathbf{c}^l := A_{x_l} \cdots A_{x_1}\mathbf{c} \tag{2.3}$$

and observe that since $A_0, \ldots, A_{p-1}$ are stochastic,

$$\begin{aligned} \operatorname{diam} \mathbf{c}^l &:= \max_{1 \leqslant j \leqslant n} \mathbf{c}^l[j] - \min_{1 \leqslant j \leqslant n} \mathbf{c}^l[j] \\ &\leqslant \operatorname{diam} \mathbf{c}^{l-1}. \end{aligned} \tag{2.4}$$

For the proof of this inequality we bound the $j$th component of $\mathbf{c}^l$ by

$$\begin{aligned} \mathbf{c}^l[j] &= \sum_{i=1}^{n} A_{x_l}[j,i]\mathbf{c}^{l-1}[i] \\ &\leqslant \sum_{i=1}^{n} A_{x_l}[j,i] \max \mathbf{c}^{l-1} \\ &= \max \mathbf{c}^{l-1}. \end{aligned}$$

Thus we conclude

$$\max \mathbf{c}^l \leqslant \max \mathbf{c}^{l-1},$$

and similarly

$$\min \mathbf{c}^l \geqslant \min \mathbf{c}^{l-1},$$

which verifies (2.4). Now, we will show that there exists a number $\beta \in [0,1)$ such that for all $l \in \mathbb{N}$

$$\operatorname{diam} \mathbf{c}^{l+k} \leqslant \beta \operatorname{diam} \mathbf{c}^l. \tag{2.5}$$

By assumption $A_{x_{l+k}} A_{x_{l+1}}$ has a positive column, say $A_{x_{l+k}} \cdots A_{x_{l+1}}[\cdot, j] > 0$, where $j$ depends on $(x_{l+k}, \ldots, x_{l+1})$. Let $\alpha_i := \alpha_i(j, x_{l+k}, \ldots, x_{l+1}) =$

$A_{x_{l+k}} \cdots A_{x_{l+1}}[i, j]$, $i = 1, \ldots, n$. Because $A_{x_{l+k}} \cdots A_{x_{l+1}}$ is a stochastic matrix, it follows that for all $i = 1, \ldots, n$

$$\mathbf{c}^{k+l}[i] \leqslant \alpha_i \mathbf{c}^l[j] + (1 - \alpha_i)\max \mathbf{c}^l = \max \mathbf{c}^l - \alpha_i(\max \mathbf{c}^l - \mathbf{c}^l[j])$$

and hence

$$\max \mathbf{c}^{k+l} \leqslant \alpha \mathbf{c}^l[j] + (1 - \alpha)\max \mathbf{c}^l,$$

where $\alpha := \min\{\alpha_i(j, y_1, \ldots, y_k) \mid i \in \{1, \ldots, n\},\ (y_1, \ldots, y_k) \in \mathbb{Z}_p^k\}$. By the same argument we obtain

$$\min \mathbf{c}^{k+l} \geqslant \alpha \mathbf{c}^l[j] + (1 - \alpha)\min \mathbf{c}^l.$$

These last two inequalities prove (2.5) with $\beta = 1 - \alpha$. Now, the sufficiency part of the theorem follows from (2.4) and (2.5).

To prove the necessity of our condition we assume for some $(x_1, \ldots, x_k) \in \mathbb{Z}_p^k$ with $k = 2^{n^2}$ that $A_{x_k} \cdots A_{x_1}$ has no positive column. We observe first that $A_{x_l} \cdots A_{x_1}[\cdot, j] > 0$ implies $A_{x_{l+1}} \cdots A_{x_1}[\cdot, j] > 0$ and conclude from this fact that for all $l \leqslant k$, $A_{x_l} \cdots A_{x_1}$ has no positive column.

Let

$$\sigma(A)[i, j] := \begin{cases} 1 & \text{if} \quad A[i, j] > 0, \\ 0 & \text{if} \quad A[i, j] = 0 \end{cases}$$

be the signum matrix of some matrix $A$, and consider $\sigma(A_{x_l} \cdots A_{x_1})$, $l = 1, \ldots, k$. Because $A_{x_l} \cdots A_{x_1}$, $l = 1, \ldots, k$ are $n \times n$ matrices without a positive column, there are less than $k$ different signum matrices. Hence, there are numbers $\nu$ and $\mu$ such that $0 < \nu < \mu < k$ and

$$\sigma(A_{x_\nu} \cdots A_{x_1}) = \sigma(A_{x_\mu} \cdots A_{x_{\nu+1}} A_{x_\nu} \cdots A_{x_1}).$$

Let $A := A_{x_\mu} \cdots A_{x_{\nu+1}}$ and $B := A_{x_\nu} \cdots A_{x_1}$. Since $A$ and $B$ are stochastic matrices, we can conclude that

$$\sigma(B) = \sigma(AB) = \sigma(\sigma(A)\sigma(B)).$$

Therefore we also have

$$\begin{aligned}\sigma(B) &= \sigma(\sigma(A)\sigma(AB))\\ &= \sigma(A^2B) = \cdots\\ &= \sigma(A^lB), \qquad l \in \mathbb{N}.\end{aligned}$$

Thus all the matrices $A^lB$, $l \in \mathbb{N}$, have no positive column, and neither does $D := \lim_{l\to\infty} A^lB$ (we assume that $\lim_{l\to\infty} A^lB$ exists; otherwise the proof would already be finished). The only way for $D\mathbf{c}$ to be a multiple of the vector $\mathbf{e} = [1,\dots,1]^T$ for all $\mathbf{c}$ is for the columns of $D$ to be multiples of $\mathbf{e}$. Therefore, since $D$ is a stochastic matrix, at least one column is positive. This contradiction proves the result. ∎

We offer some useful conditions which insure that the hypothesis of Theorem 2.1 is satisfied. For this we use the following terminology. Suppose $C$ and $R$ are subsets of $\{1,\dots,n\}$. We say $C$ *connects* the rows of $A$ in $R$, denoted by $C \to A \to R$, provided that for every $i \in R$ there is a $j \in C$ such that $A[i, j] > 0$. (This is a familiar notion; cf. [14].)

It is easy to see that when $R \to A \to S$ and $S \to B \to T$, where $A$ and $B$ are nonnegative matrices, then $R \to BA \to T$, and conversely if $R \to BA \to T$, then there is an $S$ such that $R \to A \to S$ and $S \to B \to T$. From these remarks easily follows

PROPOSITION 2.2. *Given $(y_1,\dots,y_k) \in \mathbb{Z}_p^k$, the $j$th column of $A_{y_1}\cdots A_{y_k}$ is positive if and only if there exist sets*

$$C_0 = \{j\}, C_1,\dots,C_{k-1} \subseteq C_k = \{1,\dots,n\}$$

*such that*

$$C_l \to A_{y_l} \to C_{l-1}, \qquad l = 1,\dots,k.$$

*Proof.* If such sets exist, then

$$\{j\} \to A_{y_k} \to C_{k-1} \to A_{y_{k-1}} \to C_{k-2} \to \cdots \to C_1 \to A_{y_1} \to \{1,\dots,n\}, \quad (2.6)$$

and so $\{j\} \to A_{y_1} \cdots A_{y_k} \to \{1,\ldots,n\}$. Conversely, we can assert that there exists $C_1$ such that

$$\{j\} \to A_{y_1} \cdots A_{y_{k-1}} \to C_{k-1} \to A_{y_k} \to \{1,\ldots,n\}$$

and then inductively determine $C_2,\ldots,C_k$ satisfying (2.6). ∎

From this result we obtain the following corollaries.

COROLLARY 2.3. *Let $A_0,\ldots,A_{p-1}$ be nonnegative $n \times n$ matrices such that for each $i$, $0 \leqslant i \leqslant p-1$ and each set $R \subseteq \{1,\ldots,n\}$ of consecutive integers where $|R| := (\text{cardinality of } R) > 1$ there exists a set of consecutive integers $C$ such that $C \to A_{y_i} \to R$ and $|C| < |R|$. Then there exists a $k < n$ such that for any $(y_1,\ldots,y_k) \in \mathbb{Z}_p^k$, $A_{y_1} \cdots A_{y_k}$ has a positive column.*

*Proof.* For every $(y_1,\ldots,y_n) \in \mathbb{Z}_p^k$ there is an integer $m < n$ such that there are sets of consecutive integers such that

$$\{j\} \to A_{y_{n-m}} \to C_2 \to \cdots \to C_{n-m} \to A_{y_1} \to \{1,\ldots,n\}. \qquad (2.7)$$

The "chain" in (2.7) is created by starting with the set $\{1,\ldots,n\}$ and successively applying the hypothesis until termination at a singleton occurs. The integer $m$ clearly depends on $y_1,\ldots,y_n$. We choose the smallest $m$ that occurs in this procedure, and the integer $k = n - m$ provides the conclusion of the corollary. ∎

COROLLARY 2.4. *Suppose $A_0,\ldots,A_{p-1}$ are stochastic matrices such that for each matrix $A_l$, $0 \leqslant l \leqslant p-1$,*

$$A_l[i,i]A_l[i+1,i] > 0, \qquad i = 1,\ldots,n-1,$$

*or*

$$A_l[i-1,i]A_l[i,i] > 0, \qquad i = 2,\ldots,n.$$

*Then the hypothesis of Theorem 2.1 is satisfied.*

REMARK. Corollary 2.4 is only one of many circumstances in which the hypothesis of Theorem 2.1 is satisfied. For instance, if there exist consecutive integers $l,\ldots,m$, $m - l < n - 1$, such that $A_j[i, 1 + r] > 0$ for all $j = 0,\ldots,$

$p-1$, all $r=0,1,\dots,m-1$ and all $i$ with $k_r \leqslant i \leqslant k_{r+1}$, where $k_0 < \cdots < k_{m-l+1}$ are integers with $k_0 = 1$ and $k_{m-l+1} = n$, then Theorem 2.1 is applicable. In particular, in Corollary 2.4, either $l=1$, $m=n-1$ or $l=2$, $m=n$ with $k_i = i+1$, $i=0,l,\dots,n-1$.

## 3. THE LIMITING CURVE

Given $p$ matrices $A_0,\dots,A_{p-1}$ such that $A_{x_l}\cdots A_{x_1}$ converges for all $p$-adic expansions $(x_1,x_2,\dots)\in \mathbb{Z}_p^\infty$ to a matrix of equal rows, we define a function $\pi:\mathbb{Z}_p^\infty \to \mathbb{R}^n$ through the equation

$$\mathbf{f}^{\mathrm{T}} \lim_{l\to\infty} A_{x_l}\cdots A_{x_1} = \pi(x_1,x_2,\dots) \qquad \text{for all} \quad \mathbf{f}\in\mathbb{R}^n \quad \text{with } \mathbf{f}^T\mathbf{e}=1, \tag{3.1}$$

i.e., $\pi(x_1,x_2,\dots)$ equals the rows of $\lim_{l\to\infty} A_{x_l}\cdots A_{x_1}$. For our later use, we observe that

$$[\pi(x_1,x_2,\dots)]^{\mathrm{T}} = \lim_{l\to\infty} B_{x_1}\cdots B_{x_l}\mathbf{f}. \tag{3.2}$$

Notice that the matrices in (3.2) appear in opposite order to those in (3.1).

Next, we wish to introduce a fundamental curve by $\psi:[0,1]\to\mathbb{R}^n$ by associating with any $x\in[0,1]$ its $p$-adic expansion and setting

$$\psi(x) = \lim_{l\to\infty} B_{x_1}\cdots B_{x_l}\mathbf{f}, \tag{3.3}$$

where $x=\sum_{i=1}^\infty p^{-i}x_i$ and $\mathbf{f}^T\mathbf{e}=1$. For this definition to be meaningful we must resolve any ambiguities at points with more than one $p$-adic expansion. To this end, we establish

THEOREM 3.1. *Suppose that the limit for all $p$-adic expansions $(x_1,x_2,\dots)\in\mathbb{Z}_p^\infty$ in* (3.3) *exist, and let $\mathbf{f}_i := \lim_{l\to\infty} B_i^l\mathbf{f}$, $i=0,\dots,p-1$. A function $\psi:[0,1]\to\mathbb{R}^n$ can be defined by* (3.3) *if and only if $B_j\mathbf{f}_0 = B_{j-1}\mathbf{f}_{p-1}$ for all $j=1,\dots,p-1$.*

REMARK. It is worth emphasizing here that $f_i$ is an eigenvector of $B_i$ corresponding to the eigenvalue one, that $B_i\mathbf{f}_i = \mathbf{f}_i$, and that

$$\mathbf{f}_i = \psi\left(\frac{i}{p-1}\right).$$

Because of the remarks following (2.2), the eigenvectors $\mathbf{f}_0,\dots,\mathbf{f}_{p-1}$ are uniquely defined up to a constant. If the matrices $A_0,\dots,A_{p-1}$ are stochastic, then the components of each $\mathbf{f}_0,\dots,\mathbf{f}_{p-1}$ must sum to one.

*Proof.* The real numbers $x = \sum_{i=1}^{\infty} x_i p^{-i}$, $(x_1, x_2, \dots) \in \mathbb{Z}_p^{\infty}$, which do not have a unique $p$-adic expansion are those having two terminating $p$-adic expansions of the form $(x_1,\dots,x_l, j, 0, 0, \dots)$, $1 \leqslant j \leqslant p-1$, or $(x_1,\dots,x_l, j-1, p-1, p-1, \dots)$. Thus a function can be defined by (3.3) if and only if for all $(x_1,\dots,x_l) \in \mathbb{Z}_p^l$, $l \in \mathbb{N}$, the equation

$$\lim_{k\to\infty} B_{x_1}\cdots B_{x_l}B_jB_0^k\mathbf{f} = \lim_{k\to\infty} B_{x_1}\cdots B_{x_l}B_{j-1}B_{p-1}^k\mathbf{f} \tag{3.4}$$

holds. This is equivalent to

$$B_{x_1}\cdots B_{x_l}B_j\mathbf{f}_0 = B_{x_1}\cdots B_{x_l}B_{j-1}\mathbf{f}_{p-1}, \qquad (x_1,\dots,x_l) \in \mathbb{Z}_p^l,$$

or

$$B_j\mathbf{f}_0 = B_{j-1}\mathbf{f}_{p-1}, \tag{3.5}$$

which proves the result. ∎

The compatibility conditions in Theorem 3.1 have a strong consequence.

THEOREM 3.2. *Assume we have $p$ $(n \times n)$ stochastic matrices $A_0,\dots,A_{p-1}$ such that all limits in* (2.1) *exist and such that for the transposed matrices the equations $B_j\mathbf{f}_0 = B_{j-1}\mathbf{f}_{p-1}$ $(j = 1,\dots,p-1)$ hold. Then the fundamental curve $\psi:[0,1]\to\mathbb{R}^n$ defined by the association $\sum_{i=1}^{\infty} p^{-i}x_i \mapsto \lim_{l\to\infty} B_{x_1}\cdots B_{x_l}\mathbf{f}$ is continuous, and this curve is independent of $\mathbf{f} \in \mathbb{R}^n$ with $\mathbf{f}^T\mathbf{e} = 1$.*

*Proof.* We will show that $\psi$ is left side continuous. An analogous argument proves that $\psi$ is also right continuous. Trivially $\psi$ is left side continuous at zero. Therefore it is sufficient to consider some number $t \in (0,1]$ and its $p$-adic expansion which does not terminate with zero digits. We will call this expansion $(t_1, t_2, \dots)$. Let $\mathbf{c}$ be in $\mathbb{R}^n$. Since the assumptions of Theorem 2.1 are satisfied, we can conclude from (2.5) that for any $\varepsilon > 0$ there is an $L \in \mathbb{N}$ such that

$$\max \mathbf{c}^l - \min \mathbf{c}^l < \varepsilon, \qquad l \geqslant L. \tag{3.6}$$

Furthermore, from (2.4), we know that

$$\min \mathbf{c}^L \leqslant A_{x_{L+k}} \cdots A_{x_{L+1}} \mathbf{c}^L \leqslant \max \mathbf{c}^L \tag{3.7}$$

for all $(x_{L+1}, \dots, x_{L+k}) \in \mathbb{Z}_p^k$, $k \in \mathbb{N}$. Let $a := \sum_{i=1}^{L} t_i p^{-i}$ and $b := a + p^{-L} = a + \sum_{i=L+1}^{\infty} (p-1) p^{-i}$. Then we obtain from (3.7) for any $x \in [a, b]$ that $\min \mathbf{c}^L \psi(x) \leqslant \mathbf{c}^T \psi(x) \leqslant \max \mathbf{c}^L$. Because of (3.6) and the fact that $t \in (a, b]$, $\mathbf{c}^T \psi$ is left-side continuous at $t$. Since $t$ and the control polygon $\mathbf{c}$ were arbitrary, $\psi$ is left-side continuous over $[0,1]$. ∎

It is easy to check that a function $\psi$ defined by (2.1) or (3.3) satisfies the refinement equations (1.1). In the sequel, we will consider these equations in detail and study the questions of existence, uniqueness, and smoothness of solutions.

## 4. THE REFINEMENT EQUATIONS

Let $B_0, \dots, B_{p-1}$ be some $n \times n$ matrices and $\psi: [0,1] \to \mathbb{R}^n$ a solution to the refinement equations (1.1). Then $\psi$ is essentially defined by its value at any point $y \in [0,1]$. To see this, we first observe that through successive applications of the refinement equations we obtain

LEMMA 4.1. *Let $x \in [0,1]$ have the p-adic expansion $(x_1, x_2, \dots)$, and $\hat{x} \in [0,1]$ the p-adic expansion $(x_{l+1}, x_{l+2}, \dots)$. Then the equation*

$$\psi(x) = B_{x_1} \cdots B_{x_l} \psi(\hat{x})$$

*holds.*

From this lemma follows

COROLLARY 4.1. *Let $\psi$ and $\phi:[0,1]\to\mathbb{R}^n$ be two continuous solutions of the refinement equations* (1.1) *where $\psi(y)=\phi(y)$ for some $y\in[0,1]$. Then*

$$\psi(x)=\lim_{l\to\infty} B_{x_1}\cdots B_{x_l}\psi(y), \tag{4.1}$$

*where*

$$x=\sum_{i=1}^{\infty} x_i p^{-i},$$

*and*

$$\psi=\phi.$$

Furthermore, Corollary 4.2 also shows how one can determine the dimension of the linear space $\Psi := \{\mathbf{c}^T\psi \,|\, \mathbf{c}\in\mathbb{R}^n\}$ of all functions defined by a control polygon and the given subdivision method; namely, we have

COROLLARY 4.2. *Let S be the smallest common invariant subspace of $B_0,\dots,B_{p-1}$ that contains the vector $\psi(y)$ for some $y\in[0,1]$. Then*

$$\dim S=\dim\left\{\mathbf{c}^T\psi \middle| \mathbf{c}\in\mathbb{R}^n\right\}.$$

*Proof.* According to the refinement equation, the span of the fundamental curve is a common invariant subspace of $B_0,\dots,B_{p-1}$. By Corollary 4.1, if S is any invariant subspace containing $\psi(y)$, it contains the fundamental curve. Thus $S=\operatorname{span}\{\psi(x)\,|\,x\in[0,1]\}$. Since the dimension of the linear span of any curve $\psi$ in $\mathbb{R}^n$ is $\dim\{\mathbf{c}^T\psi\,|\,\mathbf{c}\in\mathbb{R}^n\}$, the result follows. ∎

We have already pointed out that any curve which is generated by the subdivision procedure (2.1) or (3.3) satisfies the functional equation. We give a partial converse to this result in

COROLLARY 4.3. *Let $\psi:[0,1]\to\mathbb{R}^n$ be a continuous curve satisfying the functional equation such that $\mathbf{e}^T\psi(x)=1$, $x\in[0,1]$, and $\dim\{\mathbf{c}^T\psi\,|\,\mathbf{c}\in\mathbb{R}^n\}=n$. Then the subdivision procedure* (2.1) *or* (3.3) *converges to $\psi(x)$.*

*Proof.* From Equation (4.1) it follows that

$$\lim_{l \to \infty} B_{x_1} \cdots B_{x_k} \mathbf{f} = \psi(x) \mathbf{e}^T \mathbf{f} \tag{4.2}$$

for all vectors $\mathbf{f} \in \operatorname{span}\{\psi(y) \mid y \in [0,1]\}$, which proves (3.1) and (2.1). ∎

## 5. CONTINUITY AND DIFFERENTIABILITY

First we will describe conditions which insure that there exists a continuous solution to the refinement equations (1.1).

THEOREM 5.1. *There is a continuous nontrivial solution* $\psi : [0,1] \to \mathbb{R}^n$ *for the refinement equations* (1.1) *if and only if*:

(a) *There are eigenvectors* $\mathbf{f}_i$ $(i = 0, \ldots, p-1)$ *such that* $B_i \mathbf{f}_i = \mathbf{f}_i$ *and* $B_{j-1} \mathbf{f}_{p-1} = B_j \mathbf{f}_0$, $j = 1, \ldots, p-1$.

*Let* $D := \{ B_{x_1} \cdots B_{x_l} \mathbf{f}_0 \mid (x_1, \ldots, x_l) \in \mathbb{Z}_p^l,\ l \in \mathbb{N} \}$, *and* $S$ *be the span of* $D$. *Suppose* $\|\cdot\|$ *is any norm on* $\mathbb{R}^n$. *Then there is a common invariant subspace* $S^1$ *of* $B_0, \ldots, B_{p-1}$ *with the following properties*:

(b) *Given any* $\alpha$, $0 < \alpha < 1$, *there exists a positive integer* $L$ *such that* $\|B_{x_1} \cdots B_{x_L} \mathbf{v}\| \leqslant \alpha \|\mathbf{v}\|$ *for all* $\mathbf{v} \in S^1$ *and* $(x_1, \ldots, x_L) \in \mathbb{Z}_p^L$.
(c) $S = S^1 \oplus \operatorname{span} \mathbf{f}_0$.
(d) $\mathbf{f}_i - \mathbf{f}_0 \in S^1$.

REMARK. Observe that $D = \operatorname{span}\{\psi(x) \mid x \in [0,1]\}$ and in addition D spans the minimal common invariant subspace of $B_0, \ldots, B_{p-1}$ that contains $\mathbf{f}_0$. Also, as we will see later, (c) and (d) are equivalent to the fact that all constant functions $\pi_0$ are contained in $\{\mathbf{c}^T \psi \mid \mathbf{c} \in \mathbb{R}^n\}$; cf. Theorem 7.3.

*Proof.* We begin the proof by showing (a)–(d) follow when the refinement equation has a nontrivial continuous solution. For this purpose, let

$$\mathbf{f}_i := \psi\left(\frac{i}{p-1}\right).$$

Then (a) follows from Lemma 4.1, Corollary 4.1, and Theorem 3.1 because of the equation

$$\psi(x) = \lim_{l \to \infty} B_{x_1} \cdots B_{x_l} \psi(y), \qquad x = (x_1, x_2, \dots). \tag{5.1}$$

Therefore any nontrivial continuous solution of the refinement equation is everywhere nonvanishing, in particular, at $i/(p-1)$, and so $\mathbf{f}_i \neq 0$, $i = 0, 1, \dots, p-1$.

In order to prove (b) through (d) we introduce the set $D^1 := \{\psi(x) - \psi(y) \mid x, y \in [0,1]\}$ and set $S^1 := \operatorname{span} D^1$. Invoking Lemma 4.1, we see that $S^1$ is a common invariant subspace of $B_0, \dots, B_{p-1}$. Now, to prove (b) we observe the following facts. Let $\varepsilon > 0$, and choose a $\delta > 0$ such that

$$\forall x, y \in [0,1]: \quad |x - y| < \delta \quad \Rightarrow \quad \|\psi(x) - \psi(y)\| < \varepsilon.$$

From Lemma 4.1 we obtain for all $\hat{x}, \hat{y} \in [0,1]$ and all $(x_1, \dots, x_L) \in \mathbb{Z}_p^L$ the equation

$$B_{x_1} \cdots B_{x_L}[\psi(\hat{x}) - \psi(\hat{y})] = \psi(x) - \psi(y),$$

where

$$x = \sum_{j=1}^{L} x_j p^{-j} + p^{-L}\hat{x} \quad \text{and} \quad y = \sum_{j=1}^{L} x_j p^{-j} + p^{-L}\hat{y}.$$

Assuming that $p^{-L} < \delta$ gives $|x - y| < \delta$ and so $\|B_{x_1} \cdots B_{x_L}[\psi(\hat{x}) - \psi(\hat{y})]\| < \varepsilon$.

Now, let $\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subset D^1$ be a basis of $S^1$, and introduce the norm $|\sum \nu_i \mathbf{v}_i| := \sum |\nu_i|$ on $S^1$. Recall that since all norms on $\mathbb{R}^n$ are equivalent, there is a constant $c$ such that $|\cdot| < c\|\cdot\|$. We define $\varepsilon := \alpha/c$ and let $L$ be as above. Then we compute the bounds

$$\begin{aligned} \left\| B_{x_1} \cdots B_{x_L}\left(\sum \nu_i \mathbf{v}_i\right) \right\| &\leqslant \sum |\nu_i| \, \|B_{x_1} \cdots B_{x_L} \mathbf{v}_i\| \\ &< \varepsilon \sum |\nu_i| < \varepsilon c \|\mathbf{v}\| \\ &= \alpha \|\mathbf{v}\|, \end{aligned}$$

which proves (b).

We will demonstrate that (c) holds by first observing that $S^1 \subset S = \operatorname{span} D$, and, because of (b), $\mathbf{f}_0 \in S \setminus S'$. It remains to prove $\dim S^1 \geqslant \dim S - 1$. For this purpose, choose a basis $\{\mathbf{u}_0, \ldots, \mathbf{u}_m\} \subset D$ for $S^1$. Then $\mathbf{u}_1 - \mathbf{u}_0, \ldots, \mathbf{u}_m - \mathbf{u}_0$ are linearly independent and contained in $D^1$, which proves (c). Finally, by definition

$$\mathbf{f}_i - \mathbf{f}_0 = \psi\left(\frac{i}{p-1}\right) - \psi(0) \in D^1 \subset S^1$$

and so (d) follows.

For the converse we assume (a) through (d) are valid. We define $\psi(0) := \mathbf{f}_0$, $\psi(1) := \mathbf{f}_{p-1}$, and at all other $p$-adic points ($x = \sum_{j=1}^{l} x_j p^{-j}$, $x_l \neq 0$) set $\psi(x) := B_{x_1} \cdots B_{x_l}\mathbf{f}_0$, which by (a) equals $B_{x_1} \cdots B_{x_l - 1}\mathbf{f}_{p-1}$. One easily verifies that $\psi$ (at $p$-adic points) satisfies the refinement equations (1.1) and the conclusion of Lemma 4.1 remains valid for $p$-adic points.

Next, we will prove that $\psi$ is *bounded*. For this purpose, we observe that for all $(x_1, \ldots, x_m) \in \mathbb{Z}_p^m$, $m \in \mathbb{N}$ the following equation holds:

$$\begin{aligned} B_{x_1} \cdots B_{x_m}\mathbf{f}_0 &= B_{x_1} \cdots B_{x_m} B_0 \cdots B_0 \mathbf{f}_0 = B_{x_1} \cdots B_{x_{kL}}\mathbf{f}_0 \\ &= P_1 \cdots P_k \mathbf{f}_0, \end{aligned}$$

where we set $x_i = 0$, $i > m$, and $P_i = B_{x_{(i-1)L+1}} \cdots B_{x_{iL}}$, $i = 1, \ldots, k$. Therefore, we get

$$B_{x_1} \cdots B_{x_m}\mathbf{f}_0 - \mathbf{f}_0 = \sum_{l=0}^{k-1} P_1 \cdots P_l (P_{l+1}\mathbf{f}_0 - \mathbf{f}_0).$$

Thus (b) allows us to estimate the sum by $M/(1-\alpha)$, where $M = \max\{\|B_{y_1} \cdots B_{y_L}\mathbf{f}_0 - \mathbf{f}_0\| \mid (y_1, \ldots, y_L) \in \mathbb{Z}_p^L\}$. Hence $\|\psi\| \leqslant M/(1-\alpha) + \|\mathbf{f}_0\|$ as claimed.

Now, we will show that $\psi$ is uniformly continuous at $p$-adic points. To this end, we choose $\varepsilon > 0$ and an integer $k \in \mathbb{N}$ such that $\alpha^k < \varepsilon/2\|\psi\|$. Now, consider any two $p$-adic numbers $x, y$ where $|x - y| < p^{-Lk}$. Then their $p$-adic expansions are identical for the first $kL$ digits. Thus, if we set $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, then $x_i = y_i$, $1 \leqslant i \leqslant kL$. Let $\hat{x}, \hat{y}$ be the numbers which have the $p$-adic expansion $(x_{kL+1}, x_{kL+2}, \ldots)$ and

$(y_{kL+1}, y_{kL+2}, \ldots)$, respectively. Then by our definition of $\psi$ on $p$-adic numbers there hold the inequalities

$$\begin{aligned}\|\psi(x) - \psi(y)\| &= \left\|B_{x_1} \cdots B_{x_{kL}}[\psi(\hat{x}) - \psi(\hat{y})]\right\| \\ &\leqslant \alpha^k \|\psi(\hat{x}) - \psi(\hat{y})\| \\ &\leqslant \varepsilon,\end{aligned}$$

which proves the uniform continuity of $\psi$ at all $p$-adic points. Thus it can be extended to a continuous function over $[0,1]$ that necessarily satisfies the refinement equations. This completes the proof. ■

Let $\psi: [0,1] \to \mathbb{R}^n$ be a differentiable solution to the refinement equations (1.1). Then the derivative $\psi'$ satisfies the refinement equations

$$\psi'(x) = pB_i\psi'(px - i), \quad x \in \left[\frac{i}{p}, \frac{i+1}{p}\right], \qquad i = 0, \ldots, p-1. \quad (5.2)$$

This is again a refinement equation for the matrices $pB_i$, $i = 0, 1, \ldots, p-1$. We will use this fact in combination with Theorem 5.1 to prove a condition on $B_0, \ldots, B_{p-1}$ equivalent to the existence of a $d$ times differentiable function that satisfies (1.1).

THEOREM 5.2. *There is a $d$ times continuously differentiable function $\psi: [0,1] \to \mathbb{R}^n$ (where $\psi^{(0)}, \ldots, \psi^{(d)}$ are not identically zero) solving the refinement equations* (4.1) *if and only if*

(a) *There are eigenvectors $\mathbf{f}_i^{(\delta)}$, $i = 0, \ldots, p-1$ and $\delta = 0, \ldots, d$, such that $p^\delta B_i \mathbf{f}_i^{(\delta)} = \mathbf{f}_i^{(\delta)}$ and $B_j \mathbf{f}_0^{(\delta)} = B_{j-1}\mathbf{f}_{p-1}^{(\delta)}$, $j = 1, \ldots, p-1$.*

*Let $D := \{B_{x_1} \cdots B_{x_l}\mathbf{f}_0 \mid (x_1, \ldots, x_l) \in \mathbb{Z}_p^l,\ l \in \mathbb{N}\}$, $S^{(0)} = \operatorname{span} D$, and suppose $\|\cdot\|$ is any norm on $\mathbb{R}^n$. Then there are common invariant subspaces $S^{(\delta+1)}$ $(\delta = 0, \ldots, d)$ of $B_0, \ldots, B_{p-1}$ such that:*

(b) *Given any $\alpha$, $0 < \alpha < 1$, there exists an integer $L$ such that $\|(p^\delta B_{x_1}) \cdots (p^\delta B_{x_L})\mathbf{v}\| < \alpha\|\mathbf{v}\|$ for all $\mathbf{v} \in S^{(\delta+1)}$ and $(x_1, \ldots, x_L) \in \mathbb{Z}_p^L$.*

(c) $S^{(\delta)} = S^{(\delta+1)} \oplus \operatorname{span} \mathbf{f}_0^{(\delta)}$, $\delta = 0, 1, \ldots, d$.

(d) $\mathbf{f}_i^{(\delta)} - \mathbf{f}_0^{(\delta)} \in S^{(\delta+1)}$, $\delta = 0, 1, \ldots, d$.

(e) $\mathbf{f}_i^{(\delta)} \in \mathbf{f}_0^{(\delta)} + \dfrac{i}{p-1}\mathbf{f}_0^{(\delta+1)} + S^{(\delta+2)}$, $\delta = 0, 1, \ldots, d-1$.

REMARK. As we will see later, (d) and (e) are equivalent to the fact that all polynomials of degree $\pi_d$ are contained in $\{\mathbf{c}^T\psi \mid \mathbf{c} \in \mathbb{R}^n\}$; cf. Theorem 7.3.

*Proof.* It is sufficient to prove the theorem for $d = 1$, since the general case will follow from repeated applications of the remarks in the paragraph preceding Theorem 5.2.

First, we assume the refinement equation has a differentiable solution $\psi$. Then $\psi'$ satisfies the refinement equations (5.2), and from Theorem 5.1 we obtain common invariant subspaces $S^{(0)}$, $S^{(1)}$, $V^{(0)}$, and $V^{(1)}$ of $B_0, \ldots, B_{p-1}$ satisifying the conditions

$$S^{(0)} = \operatorname{span}\{\psi(x) \mid x \in [0,1]\}, \qquad S^{(1)} = \operatorname{span}\{\psi(x) - \psi(y)\ x, y \in [0,1]\},$$

$$S^{(0)} = S^{(1)} \oplus \operatorname{span} \mathbf{f}_0^{(0)},$$

and

$$V^{(0)} = \operatorname{span}\{\psi'(x) \mid x \in [0,1]\},$$

$$V^{(1)} = \operatorname{span}\{\psi'(x) - \psi'(y) \mid x, y \in [0,1]\},$$

$$V^{(0)} = V^{(1)} \oplus \operatorname{span} \mathbf{f}_0^{(1)}.$$

Since

$$\operatorname{span}\{\psi'(x) \mid x \in [0,1]\} = \operatorname{span}\{\psi(x) - \psi(y) \mid x, y \in [0,1]\} = S^{(1)}, \quad (5.3)$$

it follows that $V^{(0)} = S^{(1)}$, and therefore, by setting $S^{(2)} = V^{(1)}$, (a) through (d) are established.

To prove (e) we express $\psi$ as

$$\psi(x) = \psi(0) + x\psi'(0) + \mathbf{d}_2(x), \quad (5.4)$$

where $\lim_{x \to 0}(1/x)\mathbf{d}_2(x) = 0$, which is a consequence of the fact that $\psi \in C^1[0,1]$. In particular, for $x = i/(p-1)$ we get

$$\mathbf{f}_i = \psi\left(\frac{i}{p-1}\right) = \mathbf{f}_0 + \frac{i}{p-1}\mathbf{f}_0^{(1)} + \mathbf{d}_2\left(\frac{i}{p-1}\right), \quad (5.5)$$

and from the refinement equations follows

$$\psi\left(p^{-l}\frac{i}{p-1}\right)=B_0^l\mathbf{f}_i=\mathbf{f}_0+p^{-l}\frac{i}{p-1}\mathbf{f}_0^{(1)}+B_0^l\mathbf{d}_2\left(\frac{i}{p-1}\right).$$

Thus comparing with (5.4), we get

$$B_0^l\mathbf{d}_2\left(\frac{i}{p-1}\right)=\mathbf{d}_2\left(p^{-l}\frac{i}{p-1}\right)$$

and so

$$\lim_{l\to\infty}p^lB_0^l\mathbf{d}_2\left(\frac{i}{p-1}\right)=0.$$

Furthermore, since $\mathbf{f}_i-\mathbf{f}_0\in S^{(1)}$ and $S^{(1)}=S^{(2)}\oplus\operatorname{span}\mathbf{f}_0^{(1)}$, we conclude from (5.5) that

$$\mathbf{d}_2\left(\frac{i}{p-1}\right)=\mathbf{v}_i+\lambda_i\mathbf{f}_0^{(1)}\qquad\text{for some}\quad \mathbf{v}_i\in S^{(2)}\text{ and }\lambda_i\in\mathbb{R}.$$

Thus from (b) (for $\delta=1$) we get $\lambda_i=0$ and so

$$\mathbf{d}_2\left(\frac{i}{p-1}\right)\in S^{(2)}.$$

Therefore, (5.5) implies (e).

For the proof of the converse, we assume (a) through (e). Thus from Theorem 5.1 we know that there are two continuous functions $\psi:[0,1]\to S^{(0)}$ and $\chi:[0,1]\to S^{(1)}$ solving the functional equations

$$\psi(x)=B_i\psi(px-i),\quad x\in\left[\frac{i}{p},\frac{i+1}{p}\right],\qquad i=0,\dots,p-1,$$

$$\psi(0)=\mathbf{f}_0$$

and

$$\chi(x)=pB_i\chi(px-i),\quad x\in\left[\frac{i}{p},\frac{i+1}{p}\right],\qquad i=0,\dots,p-1,$$

$$\chi(0)=\mathbf{f}_0^{(1)}.$$

As we have already pointed out, $\psi$ maps into $S^{(0)}$ because of Corollary 4.2 and the fact that $S^{(0)}$ is invariant under $B_0,\ldots,B_{p-1}$, and by the analogous argument $\chi$ maps onto $S^{(1)}$. We will show that $\psi$ is differentiable and $\psi'=\chi$. First we will prove that

$$\psi(x)=\mathbf{f}_0+x\mathbf{f}_0^{(1)}+\mathbf{d}_2(x) \tag{5.6}$$

holds for some continuous function $\mathbf{d}_2$ such that $\mathbf{d}_2:[0,1]\to S^{(2)}$.

For the proof of (5.6) we observe that since $\psi$ maps into $S^{(0)}$ and $S^{(0)}=\operatorname{span}\mathbf{f}_0\oplus\operatorname{span}\mathbf{f}_0^{(1)}\oplus S^{(2)}$, there are three functions $\lambda_0,\lambda_1:[0,1]\to\mathbb{R}$ and $\mathbf{d}_2:[0,1]\to S^{(2)}$ such that

$$\psi(x)=\lambda_0(x)\mathbf{f}_0+\lambda_1(x)\mathbf{f}_0^{(1)}+\mathbf{d}_2(x). \tag{5.7}$$

These functions are continuous because of the uniqueness of the representation (5.7). Specifically, choose $\mathbf{c},\mathbf{v}\in\mathbb{R}^n$, where $\mathbf{c}^T\mathbf{f}_0=\mathbf{v}^T\mathbf{f}_0^{(1)}=1$, $\mathbf{c}\perp S^{(1)}$, and $\mathbf{v}\perp S^{(2)}$. Then it follows that

$$\lambda_0(x)=\mathbf{c}^T\psi(x),$$

$$\lambda_1(x)=\mathbf{v}^T\psi(x)-\lambda_0(x)\mathbf{v}^T\mathbf{f}_0,$$

and

$$\mathbf{d}_2(x)=\psi(x)-\lambda_0(x)\mathbf{f}_0-\lambda_1(x)\mathbf{f}_0^{(1)},$$

which establishes the continuity of $\lambda_0(x)$, $\lambda_1(x)$, and $\mathbf{d}_2(x)$.

Furthermore, the planar curve $\lambda(x)=[\lambda_0(x),\lambda_1(x)]^T$ satisfies the functional equation

$$\lambda(x)=\begin{bmatrix}1 & 0\\ i/p & 1/p\end{bmatrix}\lambda(px-i),\quad i=0,\ldots,p-1,\qquad x\in\left[\frac{i}{p},\frac{i+1}{p}\right], \tag{5.8}$$

with the initial condition

$$\lambda(0)=\begin{bmatrix}1\\0\end{bmatrix}. \tag{5.9}$$

The latter claim (5.9) follows immediately from the defining equation (5.7). On the other hand, the proof of (5.8) makes use of the refinement equations (1.1) and assumptions (d) and (e), which give (for $\delta = 0$ and $\delta = 1$, respectively)

$$\begin{aligned}\psi(x) &= B_i \psi(px - i)\\ &= B_i\big(\lambda_0(px-i)\mathbf{f}_0 + \lambda_1(px-i)\mathbf{f}_0^{(1)} + \mathbf{d}_2(px-i)\big)\\ &= B_i\left(\lambda_0(px-i)\left(\mathbf{f}_i - \frac{i}{p-1}\mathbf{f}_i^{(1)}\right) + \lambda_1(px-i)\mathbf{f}_i^{(1)}\right) + \mathbf{s}_2,\end{aligned}$$

where $\mathbf{s}_2$ is some element in $S^{(2)}$. Simplifying further, we get

$$\begin{aligned}\psi(x) &= \lambda_0(px-i)\mathbf{f}_i + \frac{1}{p}\left(\lambda_1(px-i) - \frac{i}{p-1}\lambda_0(px-i)\right)\mathbf{f}_i^{(1)} + \mathbf{s}_2\\ &= \lambda_0(px-i)\mathbf{f}_0 + \left(\frac{i}{p}\lambda_0(px-i) + \frac{1}{p}\lambda_1(px-i)\right)\mathbf{f}_0^{(1)} + \mathbf{s}_2.\end{aligned}$$

Comparing this equation with (5.7), we see that (5.8) follows directly. Since obviously $\mu(x) = [1, x]^T$ is also a solution of (5.8) and (5.9), we can invoke Corollary 4.2 to conclude that $\lambda = \mu$, thereby proving (5.6).

Now, we are ready to show that the difference quotient

$$\delta(x, y) := \frac{\psi(y) - \psi(x)}{y - x}, \qquad y \neq x, \quad x, y \in [0, 1],$$

approaches $\chi(x)$ as $y$ tends to $x$ where $x$ is fixed. For the proof let $\varepsilon$ be any small positive number. Because $\chi$ is uniformly continuous in $[0,1]$, there is a $\gamma > 0$ such that $|y - x| < \gamma$ implies $|\psi(y) - \psi(x)| < \varepsilon/2$.

According to (b) we can choose an integer $L$ such that $\|p^L B_{z_1} \cdots B_{z_L}\mathbf{v}\| < \frac{1}{2}\|\mathbf{v}\|$ for all $(z_1, \ldots, z_L) \in \mathbb{Z}_p^L$ and all $\mathbf{v} \in S^{(2)}$. Because $\mathbf{d}_2$ is continuous, the set $\{\|B_{z_1} \cdots B_{z_l}[\mathbf{d}(x) - \mathbf{d}(y)]\|, \ x, y \in [0,1], \ (z_1, \ldots, z_l) \in Z_p^l, \ l < L\}$ is bounded by a number $M$. We then choose $m \in \mathbb{N}$ such that

$$2^{-m}M < \frac{\varepsilon}{2p}.$$

Now, let $|y-x| < \min\{\gamma, p^{-mL}\}$ and define $k \geqslant mL$ by

$$p^{-k-1} < |y-x| \leqslant p^{-k}.$$

We then distinguish two cases.

(1) First we assume there are $p$-adic expansions $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ of $x$ and $y$ respectively with $(x_1, \ldots, x_k) = (y_1, \ldots, y_k)$. Then follows from Lemma 4.1 that

$$\delta(x,y) = p^k B_{x_1} \cdots B_{x_k} \frac{\psi(\bar{y}) - \psi(\bar{x})}{\bar{y} - \bar{x}},$$

where

$$\bar{x} = \sum_{i=1}^{\infty} p^{-i} x_{i+k}$$

and

$$\bar{y} = \sum_{i=1}^{\infty} p^{-i} y_{i+k}.$$

Therefore, using (5.6) we obtain

$$\delta(x,y) = p^k B_{x_1} \cdots B_{x_k}\left(\mathbf{f}_0^{(1)} + \frac{\mathbf{d}_2(\bar{y}) - \mathbf{d}_2(\bar{x})}{\bar{y} - \bar{x}}\right). \tag{5.10}$$

Now, recall that by construction $\chi(\hat{x}) = p^k B_{x_1} \cdots B_{x_k} \mathbf{f}_0^{(1)}$, where $\hat{x} = \sum_{i=1}^{k} x_i p^{-i}$. So we get

$$\|\delta(x,y) - \chi(x)\| \leqslant \|\chi(\hat{x}) - \chi(x)\| + \left\| p^k B_{x_1} \cdots B_{x_k} \frac{\mathbf{d}_2(\bar{y}) - \mathbf{d}_2(\bar{x})}{\bar{y} - \bar{x}} \right\|$$

and because $|\hat{x} - x| < \gamma$, $1/(\bar{y} - \bar{x}) < p$, and $k \geqslant mL$ we can conclude $|\delta(x,y) - \chi(x)| < \varepsilon$.

(2) Secondly we consider the case where $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ are $p$-adic expansions of $x$ and $y$ respectively with $(x_1, \ldots, x_k) \neq (y_1, \ldots, y_k)$.

Without loss of generality we may assume $x < y$. Because $p^{-k-1} < y - x \leqslant p^{-k}$, the number

$$z := \sum_{i=1}^{k} x_i p^{-i} + p^{-k}$$

has both $(x_1, \ldots, x_k, (p-1), (p-1), \ldots)$ and $(y_1, \ldots, y_k, 0, 0, \ldots)$ as dyadic expansions, and case (1) can be applied to $\delta(x, z)$ and $\delta(z, y)$. Since

$$\delta(x, y) = \frac{z - x}{y - x}\delta(x, z) + \frac{y - z}{y - x}\delta(z, y),$$

we also get

$$\|\delta(x, y) - \chi(x)\| \leqslant \frac{z - x}{y - x}\|\delta(x, z) - \chi(z)\| + \frac{y - z}{y - x}\|\delta(z, y) - \chi(z)\|$$
$$+ \|\chi(x) - \chi(z)\|,$$

which can be estimated as in case (1) by

$$\|\delta(x, y) - \chi(x)\| < 2\varepsilon.$$

Thus we have shown that

$$\lim_{y \to x} \frac{1}{y - x}[\psi(y) - \psi(x)]$$

exists and equals $\chi(x)$, which proves the theorem. ∎

## 6. SOME CONSEQUENCES

In this section, we examine some consequences of Theorem 5.2. We will show that smooth solutions $\psi$ to the refinement equation (1.1) must have polynomial components. Specifically we establish the following corollaries.

COROLLARY 6.1. *Let $\psi$ be a solution of the refinement equations* (1.1). *We define $\Psi = \{\mathbf{c}^T\psi \mid \mathbf{c} \in \mathbb{R}^n\}$ and $m := \dim \Psi$. If $\psi \in C^l[0,1]$, then it follows that $\pi_l \subseteq \Psi$ provided that $l < m$, and otherwise $\pi_{m-1} = \Psi$.*

*Proof.* We conclude from the proof of Theorem 5.2 that $\psi$ has a Taylor expansion of the form

$$\psi(x) = \mathbf{f}_0 + x\mathbf{f}_0^{(1)} + \cdots + \frac{x^l}{l!}\mathbf{f}_0^{(l)} + \mathbf{d}_{l+1}(x),$$

where $\mathbf{d}_{l+1}:[0,1] \to S^{(l+1)}$. Given constants $p_0, \dots, p_l$, there is a control polygon $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^T\mathbf{f}_0^{(\delta)} = \delta! p_\delta$, $\delta = 0, 1, \dots, l$, and $\mathbf{c}$ is orthogonal to $S^{(l+1)}$. This implies that $\mathbf{c}^T\psi(x) = p_0 + p_1 x + \cdots + p_l x^l$, which completes the proof.

For the second claim, recall that $\dim S^{(0)} = m$. Thus we further conclude from Theorem 5.2 that $\dim S^{(1)} = m - 1, \dots, \dim S^{(m)} = 0$, and thus $\mathbf{f}_0^{(m)} = \cdots = \mathbf{f}_0^{(l)} = \mathbf{d}_{l+1} = \mathbf{0}$. Therefore $\psi$ must be the polynomial

$$\mathbf{f}_0 + x\mathbf{f}_0^{(1)} + \cdots + \frac{x^{m-1}}{(m-1)!}\mathbf{f}_0^{(m-1)}. \qquad \blacksquare$$

There is, in fact, a stronger version of this corollary, which we describe next.

COROLLARY 6.2. *Suppose that there is a control polygon* $\mathbf{b} \in \mathbb{R}^n$ *where* $\mathbf{b}^T\psi \in C^d[0,1]$ *and* $\mathbf{b}^T\psi^{(d)}$ *is not identically zero. Then* $\pi_d \subseteq \Psi = \{\mathbf{c}^T\psi \,|\, \mathbf{c} \in \mathbb{R}^n\}$.

*Proof.* Let $M := \operatorname{span}\{\mathbf{b}^T B_{x_1} \cdots B_{x_l})^T \,|\, (x_1, \dots, x_l) \in \mathbb{Z}_p^l,\ l \in \mathbb{N}\}$. Note that, because of (1.3), all control polygons in $M$ define a differentiable function; i.e., $\mathbf{a}^T\psi \in C^d[0,1]$ for all $\mathbf{a} \in M$. Furthermore, M is an invariant subspace of $B_0^T, \dots, B_{p-1}^T$. Therefore, a restriction of the refinement equations (1.1) to M can be considered. We state this observation as a lemma, since it will be used often.

LEMMA 6.3. *Let $C$ be a common invariant subspace of $B_0^T, \dots, B_{p-1}^T$, and $m$ its dimension. We introduce subspace coordinates in $C$ by choosing an $m \times n$ matrix $P$ of rank $m$ and an $n \times m$ matrix $Q$ whose range is $C$ such that* $QP\mathbf{d} = \mathbf{d}$, $\mathbf{d} \in C$. *Furthermore, we define* $\hat{B}_k = Q^T B_i P^T$, $i = 0, \dots, p-1$, *and* *introduce the curve* $\hat{\psi} = Q^T\psi$. *Then* $\hat{\psi}$ *is a fundamental curve in* $\mathbb{R}^m$ *and satisfies the refinement equations*

$$\hat{\psi}(x) = \hat{B}_i\hat{\psi}(px - i), \quad x \in \left[\frac{i}{p}, \frac{i+1}{p}\right], \qquad i = 0, \dots, p-1. \tag{6.1}$$

*Proof.* First of all we obtain from (1.1) that

$$\hat{\psi}(x) = Q^T B_i \psi(px - i) \qquad \text{for} \quad x \in \left[\frac{i}{p}, \frac{i+1}{p}\right]. \tag{6.2}$$

Since $B_i^T Q(\mathbb{R}^m) \subseteq C$, we have $QPB_i^TQ = B_i^TQ$, or equivalently $Q^TB_i = Q^TB_iP^TQ^T$. Hence, we get from (6.2)

$$\begin{aligned}\hat{\psi}(x) &= Q^TB_iP^TQ^T\psi(px - i)\\ &= \hat{B}_i\hat{\psi}(px - i),\end{aligned}$$

which proves the lemma. ∎

Returning to the proof of Corollary 6.2, we choose any $P, Q$ as above in Lemma 6.3 relative to the invariant subspace $M$ defined earlier. Also, let $\hat{\mathbf{a}} = P\mathbf{a}$; then

$$\hat{\mathbf{a}}^T\hat{\psi} = (QP\mathbf{a})^T\psi = \mathbf{a}^T\psi, \qquad \mathbf{a} \in M.$$

Hence, in particular, $\hat{\psi} \in C^d[0,1]$, since the range of $Q$ is $M$. For the same reason $\hat{\psi}^{(d)} \neq \mathbf{0}$, since otherwise $\mathbf{b}^T\psi^{(d)} = \mathbf{0}$, which would contradict the hypothesis. ∎

Remark. For subsequent application of Lemma 6.3 we always set $P = Q^T$ and choose $Q$ so that $QQ^T$ is the orthogonal projection of $\mathbb{R}^n$ onto $C$ and $Q^TQ = I$. In this case, we can see that the range of $Q$ is a common invariant subspace of $B_0^T, \dots, B_{p-1}^T$ if and only if $QQ^TB_iQ = B_i^TQ$, $i = 0, 1, \dots, p-1$.

There are two interesting consequences of Corollary 6.2, namely, (i) if $\{\mathbf{c}^T\psi \mid \mathbf{c} \in \mathbb{R}^n\}$ contains at least one polynomial of exact degree $d$, then it also contains all polynomials up to degree $d$, and consequently (ii) under no circumstances does $\{\mathbf{c}^T\psi \mid \mathbf{c} \in \mathbb{R}^n\}$ contain a polynomial of exact degree $n$ or higher, because the dimension of $\{\mathbf{c}^T\psi \mid \mathbf{c} \in \mathbb{R}\}$ cannot be higher than $n$.

## 7. POLYNOMIAL COMPONENTS

In this section we further explore the theme of polynomial components in $\{\mathbf{c}^T\psi \mid \in \mathbb{R}^n\}$. It is our goal to provide necessary and sufficient conditions on $B_0, \ldots, B_{p-1}$ to insure that $\pi_d \subseteq \{\mathbf{c}^T\psi \mid \mathbf{c} \in \mathbb{R}^n\}$. We accomplish this by a series of successively more general results.

THEOREM 7.1. *There exists a solution $\psi:[0,1] \to \mathbb{R}^n$ of the refinement equations* (1.1) *where the components of $\psi$ span $\pi_{n-1}$ if and only if there are $p$ $(n \times n)$ matrices $U_i$ $(i = 0, \ldots, p-1)$ such that*

$$B_i = U_i D_n U_i^{-1} \quad \text{and} \quad U_i = U_0 T_n\left(\frac{i}{p-1}\right),$$

*where*

$$D_n := \operatorname{diag}\left(1, \frac{1}{p}, \ldots, \frac{1}{p^{n-1}}\right) \quad \text{and} \quad (T_n(a))_{ij} := (-a)^{i-j}\binom{i}{j},$$

$$i, j = 0, \ldots, n-1.$$

*Proof.* Let

$$\beta(x) := [1, x, \ldots, x^{n-1}]^T,$$

and consider first the possibility that $\pi_{n-1} = \{\mathbf{c}^T\psi \mid \mathbf{c} \in \mathbb{R}^n\}$. Then there exists a nonsingular $n \times n$ matrix $P$ such that $\psi(x) = P\beta(x)$. Inserting this equation into (1.1) when $i = 0$ yields the identity

$$P\beta(x) = B_0 P\beta(px),$$

or equivalently

$$\beta(x) = P^{-1}B_0 P\beta(px).$$

From this we conclude that

$$P^{-1}B_0 P = D_n,$$

which means

$$B_0 P = P D_n.$$

This suggests setting

$$U_i = P T_n\left(\frac{i}{p-1}\right),$$

so that we have shown $B_0 = U_0^{-1} D_n U_0$. It remains to show that $B_i = U_i^{-1} D_n U_i$ for all $i = 1, \ldots, p-1$. To this end, one verifies easily that $\beta(x-a) = T(a)\beta(x)$ for any real number $a$. Thus, in particular, we obtain

$$\psi(x) = U_i \beta\left(x - \frac{i}{p-1}\right).$$

Inserting this equation into (1.1) yields for all $x \in \mathbb{R}$

$$U_i \beta\left(x - \frac{i}{p-1}\right) = B_i U_i \beta\left(p\left(x - \frac{i}{p-1}\right)\right),$$

or equivalently, for all $y \in \mathbb{R}$,

$$\beta(y) = U_i^{-1} B_i U_i \beta(py),$$

which yields $U_i^{-1} B_i U_i = D_n$ thereby completing the proof of the necessity of the conditions on $B_0, \ldots, B_{p-1}$.

To prove the sufficiency of the theorem we will assume that the conditions on $B_0, \ldots, B_{p-1}$ hold. Since

$$\beta\left(x - \frac{i}{p-1}\right) = D_n \beta\left(p\left(x - \frac{i}{p-1}\right)\right)$$

we obtain by our assumption that

$$\beta\left(x - \frac{i}{p-1}\right) = U_i^{-1} B_i U_i \beta\left(p\left(x - \frac{i}{p-1}\right)\right),$$

which simplifies to

$$U_i T\left(\frac{i}{p-1}\right)\beta(x) = B_i U_i T\left(\frac{i}{p-1}\right)\beta(px-i).$$

Letting $\psi(x) := U_0\beta(x)$ finally gives

$$\psi(x) = B_i\psi(px-i), \quad x \in \left[\frac{i}{p}, \frac{i+1}{p}\right], \qquad i = 0,\dots,p-1,$$

which completes the proof. ∎

REMARK. We observe that when the matrices satisfy the conditions of Theorem 7.1 and $\mathbf{e}^T\psi \equiv 1$, the subdivision scheme converges because of Corollary 4.3.

Next, we will prove a stronger version of Theorem 7.1, namely,

THEOREM 7.2. *Assume that* $\psi: [0,1] \to \mathbb{R}^n$ *is a continuous solution of the refinement equations* (1.1) *where* $\dim\{\mathbf{c}^T\psi \,|\, \mathbf{c} \in \mathbb{R}^n\} = n$. *Then* $\pi_{d-1} \subseteq \{\mathbf{c}^T\psi \,|\, \mathbf{c} \in \mathbb{R}^n\}$ *if and only if there are* $p$ $(d \times d)$ *matrices* $V_0, \dots, V_{p-1}$ *and an* $n \times d$ *matrix* $Q$ *of rank* $d$ *such that*

$$Q^T B_i Q = V_i D_d V_i^{-1}, \qquad i = 0,1,\dots,p-1, \tag{i}$$

$$Q^T B_i Q Q^T \mathbf{v} = Q^T B_i \mathbf{v}, \qquad i = 0,1,\dots,p-1, \quad \mathbf{v} \in \mathbb{R}^n, \tag{ii}$$

*and*

$$V_i = V_0 T_d\left(\frac{i}{p-1}\right), \qquad i = 0,1,\dots,p-1. \tag{iii}$$

*Proof.* Let $C := \{\mathbf{c} \,|\, \mathbf{c}^T\psi \in \pi_{d-1},\ \mathbf{c} \in \mathbb{R}^n\}$. Obviously $C$ is a linear subspace whose dimension equals $d$; otherwise, there would be two different control polygons $\mathbf{a}$ and $\mathbf{b}$ where $\mathbf{a}^T\psi = \mathbf{b}^T\psi$, and this would contradict the assumption that $\dim\{\mathbf{c}^T\psi \,|\, \mathbf{c} \in \mathbb{R}^n\} = n$. Also, C is a common invariant subspace of $B_0^T, \dots, B_{p-1}^T$ because of (1.3). Therefore, we can use the remark following Lemma 6.3 and obtain a fundamental curve $\hat{\psi} = Q^T\psi$ on $\mathbb{R}^d$ which satisfies the functional equation (1.1) for the matrices $\hat{B}_i = Q^T B_i Q$, $i = 0, 1, \ldots, p-1$. Also, since $\operatorname{span}\{\mathbf{b}^T \psi \mid b \in \mathbb{R}^d\} = \operatorname{span}\{(Q\mathbf{b})^T \psi \mid \mathbf{b} \in \mathbb{R}^d\} = \dim\{\mathbf{c}^T \psi \mid \mathbf{c} \in C\} = \pi_{d-1}$, we can apply Theorem 7.1 (for $n = d$) and obtain $p$ $(d \times d)$ matrices $V_i$, $i = 0, \ldots, p-1$, such that

$$\hat{B}_i = V_i D_d V_i^{-1} \quad \text{and} \quad V_i = V_0 T_d\left(\frac{i}{p-1}\right).$$

This gives us equations (i) and (iii); Equation (ii) comes from the fact that $C$ is a common invariant subspace of $B_i^T$, $i = 0, 1, \ldots, p-1$.

The converse is also a consequence of Theorem 7.1, and the reasoning is as follows: Using (i) and (iii), we conclude from Theorem 7.1 that the functional equation (1.1) corresponding to the matrices $\hat{B}_i = Q^T B_i Q$, $i = 0, 1, \ldots, p-1$, has a solution $\phi$ in $\mathbb{R}^d$ which spans $\pi_{d-1}$. However, by (ii), $Q^T \psi$ also solves the same functional equation, as was shown in Lemma 6.3. We conclude from (i) that $Q^T B_0 Q$ has only one eigenvector corresponding to eigenvalue one. Therefore, since $\phi(0) \neq \mathbf{0}$ and $\psi(0) \neq \mathbf{0}$, we conclude by Corollary 4.2 that there is a $\lambda \in \mathbb{R} \setminus \{0\}$ such that $Q^T \psi = \lambda \phi$. Therefore it follows that $\pi_{d-1} \subseteq \operatorname{span}\{\mathbf{c}^T \psi \mid \mathbf{c} \in \mathbb{R}^n\}$, which finishes the proof. ∎

Finally, we will present the most general version of Theorem 7.1 available.

THEOREM 7.3. *Assume that* $\psi : [0,1] \to \mathbb{R}^n$ *is a nontrivial solution of the refinement equations* (1.1). *Let* $S := \operatorname{span} \psi[0,1]$ *be the smallest common invariant subspace of* $B_0, \ldots, B_{p-1}$ *containing* $\psi(0)$, *and set* $m = \dim S$. *Then* $\pi_{d-1} \subseteq \{\mathbf{c}^T \psi \mid \mathbf{c} \in \mathbb{R}^n\}$ *if and only if there are* $p$ $(d \times d)$ *matrices* $V_i$, $i = 0, \ldots, p-1$, *and an* $n \times d$ *matrix* $Q$ *of rank* $d$ *such that Equations* (i)–(iii) *of Theorem* 7.2 *hold, but* (ii) *for* $\mathbf{v} \in S$ *only, and* $\operatorname{range} Q \subseteq S$.

*Proof.* Let us first show that (i)–(iv) imply that the span of the fundamental curve $\psi$ contains $\pi_{d-1}$. Use the same argument employed in Theorem 7.2 to obtain a fundamental curve $\phi$ in $\mathbb{R}^d$ whose span contains $\pi_{d-1}$ such that for some $\lambda \in \mathbb{R}$, $Q^T \psi = \lambda \phi$. Condition (iv) insures that $\lambda \neq 0$, and so we get $\pi_{d-1} \subseteq \{\mathbf{c}^T \psi \mid \mathbf{c} \in \mathbb{R}^n\}$.

For the converse, let $Q_1$ be an $n \times m$ matrix whose columns form an orthonormal basis of S. Since S is a common invariant subspace of $B_0, \ldots, B_{p-1}$, we get

$$Q_1^T B_i \mathbf{v} = Q_1^T B_i Q_1 Q_1^T \mathbf{v}, \qquad i = 0, 1, \ldots, p-1, \quad \mathbf{v} \in S, \tag{7.1}$$

and that the curve $\hat{\psi} := Q^T\psi$ satisfies the functional equation (1.1) for the $m \times m$ matrices $\hat{B}_i := Q_1^T B_i Q_1$. By construction the span of $\hat{\psi}$ is $\mathbb{R}^m$, and therefore we can apply Theorem 7.2 (for $n = m$) and conclude there is an $m \times d$ matrix $Q_2$ of rank $d$ and $p$ $(d \times d)$ matrices $V_i$, $i = 0, 1, \dots, p-1$, such that

$$Q_2^T \hat{B}_i Q_2 = V_i D_d V_i^{-1}, \qquad i = 0, 1, \dots, p-1 \tag{7.2}$$

$$Q_2^T \hat{B}_i = Q_2^T \hat{B}_i Q_2 Q_2^T, \qquad i = 0, 1, \dots, p-1, \tag{7.3}$$

and

$$V_i = V_0 T_d\left(\frac{i}{p-1}\right), \qquad i = 0, 1, \dots, p-1. \tag{7.4}$$

Setting $Q = Q_1 Q_2$ we see that (7.2) gives Equation (i), while Equation (ii) follows from (7.1) and (7.3). Since $\operatorname{range} Q \subseteq \operatorname{range} Q_1 \subseteq S$, condition (iv) is obvious and thus the proof is complete. ■

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