The Polytope of Degree Sequences

Uri N. Peled and Murali K. Srinivasan
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Box 4348
Chicago, Illinois 60680

Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Robert E. Bixby

ABSTRACT

A nonnegative integer sequence $(d_1, d_2, \dots, d_n)$ is called a *degree sequence* if there exists a simple graph on the vertex set $V = \{1, 2, \dots, n\}$ such that $\deg(i) = d_i$ for all i. The degree sequence of a threshold graph is a *threshold sequence*. Let $D_n = \text{Convex Hull } \{(x_1, x_2, \dots, x_n) | (x_1, \dots, x_n) \text{ is a degree sequence}\}$. It is proved that: (1) A degree sequence f is an extreme point of D_n if and only if f is a threshold sequence. (2) Two threshold sequences f and g are adjacent extreme points of D_n if and only if f can be obtained from g by either adding 1 to two components of g or subtracting 1 from two components of g. (3) D_n is determined by the following system of inequalities:

$$\sum_{i \in S} x_i - \sum_{i \in T} x_i \leqslant |S|(n - 1 - |T|)$$

for all sets S, T with $\varnothing \neq S \cup T \subseteq \{1, 2, \dots, n\}$, $S \cap T = \varnothing$. Moreover, this system is totally dual integral. Furthermore, for $n \geqslant 4$, (S, T) determines a facet of D_n if and only if either $|S \cup T| = 1$ or else $S \neq \varnothing$, $T \neq \varnothing$, $|S \cup T| \neq n - 1$, $n - 2$. (4) f is a threshold sequence if and only if the only degree sequences majorizing f in the sense of Hardy, Littlewood, and Pólya are the rearrangements of f. Consequently, every degree sequence is a convex combination of isomorphic threshold sequences (i.e., threshold sequences that are rearrangements of each other).

1. INTRODUCTION

A nonnegative integer sequence $d = (d_1, d_2, \dots, d_n)$ is called a *degree sequence* if there exists a (simple) graph $G = (V, E)$ on the vertex set

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$V=\{1,2,\ldots,n\}$ such that $\deg(i)=d_i$ for all i. G is said to be a *realization* of d. Degree sequences have been thoroughly studied, and the following are three well-known characterizations of nonincreasing degree sequences ($d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$):

(1) $\sum_{i=1}^{n} d_i$ is even and $(d_1,\ldots,d_n)$ satisfies the Erdös-Gallai inequalities $\sum_{i=1}^{k} d_i \leqslant k(k-1)+\sum_{i=k+1}^{n}\min(d_i,k)$, $k=1,\ldots,n$ [2, 5, 16];

(2) $\sum_{i=1}^{n} d_i$ is even and $(d_1,\ldots,d_n)$ is majorized (Definition 5.1 below) by its corrected conjugate sequence $(\bar{d}_1,\ldots,\bar{d}_n)$ given by

$$\bar{d}_k = |\{i<k \mid d_i \geqslant k-1\}| + |\{i>k \mid d_i \geqslant k\}| \qquad [2];$$

(3) $(d_2-1, d_3-1,\ldots,d_{d_1+1}-1, d_{d_1+2},\ldots,d_n)$ is also a degree sequence. This criterion, when iterated, gives rise to the Havel-Hakimi algorithm [8, 13].

Let $\mathrm{DS}_n=\{(x_1,\ldots,x_n) \mid (x_1,\ldots,x_n) \text{ is a degree sequence}\}$. The *polytope of degree sequences* is $D_n = \text{Convex Hull}(\mathrm{DS}_n)$. In [15], Koren proved that:

(1) A degree sequence is an extreme point of D_n if and only if it has a unique realization as a labeled graph (see also [9]);

(2) D_n is determined by the following system of linear inequalities:

$$\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant |S|(n-1-|T|)$$

for all sets S, T such that $\varnothing \neq S\cup T \subseteq \{1,2,\ldots,n\}$, $S\cap T=\varnothing$.

It can be shown that for real $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$, the above inequalities are equivalent to the Erdös-Gallai inequalities. Therefore testing membership in D_n is reduced to sorting and verifying the Erdös-Gallai inequalities.

Koren also characterized the uniquely realizable nonincreasing degree sequences by the property $d=\bar{d}$. The proofs in [15] rely on the Erdös-Gallai inequalities.

In this paper we study the properties of D_n using threshold graphs. In Sections 2 and 4 we reproduce the results of [15] using linear programming duality and the structure of threshold graphs and show that the above system of linear inequalities is totally dual integral. This approach fits naturally the modern trend of using LP duality and some combinatorics to obtain polyhedral results [20]. We also determine the facets of D_n in Section 4. In Section 3 we study the problem of adjacency of extreme points of D_n and show that two extreme points $(f_1,\dots,f_n)$ and $(g_1,\dots,g_n)$ of D_n with realizations $G_1=(V,E_1)$ and $G_2=(V,E_2)$ (which are necessarily unique) are adjacent if and only if $|E_1 \oplus E_2|=1$ ($\oplus$ denotes the symmetric difference).

Finally, in Section 5 we show that the nonmajorizable degree sequences (more precisely, degree sequences d that cannot be majorized by other degree sequences except for rearrangements of d) are precisely the degree sequences of threshold graphs. From a well-known result in the theory of majorization it then follows that every degree sequence is a convex combination of degree sequences of isomorphic threshold graphs.

The following definitions are used in this paper. All graphs considered are finite, undirected, and without loops or parallel edges. $[a,b]$ denotes an edge joining two vertices a and b, and $N(a)=\{b \mid [a,b] \text{ is an edge}\}$ is the set of neighbors of a. $N(W)=\bigcup\{N(a) \mid a \in W\}$ for $W \subseteq V$. A *clique* is a set of pairwise adjacent vertices, and a *stable set* is a set of pairwise nonadjacent vertices. The size of the largest clique of a graph $G=(V,E)$ is written as $\omega(G)$. The vertex set of each graph discussed here is assumed to be $\{1,2,\dots,n\}$, for some n. We denote by u_i the ith unit vector.

Threshold graphs were introduced by Chvátal and Hammer [3] as a class of graphs for which there is a simple method of distinguishing stable sets from nonstable sets. A graph with vertex set $\{1,2,\dots,n\}$ is a threshold graph if there exist real weights $w_1, w_2,\dots,w_n$, t such that the 0-1 solutions of the inequality $\sum_{i=1}^{n} w_i x_i \leqslant t$ are precisely the characteristic vectors of the stable sets of the graph. The degree sequence of a threshold graph is called a *threshold sequence*.

Threshold graphs have many characterizations. We present the two that are used in the sequel.

THEOREM 1.1 [3]. *A graph is threshold if and only if it has no induced subgraphs isomorphic to $2K_2$, P_3, or C_4, as shown in Figure* 1.

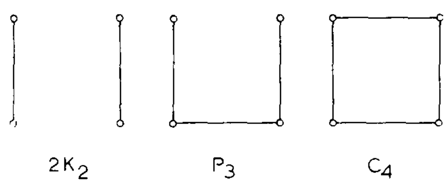

FIG. 1. The forbidden induced subgraphs of threshold graphs.

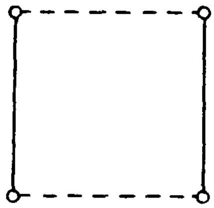

FIG. 2. The forbidden configuration of threshold graphs.

Equivalently, Theorem 1.1 can be stated as:

COROLLARY 1.2. *A graph is threshold if and only if it does not contain the configuration shown in Figure 2, where solid lines represent edges, dotted lines represent non-edges, and the absence of a line allows for the possibility of the edge existing or not.*

THEOREM 1.3 [3]. *A graph $G = (V, E)$ is threshold if and only if there is a partition of V into disjoint sets K and I (possibly empty) such that*

(1) *K is a clique;*
(2) *I is a stable set;*
(3) *there exists an ordering $u_1, u_2, \dots, u_s$ of the vertices of I such that*

$$N(u_1) \subseteq N(u_2) \subseteq \cdots \subseteq N(u_s),$$

or equivalently, there exists an ordering $v_1, v_2, \dots, v_k$ of the vertices of K such that

$$N(v_1) \cap I \subseteq N(v_2) \cap I \subseteq \cdots \subseteq N(v_k) \cap I.$$

It is shown in [7] (see also [10]) that we can always choose K with $|K| = \omega(G)$. A partition satisfying (1) and (2) of Theorem 1.3 is called a *split partition.*

THEOREM 1.4 [7] (see also [9]). *A nonnegative integer sequence d is a threshold sequence if and only if $d = \bar{d}$.*

In light of Theorem 1.4, Koren's results about the extreme points of D_n can be interpreted to mean that the extreme points of D_n are precisely the threshold sequences. For further characterizations and properties of threshold graphs and threshold sequences see [1, 3, 6, 7, 9, 11, 14, 18, 19].

2. EXTREME POINTS

We first review a few facts about the structure of threshold graphs.

LEMMA 2.1. *Let $G=(V,E)$ be a threshold graph with split partition $V=K\cup I$, $|K|=\omega(G)$. Then there exists a $v\in K$ that is not adjacent to any vertex in I.*

Proof. Enumerate the vertices of K as $v_1, v_2, \dots, v_k$, so that

$$N(v_1)\cap I\subseteq N(v_2)\cap I\subseteq \cdots \subseteq N(v_k)\cap I.$$

Then v_1 is not adjacent to any vertex in I. For if $u\in N(v_1)\cap I$, then $K\cup u$ is a larger clique, contradicting our choice of K. ∎

We now introduce a partition of the vertices of a threshold graph, which is very similar to the degree partition of [7].

DEFINITION 2.2. Let $G=(V,E)$ be a threshold graph with a split partition $V=K\cup I$, $|K|=\omega(G)$.

If the edges in K are omitted, one obtains a bipartite graph $B=(K,I;F)$. Let $k_0<k_1<\cdots<k_p$ be the distinct degrees in B of the vertices in K. By Lemma 2.1, $k_0=0$. Let $s_1<s_2<\cdots<s_p$ be the distinct nonzero degrees in B of the vertices of I (by Theorem 1.3, the number of distinct positive degrees in I is p). Set $s_0=0$, and define for $i=0,1,\dots,p$

$$K_i=\{v\in K|\deg_B(v)=k_i\},$$

$$I_i=\{v\in I|\deg_B(v)=s_i\}.$$

By this definition all the K_i and I_i are nonempty, with the possible exception of I_0. See Figure 3.

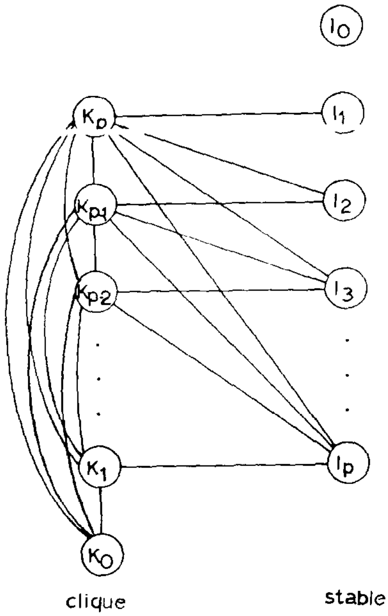

FIG. 3. A typical threshold graph. A line between K_i and I_j (or between K_i and K_j) indicates that every vertex of K_i is adjacent to every vertex of I_j (K_j).

It follows from Definition 2.2 that

$$N(K_i) \cap I = I_{p-i+1} \cup \cdots \cup I_p,$$
$$N(I_i) = K_{p-i+1} \cup \cdots \cup K_p$$

for $i = 0, 1, \ldots, p$.

REMARK. The fact that $K_0 = \{v \in K \mid N(v) \cap I = \emptyset\} \neq \emptyset$ was derived from the choice of K to satisfy $|K| = \omega(G)$. It is needed in some, but not all of the proofs below. We assume it in all cases for uniformity.

Let $(c_1, c_2, \ldots, c_n) \in \mathbf{R}^n$. Consider the combinatorial optimization problem

$$\max \sum_{i=1}^{n} c_i x_i$$
$$\text{s.t. } (x_1, x_2, \ldots, x_n) \in D_n$$
$$[\text{or equivalently, } (x_1, x_2, \ldots, x_n) \in \mathrm{DS}_n].$$

The following very simple algorithm produces an optimal solution:

ALGORITHM 2.3.

$V \leftarrow \{1,2,\dots,n\}$;
$E \leftarrow \varnothing$;
for all $i, j,\ i \neq j$, if $c_i + c_j \geqslant 0$, $E \leftarrow E \cup [i, j]$;
$(x_1, x_2, \dots, x_n) \leftarrow$ degree sequence of $G = (V, E)$.

REMARK 2.4. It is clear that the algorithm is correct: adding the edge $[i, j]$ adds $c_i + c_j$ to the objective function.

REMARK 2.5. If $c_i + c_j = 0$ for some $i, j,\ i \neq j$, we have the freedom of not adding the edge $[i, j]$ and still obtaining an optimal solution. This freedom is enough to obtain all graphs with optimal degree sequences.

REMARK 2.6. In particular, the following conditions are equivalent:

(a) $c_i + c_j \neq 0$ for all $i, j,\ i \neq j$;
(b) there is only one optimal degree sequence;
(c) there is only one graph with an optimal degree sequence.

REMARK 2.7. The algorithm can be thought of as the greedy algorithm on the uniform matroid on the edges with edge costs $c_i + c_j$. The algorithm can also be interpreted as the "greedy algorithm" described in [22, Section 19.2] applied to D_n. In [22] it is proved that the "greedy algorithm" on a polymatroid works for all $c \in \mathbf{R}^n$. Here it works on D_n, which is not a polymatroid.

THEOREM 2.8. *A degree sequence is an extreme point of D_n if and only if it is a threshold sequence.*

Proof. "Only if": Every extreme point of D_n is the unique optimum for some objective function $\sum_{i=1}^{n} c_i x_i$, so Algorithm 2.3 must produce it on input $(c_1, \dots, c_n)$. But the algorithm always produces threshold graphs. Indeed, write $V = K \cup I$ where $K = \{i \mid c_i \geqslant 0\}$, $I = \{i \mid c_i < 0\}$. Then K and I satisfy the condition of Theorem 1.3.

"If": Let $d = (d_1, d_2, \dots, d_n)$ be a threshold sequence and $G = (V, E)$ be a realization of d. Let K_i, I_i, $0 \leqslant i \leqslant p$, be as in Definition 2.2. Choose

$c=(c_1,c_2,\ldots,c_n)$ as follows (see Figure 4):

$$c_i=\begin{cases}2j+1, & i\in K_j,\\ -2p-2+2j & i\in I_j.\end{cases}$$

It is easy to check that Algorithm 2.3 with input $(c_1,c_2,\ldots,c_n)$ outputs precisely the graph $G=(V,E)$ and its degree sequence $d=(d_1,\ldots,d_n)$. Moreover, by Remark 2.6, d is the only optimal sequence. Therefore d is an extreme point of D_n. ■

COROLLARY 2.9 [8, 13]. *A degree sequence has a unique realization if and only if it is a threshold sequence.*

Proof. The "if" part follows from Theorem 2.4 and Remark 2.6. For the "only if" part, let G be a nonthreshold graph. Then by Corollary 1.2, G has a forbidden configuration (see Figure 5), and an interchange gives a different graph with the same degree sequence. ■

REMARK 2.10. The "if" part also follows easily from the forbidden configuration characterization of threshold graphs and from the fact that all realizations of a degree sequence can be obtained from each other by

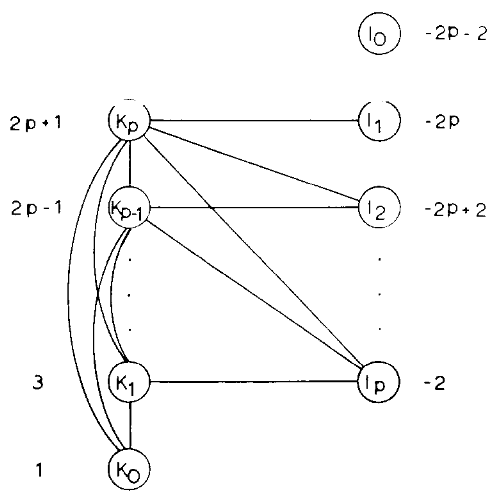

FIG. 4. Illustrating the objective function for the "if" part of Theorem 2.8.

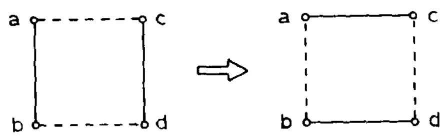

FIG. 5. An interchange.

interchanges along such configurations (see Berge [2, Chapter 8, Theorem 5]).

REMARK 2.11. A threshold sequence can be recognized and its realization constructed in time $O(n \log n)$ [3].

REMARK 2.12. The number of extreme points of D_n (i.e., the number of labeled n-vertex threshold graphs) has been determined in [1].

3. ADJACENCY

In this section we develop criteria for two extreme points of D_n to be adjacent and produce a formula for the number of adjacent threshold sequences to a given threshold sequence.

LEMMA 3.1. *Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be two threshold graphs with degree sequences f and g, respectively. If $|E_1 \oplus E_2| \geq 2$, then f and g are not adjacent extreme points of D_n.*

Proof. Assume that f and g are adjacent extreme points of D_n. Then by Theorem 2.8 and the definition of adjacency there exists a $c = (c_1, c_2, \ldots, c_n)$ such that $c \cdot x = \sum_{i=1}^n c_i x_i$ is maximized over the threshold sequences only by f and g. Algorithm 2.3 on input c will produce one of T_1, T_2, say T_1. Let $e = [i, j] \in E_1 \oplus E_2$. Then $c_i + c_j = 0$ (for otherwise e must be in all or none of the optimal graphs), and hence $e \in E_1$. We assert that $T_3 = (V, E_3)$, where $E_3 = E_1 - e$ is a threshold graph. This is a contradiction, since T_3 is different from both T_1 and T_2, whence its degree sequence h differs from both f and g by Corollary 2.9, and h maximizes $c \cdot x$. To prove the assertion, assume T_3 is nonthreshold. Then by Corollary 1.2 there exist vertices k, l with $[i, k], [j, l] \in E_3$ and $[i, j], [k, l] \notin E_3$. Then $[i, k], [j, l] \in E_1$ and $[k, l] \notin E_1$.

So $c_i + c_k \geqslant 0$, $c_j + c_l \geqslant 0$, $c_k + c_l < 0$, and $c_i + c_j = 0$. Then $c_i + c_j + c_k + c_l \geqslant 0 > c_i + c_j + c_k + c_l$, a contradiction. ∎

To prove the converse of Lemma 3.1, we determine in the next three lemmas which single edges can be added to or dropped from a given threshold graph to obtain another one.

LEMMA 3.2. *Let $G = (V, E)$ be a threshold graph with a split partition $V = K \cup I$ as in Definition* 2.2.

(a) *Let $x, y \in I$. Then $(V, E \cup [x, y])$ is a threshold graph if and only if $x, y \in I_p$ and $|K_0| = 1$ (i.e., $\deg x = \deg y = |K| - 1$).*

(b) *Let $x, y \in K$. Then $(V, E - [x, y])$ is a threshold graph if and only if*

$$\begin{array}{lll} (x, y \in K_0) & & \textit{if} \quad |K_0| \geqslant 2, \\ (x \in K_0, \quad y \in K_1) \quad \textit{or} \quad (x \in K_1, \quad y \in K_0) & & \textit{if} \quad |K_0| = 1. \end{array}$$

Proof. (a), "only if": First assume that $x \in I_i$, $y \in I_j$, *and* $i < j$. Since K_0 is nonempty, there exists a $z \in K$ with $[z, y] \notin E$. As x has more nonneighbors than y, there exists a $w \in K$, $z \neq w$, such that $[x, w] \notin E$ (see Figure 6). Adding edge $[x, y]$ closes a forbidden configuration.

Therefore, $x, y \in I_i$, for some i. Now assume that $i < p$. Vertices x and y have the same degree and so, by Theorem 1.3, they have the same neighbors in K. By our assumption there exist $w, z \in K$, $w \neq z$, such that both w and z are nonneighbors of both x and y. We reach the same contradiction as above. Thus $x, y \in I_p$. Finally, $|K_0| = 1$, as assuming $|K_0| \geqslant 2$ gives rise to the same contradiction once more.

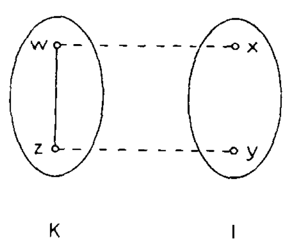

FIG. 6. Illustrating the proof of Lemma 3.2.

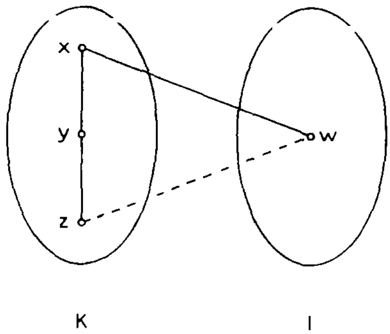

FIG. 7. Illustrating the proof of Lemma 3.2.

"If": Let $K_0 = \{z\}$. Replace K by $K' = K - \{z\} \cup \{x, y\}$ and I by $I' = I - \{x, y\} \cup \{z\}$. It is easy to check that $G' = (K' \cup I', E \cup [x, y])$ satisfies the condition of Theorem 1.3, proving that G' is threshold.

(b), "Only if":

Case 1: $|K_0| \geq 2$. Assume that $x \in K_i$, $i \geq 1$, $y \in K_j$, $j \geq 0$, $i \geq j$. This means that there exists a $z \in K_0$, $z \neq y$, and a $w \in I$ such that $[x, w] \in E$ (Figure 7). Dropping $[x, y]$ opens up a forbidden configuration. Therefore, $x, y \in K_0$.

Case 2: $|K_0| = 1$. Let $K_0 = \{z\}$. First assume that $x \in K_i$, $y \in K_j$, $i, j \geq 1$. There exists a $w \in I$ with $[x, w] \in E$. We reach the same contradiction as in case 1.

Now assume, without loss of generality, that $y \in K_0$ (hence $y = z$) and $x \in K_i$, $i \geq 2$. Choose $z_1 \in K_1$ and $w \in I$ with $[x, w] \in E$ and $[z_1, w] \notin E$ (Figure 8).

Dropping $[x, y]$ opens up a forbidden configuration. Therefore,

$$(x \in K_0, \quad y \in K_1) \quad \text{or} \quad (x \in K_1, \quad y \in K_0).$$

"If":

Case 1: $|K_0| \geq 2$. Replace K by $K' = K - \{x, y\}$, and I by $I' = I \cup \{x, y\}$.

Case 2: $|K_0| = 1$. Assume, without loss of generality, that $x \in K_1$, $y \in K_0$. Replace K by $K' = K - \{y\}$, and I by $I' = I \cup \{y\}$.

In both cases it is easy to check that $G = (K' \cup I', E - [x, y])$ satisfies the condition of Theorem 1.3, proving that G' is threshold. ∎

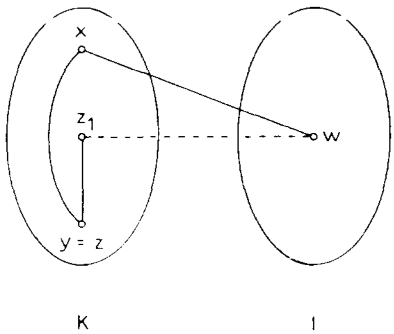

FIG. 8. Illustrating the proof of Lemma 3.2.

LEMMA 3.3. *Let $G = (V, E)$ be a threshold graph with split partition $V = K \cup I$ as in Definition 2.2.*

(a) *Let $x \in I_r$ and $y \in K_s$, $[x, y] \notin E$. Then $(V, E \cup [x, y])$ is threshold if and only if $r + s = p$.*
(b) *Let $x \in I_r$ and $y \in K_s$, $[x, y] \in E$. Then $(V, E - [x, y])$ is threshold if and only if $r + s = p + 1$.*

Proof. (a), "Only if": Since $[x, y] \notin E$, $r + s \leqslant p$. Assume that $r + s \leqslant p - 1$. Pick $z \in I_{r+1}$ and $w \in K_{p-r}$. Then $[x, w] \notin E$, $[z, y] \notin E$ and $[z, w] \in E$ (Figure 9). Adding edge $[x, y]$ closes a forbidden configuration. Hence $r + s = p$.
"If": One can easily verify that $(K \cup I,\ E \cup [x, y])$ continues to satisfy the condition of Theorem 1.3.
(b): Similar to (a). ■

LEMMA 3.4. *Let $G = (V, E)$ be a threshold graph and $[x, y] \notin E$. If $G' = (V, E \cup [x, y])$ is also a threshold graph, then G has a split partition $V = K \cup I$ with $|K| = \omega(G)$ such that $(x \in I,\ y \in K)$ or $(x \in K,\ y \in I)$.*

Proof. Let $V = K \cup I$ be a split partition of G as in Definition 2.2. If $x, y \in I$ and G' is threshold, then by Lemma 3.2, $x, y \in I_p$ and $|K_0| = 1$. Let $K_0 = \{z\}$. Replacing K by $K - \{z\} \cup \{x\}$ and I by $I - \{x\} \cup \{z\}$, we get another split partition with the required properties. ■

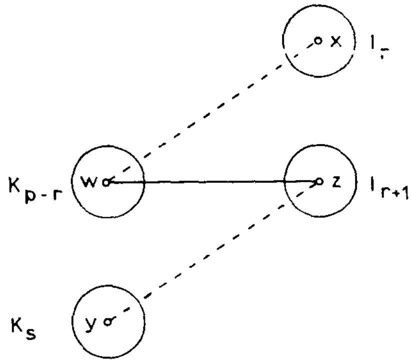

FIG. 9. Illustrating the proof of Lemma 3.3.

THEOREM 3.5. *Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be the (unique) realizations of the threshold sequences f and g, respectively. Then f and g are adjacent extreme points of D_n if and only if $|E_1 \oplus E_2| = 1$.*

Proof. The "only if" direction is Lemma 3.1.

"If": Let $E_1 \oplus E_2 = \{e\}$. Without loss of generality, $E_1 = E_2 \cup \{e\}$. From Lemma 3.4 we can assume that T_2 has a split partition $V = K \cup I$ such that $x \in I$, $y \in K$, where $[x, y] = e$. Let $x \in I_r$ and $y \in K_s$. By Lemma 3.3, $r + s = p$. Run Algorithm 2.3 on the input $c = (c_1, c_2, \ldots, c_n)$ defined as follows (see Figure 10):

$$
\begin{aligned}
c_i &= -2p + 2r - 2, && i = x,\\
c_i &= 2p - 2r + 2, && i = y,\\
c_i &= -2p + 2j - 2, && i \in I_j, \quad j \geqslant r + 1,\\
c_i &= -2p + 2j - 4, && i \in I_j, \quad i \neq x, \quad j \leqslant r,\\
c_i &= 2j + 1, && i \in K_j, \quad i \neq y, \quad j \leqslant p - r,\\
c_i &= 2j + 3, && i \in K_j, \quad j \geqslant p - r + 1.
\end{aligned}
$$

It is easy to verify that the algorithm outputs T_1 and f. Furthermore, since $c_i + c_j = 0$ if and only if $\{i, j\} = \{x, y\}$, the only other optimal solution is g by Remark 2.5. Therefore, f and g are adjacent extreme points of D_n. ∎

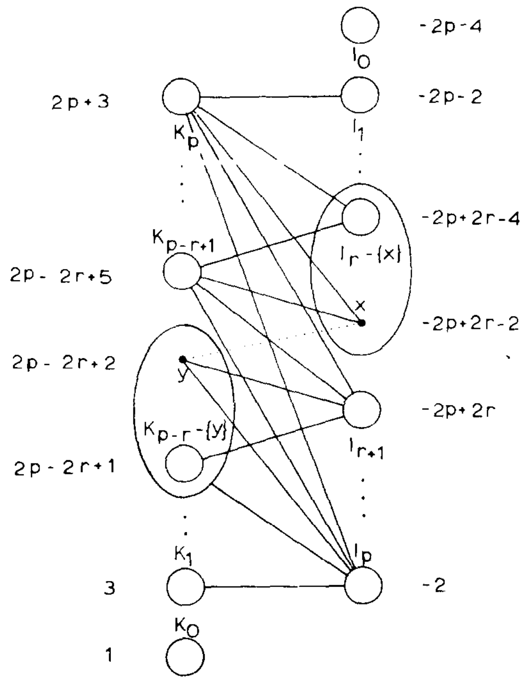

FIG. 10. Weights in the proof of Theorem 3.5.

We now develop a condition for adjacency of two threshold sequences that does not refer to their realizations as threshold graphs.

LEMMA 3.6. *Let $f=(f_1,\dots,f_n)$ be a threshold sequence with realization $T=(V,E_1)$, and $g=(g_1,\dots,g_n)$ be a degree sequence with realization $G=(V,E_2)$ such that $f=g+u_i+u_j$ for some $i\neq j$ (u_t is the* tth *unit vector.) Then $[i,j]\in E_1$.*

Proof. The proof is by induction on $\min(g_i,g_j)$. Let q be the number of positive components of g.

Basis: $\min(g_i,g_j)=0$. Without loss of generality, $g_j=0$. The number of positive components of f is $q+1$ $[q+2]$ if g_i is positive [zero]. The largest degree in G is at most $q-1$, and the largest degree in T is q $[q+1]$ by Theorem 1.3.

Since $\max_k g_k \leqslant q-1$, we have $f_k \leqslant q-1$ for $k \neq i, j$. Therefore, q $[q+1] = \max(f_i, f_j)$, so one of the nonisolated vertices i, j of T has largest degree in T. Therefore, $[i, j] \in E_1$ by Theorem 1.3.

Induction step: $\min(g_i, g_j) > 0$. In this case the number of positive components of f is also q. If f_i or f_j is $q-1$, then clearly $[i, j] \in E_1$. If not, $f_l = q-1$ for some $l \neq i, j$ by Theorem 1.3. Then $g_l = q-1$, so l is adjacent to all nonisolated vertices in both T and G. Delete l from both T and G, and apply induction to the resulting induced subgraphs. ∎

REMARK. One might wish to generalize Lemma 3.6 by relaxing one of its assumptions. Instead of assuming that $f = g + u_i + u_j$ for some $i \neq j$, we wish to assume only $f_k \geqslant g_k$ for all k, and to conclude that by dropping zero or more edges in the (unique) realization of f we can obtain some realization of g. Under the additional assumptions that (a) g is a threshold sequence and (b) $f_1 \geqslant \cdots \geqslant f_n$ and $g_1 \geqslant \cdots \geqslant g_n$, the conclusion is true [8]. However, (a) is not enough without (b), as can be seen from the example $f = (5,5,5,4,4,3)$, $g = (4,1,0,2,2,3)$.

THEOREM 3.7. *Let $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ be threshold sequences. Then they are adjacent extreme points of D_n if and only if there exist indices $i \neq j$ such that $f - g = \pm(u_i + u_j)$.*

Proof. The "only if" part follows easily from Theorem 3.5.

"If": Assume without loss of generality that $f_k = g_k + 1$, $k = i, j$. From Lemma 3.6, $[i, j] \in E_1$, where $T_1 = (V, E_1)$ is the realization of $(f_1, \ldots, f_n)$. From the uniqueness of realizations of threshold sequences, it follows that $T_2 = (V, E_1 - [i, j])$ is the realization of $(g_1, g_2, \ldots, g_n)$, and the result follows from Theorem 3.5. ∎

Below we obtain the threshold sequences adjacent to a given one without referring to its realization. For this it is convenient to use the following definition.

DEFINITION 3.7 [7]. Let $f = (f_1, \ldots, f_n)$ be any degree sequence. Let $\delta_1 < \delta_2 < \cdots < \delta_m$ be the distinct nonzero integers occurring in f. Set $\delta_0 = 0$. The *degree partition* of f is defined as $D = (D(0), \ldots, D(m))$, where

$$D(i) = \{j \mid f_j = \delta_i\}, \qquad 0 \leqslant i \leqslant m.$$

we put

$$d_i = |D(i)|.$$

If f is a threshold sequence, then D is closely related to Definition 2.2. Indeed, if $G = (V, E)$ is the realization of f, let $P = (I_0, \dots, I_p, K_0, \dots, K_p)$, where K_i, I_i are as in Definition 2.2. A little reflection shows that if $|K_0| > 1$, then $D = P$, and if $|K_0| = 1$, then D is obtained from P by merging K_0 with I_p. Thus, if $|K_0| = 1$, then

$$D(i) = \begin{cases} I_i, & 0 \leqslant i \leqslant p-1, \\ K_0 \cup I_p, & i = p, \\ K_{i-p}, & p+1 \leqslant i \leqslant m, \end{cases}$$

$$m = 2p,$$

and if $|K_0| > 1$, then

$$D(i) = \begin{cases} I_i, & 0 \leqslant i \leqslant p, \\ K_{i-p-1}, & p+1 \leqslant i \leqslant m, \end{cases}$$

$$m = 2p + 1.$$

THEOREM 3.8. *Let $f = (f_1, f_2, \dots, f_n)$ be a threshold sequence. Then the extreme points $f' = (f_1', f_2', \dots, f_n')$ of D_n adjacent to f are given by*

(a) $f' = f + u_i + u_j$, *where*

$$i \in D(r), \qquad j \in D(m-r), \qquad 0 \leqslant r \leqslant \left\lfloor \frac{m-1}{2} \right\rfloor$$

or

$$i, j \in D\left(\left\lfloor \frac{m+1}{2} \right\rfloor\right), \qquad i \neq j, \quad m \text{ even};$$

(b) $f' = f - u_i - u_j$, *where*

$$i \in D(r), \qquad j \in D(m-r+1), \qquad 1 \leqslant r \leqslant \left\lfloor \frac{m}{2} \right\rfloor$$

or

$$i, j \in D(\lfloor (m+1)/2 \rfloor), \qquad i \neq j, \quad m \text{ odd}.$$

Proof. Let $T = (V, E)$ be the realization of f. Case (a) corresponds to adding an edge to E, and case (b) corresponds to dropping an edge from E. We prove the result for case (a). Case (b) is similar.

First assume that m is odd. From Lemmas 3.2 and 3.3 we know that we can add an edge between I_r and K_{p-r} where $0 \leqslant r \leqslant p$ and no other edges. From Definition 3.7, this is the same as adding an edge between $D(r)$ and $D(m-r)$, $0 \leqslant r \leqslant \lfloor (m-1)/2 \rfloor$.

Now assume that m is even. It follows from Lemmas 3.2 and 3.3 that we can add an edge between

(1) I_r and K_{p-r}, $0 \leqslant r \leqslant p-1$,
(2) I_p and K_0,
(3) any two vertices in I_p.

As m is even, $|K_0| = 1$ and $K_0 \cup I_p = D(\lfloor (m+1)/2 \rfloor)$. The result thus follows. ∎

Using Theorem 3.8 we obtain the following result. We thank K. N. Srikanth for raising this question with one of us (M.S.).

THEOREM 3.9. *Let $f = (f_1, \ldots, f_n)$ be a threshold sequence with a degree partition as in Definition 3.7. Then the number of extreme points of D_n adjacent to f is given by the expression*

$$\sum_{i=0}^{\lfloor (m-1)/2 \rfloor} d_i d_{m-i} + \sum_{i=1}^{\lfloor m/2 \rfloor} d_i d_{m-i+1} + \binom{d_q}{2},$$

where $q = \lfloor (m+1)/2 \rfloor$.

We can now determine the threshold sequences with the largest and smallest number of adjacent sequences.

THEOREM 3.10. *Let $f = (f_1, f_2, \ldots, f_n)$ be a threshold sequence with a degree partition as in Definition 3.7.*

(a) *The number of extreme points adjacent to f is a maximum, equal to* $\binom{n}{2}$, *if and only if*

$$f_i = 0 \quad \text{for all } i \qquad (\text{stable})$$

or

$$f_i = n - 1 \quad \text{for all } i \qquad (\text{clique})$$

or

$$f_i = \begin{cases} n-1 & \text{for some } j, \\ 1 & \text{for } \ i \neq j \end{cases} \qquad (\text{star})$$

or

$$f_i = \begin{cases} 0 & \text{for some } j, \\ n-2 & \text{for } \ i \neq j \end{cases} \qquad (\text{costar}).$$

(b) *The number of extreme points adjacent to f is a minimum, equal to n, if and only if*

$$d_i = \begin{cases} 1, & i \neq 0, \lfloor (m+1)/2 \rfloor, \\ 2, & i = \lfloor (m+1)/2 \rfloor, \end{cases}$$

$$d_0 = n - \sum_{i=1}^{m} d_i.$$

Proof. Let $G = (V, E)$ be a realization of f. Let $V = K \cup I$ be as in Definition 2.2.

(a): The number of ways in which we can add or drop an edge preserving thresholdness is at most $\binom{n}{2}$. It is equal to $\binom{n}{2}$ if and only if every present edge can be dropped and every absent edge can be added. From Lemma 3.2 it follows that $p \leqslant 1$.

Case 1: $p = 1$. It follows easily from Lemma 3.2 that in this case $|K_1| = 1$, $|K_0| = 1$, and $I_0 = \varnothing$. This implies that G is a star.

Case 2: $p = 0$. We can similarly show that $\min(|K_0|, |I_0|) \leqslant 1$. This corresponds to G being a clique, a graph with no edges, or the complement of a star.

(b): Consider the parity of m.

Case 1: m is even. In this case $m = 2p$ and $d_p \geqslant 2$. From Theorem 3.9, the number of extreme points adjacent to f is

$$\begin{aligned}
&\sum_{i=0}^{p-1} d_i d_{2p-i} + \sum_{i=1}^{p} d_i d_{2p-i+1} + \binom{d_p}{2} \\
&\quad = d_0 d_{2p} + \sum_{i=1}^{p-1} d_i d_{2p-i} + \sum_{i=1}^{p} d_i d_{2p-i+1} + \binom{d_p}{2} \\
&\quad \geqslant d_0 + (p-1) + (p+1) + 1 \\
&\quad = d_0 + 2p + 1.
\end{aligned}$$

This lower bound is achieved if and only if

$$d_p = 2, \qquad d_i = 1, \quad i \neq 0, p, \qquad d_0 = n - \sum_{i=1}^{2p} d_i.$$

The value of the bound is then $d_0 + 2p - 1 = (n - \sum_{i=1}^{2p} d_i) + 2p + 1 = n$.

Case 2: m is odd. In this case m = 2p + 1 and $d_{p+1} \geqslant 2$. *The rest of the analysis is similar to case* 1. ■

4. LINEAR DESCRIPTION AND FACETS

In this section we use linear programming duality and Theorem 1.3 to provide a linear description of D_n.

LEMMA 4.1. *Let S, T be subsets of* $\{1,2,\dots,n\}$ *such that* $S \cap T = \varnothing$. *Then the inequality*

$$\sum_{i \in S} x_i - \sum_{i \in T} x_i \leqslant |S|(n - 1 - |T|) \tag{4.1}$$

is valid for D_n. *Moreover, the degree sequence of* $G = (V, E)$ *satisfies* (4.1) *with equality if and only if*

(a) $i, j \in S,\ i \neq j \Rightarrow [i, j] \in E$,
(b) $i, j \in T,\ i \neq j \Rightarrow [i, j] \notin E$,
(c) $i \in S,\ j \in \overline{S \cup T} \Rightarrow [i, j] \in E$,
(d) $i \in T,\ j \in \overline{S \cup T} \Rightarrow [i, j] \notin E$ (*see Figure* 11).

Proof. To show the validity of (4.1), we may assume that x is the degree sequence of $G = (V, E)$. Set $C = \{[i, j] \mid i \in S,\ j \in T\}$. It is easy to see that

(e) $\sum_{i \in S} x_i \leqslant |S|(n - 1 - |T|) + |C|$, equality holding if and only if (a) and (c) hold,

(f) $-\sum_{i \in T} x_i \leqslant -|C|$, equality holding if and only if (b) and (d) hold.

By adding (e) and (f) we obtain (4.1) and the "moreover" part. ■

THEOREM 4.2. *D_n is determined by the linear inequalities*

$$\sum_{i \in S} x_i - \sum_{i \in T} x_i \leqslant |S|(n - 1 - |T|) \tag{4.2}$$

for all sets S, T such that $\varnothing \neq S \cup T \subseteq \{1, 2, \ldots, n\}$, $S \cap T = \varnothing$.

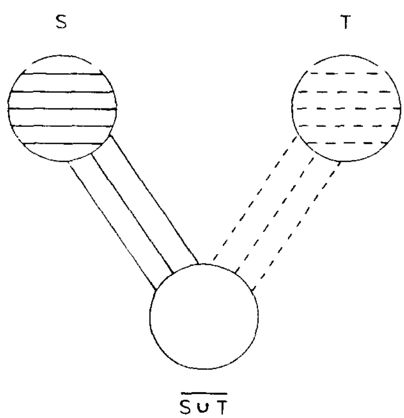

FIG. 11. S: clique; all edges between S and $\overline{S \cup T}$ present. T: stable; all edges between T and $\overline{S \cup T}$ absent.

Proof. Let $c=(c_1,c_2,\ldots,c_n)\in\mathbf{R}^n$, and consider the linear program

$$\max \sum_{i=1}^{n} c_i x_i$$

$$\text{subject to (4.2).} \tag{4.3}$$

Since it is known that Algorithm 2.3 maximizes $\sum_{i=1}^{n} c_i x_i$ over D_n, the theorem will be proved if it is shown that the solution produced by the algorithm on input c is an optimal solution of (4.3). Denote the graph produced by Algorithm 2.3 by $G=(V,E)$, and its degree sequence by d. By Lemma 4.1, d is a feasible solution of (4.3).

The dual of (4.3) is:

$$\min \sum_{\substack{\varnothing \neq S\cup T\subseteq\{1,\ldots,n\}\\ S\cap T=\varnothing}} |S|(n-1-|T|)y_{S,T} \tag{4.4}$$

subject to

$$\sum_{\substack{\varnothing \neq S\cup T\subseteq\{1,\ldots,n\}\\ S\cap T=\varnothing\\ i\in S}} y_{S,T} - \sum_{\substack{\varnothing \neq S\cup T\subseteq\{1,\ldots,n\}\\ S\cap T=\varnothing\\ i\in T}} y_{S,T} = c_i, \qquad i=1,\ldots,n,$$

$$y_{S,T}\geqslant 0 \qquad \text{for all } S,T \text{ such that} \quad \varnothing\neq S\cup T\subseteq\{1,\ldots,n\}, \quad S\cap T=\varnothing.$$

To show that d is an optimal solution of (4.3), it suffices to produce a feasible solution y to (4.4) that satisfies in addition the complementary slackness conditions

$$y_{S,T}>0 \quad \text{only if} \quad \sum_{i\in S} d_i - \sum_{i\in T} d_i = |S|(n-1-|T|). \tag{4.5}$$

(This uses only weak duality; strong duality asserts that if in fact d is optimal to (4.3), then such a y exists; we do not need this result, as we construct y explicitly.)

Set $K=\{i\mid c_i\geqslant 0\}$ and $I=\{i\mid c_i<0\}$, and put $r=|I|$. We may assume without loss of generality that $I=\{1,\ldots,r\}$, $K=\{r+1,\ldots,n\}$,

$$c_{r+1}\leqslant c_{r+2}\leqslant\cdots\leqslant c_n \quad \text{and} \quad |c_r|\leqslant|c_{r-1}|\leqslant\cdots\leqslant|c_1|.$$

From the nature of Algorithm 2.3 it is clear that

$$\begin{gathered} N(1) \subseteq N(2) \subseteq \cdots \subseteq N(r) \\ N(r+1) \cap I \subseteq N(r+2) \cap I \subseteq \cdots \subseteq N(n) \cap I. \end{gathered} \tag{4.6}$$

Consider the following pairs of sets (S_i, T_i), $1 \leqslant i \leqslant n$:

$$\left.\begin{aligned} S_i &= \{i, i+1, \dots, n\} \\ T_i &= I - N(i) \end{aligned}\right\} \qquad r+1 \leqslant i \leqslant n \tag{4.7}$$

and

$$\left.\begin{aligned} S_i &= N(i) \\ T_i &= \{1, 2, \dots, i\} \end{aligned}\right\} \qquad 1 \leqslant i \leqslant r.$$

Using Lemma 4.1 and (4.6), it can easily be verified that for the above choice of sets S_i, T_i, $i = 1, \dots, n$, the sequence d satisfies the corresponding inequality in (4.2) with equality (see Figure 12). We will find a feasible solution y of (4.4) such that $y_{S,T} > 0$ only if $(S, T) = (S_i, T_i)$ for some i. The above two statements will then guarantee (4.5).

Since $S_i \subseteq K$, $T_i \subseteq I$ for $1 \leqslant i \leqslant n$ and we imposed the condition $y_{S,T} = 0$ unless $(S, T) = (S_i, T_i)$ for some $1 \leqslant i \leqslant n$, the constraints (4.4) to be satisfied

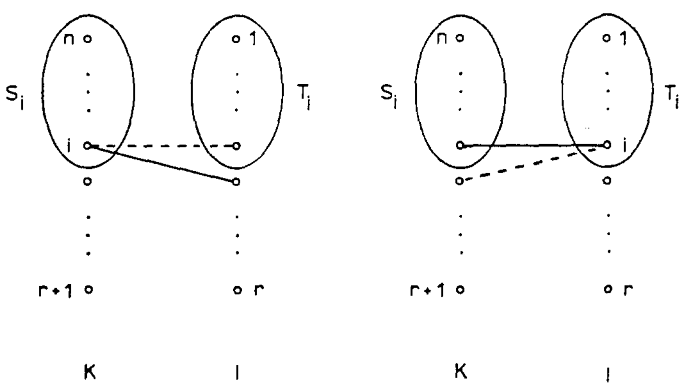

FIG. 12. Illustrating the choice of dual variables in the proof of Theorem 4.2.

by y_{S_i,T_i}, $1 \leqslant i \leqslant n$, read

$$\begin{aligned}
&\sum_{j \in T_i} y_{S_i,T_i} = |c_j|, && j = 1,\dots,r \\
&\sum_{j \in S_i} y_{S_i,T_i} = c_j, && j = r+1,\dots,n \qquad (4.8)\\
&y_{S_i,T_i} \geqslant 0, && i = 1,\dots,n.
\end{aligned}$$

For each $j \in \{1,\dots,n\}$, let P_j be the set of all indices i such that y_{S_i,T_i} appears in the jth equation of (4.8). Then for $j \in I$ we have, by (4.6) and (4.7),

$$\begin{aligned}
P_j &= \{i | j \in T_i\} = \{i \in I | i \geqslant j\} \cup \{i \in K | [i,j] \notin E\} \\
&= \{i \in I | i \geqslant j\} \cup \{i \in K | c_i < |c_j|\}, \qquad (4.9)
\end{aligned}$$

and for $j \in K$, again by (4.6) and (4.7),

$$\begin{aligned}
P_j &= \{i \in K | j \in S_i\} = \{i \in K | i \leqslant j\} \cup \{i \in I | [i,j] \in E\} \\
&= \{i \in K | i \leqslant j\} \cup \{i \in I | c_j \geqslant |c_i|\}. \qquad (4.10)
\end{aligned}$$

Define

$$\tilde{c}_t = \begin{cases} c_t, & t \in K, \\ |c_t|, & t \in I. \end{cases}$$

Let $\pi(1),\dots,\pi(n)$ be the rearrangement of $1,\dots,n$ such that

(a) $\tilde{c}_{\pi(1)} \leqslant \tilde{c}_{\pi(2)} \leqslant \cdots \leqslant \tilde{c}_{\pi(n)}$;
(b) if $\tilde{c}_{\pi(i)} = \tilde{c}_{\pi(j)}$, $i \neq j$, then

$$(i-j)[\pi(i)-\pi(j)] \begin{cases} > 0 & \text{if} \quad \pi(i), \pi(j) \in K, \\ < 0 & \text{if} \quad \pi(i), \pi(j) \in I, \end{cases}$$

$$\pi(i) < \pi(j) \qquad \text{if} \quad \pi(i) \in I, \quad \pi(j) \in K.$$

Then $P_{\pi(j)} = \{\pi(1),\dots,\pi(j)\}$ by (4.9) and (4.10).

We can now rearrange the n equations (4.8) so that the tth new equation is the $\pi(t)$th equation of (4.8), and also arrange the n variables $y_{S_1,T_1},\dots,y_{S_n,T_n}$

so that the tth new variable is $Y_{S_{\pi(t)}, T_{\pi(t)}}$. Then the new system has the form $Ay = b$ where the coefficient matrix A has zeros above the main diagonal and ones everywhere else, and the components of b are nonnegative and nondecreasing. This system has a unique, nonnegative solution y. This completes the proof.

COROLLARY 4.3. *The system* (4.2) *of linear inequalities is totally dual integral; that is, whenever $c_1, \dots, c_n$ are integers, the dual of* (4.3) *has an optimum solution in integers.*

For more information on totally dual integral systems, see [4, 20, 21].

In order to identify the facets of D_n, we need to know its dimension and that of a related polytope. For $m \geqslant 1$, $n \geqslant 1$, let $K_{m,n}$ be the complete bipartite graph with bipartition $\{1, 2, \dots, m\}$, $\{m+1, m+2, \dots, m+n\}$, and let $D_{m,n}$ be the convex hull of the degree sequences of the spanning subgraphs of $K_{m,n}$ (just as D_n is the convex hull of the degree sequences of the spanning subgraphs of K_n, the complete graph on $\{1, \dots, n\}$).

LEMMA 4.3

(a) $\dim D_1 = 0$, $\dim D_2 = 1$;
(b) $\dim D_{m,n} = m + n - 1$ *for* $m \geqslant 1$, $n \geqslant 1$;
(c) $\dim D_n = n$ *for* $n \geqslant 3$.

Proof. (a): Clear.

(b): Since $D_{m,n}$ lies on the hyperplane $\sum_{i=1}^{m} x_i = \sum_{i=m+1}^{m+n} x_i$, we have $\dim D_{m,n} \leqslant m + n - 1$. To prove the opposite inequality, consider any spanning tree of $K_{m,n}$ and let A be its edge-vertex incidence matrix. The $m + n - 1$ rows of A are linearly independent, and each of them lies in $D_{m,n}$. Hence $\dim D_{m,n} \geqslant m + n - 1$.

(c): Koren [15] proved (c) by showing that D_n has interior points for $n \geqslant 3$. We prove it as follows: for n odd, let H be any Hamiltonian cycle in K_n, and for n even, let H consist of any cycle through $n - 1$ vertices and a single edge at the remaining vertex. Let A be the edge-vertex incidence matrix of H. In each case A is nonsingular and its rows lie in D_n. Hence $\dim D_n \geqslant n$. ■

THEOREM 4.4. *For $n \geqslant 3$, the facets of D_n are given by:*

(a) $x_i \geqslant 0$, $i = 1, \dots, n$ (*only if* $n \geqslant 4$).
(b) $x_i \leqslant n - 1$, $i = 1, \dots, n$ (*only if* $n \geqslant 4$).
(c) $\sum_{i \in S} x_i - \sum_{i \in T} x_i \leqslant |S|(n - 1 - |T|)$ *for all sets* S, T *such that* $S \neq \varnothing$, $T \neq \varnothing$, $S \cap T = \varnothing$, $S \cup T \subseteq \{1, \dots, n\}$, $|S \cup T| = 2, 3, \dots, n-3, n$.

Proof. By Theorem 4.2, every facet of D_n has the form

$$\sum_{i\in S} x_i - \sum_{i\in T} x_i \leqslant |S|(n-1-|T|) \tag{4.11}$$

for some sets S,T with $\varnothing \neq S\cup T \subseteq \{1,\dots,n\}$, $S\cap T=\varnothing$.

Let $D_{S,T}$ denote the set of degree sequences satisfying (4.11) with equality. Thus by (c) of Lemma 4.3, (4.11) is a facet of D_n if and only if $\dim D_{S,T} = n-1$. From Lemma 4.1 it follows that $\dim D_{S,T} = \dim D_{|S|,|T|} + \dim D_{|\overline{S\cup T}|}$, where $\dim D_{|S|,|T|}$ means 0 if either S or T vanishes, and $\dim D_0$ is 0. Thus for $S=\varnothing$, (4.11) is a facet if and only if $|\bar{T}| = n-1$ and $|\bar{T}|\geqslant 3$, which corresponds to (a). Similarly, for $T=\varnothing$, (4.11) is a facet if and only if $|\bar{S}|=n-1$ and $|\bar{S}|\geqslant 3$, which corresponds to (b). Finally, for $S\neq\varnothing$, $T\neq\varnothing$, (4.11) is a facet if and only if $|S|+|T|-1+\dim D_{|\overline{S\cup T}|} = n-1$, or equivalently $\dim D_{|\overline{S\cup T}|} = |\overline{S\cup T}|$, which holds if and only if $|\overline{S\cup T}| = 0,3,4,\dots,n-2$ by Lemma 4.3. This case corresponds to (c). ■

5. THRESHOLD SEQUENCES AND MAJORIZATION

Definition 5.1. For any real sequence $f=(f_1,\dots,f_n)$, let $f_{[1]}\geqslant f_{[2]}\geqslant \cdots \geqslant f_{[n]}$ denote the components of f sorted in a nonincreasing order. We say that f *majorizes* g, denoted $f\succeq g$, when $\sum_{i=1}^k f_{[i]} \geqslant \sum_{i=1}^k g_{[i]}$ holds for each $k=1,\dots,n$, with equality holding for $k=n$. We say that f *strictly majorizes* g, denoted $f\succ g$, when $f\succeq g$ and at least one of the above inequalities for $k=1,\dots,n-1$ is strict. This means that the common sum of the components of f and g is distributed more evenly in g than in f.

Remark 5.2.

(*a*) *If f and g majorize each other, then f is a rearrangement of g, and conversely.*

(*b*) *If $f\succ g$, then f is not a rearrangement of g.*

(*c*) *Majorization and strict majorization are transitive.*

(*d*) *For a wealth of characterizations and applications of majorization, see* [12, 17]. *Reference* [17] *denotes majorization by* $\succ$, *not by* $\succeq$.

Definition 5.3. If $f=(f_1,f_2,\dots,f_n)$ be an integer sequence, and assume that $f_i\geqslant f_j+2$ for some i, j. Then the sequence $g=f-u_i+u_j$ is said

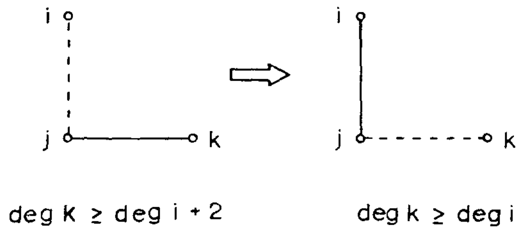

FIG. 13. A just rotation.

to be obtained from f by a *unit transformation from i to j* (u_t is the tth unit vector). In that case, clearly $f \succ g$. Conversely,

THEOREM 5.4 [12, 17]. *If f and g are integer sequences such that $f \succeq g$, then some rearrangement of g can be obtained from f by a sequence (possibly empty) of unit transformations.*

DEFINITION 5.5. Let $G = (V, E)$ be a graph, and let $i, j, k \in V$ be such that $\deg k \geqslant \deg i + 2$, $[i, j] \notin E$, $[j, k] \in E$. Then the operation of dropping from G the edge $[j, k]$ and adding the edge $[i, j]$ is called a *just rotation from k to i* (see Figure 13).

The adjective "just" is used in the sense of the "rich" vertex (k) giving some of its degree to the "poor" one (i). Thus, the degree sequence of G strictly majorizes the resulting degree sequence.

LEMMA 5.6. *A just rotation cannot result in a threshold graph.*

Proof. Note that $\deg k > \deg i$ in the resulting graph, yet i has a neighbor j that is not a neighbor of k. This would contradict Theorem 1.3 if the resulting graph were threshold. ∎

LEMMA 5.7. *Let d be the degree sequence of a graph $G = (V, E)$, and let d' result from d by a unit transformation. Then some just rotation in G results in a graph with degree sequence d'.*

Proof. Let d' result from d by a unit transformation from k to i. Then $d_k \geqslant d_i + 2$ in G, and therefore there is a vertex j such that $[i, j] \notin E$,

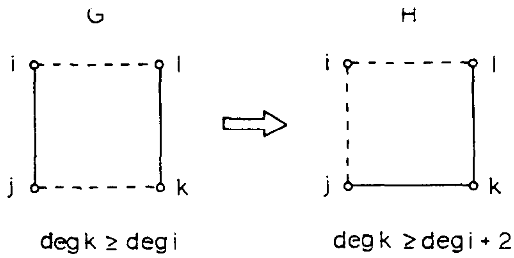

FIG. 14. Illustrating the "if" part of Theorem 5.8.

$[j, k] \in E$. Hence G has a just rotation from k to i, resulting in a graph with degree sequence d'. ■

We now characterize the threshold sequences by the property of nonmajorizability.

THEOREM 5.8. *A degree sequence d is threshold if and only if d is not strictly majorized by any degree sequence.*

Proof. "If": Assume d is the degree sequence of the nonthreshold graph $G = (V, E)$. By Corollary 1.2, G has vertices i, j, k, l, with $[i, j], [l, k] \in E$, $[i, l], [k, j] \notin E$ (Figure 14).

Without loss of generality, $\deg i \leqslant \deg k$ in G. Let the graph H result from G by deleting edge $[i, j]$ and adding edge $[j, k]$. The reverse operation is then a just rotation in H; hence the degree sequence of H strictly majorizes d.

"Only if": Assume $f \succ d$, and let G realize f. By Theorem 5.4, there are sequences $f^{(0)}, f^{(1)}, \ldots, f^{(t)}$ such that $f^{(0)} = f$, $f^{(t)}$ is a rearrangement of d, and some unit transformation on $f^{(i)}$ produces $f^{(i+1)}$ for $i = 0, \ldots, t-1$. By Lemma 5.7 it follows that some sequence of just rotations in G results in a graph with degree sequence $f^{(t)}$. By Lemma 5.6, $f^{(t)}$ is a nonthreshold sequence. Therefore, d is a nonthreshold sequence. ■

COROLLARY 5.9. *Every degree sequence is a convex combination of the rearrangements of a single threshold sequence (i.e., of the degree sequences of isomorphic threshold graphs).*

Proof. Let d be a degree sequence. If d is threshold, there is nothing to prove. Otherwise, d is strictly majorized by some degree sequence d' by

Theorem 5.8. Continue this process as long as possible. The progression $d, d', d'', \ldots$ has no repetitions by Remark 5.2, because each term is strictly majorized by the next one. It must terminate, because each of its terms is a sequence of n nonnegative integers with the same sum. By Theorem 5.8, the last sequence f is threshold and it majorizes d by transitivity. The result then follows from the well-known fact that f majorizes d if and only if d is a convex combination of the rearrangements of f [12, 17]. ■

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