q-Analogues of t-Designs and Their Existence

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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ABSTRACT

A t-$[v, k, \lambda]$ design in a vector space of dimension v over a finite field is a family of k-subspaces such that each t-subspace is contained in precisely λ elements of this family. They may be considered as a generalization of a spread in a projective space. It is shown that for given t, v, k a t-$[v, k, \lambda]$ design exists for all sufficiently large λ provided the necessary parametric conditions are satisfied. The result is proved by solving a much more general question. Analogues of these results for affine spaces are also proved. We also describe a reciprocity relation for the number of distinct t-$[v, k, \lambda]$ designs in a vector space, for given t, v, and k. This relation is similar to the one obtained by Shrikhande and Singhi for t-(v, k, λ) designs and by the authors for orthogonal arrays.

1. STATEMENT OF THEOREMS

Throughout the paper we will assume that V is a vector space of dimension v over a field GF(q). For any $0 \leqslant l \leqslant v$, $\mathbb{P}_l = \mathbb{P}_l(V)$ will denote

*This paper was written while the second author was visiting The Ohio State University, U.S.A.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

the set of all l-subspaces (subspaces of dimension l) of V. In particular $\mathbb{P}_1(V)$ is the projective space of dimension $v-1$. A *signed* t-$[v,k,\lambda]$ *design* in V is a function $f:\mathbb{P}_k(V)\to\mathbb{Z}$ (set of all integers) such that for all $T\in\mathbb{P}_t(V)$,

$$\sum f(B)=\lambda,$$

where the sum is over all $B\in\mathbb{P}_k(V)$ containing T. Such a function is said to be a t-$[v,k,\lambda]$ *design* in V if $f(B)\geqslant 0$ for all $B\in\mathbb{P}_k(V)$.

Clearly we can also think of a t-$[v,k,\lambda]$ design as a family f of k-subspaces of a vector space V such that each t-subspace is contained in precisely λ elements of f. In this sense t-$[v,k,\lambda]$ designs are q-analogues of usual t-(v,k,λ) designs for sets. Also t-$[v,k,\lambda]$ designs with $t=1$ and $\lambda=1$ are precisely classical spreads in a projective space of dimension $v-1$. Thus, general t-$[v,k,\lambda]$ designs may be also considered as higher dimensional analogues of spreads. For any two integers l and m, we denote by $\begin{bmatrix} l \\ m\end{bmatrix}$ the q-ary binomial coefficient,

$$\begin{bmatrix} l \\ m\end{bmatrix}=\frac{(q^l-1)(q^{l-1}-1)\cdots(q^{l-m+1}-1)}{(q^m-1)(q^{m-1}-1)\cdots(q-1)}.$$

In Section 2 we will prove the following theorems.

THEOREM 1.1. *Let* t-$[v,k,\lambda]$ *be any quadruple of integers,* $v\geqslant k\geqslant t\geqslant 0$. *A signed* t-$[v,k,\lambda]$ *design exists if and only if for all* $0\leqslant w\leqslant t$

$$\lambda\begin{bmatrix} v-w \\ t-w\end{bmatrix}\equiv 0 \bmod \begin{bmatrix} k-w \\ t-w\end{bmatrix}. \tag{0}$$

THEOREM 1.2. *There exists a function* $\lambda_0(t,v,k)$ *such that for all* t-$[v,k,\lambda]$ *with* $\lambda\geqslant\lambda_0(t,v,k)$ *a* t-$[v,k,\lambda]$ *design exists if and only if the above condition* (0) *is satisfied for all* $0\leqslant w\leqslant t$.

Theorems 1.1 and 1.2 may be considered as q-analogues of well-known results on t-(v,k,λ) designs due to Wilson [5] and Graver and Jukart [1]. Our proofs of these results are in similar style to that of Wilson. We actually consider a much more general question. For a function $F:\mathbb{P}_t(V)\to\mathbb{Z}$ an *integral k-realization* of F is a function $f:\mathbb{P}_k(V)\to\mathbb{Z}$ such that for all $T\in\mathbb{P}_t$

$$\sum f(B)=F(T),$$

where the sum is over all k-subspaces B containing T.

Theorem 2.4 gives necessary and sufficient conditions for the existence of such a realization. Theorem 1.1 and Theorem 1.2 follow immediately from Theorem 2.4. We note that our conditions in Theorem 2.4 are quite different from Wilson's condition [5, Theorem 3] in the case of sets. As noted in Remark 2.6, exact q-analogues of Wilson's conditions, though necessary, are not sufficient for the existence of integral k-realization. The main reason for this seems to be the fact that for any subspace $W \subseteq V$ the set $V \setminus W$ is not a subspace. This difficulty is overcome by considering the affine spaces. Thus, in the process we also prove the affine analogues of these results, which we describe now.

Fix a $(v-1)$-subspace H of V. Define $\mathbb{A}_l = \mathbb{A}_l(V, H) = \mathbb{P}_l(V) \setminus \mathbb{P}_l(H)$ for $1 \leqslant l \leqslant v$ and $\mathbb{A}_0 = \mathbb{A}_0(V, H) = \mathbb{P}_0(V)$. In particular, $\mathbb{A}_1$ is the usual affine space of dimension $v-1$. A *signed affine* t-$[v, k, \lambda]$ *design* in V is a function $f: \mathbb{A}_k \to \mathbb{Z}$ such that for all $T \in \mathbb{A}_t$

$$\sum f(B) = \lambda,$$

where the sum is over all $B \in \mathbb{A}_k$ which contain T. Such an f is an *affine* t-$[v, k, \lambda]$ *design* if $f(B) \geqslant 0$ for all $B \in \mathbb{A}_k$. For $F: \mathbb{A}_t \to \mathbb{Z}$, *integral k-realization* of F over the affine space is defined as in the projective case.

THEOREM 1.3. *Let t-$[v, k, \lambda]$ be a quadruple of integers, $v \geqslant k \geqslant t \geqslant 0$. A signed affine t-$[v, k, \lambda]$ design exists if and only if for all $1 \leqslant w \leqslant t$*

$$\lambda \begin{bmatrix} v-w \\ t-w \end{bmatrix} \equiv 0 \bmod \begin{bmatrix} k-w \\ t-w \end{bmatrix}.$$

THEOREM 1.4. *There exists a function $\lambda_0'(v, k, t)$ such that for any quadruple t-$[v, k, \lambda]$ of integers $v \geqslant k \geqslant t \geqslant 0$, $\lambda \geqslant \lambda_0'(v, k, t)$, an affine t-$[v, k, \lambda]$ design exists if and only if for all $1 \leqslant w \leqslant t$*

$$\lambda \begin{bmatrix} v-w \\ t-w \end{bmatrix} \equiv 0 \bmod \begin{bmatrix} k-w \\ t-w \end{bmatrix}.$$

Now, for a given t, v, and k, $v \geqslant k \geqslant t \geqslant 0$, let $S_{k,t}^v$ be a function from the set of nonnegative integers to itself, defined by $S_{k,t}^v(\lambda) =$ the number of distinct functions $f: \mathbb{P}_k(V) \to \mathbb{Z}$ which are t-$[v, k, \lambda]$ designs, where λ is any given nonnegative integer.

The following Theorem 1.5 and Theorem 1.6, which describe a reciprocity relation satisfied by $S_{k,t}^v$, can be easily obtained from the theorems of

Stanley and Hochster (see [4, Chapter 1]) on toroidal monoids (monoids of nonnegative integral solutions of linear equations), in a way similar to the reciprocity relations derived for the usual t-(v, k, λ) designs in [3] and for orthogonal arrays in [2]. We omit the proofs, since they are quite simple and essentially the same as those in [3, 2]. We also note here that an analogue of 3.3(a) in [3] or Lemma 4.2 in [2] for $v \geqslant k+t$ can be obtained from Theorem 2.5 in essentially the same way as the proof of Lemma 4.2 in [2]. For various concepts and definitions see [4] or [3].

THEOREM 1.5. *The function $S_{k,t}^{v}$ is a quasipolynomial. Further, if N is a quasiperiod and P_i, $0 \leqslant i \leqslant N-1$, are associated polynomials, then*

(i) *t-$[v, k, N]$ satisfy the necessary parametric conditions for the existence of designs, given by Equation* (0).
(ii) *$P_i \neq 0 \Leftrightarrow t$-$[v, k, i]$ satisfy the conditions* (0).

Using Theorem 1.5, the domain of quasipolynomial $S_{k,t}^{v}$ can be uniquely extended to all integers (see [4] or [3] for details).

THEOREM 1.6. *Given $t \leqslant k \leqslant v$, $v \geqslant k+t$, the quasipolynomial $S_{k,t}^{v}$ satisfies the following reciprocity relations:*

$$S_{k,t}^{v}(-1) = S_{k,t}^{v}(-2) = \cdots = S_{k,t}^{v}(-\rho+1) = 0$$

and

$$S_{k,t}^{v}(\lambda) = (-1)^{d} S_{k,t}^{v}(-\rho-\lambda),$$

where

$$d = \begin{bmatrix} v \\ k \end{bmatrix} - \begin{bmatrix} v \\ t \end{bmatrix}$$

and

$$\rho = \begin{bmatrix} v-t \\ k-t \end{bmatrix}.$$

We note here that from results of Section 2 it follows that for $v \leqslant k+t$, $S_{k,t}^{v}(\lambda)$ is either 0 or 1. Hence, in that case also, Theorem 1.6 is trivially true.

Finally we remark that methods of this paper may be used to prove similar results for other partially ordered sets, in particular for perfect matroid designs.

2. PROOFS

Let $F : \mathbb{P}_t(V) \to \mathbb{Z}$ be any function. In this section we study the question of the existence of integral k-realizations of F described in Section 1. We first fix some notation. For integers $k \geqslant t$ and m, l, define

$$c(k,t,m,l) = q^{(k-t+m-l)m}\begin{bmatrix} l \\ m \end{bmatrix}\begin{bmatrix} k-l \\ t-m \end{bmatrix}.$$

In particular, $c(k,t,m,l) = 0$ if $m > l$ or $t - m > k - l$, and

$$c(k,t,m,m) = q^{(k-t)m}\begin{bmatrix} k-m \\ t-m \end{bmatrix} \neq 0.$$

We also note here that for any k-subspace K of V and a $(k-l)$-subspace K_1 of K, $c(k,t,m,l)$ is the number of t-subspaces T of K satisfying the condition that the dimension of $T \cap K_1$ is $t - m$.

Let $d(k,t,m,l)$ be the unique rational number defined by the equations

$$d(k,t,m,l) = 0 \qquad \text{if} \quad l > m,$$

$$\sum_i d(k,t,m,i)c(k,t,i,l) = \delta_{m,l}, \tag{1}$$

where $\delta_{ml} = 1$ if $m = l$ and 0 if $m \neq l$. Note that the sum in (1) for any k, t, m, l is a finite sum over $m \leqslant i \leqslant l$. The values of $d(k,t,m,l)$ can be found in the same way as one inverts a triangular matrix or by considering $d(k,t,-,-)$ as an inverse of $c(k,t,-,-)$ in the incidence algebra of the partially ordered set $(\mathbb{Z}, \leqslant)$.

Now let W be any subspace of V, $\dim W = w$. Fix a complimentary subspace W' of W, i.e., $W' \subseteq V$, $W' \cap W = 0$, $\dim W' = v - w$. For any subspace U of V, $U \supseteq W$, define $\sigma_{W'}(U)$ to be the subspace $U \cap W'$. It can be easily seen that $\sigma_{W'}$ is a one-one onto map from the set of all subspaces U of V, $U \supset W$, to the set of all subspaces of W'. In particular, $\sigma_{W'}(U) \in \mathbb{P}_l(W')$, $l = \dim U - w$.

We now prove a simple lemma, which is one of the main tools for studying integral k-realizations. In the sequel we will often write $\mathbb{P}_i$ for $\mathbb{P}_i(V)$, $0 \leqslant i \leqslant v$.

LEMMA 2.1. *Let $f: \mathbb{P}_k \to \mathbb{Q}$ and $F: \mathbb{P}_t \to \mathbb{Q}$ be two rational valued functions such that for all $T \in \mathbb{P}_t$,*

$$\sum f(K) = F(T)$$

where the sum is over all $K \supseteq T$. Then for any w-subspace W of V, $0 \leqslant w \leqslant t$, and any $(v - t + w)$-subspace U of V, $U \supseteq W$,

$$\sum f(K) = \sum d(k - w, t - w, 0, t - \dim U \cap T) F(T), \tag{2}$$

where the sum on the left side is over all $W \subseteq K \subseteq U$ and the sum on the right side is over all $T \supseteq W$, $T \in \mathbb{P}_t$.

Proof. We first note that by considering $\sigma_W(V)$, $\sigma_W(U)$, $\sigma_W F \sigma_W^{-1}$, $\sigma_W f \sigma_W^{-1}$ in place of V, U, F, and f respectively, if necessary, we can assume that $\dim W = 0$. Thus, we have only to prove that

$$\sum f(K) = \sum d(k, t, 0, t - \dim U \cap T) F(T) \tag{3}$$

where the sum on the left side is over all k-subspaces $K \subseteq U$, and the sum on the right side is over all $T \in \mathbb{P}_t$. This follows easily using the equations (1) and the following equations for $0 \leqslant g \leqslant t$:

$$\sum\nolimits_g F(T) = \sum_{h=0}^{t} c(k, t, g, h) \sum\nolimits_h f(K)$$

where $\sum_g$ denotes the sum of all $F(T)$, $T \in \mathbb{P}_t$, with $t - \dim U \cap T = g$, and $\sum_h$ denotes the sum of all $f(K)$, $K \in \mathbb{P}_k$, with $k - \dim U \cap K = h$. ∎

COROLLARY 2.2. *Let $k \geqslant t$, $v \leqslant k + t$, and $F: \mathbb{P}_t \to \mathbb{Q}$ be any function. Suppose $f: \mathbb{P}_k \to \mathbb{Q}$ is any function such that for all $T \in \mathbb{P}_t$*

$$\sum f(K) = F(T)$$

where the sum is over all $K \supseteq T$. Then for all $K \in \mathbb{P}_k$ and any w-subspace W of K, $w = k + t - v$,

$$f(K) = \sum d(v - t, v - k, 0, k - \dim K \cap T) F(T), \tag{4}$$

where the sum is over all T containing W.

Proof. Let K and W be as described above. Taking $U = K$ and using Equation (2) for W, we have

$$f(K) = \sum_T d(v-t, v-k, 0, k - \dim K \cap T) F(T).$$

This completes the proof. ■

Corollary 2.3. *Let $k \geqslant t$, $v = k + t$, and $F : \mathbb{P}_t \to \mathbb{Q}$ be any function. Then there exists a unique function $f : \mathbb{P}_k \to \mathbb{Q}$ such that*

$$\sum f(K) = F(T),$$

where the sum is over all $K \supseteq T$. Further, f is given by Equation (4), with $W = \langle 0 \rangle$, the 0-subspace.

Proof. For each $f : \mathbb{P}_k \to \mathbb{Q}$, define a function $F_f : \mathbb{P}_t \to \mathbb{Q}$, where for all $T \in \mathbb{P}_t$, $\Sigma f(K) = F_f(T)$, where the sum is over all $K \supseteq T$. Now Corollary 2.2 shows that the map $f \to F_f$ is a one-one map from the vector space of rational valued functions on $\mathbb{P}_k$ to the vector space of rational valued functions on $\mathbb{P}_t$. Since the dimensions of these two vector spaces are both equal to

$$\begin{bmatrix} v \\ k \end{bmatrix} = \begin{bmatrix} v \\ t \end{bmatrix},$$

it follows that the above map is onto. This proves existence of the required f. The uniqueness follows from the previous corollary. ■

We note that Corollary 2.2 essentially answers the question of existence of integral k-realizations for $v \leqslant k + t$. We now state the main theorems of this section.

Theorem 2.4. *Let $k \geqslant t$ and $v \geqslant k + t$. Let V be a v-dimensional vector space over a finite field. Let $F : \mathbb{P}_t \to \mathbb{Z}$ be a function. Then F has an integral k-realization f if and only if the following holds: For any w-subspace W of V, $0 \leqslant w \leqslant t$, and $(v - t + w)$-subspace U of V, $U \supseteq W$,*

$$\sum d(k-w, t-w, 0, t - \dim U \cap T) F(T) \tag{5}$$

is an integer, where the sum is over all $T \in \mathbb{P}_t$, $T \supseteq W$.

THEOREM 2.5. *Let $k \geqslant t$ and $v \geqslant k+t$. Let $F : \mathbb{A}_t \to \mathbb{Z}$ be an function. F has an integral k-realization f or the affine space if and only if the following holds: For any w-subspace W of V, $W \in A_w$, $1 \leqslant w \leqslant t$, and $(v-t+w)$-subspace U of V, $U \supseteq W$,*

$$\sum d(k-w, t-w, 0, t - \dim U \cap T)F(T)$$

is an integer, where the sum is over all $T \in \mathbb{P}_t$, $T \supseteq W$.

Proof of Theorems 2.4 and 2.5. We first note that the necessity part in both the theorems follows from Lemma 2.1. We will prove the sufficiency part for both theorems together by double induction on v and t. We first note that the sufficiency of the given condition in Theorem 2.4 follows easily for $v = k+t$ from Corollary 2.3. Similarly, sufficiency for $t = 0$ in both theorems can be seen easily; in fact, given $F : \mathbb{P}_0 \to \mathbb{Z}$ or $(F : \mathbb{A}_0 \to \mathbb{Z})$ and any $K \in \mathbb{P}_k$ (or $\mathbb{A}_k$), the function $f : \mathbb{P}_k \to \mathbb{Z}$ (or $f : \mathbb{A}_k \to \mathbb{Z}$) defined by $f(S) = F(\langle 0 \rangle)$ if $S = K$ and $f(S) = 0$ otherwise, serves the purpose.

Now suppose we are given $F : \mathbb{A}_t \to \mathbb{Z}$ satisfying the given condition in Theorem 2.5, with $t > 0$, $v \geqslant k+t$. Note that H is a $(v-1)$-subspace of V and $\mathbb{A}_t = \mathbb{A}_t(V, H) = \mathbb{P}_t(V) \backslash \mathbb{P}_t(H)$. Now, let $x \in V$, $x \neq 0$, and $\langle x \rangle$ be the subspace of V generated by x, and let $\langle x \rangle'$ be a $(v-1)$-dimensional subspace such that $\langle x \rangle \cap \langle x \rangle' = \langle 0 \rangle$. For any l, $1 \leqslant l \leqslant v$, let $\mathbb{A}_{lx}$ be the set of all l-subspaces $W \in \mathbb{A}_l$ containing x, and let $F_1 = \sigma_{\langle x \rangle}(F|\mathbb{A}_{tx})\sigma_{\langle x \rangle}^{-1}$. Now if $x \in V \backslash H$, it can be easily seen from the given conditions for F that F_1 is a map from $\mathbb{P}_{t-1}(\langle x \rangle')$ to $\mathbb{Z}$ satisfying conditions of Theorem 2.4 with V, F, k, t replaced by $\langle x \rangle', F_1, k-1, t-1$ respectively and $\dim\langle x \rangle = v - 1 \geqslant (k-1) + (t-1)$. Hence, by induction we can assume that there exists a function $f_x : \mathbb{P}_{k-1}(\langle x \rangle') \to \mathbb{Z}$ such that for all $T \in \mathbb{P}_{t-1}(\langle x \rangle')$

$$\sum f_x(K) = F_1(T), \tag{6}$$

where the sum is over all $K \in \mathbb{P}_{k-1}(\langle x \rangle')$ containing T. Similarly, if $x \in H$, it can be easily seen that F_1 is a map from $\mathbb{A}_{t-1}(\langle x \rangle', H')$ to $\mathbb{Z}$, $H' = \langle x \rangle' \cap H$, satisfying the given conditions of Theorem 2.5 with V, F, k, t replaced by $\langle x \rangle', F_1, k-1, t-1$, and where $(v-1) = \dim\langle x \rangle' > (k-1)+(t-1)$. Hence, by induction, again we can find an $f_x : \mathbb{A}_{k-1}(\langle x \rangle', H') \to \mathbb{Z}$ such that for all $T \in \mathbb{A}_{t-1}(\langle x \rangle', H')$

$$\sum f_x(K) = F_1(T), \tag{7}$$

where the sum is over all $K \in A_{k-1}(\langle x \rangle', H')$ containing T.

It can be easily checked that for each $x \in V$, $\sigma_{\langle x\rangle}^{-1} f_x \sigma_{\langle x\rangle}$ is a map from $\mathbf{A}_{kx}$ to $\mathbf{Z}$. Let

$$f = \sum_{x \in V \setminus H} \left(\sigma_{\langle x\rangle}^{-1} f_x \sigma_{\langle x\rangle}\right) - (q-1) \sum_{x \in H} \left(\sigma_{\langle x\rangle}^{-1} f_x \sigma_{\langle x\rangle}\right).$$

Using the fact that for any $T \in \mathbf{A}_t$

$$|T \cap H| = \frac{q^{t-1}-1}{q-1} \quad \text{and} \quad |T \setminus (T \cap H)| = q^{t-1}$$

and (6) and (7) above, it can be easily seen that the map $f: \mathbf{A}_k \to \mathbf{Z}$ is a k-realization for $F: \mathbf{A}_t \to \mathbf{Z}$.

Now suppose that we are given a function $F: \mathbf{P}_t \to \mathbf{Z}$ satisfying the conditions of Theorem 2.4. Let H be a $(v-1)$-subspace of V, and let $F_1 = F|(\mathbf{A}_t(V, H))$. From the conditions on F it can be easily seen that conditions of Theorem 2.5 are satisfied with F replaced by F_1. Hence, using the previous part of the proof, we can find a function $f_1: \mathbf{A}_k(V, H) \to \mathbf{Z}$ such that for all $T \in \mathbf{A}_t$,

$$\sum f_1(K) = F_1(T), \tag{8}$$

where the sum is over all $K \in \mathbf{A}_k$, $K \supseteq T$. Let $f': \mathbf{P}_k \to \mathbf{Z}$ be defined by $f'(K) = f_1(K)$ if $K \in \mathbf{A}_k(V, H)$, and $f'(K) = 0$ otherwise, let $F': \mathbf{P}_t \to \mathbf{Z}$ be defined by

$$F'(T) = \sum f'(K),$$

where the sum is over all $K \in \mathbf{P}_k$ containing T. From the necessity part of Theorem 2.4 it follows that the numbers defined in (5) with F replaced by F' are integers. Also, using (8), $F'(T) = F(T)$ for all $T \in \mathbf{A}_t$. From this it follows that if $F_2 = (F - F')|\mathbf{P}_t(H)$, the conditions of Theorem 2.4 are satisfied with V and F replaced by H and F_2. Since $\dim H = v - 1$, using induction we can assume that there exists $f_2: \mathbf{P}_k(H) \to \mathbf{Z}$ such that for all $T \in \mathbf{P}_t(H)$

$$\sum f_2(K) = F_2(T), \tag{9}$$

where the sum is over all $K \in \mathbf{P}_k(H)$ containing T. Now using (8) and (9) it can be easily seen that $f: \mathbb{P}_k(V) \to \mathbb{Z}$ defined by

$$f(K) = \begin{cases} f_1(K) & \text{if} \quad K \in \mathbb{A}_k(V, H), \\ f_1(K) + f_2(K) & \text{if} \quad K \in \mathbb{P}_k(H) \end{cases}$$

is an integral k-realization for F. This completes the proof of sufficiency for both the theorems by induction. ■

REMARK 2.6. Let $F: \mathbb{P}_t \to \mathbb{Z}$ and $f: \mathbb{P}_k \to \mathbb{Z}$ be two functions such that f is an integral k-realization for F. It can be easily seen that for any subspace W of V, $\dim W = w \leqslant t$,

$$\sum_{W \subseteq T} F(T) \equiv 0 \bmod \begin{bmatrix} k-w \\ t-w \end{bmatrix} \tag{10}$$

In fact, the left side of (10) is precisely

$$\begin{bmatrix} k-w \\ t-w \end{bmatrix} \sum_{W \subseteq K} f(K).$$

Thus, a necessary condition for the existence of an integral k-realization for a given F is that (10) is satisfied for all w-subspaces W. We note that the condition (10) is precisely the q-analogue of the necessary and sufficient condition in Theorem 3 of Wilson [5]. However, the following example shows that these conditions are not sufficient for existence of an f. Let $t = 1$, and $T \in \mathbb{P}_1(V)$ be a given element of V. Define $F: \mathbb{P}_1(V) \to \mathbb{Z}$ by $F(S) = (q^K - 1)/(q-1)$ if $S = T$ and $F(S) = 0$ otherwise. F clearly satisfies (10). However, for this F, no f satisfying (5) can exist for $v \geqslant k+1$, as F does not satisfy the conditions of Theorem 2.4. However, there may be a way of expressing necessary and sufficient conditions in Theorem 2.4 which is combinatorially more enlightening.

Proof of Theorem 1.1. For $0 \leqslant l \leqslant v$, let $j_l: \mathbb{P}_l(V) \to \mathbb{Z}$ be defined by $j_l(B) = 1$ for all $B \in \mathbb{P}_l(V)$. Now, clearly, for all $T \in \mathbb{P}_t$,

$$\sum_{T \subseteq K} j_k(K) = \begin{bmatrix} v-t \\ k-t \end{bmatrix} j_t(T).$$

From this, using Lemma 2.1 for

$$f = j_k \quad \text{and} \quad F = \begin{bmatrix} v-t \\ k-t \end{bmatrix} j_t,$$

we have, for any w-subspace W and $(v-t+w)$-subspace U, $0 \leqslant w \leqslant t$,

$$\begin{bmatrix} v-t \\ k-w \end{bmatrix} = \sum j_k(K) = \begin{bmatrix} v-t \\ k-t \end{bmatrix} \sum d(k-w, t-w, 0, t - \dim U \cap T) j_t(T),$$

where the sums are as described in Lemma 2.1. Thus,

$$\frac{\begin{bmatrix} v-t \\ k-w \end{bmatrix}}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}} = \sum d(k-w, t-w, 0, t - \dim U \cap T) j_t(T).$$

From this it is clear that for $v \geqslant k+t$, λj_t satisfies the conditions of Theorem 2.4 if and only if

$$\frac{\lambda \begin{bmatrix} v-t \\ k-w \end{bmatrix}}{\begin{bmatrix} v-t \\ k-t \end{bmatrix}} \text{ is an integer for all } 0 \leqslant w \leqslant t.$$

Using

$$\sum \begin{bmatrix} (v-w)-(k-w) \\ (t-w-i) \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} = \begin{bmatrix} v-w \\ t-w \end{bmatrix},$$

$$\sum (-1)^i \begin{bmatrix} v-w \\ t-w-i \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} = \begin{bmatrix} (v-w)-(k-w) \\ t-w \end{bmatrix}$$

for $0 \leqslant w \leqslant t$ (sums are over $0 \leqslant i \leqslant t-w$), it can be easily seen that the above conditions are equivalent to the conditions (0) in the statement of Theorem 1.1. Using Theorem 2.4, the proof of Theorem 1.1 is now complete for $v \geqslant k+t$. For $v \leqslant k+t$, from the above discussion and Corollary 2.2 one can easily see that the only signed t-$[v, k, \lambda]$ design is the trivial design

$$\lambda\left(\begin{bmatrix} v-t \\ k-t \end{bmatrix}^{-1}\right) j_k,$$

which is indeed a t-$[v, k, \lambda]$ design if the given conditions are satisfied. ∎

Proof of Theorem 1.2. Let t, v, k be given, $v \geqslant k \geqslant t \geqslant 0$. For each λ satisfying the conditions (0), let $f_\lambda : \mathbb{P}_k[V] \to \mathbb{Z}$ be a signed t-$[v, h, \lambda]$ design given by Theorem 1.1. Let

$$\lambda_0(v, k, t) = \begin{bmatrix} v-t \\ k-t \end{bmatrix} \max\left\{ f_\lambda(K) \,\middle|\, K \in \mathbb{P}_k,\ 0 \leqslant \lambda < \begin{bmatrix} v-t \\ k-t \end{bmatrix},\ \lambda \text{ satisfying } (0) \right\}.$$

Now given any $\lambda \geqslant \lambda_0(v, k, t)$, let

$$\lambda = m\begin{bmatrix} v-t \\ k-t \end{bmatrix} + \lambda',\ 0 \leqslant \lambda' < \begin{bmatrix} v-t \\ k-t \end{bmatrix}.$$

It can be easily seen that λ satisfies the conditions (0) if and only if λ' also satisfies them. It follows easily that $f = mj_k + f_{\lambda'}$ is a signed t-$[v, k, \lambda]$ design with $f(K) \geqslant 0$ for all $K \in \mathbb{P}_k$. This completes the proof. ∎

Theorems 1.3 and 1.4 can similarly be deduced from Theorem 2.5.

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Received 9 February 1988; final manuscript accepted 3 March 1988