On Cutting-Plane Proofs in Combinatorial Optimization

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To Alan J. Hoffman with respect and affection.

Submitted by Alexander Schrijver

ABSTRACT

Gomory's cutting-plane technique can be viewed as a recursive procedure for proving the validity of linear inequalities over the set of all integer vectors in a prescribed polyhedron. The number of rounds of cutting planes needed to obtain all valid linear inequalities is known as the rank of the polyhedron. We prove that polyhedra featured in popular formulations of the stable-set problem, the set-covering problem, the set-partitioning problem, the knapsack problem, the bipartite-subgraph

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problem, the maximum-cut problem, the acyclic-subdigraph problem, the asymmetric traveling-salesman problem, and the traveling-salesman problem have arbitrarily high rank. In particular, we prove conjectures of Barahona, Grötschel, and Mahjoub; Chvátal; Grötschel and Pulleyblank; and Jünger.

1. INTRODUCTION

How do you certify that there are no k pairwise nonadjacent vertices in a prescribed graph? How do you certify that there is no acyclic subgraph with k arcs in a prescribed directed graph? How do you certify that there is no tour of length at most k in a prescribed instance of the traveling-salesman problem? One way is to first state the problem as an integer linear-programming problem and then solve it by a cutting-plane algorithm of the kind designed by Gomory (1958, 1960, 1963): like all algorithms, the cutting-plane algorithm must provide a certificate when it solves the problem. The subject of this paper is the structure of the cutting-plane certificates for several popular formulations of combinatorial optimization problems. Our results state that, for certain instances of these formulations, all cutting-plane certificates must be complex in a sense.

To make this statement more precise, consider a polyhedron P in the n-dimensional Euclidean space $\mathbf{R}^n$; as usual, let $\mathbf{Z}^n$ denote the set of all vectors in $\mathbf{R}^n$ all of whose components are integers; in addition, set

$$P' = \{x \in P : \mathbf{a}^T\mathbf{x} \leqslant b \text{ whenever } \mathbf{a} \in \mathbf{Z}^n,\ b \in \mathbf{Z},$$
$$\text{and } \max\{\mathbf{a}^T\mathbf{x} : \mathbf{x} \in P\} < b + 1\}.$$

Trivially, $P \cap \mathbf{Z}^n \subseteq P'$; if we define $P^{(0)} = P$ and, recursively,

$$P^{(j)} = (P^{(j-1)})'$$

for all positive integers j, then $P \cap \mathbf{Z}^n \subseteq P^{(j)}$ for all nonnegative integers j. Hence, letting P_I denote the convex hull of $P \cap \mathbf{Z}^n$, we have $P_I \subseteq P^{(j)}$ for all nonnegative integers j.

Theorem 1.1. *For every bounded polyhedron P there is a nonnegative integer j such that $P^{(j)} = P_I$.*

The validity of Theorem 1.1 follows from Gomory's own analysis of his cutting-plane algorithms; an alternative proof was given by Chvátal (1973b).

As pointed out by Schrijver (1980), the assumption that P is bounded cannot be dropped: for instance, if c is an irrational number and P in $\mathbf{R}^2$ is defined by $x_2 = cx_1$, then $P^{(j)} = P$ for all j but P_I includes only the origin. However, Schrijver (1980) also proved that this assumption can be replaced by another: rather than assuming that P is bounded, we may assume that P is rational (that is, P consists of all solutions of some system $\mathbf{Ax} \leqslant \mathbf{b}$ of linear inequalities such that all entries of $\mathbf{A}$ and $\mathbf{b}$ are integers).

THEOREM 1.2. *For every rational polyhedron P there is a nonnegative integer j such that $P^{(j)} = P_I$.*

Incidentally, Theorem 1.1 follows easily from Theorem 1.2.

The *rank* of a polyhedron P is the smallest j such that $P^{(j)} = P_I$. Our results provide lower bounds on the rank of polyhedra featured in popular formulations of

(1) the stable-set problem (Theorems 3.2 and 3.3),
(2) the set-covering problem (Theorem 3.4),
(3) the set-partitioning problem (Theorem 3.5),
(4) the knapsack problem (Theorem 3.6),
(5) the bipartite-subgraph problem (Theorem 4.1),
(6) the maximum-cut problem (Theorem 4.3),
(7) the acyclic-subdigraph problem (Theorem 5.1),
(8) the asymmetric traveling-salesman problem (Theorem 8.1),
(9) the traveling-salesman problem (Theorems 8.3 and 8.4).

In particular, we prove conjectures of Barahona, Grötschel, and Mahjoub (Theorem 4.1), Chvátal (Theorem 8.3), Grötschel and Pulleyblank (Theorem 8.5), and Jünger (Theorem 5.5), and answer a question of Schrijver (Theorem 3.3).

We still have not explained the relationship between the rank of polyhedra and the complexity of cutting-plane certificates; discussion of this relationship is postponed till Section 6, where cutting-plane proofs will be defined.

We close the present section with a few comments on terminology and notation.

Let P be a polyhedron in R^n, and let $\mathbf{a}^T\mathbf{x} \leqslant b$ be an inequality valid over $P \cap \mathbf{Z}^n$. The *depth* of this inequality relative to P is the smallest d such that $\mathbf{a}^T\mathbf{x} \leqslant b$ is valid over $P^{(d)}$. Thus the rank of P is the maximum depth, relative to P, of an inequality valid over $P \cap \mathbf{Z}^n$. In fact, our lower bounds on the rank of P are typically presented by exhibiting an inequality valid over $P \cap \mathbf{Z}^n$ and establishing a lower bound on its depth relative to P.

We let $\mathbf{Z}_+$ denote the set of all nonnegative integers. Whenever convenient, we ignore the artificial distinction between real vectors with components subscripted by elements of a finite set V on the one hand and mappings from V to $\mathbf{R}$ on the other hand; we let $\mathbf{R}^V$ denote the set of these objects (and similarly for $\mathbf{Z}^V$ and $\mathbf{Z}_+^V$). We let $\mathbf{e}$ denote the vector with all components equal to 1 (and length understood from the context), and we let e denote the base of natural logarithms 2.71828...; to avoid overworking this letter, we let the mnemonic h denote a typical edge of a graph. Each edge of an undirected graph is a set of two vertices; thus, for instance, the statement "edge h has precisely one endpoint in the set W" can be recorded as "$|h \cap W| = 1$". All three expressions

$$\sum_{i \in S} x_i, \qquad \sum (x_i : i \in S), \quad \text{and} \quad \mathbf{x}(S)$$

have the same meaning and are used interchangeably. The natural logarithm of t is $\ln t$, and the binary logarithm of t is $\log_2 t$. As usual, $\lfloor t \rfloor$ and $\lceil t \rceil$ denote t rounded down and up, respectively, to the nearest integer.

2. FOUR LEMMAS

Our first lemma will be used four times in the following three sections. It guarantees that certain points lie in $P^{(j)}$ for all small values of j. These points are on the half line that starts at a point $\mathbf{u}$ in P and goes in the direction opposite to a vector $\mathbf{v}$.

LEMMA 2.1. *Let P be a rational polyhedron in $\mathbf{R}^n$; let $\mathbf{u}$ and $\mathbf{v}$ be points in $\mathbf{R}^n$; let $m_1, m_2, \ldots, m_d$ be positive numbers; write*

$$\mathbf{x}^{(j)} = \mathbf{u} - \left(\sum_{i=1}^{j} \frac{1}{m_i} \right) \mathbf{v}. \qquad (j = 0, 1, \ldots, d).$$

If $\mathbf{u} \in P$ and if, for all $j = 1, \ldots, d$, every inequality $\mathbf{a}^T\mathbf{x} \leqslant b$ valid over $P \cap \mathbf{Z}^n$ with $\mathbf{a} \in \mathbf{Z}^n$ and $\mathbf{a}^T\mathbf{v} < m_j$ satisfies $\mathbf{a}^T\mathbf{x}^{(j)} \leqslant b$, then $\mathbf{x}^{(j)} \in P^{(j)}$ for all $j = 0, 1, \ldots, d$.

Proof. By induction on j; note that $\mathbf{x}^{(0)} = \mathbf{u}$ and that $\mathbf{u} \in P^{(0)}$ by assumption. Now assume that $\mathbf{x}^{(j-1)} \in P^{(j-1)}$ for some positive integer j not exceeding d. If $\mathbf{x}^{(j)} \notin P^{(j)}$ then $\mathbf{a}^T\mathbf{x}^{(j)} > b$ for some $\mathbf{a}$ and b such that

$$\mathbf{a} \in \mathbf{Z}^n, \qquad b \in \mathbf{Z}, \quad \text{and} \quad \max\{\mathbf{a}^T\mathbf{x} : \mathbf{x} \in P^{(j-1)}\} < b + 1. \tag{2.1}$$

Hence we only need show that $\mathbf{a}^T\mathbf{x}^{(j)} \leqslant b$ whenever (2.1) holds. For this purpose, we may assume that $\mathbf{a}^T\mathbf{v} \geqslant m_j$ (otherwise the desired conclusion is guaranteed by assumption). Since $\mathbf{a}^T\mathbf{x} < b + 1$ is valid over $P^{(j-1)}$, we have

$$\mathbf{a}^T\mathbf{x}^{(j)} = a^T\mathbf{x}^{(j-1)} - \frac{1}{m_j}\mathbf{a}^T\mathbf{v} < (b+1) - 1 = b,$$

as claimed. ∎

On several occasions, we shall establish a lower bound on the rank of a polyhedron T indirectly, by transforming another polyhedron S into a subset of T. Lemma 2.3 spells out conditions which are sufficient to guarantee that the rank of T is at least the rank of S. The first part of the argument will be used on its own twice in Section 8; we set it out separately as Lemma 2.2.

LEMMA 2.2. *Let* $\mathbf{C}$ *be an integer matrix of size* $t \times s$, *and let* $\mathbf{d} \in \mathbf{Z}^t$; *let* $f: \mathbf{R}^s \to \mathbf{R}^t$ *be defined by* $f(\mathbf{x}) = \mathbf{C}\mathbf{x} + \mathbf{d}$; *let* S *be a polyhedron in* $\mathbf{R}^s$, *and let* T *be a polyhedron in* $\mathbf{R}^t$. *If* $f(S) \subseteq T$ *then*

$$f(S^{(i)}) \subseteq T^{(i)} \tag{2.2}$$

for all nonnegative integers i.

Proof. By induction on i; note that (2.2) with $i = 0$ holds by assumption. Now assume that (2.2) holds for some nonnegative integer i, and consider an arbitrary $\mathbf{x}^*$ in $S^{(i+1)}$; we need only show that $f(\mathbf{x}^*) \in T^{(i+1)}$. Since $\mathbf{x}^* \in S^{(i)}$, we have $f(\mathbf{x}^*) \in T^{(i)}$ by (2.2); hence our task reduces to showing that $\mathbf{a}^T f(\mathbf{x}^*) \leqslant b$ whenever $\mathbf{a} \in \mathbf{Z}^t$, $\mathbf{b} \in \mathbf{Z}$, and

$$\max\{\mathbf{a}^T\mathbf{y} : \mathbf{y} \in T^{(i)}\} < b + 1.$$

The last inequality combined with (2.2) guarantees that

$$\max\{\mathbf{a}^T(\mathbf{C}\mathbf{x}+\mathbf{d}):\mathbf{x}\in S^{(i)}\} < b+1;$$

since $\mathbf{C}$ and $\mathbf{d}$ are integral, it follows that

$$\max\{\mathbf{a}^T(\mathbf{C}\mathbf{x}+\mathbf{d}):\mathbf{x}\in S^{(i+1)}\} \leqslant b;$$

in particular, $\mathbf{a}^T f(\mathbf{x}^*) \leqslant \mathbf{b}$. ■

LEMMA 2.3. *Let the assumptions of Lemma 2.2 hold; in addition, let f be one-to-one and let $f(S\cap \mathbf{Z}^s)\supseteq T\cap \mathbf{Z}^t$. Then the rank of T is at least the rank of S.*

Proof. Since $f(S\cap\mathbf{Z}^s)\supseteq T\cap\mathbf{Z}^t$, and f is a one-to-one linear function, we have

$$f^{-1}(T_I)\subseteq S_I. \tag{2.3}$$

Now let r be the rank of S; we only need show that the rank of T is at least r. If $r=0$, then the desired conclusion is trivial; if $r>0$, then there is a point $\mathbf{x}$ in $S^{(r-1)}-S_I$ and (2.2), (2.3) guarantee that $f(\mathbf{x})\in T^{(r-1)}-T_I$. ■

Recall that a *face* of a polyhedron P is the intersection of P with a hyperplane $\{\mathbf{x}:\mathbf{a}^T\mathbf{x}=b\}$ such that

$$b=\max\{\mathbf{a}^T\mathbf{x}:\mathbf{x}\in P\}.$$

LEMMA 2.4. *If S is a face of T, then the rank of T is at least the rank of S.*

Proof. Since $S\subseteq T$, we have trivially

$$S^{(i)}\subseteq T^{(i)} \qquad \text{for all nonnegative integers } i. \tag{2.4}$$

In addition, we claim that

$$S\cap T_I\subseteq S_I. \tag{2.5}$$

To justify (2.5), consider an arbitrary $\mathbf{x}$ in $S\cap T_I$. By definition, $\mathbf{x}$ is a convex combination of points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ in $T \cap \mathbf{Z}^n$. Since $\mathbf{a}^T\mathbf{x} = b$ and $\mathbf{a}^T\mathbf{x}^{(i)} \leqslant b$ for all i, we must have $\mathbf{a}^T\mathbf{x}^{(i)} = b$ for all i. To put it differently, $\mathbf{x}^{(i)} \in S$ for all i, and so $\mathbf{x} \in S_I$.

The rest is straightforward. Let r be the rank of S; we only need show that the rank of T is at least r. If $r = 0$ then the desired conclusion is trivial; if $r > 0$, then there is a point $\mathbf{x}$ in $S^{(r-1)} - S_I$ and (2.4), (2.5) guarantee that $\mathbf{x} \in T^{(r-1)} - T_I$. ∎

3. STABLE SETS, SET COVERING, SET PARTITIONING, AND THE KNAPSACK PROBLEM

Let G be an undirected graph with vertex set V; let $\mathbf{C}$ denote the family of all cliques in G; let P denote the polyhedron in $\mathbf{R}^V$ defined by

$$\begin{aligned} \mathbf{x}(C) &\leqslant 1 \qquad \text{for all } C \text{ in } \mathbf{C}, \\ \mathbf{x}_v &\geqslant 0 \qquad \text{for all } v \text{ in } V. \end{aligned} \tag{3.1}$$

The problem of maximizing a linear function over $P \cap \mathbf{Z}^V$ is known as the *maximum-weight stable-set problem*, and P_I is called the *stable-set polytope* of G. (These terms are explained by observing that an integer vector belongs to P if and only if it is the incidence vector of a set of pairwise nonadjacent vertices; such sets are called *stable*.) The stable-set polytope has been studied by Chvátal (1973b, 1975), Padberg (1973, 1977), Nemhauser and Trotter (1974), Trotter (1975), Wolsey (1976), Balas and Zemel (1977), Boulala and Uhry (1979), Ben Rebea (1981), Giles and Trotter (1981), Fonlupt and Uhry (1982), Sbihi and Uhry (1984), and Gerards and Schrijver (1986).

Our first task in this section is to show that the rank of (3.1) may increase as fast as a linear function of the number of vertices of G [the lower bound established previously by Chvátal (1973b) is only logarithmic in the number of vertices]. We begin with a lemma, in which $\alpha(G)$ stands for the largest number of pairwise nonadjacent vertices in G.

LEMMA 3.1. *Let G be a graph with n vertices and let k, s be positive integers. If $k < s$ and if every subgraph of G with s vertices is k-colorable, then the depth of $\mathbf{e}^T\mathbf{x} \leqslant \alpha(G)$ relative to* (3.1) *is at least*

$$\frac{s}{k} \ln \frac{n}{k\alpha(G)}.$$

Proof. Writing

$$\mathbf{x}^{(j)} = \frac{1}{k}\left(\frac{s}{s+k}\right)^j \mathbf{e},$$

we only need show that $\mathbf{x}^{(j)} \in P^{(j)}$ for all j: if $j < (s/k)\ln[n/k\alpha(G)]$ then

$$\mathbf{e}^T\mathbf{x}^{(j)} = \frac{n}{k}\left(\frac{s}{s+k}\right)^j \geqslant \frac{n}{k}e^{-jk/s} > \alpha(G).$$

To show that $\mathbf{x}^{(j)} \in P^{(j)}$ for all j, we only need verify the assumptions of Lemma 2.1 with

$$\mathbf{u} = \frac{1}{k}\mathbf{e}, \qquad \mathbf{v} = \mathbf{e}, \quad \text{and} \quad m_j = s\left(\frac{s+k}{s}\right)^j.$$

Since G contains no clique with more than k vertices, we have $\mathbf{u} \in P$. Now consider an arbitrary inequality $\mathbf{a}^T\mathbf{x} \leqslant b$ valid over $P \cap \mathbf{Z}^V$ and such that $\mathbf{a} \in \mathbf{Z}^V$, $\mathbf{a}^T\mathbf{e} < m_j$; we only need to verify that $\mathbf{a}^T\mathbf{x}^{(j)} \leqslant b$. For this purpose, note that

$$b \geqslant \max\{\mathbf{a}^T\mathbf{x} : \mathbf{x} \in P \cap \mathbf{Z}^V\} \geqslant \frac{1}{k}\max\left\{\sum_{i \in S} a_i : |S| \leqslant s\right\}. \tag{3.2}$$

We may assume that $\mathbf{a}^T\mathbf{e} > 0$ (otherwise $\mathbf{a}^T\mathbf{x}^{(j)} \leqslant 0$ and we are done, as $b \geqslant 0$); if $\mathbf{a}$ has at most s positive components, then (3.2) implies $b \geqslant (1/k)\mathbf{a}^T\mathbf{e} \geqslant \mathbf{a}^T\mathbf{x}^{(j)}$; if $\mathbf{a}$ has at least s positive components, then (3.2) implies

$$b \geqslant \frac{s}{k} > \frac{s}{km_j}\mathbf{a}^T\mathbf{e} = \mathbf{a}^T\mathbf{x}^{(j)}. \tag{3.3}$$ ■

THEOREM 3.2. *There are a positive integer c, a positive number ε, and arbitrarily large graphs G such that G has n vertices and cn edges and the depth of $\mathbf{e}^T\mathbf{x} \leqslant \alpha(G)$ relative to* (3.1) *is at least εn.*

Proof. Erdös (1962) proved that for every positive t there are a positive integer c, a positive number δ, and arbitrarily large graphs G such that G

has n vertices and cn edges, $\alpha(G) < tn$, and every subgraph of G with at most δn vertices is 3-colorable. (In fact, he proved that for all sufficiently large n, at least 99% of all graphs with n vertices and cn edges have the last two properties.) Any t smaller than $\frac{1}{3}$ will do for our purpose: we only need set $k = 3$ and $s = \lfloor \delta n \rfloor$ in Lemma 3.1. ∎

A graph is called *claw-free* if it does not contain three pairwise nonadjacent vertices with a common neighbor. Giles and Trotter (1981) gave examples of claw-free graphs G such that (3.1) has rank at least two; Schrijver asked whether this lower bound can be replaced by arbitrarily large numbers; our next theorem provides an affirmative answer.

THEOREM 3.3. *There are arbitrarily large graphs G such that G has n vertices, $\alpha(G) = 2$, and the depth of $\mathbf{e}^T\mathbf{x} \leqslant 2$ relative to* (3.1) *exceeds* $\frac{1}{3}\ln n$.

Proof. Erdös (1961) has proved that there are a positive c and arbitrarily large graphs G such that G has n vertices, $\alpha(G) = 2$, and every clique in G has at most $cn^{1/2}\ln n$ vertices. We only need apply Lemma 3.1 with k equal to the largest number of vertices in a clique G and with $s = k + 1$. ∎

As we shall observe later (Theorem 9.1), the lower bounds of Theorem 3.2 and 3.3 cannot be improved beyond a constant factor.

Next let $\mathbf{A}$ be a zero-one matrix of size $m \times n$, and let P denote the polyhedron in $\mathbf{R}^n$ defined by

$$\mathbf{A}\mathbf{x} \geqslant \mathbf{e}, \qquad \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{e} \tag{3.4}$$

The problem of minimizing a linear function over $P \cap \mathbf{Z}^n$ is known as the *set-covering problem*, and P_I is called the *set-covering polytope*. (These terms come from interpreting the jth column of $\mathbf{A}$ as the incidence vector of a subset S_j of the "ground set" $\{1,2,\ldots,m\}$, and calling a set J of subscripts a *cover* if the union of all S_j with $j \in J$ is the ground set.) The set-covering polytope has been studied by Balas (1980), Balas and Ng (1985), Sassano (1985), and Cornuéjols and Sassano (1986). Our objective is to show that the rank of (3.4) may increase as fast as a linear function of the number of positive entries in $\mathbf{A}$.

THEOREM 3.4. *There are a positive number ε and arbitrarily large zero-one matrices $\mathbf{A}$ of size $m \times n$ such that $m \geqslant n$, each row of $\mathbf{A}$ has precisely two ones, and the polyhedron defined by* (3.4) *has rank at least εm.*

Proof. Given a graph G, consider the transpose $\mathbf{A}$ of the incidence matrix of G: rows of $\mathbf{A}$ correspond to edges of G, columns of $\mathbf{A}$ correspond to vertices of G, and each row of $\mathbf{A}$ is the incidence vector of the corresponding edge of G. Let S denote the polyhedron defined by (3.1), and let T denote the polyhedron defined by (3.4); let $f: \mathbf{R}^V \to \mathbf{R}^V$ be defined by $f(\mathbf{x}) = \mathbf{e} - \mathbf{x}$. The desired conclusion follows from Lemma 2.3 and Theorem 3.2. ∎

Now let $\mathbf{A}$ be a zero-one matrix of size $m \times n$, and let P denote the polyhedron in $\mathbf{R}^n$ defined by

$$\mathbf{A}\mathbf{x} = \mathbf{e}, \qquad \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{e}. \tag{3.5}$$

The problem of minimizing a linear function over $P \cap \mathbf{Z}^n$ is known as the *set-partitioning problem*, and P_I is called the *set-partitioning polytope*. (These terms come from interpreting once again the jth column of $\mathbf{A}$ as the incidence vector of a subset S_j of the "ground set" $\{1, 2, \dots, m\}$, and calling a set J of subscripts a *partition* if the sets S_j with $j \in J$ are pairwise disjoint and their union is the ground set.) The set-partitioning polytope has been studied by Balas (1977), Johnson (1980), and others. Again, our objective is to show that the rank of (3.5) may increase as fast as a linear function in the number of positive entries in $\mathbf{A}$.

THEOREM 3.5. *There are a positive constant ε and arbitrarily large zero-one matrices* $\mathbf{A}$ *of size $m \times n$ such that $m \leqslant n \leqslant 2m$, each row of* $\mathbf{A}$ *has precisely three ones, and the polyhedron defined by* (3.5) *has rank at least εm.*

Proof. Given a zero-one matrix $\mathbf{A}$ of size $m \times n$ such that $\mathbf{A}\mathbf{e} = 2\mathbf{e}$, let S denote the polyhedron in $\mathbf{R}^n$ defined by (3.4), let T denote the polyhedron in $\mathbf{R}^{n+m}$ defined by

$$\mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{e}, \qquad \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{e}, \quad \mathbf{0} \leqslant \mathbf{y} \leqslant \mathbf{e},$$

and define $f: \mathbf{R}^n \to \mathbf{R}^{n+m}$ by

$$f(\mathbf{x}) = \begin{bmatrix} \mathbf{e} - \mathbf{x} \\ \mathbf{A}\mathbf{x} - \mathbf{e} \end{bmatrix}.$$

The desired conclusion follows from Lemma 2.3 and Theorem 3.4. ∎

Finally, let $\mathbf{a}$ be a vector in $\mathbf{Z}_+^n$ and let b be an integer; let P be the polyhedron in $\mathbf{R}^n$ defined by

$$\mathbf{a}^T\mathbf{x} \leqslant b, \qquad \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{e}. \tag{3.6}$$

The problem of maximizing a linear function over $P \cap \mathbf{Z}^n$ is known as as the *zero-one knapsack problem*, and P_I is called the *knapsack polytope*. The knapsack polytope has been studied by Balas (1975), Hammer, Johnson, and Peled (1975), Padberg (1975), Wolsey (1975), Balas and Zemel (1978), Johnson (1980), Hammer and Peled (1982), and others. We are going to show that the rank of (3.6) may increase as fast as a linear function of the number of variables even when b and the components of $\mathbf{a}$ are relatively small.

THEOREM 3.6. *There is a positive constant ε such that, for arbitrarily large n, there are a vector $\mathbf{a}$ in $\mathbf{Z}_+^n$ and a positive integer b with the following properties: each component of $\mathbf{a}$ as well as b is less than 4^n, and the rank of the polyhedron defined by* (3.6) *is at least εn.*

Proof. By Lemma 2.4, we only need prove a modified statement, where the polyhedron defined by (3.6) is replaced by the polyhedron defined by

$$\mathbf{a}^T\mathbf{x} = b, \qquad \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{e}. \tag{3.7}$$

Given a zero-one matrix $\mathbf{A}$ consisting of rows $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(m)}$ such that each $\mathbf{a}^{(i)}$ has precisely three ones, define

$$\mathbf{a}^T = \sum_{i=0}^{m-1} 4^i \mathbf{a}^{(i+1)}, \qquad b = \sum_{i=0}^{m-1} 4^i.$$

Let S denote the polyhedron (3.5), and let T denote the polyhedron (3.7).

Since $\mathbf{a}^T\mathbf{x} = b$ is a linear combination of $\mathbf{A}\mathbf{x} = \mathbf{e}$, we have $S \subseteq T$. Hence we only need prove that

$$T \cap \mathbf{Z}^n \subseteq S \cap \mathbf{Z}^n, \tag{3.8}$$

for then the desired conclusion will follow from Lemma 2.3 [with $f(\mathbf{x}) = \mathbf{x}$] and Theorem 3.5. To prove (3.8), consider an arbitrary $\mathbf{x}$ in $T \cap \mathbf{Z}^n$: we have

$$\sum_{i=0}^{m-1} (\mathbf{a}^{(i+1)}\mathbf{x}) 4^i = \sum_{i=0}^{m-1} 4^i.$$

Since each $\mathbf{a}^{(i+1)}$ has precisely three ones, each $\mathbf{a}^{(i+1)}\mathbf{x}$ is one of the integers $0, 1, 2, 3$; since the 4-ary expansion of any nonnegative integer is unique, we conclude that $\mathbf{a}^{(i+1)}\mathbf{x} = 1$ for all $i = 0, 1, \ldots, m-1$, and so $\mathbf{x} \in S \cap \mathbf{Z}^n$. ■

Let us conclude this section with another look at the stable set polytope. Consider an undirected graph G with vertex set V; for each subset S of V, let $\alpha(S)$ denote the largest number of pairwise nonadjacent vertices in S; let P denote the polyhedron in $\mathbf{R}^V$ defined by

$$\begin{aligned} \mathbf{x}(S) &\leqslant \alpha(S) && \text{for all subsets } S \text{ of } V, \\ x_v &\geqslant 0 && \text{for all } v \text{ in } V. \end{aligned} \tag{3.9}$$

Obviously, P_I is the stable-set polytope of G. Even though (3.9) approximates the stable-set polytope better than (3.1) does, the rank of (3.9) can still exceed any constant.

THEOREM 3.7. *There are a positive constant c and arbitrarily large graphs G with n vertices such that the rank of* (3.9) *exceeds* $c(n/\ln n)^{1/2}$.

Proof. To construct G, let p_k denote the kth prime; for an arbitrary but fixed positive integer m, take pairwise disjoint sets $V_1, V_2, \ldots, V_m$ such that $|V_k| = 2p_k + 1$ for all k, and let V denote the union of these sets. Then let each V_k induce a chordless cycle of length $2p_k + 1$, and make every two vertices in distinct sets V_k adjacent. [The same construction has been used by Chvátal, Garey, and Johnson (1978) for a different purpose.]

Now write $b = p_1 p_2 \cdots p_m$ and $a_k = b/p_k$ for all k. On the one hand, it is not difficult to see that

$$\sum_{k=1}^{m} a_k \sum (x_v : v \in V_k) \leqslant b$$

defines a facet of P_I, and so it (or its positive multiple) must be induced in every system of linear inequalities that defines P_I. On the other hand, Remark 1 after Theorem 10 in Cook, Gerards, Schrijver, and Tardos (1986) guarantees that $P^{(j)}$ can be defined by a system of linear inequalities with integer coefficients whose absolute values do not exceed n^j. Hence the rank of P is at least $(\ln a_1)/(\ln n)$.

Finally, observe that $a_1 = \frac{1}{2}b > \frac{1}{2}m! > \frac{1}{2}(m/e)^m$, and so $\ln a_1 > c_1 m \ln m$ for some positive constant c_1. The prime-number theorem asserts that $p_k \sim k \ln k$, and so

$$n = \sum_{k=1}^{m} (2p_k + 1) < c_2 \sum_{k=1}^{m} k \ln k \leqslant c_2 \ln m \sum_{k=1}^{m} k < c_3 m^2 \ln m$$

for some positive constants c_2, c_3. Hence $m > c_4(n/\ln n)^{1/2}$ for some positive constant c_4, and the desired result follows. ∎

4. THE BIPARTITE-SUBGRAPH PROBLEM AND THE MAXIMUM-CUT PROBLEM

Let G be a complete graph with vertex set V and edge set E; let $\boldsymbol{O}$ denote the family of all odd circuits in G; let P denote the polyhedron in $\mathbf{R}^E$ defined by

$$\begin{aligned} \mathbf{x}(C) &\leqslant |C| - 1 \qquad \text{for all } C \text{ in } \boldsymbol{O}, \\ 0 \leqslant x_h &\leqslant 1 \qquad \text{for all } h \text{ in } E. \end{aligned} \tag{4.1}$$

The problem of maximizing a linear function over $P \cap \mathbf{Z}^E$ is known as the *bipartite-subgraph problem*, and P_I is called the *bipartite-subgraph polytope*. (These terms are explained by observing that an integer vector belongs to P if and only if it is the incidence vector of a bipartite subgraph of G.) The bipartite-subgraph polytope has been studied by Barahona, Grötschel, and Mahjoub (1985) and Gerards (1985). In particular, Barahona, Grötschel, and Mahjoub (1985) conjectured that, for each complete subgraph H of G such that H has $2k+1$ vertices, the depth of the inequality

$$\mathbf{x}(H) \leqslant k(k+1) \tag{4.2}$$

relative to (4.1) grows linearly with k. We shall prove this conjecture. Straightforward induction on k shows that the depth of (4.2) relative to (4.1) relative to (4.1) is at most $k-1$; the lower bound is provided by the following theorem.

Theorem 4.1. *The depth of* (4.2) *relative to* (4.1) *is at least* $\frac{1}{2}(k-1)$.

In order to prove this theorem, we first establish an auxiliary result.

THEOREM 4.2. *Let (V, E) be a complete graph, and let $a: E \to \mathbf{Z}$ be a function; write $m = a(E)$. If $m > 0$, then there is a subset S of V such that*

$$\sum (a(h): h \in E,\ |h \cap S| = 1) \geqslant \frac{m}{2} + \left(\frac{m}{8}\right)^{1/2} - \frac{1}{4}. \tag{4.3}$$

Proof. Write $n = |V|$; we shall proceed by induction on n and distinguish between two cases.

Case 1: $a(h^*) \leqslant 0$ for some h^*. In this case, we identify the two endpoints of h^* and use the induction hypothesis. Formally, let u and v be the two endpoints of h^*, and let w be a point outside V; define

$$V^* = (V - \{u, v\}) \cup \{w\},$$

$$E^* = \{h : h \subseteq V^*,\ |h| = 2\},$$

$$a^*(h) = a(h) \qquad \text{whenever} \quad h \in E \cap E^*,$$

$$a^*(wx) = a(ux) + a(vx) \qquad \text{whenever} \quad x \in V^*, \quad x \neq w,$$

$$m^* = m - a(uv).$$

By the induction hypothesis, there is a subset S of V^* such that

$$\sum (a^*(h): h \in E^*,\ |h \cap S| = 1) \geqslant \frac{m^*}{2} + \left(\frac{m^*}{8}\right)^{1/2} - \frac{1}{4}. \tag{4.4}$$

Without loss of generality, we may assume that $S \subseteq V$ (to put it differently, $w \notin S$); now the left-hand side of (4.4) equals the left-hand side of (4.3). As $m^* \geqslant m$, the right-hand side of (4.4) is at least the right-hand side of (4.3).

Case 2: $a(h) \geqslant 1$ for all h. In this case, averaging over all subsets S of V such that $|S| = \lfloor n/2 \rfloor$, we observe that at least one of them satisfies

$$\sum (a(h): h \in E,\ |h \cap S| = 1) \geqslant \frac{\lfloor n/2 \rfloor \lceil n/2 \rceil}{\binom{n}{2}} m \geqslant \frac{n+1}{2n} m. \tag{4.5}$$

Since

$$m \geqslant \binom{n}{2} \quad \text{and} \quad n \geqslant 2,$$

we have

$$\frac{m}{2n} - \left(\frac{m}{8}\right)^{1/2} = \left[\left(\frac{m}{2n}\right)^{1/2} - \left(\frac{n}{16}\right)^{1/2}\right]^2 - \frac{n}{16}$$

$$\geqslant \left[\left(\frac{n-1}{4}\right)^{1/2} - \left(\frac{n}{16}\right)^{1/2}\right]^2 - \frac{n}{16} \geqslant -\frac{1}{4},$$

and so the right-hand side of (4.5) is at least the right-hand side of (4.3). ∎

Proof of Theorem 4.1. Writing

$$\mathbf{x}(j) = \frac{1}{2}\left(1 + \frac{1}{4j+3}\right)\mathbf{e},$$

we only need show that $\mathbf{x}^{(j)} \in P^{(j)}$ for all j: if $j < \frac{1}{2}(k-1)$ then

$$\mathbf{x}^{(j)}(H) = \frac{1}{2}\left(1 + \frac{1}{4j+3}\right)\binom{2k+1}{2} > k(k+1).$$

To show that $\mathbf{x}^{(j)} \in P^{(j)}$ for all j, we only need verify the assumptions of Lemma 2.1 with $\mathbf{u} = \frac{2}{3}\mathbf{e}$, $\mathbf{v} = \mathbf{e}$, and $m_j = \frac{1}{2}(4j+3)(4j-1)$. It is easy to see that $\mathbf{u} \in P$. Now consider an arbitrary inequality $\mathbf{a}^T\mathbf{x} \leqslant b$ valid over $P \cap \mathbf{Z}^E$, and such that $\mathbf{a} \in \mathbf{Z}^E$, $\mathbf{a}^T\mathbf{e} < m_j$ for some j; we need only verify that $\mathbf{a}^T\mathbf{x}^{(j)} \leqslant b$. For this purpose, we may assume that $\mathbf{a}^T\mathbf{e} \geqslant 1$ (else $\mathbf{a}^T\mathbf{x}^{(j)} \leqslant 0$ and we are done, as $b \geqslant 0$). Now write

$$A = 4j+3, \qquad B = [(4j+3)(4j-1)]^{1/2},$$

and observe that

$$A + B > 2B = (8m_j)^{1/2},$$

$$A - B \leqslant 2\frac{4j+3}{4j+1} \leqslant \frac{14}{5} < 8^{1/2}$$

[here we use the inequality $1-(1-t)^{1/2} \leqslant t/(2-t)$ with $t = 4/(4j+3)$]. Since $1 \leqslant \mathbf{a}^T\mathbf{e} < m_j$, we conclude that

$$\left[(8\mathbf{a}^T\mathbf{e})^{1/2} - A - B\right]\left[(8\mathbf{a}^T\mathbf{e})^{1/2} - A + B\right] < 0;$$

that is,

$$2\mathbf{a}^T\mathbf{e} - (4j+3)(2\mathbf{a}^T\mathbf{e})^{1/2} + (4j+3) < 0. \tag{4.6}$$

Finally, Theorem 4.2 guarantees that

$$b \geqslant \frac{\mathbf{a}^T\mathbf{e}}{2} + \left(\frac{\mathbf{a}^T\mathbf{e}}{8}\right)^{1/2} - \frac{1}{4};$$

this bound combined with (4.6) yields

$$b > \tfrac{1}{2}\mathbf{a}^T\mathbf{e}\left(1 + \frac{1}{4j+3}\right) = \mathbf{a}^T\mathbf{x}^{(j)},$$

and the proof is completed. ∎

Now let $\boldsymbol{P}$ denote the set of all ordered pairs (B, C) such that C is (the edge set of) a circuit in G and B is a subset of C such that $|B|$ is odd; let Q denote the polyhedron in $\mathbf{R}^E$ defined by

$$\begin{aligned} \mathbf{x}(B) - \mathbf{x}(C-B) \leqslant |B| - 1 \qquad & \text{for all } (B, C) \text{ in } \boldsymbol{P}, \\ 0 \leqslant x_h \leqslant 1 \qquad & \text{for all } h \text{ in } E. \end{aligned} \tag{4.7}$$

The problem of maximizing a linear function over $Q \cap \mathbf{Z}^E$ is known as the *maximum-cut problem*, and Q_I is called the *cut polytope*. (These terms are explained by observing that an integer vector belongs to Q if and only if it intersects each circuit in G in an even number of edges, and thus it is the incidence vector of a set of all the edges that have precisely one endpoint in some fixed subset S of V; such sets of edges are called *cuts*.) The cut polytope has been studied by Barahona and Mahjoub (1986).

THEOREM 4.3. *The depth of* (4.2) *relative to* (4.7) *is at least* $\frac{1}{2}(k-1)$.

Proof. Identical with the proof of Theorem 4.1, except that Q is substituted for P. ∎

5. THE ACYCLIC-SUBDIGRAPH PROBLEM

Let G be a complete directed graph with vertex set V and edge set D (so that D consists of all ordered pairs of distinct elements of V); let $\mathbf{C}$ denote the family of all (edge-sets of) directed circuits in G (including directed circuits of length two); let P denote the polytope in $\mathbf{R}^D$ defined by

$$\begin{aligned} \mathbf{x}(C) &\leqslant |C|-1 \qquad \text{for all } C \text{ in } \mathbf{C}, \\ 0 \leqslant x_h &\leqslant 1 \qquad\qquad \text{for all } h \text{ in } D. \end{aligned} \tag{5.1}$$

The problem of maximizing a linear function over $P \cap \mathbf{Z}^D$ is known as the *acyclic-subdigraph problem*, and P_I is called the *acyclic-subdigraph polytope*. (These terms are explained by observing that an integer vector belongs to P if and only if it is the incidence vector of an acyclic subdigraph of G.) The acyclic-subdigraph polytope has been studied by Grötschel, Jünger, and Reinelt (1985).

In our next theorem, as elsewhere in this paper, we let n stand for $|V|$.

THEOREM 5.1. *For all sufficiently large* n, *the rank of the polytope defined by* (5.1) *exceeds* $10^{-8}n^{3/2}$.

Before proving this theorem, we shall establish three auxiliary results. There, as usual, $\|\mathbf{x}\|_p$ denotes the l_p-norm of a vector $\mathbf{x}$.

LEMMA 5.2. *For every integer matrix consisting of rows* $\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^r$ *and columns* $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^s$, *we have*

$$\sum_{i=1}^{r} \|\mathbf{a}^i\|_2 + \sum_{j=1}^{s} \|\mathbf{b}^j\|_2 \geqslant \left(\sum_{j=1}^{s} \|\mathbf{b}^j\|_1 \right)^{3/4}.$$

Proof. First, let us show that

$$\sum_{i=1}^{r} \|\mathbf{a}^i\|_2 + \sum_{j=1}^{s} \|\mathbf{b}^j\|_2 \geqslant k^{-1/2} \sum_{j=1}^{k} \|\mathbf{b}^j\|_1 + \sum_{j=1}^{s} \|\mathbf{b}^j\|_1^{1/2}$$

for all $k = 1, 2, \ldots, s$. For this purpose, let $\tilde{\mathbf{a}}^i$ denote the vector consisting of the first k components of $\mathbf{a}^i$. Since $\|\mathbf{x}\|_2 \geqslant k^{-1/2}\|\mathbf{x}\|_1$ whenever $\mathbf{x} \in \mathbf{R}^k$, we have

$$\sum_{i=1}^{r} \|\mathbf{a}^i\|_2 \geqslant \sum_{i=1}^{r} \|\tilde{\mathbf{a}}^i\|_2 \geqslant k^{-1/2} \sum_{i=1}^{r} \|\tilde{\mathbf{a}}^i\|_1 = k^{-1/2} \sum_{j=1}^{k} \|\mathbf{b}^j\|_1.$$

Now observe that $\|\mathbf{x}\|_2 \geqslant \|\mathbf{x}\|_1^{1/2}$ whenever $\mathbf{x} \in \mathbf{Z}^r$.

We shall complete the proof of the Lemma by justifying the following claim: for every nonincreasing sequence $c_1, c_2, \ldots, c_s$ of nonnegative reals there is a subscript k such that $1 \leqslant k \leqslant s$ and

$$k^{-1/2} \sum_{j=1}^{k} c_j + \sum_{j=1}^{s} c_j^{1/2} \geqslant \left(\sum_{j=1}^{s} c_j \right)^{3/4}$$

This will be done by induction on s. Writing $d = \sum_{j=1}^{s} c_j$, note that $c_s \leqslant d/s$. Hence we may assume that $c_s < d^{1/2}$ (for otherwise the desired conclusion holds with $k = s$). By the induction hypothesis, there is a subscript k such that $1 \leqslant k \leqslant s - 1$ and

$$k^{-1/2} \sum_{j=1}^{k} c_j + \sum_{j=1}^{s-1} c_j^{1/2} \geqslant \left(\sum_{j=1}^{s-1} c_j \right)^{3/4}$$

Hence we only need show that

$$(d - c_s)^{3/4} + c_s^{1/2} \geqslant d^{3/4}.$$

This inequality follows from observing that

$$d^{3/4} - (d - c_s)^{3/4} \leqslant \tfrac{3}{4}(d - c_s)^{-1/4} \cdot c_s \leqslant \frac{3}{4} \left(\frac{2}{d} \right)^{1/4} c_s$$

and that

$$\frac{3}{4}\left(\frac{2}{d}\right)^{1/4} c_s^{1/2} < \frac{3}{4} 2^{1/4} < 1. \qquad \blacksquare$$

LEMMA 5.3. *Let n be a positive integer, and let $[n]$ denote the set $\{1,2,\dots,n\}$. For all* $\mathbf{x}$ *in* $\mathbf{R}^n$, *we have*

$$2^{-n}\sum\left(|\mathbf{x}(S)| : S \subseteq [n]\right) \geq 2^{-3/2}\|\mathbf{x}\|_2. \tag{5.2}$$

Proof. We shall rely on the inequality

$$2^{-n}\sum\left(|\mathbf{x}(S) - \mathbf{x}([n]-S)| : S \subseteq [n]\right) \geq 2^{-1/2}\|\mathbf{x}\|_2, \tag{5.3}$$

proved by Szarek (1976); see also pp. 138–139 in Devroye and Györfi (1985). To derive (5.2) from (5.3), we only need observe that

$$\sum\left(|\mathbf{x}(S)| : S \subseteq [n]\right) = \tfrac{1}{2}\sum\left(|\mathbf{x}(S)| + |\mathbf{x}([n]-S)| : S \subseteq [n]\right)$$

and use the inequality $|a| + |b| \geq |a+b|$ with $a = \mathbf{x}(S)$, $b = -\mathbf{x}([n]-S)$. $\blacksquare$

An *oriented graph* is a directed graph H such that, for each pair of distinct vertices v, w of H, at most one of vw, wv is an edge of H.

THEOREM 5.4. *Let H be an oriented graph with vertex set V and edge set E; let $a: E \to \mathbf{Z}$ be a function; write $m = a(E)$. If $m \geq 0$, then there is an acyclic subset E^* of E with*

$$a(E^*) \geq \tfrac{1}{2}m + \tfrac{1}{27}m^{3/4}.$$

Proof. Extend the domain of a to $V \times V$ by writing $a(vw) = 0$ whenever $vw \notin E$; for each ordered pair v, w of vertices of H, write

$$\mathbf{x}^v(w) = a(vw) - a(wv);$$

for each vertex v of H, let $\mathbf{x}^v$ denote the vector with components $\mathbf{x}^v(w)$.

Applying Lemma 5.2 to the skew-symmetric matrix with columns $\mathbf{x}^v$, we find that

$$\sum_{v \in V} \|\mathbf{x}^v\|_2 \geqslant \tfrac{1}{2}\left(\sum_{v \in V} \|\mathbf{x}^v\|_1 \right)^{3/4} \geqslant \tfrac{1}{2}(2m)^{3/4}. \tag{5.4}$$

By Lemma 5.3, we have

$$\sum \left(|\mathbf{x}^v(S)| : S \subseteq V - \{v\} \right) \geqslant 2^{n-2.5} \|\mathbf{x}^v\|_2$$

for all v; now (5.4) guarantees that

$$\sum_{v \in V} \sum \left(|\mathbf{x}^v(S)| : S \subseteq V - \{v\} \right) \geqslant 2^{n-3.5}(2m)^{3/4}.$$

Hence there is a set S such that

$$\sum \left(|\mathbf{x}^v(S)| : v \notin S \right) \geqslant 2^{-11/4} m^{3/4} > \tfrac{4}{27} m^{3/4}.$$

It follows that there is a subset R of $V - S$ such that

$$\left| \sum (\mathbf{x}^v(S) : v \in R) \right| \geqslant \tfrac{2}{27} m^{3/4}.$$

To put it differently, there are disjoint subsets R, S of V such that

$$|a(R \times S) - a(S \times R)| \geqslant \tfrac{2}{27} m^{3/4};$$

switching R and S if necessary, we may assume that

$$a(R \times S) - a(S \times R) \geqslant \tfrac{2}{27} m^{3/4}. \tag{5.5}$$

The rest is routine. Each linear order $<$ on V defines an acyclic subset E^0 of E: set $uv \in E^0$ if and only if $uv \in E$ and $u < v$. Clearly, there is a pair of linear orders on V such that, for any pair u, v of distinct vertices, we have

$$u < v \quad \text{in both orders} \quad \text{if and only if} \quad u \in R, \quad v \in S.$$

If E_1^0, E_2^0 denote the two acyclic subsets of E defined by these two orders, then trivially

$$a(E_1^0)+a(E_2^0)=a(E)+a(R\times S)-a(S\times R);$$

now (5.5) guarantees that at least one of E_1^0, E_2^0 has the property required of E^*. ∎

A *tournament* is a directed graph H such that, for each pair of distinct vertices v, w of H, precisely one of vw, wv is an edge of H. Spencer (1971) proved that, for some positive constant c, every tournament with n vertices contains an acyclic set of at least $\frac{1}{2}\binom{n}{2}+cn^{3/2}$ edges. Note that this theorem is a special case of Theorem 5.4; conversely, our proof of Theorem 5.4 relies in part on the ideas used by Spencer.

Proof of Theorem 5.1. Spencer (1980) proved that, for some constant c and for all sufficiently large n, there is a tournament in which every acyclic set of edges has size at most

$$\tfrac{1}{2}\binom{n}{2}+cn^{3/2};$$

subseqently, de la Vega (1983) proved that we may set $c=1.73$. (In fact, both Spencer and de la Vega proved that a randomly chosen tournament is very likely to have the property.) Let $\mathbf{v}$ be the incidence vector of the edge set of such a tournament. Writing

$$\mathbf{x}^{(j)}=\left[\tfrac{1}{2}+\tfrac{1}{3}(3^{10}j+8)^{-1/3}\right]\mathbf{v},$$

we need only show that $\mathbf{x}^{(j)}\in P^{(j)}$ for all j: if $j\leqslant 10^{-8}n^{3/2}$ then

$$\mathbf{v}^T\mathbf{x}^{(j)}=\binom{n}{2}\left[\tfrac{1}{2}+\tfrac{1}{3}(3^{10}j+8)^{-1/3}\right]>\binom{n}{2}\left(\tfrac{1}{2}+3.48n^{-1/2}\right),$$

and so $\mathbf{x}^{(j)}\notin P_I$.

To show that $\mathbf{x}^{(j)}\in P^{(j)}$ for all j, we need only verify the assumptions of Lemma 2.1 with $\mathbf{u}=\frac{2}{3}\mathbf{v}$ and

$$m_j=\frac{3(3^{10}j+8)^{1/3}(3^{10}(j-1)+8)^{1/3}}{(3^{10}j+8)^{1/3}-(3^{10}(j-1)+8)^{1/3}}.$$

Let us note at once that

$$(3^{10}j+8)^{1/3}-[3^{10}(j-1)+8]^{1/3}\geqslant 3^9(3^{10}j+8)^{-2/3},$$

and so

$$m_j\leqslant 3^{-8}(3^{10}j+8)[3^{10}(j-1)+8]^{1/3}<3^{-8}(3^{10}j+8)^{4/3};$$

hence

$$\tfrac{1}{3}(3^{10}j+8)^{-1/3}<\tfrac{1}{27}m_j^{-1/4}. \tag{5.6}$$

Obviously, $\mathbf{u}\in P$. Now consider an arbitrary inequality $\mathbf{a}^T\mathbf{x}\leqslant b$ valid over $P\cap \mathbf{Z}^D$ and such that $\mathbf{a}\in\mathbf{Z}^D$, $\mathbf{a}^T\mathbf{v}<m_j$; we only need verify that $\mathbf{a}^T\mathbf{x}^{(j)}\leqslant b$. If $\mathbf{a}^T\mathbf{v}\leqslant 0$ then $\mathbf{a}^T\mathbf{x}^{(j)}\leqslant 0$ and we are done as $b\geqslant 0$; if $\mathbf{a}^T\mathbf{v}>0$ then

$$\mathbf{a}^T\mathbf{x}^{(j)}<\left(\tfrac{1}{2}+\tfrac{1}{27}m_j^{-1/4}\right)\mathbf{a}^T\mathbf{v}<\tfrac{1}{2}\mathbf{a}^T\mathbf{v}+\tfrac{1}{27}(\mathbf{a}^T\mathbf{v})^{3/4}\leqslant b$$

by (5.6) and Theorem 5.4. ∎

Finally, let s_i, t_i $(i=1,2,\dots,k)$ be distinct points in V; let $\mathbf{v}$ denote the incidence vector of the set of the k edges s_i, t_i, and let $\mathbf{w}$ denote the incidence vector of the set of the $k(k-1)$ edges t_is_j for which $i\neq j$. It is easy to see that the inequality

$$(\mathbf{v}+\mathbf{w})^T\mathbf{x}\leqslant k(k-1)+1 \tag{5.7}$$

is valid over $P\cap\mathbf{Z}^D$; this inequality was introduced by Grötschel, Jünger, and Reinelt (1985) under the name of "simple k-fence inequality". [Actually, Grötschel, Jünger, and Reinelt proved that (5.7) defines a facet of P_I; however, this fact is irrelevant to our discussion.] Jünger publicized (but never published) the conjecture that the depth of (5.7) relative to (5.1) tends to infinity with k; we are about to prove this conjecture.

THEOREM 5.5. *The depth of* (5.7) *relative to* (5.1) *is at least* $\log_2 k-1$.

Proof. Writing

$$\mathbf{x}^{(j)}=\mathbf{w}+2^{-(j+1)}\mathbf{v},$$

we need only show that $\mathbf{x}^{(j)} \in P^{(j)}$ for all j: if $j < \log_2 k - 1$ then

$$(\mathbf{w}+\mathbf{v})^T\mathbf{x}^{(j)} = \mathbf{w}^T\mathbf{w} + 2^{-(j+1)}\mathbf{v}^T\mathbf{v} = k(k-1) + 2^{-(j+1)}k > k(k-1)+1,$$

and so $\mathbf{x}^{(j)}$ fails to satisfy (5.7).

To show that $\mathbf{x}^{(j)} \in P^{(j)}$ for all j, we need only verify the assumption of Lemma 2.1 with $\mathbf{u} = \mathbf{w} + \frac{1}{2}\mathbf{v}$ and $m_j = 2^{j+1}$. Obviously, $\mathbf{u} \in P$. Now consider an arbitrary inequality $\mathbf{a}^T\mathbf{x} \leqslant b$ valid over $P \cap \mathbf{Z}^D$ and such that $\mathbf{a} \in \mathbf{Z}^D$, $\mathbf{a}^T\mathbf{v} < m_j$; we need only verify that $\mathbf{a}^T\mathbf{x}^{(j)} \leqslant b$. If $\mathbf{a}^T\mathbf{v} \leqslant 0$ then let $\mathbf{d}$ be the zero vector; if $\mathbf{a}^T\mathbf{v} > 0$ then let $\mathbf{d}$ be the incidence vector of any single edge s_it_i such that $a(s_it_i) \geqslant 1$. In either case, $\mathbf{w}+\mathbf{d} \in P \cap \mathbf{Z}^D$, and so

$$\mathbf{a}^T\mathbf{x}^{(j)} = \mathbf{a}^T\mathbf{w} + \frac{1}{m_j}\mathbf{a}^T\mathbf{v} \leqslant \mathbf{a}^T(\mathbf{w}+\mathbf{d}) \leqslant b,$$

as desired. ■

The lower bound of Theorem 5.5 can be improved to $\lceil \log_2(k-1) \rceil$; for a proof, see Hartmann (1988). Straightforward induction on k shows that this improved bound is best possible.

6. CUTTING-PLANE PROOFS

Arguments of this section rely on the duality theorem of linear programming in the following strong form: if a system $\mathbf{A}\mathbf{x} \leqslant \mathbf{b}$ of linear inequalities in n variables has a solution and if each of its solutions satisfies a linear inequality $\mathbf{c}^T\mathbf{x} \leqslant d$, then there is a nonnegative vector $\mathbf{y}$ with at most n positive components such that $\mathbf{y}^T\mathbf{A} = \mathbf{c}^T$ and $\mathbf{y}^T\mathbf{b} \leqslant d$.

Let m and M be positive integers; let

$$\mathbf{a}_i^T\mathbf{x} \leqslant b_i \qquad (i = 1, 2, \ldots, m), \tag{6.1}$$

$$\mathbf{a}_i^T\mathbf{x} \leqslant b_i \qquad (i = m+1, m+2, m+M) \tag{6.2}$$

be sequences of linear inequalities in n variables such that $\mathbf{a}_i \in \mathbf{Z}^n$ whenever $m < i \leqslant m + M$; let

$$w_{ik} \qquad (m < i \leqslant m+M, \quad 1 \leqslant k < i) \tag{6.3}$$

be nonnegative numbers such that

$$\mathbf{a}_i = \sum_{k=1}^{i-1} w_{ik}\mathbf{a}_k \quad \text{and} \quad b_i \geqslant \left\lfloor \sum_{k=1}^{i-1} w_{ik} b_k \right\rfloor \qquad \text{for all } i.$$

Straightforward induction on i shows that every integer solution of (6.1) satisfies all the inequalities (6.2). We shall refer to the sequence (6.2), along with the numbers (6.3), as a *cutting-plane proof* of $\mathbf{a}_{m+M}^T\mathbf{x} \leqslant b_{m+M}$ from (6.1) and say that M is the *length* of this proof.

Now let P denote the polyhedron in $\mathbf{R}^n$ defined by (6.1). If there is a cutting-plane proof of an inequality $\mathbf{a}^T\mathbf{x} \leqslant b$ from (6.1), then, as we have just observed, $\mathbf{a}^T\mathbf{x} \leqslant b$ holds for all $\mathbf{x}$ in $P \cap \mathbf{Z}^n$. The converse is true as soon as P is rational and $P \cap \mathbf{Z}^n$ is nonempty (actually, the second assumption can be dropped): this is guaranteed by the following theorem in conjunction with Theorem 1.2.

THEOREM 6.1. *Let* $\mathbf{a}_i \in \mathbf{Z}^n$ *whenever* $i = 1, 2, \dots, m$; *let* (6.1) *have an integer solution; let* $\mathbf{a}^T\mathbf{x} \leqslant b$ *have depth at most* d *relative to* (6.1), *and let* $\mathbf{a} \in \mathbf{Z}^n$, $b \in \mathbf{Z}$. *Then there is a cutting-plane proof of* $\mathbf{a}^T\mathbf{x} \leqslant b$ *from* (6.1) *of length at most* $(n^{d+1} - 1)/(n-1)$.

Proof. Let (6.1) satisfy the assumptions of the theorem; let P denote the polyhedron defined by (6.1). We propose to prove the following statement by induction on d: if $\mathbf{a} \in \mathbf{Z}^n$, $b \in \mathbf{Z}$, and if all $\mathbf{x}$ in $P^{(d)}$ satisfy $\mathbf{a}^T\mathbf{x} < b + 1$, then there is a cutting-plane proof of $\mathbf{a}^T\mathbf{x} \leqslant b$ from (6.1) of length at most $(n^{d+1} - 1)/(n-1)$.

In case $d = 0$, the desired conclusion follows at once from the duality theorem. Now let d be a positive integer. Since P is a rational polyhedron, Theorem 1 of Schrijver (1980) guarantees that $P^{(d)}$ is a rational polyhedron; hence $P^{(d)}$ consists of all solutions of some system

$$\tilde{\mathbf{a}}_i^T\mathbf{x} \leqslant \tilde{b}_i \qquad (i = 1, 2, \dots, \tilde{m})$$

such that, for all $i = 1, 2, \dots, \tilde{m}$, we have $\tilde{\mathbf{a}}_i \in \mathbf{Z}^n$, $\tilde{b}_i \in \mathbf{Z}$, and $\tilde{\mathbf{a}}_i^T\mathbf{x} < \tilde{b}_i + 1$ whenever $\mathbf{x} \in P^{(d-1)}$. By the induction hypothesis, there is a cutting-plane proof of each $\tilde{\mathbf{a}}_i^T\mathbf{x} \leqslant \tilde{b}_i$ from (6.1) of length at most $(n^d - 1)/(n-1)$; by the duality theorem, there are a set I of at most n subscripts i and a set of positive numbers y_i $(i \in I)$ such that

$$\sum_{i \in I} y_i \tilde{\mathbf{a}}_i = \mathbf{a}, \qquad \sum_{i \in I} y_i \tilde{b}_i < b + 1.$$

It follows that there is a cutting-plane proof of $\mathbf{a}^T\mathbf{x} \leqslant b$ from (6.1) of length at most $1 + n(n^d - 1)/(n-1)$, which is the desired conclusion. ∎

Finally, we are ready to relate rank to computational complexity. For example, let $f(n)$ denote the smallest t such that, for every graph G with n vertices, there is a cutting-plane proof of $\mathbf{e}^T\mathbf{x} \leqslant \alpha(G)$ from (3.1) of length at most t. There are reasons to conjecture that $f(n)$ grows faster than every polynomial in n. (Analogous conjectures can be made about the set-covering problem, the set-partitioning problem, the knapsack problem, the bipartite-subgraph problem, the maximum-cut problem, and the acyclic-subdigraph problem.) These reasons stem from the theory of NP-completeness, and we shall not elaborate on them; the interested reader is referred to Cook, Coullard, and Turán (1987) or to Chapter 23 of Schrijver (1986).

The conjecture that $f(n)$ grows faster than every polynomial in n implies (by virtue of Theorem 6.1) that the depth of $\mathbf{e}^T\mathbf{x} \leqslant \alpha(G)$ relative to (3.1) cannot be bounded from above by any constant; from this point of view, Theorem 3.2 can be seen as evidence supporting the conjecture. (Analogous comments can be made on Theorems 3.4, 3.5, 3.6, 4.1, 4.3, and 5.1.) However, the strength of this evidence is questionable: inequalities of large depth may admit short cutting-plane proofs. For example, there are arbitrarily large graphs G with n vertices and with $\alpha(G) = 2$ such that the depth of $\mathbf{e}^T\mathbf{x} \leqslant 2$ relative to (3.1) exceeds $\frac{1}{3}\ln n$ and yet there is a cutting-plane proof of $\mathbf{e}^T\mathbf{x} \leqslant 2$ from (3.1) of length at most $\binom{n}{2}$: the first property is guaranteed by Theorem 3.3, and the second property is guaranteed by the following result.

THEOREM 6.2. *For every graph G with n vertices, there is a cutting-plane proof of* $\mathbf{e}^T\mathbf{x} \leqslant \alpha(G)$ *from* (3.1) *of length at most* $\binom{n}{\alpha(G)}$.

Proof. For every subset W of V, let $\alpha(W)$ denote the largest number of pairwise nonadjacent vertices in W. We propose to prove by induction on $|W|$ that there is a cutting-plane proof of

$$\sum (x_v : v \in W) \leqslant \alpha(W) \tag{6.4}$$

from (3.1) of length at most $\binom{|W|}{\alpha(W)}$.

For this purpose, we may assume that $1 < \alpha(W) < |W|$, for otherwise the desired conclusion is trivial. Since $\alpha(W) < |W|$, there is a vertex w in W such that $\alpha(W - \{w\}) = \alpha(W)$. Write $W_1 = W - \{w\}$, and let W_0 denote the set of all vertices in W_1 are not adjacent to w. Since $\alpha(W_0) \leqslant \alpha(W) - 1$, there is a set W_2 such that $W_0 \subseteq W_2 \subset W_1$ and $\alpha(W_2) = \alpha(W) - 1$. By the induction hypothesis, there are a cutting-plane proof of

$$\sum (x_v : v \in W_1) \leqslant \alpha(W) \tag{6.5}$$

from (3.1) of length at most $\binom{|W_1|}{\alpha(W)}$ and a cutting-plane proof of

$$\sum (x_v : v \in W_2) \leqslant \alpha(W) - 1 \tag{6.6}$$

from (3.1) of length at most

$$\binom{|W_2|}{\alpha(W) - 1}.$$

Note that

$$\binom{|W_1|}{\alpha(W)} + \binom{|W_2|}{\alpha(W) - 1} < \binom{|W| - 1}{\alpha(W)} + \binom{|W| - 1}{\alpha(W) - 1} = \binom{|W|}{\alpha(W)},$$

since $|W_2| < |W| - 1$ and $\alpha(W) > 1$; note also that the maximum of $\Sigma(x_v : v \in W)$ subject to (3.1), (6.5), and (6.6) is strictly less than $\alpha(W) + 1$. Hence the desired cutting-plane proof of (6.4) from (3.1) can be obtained by concatenating a cutting-plane proof of (6.5) from (3.1), a cutting-plane proof of (6.6) from (3.1), and the inequality (6.4). ∎

7. TWO MORE LEMMAS

If every cutting-plane proof of $\mathbf{c}^T\mathbf{x} \leqslant d$ from $\mathbf{Ax} \leqslant \mathbf{b}$ must use many inequalities from $\mathbf{Ax} \leqslant \mathbf{b}$, then it must be long; in that case, Theorem 6.1 guarantees that the depth of $\mathbf{c}^T\mathbf{x} \leqslant d$ relative to $\mathbf{Ax} \leqslant \mathbf{b}$ is large. Our next lemma refines this observation. [The lemma deals just with cutting-plane proofs, but the argument applies as well to more general proof systems; for example, see Cook, Kannan, and Schrijver (1986).]

LEMMA 7.1. *Let*

$$\mathbf{a}_i^T\mathbf{x}+\mathbf{b}_i^T\mathbf{y} \leqslant c_i \qquad (i=1,2,\ldots,m) \tag{7.1}$$

be a system of linear inequalities such that $\mathbf{a}_i \in \mathbf{Z}^p$, $\mathbf{b}_i \in \mathbf{Z}^q$, $c_i \in \mathbf{Z}$ *for all* i; *let*

$$\mathbf{a}^T\mathbf{x}+\mathbf{b}^T\mathbf{y} \leqslant c \tag{7.2}$$

be a linear inequality such that $\mathbf{a} \in \mathbf{Z}^p$, $\mathbf{b} \in \mathbf{Z}^q$, $c \in \mathbf{Z}$. *Let* $\mathbf{y}^*$ *be a vector in* $\mathbf{Z}^q$ *such that the system*

$$\mathbf{a}_i^T\mathbf{x} \leqslant c_i - \mathbf{b}_i^T\mathbf{y}^* \qquad (i=1,2,\ldots,m) \tag{7.3}$$

has an integer solution and such that, for at least t *distinct choices of the subscript* k, *the system*

$$\mathbf{a}_i^T\mathbf{x} \leqslant c_i - \mathbf{b}_i^T\mathbf{y}^* \quad (i=1,2,\ldots,m;\ i \neq k), \qquad \mathbf{a}^T\mathbf{x} > c - \mathbf{b}^T\mathbf{y}^*$$

has an integer solution. Then every cutting-plane proof of (7.2) *from* (7.1) *has length at least* $(t-1)/(p-1)$, *and the depth of* (7.2) *relative to* (7.1) *is at least* $(\log t/\log p)-1$.

Proof. First, we propose to show that every cutting-plane proof of

$$\mathbf{a}^T\mathbf{x} \leqslant c - \mathbf{b}^T\mathbf{y}^* \tag{7.4}$$

from (7.3) has length at least $(t-1)/(p-1)$ and that the depth of (7.4) relative to (7.3) is at least $(\log t/\log p)-1$.

For this purpose, consider the shortest cutting-plane proof of (7.4) from (7.3): this proof consists of a sequence of M linear inequalities along with certain numbers w_{ik} ($m < i \leqslant m+M$, $1 \leqslant k < i$). Let N denote the total number of the numbers w_{ik} that are positive. On the one hand, for each k such that $m < k < m+M$ there must be at least one i such that $k < i \leqslant m+M$ and $w_{ik} > 0$ (else the inequality subscripted by k could be dropped, contradicting the minimality of M); by assumption, $w_{ik} > 0$ for at least t subscripts k with $k \leqslant m$; hence $N \geqslant M-1+t$. On the other hand, the duality theorem (in the strong form quoted at the beginning of Section 6) allows us to assume that, for each i such that $m < i \leqslant m+M$, there are at most p positive numbers w_{ik}; hence $N \leqslant pM$. Comparing the two bounds on

N, we conclude that $M \geqslant (t-1)/(p-1)$. Next, if d denotes the depth of (7.4) relative to (7.3), then Theorem 6.1 yields

$$\frac{t-1}{p-1} \leqslant M \leqslant \frac{p^{d+1}-1}{p-1},$$

and so $d \geqslant (\log t / \log p) - 1$.

To complete the proof of the lemma, note that every cutting-plane proof of (7.2) from (7.1) reduces to a cutting-plane proof of (7.4) from (7.3): each inequality $\mathbf{a}_i^T\mathbf{x} + \mathbf{b}_i^T\mathbf{y} \leqslant c_i$ reduces to $\mathbf{a}_i^T\mathbf{x} \leqslant c_i - \mathbf{b}_i^T\mathbf{y}^*$ and the numbers w_{ik} remain unchanged. In addition, let P denote the polyhedron in $\mathbf{R}^{p+q}$ defined by (7.1), and let P_0 denote the polyhedron in $\mathbf{R}^p$ defined by (7.3); straightforward induction on j shows that

$$P_0^{(j)} \subseteq \left\{ \mathbf{x} \in \mathbf{R}^p : [\mathbf{x}, \mathbf{y}^*]^T \in P^{(j)} \right\}.$$

Now the desired conclusion follows. ■

LEMMA 7.2. *If S denotes the set of all vectors $[y_1, y_2, \ldots, y_k]$ such that*

$$\sum_{i \in A} y_i + \sum_{i \notin A} (1 - y_i) \geqslant \tfrac{1}{2} \qquad \textit{whenever} \quad A \subseteq \{1,2,\ldots,k\},$$

$$0 \leqslant y_i \leqslant 1 \qquad \textit{whenever} \quad i = 1,2,\ldots,k$$

then $\frac{1}{2}\mathbf{e} \in S^{(k-1)}$.

Proof. Let F_j denote the set of all vectors $\mathbf{y}$ of length k such that j components of $\mathbf{y}$ equal $\frac{1}{2}$ and each of the remaining $k-j$ components is 0 or 1. We propose to prove by induction on j that

$$F_j \subseteq S^{(j-1)} \tag{7.5}$$

for all $j = 1,2,\ldots,k$. The case of $j = 1$ is trivial; now assume that (7.5) holds for some positive integer j such that $j < k$; let $\mathbf{y}^*$ be an arbitrary vector in F_{j+1}; we only need prove that $\mathbf{y}^* \in S^{(j)}$. For this purpose, consider an arbitrary inequality $\mathbf{a}^T\mathbf{y} < b + 1$ valid over $S^{(j-1)}$ and such that $\mathbf{a} \in \mathbf{Z}^k$,

$b \in \mathbf{Z}$; our task reduces to proving that $\mathbf{a}^T\mathbf{y}^* \leqslant b$. Since $\mathbf{y}^*$ belongs to the convex hull of F_j, the induction hypothesis guarantees that $\mathbf{y}^* \in S^{(j-1)}$, and so $\mathbf{a}^T\mathbf{y}^* < b + 1$. Hence we may assume that $\mathbf{a}^T\mathbf{y}^*$ is not an integer (otherwise we are done). This assumption implies that $y_i^* = \frac{1}{2}$ and $a_i \neq 0$ for some i. Replacing the ith component of $\mathbf{y}^*$ by 0, we obtain a vector $\mathbf{y}'$; replacing the ith component by 1, we obtain a vector $\mathbf{y}''$. Note that $\max(\mathbf{a}^T\mathbf{y}', \mathbf{a}^T\mathbf{y}'') \geqslant \mathbf{a}^T\mathbf{y}^* + \frac{1}{2}$; since $\mathbf{y}', \mathbf{y}'' \in F_j$, the induction hypothesis guarantees that $\mathbf{y}', \mathbf{y}'' \in S^{(j-1)}$, and so $\max(\mathbf{a}^T\mathbf{y}', \mathbf{a}^T\mathbf{y}'') < b + 1$. Hence $\mathbf{a}^T\mathbf{y}^* < b + \frac{1}{2}$; since $2\mathbf{a}^T\mathbf{y}^*$ is an integer, we conclude that $\mathbf{a}^T\mathbf{y}^* \leqslant b$. ∎

8. THE TRAVELING-SALESMAN PROBLEM

As in Section 5, let G denote a complete directed graph with vertex set V and edge set D; call a subset W of V *proper* if $W \neq \varnothing$, $W \neq V$; let P denote the polyhedron in $\mathbf{R}^D$ defined by

$$\begin{aligned}
&\mathbf{x}(\{v\} \times (V - \{v\})) = 1 && \text{for all } v \text{ in } V,\\
&\mathbf{x}((V - \{v\}) \times \{v\}) = 1 && \text{for all } v \text{ in } V,\\
&\mathbf{x}[(W \times (V - W)) \cup ((V - W) \times W)] \geqslant 2 && \text{for all proper subsets } W \text{ of } V, \qquad (8.1)\\
&0 \leqslant x_h \leqslant 1 && \text{for all } h \text{ in } D.
\end{aligned}$$

The problem of minimizing a linear function over $P \cap \mathbf{Z}^D$ is known as the *asymmetric traveling-salesman problem*, and P_I is called the *asymmetric traveling-salesman polytope*. (These terms are explained by observing that an integer vector belongs to P if and only if it is the incidence vector of a directed Hamiltonian circuit in G.) The asymmetric traveling-salesman polytope has been studied by Grötschel and Padberg (1975), Grötschel (1977), Grötschel and Wakabayashi (1981a, 1981b), and Balas (1987).

We shall prove that the rank of P is at least $\lfloor n/8 \rfloor$ by exhibiting (whenever $n \geqslant 8$) a linear inequality valid over P_I whose depth relative to P is at least $\lfloor n/8 \rfloor$. To describe the inequality, write $k = \lfloor n/8 \rfloor$ and $r = n - 8k$; label the vertices in V as

$$a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i \quad (i = 1, 2, \ldots, k), \qquad w_j \quad (j = 1, 2, \ldots, r);$$

set $w_0 = e_k$ and $w_{r+1} = a_1$. Let $D_{1/2}$ denote the set consisting of the edges

$$a_ib_i, a_ih_i, g_if_i, g_ih_i, b_ic_i, d_ic_i, f_ie_i, d_ie_i, b_if_i, f_ib_i \qquad (i = 1,2,\dots,k);$$

let D_1 denote the set consisting of all the edges

$$h_id_i \quad (i = 1,2,\dots,k), \qquad c_ig_{i+1}, e_ia_{i+1} \quad (i = 1,2,\dots,k-1),$$
$$c_kg_1, w_jw_{j+1} \quad (j = 0,1,\dots,r).$$

For illustration with $n = 27$, see Figure 1.

It is easy to check that the directed graph with vertex set V and edge set $D_{1/2} \cup D_1$ contains no directed Hamiltonian circuit; to put it differently, the inequality

$$\mathbf{x}(D_{1/2} \cup D_1) \leqslant n - 1 \tag{8.2}$$

is valid over $P \cap \mathbf{Z}^D$.

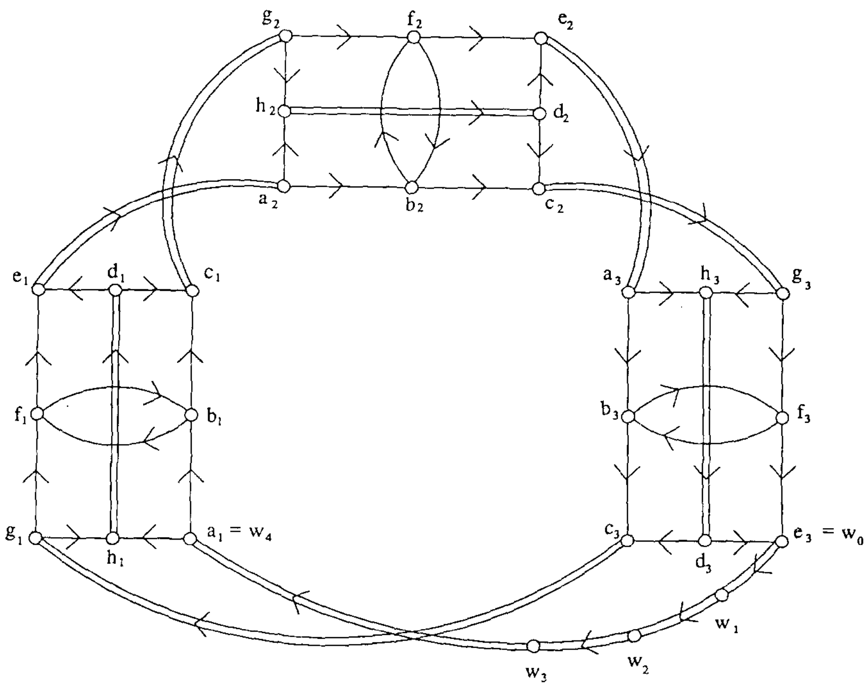

FIG. 1. Double links indicate edges in D_1; single links indicate edges in $D_{1/2}$.

THEOREM 8.1. *The depth of* (8.2) *relative to* (8.1) *is at least* $\lfloor n/8 \rfloor$. *Furthermore, every cutting-plane proof of* (8.2) *from* (8.1) *has length at least* $2^{n/8}/3n$.

Before proving this thoerem, we shall establish an auxiliary result. Define a vector $\mathbf{x}^*$ in $\mathbf{R}^D$ by setting

$$x_h^* = \begin{cases} 1 & \text{if} \quad h \in D_1 \\ \frac{1}{2} & \text{if} \quad h \in D_{1/2} \\ 0 & \text{if} \quad h \notin D_1 \cap D_{1/2}. \end{cases}$$

LEMMA 8.2. *If* P *denotes the polyhedron defined by* (8.1), *then* $\mathbf{x}^* \in P^{(k-1)}$.

Proof. First, let us define a certain mapping $f: \mathbf{R}^k \to \mathbf{R}^D$. For this purpose, write

$$A_i = \{a_i b_i, b_i f_i, f_i e_i, g_i h_i, d_i c_i\},$$

$$B_i = \{a_i h_i, d_i e_i, g_i f_i, f_i b_i, b_i c_i\}.$$

For each $\mathbf{y}$ in $\mathbf{R}^k$, write $\mathbf{y} = [y_1, y_2, \dots, y_k]^T$ and let $f(\mathbf{y})$ denote the vector $\mathbf{x}$ defined by

$$x_h = \begin{cases} 1 & \text{if} \quad h \in D_1, \\ y_i & \text{if} \quad h \in A_i, \\ 1 - y_i & \text{if} \quad h \in B_i, \\ 0 & \text{if} \quad h \notin D_1 \cup D_{1/2}, \end{cases}$$

Note that $\mathbf{x}^* = f(\frac{1}{2}\mathbf{e})$. Hence with S as in Lemma 7.2, we need only show that

$$f(S) \subseteq P, \tag{8.3}$$

for then the desired conclusion follows from Lemma 2.2 (with $T = P$) and Lemma 7.2. To prove (8.3), consider an arbitrary $\mathbf{y}$ in S. We need only show that $f(\mathbf{y})$ in place of $\mathbf{x}$ satisfies

$$\mathbf{x}[(W \times (V - W)) \cup ((V - W) \times W)] \geqslant 2 \tag{8.4}$$

for all proper subsets W of V: it is easy to see that $f(\mathbf{y})$ in place of $\mathbf{x}$ satisfies all the remaining constraints in (8.1) simply because $\mathbf{0} \leqslant \mathbf{y} \leqslant \mathbf{e}$.

For each subset B of $\{1,2,\ldots,k\}$, let $\mathbf{y}_B$ denote the incidence vector of B and define

$$W_B = \left(\bigcup_{i \in B} \{g_i, h_i, d_i, c_i\} \right) \cup \left(\bigcup_{i \notin B} \{g_i, f_i, b_i, c_i\} \right). \tag{8.5}$$

Observe that $f(\mathbf{y}_B)$ is the incidence vector of a subgraph of G that consists of two directed circuits, one spanning W_B and the other spanning $V - W_B$. Hence $f(\mathbf{y}_B)$ in place of $\mathbf{x}$ satisfies all the constraints (8.4) except for

$$\mathbf{x}\big[(W_B \times (V - W_B)) \cup ((V - W_B) \times W_B)\big] \geqslant 2. \tag{8.6}$$

Since $\mathbf{y}$ is a convex combination of zero-one vectors, and since f is a linear function, we conclude that $f(\mathbf{y})$ in place of $\mathbf{x}$ satisfies all the constraints (8.4) except possibly some of the constraints (8.6). Now we only need observe that the left-hand side of (8.6) with $f(\mathbf{y})$ in place of $\mathbf{x}$ is at least

$$4 \sum_{i \in B} (1 - y_i) + 4 \sum_{i \notin B} y_i,$$

which is at least 2, as $\mathbf{y} \in S$. ■

Proof of Theorem 8.1. The first assertion follows at once from Lemma 8.2; to prove the second assertion, we shall use Lemma 7.1. For each subset B of $\{1,2,\ldots,k\}$, let $\mathbf{y}_B$ denote the incidence vector of B and define W_B by (8.5). As in the proof of Lemma 8.2, observe that $f(\mathbf{y}_B)$ in place of $\mathbf{x}$ satisfies all the constraints (8.1) except for (8.6).

Hence Lemma 7.1 with $t = 2^k$ and $p = 10k + 1$ [counting the $10k$ edges in $D_{1/2}$ plus an additional edge, say $d_1 f_1$, to assure solvability of (7.3) in integers] guarantees that every cutting-plane proof of (8.2) from (8.1) has length at least $(2^k - 1)/10k$. ■

Next, let $\tilde{G}$ denote a complete undirected graph with vertex set V and edge-set E; let Q denote the polyhedron in $\mathbf{R}^E$ defined by

$$\begin{aligned} \sum (x_h : v \in h) &= 2 && \text{for all } v \text{ in } V, \\ \sum (x_h : |h \cap W| = 1) &\geqslant 2 && \text{for all proper subsets } W \text{ of } V, \\ 0 \leqslant x_h &\leqslant 1 && \text{for all } h \text{ in } E. \end{aligned} \tag{8.7}$$

The problem of minimizing a linear function over $Q \cap \mathbf{Z}^E$ is known as the *traveling-salesman problem*, and Q_I is called the *traveling-salesman polytope*. (These terms are explained by observing that an integer vector belongs to Q if and only if it is the incidence vector of a Hamiltonian circuit in $\tilde{G}$.) The traveling-salesman polytope has been studied by Dantzig, Fulkerson, and Johnson (1954), Chvátal (1973a), Maurras (1975), Grötschel and Padberg (1979a, 1979b, 1985), Grötschel and Pulleyblank (1986), and others.

In particular, Chvátal (1973a) conjectured that the rank of Q tends to infinity with n. We shall prove this conjecture by exhibiting a linear inequality valid over Q_I whose depth relative to Q is at least $\lfloor n/8 \rfloor$. To describe this inequality, let $D_{1/2}$ and D_1 be as in (8.2); write

$$E_i = \{\{v, w\} : (v, w) \in D_i \text{ or } (w, v) \in D_i \text{ or both}\}.$$

It is easy to check that no Hamiltonian circuit in the graph with vertex set V and edge set $E_{1/2} \cup E_1$ includes all the edges of E_1; to put it differently; the inequality

$$\mathbf{x}(E_{1/2}) + 2\mathbf{x}(E_1) \leqslant (n-1) + |E_1| \tag{8.8}$$

is valid over $Q \cap \mathbf{Z}^E$. [This inequality is closely related to a sequence $G_1, G_2, G_3, \ldots$ of hypo-Hamiltonian graphs constructed by Chvátal (1973c); there, G_k has $8k+2$ vertices. If V and E^* denote the vertex set and the edge set, respectively, of G_k and if Q is defined by (8.7), then $\mathbf{x}(E) \leqslant n-1$ is valid over $Q \cap \mathbf{Z}^E$, and its depth relative to (8.7) is at least k. The first of these assertions amounts to saying that G_k is not Hamiltonian; the second follows from a variation on the proof of Theorem 8.3.]

THEOREM 8.3. *The depth of* (8.8) *relative to* (8.7) *is at least* $\lfloor n/8 \rfloor$. *Furthermore, every cutting-plane proof of* (8.8) *from* (8.7) *has length at least* $2^{n/8}/3n$.

Proof. Let D and P be as in the beginning of this section. For each $\mathbf{x}$ in $\mathbf{R}^D$, let $f(\mathbf{x})$ denote the vector $\mathbf{y}$ in $\mathbf{R}^E$ defined by

$$y_{\{u,v\}} = x_{(u,v)} + x_{(v,u)}.$$

With $k = \lfloor n/8 \rfloor$ and $\mathbf{x}^*$ as in Lemma 8.2, we have $f(\mathbf{x}^*) \in Q^{(k-1)}$ by Lemma 2.2 (with P in place of S and Q in place of T) and Lemma 8.2. Since $f(\mathbf{x}^*)$ fails to satisfy (8.8), the first assertion follows.

To prove the second assertion, define W_B by (8.5) and observe that the graph with vertex set V and edge set $E_{1/2} \cup E_1$ has a subgraph consisting of two circuits, one spanning W_B and the other spanning $V - W_B$. This subgraph includes all the edges of E_1, and the incidence vector of its edge set satisfies all the constraints (8.7) except for

$$\sum (x_h : |h \cap W_B| = 1) \geq 2.$$

Hence Lemma 7.1 with $t = 2^k$ and $p = 9k + 1$ [counting the $9k$ edges in $E_{1/2}$ plus an additional edge, say $d_1 f_1$, to assure solvability of (7.3) in integers] guarantees that every cutting-plane proof of (8.8) from (8.7) has length at least $(2^k - 1)/9k$. ■

A large class of linear inequalities over $Q \cap \mathbf{Z}^E$ has been introduced by Grötschel and Pulleyblank (1986); we are going to describe this class now. For each subset W of V, set

$$E(W) = \{h : h \subseteq W,\ |h| = 2\}.$$

Let $H_1, H_2, \ldots, H_r$ and $T_1, T_2, \ldots, T_s$ be nonempty subsets of W; set

$$V^* = H_1 \cup H_2 \cup \cdots \cup H_r \cup T_1 \cup T_2 \cup \cdots \cup T_s,$$

$$E^* = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_r) \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_s).$$

Sets H_i will be called *handles*, and sets T_j will be called *teeth*. The graph G^* with vertex set V^* and edge set E^* is called a *clique tree* if

(1) the handles are pairwise disjoint and the teeth are pairwise disjoint,

(2) for each handle, the number of teeth it intersects is odd and at least three,

(3) each tooth includes at least two vertices, at most $n - 2$ vertices, and at least one vertex that belongs to no handle,

(4) G^* is connected,

(5) if a handle H_i and a tooth T_j intersect then $H_i \cap T_j$ is a cutset of G^*.

Finally, let t_j denote the number of handles that intersect T_j. Grötschel and Pulleyblank proved that the inequality

$$\sum_{i=1}^{r} \mathbf{x}(E(H_i)) + \sum_{j=1}^{s} \mathbf{x}(E(T_j)) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2} \quad (8.9)$$

is valid over $Q \cap \mathbf{Z}^E$. [In fact, they proved that (8.9) defines a facet of Q_I; however, this fact is irrelevant to our discussion.] Inequalities (8.9) are called *clique-tree inequalities*; by the *clique-tree polytope*, we shall mean the polytope in $\mathbf{R}^E$ defined by all the clique-tree inequalities and by the constraints

$$\begin{aligned} \sum (x_h : v \in h) &= 2 \qquad \text{for all } v \text{ in } V, \\ 0 \leqslant x_h &\leqslant 1 \qquad \text{for all } h \text{ in } E. \end{aligned} \tag{8.10}$$

It is easy to see that the polytope in $\mathbf{R}^E$ defined by (8.10) and by

$$\mathbf{x}(E(W)) \leqslant |W| - 1 \qquad \text{for all subsets } W \text{ of } V \text{ such that} \quad 2 \leqslant |W| \leqslant n-2 \tag{8.11}$$

is precisely the polytope Q defined by (8.7); in addition, (8.11) is a special case of (8.9) with $r = 0$, $s = 1$, $T_1 = W$. Hence the clique-tree polytope approximates the traveling-salesman polytope better than (8.7) does; nevertheless, an analogue of Theorem 8.3 holds true when (8.7) is replaced by the clique-tree polytope and the lower bounds are slightly weakened.

THEOREM 8.4. *The depth of* (8.8) *relative to the clique-tree polytope is at least*

$$\frac{1}{48}\left(\frac{n}{\log_2 n}\right) - 2.$$

Furthermore, every cutting-plane proof of (8.8) *from the clique-tree polytope has length at least* $n^{-3}2^{n/12}$.

Proof. Let F^* denote the family of all 2^k sets W_B defined by (8.5). In the proof of Theorem 8.3, we observed that every cutting-plane proof of (8.8) from (8.7) must use all the inequalities

$$\sum (x_h : |h \cap W_B| = 1) \geqslant 2 \qquad \text{with} \quad W_B \in F^*. \tag{8.12}$$

We propose to show that

$$\text{every clique-tree inequality admits a cutting-plane proof from (8.7) that uses at most } n\cdot 2^{2k/3} \text{ of the inequalities (8.12).} \tag{8.13}$$

As soon as (8.13) is proved, the rest follows easily: by (8.13), every cutting-plane proof of (8.8) from the clique-tree polytope that uses M clique-tree inequalities can be converted into a cutting-plane proof of (8.8) from (8.7) that uses at most $Mn\cdot 2^{2k/3}$ of the inequalities (8.12). Since all 2^k inequalities must be used, we must have $M \geqslant n^{-1}2^{k/3}$. Now we only need apply Lemma 7.1 with $t = n^{-1}2^{k/3}$ and $p = n(n-1)/2$.

Our proof of (8.13) begins with two preliminary observations. First, let T be an arbitrary subset of V that contains at least one member of F^*. By definition,

$$g_i, c_i \in T \qquad \text{for all} \quad i = 1, 2, \ldots, k,$$

and there is a subset S of $\{1, 2, \ldots, k\}$ such that

$$f_i, b_i \in T \quad \text{whenever } i \in S \quad \text{and} \quad h_i, d_i \in T \quad \text{whenever } i \notin S.$$

If R denotes the set of subscripts i such that

$$h_i, d_i \in T, \quad i \in S \quad \text{or} \quad f_i, b_i \in T, \quad i \notin S,$$

that T contains precisely $2^{|R|}$ members of F^*. Since $|R| \leqslant \frac{1}{2}(|T| - 4k)$, we conclude that

$$\text{every set of size } t \text{ contains at most } 2^{(t-4k)/2} \text{ members of } F^*. \tag{8.14}$$

Next, consider a clique tree with handles $H_1, H_2, \ldots, H_r$ and teeth $T_1, T_2, \ldots, T_s$; we wish to point out that a large tooth cannot intersect too many handles. For this purpose, fix a subscript j, and let I denote the set of all subscripts i such that $H_i \cap T_j \neq \varnothing$. By definition, each H_i meets at least three teeth; letting i run through I, we obtain $2|I|$ teeth distinct from T_j; since every nonempty intersection of a handle and a tooth is a cutset of the clique tree, these $2|I|$ teeth are pairwise distinct. Since every tooth includes at least two nodes and since every two teeth are disjoint, we conclude that $4|I| \leqslant n - |T_j|$. To put it differently,

$$\text{every tooth of size } t \text{ meets at most } (n-t)/4 \text{ handles.} \tag{8.15}$$

Finally, let F denote the family of all nonempty sets that have either the form $H_i \cap T_j$ for some subscripts i and j or the form $T_j - \bigcup(H_i : i \in J)$ for some subscript j and some (possibly empty) family J of subscripts. In proving their Theorem 3.7, Grötschel and Pulleyblank (1986) have shown that the clique-tree inequality (8.9) has a cutting-plane proof from the system

$$\begin{aligned} \sum(x_h : v \in h) &= 2 && \text{for all } v \text{ in } V, \\ \sum(x_h : |h \cap W| = 1) &\geqslant 2 && \text{for all } W \text{ in } F, \\ 0 \leqslant x_h &\leqslant 1 && \text{for all } h \text{ in } E. \end{aligned}$$

This fact reduces our task of proving (8.13) to the task of proving that

$$|F \cap F^*| \leqslant n \cdot 2^{2k/3}. \tag{8.16}$$

To prove (8.16), note that each member of F is a subset of some tooth and that each member of F^* includes g_1. Since every two teeth are disjoint, we conclude that there is a single tooth, say T_j, that contains all the members of $F \cap F^*$. If $|T_j| \leqslant 16k/3$, then (8.14) guarantees that

$$|F \cap F^*| \leqslant 2^{2k/3};$$

if $|T_j| \geqslant 16k/3$, then (8.15) guarantees that

$$|F \cap F^*| \leqslant \frac{2k}{3} + \frac{7}{4} + 2^{7/4} \cdot 2^{2k/3};$$

in either case, (8.16) holds with room to spare. ■

The last theorem of this section bears on a remark made by Grötschel and Pulleyblank (1986) at the end of their paper:

Clique tree inequalities have several interesting properties with respect to the rank function defined by Chvátal (1973b). For example, a clique tree with more than one handle is of rank at least two, and it appears that as the number of handles increases, so too does the rank although not linearly. This will be treated in a subsequent paper.

THEOREM 8.5. *For every positive integer k there is a clique tree G^* with $5k+1$ vertices such that, for every complete undirected graph $\tilde{G}$ that contains G^*, the depth of the corresponding clique-tree inequality* (8.9) *relative to* (8.7) *is at least* $[(k-1)/(4+2\log_2 k)]-1$ *and every cutting-plane proof of* (8.9) *from* (8.7) *has length at least* $2^k/40k^2$.

Proof. The vertices of G^* are

$$a_i, b_i, c_i, d_i, e_i \quad (i=1,2,\dots,k) \quad \text{and} \quad f,$$

its handles are

$$\{b_i, c_i, d_i\} \qquad (i=1,2,\dots,k),$$

and its teeth are

$$\{a_i, b_i\}, \{d_i, e_i\} \quad (i=1,2,\dots,k) \quad \textit{and} \quad \{c_1, c_2, \dots, c_k, f\}.$$

For illustration with $k=5$, see Figure 2.

Now let $\tilde{G}$ be any complete undirected graph whose vertex set V includes all the vertices of G^*; enumerate all the vertices of $\tilde{G}-G^*$ as $w_1, w_2, \dots, w_r$; let E_1 denote the set of edges

$$e_k w_1, w_1 w_2, \dots, w_{r-1} w_r, w_r a_1$$

(so that $E_1=\varnothing$ in case $r=0$); let $E_{1/2}$ denote the set of all the edges of $\tilde{G}$ that have both endpoints in G^*. For each subset S of $\{1,2,\dots,k\}$ such that $|S| \geqslant 2$, set

$$W_S = \{f\} \cap \{c_i : i \in S\}$$

and observe that $\tilde{G}$ has a subgraph F_S with the following properties:

(1) F_S consists of two circuits, one spanning W_S and the other spanning $V-W_S$,

(2) F_S uses edges $a_i b_i, b_i d_i, d_i e_i$ whenever $i \in S$, and it uses edges $a_i b_i, b_i c_i, c_i d_i, d_i e_i$ whenever $i \notin S$,

(3) F_S uses only edges of $E_{1/2} \cup E_1$, and it uses all the edges of E_1.

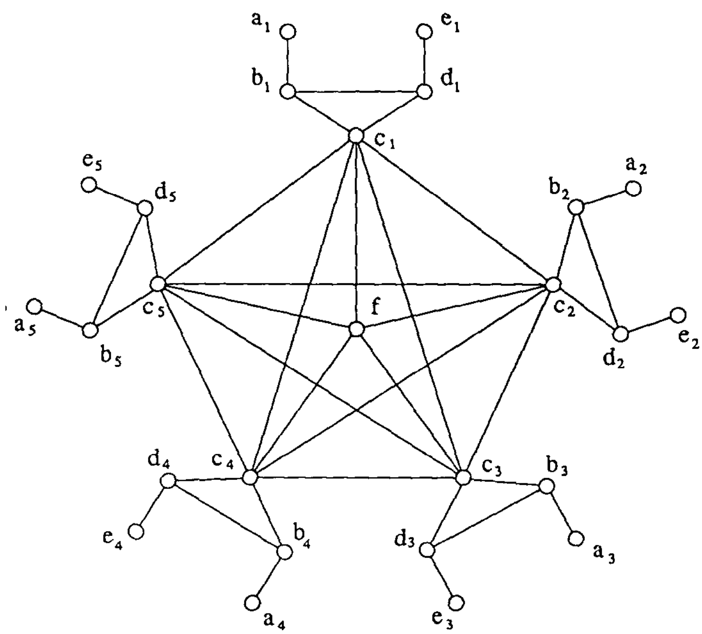

FIG. 2.

Note that F_S and G^* share $4k+1$ edges; note also that the right-hand side of (8.9) is $4k$. Hence Lemma 7.1 with

$$t = 2^k - (k+1) \quad \text{and} \quad p = \binom{5k+1}{2}$$

guarantees the desired conclusion: observe that $t \geqslant \frac{3}{8}2^k + 1$ whenever $k \geqslant 3$ and that $p \leqslant 15k^2$ for all k. ∎

9. AN UPPER BOUND ON DEPTH

We have already noted that the lower bound of Theorem 4.1 (and hence also the lower bound of Theorem 4.3) is best possible within a factor of two. To show that a few other lower bounds derived earlier in this paper are best possible within a constant factor, we only need appeal to the following theorem.

THEOREM 9.1. *Let* $\mathbf{A}$ *be a matrix of size* $m \times n$ *with all entries nonnegative, and let* $\mathbf{b} \in \mathbf{R}^m$, $\mathbf{c} \in \mathbf{Z}_+^n$, $d \in \mathbf{Z}_+$. *If* $d < \mathbf{c}^T\mathbf{e}$ *and if all integer solutions* $\mathbf{x}$ *of*

$$\mathbf{A}\mathbf{x} \leqslant \mathbf{b}, \qquad \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{e} \tag{9.1}$$

satisfy $\mathbf{c}^T\mathbf{x} \leqslant d$, *then the depth of* $\mathbf{c}^T\mathbf{x} \leqslant d$ *relative to* (9.1) *is at most* $\mathbf{c}^T\mathbf{e} - d$; *if* $\mathbf{c}^T\mathbf{e} \geqslant 2d+1$, *then the upper bound can be replaced by*

$$d + 1 + \left\lceil (2d+1)\ln \frac{\mathbf{c}^T\mathbf{e}}{2d+1} \right\rceil .$$

Proof. First, we propose to show that, with P defined by (9.1),

$$\text{for each } \mathbf{v} \text{ in } \mathbf{Z}_+^n \text{ such that } \mathbf{v} \leqslant \mathbf{c} \text{ and } \mathbf{v}^T\mathbf{e} \leqslant d+1, \text{ the inequality } \mathbf{v}^T\mathbf{x} \leqslant d \text{ is valid over } P'. \tag{9.2}$$

If $\mathbf{v}^T\mathbf{e} \leqslant d$, then the conclusion is immediate (since $\mathbf{x} \leqslant \mathbf{e}$ whenever $\mathbf{x} \in P$); hence we may assume that $\mathbf{v}^T\mathbf{e} = d+1$. Now we only need show that

$$\max\{\mathbf{v}^T\mathbf{x} : \mathbf{x} \in P\} < \mathbf{v}^T\mathbf{e}.$$

If this inequality fails then P must include a point $\mathbf{x}^*$ such that $x_j^* = 1$ whenever $v_j \geqslant 1$. Let $\tilde{\mathbf{x}}$ be the zero-one vector of length n defined by $\tilde{x}_j = 1$ if and only if $v_j \geqslant 1$. Since all entries of $\mathbf{A}$ are nonnegative and since $\tilde{\mathbf{x}} \leqslant \mathbf{x}^*$, we have $\tilde{\mathbf{x}} \in P$, and so $\mathbf{c}^T\tilde{\mathbf{x}} \leqslant d$ by assumption. However, $\mathbf{c}^T\tilde{\mathbf{x}} \geqslant \mathbf{v}^T\tilde{\mathbf{x}} = \mathbf{v}^T\mathbf{e} = d+1$, a contradiction.

If $d = 0$, then (9.2) implies that $\mathbf{c}^T\mathbf{x} \leqslant 0$ is valid over P'; hence we may assume that $d \geqslant 1$. Now let us define a sequence $n_0, n_1, n_2, \ldots$ of integers by setting $n_0 = d+1$ and letting each n_k with $k \geqslant 1$ be the largest integer smaller than $(d+1)n_{k-1}/d$. We propose to show that

$$\text{for each } \mathbf{v} \text{ in } \mathbf{Z}_+^n \text{ such that } \mathbf{v} \leqslant \mathbf{c} \text{ and } \mathbf{v}^T\mathbf{e} \leqslant n_{k-1}, \text{ the inequality } \mathbf{v}^T\mathbf{x} \leqslant d \text{ is valid over } P^{(k)}. \tag{9.3}$$

For this purpose, we shall use induction on k; note that (9.3) with $k = 1$ is just (9.2). Now assume that (9.3) holds for some positive integer k, and consider an arbitrary $\mathbf{v}$ in $\mathbf{Z}_+^n$ such that $\mathbf{v} \leqslant \mathbf{c}$ and $\mathbf{v}^T\mathbf{e} \leqslant n_k$; we have to show that $\mathbf{v}^T\mathbf{x} \leqslant d$ holds over $P^{(k+1)}$. If $\mathbf{v}^T\mathbf{e} \leqslant n_{k-1}$, then (9.3) guarantees that $\mathbf{v}^T\mathbf{x} \leqslant d$ holds over $P^{(k)}$; hence we may assume that $\mathbf{v}^T\mathbf{e} > n_{k-1}$. Let S consist of all ordered pairs i, j of integers such that $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant v_j$. Next, let $\mathbf{x}^*$ be an arbitrary but fixed vector in $P^{(k)}$; write $y_{ij} = x_j^*$ whenever $ij \in S$. For each subset T of S such that $|T| = n_{k-1}$, the assumption (9.3) guarantees that

$$\sum_{ij \in T} y_{ij} = \sum_{j=1}^{n} |\{i : ij \in T\}| x_j^* \leqslant d;$$

now symmetry implies that

$$\sum_{ij \in S} y_{ij} \leqslant d \cdot \frac{|S|}{n_{k-1}}.$$

Hence

$$\mathbf{v}^T \mathbf{x}^* = \sum_{ij \in S} y_{ij} \leqslant d \cdot \frac{|S|}{n_{k-1}} \leqslant d \cdot \frac{n_k}{n_{k-1}} < d + 1.$$

Since $\mathbf{x}^*$ was an arbitrary vector in $P^{(k)}$, we conclude that

$$\max\{\mathbf{v}^T\mathbf{x} : \mathbf{x} \in P^{(k)}\} < d + 1,$$

and so $\mathbf{v}^T\mathbf{x} \leqslant d$ holds over $P^{(k+1)}$.

Now we only need prove that the smallest k with $n_{k-1} \geqslant \mathbf{c}^T\mathbf{e}$ satisfies $k \leqslant \mathbf{c}^T\mathbf{e} - d$ and, in case $\mathbf{c}^T\mathbf{e} \geqslant 2d + 1$,

$$k \leqslant d + 1 + \left\lceil (2d+1) \ln \frac{\mathbf{c}^T\mathbf{e}}{2d+1} \right\rceil.$$

For this purpose, we first use induction on k to show that $n_k \geqslant d + 1 + k$ for all k. Then, observing that

$$(2d+1)\left(1 + \frac{1}{2d}\right)^{k-d} + 1 < \frac{d+1}{d}(2d+1)\left(1 + \frac{1}{2d}\right)^{k-1-d}$$

whenever $k > d$, we use further induction on k to show that

$$n_k \geqslant (2d+1)\left(1+\frac{1}{2d}\right)^{k-d} \qquad \text{for all} \quad k = d, d+1, d+2, \ldots .$$

Since

$$1 + \frac{1}{2d} \geqslant \exp\frac{1}{2d+1},$$

the desired conclusion follows. ■

By Theorem 9.1, the lower bound of Theorem 3.2 cannot be improved by more than a constant factor, and the lower bound of Theorem 3.3 cannot be improved to more than $5 \ln n$. Furthermore, Theorem 9.1 guarantees that the depth of (8.2) relative to the polyhedron in $\mathbf{R}^D$ defined by

$$\begin{aligned}
\mathbf{x}(\{v\} \times (V - \{v\})) &\leqslant 1 && \text{for all } v \text{ in } V,\\
\mathbf{x}((V - \{v\}) \times \{v\}) &\leqslant 1 && \text{for all } v \text{ in } V,\\
\mathbf{x}(W \times W) &\leqslant |W| - 1 && \text{for all proper subsets } W \text{ of } v,\\
0 \leqslant x_h &\leqslant 1 && \text{for all } h \text{ in } D
\end{aligned} \tag{9.4}$$

is at most $(5n/8)+1$, and that the depth of (8.8) relative to the polyhedron in $\mathbf{R}^E$ defined by

$$\begin{aligned}
\textstyle\sum (x_h : v \in h) &\leqslant 2 && \text{for all } v \text{ in } V,\\
\mathbf{x}(W) &\leqslant |W| - 1 && \text{for all proper subsets } W \text{ of } V,\\
0 \leqslant x_h &\leqslant 1 && \text{for all } h \text{ in } E
\end{aligned} \tag{9.5}$$

is at most $(n/2)+1$. It follows that the lower bound of Theorem 8.1 cannot be improved to more than $(5n/8)+1$ [since all solutions of (8.1) are solutions of (9.4)] and that the lower bound of Theorem 8.3 cannot be improved to more than $(n/2)+1$ [since all solutions of (8.7) are solutions of (9.5)].

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