Fisher Type Inequalities for Euclidean *t*-Designs

P. Delsarte
Philips Research Laboratory Brussels
Av. Van Becelaere 2, Box 8
B-1170 Brussels, Belgium

and

J. J. Seidel
Department of Mathematics and Computing Science
Eindhoven University of Technology
5600 MB Eindhoven, The Netherlands

Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

ABSTRACT

The notion of a Euclidean *t*-design is analyzed in the framework of appropriate inner product spaces of polynomial functions. Some Fisher type inequalities are obtained in a simple manner by this method. The same approach is used to deal with certain analogous combinatorial designs.

1. INTRODUCTION

The notions of spherical *t*-designs [3] and of cubature formulae of strength *t* for the sphere [5] deal with some order-*t* approximations of the whole unit sphere in Euclidean space by a finite subset of it. Measures of strength *t* provide a general setting for such approximations [9]. This paper is concerned with the case where the measure has a finite discrete support, distributed over a certain number of concentric spheres. In other words, it deals with finite weighted sets of strength *t* in a *d*-dimensional Euclidean space [9]. By analogy with some related concepts, such configurations are referred to as

Euclidean t-designs. The main purpose of the paper is to derive *Fisher type inequalities*, i.e., lower bounds for the cardinality of a Euclidean *t*-design depending only on the "parameters."

There are formal analogies between Euclidean *t*-designs and some generalizations of combinatorial *t*-designs that admit several block sizes [8, 11, 15]. Both cases are indeed concerned with approximations of reference measures by some other measures having a smaller support. The paper considers also, with less details, the question of Fisher type inequalities for the combinatorial (or Boolean) case, which is seen to be more difficult than the Euclidean case.

The basic idea of our method is to characterize the design property in terms of an identity between two inner products (corresponding to both measures) over a well-defined function space. This idea proves to be very efficient; it allows one to derive a variety of Fisher type inequalities, generalizing previous results, in a transparent and economical manner.

Section 2 gives the definitions of spherical *t*-designs, cubature formulae of strength *t* for the unit sphere, weighted sets of strength *t* in Euclidean space, and measures of strength *t* in Euclidean space, which have increasing generality. The *test functions* for all these definitions are the real polynomials in *d* variables of total degree less than or equal to *t*.

Section 3 contains some useful results concerning the real linear spaces of homogeneous polynomial functions, of a given degree, restricted to *spherical sets* (which are unions of concentric spheres, centered at the origin). In particular, the dimension of the relevant space of test functions is explicitly obtained, in terms of the parameter *p* counting the distinct radii of the Euclidean design.

By taking as test functions all homogeneous polynomials of degree *j*, one obtains the definition of a weighted set *admitting index j*. (Thus, strength *t* means indices $1, 2, \ldots, t$.) Section 4 is devoted to this subject. In particular, antipodal weighted sets are introduced, and a useful characterization of the presence of a collection of indices is given in terms of certain inner product identities.

Fisher type inequalities are derived in Section 5. First, a general result is given in the case of weighted sets admitting a collection *T* of indices *j* of the form $T = A + A$, for a suitable set *A* of nonnegative integers. Then, this result is applied to weighted sets of even strength *t* and to antipodal weighted sets of odd strength *t*. The bounds thus obtained are well-defined sums of binomial coefficients, involving only the dimension *d*, the strength *t*, and the number of radii *p*.

Section 6 is concerned with a Boolean (or combinatorial) analog of the theory. The combinatorial definition of an "index" is first translated into an algebraic characterization similar to that of the Euclidean counterpart. Then, the properties of the relevant spaces of test functions are examined. Finally,

some Fisher type inequalities are obtained, generalizing the Ray-Chaudhuri–Wilson bound for classical *t*-designs [10, 14]. Some interesting open problems are suggested in this section.

2. DEFINITIONS

For an integer d, with $d \geqslant 2$, let $E = \mathbf{R}^d$ denote the d-dimensional real *Euclidean space*, equipped with the 2-norm $\|\cdot\|$. The *unit sphere* S is defined to be the set

$$S = \{ x \in E : \|x\| = 1 \}. \tag{2.1}$$

The standard normalized angular measure over S will be denoted by $d\sigma$. (The total measure of S equals unity.) Given a nonnegative integer j, the linear space of the *real polynomials*, in d variables, *of total degree* $\leqslant j$ will be denoted by $\mathrm{Pol}_j(E)$.

DEFINITION 2.1. A finite nonempty subset X of the unit sphere S is said to be a *spherical t-design*, for a nonnegative integer t, if it satisfies

$$\sum_{x \in X} f(x) = |X| \int_S f(x) \, d\sigma(x), \qquad \text{all} \quad f \in \mathrm{Pol}_t(E). \tag{2.2}$$

Spherical *t*-designs have been introduced as a setting for various combinatorial structures, such as few-angle sets, association schemes, and related topics [3]. Some nice examples arise from representations of finite groups [1, 4].

A generalization is obtained by admitting "weights." A *weight function* on X is simply a mapping w from X into the set $\mathbf{R}_+^*$ of positive real numbers. We use the notation $w(X)$ for the total weight of X, i.e., the sum of the weights $w(x)$ for $x \in X$.

DEFINITION 2.2. A finite weighted set (X, w), with $X \subset S$ and $w : X \to \mathbf{R}_+^*$, is said to provide a *cubature formula of strength t for the unit sphere* if it satisfies

$$\sum_{x \in X} f(x) w(x) = w(X) \int_S f(x) \, d\sigma(x), \qquad \text{all} \quad f \in \mathrm{Pol}_t(E). \tag{2.3}$$

Interesting spherical designs and cubature formulae arise from finite subgroups of the orthogonal group of S. The orbits of such a subgroup G are spherical t-designs for a certain t (depending on G), and suitable weighted combinations of these orbits may provide cubature formulae of strength t' larger than t; see [5].

By definition, a cubature formula of strength t yields an order-t approximation of the unit sphere S by a finite subset $X \subset S$, in the sense that a weighted average of $f(x)$ over X equals the natural average of $f(x)$ over S for all polynomials f of degree not exceeding t. The paper deals with an extension of that concept, where the constraint of "constant radius" is relaxed.

Consider a *finite weighted set* (Y, w), where Y is a finite nonempty subset of E, not containing the origin, and w is a strictly positive weight function defined on Y. We use the *polar coordinates* of nonzero vectors $y \in E$; thus we write $y = rx$, with $r \in \mathbf{R}_+^*$ and $x \in S$, hence $r = \|y\|$. The *radial support* R of the set Y is the set of its radii with respect to the origin, i.e.,

$$R = \{ \|y\| : y \in Y \}. \tag{2.4}$$

The *spherical components* of Y are its subsets $Y_r = Y \cap rS$, with $r \in R$, where rS is the sphere of radius r (and center 0) in E. The *weight of* Y *on* rS is defined by

$$w(Y_r) = \sum_{y \in Y_r} w(y), \quad \text{with} \quad Y_r = Y \cap rS. \tag{2.5}$$

The *spherical support* of Y is the union of the concentric spheres (centered at the origin) that contain at least one point of Y; hence it is the set RS given by

$$RS = \bigcup_{r \in R} rS. \tag{2.6}$$

We are now in a position to introduce the main subject of this paper. The following definition of a weighted set of strength t is exactly equivalent to the definition given originally in [9], although it is expressed in slightly different terms.

DEFINITION 2.3.   A finite weighted set (Y, w), with $Y \subset E$, $0 \notin Y$, and $w : Y \to \mathbf{R}^*_+$, is said to have *strength t* if it satisfies

$$\sum_{y \in Y} f(y) w(y) = \int_{RS} f(y)\, d\mu(y), \qquad \text{all} \quad f \in \mathrm{Pol}_t(E), \qquad (2.7)$$

where $d\mu$ is the measure, of support RS, defined by

$$d\mu(y) = w(Y_r)\, d\sigma(x), \qquad \text{for} \quad y = rx, \quad x \in S, \quad r \in R. \qquad (2.8)$$

Although this definition characterizes the same type of approximation property as Definitions 2.1 and 2.2, it should be noted that the reference measure $d\mu$ depends here on the weighted set under consideration, through its radial factor $w(Y_r)$. For the sake of clarity, let us write down (2.7) in the more explicit form

$$\sum_{y \in Y} f(y) w(y) = \sum_{r \in R} w(Y_r) \int_S f(rx)\, d\sigma(x). \qquad (2.9)$$

As a further generalization, we briefly mention the concept of a measure of strength t defined originally in [9], in different terms. Let $B = \{ y \in E : \|y\| \leqslant 1 \}$ denote the closed unit ball in E. The definition below confronts a given measure $d\xi$ on the Euclidean space E (having appropriate integrability properties) with the measure $d\xi^*$ defined by

$$d\xi^*(rx) = d\xi(rB)\, d\sigma(x). \qquad (2.10)$$

DEFINITION 2.4.   A measure $d\xi$ on E is said to have *strength t* if it satisfies

$$\int_E f(y)\, d\xi(y) = \int_E f(y)\, d\xi^*(y), \qquad \text{all} \quad f \in \mathrm{Pol}_t(E). \qquad (2.11)$$

Specific properties of some remarkable lattices can be expressed in this framework [9]. Such applications are concerned with infinite discrete measures $d\xi$. It is easily seen that Definition 2.4 amounts to Definition 2.3 in the case of a finite discrete measure $d\xi$.

REMARK.   The equivalence between Definition 2.4 and the definition given in [9] can be explained as follows. For any homogeneous polynomial f

of degree j, the identity (2.11) reads

$$\int_E f(y)\,d\xi(y) = \int_0^\infty r^j d\xi(rB) \int_S f(x)\,d\sigma(x). \qquad (2.12)$$

Thus, the $d\xi$-average of f and the $d\sigma$-average of f are equal within a factor depending only on the degree j, for $j = 0, 1, \ldots, t$, which is the definition of the strength-t property in [9].

In the sequel we shall restrict our attention to Definition 2.3 (and some related concepts). By analogy with the terminologies of combinatorial t-design [6] and of spherical t-design [3], we suggest the name *Euclidean t-design* to mean a finite weighted (or nonweighted) set of strength t in a Euclidean space (Definition 2.3).

3.   POLYNOMIAL FUNCTIONS OVER SPHERICAL SETS

To derive bounds for Euclidean t-designs (and related configurations) we shall use some properties of linear spaces consisting of polynomial functions defined over a *spherical set* RS. The definition of RS is given in (2.6), with R denoting any finite set of positive real numbers. The *number of radii*,

$$p = |R|, \qquad (3.1)$$

will often occur in the results below.

As usual, let us denote by $\mathrm{Hom}_j(E)$ the linear space of *real homogeneous polynomials* (in d variables) of total *degree* j (where j is any nonnegative integer). Since the monomials are linearly independent, the dimension of that space is given by the binomial coefficient

$$\dim \mathrm{Hom}_j(E) = \binom{d+j-1}{d-1}. \qquad (3.2)$$

Recall that the space $\mathrm{Pol}_j(E)$ involved in the definitions of Section 1 is the sum

$$\mathrm{Pol}_j(E) = \sum_{i=0}^{j} \mathrm{Hom}_i(E). \qquad (3.3)$$

Of course, (3.3) is a *direct sum*.

In the sequel we shall frequently use *restriction mappings* for function spaces. Given a linear space $F(M)$ consisting of real-valued functions defined over a set M, and given a subset N of M, we shall denote by $F(N)$ the homomorphic image of $F(M)$ obtained by restricting all functions in $F(M)$ to the domain N. For example, one obtains the spaces $\mathrm{Hom}_j(RS)$ and $\mathrm{Pol}_j(RS)$ from $\mathrm{Hom}_j(E)$ and $\mathrm{Pol}_j(E)$ by restriction to the spherical set RS.

We first give an elementary *inclusion lemma* and then deduce an important *decomposition theorem* which plays a crucial rôle in some derivations of Section 5.

LEMMA 3.1. *For any spherical set RS with p distinct radii, one has the vector space inclusion*

$$\mathrm{Hom}_j(RS) \subset \sum_{i=1}^{p} \mathrm{Hom}_{j+2i}(RS). \tag{3.4}$$

Proof. Given a homogeneous polynomial $f \in \mathrm{Hom}_j(E)$, consider the obvious identity

$$f(y) \prod_{r \in R} \left(r^2 - \|y\|^2 \right) = 0, \qquad \text{for} \quad y \in RS. \tag{3.5}$$

By expansion of the R-product, it allows one to express the value $f(y)$ as a constant linear combination of the p values $\|y\|^{2i}f(y)$, with $i = 1, 2, \ldots, p$, for y varying over RS. ∎

THEOREM 3.2. *For any nonnegative integer j, one has the direct sum decomposition*

$$\mathrm{Pol}_j(RS) = \sum_{i=0}^{2p-1} \mathrm{Hom}_{j-i}(RS), \tag{3.6}$$

where $p = |R|$, with the convention $\mathrm{Hom}_k(E) = \{0\}$ for $k < 0$.

Proof. In view of (3.3), the identity (3.6) is immediate from Lemma 3.1, since (3.4) shows that $\mathrm{Hom}_{j-k}(RS)$ is a subspace of the right hand side of

(3.6) when $k \geqslant 2p$. To establish that (3.6) is a direct sum, it suffices to prove the dimension property

$$\dim \operatorname{Pol}_j(RS) = \sum_{i=0}^{2p-1} \dim \operatorname{Hom}_{j-i}(RS). \tag{3.7}$$

Consider the restriction homomorphism $\phi: \operatorname{Pol}_j(E) \to \operatorname{Pol}_j(RS)$. Its kernel consists of the polynomials $f(y)$ of degree j that vanish over each sphere rS with $r \in R$. Therefore, applying Hilbert's *Nullstellensatz* [13], one obtains

$$\operatorname{Ker} \phi = \operatorname{Pol}_{j-2p}(E) \prod_{r \in R} \left(r^2 - \|y\|^2 \right). \tag{3.8}$$

Since $\dim \operatorname{Pol}_j(RS) = \dim \operatorname{Pol}_j(E) - \dim \operatorname{Ker} \phi$, one deduces the desired result (3.7) by use of (3.8), owing to the fact that (3.3) is a direct sum decomposition and that $\operatorname{Hom}_i(RS)$ is isomorphic to $\operatorname{Hom}_i(E)$. ■

REMARK. It follows from Theorem 3.2 that the right hand side of the inclusion relation (3.4) is a direct sum. In fact, this property is exactly equivalent to Theorem 3.2.

For future use let us now consider the *multiplicative properties* of polynomial spaces. The homogeneous spaces clearly satisfy the identity

$$\operatorname{Hom}_i(E)\operatorname{Hom}_j(E) = \operatorname{Hom}_{i+j}(E) \tag{3.9}$$

for $i \geqslant 0$, $j \geqslant 0$. In this context, the product FG of two function spaces F and G is defined as the linear space spanned by the products fg with $f \in F$ and $g \in G$. The same identity (3.9) holds if Hom is replaced by Pol.

These properties can be generalized as follows. Given a finite nonempty set I of nonnegative integers, define the function space $\operatorname{Pol}_I(E)$ to be the sum

$$\operatorname{Pol}_I(E) = \sum_{i \in I} \operatorname{Hom}_i(E). \tag{3.10}$$

Note that this is a direct sum. The simple cases $I = \{i\}$ and $I = \{0, 1, \ldots, i\}$

yield $\text{Pol}_i = \text{Hom}_i$ and $\text{Pol}_I = \text{Pol}_i$, respectively. As a straightforward consequence of (3.9) one obtains the general result

$$\text{Pol}_I(E)\text{Pol}_J(E) = \text{Pol}_{I+J}(E), \tag{3.11}$$

where the set $I + J$ consists of the sums of integers $i + j$ with $i \in I$ and $j \in J$. Note that the identities (3.9), (3.10), and (3.11) remain valid when the set E is replaced by any subset, because the restriction mapping is an algebra homomorphism.

4. INDICES OF WEIGHTED SETS

Although we are mainly interested in Euclidean *t*-designs (Definition 2.3), we now consider a weakening of the defining property (2.7); it involves the notion of "indices" (used in a sense similar to that in [12]) instead of the notion of strength.

DEFINITION 4.1. Given a nonnegative integer j, a finite weighted set (Y, w), with $Y \subset E$, $0 \notin Y$, and $w : Y \to \mathbf{R}_+^*$, is said to admit the *index j* if it satisfies

$$\sum_{y \in Y} f(y)w(y) = \int_{RS} f(y)\, d\mu(y), \qquad \text{all} \quad f \in \text{Hom}_j(E). \tag{4.1}$$

Note that (Y, w) admits always the index 0. It results from (3.3) that (Y, w) is a Euclidean *t*-design (i.e., has strength t) if and only if it admits the indices $j = 1, 2, \ldots, t$. The following theorem, which is an immediate consequence of Lemma 3.1, says that some indices cannot be present without some others.

THEOREM 4.2. *If a weighted set (Y, w) having p radii admits the indices $j + 2, j + 4, \ldots, j + 2p$, then it admits the indices $j, j - 2, \ldots$.*

COROLLARY 4.3. *If (Y, w) admits the $2p$ consecutive indices t, $t - 1, \ldots, t - 2p + 1$, with $p = |R|$, then it has strength t.*

A weighted set (Y, w) is said to be *antipodal* if it satisfies $Y = -Y$ and $w(y) = w(-y)$ for all $y \in Y$. (In fact, as far as the question of bounds is concerned, the second condition is not really restrictive.) Define the set Y^* as

any subset of Y that contains exactly one element of each antipodal pair $\{y, -y\}$. Of course, $|Y|$ equals $2|Y^*|$. The following result will be useful to derive a bound for antipodal Euclidean t-designs.

THEOREM 4.4.   *The antipodal weighted set (Y, w) admits all odd indices and admits exactly the same even indices as its weighted subset (Y^*, w).*

Proof.   This readily follows from the definitions by use of the fact that a homogeneous polynomial $f \in \mathrm{Hom}_j(E)$ satisfies $f(-y) = (-1)^j f(y)$. Details are omitted.   ∎

For future use let us finally express the defining property (4.1) of the indices in terms of the *inner products* $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ associated with the weight function w and with the reference measure $d\mu$. These inner products are given by

$$[f, g] = \sum_{y \in Y} f(y)g(y)w(y), \tag{4.2}$$

$$\langle f, g \rangle = \int_{RS} f(y)g(y) \, d\mu(y), \tag{4.3}$$

where f and g vary over an appropriate linear space of square integrable functions.

THEOREM 4.5.   *Given a finite set T of nonnegative integers, assume the function space $\mathrm{Pol}_T(E)$ to factorize as the product of two function spaces F and G. Then the weighted set (Y, w) admits the set T of indices if and only if the corresponding inner products satisfy*

$$[f, g] = \langle f, g \rangle, \quad \text{all} \quad f \in F, \quad g \in G. \tag{4.4}$$

Proof.   The defining property of T is $[1, h] = \langle 1, h \rangle$ for all $h \in \mathrm{Pol}_T(E)$. Since $[1, fg] = [f, g]$ and $\langle 1, fg \rangle = \langle f, g \rangle$, this implies (4.4) in view of the property $\mathrm{Pol}_T(E) = FG$. Conversely, suppose (4.4) to be satisfied. Since the elements of $\mathrm{Pol}_T(E)$ are sums of products $h = fg$ with $f \in F$ and $g \in G$, it follows from (4.4) that $[1, h]$ equals $\langle 1, h \rangle$ for all h in $\mathrm{Pol}_T(E)$.   ∎

5.   LOWER BOUNDS

This section is concerned with the problem of finding a lower bound for the cardinality $|Y|$ of a weighted set (Y, w) that admits a *specified set T of indices*. By analogy with the celebrated Fisher inequality for combinatorial block designs, such a bound is called a *Fisher type inequality* when it depends only on the "parameters," which are the dimension, the indices, and the number of radii.

Our method applies to the important case where the set T can be written in the form

$$T = A + A, \tag{5.1}$$

where A is any finite set of nonnegative integers. This simply means that the indices in T are the sums of two elements of A. It is easily seen that A is uniquely determined from T (when it exists).

Before giving general results we now mention some simple but significant examples of sets T enjoying the property (5.1). For a positive integer e, we consider the pairs (T, A) given by

$$T = \{2e\}, \qquad\qquad A = \{e\}, \tag{5.2}$$

$$T = \{0, 1, 2, \ldots, 2e\}, \quad A = \{0, 1, \ldots, e\}, \tag{5.3}$$

$$T = \{0, 2, 4, \ldots, 2e\}, \quad A = \{0, 2, \ldots, e\}, \qquad e \text{ even}, \tag{5.4}$$

$$T = \{2, 4, 6, \ldots, 2e\}, \quad A = \{1, 3, \ldots, e\}, \qquad e \text{ odd}. \tag{5.5}$$

The case (5.2) needs no comment. The case (5.3) means that (Y, w) has strength $t = 2e$. The cases (5.4) and (5.5) are relevant to the question of an antipodal weighted set (Y, w) of strength $t = 2e + 1$. Indeed, according to Theorem 4.4, this property is characterized exactly by the fact that (Y^*, w) admits the indices $2, 4, \ldots, 2e$. [The distinction between (5.4) and (5.5) is immaterial in that respect.]

The following theorem shows why the assumption (5.1) is especially attractive. It is a straightforward consequence of Theorem 4.5, based on the identity (3.11).

THEOREM 5.1.   *The weighted set* (Y, w) *admits the set of indices* $T = A + A$ *if and only if*

$$[f, g] = \langle f, g \rangle, \qquad \text{all } f \text{ and } g \text{ in } \mathrm{Pol}_A(E). \tag{5.6}$$

We are now in a position to prove rather general lower bounds (Theorem 5.2 and Corollary 5.3), from which we can deduce explicit Fisher type inequalities in some interesting special cases (Theorem 5.4).

THEOREM 5.2.   *If a weighted set* (Y, w) *admits the set of indices* $T = A + A$, *then the cardinality of* Y *is bounded from below by*

$$|Y| \geqslant \dim \mathrm{Pol}_A(RS). \tag{5.7}$$

Proof.   The inner product $\langle \cdot, \cdot \rangle$, defined in (4.3), is nonsingular over the space $\mathrm{Pol}_A(RS)$. Indeed, for a given polynomial $f \in \mathrm{Pol}_A(E)$, the squared norm $\langle f, f \rangle$ is strictly positive unless $f(y)$ vanishes for all $y \in RS$, since RS is the exact support of the measure $d\mu$. Therefore, the identity (5.6) implies the natural isomorphism

$$\mathrm{Pol}_A(Y) \cong \mathrm{Pol}_A(RS). \tag{5.8}$$

Let $\mathbf{R}^Y$ denote the real vector space of all mappings from the finite set Y to the field $\mathbf{R}$. Since $\mathbf{R}^Y$ has dimension $|Y|$ and since it includes $\mathrm{Pol}_A(Y)$ as a subspace, one immediately deduces (5.7) from (5.8).   ■

COROLLARY 5.3.   *If an antipodal weighted set* (Y, w) *admits a set* T *of even indices of the form* $T = A + A$, *then the cardinality of* Y *is bounded from below by*

$$|Y| \geqslant 2 \dim \mathrm{Pol}_A(RS). \tag{5.9}$$

Proof.   Apply Theorems 4.4 and 5.2.   ■

It seems difficult to obtain general explicit formulae for the dimension of $\mathrm{Pol}_A(RS)$. However, the special cases mentioned above can be solved with the help of the results of Section 3. The outcome is the following.

THEOREM 5.4.   *If (Y, w) admits the index $2e$, then*

$$|Y| \geqslant \binom{d + e - 1}{d - 1}.$$

(5.10)

If (Y, w) has strength $2e$ and has p radii, then

$$|Y| \geqslant \sum_{i=0}^{2p-1} \binom{d + e - i - 1}{d - 1}.$$

(5.11)

If (Y, w) is antipodal, has strength $2e + 1$ and has p radii, then

$$|Y| \geqslant 2 \sum_{i=0}^{p-1} \binom{d + e - 2i - 1}{d - 1}.$$

(5.12)

Proof.   Let us examine only the third case (the others being similar and simpler). The bound (5.12) is obtained by applying Corollary 5.3 to the set $A = \{e, e - 2, \ldots, q\}$, where q is the parity of e [see (5.4) and (5.5)]. From Theorem 3.2 it follows that $\text{Pol}_A(RS)$ is the direct sum of the spaces $\text{Hom}_j(RS)$ with $j = e, e - 2, \ldots, e - 2p + 2$. Since $\text{Hom}_j(RS)$ is isomorphic to $\text{Hom}_j(E)$, its dimension equals the binomial coefficient (3.2). Collecting the results, one obtains (5.12).   ∎

Theorem 5.4 generalizes some previous results (and its proof is simpler than previous methods). The bounds (5.11) and (5.12) for spherical t-designs $[p = 1$ and $w(y) = 1]$ are given in [3]. In this case, several examples are known where the bounds are tight. Note that a weighted set admitting the index 2 is essentially equivalent to a *eutactic star* [9, 12]; the bound (5.10) reduces to the expected inequality $|Y| \geqslant d$. An interesting special case of (5.11) is given in [9], namely

$$|Y| \geqslant \binom{d + e}{d} \qquad \text{when} \quad 2p \geqslant e + 1.$$

(5.13)

Unfortunately, in the case $p > 1$ we do not have good examples of Euclidean t-designs with "small cardinalities." The Fisher type inequalities of Theorem 5.4 are likely not to be achievable, except for some very special values of d, t, and p (see conjecture 3.4 of [9] in that respect).

6. BOOLEAN ANALOGS

There exists a strong analogy between the theory of combinatorial t-designs and the theory of spherical t-designs, which is especially relevant to the subject of lower bounds, including Fisher type inequalities [2]. The question naturally arises whether this can be extended further to an analogy between "Boolean designs with several block sizes," as considered e.g. in [8, 11, 15], and "Euclidean designs with several radii," as considered above (roughly speaking). We now examine some Boolean (or combinatorial) counterparts of the preceding definitions and results. The theory proves to be more difficult (and less efficient) in the Boolean case than in the Euclidean case. This may be explained by the fact that all Euclidean spheres of a given dimension are essentially equivalent whereas the structure of a "Boolean sphere" depends strongly on its "radius" (block size).

Let V be a finite nonempty set of v elements, called *points*. We denote by E the set of subsets of V; hence, $|E| = 2^v$. We consider a nonempty subset Y of E, whose elements are called *blocks*, together with a strictly positive weight function w defined over Y. The following definition of an "index" is a suitable analog of Definition 4.1 (see Theorem 6.2 below).

DEFINITION 6.1.   Given an integer j, with $0 \leqslant j \leqslant v$, the weighted set (Y, w) is said to admit the *index* j if the sum of the weights $w(y)$ over all blocks $y \in Y$ that include a set $z \in E$ with $|z| = j$ is independent of the choice of z.

When w is integer-valued (or, equivalently, rational-valued), this definition means exactly that (Y, w) is a *j-wise balanced design* [8, 15]. Positive integer weights are indeed naturally interpreted as block repetition numbers. (Note that the term "index" is sometimes used in design theory with a completely different meaning.)

Consider now the *zeta function* $\zeta : E \times E \to \mathbf{R}$ of the Boolean lattice E. It is given by

$$\zeta(z, y) = 1 \quad \text{if} \quad z \subset y, \qquad \zeta(z, y) = 0 \quad \text{otherwise.} \qquad (6.1)$$

For a given $z \in E$, define the function $f_z : E \to \mathbf{R}$ by $f_z(y) = \zeta(z, y)$. Note that f_z can be represented as a squarefree monomial, in v variables, of degree $|z|$. For an integer j, with $0 \leqslant j \leqslant v$, we define the *homogeneous space* $\mathrm{Hom}_j(E)$ to be the linear span of all functions f_z with $|z| = j$.

Next, let us introduce some notations analogous to (2.4), (2.5), and (2.6), namely

$$K = \{\, |y|: y \in Y \,\},\qquad\qquad (6.2)$$

$$m(k) = |E_k|^{-1} \sum_{y \in Y \cap E_k} w(y),\qquad\qquad (6.3)$$

$$E_K = \bigcup_{k \in K} E_k,\qquad\qquad (6.4)$$

where $E_k = \{\, y \in E : |y| = k \,\}$ for $k = 0, 1, \dots, v$. Thus K is the *set of block sizes of Y*, and E_K is the *"spherical support"* of Y. By a simple counting argument one can prove the following result.

THEOREM 6.2.   *The weighted set* (Y, w) *admits the index j if and only if it satisfies*

$$\sum_{y \in Y} f(y)w(y) = \sum_{y \in E_K} f(y)m(|y|), \qquad all \quad f \in \mathrm{Hom}_j(E). \quad (6.5)$$

To imitate the theory of the Euclidean case, one should have discrete counterparts of the properties of homogeneous spaces given in Section 3. First, one can prove the inclusion property

$$\mathrm{Hom}_j(E_K) \subset \sum_{i=1}^{p} \mathrm{Hom}_{j+i}(E_K),\qquad\qquad (6.6)$$

with $p = |K|$ the *number of block sizes*. [By convention, $\mathrm{Hom}_j(E) = \{0\}$ for $j > v$.] The argument is based on a suitable analog of (3.5), namely

$$\zeta(z, y) \prod_{k \in K} (k - |z| - |y \cap \bar{z}|) = 0, \qquad for \quad y \in E_K,\qquad (6.7)$$

where $\bar{z}$ denotes the complement of z in V. One easily sees that, because z and $\bar{z}$ are disjoint, the function g_i defined by $g_i(y) = \zeta(z, y)|y \cap \bar{z}|^i$ belongs to the sum of the spaces $\mathrm{Hom}_{j+l}(E)$ with $l = 1, 2, \dots, i$, where $j = |z|$. By expansion of the K-product in (6.7) one then obtains the desired result (6.6).

Let $\mathrm{Pol}_j(E)$ be the linear space generated by the functions $f_z = \zeta(z, \cdot)$ with $|z| \leqslant j$. [Formally, the definition is the same as in (3.3).] The property (6.6) can be expressed in the form of the identity

$$\mathrm{Pol}_j(E_K) = \sum_{i=0}^{p-1} \mathrm{Hom}_{j-i}(E_K). \tag{6.8}$$

In contrast with the statement of Theorem 3.2, we do not claim that (6.8) is a direct sum decomposition, although this may be true in most cases. The difficulty is due to the fact that the *Nullstellensatz* does not apply here.

In the nontrivial situations, the vector space $\mathrm{Hom}_j(E_K)$ is isomorphic to $\mathrm{Hom}_j(E)$. In fact, as shown in [7], the dimension of $\mathrm{Hom}_j(E_k)$ is given by

$$\dim \mathrm{Hom}_j(E_k) = \binom{v}{j}, \quad \text{if} \quad j \leqslant k \leqslant v - j. \tag{6.9}$$

The dimension of $\mathrm{Pol}_j(E_K)$ is an open problem which could be interesting (see Theorem 6.3); of course, (6.8) and (6.9) yield an upper bound, but this is not sufficient for our purpose.

Instead of (3.9), one can prove the more complicated identity

$$\mathrm{Hom}_i(E)\mathrm{Hom}_j(E) = \sum_{l=i \vee j}^{i+j} \mathrm{Hom}_l(E), \tag{6.10}$$

with $i \vee j = \max(i, j)$; it expresses the fact that the cardinality l of the union of an i-set and a j-set belongs to the interval $i \vee j \leqslant l \leqslant i + j$. The "spread" of the product (6.10) is an important discrepancy with the Euclidean case. The reader can easily find out a counterpart of the general property (3.11), based on (6.10). Here we consider only two interesting simple examples, namely

$$\mathrm{Hom}_j(E)\mathrm{Hom}_j(E) = \sum_{l=j}^{2j} \mathrm{Hom}_l(E), \tag{6.11}$$

$$\mathrm{Pol}_j(E)\mathrm{Pol}_j(E) = \mathrm{Pol}_{2j}(E). \tag{6.12}$$

Using the identities (6.9), (6.11), (6.12), and applying the same technique as in Section 5, one obtains the following "Fisher type inequalities."

THEOREM 6.3. *If* (Y, w) *admits the indices* $j = e, e+1, \ldots, 2e$ *and if at least one of the block sizes* $k \in K$ *satisfies* $e \leqslant k \leqslant v - e$, *then*

$$|Y| \geqslant \binom{v}{e}. \tag{6.13}$$

If (Y, w) *admits the indices* $j = 0, 1, \ldots, 2e$, *then*

$$|Y| \geqslant \dim \mathrm{Pol}_e(E_K). \tag{6.14}$$

When using Theorem 6.3 one has to remember that (Y, w) automatically admits the index j if it admits the indices $j+1, j+2, \ldots, j+p$, where p is the number of block sizes; this follows directly from (6.6). The first bound, (6.13), is a generalization of the Ray-Chaudhuri–Wilson inequality for combinatorial $2e$-designs [10, 14]. The second bound, (6.14), is less explicit. In case (6.8) is a direct sum, and if at least one of the block sizes k satisfies $e \leqslant k \leqslant v - e$, then (6.14) becomes

$$|Y| \geqslant \sum_{i=0}^{p-1} \binom{v}{e-i}. \tag{6.15}$$

It should be mentioned that our method does not lead to a Fisher type inequality in the case of a single index j (except of course when $p = 1$), generalizing Ryser's inequality $|Y| \geqslant v$ for $j = 2$; see [11].

REFERENCES

1 E. Bannai, Spherical designs and group representations, *Contemp. Math.* 34:95–107 (1984).
2 P. Delsarte, Hahn polynomials, discrete harmonics, and *t*-designs, *SIAM J. Appl. Math.* 34:157–166 (1978).
3 P. Delsarte, J. M. Goethals, and J. J. Seidel, Spherical codes and designs, *Geom. Dedicata* 6:363–388 (1977).
4 J. M. Goethals and J. J. Seidel, Spherical designs, *Proc. Sympos. Pure Math.* 34:255–272 (1979).
5 J. M. Goethals and J. J. Seidel, Cubature formulae, polytopes, and spherical designs, in *The Geometric Vein*, The Coxeter Festschrift (C. Davis, B. Grünbaum, and F. A. Sherk, Eds.), Springer, Berlin, 1982, pp. 203–218.
6 D. R. Hughes, On *t*-designs and groups, *Amer. J. Math.* 87:761–778 (1965).
7 W. M. Kantor, On incidence matrices of finite projective and affine spaces, *Math. Z.* 124:315–318 (1972).

8   E. S. Kramer, Some results on *t*-wise balanced designs, *Ars Combin.* 15:179–192 (1983).

9   A. Neumaier and J. J. Seidel, Discrete measures for spherical designs, eutactic stars and lattices, *Nederl. Akad. Wetensch. Proc. Ser. A* 91 (*Indag. Math.* 50):321–334 (1988).

10 D. K. Ray-Chaudhuri and R. M. Wilson, On *t*-designs, *Osaka J. Math.* 12:737–744 (1975).

11 H. J. Ryser, New types of combinatorial designs, in *Actes Congrès Internat. Math.*, Tome 3, 1970, pp. 235–239.

12 J. J. Seidel, Eutactic stars, *Colloq. Math. Soc. János Bolyai* 18:983–999 (1976).

13 B. L. van der Waerden, *Modern Algebra*, Vol. II, Ungar, New York, 1964.

14 R. M. Wilson, On the theory of *t*-designs, in Enumeration and Designs (D. M. Jackson and S. A. Vanstone, Eds.), Academic Press, New York, 1984, pp. 19–49.

15 D. R. Woodall, The λ-μ problem, *J. London Math. Soc.* (2) 1:509–519 (1969).