Conjugate Duality for Convex Programs: A Geometric Development

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To Alan J. Hoffman, who has inspired me by the beauty and ingenuity of his mathematics and the warmth of his friendship, on the occasion of his 65th birthday.

Submitted by Uriel G. Rothblum

ABSTRACT

This expository note develops Rockafellar's (1968, 1970) conjugate-duality theory for convex programs using Gould's (1972) geometric formulation of the dual program. This approach makes the duality theory transparent geometrically and motivates the use of conjugate functions.

INTRODUCTION

The purpose of this expository note is to sketch how Rockafellar's (1968, 1970) elegant reformulation and generalization of Fenchel's (1951) conjugate-duality theory for convex programs can be developed by means of a natural geometric formulation of the dual program suggested by Gould (1972).[1] This approach has the advantages of making the duality theorem transparent geometrically and of motivating the definition and use of conjugate functions. When specialized to linear programs with a convexity constraint, this geometric formulation of duality is a variant of one proposed by Dantzig (1963).

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[1]In the notation of this paper, Gould's primal program entails minimizing $c(x)$ subject to $g(x) \geqslant p$ with c convex and g concave, while his dual program involves maximizing (1) subject to (2) and $\pi \geqslant 0$. Gould discusses several alternate forms of the dual, but not the now familiar one of Rockafellar given in (5).

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PRIMAL PROGRAM

Consider the *primal program* of choosing $x \in \Re^n$ that minimizes

$$c(x, p),$$

where c is a $+\infty$ or real-valued convex function on $\Re^{n+m}$ and $p \in \Re^m$. Let C be the *projection* of c, i.e.,

$$C(q) \equiv \inf_x c(x, q) \qquad \text{for} \quad q \in \Re^m.$$

It is well known that C is convex.

DUAL PROGRAM

The *dual program* is that of choosing $(\pi, \mu) \in \Re^{m+1}$ to maximize the linear function

$$\langle \pi, p \rangle - \mu \tag{1}$$

subject to the linear inequalities

$$\langle \pi, q \rangle - \mu \leqslant C(q) \qquad \text{for all} \quad q \in \Re^m, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\Re^m$. The geometric interpretation of the problem, illustrated in Figure 1, is that of choosing, from among all affine functions $\langle \pi, \cdot \rangle - \mu$ minorizing $C(\cdot)$, one whose value at p is maximum.

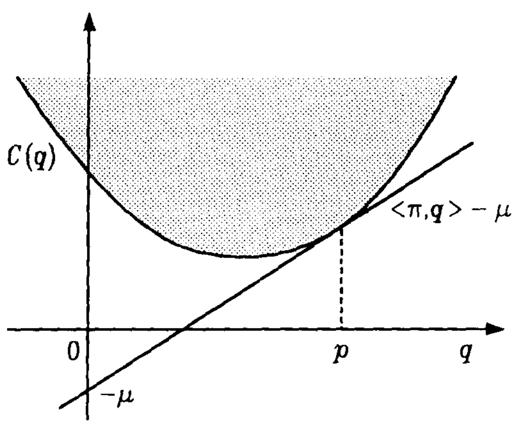

FIG. 1.

If the supremum of (1) subject to (2) equals $C(p)$, we say C is *closed* at p. In this connection, we adopt the usual convention that if the inequalities (2) have no solution, which is so if and only if C is $-\infty$ somewhere, then the supremum of (1) subject to (2) is $-\infty$. It should also be emphasized that because of the convexity of C, the condition that C is closed at p is normally —but not always—satisfied, and so is merely a regularity condition on C. For example, as discussed by Rockafellar (1970, pp. 51, 52, 102), if C is *proper*, i.e., somewhere finite and nowhere $-\infty$, then C is *closed*, i.e., everywhere closed, if and only if C is lower semicontinuous. Also, C is closed at p if $C(p) = -\infty$. On the other hand, C is not closed at p if $C(p) = +\infty$ and C is $-\infty$ somewhere, in which case neither the primal nor dual program is *feasible*.

Thus, C is closed at p if and only if the infimum of the primal objective function equals the supremum of the dual objective function, which expresses the main fact of the duality theorem of convex programming. However, this formulation of duality conceals the fact that the dual of the dual is the primal. We now develop a more convenient form of the dual program which reveals this fact and also introduces conjugate functions in a natural way.

ROCKAFELLAR'S SYMMETRIC CONJUGATE-DUALITY THEORY

For each fixed $\pi \in \Re^m$, one maximizes (1) subject to (2) by putting $\mu \equiv C^*(\pi)$, where

$$C^*(\pi) \equiv \sup_q \left[\langle \pi, q \rangle - C(q)\right].$$

The function C^*, being the supremum of affine functions, is convex and is called the *conjugate* of C. The dual program then reduces to choosing $\pi \in \Re^m$ that maximizes

$$\langle \pi, p \rangle - C^*(\pi). \tag{3}$$

Now the supremum of (3) over π is $C^{**}(p)$, where C^{**} is the conjugate of C^*, i.e.,

$$C^{**}(p) \equiv \sup_\pi \left[\langle \pi, p \rangle - C^*(\pi)\right].$$

Hence, $C(p) \geqslant C^{**}(p)$, and

$$C(p) = C^{**}(p) \tag{4}$$

if and only if C is closed at p.

Following Rockafellar (1968, 1970) and denoting by c^* the conjugate of c, we can simplify the dual program further by substituting $C^*(\pi) = c^*(0, \pi)$ in (3), which is justified because

$$C^*(\pi) = \sup_q \left[\langle \pi, q \rangle - C(q)\right] = \sup_{x,q} \left[\langle 0, x \rangle + \langle \pi, q \rangle - c(x, q)\right] = c^*(0, \pi).$$

Thus, on putting $p = 0$, it follows from (4) that

$$\inf_x c(x, 0) = \sup_\pi - c^*(0, \pi) \tag{5}$$

if and only if C is closed at 0. In this event, if c is closed and proper, then the dual of the dual is the primal. To see this, multiply (5) by -1, interchange 0 and the other variable in each program, and use the fact, shown above, that $c = c^{**}$. Thus from what was shown above, (5) holds if and only if $\inf_\pi c^*(\cdot, \pi)$ is closed at 0. These are the central ideas of Rockafellar's *symmetric duality theory* (1968, 1970). As Rockafellar has shown, the duality theories for most special convex programs, including linear and convex quadratic programs, are easy specializations of these results.

APPLICATION TO LINEAR-PROGRAMMING DUALITY THEORY

As an application of these ideas, consider the primal linear program of minimizing cx subject to $Ax = b$, $x \geqslant 0$. Following Rockafellar (1968, 1970), convert this program to the form of the primal program discussed above by setting $c(x, q) = cx$ if $Ax = b + q$, $x \geqslant 0$ and setting $c(x, q) = +\infty$ otherwise. This reduces the given primal linear program to choosing x to minimize $c(x, 0)$. Also $c^*(0, \pi) = \sup_{Ax = b+q,\ x \geqslant 0}(\pi q - cx) = \sup_{x \geqslant 0}(\pi A - c)x - \pi b$, which equals $-\pi b$ if $\pi A \leqslant c$ and $+\infty$ otherwise. Thus the dual program of choosing π to maximize $-c^*(0, \pi)$ reduces to the familiar dual linear program of choosing π to maximize πb subject to $\pi A \leqslant c$. In that event, either both programs are infeasible or $C(0)$ is closed, and in the latter event (5) holds, which is the familiar duality theorem of linear programming.

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