An Application of Submodular Flows

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

Extending theorems of Rado and Lovász, we introduce a new framework for problems concerning supermodular functions and graphs. Among the applications is an optimization problem for finding a minimum-cost subgraph H of a digraph $G=(V,E)$ such that H contains k disjoint paths from a fixed node of G to any other node. Another consequence is a characterization for graphs having a branching that meets all directed cuts. A theorem of Vidyasankar on optimal covering by arborescences and a matroid intersection theorem of Gröflin and Hoffman are also shown to be special cases.

1. INTRODUCTION

Generalizing Hall's theorem, R. Rado [23] proved a theorem concerning bipartite graphs and matroid rank functions. L. Lovász [20] found another generalization of Hall's theorem including bipartite graphs and a special type of supermodular function. The aim of the present note is to exhibit a common generalization of these results and to show various applications of the model introduced.

Among these applications is an extension of a theorem of Vidyasankar on the minimum number of arborescences covering the arcs of a directed graph. A theorem of Gröflin and Hoffman on matroid intersection is also a consequence. As a new result we derive a characterization for digraphs having a branching that meets all directed cuts. With the help of our model the following optimization problem can be solved: Given a digraph $G=(V,E)$

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with a specified node r and a cost function on the arcs, find a minimum-cost subgraph H of G such that for any node $v \in V$, H contains k openly disjoint paths from r to v.

Since the early results of Rado and Lovász, many different models have been introduced to provide general frameworks for problems including graphs and sub- (or super-) modular functions. Hoffman and Schwartz introduced lattice polyhedra (for a survey, see [16]); Edmonds and Giles [5] defined submodular flows. The concept of polymatroidal flows is due to Lawler and Martel [19] and to Hassin [17]. "Independent flows" have been investigated by Fujishige [14], and "kernel systems" by Frank [7]. A very general model has been devised by Schrijver [25]. The reader can find an excellent survey on the relationship of these models in [24].

In our approach we rely on submodular flows. See [3, 5, 10, 12, 27, 31] for details.

It is easy to derive Rado's theorem from the theory of submodular flows, but it was not known if Lovász's theorem is also a consequence. In this paper we show this.

Throughout we work with a bipartite graph $G = (A, B; E)$, where E denotes the edge set and A and B form the two-partition of the node set. For $F \subseteq E$, $X \subseteq A$ we use the notation $\Gamma_F(X) := \{v \in B,\ uv \in F \text{ for some } u \in X\}$. $d_F(X)$ denotes the number of edges in F incident to X. Γ_E and d_E are abbreviated by Γ and d, respectively.

We do not distinguish between a one-element set and its element. For a function $g: S \to R$ we use the notation $g(X) := \Sigma(g(v): v \in X)$, where $X \subseteq S$. The incidence vector of a subset $X \subseteq S$ is denoted by $\mathscr{X}(X)$.

Let S be a finite ground set. Two subsets X and Y are called *intersecting* if $X - Y$, $Y - X$ and $X \cap Y$ are nonempty. If, in addition, $S - (X \cup Y)$ is nonempty, then X and Y are called *crossing*. A family $\mathscr{F}$ of subsets is called *laminar* if it contains no intersecting subsets. Let $b: 2^S \to R \cup \{\infty\}$ be a set function with $b(\varnothing) = 0$. We say that b is *submodular* on subsets X and Y if $b(X) + b(Y) \geqslant b(X \cap Y) + b(X \cup Y)$. b is called an *intersecting* (crossing) submodular function if b is submodular on every intersecting (crossing) pair of subsets. If b is submodular on every pair, b is *fully submodular*. An integer-valued, nonnegative, finite, monotone increasing fully submodular function is called a *polymatroid function*. A polymatroid function is a matroid rank function if its value on singletons is 0 or 1. If $-p$ is an intersecting submodular function, p is called *intersecting supermodular*.

THEOREM 1.1 (R. Rado [23]). *Let $G = (A, B; E)$ be a bipartite graph and M a matroid on B with rank function r. There exists a subset $F \subseteq E$ of edges such that $d_F(v) = 1$ for $v \in A$ and $r(\Gamma_F(Y)) \geqslant |Y|$ for every $Y \subseteq A$ if* *and only if*

$$r(\Gamma_E(Y)) \geqslant |Y| \qquad \textit{for every} \quad Y \subseteq A. \tag{1.1}$$

THEOREM 1.2 (L. Lovász [20]). *Let $G=(A, B; E)$ be a bipartite graph and $p: 2^A \to Z \cup \{-\infty\}$ an intersecting supermodular function for which $p(v) \geqslant 0$ for $v \in A$ and*

$$p(X)+p(Y) \geqslant p(X \cup Y) \qquad \textit{for} \quad X, Y \subseteq A, \quad X \cap Y = \varnothing. \tag{1.2}$$

There exists a subset $F \subseteq E$ of edges such that $d_F(v)=p(v)$ for every $v \in A$ and $|\Gamma_F(X)| \geqslant p(X)$ for every $X \subseteq A$ if and only if

$$|\Gamma_E(X)| \geqslant p(X) \qquad \textit{for every} \quad X \subseteq A. \tag{1.3}$$

A common generalization we are going to prove is as follows.

THEOREM 1.3. *Let $G=(A, B; E)$ be a bipartite graph and $p: 2^A \to Z \cup \{-\infty\}$ an intersecting supermodular function (not necessarily satisfying (1.2)). Let furthermore r be a matroid rank function on B, and $g: A \to Z_+ \cup \{\infty\}$ an arbitrary function. There exists a subset $F \subseteq E$ of edges such that $d_F(v) \leqslant g(v)$ for every $v \in A$ and $r(\Gamma_F(X)) \geqslant p(X)$ for every $X \subseteq A$ if and only if*

$$p(Y) \leqslant r(\Gamma_E(X)) + g(Y-X) \tag{1.4}$$

holds whenever $X \subseteq Y \subseteq A$.

If $g \equiv 1$ and $p(X) :\equiv |X|$, then $p(Y)=p(X)+|Y-X| \leqslant r(\Gamma_E(X)) + g(Y-X)$, so (1.1) implies (1.4) and Rado's theorem follows.

If $g(v)=p(v)$ for $v \in A$, p satisfies (1.2), and r is the cardinality function, then $p(X)+g(Y-X)=p(X)+\Sigma(p(v): v \in Y-X) \geqslant p(Y)$, and therefore (1.3) implies (1.4): $|\Gamma_E(X)|+g(Y-X) \geqslant |\Gamma_E(X)|+p(Y)-p(X) \geqslant p(Y)$. Thus Lovász's theorem follows.

The sufficiency of (1.4) will follow from a more general result. To see the necessity let $F \subseteq E$ satisfy the requirement. Then we have $p(Y) \leqslant r(\Gamma_F(Y)) \leqslant r(\Gamma_F(X)) + r(\Gamma_F(Y-X)) \leqslant r(\Gamma_F(X)) + |\Gamma_F(Y-X)| \leqslant r(\Gamma_E(X)) + g(Y-X)$.

Note that Welsh [30] extended Rado's theorem by replacing the matroid rank function r by a submodular function, and therefore a common generalization of Welsh's theorem and of Lovász's theorem would be desirable. However, if in Theorem 1.3 r is replaced by a submodular function, the matroid matching problem becomes a special case, and therefore the problem is NP-hard [18, 21].

Let us briefly summarize some notation and results on submodular flows. Let $G=(V,E)$ be a digraph, and b a crossing submodular function on 2^V. Let $f: E \to R \cup \{-\infty\}$, $g: E \to R \cup \{\infty\}$ be functions. Let $c: E \to R$ be a cost function. For a vector $x: E \to R$ and a subset $A \subseteq V$ let us denote $\rho_x(A) = \Sigma(x(uv): uv \in E,\ uv \text{ enters } A)$, $\delta_x(A) := \rho_x(V-A)$, and $\lambda_x(A) := \rho_x(A) - \delta_x(A)$. For $F \subseteq E$, $\rho_F(A)$ denotes the number of edges entering A, and $\delta_F(A) := \rho_F(V-A)$. The functions ρ_E and δ_E are abbreviated by ρ and δ, respectively.

An x is called a *submodular flow* if

$$f \leqslant x \leqslant g \quad \text{and} \quad \lambda_x(A) \leqslant b(A) \quad \text{for every subset } A \text{ of } V. \tag{1.5}$$

The linear system in (1.5) is called a *submodular flow system*. Note that if p is a crossing supermodular function, then

$$f \leqslant x \leqslant g \quad \text{and} \quad \lambda_x(A) \geqslant p(A) \quad \text{for every subset } A \text{ of } V \tag{1.5$'$}$$

is also a submodular flow system. [Indeed, $\lambda_x(A) \geqslant p(A)$ is equivalent to $\lambda_x(\bar{A}) \leqslant -p(A)$, and $b(X) := -p(\bar{X})$ is crossing submodular.]

The main theorem on submodular flows is as follows.

THEOREM 1.4 (J. Edmonds and R. Giles [5]). *The linear system*

$$\{ f \leqslant x \leqslant g \quad \textit{and} \quad \lambda_x(X) \leqslant b(X) \quad \textit{for every subset } X \textit{ of } V \} \tag{i}$$

is totally dual integral. Consequently, if f, g, and b are integer-valued, the linear program $\max(cx: x$ *satisfies* (i)) *has an integral optimal solution (provided it has an optimal solution). If c is integer-valued, the dual linear program has an integer-valued optimal solution y (if it has an optimal solution). If, in addition, b is an intersecting submodular function, y can be* *chosen in such a way that the subsets corresponding to positive dual variables form a laminar family.*

2. THE MODEL

Let $p: 2^S \to R \cup \{-\infty\}$ be an intersecting supermodular function and r a rank function of a matroid on S such that

$$p(\varnothing) = 0, \qquad p \leqslant r, \qquad p(s) \geqslant 0 \quad \text{for every } s \in S. \tag{2.0}$$

Let us call a subset $T \subseteq S$ a *supporting set* if

$$r(T \cap X) \geqslant p(X) \qquad \text{for every} \quad X \subseteq S. \tag{2.1}$$

By the assumption, S is supporting, and since r is monotone, a superset of a supporting set is also supporting. What is the minimum cardinality of a supporting set? Or, more generally, given a nonnegative cost function $c: S \to R_+$, find a supporting set of minimum cost.

THEOREM 2.1. *Let p and r satisfy* (2.0). *The minimum cost $c(T)$ of a supporting set T is*

$$\max\Big(\sum w(A)h(A) : \sum (w(A)\mathscr{X}(A) : A \subseteq S) \leqslant c,\ w \geqslant 0\Big), \tag{2.2}$$

where $w: 2^S \to R_+$ and $h(A) = \max(p(Y) - r(Y - A) : Y \supseteq A)$. If c is integer-valued, w can be chosen integer-valued.

Let us show that this result immediately implies a theorem of Gröflin and Hoffman [15]. Let r_1 and r_2 be the rank functions of two matroids on the same groundset S, and suppose that $r_1(S) = r_2(S) = r_{12}(S) = k$, where $r_{12}(B)$ denotes the maximum cardinality of a common independent subset of B $(B \subseteq S)$. By Edmonds's matroid intersection theorem,

$$r_{12}(B) = \min_{Z \subseteq B} \big[r_1(Z) + r_2(B - Z)\big]. \tag{$*$}$$

THEOREM (H. Gröflin and A. Hoffman [15]). *Given a nonnegative cost function $c: S \to R_+$, the minimum cost of a common basis is*

$$\max\left(\sum w(A) r_{12}(S-A) : \sum (w(A)\mathscr{X}(A) : A \subseteq S) \leqslant c,\ w \geqslant 0\right),$$

where $w: 2^S \to R_+$. If c is integer-valued, w can be chosen integer-valued.

Proof. Apply Theorem 2.1 with the choice $r(X) := r_1(X)$, $p(X) := k - r_2(S - X)$, and observe that $h(A) = r_{12}(S - A)$ follows from $(*)$. ∎

Theorem 2.1 has the slight drawback that it includes a certain function h. For the special case when c is 0-1-valued, the theorem can be formulated in a stronger form that does not use the function h (but the maximization formula becomes more complicated). Assume that $c(s)$ is 0 on the elements of a subset N of S, and 1 otherwise.

THEOREM 2.2. *Let p and r satisfy* (2.0), *and let c be 0-1-valued. The minimum cost of a supporting set is*

$$\max_{\mathscr{F}} \sum_{Y \in \mathscr{F}} \left[p(Y) - r\left((N \cap Y) \cup \bigcup \{Z : Z \in \mathscr{F},\ Z \subset Y\}\right)\right], \tag{2.3}$$

where the maximum is taken over all laminar families $\mathscr{F}$ of distinct subsets of S.

When N is empty Theorem 2.2 specializes to

COROLLARY 2.3. *The minimum cardinality of a supporting set is*

$$\max_{\mathscr{F}} \sum_{Y \in \mathscr{F}} \left[p(Y) - r\left(\bigcup \{Z : Z \in \mathscr{F},\ Z \subset Y\}\right)\right] \tag{2.4}$$

where the maximum is taken over all laminar families $\mathscr{F}$ of distinct subsets of S.

Proof of Theorems 2.1 and 2.2. We prove these theorems simultaneously.

To see that max $\leqslant$ min in Theorem 2.1, let T be a supporting set. For any two subsets $X \subseteq Y$ of S we have $|T \cap X| + r(Y - X) \geqslant r(T \cap Y) \geqslant p(Y)$. Thus

$$|T \cap X| \geqslant p(Y) - r(Y - X) \tag{2.5}$$

and hence $|T \cap X| \geq h(X)$. From this $c(T) \geq \Sigma_{t \in T}\Sigma(w(A): t \in A) = \Sigma_{A \subseteq S}|T \cap A|w(A) \geq \Sigma_{A \subseteq S}h(A)w(A)$, and max $\leq$ min follows in (2.2).

The max $\leq$ min direction in Theorem 2.3 follows similarly. Indeed, let T be a supporting set and let $T' = T - N$. Now, for any two subsets $X \subseteq Y$ of S, we have $|T' \cap X| + r((N \cap Y) \cup (Y - X)) \geq r(T \cap Y) \geq p(Y)$ and $|T' \cap X| \geq p(Y) - r((N \cap Y) \cup (Y - X))$, from which max $\leq$ min in (2.3) easily follows.

To see the equalities, let us consider the polyhedron Q of vectors $x \in R^S$ satisfying

$$x(X) \geq p(Y) - r(Y - X) \qquad \text{for every} \quad X \subseteq Y\ (\subseteq S). \tag{2.6}$$

CLAIM 1. *The 0-1 vectors in Q are precisely the incidence vectors of supporting sets.*

Proof. By (2.5), obviously $\mathscr{X}_T \in Q$ for every supporting set T. Conversely, let $x = \mathscr{X}_T$ be in Q for some $T \subseteq S$, and let $X \subseteq S$. Apply (2.6) by substituting $X - T$ for X and X for Y. We obtain $x(X - T) \geq p(X) - r(X \cap T)$. Since $x(X - T) = 0$, (2.1) holds and the claim follows. ∎

CLAIM 2. *The linear system in (2.6) is a submodular flow system.*

Proof. Let S' and S'' be two disjoint copies of S, and let $V = S' \cup S''$. Where $X \subseteq S$ we adopt the notation X' and X'' for the corresponding subsets of S' and S'', respectively. Define $E = \{s''s' : s \in S\}$. The elements of E and S are in 1-1 correspondence, and we identify R^S and R^E. For $X, Y \subseteq S$ let $p_1(Y' \cup X'') := p(Y) - r(X)$ if $X \subseteq Y$, and $= -\infty$ otherwise.

Now p_1 is an intersecting supermodular function, and

$$\rho_x(Z) - \delta_x(Z) \geq p_1(Z) \qquad \text{for every} \quad Z \subseteq V \tag{2.7'}$$

is a submodular flow system. Since $\delta(Z) = 0$ whenever $p_1(Z)$ is finite, (2.7′) is equivalent to

$$\rho_x(Z) \geq p_1(Z) \qquad \text{for every} \quad Z \subseteq V, \tag{2.7}$$

which is, in turn, equivalent to (2.6) ∎

Let $\mathscr{R}$ denote the family $\{Z \subseteq V: p_1(Z) \text{ is finite}\}$. Let $c: E \to Z_+$ be a nonnegative integral cost function. Let us consider the following dual pair of linear programs:

$$\min(cx: x \geqslant 0,\ \rho_x(Z) \geqslant p_1(Z) \quad \text{for} \quad Z \in \mathscr{R}), \tag{2.8}$$

$$\max\Bigg(\sum_{Z \in \mathscr{R}} p_1(Z) z(Z): z \geqslant 0,$$

$$\sum(z(Z): e \text{ enters } Z) \leqslant c(e) \qquad \text{for every } e \in E\Bigg), \tag{2.9}$$

where $z: \mathscr{R} \to R_+$.

By Theorem 1.4 there is an integral solution x_0 to (2.8) and an integral solution z_0 to (2.9) such that $cx_0 = p_1 z_0$ and the family $\mathscr{F}_1 = \{Z \in \mathscr{R}: z_0(Z) > 0\}$ is laminar. If there is more than one such x_0, assume that x_0 has minimum component sum.

Claim 3. *x_0 is a 0-1 vector.*

Proof. Indirectly, let us suppose that a component of x_0, say $x_0(i)$, is greater than 1. Since $c \geqslant 0$ and x_0 is optimal, on decreasing $x_0(i)$ by 1 we obtain a vector which is not a solution to (2.8). This and the fact that p_1 is integer-valued imply that there must be a set $Z = Y' \cup X''$ such that $X \subseteq Y$, $i \in Y - X$, and $\rho_{x_0}(Z) = p_1(Z)$. Lt $X_1 = X + i$ and $Z_1 = Y' \cup X_1''$. Then $p_1(Z_1) = p(Y) - r(X_1) \geqslant p(Y) - r(X) - 1 = p_1(Z) - 1 = \rho_{x_0}(Z) - 1$, and we have $\rho_{x_0}(Z_1) = \rho_{x_0}(Z) - x_0(i) \leqslant \rho_{x_0}(Z) - 2 = p_1(Z) - 2 < p_1(Z_1)$, contradicting the fact that x_0 satisfies (2.8). [Here we have exploited the fact that r is a matroid rank function and so $r(X+i) \leqslant r(X) + 1$.] ■

Let T denote the set for which $\mathscr{X}_T = x_0$. For every $Z = Y' \cup X'' \in \mathscr{F}_1$ define $w(A) = \Sigma(z_0(Z): Z = Y' \cup X'',\ A = Y - X)$. Then $\Sigma(w(A)\mathscr{X}(A): A \subseteq S) \leqslant c$ and $\Sigma(w(A)h(A): A \subseteq S) \geqslant \Sigma_{Z \in \mathscr{R}} p_1(Z) z_0(Z) = c(T)$, from which Theorem 2.1 follows.

To see Theorem 2.2 recall that $c(s) = 0$ if $s \in N$ and $= 1$ otherwise. We can assume that every edge $s''s'$ ($s \notin N$) enters a set $Z \in \mathscr{F}_1$. For otherwise, let $Z = \{s'\}$ and revise z_0 by increasing it from 0 to 1. Since $p(s) \geqslant 0$, the revised z_0 is another optimal solution to (2.9).

Let $Z_0 = Y_0' \cup X_0''$ be an arbitrary member of $\mathscr{F}_1$, and let $Z_i = Y_i' \cup X_i''$ ($i = 1, 2, \ldots, k$) denote the maximal sets of $\mathscr{F}_1$ for which $Z_i \subset Z_0$. For every $s \in S - N$ the edge $s''s'$ enters precisely one set in $\mathscr{F}_1$. Therefore we have $X_0 = \bigcup_{i=1}^k Y_i \cup (Y_0 \cap N)$ and $\rho_{x_0}(Z_0) = p_1(Z_0) = p(Y_0) - r(X_0)$.

Consequently, T and the family $\mathscr{F} = \{Y : Y' \cup X'' \in \mathscr{F}_1$ for a certain $X \subseteq S\}$ satisfy $c(T) = |T - N| = \sum_{Y \in \mathscr{F}}[p(Y) - r((N \cap Y) \cup \bigcup(Z : Z \in \mathscr{F}, Z \subset Y))]$. ■

REMARK 1. In [11] a strongly polynomial algorithm was developed to solve optimization problems (and their linear-programming duals) over an (integral) submodular flow polyhedron in the 0-1 unit cube. That algorithm needed an oracle (A) that can minimize certain submodular functions.

Let $T \subseteq S$ be a subset, and define $p_T(X, Y) := p(Y) - r(X) - |(Y - X) \cap T|$ for $X \subseteq Y \subseteq S$. One can easily show that in our case oracle (A) is available if for any subset $T \subseteq S$ and for any two elements $a, b \in S$ one can maximize $p_T(X, Y)$ over subsets $X \subseteq Y$ for which (1) $a \in X$ and $b \in Y - X$, (2) $a \in X$ and $b \in S - Y$, (3) $a \in Y - X$ and $b \in S - Y$. (These are three distinct oracles.)

REMARK 2. One can ask if there is an extension of these results for the case when the matroid in question is replaced by an (integral) polymatroid. That is, given a polymatroid function b (i.e. a submodular, monotone increasing, nonnegative integer-valued function on 2^S) and an intersecting supermodular function p such that $b \geqslant p$, find a subset T of minimum cardinality for which $b(T \cap X) \geqslant p(X)$ for every $X \subseteq S$. (Again call such a T supporting.) This problem however is at least as difficult as the matroid matching problem; therefore it cannot be solved in polynomial time [18, 21]. Indeed, let b be a 2-polymatroid function, that is, $b(s) = 2$ for every $s \in S$. Let k be a positive integer, and define $p(X) := 2k$ if $X = S$, $:= 0$ if $X = \varnothing$, and $:= -\infty$ otherwise.

Since $b(T) \leqslant 2|T|$, the minimum cardinality of a supporting set is at least k. Thus if one can find a minimum cardinality supporting set, one is able to decide:

$$\text{is there a subset } X \subseteq S \text{ of } k \text{ elements for which } b(X) = 2|X|? \qquad (*)$$

$(*)$ is an equivalent formulation of the matroid matching problem.

3. APPLICATIONS TO BIPARTITE GRAPHS

Let $G = (A, B; E)$ be a simple bipartite graph with no isolated nodes, and $N \subset E$ a specified subset of edges. Let $p : 2^A \to \mathbf{Z} \cup \{-\infty\}$ be an intersecting supermodular function, and M a matroid on B with rank function r. Let

$c: E \to R_+$ be a nonnegative cost function. Call a subset $R \subseteq E$ *supporting* if $r(\Gamma_R(X)) \geqslant p(X)$ for every $X \subseteq A$. We say that R is *supporting with* N if $R \cup N$ is supporting. Let us suppose that E is supporting.

THEOREM 3.1. *In a bipartite graph $G = (A, B; E)$ the minimum cost $c(R)$ of a supporting set $R \subseteq E$ is* $\max(\Sigma\{w(Z)[p(Z \cap A) - r(Z \cap B)]: Z \subseteq A \cup B\}: \Sigma[w(Z): u \in Z \cap A,\ v \in B - Z] \leqslant c(uv)$ *for every* $uv \in E,\ w \geqslant 0)$. *Furthermore, if c is integer-valued, w can be chosen integer-valued.*

Proof. First, define a matroid M_1 on E as follows. For $F \subseteq E$ let $r_1(F) = r(X)$, where $X \subseteq B$ is the set of elements in B incident to F. Second, define $p_1: 2^E \to Z \cup \{-\infty\}$ by $p_1(F) = p(X)$ if F is the set of edges incident to the elements of a certain $X \subseteq A$, $= \max(0, p(s))$ if $F = \{s\}$ $(s \in E)$, and $= -\infty$ otherwise. Now Theorem 2.1 applies to p_1, r_1, and E, and Theorem 3.1 follows. ∎

Analogously Theorem 2.2 implies:

THEOREM 3.2. *In a bipartite graph $G = (A, B; E)$ the minimum cardinality of a set R supporting with N is equal to* $\max_{\mathscr{F}} \Sigma_{Y \in \mathscr{F}}[p(Y) - r(\Gamma_N(Y) \cup \Gamma(\bigcup\{Z: Z \in \mathscr{F},\ Z \subset Y\}))]$, *where the maximum is taken over all laminar families $\mathscr{F}$ of distinct subsets of A.*

One can be interested in finding a supporting set that meets some degree constraints at the nodes of A. Let $g: A \to Z \cup \{\infty\}$ be a function.

THEOREM 3.3. *Let G, N, r, and p be as before. There exists a subset R of E supporting with N such that $d_R(v) \leqslant g(v)$ for every $v \in A$ if and only if*

$$p(Y) \leqslant g(Y - X) + r(\Gamma_N(Y) \cup \Gamma(X)) \tag{3.1}$$

holds whenever $X \subseteq Y \subseteq A$.

Proof. Necessity: Let R be supporting with N. Then $p(Y) \leqslant r(\Gamma_{R \cup N}(Y)) \leqslant r(\Gamma_N(Y) \cup \Gamma(X) \cup \Gamma_R(Y - X)) \leqslant r(\Gamma_N(Y) \cup \Gamma(X)) + |\Gamma_R(Y - X)| \leqslant r(\Gamma_N(Y) \cup \Gamma(X)) + g(Y - X)$.

To see the sufficiency, we can obviously assume that g is finite everywhere. Observe that if there is a solution with respect to g, then this solution is good with respect to any (componentwise) bigger g'. So we can suppose that g is minimal in the sense that (3.1) is not true if we decrease any positive component of g.

CLAIM 1. *$g(v) \leqslant r(\Gamma(v)) - r(\Gamma_N(v))$ for every $v \in A$.*

Proof. Suppose $g(v) > r(\Gamma(v)) - r(\Gamma_N(v))$ for some $v \in A$. Revise g by reducing $g(v)$ to $g'(v) := r(\Gamma(v)) - r(\Gamma_N(v))$. Now there is a Y and X ($X \subseteq Y \subseteq A$, $v \in Y - X$) for which $p(Y) > g'(Y - X) + r(\Gamma_N(Y) \cup \Gamma(X))$. By submodularity we have $r(\Gamma(v)) + r(\Gamma_N(Y) \cup \Gamma(X)) \geqslant r(\Gamma(v) \cap [\Gamma_N(Y) \cup \Gamma(X)]) + r(\Gamma(v) \cup [\Gamma_N(Y) \cup \Gamma(X)]) \geqslant r(\Gamma_N(v)) + r(\Gamma_N(Y) \cup \Gamma(X + v))$. Therefore, $p(Y) > g'(Y - X) + r(\Gamma_N(Y) \cup \Gamma(X)) = g(Y - (X + v)) + r(\Gamma(v)) - r(\Gamma_N(v)) + r(\Gamma_N(Y) \cup \Gamma(X)) \geqslant g(Y - (X + v)) + r(\Gamma_N(Y) \cup \Gamma(X + v))$. This shows that Y and $X' = X + v$ violate (3.1), a contradiction. ∎

Observe that increasing an intersecting supermodular function on singletons results in an intersecting supermodular function. So we can suppose that p is maximal in the sense that (3.1) is not true if we increase p on any singleton.

CLAIM 2. *$p(v) = g(v) + r(\Gamma_N(v))$ for every $v \in A$.*

Proof. Applying (3.1) to $Y = \{v\}$ and $X = \varnothing$, we obtain that $p(v) \leqslant g(v) + r(\Gamma_N(v))$ for every $v \in A$. Suppose indirectly that we have strict inequality for a certain v. Revise p by increasing the value $p(v)$ to $p'(v) := g(v) + r(\Gamma_N(v))$.

Now (3.1) can be violated (with respect to p') only if either $Y = \{v\}$ and $X = \varnothing$ or else $Y = \{v\}$ and $X = \{v\}$. The first case would mean that $p'(v) > g(v) + r(\Gamma_N(v))$, contradicting the definition of $p'(v)$. The second case would mean that $p'(v) > r(\Gamma_N(v) \cup \Gamma(v)) = r(\Gamma(v)$, that is, $g(v) + r(\Gamma_N(v)) > r(\Gamma(v))$, contradicting Claim 1. ∎

Specializing (3.1) to $X = Y$, we obtain $p(Y) \leqslant r(\Gamma(Y))$ for every $Y \subseteq A$, so Theorem 3.1 applies. Let R ($\subseteq E - N$) be supporting with N. Then $r(\Gamma_{N \cup R}(v)) \geqslant p(v) = g(v) + r(\Gamma_N(v))$; therefore $d_R(v) \geqslant g(v)$ and $|R| \geqslant g(A)$. That is, the minimum in Theorem 3.2 is at least $g(A)$, and it is exactly $g(A)$ if and only if R satisfies $d_R(v) \leqslant g(v)$ for every $v \in A$.

Let us assume that the minimum in question is greater than $g(A)$. By Theorem 3.2 there is a laminar family $\mathscr{F}$ for which $g(A) < \sum_{Y \in \mathscr{F}}[p(Y) - r(\Gamma_N(Y) \cup \Gamma(\bigcup\{Z: Z \in \mathscr{F},\ Z \subset Y\}))]$. Hence $0 < p(Y) - r(\Gamma_N(Y) \cup \Gamma(X)) - g(Y - X)$ holds for at least one member of $\mathscr{F}$, where X denotes $X := \bigcup(Z: Z \in \mathscr{F},\ Z \subset Y)$. This contradicts (3.1). ∎

Observe that Theorem 1.3 formulated in the Introduction is Theorem 3.3 specialized to $N = \varnothing$.

REMARK. A. Schrijver [26] proved the following theorem: Given two nonnegative, intersecting supermodular functions p_1 and p_2 on 2^S, the elements of S can be partitioned into k color classes so that each subset $X \subseteq S$ intersects at least $\max(p_1(X), p_2(X))$ color classes if and only if $p_i(X) \leqslant \min(k, |X|)$ $(i = 1, 2)$.

This theorem can be slightly reformulated in term of supporting sets as follows. Let $G = (S, Z; E)$ be a complete bipartite graph with $|Z| = k$, and let p_1 and p_2 be as before. There exists a subset $R \subseteq E$ for which $d_R(v) \leqslant 1$ for every $v \in S$ and R is supporting with respect to both p_1 and p_2.

Compare this result with Theorem 3.3 (restricted to the case $N = \varnothing$). In Theorem 3.3 we have an arbitrary bipartite graph, an arbitrary g, a matroid rank function, and one supermodular function. Here the bipartite graph is complete, $g \equiv 1$, and two supermodular functions are involved. Does there exist a common generalization?

4. APPLICATIONS TO DIGRAPHS

Let $D = (V, E)$ be a directed graph. For sets $X \subseteq V$ and $F \subseteq E$ denote $O_F(X) := \{u \in V - X : \text{there is a } uv \in F \text{ with } v \in X\}$ and $I_F(X) := \{v \in X : \text{there is a } uv \in F \text{ with } u \in V - X\}$. (Letters O and I refer to the words outer and inner, respectively.) O_E and I_E are abbreviated by O and I, respectively. Let $p : 2^V \to Z \cup \{-\infty\}$ be an intersecting supermodular function such that $|O(X)| \geqslant p(X)$ for every $X \subseteq V$. Let $g : V \to Z \cup \{\infty\}$ and $c : E \to R_+$ be functions.

We call a subset $F \subseteq E$ of edges *out-supporting* if $|O_F(X)| \geqslant p(X)$ for every $X \subseteq V$. By the assumption made on p, the set E is out-supporting.

With the help of the well-known node-splitting technique (see e.g. [6, p. 24]) problems concerning out-supporting sets in directed graphs can be reduced to those concerning supporting sets in bipartite graphs. Namely, define a bipartite graph $G' = (V', V''; E')$ as follows. For a set $X \subseteq V$ we adopt the notation X' and X'' for the corresponding sets in V' and V''. The nodes $v \in V'$ and $v'' \in V''$ correspond to a node $v \in V$. Let $N = \{v''v' : v \in V\}$ and $E' := N \cup \{u''v' : uv \in E\}$. Define $p' : 2^{V'} \to Z \cup \{-\infty\}$ by $p'(X') := p(X) + |X|$. A cost function $c : E \to R_+$ can be extended to a cost function $c' : E' \to R_+$ by letting $c'(e) = 0$ if $e \in N$ and $c'(u''v') = c(uv)$ if $uv \in E$.

Observe now that $F \subseteq E$ is out-supporting (with respect to p) if and only if the corresponding F' is supporting with N (with respect to p' and $r(X) \equiv |X|$). Utilizing the construction above, we can establish the following counterparts of theorems of Section 3. We do not formulate these results in their full generality, since, unlike the bipartite case, the only interesting applications we were able to find use the free matroid (a matroid with no circuits). Therefore we state these counterparts only for the free matroid.

THEOREM 4.1. *The minimum cost of an out-supporting set of edges is* $\max\{\Sigma[w(Y,X)(p(Y)-|X|): Y \subset V,\ Y \cap X = \varnothing,\ X \subseteq O(Y)]: \Sigma[w(Y,X): Y \cup X \in \mathcal{F},\ Y \cap X = \varnothing,\ X \subseteq O(Y),\ u \in O(Y) - X,\ v \in Y] \leqslant c(uv)$ *for each edge* $uv \in E\}$. *Furthermore, if* c *is integer-valued,* w *can be chosen integer-valued.*

Proof. Suppose that the optimal solution w' in Theorem 3.1 is such that

$$\sum(|Z|: w'(Z) > 0) \qquad (*)$$

is minimum. Let us consider any set $Z \subseteq V' \cup V''$ for which $w'(Z) > 0$. Since $c'(e) = O$ for $e \in N$, we have $v'' \in Z$ whenever $v' \in Z$. We can assume that if $v'' \in Z$ and $v' \notin Z$, then v'' has a neighbor in $Z \cap V'$. For otherwise revise w' by $w'(Z - v'') := w'(Z - v'') + w'(Z)$ and $w'(Z) := 0$. The new w' would be another optimal solution, contradicting to the minimum choice of $(*)$.

Let $Y' = Z \cap V'$ and $X'' = \{v'': v'' \in Z,\ v' \notin Z\}$, and define $w(Y,X) = w'(Z)$. By Theorem 3.1 the result follows. ■

The next two results follow in an analogous way from Theorem 3.2 and 3.3, respectively.

THEOREM 4.2. *The minimum cardinality of an out-supporting set of edges is* $\max_{\mathcal{F}}(\Sigma_{Y \in F}[p(Y) - |O(\bigcup\{Z: Z \in \mathcal{F},\ Z \subset Y\}) - Y|])$, *where* $\mathcal{F}$ *is laminar.*

THEOREM 4.3. *Let* $g: V \to Z_+ \cup \{\infty\}$ *be a function. There is an out-supporting set* F *of edges for which* $\rho_F(v) \leqslant g(v)$ *for every* $v \in V$ *if and only if* $p(Y) \leqslant g(Z) + |O(Y-Z) - Y|$ *holds for every* $Y \subseteq V$ *and* $Z \subseteq I(Y)$.

By a simple construction we have

THEOREM 4.4. *Let* $g: V \to Z_+ \cup \{\infty\}$ *and* $\tilde{g}: E \to Z_+ \cup \{\infty\}$ *be two functions and* $p: 2^V \to Z \cup \{-\infty\}$ *an intersecting supermodular function such that* $\rho_{\tilde{g}}(X) \geqslant p(X)$ *for every* $X \subseteq V$. *There exists a nonnegative integer* *vector* $x: E \to Z_+$ *for which* $x \leqslant \tilde{g}$, $\rho_x(v) \leqslant g(v)$ *for every* $v \in V$, *and* $\rho_x(X) \geqslant p(X)$ *for every* $X \subseteq V$ *if and only if*

$$p(X) \leqslant g(Z) + \sum [\tilde{g}(uv): uv \in E,\ u \in V - X, \quad v \in I(X) - Z] \quad (4.1)$$

for every $X \subseteq V$ *and* $Z \subseteq I(X)$.

If $\tilde{g} \equiv \infty$, *then* (4.1) *is equivalent to*

$$p(X) \leqslant g(I(X)) \qquad \text{for every} \quad X \subseteq V. \qquad (4.2)$$

Proof. First we show that if $\tilde{g} \equiv \infty$, then (4.2) implies (4.1). Indeed, if a certain X violates (4.1), then the sum in (4.1) cannot include any term, that is, $Z = I(X)$.

The necessity of (4.1) is straightforward. To see the sufficiency it suffices to prove the result when $\tilde{g} \equiv 1$. Indeed, first delete edges with $\tilde{g}(e) = 0$. Second, if $\tilde{g}(e) = \infty$ for some $e \in E$, then we can replace $\tilde{g}(e)$ by a big enough integer [e.g. $\max p(X)$ would do]. Finally, if $\tilde{g}$ is finite and positive everywhere, then replace each edge e by $\tilde{g}(e)$ parallel edges. Obviously, if the new problem has a solution, then so does the original one. On the other hand, if a certain X and Z violate (4.1) with respect to the new problem, then the same X and Z violate (4.1) with respect to the original problem.

So suppose that $\tilde{g} \equiv 1$, and assume that (4.1) holds. We are going to find a subset F of edges for which $\rho_F(X) \geqslant p(X)$ for every $X \subseteq V$ and $\rho_F(v) \leqslant g(v)$ ($v \in V$). For each node v let $\varphi(v) = \{v_0\} \cup \{v_e: e \in E \text{ leaves } v\}$ be a set of $\delta(v) + 1$ distinct nodes. Construct a digraph $G_1 = (V_1, E_1)$ where $V_1 = \bigcup_{v \in V} \varphi(v)$ and $E_1 = \{u_e v_0: e = uv \in E\}$. (Intuitively we cut out every edge at its tail.) Let $g_1(v_0) = g(v)$ for every $v \in V$ and $g_1(v_e) = 0$ for every $e \in E$. Let us define $p_1(X_1) = p(X)$ if $X_1 = \bigcup_{v \in X} \varphi(v)$ for some $X \subseteq V$, and $= -\infty$ otherwise. By applying Theorem 4.3 to G_1, p_1, g_1 the result follows. ∎

Here we list some consequences of Theorem 4.4. Given a directed graph $G = (V, E)$, a *directed cut* is a nonempty subset of edges entering some subset X of nodes such that there are no edges leaving X. A *directed cut covering*, or simply a *cover*, is a subset of edges meeting all directed cuts. Lucchesi and Younger [22] proved that the minimum cardinality of a cover is equal to the maximum number of edge-disjoint directed cuts.

One can be interested in finding a cover that meets some upper-bound restriction on the in-degrees. Note that any minimal (not necessarily minimum) cover is a directed forest. Recall that a *branching* is a directed forest with no two edges entering the same node.

THEOREM 4.5. *There is a branching meeting all directed cuts if and only the number of components of $V - Y$ with no entering edges is at most $|Y|$ for every subset $Y \subseteq V$.*

Proof. The necessity is trivial, so we are concerned with the sufficiency. For a nonempty subset X of nodes let us define $c(X)$ to be the number of components of $V - X$ if there is no edge leaving X and $-\infty$ otherwise. Let $c(\varnothing) := 0$. It is not difficult to see that c is an intersecting supermodular function (see, e.g., Lemma 2.3 in [9]) and F is a cover if and only if $\rho_F(X) \geqslant c(X)$ for every $\varnothing \neq X \subseteq V$. We claim that (4.2) holds with respect to $p := c$, $g :\equiv 1$, and $\tilde{g} :\equiv \infty$. Indeed, if X violated (4.2), then $I(X)$ would violate the condition of the theorem. Hence Theorem 4.4 implies that there is an integer vector x for which $\rho_x(X) \geqslant c(X)$ for every $X \subseteq V$ and $\rho_x(v) \leqslant 1$ for every $v \in V$. By this second inequality x is 0-1-valued. Let $F' := \{e : x(e) = 1\}$. Now F' is a cover satisfying the indegree restriction, so a minimal cover F included in F' satisfies the requirement of the theorem. ∎

REMARK. Notice the formal analogy between Tutte's characterization of the existence of a perfect matching of undirected graphs and Theorem 4.5. Some further analogies of this type were discussed in [13].

From Theorem 4.4 one can easily derive a necessary and sufficient condition for the existence of a directed-cut covering satisfying an arbitrary upper-bound restriction on the in-degrees.

Our next corollary is about packing and covering with arborescences. The basic result in this area, due to J. Edmonds [3], is concerned with packing: In a directed graph $G = (V, E)$ there are k edge-disjoint (spanning) arborescences of root r $(r \in V)$ if and only if $\rho(X) \geqslant k$ for every $X \subseteq V - r$. A counterpart of this theorem concerning covering arborescences was proved by K. Vidyasankar [28]: A directed graph G can be covered by k spanning arborescences rooted at r if and only if there is no edge entering r, $\rho(v) \leqslant k$ for every $v \in V - r$, and $k - \rho(X) \leqslant \Sigma[k - \rho(v) : v \in I(X)]$ for every $X \subseteq V - r$. (For other results concerning arborescences, see [1, 7].)

We are going to show the following common generalization:

THEOREM 4.6. *Let $G = (V, E)$ be a directed graph, and r a specified node of G with no entering edges. Let $f : E \to Z_+$ and $q : E \to Z_+ \cup \{\infty\}$ be two functions such that $f \leqslant q$. There is a family $\mathscr{A}$ of k spanning arbores-*

cences rooted at r such that each edge e is contained in at least $f(e)$ and at most $q(e)$ members of $\mathscr{A}$ if and only if $\rho_f(v) \leqslant k$ for every $v \in V - r$ and

$$k - \rho_f(X) \leqslant \sum \left[k - \rho_f(v) : v \in A\right] + \sum \left[q(e) - f(e) : e = uv \in E,\ v \in I(X) - A,\ u \in V - X\right] \tag{4.3}$$

holds whenever $X \subseteq V - r$ and $A \subseteq I(X)$.

Proof. We are going to rely on Edmond's theorem. The required family $\mathscr{A}$ exists if and only if there is an integer vector $x: E \to Z_+$ for which

(a) $x \leqslant q - f$,
(b) $\rho_{f+x}(v) = k$ for every $v \in V - r$, and
(c) $\rho_{f+x}(X) \geqslant k$ for every $X \subseteq V - r$.

Indeed, if we have such an x, replace each edge e by $f(e) + x(e)$ (parallel) copies. By Edmonds's theorem the edge set of the resulting graph can be partitioned into k edge-disjoint arborescences. The corresponding k arborescences in G satisfy the requirements of the theorem. Conversely, if the desired family $\mathscr{A}$ of arborescences exists, then let $y(e)$ denote the number of arborescences in $\mathscr{A}$ containing e. Then $x = y - f$ satisfies (a), (b), and (c).

The existence of an x satisfying (a), (b), (c) and therefore the theorem follows if we apply Theorem 4.4 with the following choice: let $p(X) := k - \rho_f(X)$ if $\varnothing \neq X \subseteq V - r$, $= 0$ if $X = \varnothing$, and $= -\infty$ otherwise. (This p is intersecting supermodular.) Let $g(v) := k - \rho_f(v)$ $(v \in V)$ and $\tilde{g}(e) := q(e) - f(e)$ $(e \in E)$. ■

REMARK. In our derivation Theorem 4.6 has been obtained as a by-product of a more general approach. One may be interested in knowing if this result can be proved without referring to such general devices. Here we briefly outline a direct proof that deduces Theorem 4.6 from Edmonds's abovementioned theorem by an elementary construction.

Direct proof of Theorem 4.6. We can assume that $q(e) \leqslant k$ for every $e \in E$. Construct a digraph $G' = (V', E')$ as follows. Let $V' := V \cup \{v' : v \in V - r\}$. Let E' consist of the following type of edges. For each $v \in V - r$ there are k edges from v to v' and $k - \rho_f(v)$ edges from v' to v, and for every edge

$uv \in V$ there are $f(uv)$ parallel edges from u to v and $q(uv) - f(uv)$ parallel edges from u to v'.

If in G' there are k edge-disjoint spanning arborescences rooted at r, then the arborescences in G corresponding to these will satisfy the requirements. If no such a family exists, then, by Edmonds's theorem, there is a set $Y' \subseteq V' - r$ for which $\rho'(Y') < k$. Suppose that Y' is maximal. Obviously, if v' is in Y', then so is v. Furthermore, if $v \in Y'$ and $v' \notin Y'$, then there is edge $uv \in E$ with $u \notin Y'$. For otherwise $\rho'(Y' + v') \leqslant \rho'(Y')$, contradicting the maximal choice of Y'.

Consequently, Y' has the following form: $Y' = \{v, v' : v \in X - A\} \cup \{v : v \in A\}$ for some $X \subseteq V - r$ and $A \subseteq I(X)$. Now we have $k > \rho'(Y') = \Sigma_{v \in A}[k - \rho_f(v)] + \Sigma[q(uv) : u \notin X,\ \ v \in I(X) - A] + \Sigma[f(uv) : u \notin X,\ \ v \in A] = \Sigma_{v \in A}[k - \rho_f(v)] + \Sigma[q(uv) - f(uv) : u \notin X,\ \ v \in I(X) - A] + \rho_f(X)$, contradicting (4.3). ■

5. IMPROVING NETWORKS

PROBLEM A. Suppose we are given a digraph G_1 with a source s and a target t such that the maximum number of edge-disjoint paths from s to t is k. The goal is to increase this number to a specified integer K $(K > k)$ by adding certain new edges to the graph. If the possible new edges have nonnegative costs, what is the minimum total cost of new edges to be added?

This problem can easily be reduced to a minimum-cost flow problem in the union graph of the new and the original edges where the costs of the original edges are defined to be zero. (See, for example, [6].)

PROBLEM B. Now suppose we want to improve our digraph by adding edges of minimum cost so as to have K edge-disjoint paths from a source s to each other node.

One can relatively easily show that if a digraph D has K edge-disjoint paths from s to each other node but removing any edge destroys this property, then every node of D different from s has precisely K entering edges. Thus Problem B can be reduced to a weighted matroid intersection problem where the first matroid is K times the circuit matroid of the underlying undirected graph (that is, a subset of edges is independent if it is the union of K forests) while the second matroid is a partition matroid where a subset of edges is independent if it contains no more than K edges entering the same node. Since there are good algorithms for the matroid intersection problem [4], Problem B is also solvable in polynomial time.

In both problems one can be interested in openly disjoint paths rather than edge-disjoint. In the first case we can easily reduce the corresponding problem to Problem A by using the straightforward node-duplicating technique mentioned at the beginning of Section 4.

The openly-disjoint-paths counterpart of Problem B is as follows.

PROBLEM C. Given a digraph $G_1 = (V, E_1)$ and a specified source $s \in V$, add a set of edges of minimum cost to G_1 so as to have K openly disjoint paths from s to v for every $v \in V - S$.

The following version obviously includes Problem C.

PROBLEM C1. Given a digraph $G_1 = (V, E_1)$ and a specified subset S of V, add a set of edges of minimum cost to G_1 so as to have K paths from S to v for every $v \in V - S$ that are pairwise disjoint except at v.

A slight generalization is as follows.

PROBLEM C2. Let us be given a directed graph $G = (V, E)$, a subset $S \subset V$, and a nonnegative cost function $c: E \to R_+$. Find a subset F of edges of minimum cost so that

($*$) for every node $v \in V - S$ the digraph (V, F) contains K paths from S to v that are pairwise disjoint except at v.

It would be natural to try to reduce this problem to the edge-disjoint case with the help of an elementary construction. We were not able to find such a reduction. However, the model worked out in previous sections helps us.

By a version of Menger's theorem, an F satisfies ($*$) if and only if $|O_F(X)| \geqslant K$ for every $X \subseteq V - S$. Define a function $p: 2^V \to Z \cup \{-\infty\}$ by $p(X) := K$ if $\varnothing \neq X \subseteq V - S$, $= 0$ if $X = \varnothing$, and $= -\infty$ otherwise. Such a p is intersecting supermodular, and Theorem 4.1 provides a min-max formula for the minimum cost of F.

Our derivation also gives rise to a polynomial-time algorithm, since the necessary oracles mentioned in Remark 1 at the end of Section 2 can be constructed. However, it would be desirable to devise a more direct algorithm.

We can apply Theorem 4.3 to this case as well. By choosing $g(v) = K$ $(v \in V)$ one can show that the necessary and sufficient condition in Theorem 4.3 automatically holds, and then one has

COROLLARY. *Suppose that a digraph $G = (V, E)$ contains K internally node-disjoint paths from a specified node s to any other node, and G loses* *this property if we delete any of its edges. Then the in-degree* $\rho(v) = K$ *for every* $v \in V - s$.

(Note that this corollary has an easy direct proof: if a node v distinct from s has more than K entering edges, then take K internally node-disjoint paths from s to v. It is easy to see that an edge entering v which is not used by these paths can be deleted.)

Let us conclude by mentioning that various other augmentation problems occur in the literature. For further references see a recent paper of T. Watanabe and A. Nakamura [29].

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