The Class of Matrices of Zeros, Ones, and Twos with Prescribed Row and Column Sums*

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Hans Schneider

ABSTRACT

We study the class $\mathfrak{A}_2(R,S)$ of all $(0,1,2)$-matrices with a prescribed row sum vector R and column sum vector S. A $(0,1,2)$-matrix in $\mathfrak{A}_2(R,S)$ is defined to be parsimonious provided no $(0,1,2)$-matrix with the same row and column sum vectors has fewer positive entries. In a parsimonious $(0,1,2)$-matrix A there are severe restrictions on the $(0,1)$-matrix $A^{\langle 1\rangle}$ which records the positions of the 1's in A. We attempt to understand the relationships between the set of these matrices $A^{\langle 1\rangle}$ and the pair (R,S).

1. INTRODUCTION

Let r be a positive integer, and let $A=[a_{ij}]$ be an m by n matrix with $a_{ij}\in\{0,1,\dots,r\}$ $(i=1,\dots,m;\ j=1,\dots,n)$. We call A a $(0,1,\dots,r)$*-matrix*. Let

$$r_i=\sum_{j=1}^{n}a_{ij}\qquad(i=1,\dots,m)$$

and

$$s_j=\sum_{i=1}^{m}a_{ij}\qquad(j=1,\dots,n).$$

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Then

$$R=(r_1,\dots,r_m) \quad \text{and} \quad S=(s_1,\dots,s_n)$$

are the *row sum vector* and *column sum vector*, respectively, of A. We denote the sum of all the entries of A by $\sigma(A)$. Thus

$$\sigma(A)=r_1+\cdots+r_m=s_1+\cdots+s_n.$$

In this paper we study the class $\mathfrak{A}_2(R,S)$ of (0,1,2)-matrices with row sum vector R and column sum vector S. Let A be a matrix in $\mathfrak{A}_2(R,S)$. Then A admits a unique decomposition

$$A=A^{\langle 1\rangle}+2A^{\langle 2\rangle}, \tag{1.1}$$

where $A^{\langle 1\rangle}$ and $A^{\langle 2\rangle}$ are (0,1)-matrices. We call $A^{\langle 1\rangle}$ the 1-*pattern matrix* of A and $A^{\langle 2\rangle}$ the 2-*pattern matrix* of A. The (0,1)-matrix

$$A^{+}=A^{\langle 1\rangle}+A^{\langle 2\rangle} \tag{1.2}$$

indicates the pattern of positive entries of A and is called the *pattern matrix* of A. We denote the number of 1's of A by

$$\rho(A)=\sigma(A^{\langle 1\rangle}),$$

and the number of 2's of A by

$$\beta(A)=\sigma(A^{\langle 2\rangle}).$$

We denote the number of positive entries of A by

$$\tau(A)=\sigma(A^{+}).$$

It follows that

$$\tau(A)=\rho(A)+\beta(A) \tag{1.3}$$

and

$$\sigma(A)=\rho(A)+2\beta(A). \tag{1.4}$$

We define A to be *parsimonious* [2] provided

$$\tau(A) \leqslant \tau(A')$$

for all (0,1,2)-matrices A' in $\mathfrak{A}_2(R,S)$. Thus A is parsimonious if and only if A has the smallest number of positive entries among all (0,1,2)-matrices in $\mathfrak{A}_2(R,S)$. It follows from (1.3) and (1.4) that each one of the following conditions is equivalent to the parsimony of A:

(1.5) A has the largest number of 0's,
(1.6) A has the smallest number of 1's,
(1.7) A has the largest number of 2's

among all (0,1,2)-matrices in $\mathfrak{A}_2(R,S)$. We denote by

$$\mathscr{P}^{\langle 1\rangle}(R,S) = \left\{ A^{\langle 1\rangle} : A \in \mathfrak{A}_2(R,S) \text{ and } A \text{ is parsimonious} \right\}$$

the collection of the 1-pattern matrices of the parsimonious (0,1,2)-matrices in $\mathfrak{A}_2(R,S)$. We also obtain from (1.3) and (1.4) that the following three conditions are equivalent to one another:

(1.8) A has the smallest number of 0's,
(1.9) A has the largest number of 1's,
(1.10) A has the smallest number of 2's

among all (0,1,2)-matrices in $\mathfrak{A}_2(R,S)$. We define A to be *improvident* [2] provided these conditions are satisifed. For example,

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

are parsimonious and improvident matrices, respectively, in the class $\mathfrak{A}_2(R,S)$, where $R=(5,4)$ and $S=(3,2,2,2)$.

We can also characterize parsimony and improvidence in the following manner. Let $A=[a_{ij}]$ be in $\mathfrak{A}_2(R,S)$. Then by (1.4)

$$\begin{aligned} \operatorname{tr}(AA^T) &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \\ &= 4\beta(A)+\rho(A) = 2\beta(A)+\sigma(A). \end{aligned}$$

Now $\sigma(A)=\sigma(X)$ for all matrices X in $\mathfrak{A}_2(R,S)$. Hence by condition (1.7) for parsimony the matrix A is parsimonious if and only if

$$\operatorname{tr}(AA^T) \geqslant \operatorname{tr}(XX^T)$$

for all X in $\mathfrak{A}_2(R, S)$. Also, by condition (1.10) for improvidence A is improvident if and only if

$$\operatorname{tr}(AA^T) \leqslant \operatorname{tr}(XX^T)$$

for all X in $\mathfrak{A}_2(R, S)$. Thus if we view the matrices in $\mathfrak{A}_2(R, S)$ as points in Euclidean mn-dimensional space, then the matrices farthest from the origin are the parsimonious matrices, and the matrices nearest to the origin are the improvident matrices.

Finally we observe that the parsimony and the improvidence of matrices are invariant under permutations of rows and columns.

In Section 2 we first review the necessary and sufficient conditions on the integer r and the vectors R and S for the nonemptiness of the class $\mathfrak{A}_2(R, S)$ $(r = 1, 2)$. When $r = 1$, these conditions are equivalent to those of Gale [5], Ryser [8], and Ford and Fulkerson [4] for the existence of a $(0,1)$-matrix with prescribed row and column sum vectors R and S. From the Gale-Ryser-Ford-Fulkerson theorem we deduce necessary and sufficient conditions on R and S for the existence of a $(0,1,2)$-matrix A in $\mathfrak{A}_2(R, S)$ such that $A = 2A^{\langle 2 \rangle}$ (in which case A is clearly parsimonious) and for a $(0,1,2)$-matrix A such that $A = A^{\langle 1 \rangle}$ (in which case A is clearly improvident). We then discuss how one may pass from any $(0,1,2)$-matrix in $\mathfrak{A}_2(R, S)$ to any other by a sequence of simple operations called interchanges. We next show that for every parsimonious $(0,1,2)$-matrix A, the 1-pattern matrix $A^{\langle 1 \rangle}$ has the property that each 1 is either the only 1 in its row or the only 1 in its column. We call a $(0,1)$-matrix with this property a *constellation matrix*. Moreover, any constellation matrix can serve as the 1-pattern matrix of some parsimonious $(0,1,2)$-matrix. In contrast, any $(0,1)$-matrix whatsoever can serve as the 2-pattern matrix of an improvident $(0,1,2)$-matrix.

In Section 3 we obtain inequalities that relate the number $\rho(A)$ of 1's in a parsimonious $(0,1,2)$-matrix A and the number of odd row sums and odd column sums of A. Then we investigate the set of constellation matrices which can occur as the 1-pattern matrices of the parsimonious $(0,1,2)$-matrices in a class $\mathfrak{A}_2(R, S)$. In the final section we discuss some problems for future research.

We conclude this introduction with a discussion of the relationship between the class $\mathfrak{A}_2(R, S)$ of $(0,1,2)$-matrices and the class $\mathfrak{B}_2(R, S)$ of bipartite 2-multigraphs with prescribed degree sequences R and S.

Let $A = [a_{ij}]$ be a (0,1,2)-matrix in $\mathfrak{A}_2(R, S)$. We associate with A a *bipartite 2-multigraph* B as follows. The vertex set of B is partitioned into the disjoint sets

$$X = \{x_1, \dots, x_m\} \quad \text{and} \quad Y = \{y_1, \dots, y_n\}.$$

For each positive a_{ij} there is an edge $\{x_i, y_j\}$ of B with multiplicity a_{ij}. The degree of vertex x_i in B is r_i $(i = 1, \dots, m)$, and the degree of vertex y_j in B is s_j $(j = 1, \dots, n)$. Thus R and S are the degree sequences of B for the vertices in X and Y, respectively. The number of edges of B counting multiplicities equals $\sigma(A)$. We let $\mathfrak{B}_2(R, S)$ denote the class of all bipartite 2-multigraphs that arise in this way.

Let A be a matrix in $\mathfrak{A}_2(R, S)$, and let B be its associated bipartite 2-multigraph in $\mathfrak{B}_2(R, S)$. We draw B by representing each edge of multiplicity 2 by a blue edge and each edge of multiplicity 1 by a red edge. Thus $\beta(A)$ counts the number of blue edges and $\rho(A)$ counts the number of red edges in our drawing of B. The total number of edges in this drawing equals $\tau(A)$. We say that the bipartite 2-multigraph B is *parsimonious* provided its associated matrix A is parsimonious. Thus the bipartite 2-multigraph B is parsimonious if and only if its drawing contains the smallest number of edges among the drawings of all bipartite 2-multigraphs in $\mathfrak{B}_2(R, S)$.

Suppose B is parsimonious. The red edges of B, which correspond to the 1's of the constellation matrix $A^{\langle 1 \rangle}$, determine a configuration of stars. Each star has its center in either X or Y. If a star consists of a single edge, then either the vertex in X or the vertex in Y can serve as the center. This ambiguity creates difficulties, some of which we have not been able to resolve.

Because of the above association of (0,1,2)-matrices with bipartite 2-multigraphs, the results in this paper are bipartite analogues of our results in [2]. The distinction between the vertex sets X and Y in a bipartite 2-multigraph changes the essential nature of the questions considered. Hence the work reported here is not a special case of that in [2].

2. PARSIMONIOUS (0,1,2)-MATRICES

We begin with the fundamental existence theorem of Gale [5], Ryser [8,9], and Ford and Fulkerson [4].

THEOREM 2.1. *Let $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ be nonnegative integral vectors with $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. There exists a matrix in*

$\mathfrak{A}_1(R,S)$ *if and only if*

$$|I||J| + \sum_{i \notin I} r_i - \sum_{j \in J} s_j \geqslant 0 \tag{2.1}$$

for all $I \subseteq \{1,\ldots,m\}$ *and* $J \subseteq \{1,\ldots,n\}$.

The above theorem is a special case of a far more general theorem derived by Mirsky [6; 7, Theorem 11.5.1, p. 205] in which arbitrary (rather than 0 and 1) lower and upper bounds are imposed on each entry of the matrices. This theorem gives the following necessary and sufficient conditions for the nonemptiness of the class $\mathfrak{A}_2(R,S)$.

THEOREM 2.2. *Let* $R = (r_1,\ldots,r_m)$ *and* $S = (s_1,\ldots,s_n)$ *be nonnegative integral vectors with* $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. *There exists a matrix in* $\mathfrak{A}_2(R,S)$ *if and only if*

$$2|I||J| + \sum_{i \notin I} r_i - \sum_{j \in J} s_j \geqslant 0 \tag{2.2}$$

for all $I \subseteq \{1,\ldots,m\}$ *and* $J \subseteq \{1,\ldots,n\}$.

Let $R = (r_1,\ldots,r_m)$ and $S = (s_1,\ldots,s_n)$ be nonnegative integral vectors with $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. It follows from Theorem 2.1 that there exists a matrix D in the class $\mathfrak{A}_2(R,S)$ each of whose entries is 0 or 1 if and only if (2.1) holds. The matrix D is clearly improvident, as $D^{\langle 2 \rangle}$ is a zero matrix. It also follows from Theorem 2.1 that there exists a matrix E in the class $\mathfrak{A}_2(R,S)$ each of whose entries is 0 or 2 if and only if each r_i and each s_j is even and (2.2) holds. The matrix E is clearly parsimonious, as $E^{\langle 1 \rangle}$ is a zero matrix.

Let $A = [a_{ij}]$ be an m by n matrix. Suppose h, k, p, and q are integers with $1 \leqslant h < k \leqslant m$ and $1 \leqslant p < q \leqslant n$. We may obtain from A a matrix A' with the same row and column sum vectors as A by adding either

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

to the 2 by 2 submatrix of A which occurs in rows h and k and columns p and q. The matrix A' is said to be obtained from A by an (hk,pq)-*interchange.* It follows in the usual way [1, 8, 9] that the following *interchange property* holds: *Let* A *and* A^* *be any two* $(0,1,2)$-*matrices in*

$\mathfrak{A}_2(R, S)$. Then there is a sequence of interchanges which transforms A to A^* where each intermediate matrix is also a (0,1,2)-matrix in $\mathfrak{A}_2(R, S)$.

We now use interchanges to show that the 1-pattern matrix of a parsimonious (0,1,2)-matrix is highly restricted.

THEOREM 2.3. *If A is a parsimonious (0,1,2)-matrix in the class $\mathfrak{A}_2(R, S)$, then the 1-pattern matrix $A^{\langle 1\rangle}$ of A is a constellation matrix. Moreover, if the (0,1)-matrix C is a constellation matrix, then there exists a nonempty class $\mathfrak{A}_2(R, S)$ such that every matrix Z in $\mathfrak{A}_2(R, S)$ is parsimonious and has 1-pattern matrix $Z^{\langle 1\rangle}$ which is obtained from C by a permutation of rows and of columns.*

Proof. Assume there is a 1 of A whose row and column each contain an additional 1. Without loss of generality

$$\begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}$$

is a 2 by 2 submatrix of A. An appropriate interchange transforms this submatrix to one of

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \ (a=0), \qquad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \ (a=1), \qquad \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} \ (a=2)$$

and thereby transforms A to a matrix in $\mathfrak{A}_2(R, S)$ with fewer 1's than A. This contradicts the parsimony of A, and we conclude that $A^{\langle 1\rangle}$ is a constellation matrix.

Let C be a constellation matrix. Because any matrix obtained from a constellation matrix by permutations of its rows and its columns is also a constellation matrix, we may assume

$$C = \begin{bmatrix} I & O & O & O \\ O & O & U & O \\ O & V & O & O \\ O & O & O & O \end{bmatrix}, \tag{2.3}$$

where I is an identity matrix, U is a matrix with at least two 1's in each row,

V is a matrix with at least two 1's in each column, and the O's represent zero matrices of appropriate sizes. Any of these submatrices may be vacuous. Let

$$A = \begin{bmatrix} I & \bar{O} & O & O \\ \bar{O} & \bar{O} & \bar{U} & O \\ O & \bar{V} & O & O \\ O & O & O & O \end{bmatrix},$$

where $\bar{M}$ denotes the matrix obtained from a matrix M by replacing each 0 with a 2. Clearly, $A^{\langle 1 \rangle} = C$. We observe that no sequence of interchanges can alter the submatrices O and $\bar{O}$ of A.

Suppose $\mathfrak{A}_2(R,S)$ is the class that contains A. Let Z be any matrix in the class $\mathfrak{A}_2(R,S)$. It follows from the interchange property and the above observation that

$$Z = \begin{bmatrix} P & \bar{O} & O & O \\ \bar{O} & \bar{O} & \bar{K} & O \\ O & \bar{L} & O & O \\ O & O & O & O \end{bmatrix},$$

where Z and A are conformally partitioned. Because Z has the same row sum and column sum vectors as A, we see that P is a permutation matrix, that $\bar{K}$ is a matrix obtained from $\bar{U}$ by a column permutation, and that $\bar{L}$ is a matrix obtained from $\bar{V}$ by a row permutation. The theorem now follows. ∎

A $(0,1,2)$-matrix whose 1-pattern matrix is a constellation matrix need not be parsimonious. For example, the two matrices

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

are in the class $\mathfrak{A}_2(R,S)$ where $R = S = (3,3,2)$. Although $A^{\langle 1 \rangle}$ is a constellation matrix, A is not parsimonious because A' has fewer 1's.

While the 1-pattern matrix of a parsimonious $(0,1,2)$-matrix is restricted to be a constellation matrix, there is no restriction on the 2-pattern matrix of an improvident matrix.

THEOREM 2.4. *Let D be an arbitrary $(0,1)$-matrix. Then there exists an improvident $(0,1,2)$-matrix A whose 2-pattern matrix equals D.*

Proof. Let A be the matrix obtained from $2D$ by replacing each 0 with a 1. Suppose $\mathfrak{A}_2(R,S)$ is the class that contains A. Then each row of A has at least as many 1's as the corresponding row of any matrix in $\mathfrak{A}_2(R,S)$. It follows that A is improvident and that $A^{\langle 2\rangle} = D$. ■

3. CLASS COMPATIBILITY OF CONSTELLATION MATRICES

We recall that a constellation matrix C is a $(0,1)$-matrix with the property that each 1 is either the only 1 in its row or the only 1 in its column. Let p and q be integers with $p \geqslant 2$ and $q \geqslant 2$. A *horizontal constituent* $K_{1,p}$ (respectively, *vertical constituent* $K_{q,1}$) of C is the collection of the 1's in a row (respectively, 1's in a column) with exactly p 1's (respectively, q 1's). An *ambiguous constituent* $K_{1,1}$ of C is a 1 which is the only 1 in both its row and its column, and hence is a constituent with no clearly defined orientation. If we had not defined each $K_{1,1}$ as an ambiguous constituent, then we would have been forced to assign arbitrarily an orientation to each $K_{1,1}$. It is precisely this ambiguity that introduces complications. The *size* of a constituent is the number of its 1's. A constituent is *even* or *odd* according as its size is even or odd. We do not distinguish between constituents of the same type if they have the same size. The *configuration* of C is the multiset of constituents of C of all types. A collection $\mathscr{C}$ of constellation matrices is *class-compatible* provided there exists a pair (R,S) such that each C in $\mathscr{C}$ has the same configuration of constituents as some matrix in the set $\mathscr{P}^{\langle 1\rangle}(R,S)$. We then say that the pair (R,S) *affords* the collection $\mathscr{C}$. Note that the matrices in $\mathscr{C}$ may have different sizes. We seek necessary and sufficient conditions in order that a collection $\mathscr{C}$ of constellation matrices be class-compatible. A collection $\mathscr{C}$ of constellation matrices is *arithmetically compatible* provided there exist nonnegative integers ρ, λ, and ν such that each constellation matrix in $\mathscr{C}$ has ρ 1's, has λ rows with an odd number of 1's, and has ν columns with an odd number of 1's.

LEMMA 3.1. *If a collection $\mathscr{C}$ of constellation matrices is class-compatible, then $\mathscr{C}$ is also arithmetically compatible.*

Proof. Suppose the collection $\mathscr{C}$ of constellation matrices is class-compatible. Let the pair (R,S) afford $\mathscr{C}$. By the definition of parsimony each constellation matrix in $\mathscr{C}$ has the same number ρ of 1's. For any constellation matrix in $\mathscr{C}$ the number of rows (respectively, columns) with an odd number of 1's equals the number λ (respectively, ν) of odd components of R (respectively, S). Therefore $\mathscr{C}$ is arithmetically compatible. ■

The integers ρ, λ, and ν associated with an arithmetically compatible collection $\mathscr{C}$ of constellation matrices are called the *arithmetic invariants* of $\mathscr{C}$. By Lemma 3.1 a class-compatible collection of constellation matrices has arithmetic invariants.

THEOREM 3.2. *Let $\mathscr{C}$ be an arithmetically compatible collection of constellation matrices with arithmetic invariants ρ, λ, and ν. Then*

$$\rho \equiv \lambda \equiv \nu \pmod 2, \tag{3.1}$$

$$\lambda + \nu \geqslant \rho \geqslant \max\{\lambda, \nu\}. \tag{3.2}$$

Moreover, the invariant

$$d = \lambda + \nu - \rho \tag{3.3}$$

equals the number of odd constituents in any constellation matrix in $\mathscr{C}$.

Proof. Let C be any constellation matrix in $\mathscr{C}$. Each odd constituent of C contributes an odd number to each of ρ, λ, and ν, while each even constituent contributes an even number. Hence (3.1) holds. Each of the λ rows of C with an odd number of 1's contributes at least one to the total number ρ of 1's, and hence $\rho \geqslant \lambda$. Similarly, $\rho \geqslant \nu$. Let d denote the number of odd constituents of C. Consider a constituent with t 1's. If t is even, then this constituent contributes t to both of the sums $\lambda + \nu$ and $\rho + d$. If t is odd, then the constituent contributes $t+1$ to both of $\lambda + \nu$ and $\rho + d$. Hence $\lambda + \nu = \rho + d$, and the theorem follows. ∎

We now discuss the cases of equality in (3.2). It follows from Theorem 3.2 that $\lambda + \nu = \rho$ if and only if a constellation matrix in $\mathscr{C}$ has no odd constituents. Suppose that $\rho = \lambda \geqslant \nu$. Let C be a constellation matrix in $\mathscr{C}$. It follows from the proof of Theorem 3.2 that each row of C contains at most one 1 and hence C has no horizontal constituents. Conversely, if C has no horizontal constituents, then $\rho = \lambda \geqslant \nu$. Similarly, $\rho = \nu \geqslant \lambda$ if and only if C has no vertical constituents. It follows that $\rho = \lambda = \nu$ if and only if each constituent of C is ambiguous. We characterize those pairs (R, S) for which each constellation matrix in $\mathscr{P}^{\langle 1 \rangle}(R, S)$ has only ambiguous constituents.

THEOREM 3.3. *Let $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ be nonnegative integral vectors with $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. Suppose λ is the number* *of odd components of R, and ν is the number of odd components of S. Then the following assertions are equivalent*:

(3.4) *There exists a parsimonious* $(0,1,2)$*-matrix in* $\mathfrak{A}_2(R,S)$ *whose* 1*-pattern matrix has only ambiguous constituents.*

(3.5) *Every parsimonious* $(0,1,2)$*-matrix in the nonempty class* $\mathfrak{A}_2(R,S)$ *has a* 1*-pattern matrix with only ambiguous constituents.*

(3.6) (a) $\lambda = \nu$,
(b) $|I||J| + \sum_{i \notin I} \lceil r_i/2 \rceil - \sum_{j \in J} \lceil s_j/2 \rceil \geqslant 0$ *for all* $I \subseteq \{1,\dots,m\}$ *and* $J \subseteq \{1,\dots,n\}$,
(c) $|I||J| + \sum_{i \notin I} \lfloor r_i/2 \rfloor - \sum_{j \in J} \lfloor s_j/2 \rfloor \geqslant 0$ *for all* $I \subseteq \{1,\dots,m\}$ *and* $J \subseteq \{1,\dots,n\}$.

Proof. Suppose A is a parsimonious matrix in $\mathfrak{A}_2(R,S)$ whose 1-pattern matrix $A^{\langle 1 \rangle}$ has only ambiguous constituents. Let $\sigma(A^{\langle 1 \rangle}) = \rho$. Then $\rho = \lambda = \nu$. The equivalence of (3.4) and (3.5) now follows from the discussion which precedes the theorem.

For a real vector $T = (t_1,\dots,t_q)$ let

$$\lceil T/2 \rceil = \left(\lceil t_1/2 \rceil, \dots, \lceil t_q/2 \rceil \right),$$

$$\lfloor T/2 \rfloor = \left(\lfloor t_1/2 \rfloor, \dots, \lfloor t_q/2 \rfloor \right).$$

Now suppose (3.4) holds, and let A be a parsimonious matrix in $\mathfrak{A}_2(R,S)$ such that $A^{\langle 1 \rangle}$ has only ambiguous constituents. Then (3.6)(a) holds. The $(0,1)$-matrix $\frac{1}{2}(A + A^{\langle 1 \rangle})$ is in $\mathfrak{A}_1(\lceil R/2 \rceil, \lceil S/2 \rceil)$, and the $(0,1)$-matrix $\frac{1}{2}(A - A^{\langle 1 \rangle})$ is in $\mathfrak{A}_1(\lfloor R/2 \rfloor, \lfloor S/2 \rfloor)$. Hence (3.6)(b) and (c) follow from Theorem 2.1, and (3.6) holds.

Now suppose (3.6) holds. Because $r_1 + \cdots + r_m = s_1 + \cdots + s_n$ and $\lambda = \nu$, we have

$$\sum_{i=1}^{m} \lceil r_i/2 \rceil = \sum_{j=1}^{n} \lceil s_j/2 \rceil,$$

and

$$\sum_{i=1}^{m} \lfloor r_i/2 \rfloor = \sum_{j=1}^{n} \lfloor s_j/2 \rfloor.$$

Because (b) and (c) hold, it now follows from Theorem 2.1 and the bipartite analogue of the 1-factor theorem as given in [3] that there exist (0, 1)-matrices X in $\mathfrak{A}_1(\lceil R/2\rceil, \lceil S/2\rceil)$ and Y in $\mathfrak{A}_1(\lfloor R/2\rfloor, \lfloor S/2\rfloor)$ with $Y \leqslant X$ (entrywise). The matrix $X+Y$ is in the class $\mathfrak{A}_2(R, S)$ and has 1-pattern matrix $X-Y$ which has only ambiguous constituents. It follows that $X+Y$ is parsimonious, and hence (3.4) holds. ∎

Let $\mathscr{C}$ be an arithmetically compatible collection of constellation matrices with arithmetic invariants ρ, λ, and ν. The following examples show that no two of these invariants determines the other. If $R=S=(1,1)$, then $\rho=2$ and $\lambda=\nu=2$, while if $R=S=(4,1,1)$, then $\rho=4$ and $\lambda=\nu=2$. Thus λ and ν do not determine ρ. If $R=S=(1,1,1)$, then $\nu=3$ and $\rho=\lambda=3$, while if $R=(3,1,1)$ and $S=(4,1)$, then $\nu=1$ and $\rho=\lambda=3$. Thus ρ and λ do not determine ν. Similarly, ρ and ν do not determine λ. Therefore ρ, λ and ν are independent parameters defining an arithmetically compatible collection of constellation matrices. However, we do not know whether the arithmetic invariants ρ, λ, and ν also determine a class-compatible collection of constellation matrices. In other words, if $\mathscr{C}$ is an arithmetically compatible collection of constellation matrices, then we do not know whether there always exists a pair (R, S) which affords $\mathscr{C}$. If there always is such a pair (R, S), then the converse of Lemma 3.1 is true.

For example, let

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \tag{3.7}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

be constellation matrices with respective configurations $\{K_{3,1}, K_{1,2}\}$, $\{K_{1,1}, K_{2,1}, K_{1,2}\}$, and $\{K_{1,3}, K_{2,1}\}$. Then the collection $\mathscr{C} = \{C_1, C_2, C_3\}$ is arithmetically compatible with arithmetic invariants $\rho=5$ and $\lambda=\nu=3$. However, we do not know if $\mathscr{C}$ is class-compatible, that is, we have been unable to find a pair (R, S) which affords $\mathscr{C}$.

Let $\mathscr{C}$ be a collection of constellation matrices each of which has exactly d odd constituents. Then $\mathscr{C}$ is *orientable* provided there exists nonnegative integers d_h and d_v with

$$d = d_h + d_v, \tag{3.8}$$

such that each constellation matrix in $\mathscr{C}$ has at most d_h odd horizontal constituents and at most d_v odd vertical constituents. Thus $\mathscr{C}$ is orientable if and only if for each constellation matrix in $\mathscr{C}$ it is possible to assign one of the two orientations, horizontal and vertical, to each of its ambiguous constituents so that the resulting configuration has d_h odd horizontal constituents and d_v odd vertical constituents. Each of two collections $\{C_1, C_2\}$ and $\{C_2, C_3\}$ of constellation matrices in (3.7) is orientable, but the collection $\{C_1, C_2, C_3\}$ is not orientable.

We now prove the converse of Lemma 3.1 with the additional hypothesis of orientability.

THEOREM 3.4. *If an orientable collection $\mathscr{C}$ of constellation matrices is arithmetically compatible, then $\mathscr{C}$ is class-compatible.*

Proof. Let $\mathscr{C}$ be an orientable collection of constellation matrices which is arithmetically compatible with arithmetic invariants ρ, λ, and ν. We orient the ambiguous constituents of each constellation matrix C in $\mathscr{C}$ so that each C has d_h odd horizontal constituents and d_v odd vertical constituents. Now the total number of 1's in the horizontal constituents is $\nu - d_v$ and the total number of 1's in the vertical constituents is $\lambda - d_h$ for each C in $\mathscr{C}$.

Consider the matrix

$$A = \begin{bmatrix} 2J & A_h \\ A_v & O \end{bmatrix}, \tag{3.9}$$

where

$$A_h = \begin{array}{cl} \left[\begin{array}{c|c|c} 2J & O & I \\ \hline O & 2J & O \\ \hline 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 \end{array}\right] & \begin{array}{l} \}\, d_h \\ \}\, (\rho-\lambda)/2 \\ \end{array} \\ \begin{array}{ccc} \quad(\rho-\lambda)/2 & (\rho-\lambda)/2 & \quad d_h \end{array} & \end{array} \tag{3.10}$$

and

$$(3.11) \qquad A_v = \begin{array}{c} \left[\begin{array}{c|c|c} 2J & O & \begin{matrix}1\\ \vdots \\ 1\end{matrix} \\ \hline O & 2J & \begin{matrix}1\\ \vdots \\ 1\end{matrix} \\ \hline I & O & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \end{array}\right] \begin{matrix} \}\, (\rho-\nu)/2 \\ \\ \}\, (\rho-\nu)/2 \\ \\ \}\, d_v \end{matrix} \\ \underbrace{\qquad}_{d_v} \underbrace{\qquad}_{(\rho-\nu)/2} \end{array}$$

In (3.9), (3.10), and (3.11), O, I and J denote, respectively, a zero matrix, an identity matrix, and an all 1's matrix of the appropriate size. Let $\mathfrak{A}_2(R, S)$ be the class of (0,1,2)-matrices which contains A. It follows from (3.9) that every matrix B in $\mathfrak{A}_2(R, S)$ has a decomposition

$$(3.12) \qquad B = \begin{bmatrix} 2J & B_h \\ B_v & O \end{bmatrix}$$

like that of A. Let $\mathfrak{A}_2(R_h, S_h)$ and $\mathfrak{A}_2(R_v, S_v)$ be the classes which contain A_h and A_v, respectively. Then the matrix B is parsimonious if and only if both B_h and B_v are parsimonious.

Each column sum of A_h is odd, and A_h has exactly one 1 in each column. Thus a matrix B_h in $\mathfrak{A}_2(R_h, S_h)$ is parsimonious if and only if it has exactly one 1 in each column. Hence the number ρ_h of 1's in a parsimonious matrix in $\mathfrak{A}_2(R_h, S_h)$ is given by

$$\rho_h = (\rho - \lambda) + d_h = \nu - d_v$$

by (3.10), (3.3), and (3.8).

Let C be any constellation matrix in $\mathscr{C}$, and let the sizes of its horizontal constituents be

$$2p(1)+1, \ldots, 2p(d_h)+1, 2p(d_h+1), \ldots, 2p(d_h+e),$$

where e is the number of even horizontal constituents of $\mathscr{C}$. We have

$$(3.13) \qquad p(1) + p(2) + \cdots + p(d_h + e) = \frac{\rho - \lambda}{2},$$

and hence neither the sum $p(1)+\cdots+p(d_h)$ nor e exceeds $(\rho-\lambda)/2$. We define the $(0,1,2)$-matrix

$$X=\underbrace{\left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\begin{matrix}1\cdots 1 & & 2\\ & \ddots & \\ 2 & & 1\cdots 1\end{matrix}}^{p(1)+\cdots+p(d_h)}} & & O & \\ \hline & & & 1\cdots 1 & & 0\\ & O & & & \ddots & \\ & & & 0 & & 1\cdots 1\\ \hline 2 & \cdots & 2 & 0 & \cdots & 0 \end{array}\right]}_{(\rho-\lambda)/2} \begin{array}{l}\left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} d_h\\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} (\rho-\lambda)/2\\ \vphantom{1}\end{array}$$

where row i has $p(i)$ consecutive 1's for $i=1,2,\ldots,d_h+e$. The matrix

$$B_h=\left[\begin{array}{c|c|c} X & 2J-X & \begin{matrix} I \\ \hline O \\ \hline \underbrace{0\ \cdots\ 0}_{d_h}\end{matrix}\end{array}\right]\begin{array}{l}\} d_h\\ \} (\rho-\lambda)/2\\ \vphantom{0}\end{array}$$

is a parsimonious matrix in the class $\mathfrak{A}_2(R_h, S_h)$. By a similar construction we produce a parsimonious matrix B_v in the class $\mathfrak{A}_2(R_v, S_v)$. The matrix B is a parsimonious matrix in $\mathfrak{A}_2(R, S)$ of the form (3.12). Moreover, the 1-pattern matrix $B^{\langle 1\rangle}$ has the same configuration of constituents as the constellation matrix C. Therefore the collection $\mathscr{C}$ is class-compatible. ■

4. CODA

In this final section we pose several problems concerning classes of $(0,1,2)$-matrices with prescribed row and column sum vectors.

In Theorem 3.4 we showed that any arithmetically compatible collection of constellation matrices which is orientable must be class-compatible. By Lemma 3.1 any class-compatible collection of constellation matrices is arithmetically compatible. Now consider the collection $\mathscr{C}=\{C_1, C_2, C_3\}$ of constellation matrices defined in (3.7). Although $\mathscr{C}$ is arithmetically compatible, it is not orientable, and we have been unable to determine whether $\mathscr{C}$ is class-compatible. More generally, we may ask whether the converses of Lemma 3.1 and Theorem 3.4 are valid.

PROBLEM 4.1. Let $\mathscr{C}$ be a collection of constellation matrices. Does either of the following two statements hold?

(i) If $\mathscr{C}$ is arithmetically compatible, then $\mathscr{C}$ is class-compatible.
(ii) If $\mathscr{C}$ is class-compatible, then $\mathscr{C}$ is orientable.

We note that (i) and (ii) cannot both be true, because the collection $\{C_1, C_3\}$ of constellation matrices in (3.7) is arithmetically compatible but is not orientable.

Suppose (i) is valid. Then the class-compatibility of $\mathscr{C}$ may be tested easily by computing the arithmetic invariants of each constellation matrix in $\mathscr{C}$. Now suppose (ii) is valid. Then $\mathscr{C}$ is class-compatible if and only if $\mathscr{C}$ is arithmetically compatible and

$$d_h' + d_v' \leqslant \lambda + \nu - \rho,$$

where λ, ν, and ρ are the arithmetic invariants of $\mathscr{C}$, and d_h' (respectively, d_v') is the maximum number of odd horizontal (respectively, vertical) constituents among all constellation matrices in $\mathscr{C}$.

We define three binary relations on the collection of all constellation matrices as follows:

$C \sim_a C'$ provided $\{C, C'\}$ is arithmetically compatible,
$C \sim_c C'$ provided $\{C, C'\}$ is class-compatible,
$C \sim_o C'$ provided $\{C, C'\}$ is orientable.

We note that each of these binary relations is reflexive and symmetric. The relation $\sim_a$ is clearly transitive and hence is an equivalence relation. However, the relation $\sim_o$ is not transitive, because the constellation matrices C_1, C_2, and C_3 in (3.7) satisfy $C_1 \sim_o C_2$, $C_2 \sim_o C_3$, and $C_1 \not\sim_o C_3$.

PROBLEM 4.2. Is the binary relation $\sim_c$ transitive?

If (i) of Problem 4.1 holds, then $\sim_c$ is transitive; if (ii) of Problem 4.1 holds, then $\sim_c$ is not transitive.

Recall that $\rho(A)$ counts the number of 1's in the $(0,1,2)$-matrix A, and $\beta(A)$ counts the number of 2's. Let $\mathfrak{A}_2(R, S)$ be a nonempty class of $(0,1,2)$-matrices. We define

$$\tilde{\rho} = \tilde{\rho}(R, S) = \min\{\rho(A) : A \in \mathfrak{A}_2(R, S)\}$$

and

$$\bar{\rho} = \bar{\rho}(R, S) = \max\{\rho(A) : A \in \mathfrak{A}_2(R, S)\}.$$

Then $\rho(A) = \tilde{\rho}$ if and only if A is parsimonious. Also, $\rho(A) = \bar{\rho}$ if and only if A is improvident.

Now let A be a parsimonious matrix in $\mathfrak{A}_2(R, S)$. The number λ (respectively, ν) of odd horizontal (respectively, vertical) constituents of the constellation matrix $A^{\langle 1 \rangle}$ equals the number of odd components of R (respectively, S). It follows from Theorem 3.2 that $\max\{\lambda, \nu\} \leqslant \tilde{\rho} \leqslant \bar{\rho} \leqslant \lambda + \nu$.

PROBLEM 4.3.

(i) Find formulas to evaluate $\tilde{\rho}$ and $\bar{\rho}$.

(ii) Given nonnegative integral vectors R and S, find good algorithms for the construction of a parsimonious matrix and an improvident matrix in $\mathfrak{A}_2(R, S)$.

Let r be a positive integer, and let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be nonnegative integral vectors. We denote by $\mathfrak{A}_r(R, S)$ the class of all $(0, 1, \ldots, r)$-matrices with row sum vector R and column sum vector S. A theorem of Mirsky [6] gives necessary and sufficient conditions for the nonemptiness of the class $\mathfrak{A}_r(R, S)$. We generalize the notion of parsimony and define a matrix in $\mathfrak{A}_r(R, S)$ to be *parsimonious* provided it has the smallest number of positive entries among all matrices in $\mathfrak{A}_r(R, S)$. We showed in Section 1 that for $r = 2$ a matrix A is parsimonious if and only if $\operatorname{tr}(AA^T)$ is maximum. However, this characterization of parsimony is not valid in general. For example, let $r = 4$ and let $R = S = (5, 2)$. Then the class $\mathfrak{A}_r(R, S)$ consists of the two matrices

$$U = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then U is parsimonious and V is not. On the other hand,

$$\operatorname{tr}(VV^T) = 19 > 17 = \operatorname{tr}(UU^T).$$

Thus if we instead define a matrix A in $\mathfrak{A}_r(R, S)$ $(r > 2)$ to be parsimonious provided $\operatorname{tr}(AA^T)$ is maximum, we obtain a notion of parsimony which neither implies nor is implied by the one given.

In a subsequent paper we shall investigate these and other notions of parsimony as well as other properties of the class $\mathfrak{A}_r(R, S)$.

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