# Generalized Scalings Satisfying Linear Equations

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday with admiration for his scholarship and joy in life and mathematics.

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ABSTRACT

We unify and generalize a broad class of problems referred in the literature as "scaling problems," by extending the applicability of a formulation suggested recently by Bapat and Raghavan. Specifically, let $a \in R^n$, $b \in R^m$, and $C \in R^{m \times n}$ be given, where $a$ is strictly positive. A *C-scaling* of the vector $a$ is defined to be a vector $a' \in R^n$ with $a'_i = a_i \prod_{k=1}^{m} u_k^{C_{ki}}$ for some strictly positive vector $u \in R^m$. The problem of finding a C-scaling of the vector $a$ which satisfies the linear system $Cx = b$ will be called a *generalized scaling problem*. In this paper it is shown that previously studied matrix-scaling problems, (e.g., finding scalings with prespecified row sums and column sums, or finding scalings with row sums equaling the corresponding column sums, or finding scalings of multidimensional matrices with prespecified margins) are special instances of generalized scaling problems. Generalized scaling problems are reduced to convex optimization problems, and the reduction is used to characterize solutions, to develop necessary and sufficient conditions for their existence, to establish uniqueness results and to characterize approximate solutions.

## 1. INTRODUCTION

A *scaling of* an array of numbers, e.g., a vector or a matrix, is obtained by multiplying each of the elements of that array by a corresponding positive number, where some structure is imposed on the multiplying coefficients. For example, if $A = (A_{ij})$ is a (rectangular) matrix and $B = (B_{ij})$ is a matrix with




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$B_{ij} = x_i A_{ij} y_j$ for some positive vectors $x$ and $y$, we call $B$ a $(D, E)$*-scaling of* $A$. Also, if $A = (A_{ij})$ is a square matrix and $B = (B_{ij})$ satisfies $B_{ij} = x_i A_{ij}(x_j)^{-1}$, we call $B$ a $(D, D^{-1})$*-scaling of* $A$. *Scaling problems* concern the identification of scalings of given arrays which have certain characteristics. In particular, problems where the desired properties are defined via linear constraints have attracted a lot of attention. We next describe in some detail three examples of such scaling problems; see Bapat and Raghavan (1989) for a related discussion.

The first example for a scaling problem with linear constraints concerns the identification of a $(D, E)$-scaling of a given matrix that has predescribed row sums and column sums. This problem has been studied extensively in the last fifty years and has many applications, including budget allocation, data preprocessing in numerical analysis, assessment of probability distributions in statistics, social accounting, data assessment for transportation problems, computed tomography and image reconstruction, and many more; see Bacharach (1970), Schneider and Zenios (1987) and Rothblum and Schneider (1989) and references therein. For example, suppose that $A = (A_{ij})$ represents the annual budget, where $A_{ij}$ is the budget allocated to department $i$ for budget category $j$. One way of constructing a new budget given the previous one is to first determine the total allocations to each of the departments and for each of the budget categories, and then update the original budget by using multipliers for each department and for each budget category. The problem of finding the right multipliers will then become the problem of finding a $(D, E)$-scaling of $A$ where the row sums and the column sums are the new allocations to the different departments and for the various budget categories. The first paper with which we are familiar that considers the problem of finding a $(D, E)$-scaling having predescribed row sums and column sums is Kruithof (1937), which examines an application to assessment of telephone traffic and describes a heuristic solution for the computation of corresponding scalings. Independently, Sinkhorn (1964) studied doubly stochastic $(D, E)$-scalings of strictly positive square matrices. He obtained a number of interesting results including a constructive proof for the existence of such scalings. The algorithm he uses coincides with that of Kruithof. The general problem of finding a $(D, E)$-scaling with prespecified row sums and column sums was introduced and analyzed in Brualdi, Parter, and Schneider (1966) and independently in Sinkhorn (1967).

The second example concerns the problem of identifying a $(D, D^{-1})$-scaling of a given square matrix whose row sums equal the corresponding column sums; see Eaves, Hoffman, Rothblum, and Schneider (1985), Schneider and Zenios (1987), and references therein. In particular, such problems arise in trade balancing and are used routinely by the World Bank [e.g., King (1981)]. To be more specific, let $A = (A_{ij})$ represent the trade levels between the different countries, where $A_{ij}$ is the total value of the export of country $i$ to country $j$. A $(D, D^{-1})$-scaling of the matrix $A$ represents updated trade figures after corresponding annual inflation rates for each of the different countries have been taken into account. Trade balance will be achieved when the new figures yield a matrix whose row sums equal the corresponding column sums.

The third and last example concerns scaling problems of multidimensional matrices and generalizes the first example discussed above; see Bapat (1982) and Raghavan (1984, 1985). Specifically, given $A = (A_{ijk})$, where $i$, $j$, and $k$ range over finite sets, one is interested in identifying $x_i$'s, $y_j$'s, and $z_k$'s for which $B = (B_{ijk}) = (x_i y_j z_k A_{ijk})$ has one dimensional margins given by prespecified vectors $r$, $s$, and $t$, i.e.

$$\sum_{jk} B_{ijk} = r_i, \qquad \sum_{ik} B_{ijk} = s_j, \quad \text{and} \quad \sum_{ij} B_{ijk} = t_k$$

for all relevant, $i$, $j$, and $k$. This problem arises in examining the growth of tumors when the initial data are completely specified (obtained for example through an operation) and one tries to obtain updates from information that is obtained later and concerns only the marginal sums (obtained for example from three dimensional X-rays); see Herman and Lint (1976) or Lakshminarayanan and Lint (1979) for further details. Of course, the above example, which concerns multidimensional matrices of order three, can be generalized to be of higher order.

Bapat and Raghavan (1989) suggest a unified framework for the first and third examples listed above. In the current paper we extend the applicability of their approach so that it also captures the second example. Some notation will be needed to introduce the unified formulation. We call a vector $w \in R^p$ *nonnegative*, written $w \geqslant 0$, if all the coordinates of $w$ are nonnegative. We call $w$ *strictly positive*, written $w \gg 0$, if all the coordinates of $w$ are positive. Finally, we call $w$ *semipositive*, written $w > 0$, if $w \geqslant 0$ and $w \neq 0$. Throughout the remainder of this paper let $a \in R^n$, $b \in R^m$, and $C \in R^{m \times n}$ be given where $a \gg 0$. A vector $a' \in R^n$ is called a *C-scaling of a* if for some strictly positive vector $u \in R^m$

$$a'_i = a_i \left( \prod_{k=1}^{m} u_k^{C_{ki}} \right), \qquad i = 1, \dots, n. \tag{1}$$

We will consider the problem of finding $C$-scalings $a'$ of $a$ that satisfies the linear system

$$Ca' = b. \tag{2}$$

We call the problem of finding such $C$-scalings a *generalized scaling problem*, Bapat and Raghavan (1989) consider the case where the set

$$\{x \in R^n : Cx = b, x \geqslant 0\}$$

is bounded.

Of course, if $a$ is nonnegative rather than strictly positive, then for every $C$-scaling $a'$ of $a, a'_i = 0$ whenever $a_i = 0$. So, the zero coordinates of $a$ and the corresponding columns of $C$ can be dropped, and without loss of generality we may assume that $a$ is strictly positive. Still, in applications where the vector $a$ represents a matrix, the inclusion of the zero coordinates is convenient and natural because it allows the use of matrix notation. When we develop the theory of generalized scalings, we will assume that the given vector $a$ is strictly positive; but in the discussion of examples we let $a$ have zero coordinates whenever convenient.

Following Bapat and Raghavan (1989), we next demonstrate that each of the examples described above can be viewed as a generalized scaling problem. First consider the problem of finding a $(D, E)$-scaling with prespecified row sums and column sums. To avoid cumbersome notation we examine only $2 \times 3$ matrices. Let $A, A' \in R^{2\times 3}$, let $r \in R^2$, and let $c \in R^3$ be given. We will represent $A$ and $A'$ by six dimensional vectors $a = (A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23})$ and $a' = (A'_{11}, A'_{12}, A'_{13}, A'_{21}, A'_{22}, A'_{23})$, respectively. We observe that $A'$ has row sums $r_1$ and $r_2$ and column sums $c_1$, $c_2$, and $c_3$ if and only if $Ca' = b$, where

$$C = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} r_1 \\ r_2 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Also, $a'$ is a $C$-scaling $a'$ of $a$ if and only if for some multipliers $(v_1, v_2, w_1, w_2, w_3)$

$$A'_{12} = A_{12}(v_1)^1(v_2)^0(w_1)^0(w_2)^1(w_3)^0 = v_1 A_{12} w_2$$

and, in general,

$$A'_{ij} = v_i A_{ij} w_j, \qquad i = 1,2 \text{ and } j = 1,2,3.$$

So the problem of finding a $(D, E)$-scaling of $A$ with given row sums and column sums reduces to the problem of finding a $C$-scaling $a'$ of $a$ which satisfies (2) for $C$ and $b$ given above.

We next consider the problem of finding $(D, D^{-1})$-scalings with row sums equaling corresponding column sums. Again, we consider only $3 \times 3$ matrices. Let $A, A' \in R^{3\times 3}$ be represented by corresponding nine dimensional vectors $a$ and $a'$ respectively. Then the row sums of $A'$ equal the corresponding column sums if and only if $Ca' = 0$ for

$$C = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

Also, $a'$ is a $C$-scaling of $a$ if and only if for some multipliers $v_1$, $v_2$, and $v_3$

$$A'_{23} = A_{23}(v_1)^0(v_2)^1(v_3)^{-1} = v_2 A_{23}(v_3)^{-1}$$

and, in general,

$$A'_{ij} = v_i A_{ij}(v_j)^{-1}, \qquad i, j = 1,2,3.$$

So, the problem of finding a $(D, D^{-1})$-scaling of $A$ with row sums equaling the corresponding column sums reduces to the problem of finding a $C$-scaling $a'$ of $a$ which satisfies (2) with $C$ defined as above and $b = 0$. We observe that in the above example the set $\{x \in R^n : Cx = b, x \geqslant 0\}$ is unbounded; hence it does not fit the case considered in Bapat and Raghavan (1989).

Finally, we let the reader verify that the problem of scaling a multidimensional matrix of order three (or higher) with prespecified margins can also be described as a generalized scaling problem with a corresponding matrix $C$ and vector $b$. In particular, if the dimension of $A$ is $2\times 3\times 2$, we have that

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

[here $A$ will be identified with $a = (A_{111}, A_{112}, A_{121}, A_{122}, A_{131}, A_{132}, A_{211}, A_{212}, A_{221}, A_{222}, A_{231}, A_{232})$].

The first formulation of the generalized scaling problem and its relationship to $(D, E)$-scaling of matrices with prespecified row sums and column sums and to scalings of multidimensional matrices with prespecified margins appears in Bapat and Raghavan (1989). In particular, it is shown there that when $K \equiv \{x \in R^n : Cx = b, x \geqslant 0\}$ is nonempty and bounded, every strictly positive vector $a$ has a $C$-scaling $a'$ that satisfies $Ca' = b$. An instance of this problem where $C$ has a row all of whose elements are one was considered by Darroch and Ratcliff (1972), who examined scalings of probability distributions. The latter paper motivated Bapat and Raghavan (1989).

The development of the current paper relies on the identification of a convex optimization problem whose solutions are in one-to-one correspondence with the solutions to the given generalized scaling problem. In particular, generalizations and extensions of the results of Bapat and Raghavan (1989) are obtained, while proofs are simpler. In particular, solutions to generalized scaling problems are characterized and their uniqueness is established. Also, necessary and sufficient conditions for their existence are derived. Finally, approximate solutions to generalized scaling problems are defined and characterized. The development (and, in particular, the use of optimization formulations to study scaling problems) follows the approach used in Rothblum and Schneider (1989), where $(D, E)$-scalings of matrices with prespecified row sums and column sums are considered; but the optimization problem used there is different from the restriction of the one used here to their scaling problem.

Independently of the current paper, Franklin and Lorenz (1989) also extended the results of Bapat and Raghavan (1989) and obtained results that are similar to ours by considering optimization problems that are related to the duals of the problems used here. Also, Schneider (1989) formulated a matrix scaling problem that unifies the first and second examples discussed above. His analysis of his unified problem uses entropy optimization techniques that are related to the approach of Franklin and Lorenz.

We make no attempt in this paper to survey the vast literature on scaling problems, and our referencing is certainly incomplete. In fact, Campbell (personal communications to Hans Schneider) has collected over 400 references to papers which consider different kinds of scaling problems.

## 2. CHARACTERIZATION, EXISTENCE, UNIQUENESS, AND APPROXIMATION OF SOLUTIONS TO THE GENERALIZED SCALING PROBLEM

In the remainder of this paper we let $b \in R^m$ and $C \in R^{m \times n}$ be fixed. In studying the problem of finding a $C$-scaling $a'$ of a given strictly positive vector $a$ that satisfies $Ca' = b$ we will find it useful to consider the following nonlinear optimization problem:

$$\text{Program I:} \quad \min \sum_{j=1}^{n} a_j \left( \prod_{k=1}^{m} y_k^{C_{kj}} \right) - \sum_{k=1}^{m} b_k \log y_k$$

$$\text{s.t.} \quad y \gg 0.$$

We observe that the objective of Program I is not convex. However, the change of variables $\log y_i = z_i$, $i = 1,\ldots,m$, converts Program I into the following unconstrained optimization problem

$$\text{Program II:} \quad \min \sum_{j=1}^{n} a_j \exp\left( \sum_{i=1}^{m} z_i C_{ij} \right) - \sum_{k=1}^{m} b_k z_k.$$

We next show that the objective of Program II is convex.

LEMMA 1. *Let $a \in R^n$ be a strictly positive vector. Then the objective function of Program II is convex.*

*Proof.* Let $z^1, z^2 \in R^m$ and $0 < \alpha < 1$. The convexity of the exponential function and the positivity of the $a_j$'s assure that

$$\sum_{j=1}^{n} a_j \exp\left[ \sum_{i=1}^{n} \left[(1-\alpha)z^1 + \alpha z^2\right]_i C_{ij} \right]_j$$

$$= \sum_{j=1}^{n} a_j \exp\left[(1-\alpha)(z^1C)_j + \alpha(z^2C)_j\right]$$

$$\leqslant \sum_{j=1}^{n} a_j \left[(1-\alpha)\exp(z^1C)_j + \alpha \exp(z^2C)_j\right]$$

$$= (1-\alpha) \sum_{j=1}^{n} a_j \exp(z^1C)_j + \alpha \sum_{j=1}^{n} a_j \exp(z^2C)_j,$$

proving the convexity of the first term of the objective function of Program II. As the second term is linear, our proof is complete. ∎

Our next result shows that the generalized scaling problem reduces to the problem of finding optimal solutions to Program I.

THEOREM 1 (Characterization). *Let $a \in R^n$ and $u \in R^m$ be strictly positive vectors. Then the following are equivalent*:

(a) *the vector $a' \in R^n$ defined by*

$$a'_j = a_j \left( \prod_{k=1}^{m} u_k^{C_{kj}} \right) \tag{3}$$

*satisfies $Ca' = b$, and*

(b) *the vector $u$ is optimal for Program I.*

*Proof.* (a) $\Rightarrow$ (b): Assume that $a'$ defined via (3) satisfies $Ca' = b$. Consider the vector $s \in R^m$ defined by $s_i = \log u_i$, $i = 1, \ldots, m$. Then our assumption states that

$$\sum_{j=1}^{n} C_{ij} a_j \exp\left( \sum_{k=1}^{m} s_k C_{kj} \right) = b_i \qquad i = 1, \ldots, m.$$

Let $h(\cdot)$ be the objective of Program II. As

$$\frac{\partial}{\partial z_i} h(z) = \sum_{j=1}^{n} a_j C_{ij} \exp\left( \sum_{k=1}^{m} z_k C_{kj} \right) - b_i, \qquad i = 1, \ldots, m,$$

we have that $\nabla h(s) = 0$. By the convexity of $h(\cdot)$ (see Lemma 1), we conclude that $s$ is optimal for Program II, immediately implying that $u$ is optimal for Program I.

(b) $\Rightarrow$ (a): Suppose that $u$ is optimal for Program I. As the feasible set of Program I is open and its objective is differentiable, the partial derivative of its objective must equal zero at $u$, i.e., for $i = 1, \ldots, m$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial y_i} \left[ \sum_{j=1}^{n} a_j \left( \prod_{k=1}^{m} y_k^{C_{kj}} \right) - \sum_{k=1}^{m} b_k \log y_k \right] \Bigg|_{y=u} \\ &= \left[ \sum_{j=1}^{n} a_j \left( \prod_{\substack{k=1 \\ k \neq i}} y_k^{C_{kj}} \right) C_{ij} y_i^{C_{ij}-1} - b_i y_i^{-1} \right] \Bigg|_{y=u} \\ &= u_i^{-1} \left[ \sum_{j=1}^{n} C_{ij} a_j \left( \prod_{k=1}^{m} u_k^{C_{kj}} \right) - b_i \right], \end{aligned}$$

implying that $a'$ defined by (3) satisfies $Ca' = b$. ■

When considering the problem of finding $(D, E)$-scalings of a matrix $A \in R^{m \times n}$ with row sums and column sums that are, respectively, equal to the coordinates of vectors $r \in R^m$ and $c \in R^n$, Program I becomes

$$\min\left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} x_i A_{ij} y_j - \sum_{i=1}^{m} r_i(\log x_i) - \sum_{j=1}^{n} c_j(\log y_j) \right\}$$

$$\text{s.t.} \quad x \gg 0, \quad y \gg 0.$$

This is a variant of the following program first considered by Marshall and Olkin (1968) and later studied by Rothblum and Schneider (1988):

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} x_i A_{ij} y_i$$

$$\text{s.t.} \quad x \gg 0, \quad y \gg 0,$$

$$\prod_{i=1}^{m} x_i^{r_i} = \prod_{j=1}^{n} y_j^{c_j} = 1.$$

In particular, Rothblum and Schneider show that when $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$ there is a one-to-one correspondence between optimal solutions of the second program and desired scaling coefficients. Of course, Theorem 1 shows that the set of desired scaling coefficients coincides with the set of optimal solutions of the first program. Also, when the problem of finding a $(D, D^{-1})$-scaling of a square matrix $A \in R^{n \times n}$ with row sums equaling corresponding column sums is considered, Program I becomes

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_{ij} (x_j)^{-1}$$

$$\text{s.t.} \quad x \gg 0.$$

This program has been used in Eaves, Hoffman, Rothblum, and Schneider (1985). We finally mention that for the multidimensional matrix scaling problem described in the introduction, Program I becomes

$$\min\left\{ \sum_i \sum_j \sum_k x_i y_j z_k A_{ijk} - \sum_i r_i(\log x_i) - \sum_j s_j(\log y_j) - \sum_k t_k(\log z_k) \right\}$$

$$\text{s.t.} \quad x \gg 0, \quad y \gg 0, \quad z \gg 0.$$

By considering Theorem 1 with Lemma 1 and with the equivalence of Programs I and II, we see that the solution of the generalized scaling problem reduces to the solution of an unconstrained optimization problem that has a convex objective. Hence algorithms that solve convex optimization problems can be used to solve the generalized scaling problem.

Our next result characterizes existence of solutions to the generalized scaling problem.

THEOREM 2 (Existence). *Let $a \in R^n$ be a strictly positive vector. Then the following are equivalent:*

(a) *there exists a C-scaling $a'$ of $a$ that satisfies $Ca' = b$,*

(b) $\{x \in R^n : Cx = b,\ x \gg 0\} \neq \varnothing$,

(c) *there exists no $\lambda \in R^n$ for which $\lambda^T C \geqslant 0$, $b^T \lambda \leqslant 0$, and either $\lambda^T C \neq 0$ or $\lambda^T b \neq 0$, and*

(d) *Program I has an optimal solution.*

*Proof.* (a) $\Rightarrow$ (b): This implication follows immediately from the fact that every $C$-scaling of a strictly positive vector is also strictly positive.

(b) $\Rightarrow$ (c): It follows from the theorem of the alternative [e.g., Schrijver (1986, Corollary 7.1k, pp. 94–95)] that if $\{x \in R^n : Cx = b, x \gg 0\} \neq \varnothing$ then there exist no $\lambda \in R^m$ and $\mu \in R^n$ such that

$$\lambda^T C - \mu^T = 0, \qquad \mu \geqslant 0, \qquad \lambda^T b \leqslant 0 \quad \text{and} \quad \text{either } \lambda^T b \neq 0 \text{ or } \mu \neq 0,$$

i.e., there exist no $\lambda \in R^m$ satisfying the conditions spelled out in (c). So, indeed, (b) implies (c).

(c) $\Rightarrow$ (d): Assume that (c) holds, i.e.,

$$\lambda^T C \geqslant 0, \quad b^T \lambda \geqslant 0 \ \Rightarrow\ \lambda^T C = 0, \quad \lambda^T b = 0, \tag{4}$$

and we will show that Program II has an optimal solution. It will then follow immediately that Program I has an optimal solution as well.

Denote the objective function of Program II by $h(\cdot)$, i.e., $h(z) = \sum_{j=1}^n a_j \exp(zC)_j - b^T z$. A *direction of recession* of $h(\cdot)$ is defined to be a vector $d$ for which

$$\sup_{z \in R^n} \{h(z + d) - h(z)\} \leqslant 0. \tag{5}$$

As the function $h(\cdot)$ is convex (Lemma 1), we have from Rockafellar [1970, Theorem 173, p. 267] that existence of an optimal solution for Program II is implied by the assertion that every direction of recession $d$ of $h(\cdot)$ satisfies

$$h(z+d)-h(z)=0 \qquad \text{for all} \quad z \in R^n. \tag{6}$$

Thus, it suffices to show that (5) implies (6).

Assume that (5) holds. Substituting the explicit expression of $h(\cdot)$ into (5) we have that

$$\sup_{z \in R^n} \left\{ \sum_{j=1}^{n} a_j \exp\left[(z^TC)_j + (d^TC)_j\right] - \sum_{j=1}^{n} a_j \exp(z^TC)_j - b^Td \right\} \leqslant 0, \tag{7}$$

or equivalently,

$$\sum_{j=1}^{n} a_j \left\{\exp(z^TC)_j\left[\exp(d^TC)_j - 1\right]\right\} - b^Td \leqslant 0 \qquad \text{for every} \quad z \in R^n. \tag{8}$$

We next argue that $d^TC \leqslant 0$. Let $J_+ \equiv \{j=1,\dots,n:(d^TC)_j > 0\}$ and let $J_- \equiv \{j=1,\dots,n:(d^TC)_j \leqslant 0\}$, and we will show that $J_+ = \varnothing$. Suppose that $J_+ \neq \varnothing$ and $p \in J_+$. Then for each $M>0$, (8) with $z \equiv Md$ implies that

$$\begin{aligned}
&a_p \exp\left[M(d^TC)_p\right]\left[\exp(d^TC)_p - 1\right] \\
&\qquad \leqslant \sum_{j \in J_+} a_j \exp\left[M(d^TC)_j\right]\left[\exp(d^TC)_j - 1\right] \\
&\qquad \leqslant b^Td - \sum_{j \in J_-} a_j \exp\left[M(d^TC)_j\right]\left[\exp(d^TC)_j - 1\right] \\
&\qquad \leqslant b^Td + \sum_{j \in J_-} a_j,
\end{aligned}$$

implying that $a_p \exp[M(d^TC)_p][\exp(d^TC)_p - 1]$ is bounded from above in $M$. But, this conclusion is false, as $(d^TC)_p > 0$. This contradiction proves that

$J_+ = \emptyset$, i.e., $d^TC \leqslant 0$. We also have from (8), again with $z = Md$, that for each $M > 0$

$$b^Td \geqslant \sum_{j=1}^{n} a_j \exp\left[M(d^TC)_j\right]\left[\exp(d^TC)_j - 1\right].$$

As $d^TC \leqslant 0$, the right hand side of the above inequality converges to zero as $M \to \infty$, and we conclude that $b^Td \geqslant 0$. As $d^TC \leqslant 0$ and $b^Td \geqslant 0$, it now follows from (4) that $d^TC = 0$ and $b^Td = 0$, implying that (8) and (7) hold as equalities. So (6) is satisfied, thereby completing our proof that (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (a): This implication is immediate from Theorem 1. ∎

COROLLARY 1. *If there exists one strictly positive vector $a \in R^n$ which has a C-scaling $a'$ that satisfies $Ca' = b$, then every strictly positive vector has such a C-scaling.*

*Proof.* The conclusion is immediate from the equivalence of condition (a) and (b) in Theorem 2 and the fact that condition (b) is independent of the selection of the vector $a$. ∎

COROLLARY 2. *If there exists one strictly positive vector $a \in R^n$ for which Program I, or equivalently Program II, has an optimal solution, then this conclusion is shared by every strictly positive vector.*

*Proof.* The conclusion is immediate from the equivalence of conditions (d) and (b) in Theorem 2 and the fact that condition (b) is independent of the selection of the vector $a$. ∎

We next examine condition (c) of Theorem 2 for the problem of finding $(D, E)$-scalings of a matrix $A \in R^{m \times n}$ with row sums and column sums given by the coordinates of vectors $r \in R^m$ and $c \in R^n$, respectively. In this case we use the explicit form of the corresponding matrix $C$, as derived in the introduction, to conclude that condition (c) of Theorem 2 becomes:

(c′) there exist no vectors $v \in R^m$ and $w \in R^m$ for which

$$v_i + w_j \leqslant 0 \qquad \text{for all} \quad i = 1, \dots, m \text{ and } j = 1, \dots, n \text{ with } A_{ij} \neq 0,$$

$$\sum_{i=1}^{m} v_i r_i + \sum_{j=1}^{n} w_j c_j \leqslant 0,$$

and either

$$v_i + w_j < 0 \qquad \text{for some} \quad i = 1,\dots,m \text{ and } j = 1,\dots,n \text{ with } A_{ij} \neq 0$$

or

$$\sum_{i=1}^{m} v_i r_i + \sum_{j=1}^{n} w_j c_j < 0.$$

Menon and Schneider (1969) characterized the existence of $(D, E)$-scalings with prespecified row sums and column sums by a finite set of inequalities known as the Menon-Schneider conditions. These conditions assert that if for $I \subseteq \{1,\dots,m\}$ and $J \subseteq \{1,\dots,n\}$, $A_{ij} = 0$ for all pairs $(i, j) \in I \times J^c$, then $\sum_{i \in I} r_i \leqslant \sum_{j \in J} c_j$ and equality holds if and only if, in addition, $A_{ij} = 0$ for all pairs $(i, j) \in I^c \times J$ (superscript $c$ denotes the complement of a set within the relevant class). It was shown in Rothblum and Schneider (1988) that standard results from linear programming imply the equivalence of the Menon-Schneider conditions and condition (c′).

When the problem of finding $(D, D^{-1})$-scaling of a square matrix $A \in R^{n \times n}$ with row sums equaling the corresponding column sums is considered, the substitution of the explicit form of the corresponding matrix $C$ implies that condition (c) of Theorem 2 becomes:

(c″) there exist no vector $v \in R$ for which

$$v_i - v_j \leqslant 0 \qquad \text{for all} \quad i, j = 1,\dots,n \text{ with } A_{ij} \neq 0$$

with strict inequality holding for at least one pair $i$, $j$.

This condition is known to be equivalent to the assertion that if for $I \subseteq \{1,\dots,m\}$ and $J \subseteq \{1,\dots,n\}$, $A_{ij} = 0$ for all $(i, j) \in I \times J^c$, then $A_{ij} = 0$ for all $(i, j) \in I^c \times J$; see Saunders and Schneider (1979).

We observe that using standard techniques from the theory of linear inequalities—e.g., Eaves and Rothblum (1988)—one can obtain a finite set of inequalities that is equivalent to condition (c) of Theorem 2. But, unlike the cases discussed above, the resulting inequalities need not have a simple representation.

Our next result asserts uniqueness of solutions to the generalized scaling problems whenever a solution exists.

Theorem 3 (Uniqueness). *Let $a \in R^n$ be a strictly positive vector. Then $a$ has at most one C-scaling $a'$ which satisfies $Ca' = b$. Further, if $a'$ is a*

*C-scaling of $a$ which satisfies $Ca' = b$ and $u$ is a strictly positive vector in $R^m$ which satisfies*

$$a'_j = a_j\left(\prod_{k=1}^{m} u_k^{C_{kj}}\right), \qquad j = 1,\dots,n, \tag{9}$$

*then*

$$a'_j = a_j\left(\prod_{k=1}^{n} \bar{u}_k^{C_{kj}}\right), \qquad j = 1,\dots,n, \tag{10}$$

*for a strictly positive vector $\bar{u} \in R^m$ if and only if*

$$\sum_{i=1}^{m} (\log u_i)C_{ij} = \sum_{i=1}^{m} (\log \bar{u}_i)C_{ij} \qquad \textit{for each} \quad j = 1,\dots,n. \tag{11}$$

*Proof.* Assume that $u$ and $\bar{u}$ are strictly positive vectors in $R^m$ for which

$$\sum_{j=1}^{n} C_{ij}a_j\left(\prod_{k=1}^{m} u_k^{C_{kj}}\right) = \sum_{i=1}^{m} C_{ij}a_j\left(\prod_{k=1}^{m} \bar{u}_k^{C_{kj}}\right) = b_i, \qquad i = 1,\dots,m.$$

In this case, Theorem 1 implies that $u$ and $\bar{u}$ are both optimal for Program I and therefore the vectors $s$ and $\bar{s}$ in $R^m$ that are defined by $s_i = \log u_i$ and $\bar{s}_i = \log \bar{u}_i$, $i = 1,\dots,m$, are optimal for Program II. As the exponential function is strictly convex and the $a_j$'s are positive, we have that if $sC \neq \bar{s}C$, then

$$\sum_{j=1}^{n} a_j \exp\left[(2^{-1}s + 2^{-1}\bar{s})C\right]_j - b^T(2^{-1}s + 2^{-1}\bar{s})$$

$$< 2^{-1}\left\{\sum_{j=1}^{n} a_j \exp(sC)_j - b^T s\right\} + 2^{-1}\left\{\sum_{j=1}^{n} a_j \exp(\bar{s}C)_j - b^T\bar{s}\right\},$$

contradicting the optimality of $s$ and $\bar{s}$. This contradiction proves that necessarily $sC = \bar{s}C$. So for $j = 1,\dots,n$

$$\sum_{i=1}^{n} (\log u_i)C_{ij} = \sum_{i=1}^{n} s_i C_{ij} = \sum_{i=1}^{n} \bar{s}C_{ij} = \sum_{i=1}^{n} (\log \bar{u})_i C_{ij}. \tag{12}$$

On exponentiation of this equation we get that

$$a_j\left(\prod_{k=1}^{m} u_k^{C_{kj}}\right) = a_j\left(\prod_{k=1}^{m} \bar{u}_k^{C_{kj}}\right), \qquad j = 1,\dots,n.$$

In particular, we have established the uniqueness of a $C$-scaling $a'$ of $a$ that satisfies $Ca' = b$ and that if $a'$ is such a $C$-scaling and $u$ and $\bar{u}$ satisfy (9) and (10), respectively, then necessarily (11) holds [see (12)].

It remains to show that if $a'$, $u$, and $\bar{u}$ satisfy (9) and (11), then (10) must hold. This fact is immediate by exponentiating (11) to get that

$$\prod_{k=1}^{m} u_k^{C_{kj}} = \prod_{k=1}^{m} \bar{u}_k^{C_{kj}}, \qquad j = 1,\dots,n. \qquad \blacksquare$$

We have already observed that when $(D, E)$-scalings of a matrix $A \in R^{m \times n}$ with prespecified row sums and column sums is considered, vectors $u$ of multiplying coefficients are naturally partitioned into a pair of vectors $(v, w) \in R^m \times R^n$. Theorem 3 establishes uniqueness of the desired scaling when one exists and asserts that if $(v, w)$ and $(\bar{v}, \bar{w})$ are multipliers that yield the desired scaling then $v_i / w_j = \bar{v}_i / \bar{w}_j$ for each $i = 1,\dots,m$ and $j = 1,\dots,n$ with $A_{ij} \neq 0$. So, if there is a chain $(i_1, j_1),\dots,(i_q, j_q)$ of indices with $A_{i_t j_t} \neq 0$ for $t = 1,\dots,q$ and $A_{i_t j_{t+1}} \neq 0$ for $t = 1,\dots,q-1$, then for some $\alpha > 0$, we have $v_{i_t}/\bar{v}_{i_t} = \alpha$ and $w_{i_t}/\bar{w}_{i_t} = \alpha^{-1}$ for $t = 1,\dots,q$. Now, by defining an equivalence relation on $\{(i, j) : i = 1,\dots,m \text{ and } j = 1,\dots,n\}$ via the existence of chains as above from one pair to another, for each class we get uniqueness of the coordinates of $v$ and $w$ except for a scalar multiple of $v$ by a constant and of $w$ by the reciprocal of that constant. [See Menon and Schneider (1969, Theorems 3.9, 4.1) for an early derivation of these uniqueness results.] We next observe from similar arguments that when the problem of $(D, D^{-1})$-scalings with row sums equaling the corresponding column sums is considered, we have that if $v$ and $\bar{v}$ are two vectors of multiplying coefficients yielding the same desired scaling, then $v_i / v_j = \bar{v} / \bar{v}_j$ for each pair of $i, j = 1, \ldots, n$ with $A_{ij} \neq 0$. So, by partitioning $\{1, \ldots, n\}$ into classes using the usual equivalence relation defined via the accessibility relation, we have that for each class we have a constant $\alpha$, so that $\bar{v}_i = \alpha v_i$ for each $i$ in the class. [See Saunders and Schneider (1979) or Eaves, Hoffman, Rothblum, and Schneider (1985) for derivations of these uniqueness results.]

Our next result examines $C$-scalings that satisfy approximate versions of the linear system (2). We will use the notation $\| \; \|_\infty$ for the $l_\infty$ norm, i.e., for a vector $x \in R^p$, $\|x\|_\infty = \max_{1 \leqslant i \leqslant p} |x_i|$.

THEOREM 4 (Approximation). *Let $a \in R^n$ be a strictly positive vector. Then the following are equivalent*:

(a) *for every $\varepsilon > 0$ there exists a $C$-scaling $a'$ of $a$ that satisfies $\|Ca' - b\|_\infty \leqslant \varepsilon$;*

(b1) *for every $\varepsilon > 0$, $\{x \in R^n : \|Cx - b\|_\infty \leqslant \varepsilon, x \gg 0\} \neq \varnothing$;*

(b2) $\{x \in R^n : Cx = b, x \geqslant 0\} \neq \varnothing$;

(c) *there exist no $\lambda \in R^n$ for which $\lambda^T C \geqslant 0$ and $\lambda^T b < 0$; and*

(d) *the objective function of Program I is bounded from below.*

*Proof.* (a) $\Rightarrow$ (b1): Suppose that (a) holds, and let $\varepsilon > 0$ be given. Then there exists a $C$-scaling $a'(\varepsilon)$ of $a$ which satisfies $\|Ca'(\varepsilon) - b\|_\infty \leqslant \varepsilon$. Let $b'(\varepsilon) = Ca'(\varepsilon)$. As $a'(\varepsilon)$ is a $C$-scaling of $a$, the implication (a) $\Rightarrow$ (b) of Theorem 1 assures that the set $\{x \in R^n : Cx = b'(\varepsilon), x \gg 0\} \neq \varnothing$. As every $x \in R^n$ satisfying $Cx = b'(\varepsilon)$ satisfies $\|Cx - b\|_\infty = \|b'(\varepsilon) - b\|_\infty = \|Ca'(\varepsilon) - b\|_\infty \leqslant \varepsilon$, we have that $\{x \in R^n : \|Cx - b\|_\infty \leqslant \varepsilon, x \gg 0\} \neq \varnothing$.

(b1) $\Rightarrow$ (a): Suppose that (b1) holds, and let $\varepsilon > 0$ be given. Then $\{x \in R^n : \|Cx - b\|_\infty \leqslant \varepsilon, x \gg 0\} \neq \varnothing$. Let $\bar{x}(\varepsilon)$ be an element in this set and let $\bar{b}(\varepsilon) = C\bar{x}(\varepsilon)$. Then $\{x \in R^n : Cx = \bar{b}(\varepsilon), x \gg 0\} \neq \varnothing$, and the implication (b) $\Rightarrow$ (a) of Theorem 1 assures the existence of a $C$-scaling $a'$ of $a$ for which $Ca' = \bar{b}(\varepsilon)$. In particular, $\|Ca' - b\|_\infty = \|\bar{b}(\varepsilon) - b\|_\infty = \|C\bar{x}(\varepsilon) - b\|_\infty \leqslant \varepsilon$. So $a$ has a $C$-scaling satisfying $\|Ca' - b\|_\infty \leqslant \varepsilon$.

(b1) $\Leftrightarrow$ (b2): This equivalence follows from standard results about linear inequalities; see the Appendix of Rothblum and Schneider (1989).

(b2) $\Leftrightarrow$ (c): It follows from the theorem of the alternative [e.g., Schrijver (1986, Corollary 7.1d, p. 89)] that $\{x \in R^n : Cx = b, x \geqslant 0\} \neq \varnothing$ if and only if there exist no $\lambda \in R^n$ and $\mu \in R^n$ such that $\lambda^T C - \mu^T = 0$, $\mu \geqslant 0$, $\lambda^T b \leqslant 0$, and $\lambda^T b \neq 0$, i.e., there is no $\lambda \in R^n$ with $\lambda^T C \geqslant 0$ and $\lambda^T b < 0$. So, indeed, (b2) $\Leftrightarrow$ (c).

(b2) $\Rightarrow$ (d): Assume that (b2) holds. Thus, there exists a vector $x^* \in R^n$ satisfying $Cx^* = b$ and $x^* \geqslant 0$. Let $J = \{i = 1, \ldots, n : x_i^* > 0\}$. Let $C^J$ be the submatrix of $C$ consisting of the columns of $C$ that correspond to the elements of $J$. As $\{x \in R^{|J|}: C^J x = b, x \gg 0\} \neq \varnothing$, we conclude from the equivalence of (b) and (d) in Theorem 2 that the minimization problem

$$\min \sum_{j \in J} a_j \left( \prod_{k=1}^{m} u_k^{C_{kj}} \right) - \sum_{i=1}^{m} b_i u_i$$

$$\text{s.t.} \quad u \gg 0$$

attains a minimum. The corresponding (minimal) objective value clearly provides a lower bound to the objective value of Program II, as each term $a_j(\Pi_{k=1}^{m} u_k^{C_{kj}})$ for $j \in \{1, \dots, m\} \setminus J$ is positive.

(d) $\Rightarrow$ (c): Suppose that (d) holds, i.e., the objective value of Program I is bounded. To establish a contradiction, assume that there exists a vector $\lambda$ with $\lambda^T C \geqslant 0$ and $\lambda^T b < 0$. For every $M > 0$, let $h(M)$ be the vector in $R^m$ defined by $h(M)_i = \exp(-\lambda_i M)$. Then, as $\lambda^T C \geqslant 0$ and $\lambda^T b < 0$,

$$\begin{aligned} &\sum_{j=1}^{n} a_j \left( \prod_{k=1}^{m} [h(M)_k]^{C_{kj}} \right) - \sum_{i=1}^{m} b_i \log h(M)_i \\ &= \sum_{j=1}^{n} a_j \exp\left( - \sum_{k=1}^{m} \lambda_k M C_{kj} \right) + M \sum_{i=1}^{m} b_i \lambda i \\ &\leqslant \sum_{j=1}^{n} a_j + M \lambda^T b \to -\infty, \end{aligned}$$

contradicting the assertion that the objective function of Program II is bounded from below. This contradiction proves that, indeed, (d) $\Rightarrow$ (c). ∎

Conditions (a), (b1), (b2), (c) and (d) have natural representation when considering $(D, E)$-scalings having prespecified row sums and column sums or $(D, D^{-1})$-scalings with row sums equaling corresponding column sums. They are obtained by using the explicit representation of the corresponding matrices $C$. We do not include the details here.

In Rothblum and Schneider (1988) boundedness from below of the variant of Program I considered, there is shown to be equivalent to the first three conditions listed in Theorem 4 for the problem of finding $(D, E)$-scalings having prespecified row sums and column sums.

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