# Nonsingularity Criteria for General Combinatorially Symmetric Matrices

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Uriel G. Rothblum

ABSTRACT

We present two criteria for nonsingularity of matrices over general fields. The first applies to irreducible acyclic matrices with zero diagonal, the second to arbitrary combinatorially symmetric matrices. Both involve graphs of matrices and a closure operation on sets of vertices of the graphs. We focus attention not on the entries of the matrices under consideration but rather on patterns of zero entries of null vectors of the matrices. Our criteria are of theoretical rather than of computational interest.

## 1. GRAPHS AND GRAPHS OF MATRICES

By a *graph* we mean any undirected graph, which may contain loops, i.e., if graph $G$ has vertex set $V = \langle n \rangle \equiv \{1,\dots,n\}$ and edge set $E$, we allow $\{i,j\} \in E$ for $i=j$ (and call $\{i,i\}$ a *loop* of $G$). Vertex $j$ is a *neighbor* of vertex $i$ if $\{i,j\} \in E$; the set of neighbors of $i$ will be denoted by $N(i)$.

A *path of length* $k$ in $G$ is a sequence $v_0, v_1, \dots, v_k$ of vertices for which $\{v_i, v_{i+1}\} \in E$, $i = 0,1,\dots,k-1$. A path is *closed* if $v_0 = v_k$; a closed path is a *cycle* if $k \geqslant 3$ and $v_0,\dots,v_{k-1}$ are distinct. The graph $G$ is *acyclic* if it has no cycles.




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We shall say that *i is connected to j*, and write $i \sim j$, if either $i = j$ or $i \neq j$ and there exists a path from $i$ to $j$. A set of vertices is said to be *connected* if every pair of its vertices is connected. Connected acyclic graphs are trees or trees with loops. A *component* of a graph is a maximal connected set of its vertices, or, in other words, an equivalence class of the equivalence relation $\sim$.

A subset $\omega$ of $V = V(G)$ is said to be *closed in G* if the cardinality $|N(i)\backslash\omega| \neq 1$ for all $i \in V$, where $N(i)\backslash\omega$ is the complement of $\omega$ in $N(i)$. The collection of all closed sets in $G$ will be denoted by $\Omega(G)$. Since $V(G) \in \Omega(G)$ and $\Omega(G)$ is closed under taking intersections, for each $\alpha \subseteq V(G)$ we may define the closure $\bar{\alpha}$ of $\alpha$ to be the smallest closed set in $G$ containing $\alpha$. Note that if $\alpha \subseteq \beta$ then $\bar{\alpha} \subseteq \bar{\beta}$. The smallest closed set in $V(G)$ is of course $\overline{\varnothing}$.

Let $A$ be an $n \times n$ matrix over an arbitrary field $\mathbf{F}$. We define the *graph of A* to be the graph $G(A)$ with $V(G(A)) = \langle n \rangle$ and

$$E(G(A)) = \{\{i, j\} : a_{ij} \neq 0 \text{ or } a_{ji} \neq 0\}.$$

We say that $A$ is *acyclic* if $G(A)$ is an acyclic graph. If $\alpha \subseteq \langle n \rangle$, let $A[\alpha]$ be the principal submatrix of $A$ based on $\alpha$, and let $A(\alpha) = A[\alpha^c]$, where $\alpha^c = \langle n \rangle \backslash \alpha$. For $i \in \alpha$, let $(i)_\alpha$ be the component of $G(A[\alpha])$ containing $i$. Note that $G(A[\alpha])$ is just the subgraph of $G(A)$ determined by vertex set $\alpha$. A *component* of $A$ is a principal submatrix $A[\alpha]$ for which $\alpha$ is a component of $G(A)$.

## 2. PRELIMINARY RESULTS

The matrix $A$ is *combinatorially symmetric* if

$$a_{ij} \neq 0 \text{ iff } a_{ji} \neq 0, \qquad i, j \in \langle n \rangle.$$

Note that if $A$ is irreducible and acyclic, then it must also be combinatorially symmetric. For any $x \in \mathbf{F}^n$, let $\omega(x)$ be the set of indices of zero entries of $x$, i.e.,

$$\omega(x) = \{i \in \langle n \rangle : x_i = 0\}.$$

PROPOSITION 1. *Let $A \in \mathbf{F}^{n,n}$ be a combinatorially symmetric matrix. If $Ax = 0$, then $\omega(x) \in \Omega(G(A))$. If $\langle n \rangle$ is the only element of $\Omega(G(A))$, then $A$ is nonsingular.*

*Proof.* Suppose that $N(i)\backslash\omega(x) = \{j\}$ for some $j \in \langle n \rangle$. Then

$$(Ax)_i = \sum_{k \in N(i)\backslash\omega(x)} a_{ik}x_k = a_{ij}x_j.$$

Since $A$ is combinatorially symmetric, $a_{ij} \neq 0$; and since $j \notin \omega(x)$, $x_j \neq 0$. This implies that $(Ax)_i \neq 0$, a contradiction. The last statement is now obvious from the first. ∎

EXAMPLE. Both statements of Proposition 1 fail in general for matrices (over any $\mathbf{F}$) which are not combinatorially symmetric. For

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$Ax = 0$, yet $\omega(x) = \{2,3\} \notin \Omega(G(A)) = \{\langle 4 \rangle\}$. (This $A$ is even irreducible.)

PROPOSITION 2. *Let $G$ be a graph with vertex set $\langle n \rangle$, and let $\omega \in \Omega(G)$. There exists a $(0,1,-1)$ matrix $A \in \mathbf{F}^{n,n}$ for which $G(A) = G$ and $Ax = 0$ for the $(0,1)$ vector $x$ with $\omega(x) = \omega$.*

*Proof.* Let $x$ be the $(0,1)$ vector satisfying $\omega(x) = \omega$. For each $i \in \langle n \rangle$, either $N(i) \subseteq \omega$ or $|N(i)\backslash\omega| \geqslant 2$. In either case, we can choose

$$a_{ik} = \begin{cases} 1, & k \in N(i) \cap \omega, \\ 0, & k \notin N(i). \end{cases}$$

In the first case this defines all $a_{ik}$, $k = 1, \dots, h$. In the second case, we must still determine $a_{ik} \in \{0,1,-1\}$, $k \in N(i)\backslash\omega$, so that

$$\sum_{k=1}^{n} a_{ik}x_k = \sum_{k \in N(i)\backslash\omega} a_{ik} = 0 \tag{1}$$

and so that, when $A$ is completely defined,

$$a_{pq} \neq 0 \text{ or } a_{qp} \neq 0 \quad \text{iff} \quad p \text{ and } q \text{ are neighbors in } G. \tag{2}$$

Our proof (and completion of the definition of $A$) will be by induction on $i$.

$i=1$: If $|N(1)\backslash\omega|$ is even, we can choose $a_{1k} \in \{1, -1\}$ for $k \in N(1)\backslash\omega$ so that (1) is satisfied. If $|N(1)\backslash\omega|$ is odd (and at least 3), there exists a $k_1 \in N(1)\backslash\omega$ with $k_1 \geqslant 2$. Without loss of generality $k_1 = 2$, and we can define $a_{12} = 0$ and choose $a_{1k} \in \{1, -1\}$ for the other $k \in N(1)\backslash\omega$ so that (1) holds.

$i=2$: If $|N(2)\backslash\omega|$ is even, again we can choose $a_{2k} \in \{1, -1\}$ for $k \in N(2)\backslash\omega$. If $|N(2)\backslash\omega|$ is odd (and at least 3), there exists a $k_2 \in N(2)\backslash\omega$ with $k_2 \geqslant 3$. Without loss of generality $k_2 = 3$, and we can define $a_{23} = 0$ and choose $a_{2k} \in \{1, -1\}$ for the other $k \in N(2)\backslash\omega$. In either case, as 1 and 2 are neighbors in $G$, we will have $a_{21} \neq 0$ as required.

Now assume we have completely defined rows $1, 2, \dots, j-1$ of $A$ so that, for $1 \leqslant i \leqslant j-1$, (1) holds, so that

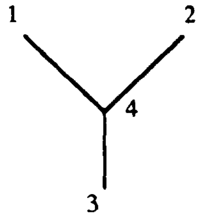
$$a_{ik} = 0 \quad \text{for } k \in N(i)\backslash\omega \qquad \text{only if} \quad k \leqslant i+1,$$

and so that, for $1 \leqslant p, q \leqslant j-1$, (2) holds.

$i=j$: If $|N(j)\backslash\omega|$ is even, again we can choose $a_{jk} \in \{1, -1\}$ for $k \in N(j)\backslash\omega$. If $|N(j)\backslash\omega|$ is odd (and at least 3), there exists a $k_j \in N(j)\backslash\omega$ with $k_j \notin \{j-1, j\}$. If there exists such a $k_j \geqslant j+1$, without loss of generality $k_j = j+1$, and we can define $a_{jj+1} = 0$ and choose $a_{jk} \in \{1, -1\}$ for the other $k \in N(j)\backslash\omega$. If $a_{jk} = 0$, $k = j+1, \dots, n$, then we must have $k_j = h \leqslant j-2$. Let $a_{jh} = 0$, and choose $a_{jk} \in \{1, -1\}$ for the other $k \in N(j)\backslash\omega$. We must have $a_{hj} \neq 0$, as $h$ and $j$ are neighbors in $G$ and $j > h+1$. For all other $1 \leqslant p < j-1$, $q = j$, (2) clearly holds. ∎

NOTE. Given a graph $G$ and $\omega \in \Omega(G)$, for any field $\mathbf{F} \neq \mathrm{GF}(2)$ we can prove that there exists a combinatorially symmetric matrix $A \in \mathbf{F}^{n,n}$ for which $G(A) = G$ and $Ax = 0$ for the $(0,1)$ vector $x$ with $\omega(x) = \omega$. The proof relies on the fact that, in any $\mathbf{F} \neq \mathrm{GF}(2)$, for any positive integer $m \geqslant 2$ there exist nonzero $a_1, \dots, a_m \in \mathbf{F}$ whose sum is zero. We cannot do this in GF(2), and the existence of an appropriate combinatorially symmetric $A$ cannot be guaranteed.

To see this, if $G$ is




then $\omega = \{4\} \in \Omega(G)$. The only combinatorially symmetric $A$ with $G(A) = G$ is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and for the vector $x$ for which $\omega(x) = \omega = \{4\}$, $Ax \neq 0$.

## 3. ACYCLIC MATRICES WITH ZERO DIAGONAL

Our principal result in this section will show that if $\mathbf{F} \neq \mathrm{GF}(2)$, an irreducible and acyclic matrix $A \in \mathbf{F}^{n,n}$ with zero diagonal is nonsingular iff $G(A)$ has no proper closed sets. Note that an irreducible acyclic matrix with zero diagonal is a matrix whose graph is a tree (without loops).

PROPOSITION 3. *Let $A, B \in \mathbf{F}^{n,n}$ be irreducible acyclic matrices with zero diagonals, and suppose $G(A) = G(B)$. Then there exist nonsingular diagonal matrices $X, Y \in \mathbf{F}^{n,n}$ for which $B = XAY$.*

*Proof.* The proof follows closely that of case 1 in Theorem 1 of [1]. ■

THEOREM 1. *Let $\mathbf{F} \neq \mathrm{GF}(2)$. Let $A \in \mathbf{F}^{n,n}$ be irreducible and acyclic, with zero diagonal. Then, for every $\omega \in \Omega(G(A))$, there exists a vector $x \in \mathbf{F}^n$ for which $\omega(x) = \omega$ and $Ax = 0$.*
*The following are thus equivalent*:

(i) *$A$ is nonsingular*,
(ii) $\overline{\varnothing} = \langle n \rangle$,
(iii) *$\langle n \rangle$ is the only element of $\Omega(G(A))$.*

*Proof.* Let $A$ be a combinatorially symmetric matrix, and let $\omega \in \Omega(G(A))$. As $\mathbf{F} \neq \mathrm{GF}(2)$, by the note after Proposition 2 there exists a combinatorially symmetric matrix $B \in \mathbf{F}^{n,n}$ and a $(0,1)$ vector $y$ for which $G(B) = G(A)$, $\omega(y) = \omega$, and $By = 0$. $B$ must also be irreducible and acyclic. By Proposition 3, there exists nonsingular diagonal matrices $X, Y$ for which $B = XAY$. Now $0 = By = XAYy$ implies $Ax = 0$ for $x = Yy$, and $\omega(x) = \omega(y) = \omega$.

That (i), (ii), and (iii) are now equivalent is immediate from the above and Proposition 1. ∎

EXAMPLES. The requirement that $A$ have zero diagonal cannot be deleted: take, over $\mathbf{F}$ of characteristic not 2,

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

The theorem also fails for general combinatorially symmetric matrices; consider, over $\mathbf{F}$ of characteristic not 2, the nonsingular matrix

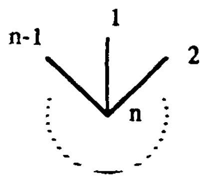
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

As corollaries of Theorem 1, we mention the following two statements. Each is easily proved independently, and each provides motivation for the results of this paper.

COROLLARY 1. *An $n \times n$ irreducible tridiagonal matrix with zero diagonal is nonsingular iff $n$ is even.*

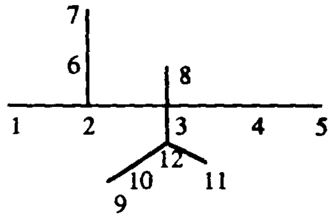
COROLLARY 2. *An irreducible matrix with zero diagonal whose graph is a star is always singular.*

A star graph has the form, for $n \geqslant 3$,




For such a graph $G$, $\overline{\varnothing} = \{n\}$. For $n = 4$, the star graph and the example matrix following Proposition 2 give a counterexample to the first statement of Theorem 1 for $\mathbf{F} = \mathrm{GF}(2)$.

EXAMPLE. It is easy to calculate using Theorem 1 that if $G$ is the graph




then any irreducible matrix $A$ for which $G(A)=G$ is nonsingular, as $\overline{\varnothing}=\langle n\rangle$ in $G$.

## 4. A SPECIAL CLOSED SET

In this section we give an algorithmic definition of a special closed set in $\langle n\rangle$ to be used in the next section for the determination of nonsingularity.

Let $A$ be an $n\times n$ matrix over an arbitrary field $\mathbf{F}$, and let $G=G(A)$. We first define a closed set $\varphi(A)\subseteq\langle\omega\rangle$ by the algorithm in Figure 1. This

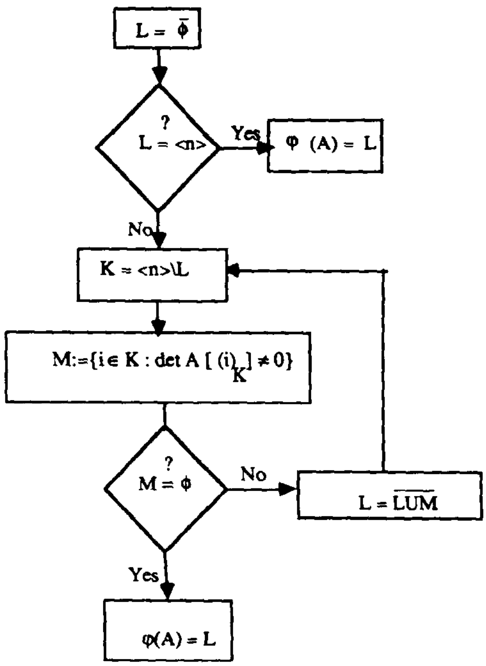



FIG. 1.

algorithm is very similar to the one for constructing $Q_\lambda(A)$ defined in [3]. Observe also that in determining $M$ we need only consider $(i)_K$ for which $|(i)_K| > 1$; any $(i)_K$ with $\det A[(i)_K] \neq 0$ will have $|(i)_K| > 1$. To see this, suppose that for some $i \in K$ we have $(i)_K = \{i\}$ and $a_{ii} \neq 0$. Then $|N(i)\setminus L = \{i\}| = 1$, a contradiction to $L = \overline{L}$.

We have $\overline{\emptyset} \neq \emptyset$ iff $N(i) = \{j\}$ for some $i, j \in \langle n \rangle$. If $G(A)$ is connected and $\overline{\emptyset} = \emptyset$, then $K = \langle n \rangle$ and

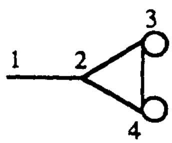
$$\varphi(A) = M = \begin{cases} \langle n \rangle, & \det A \neq 0, \\ \emptyset, & \det A = 0. \end{cases}$$

If $\overline{\emptyset} \neq \emptyset$, we do not have to calculate $\det A$. This is however of no practical value, as any reasonable computational procedure to determine $\det A$ would first reduce the problem by omitting successively rows and columns which have a single nonzero entry; the reduced matrix would have $\overline{\emptyset} = \emptyset$.

Examples. Over $\mathbf{F}$ of characteristic not two,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

$G(A) = G(B)$:




$$\overline{\emptyset} = \{2\}, \qquad K = \{1\} \cup \{3,4\}.$$

$$\det A[3,4] = 0, \qquad \varphi(A) = \overline{\emptyset} = \qquad \{2\}; \qquad \det A = 0;$$

$$\det B[3,4] = -1, \qquad \varphi(B) = \overline{\{2\} \cup \{3,4\}} = \langle 4 \rangle; \qquad \det B = 1.$$

In fact, as we show next, our algorithm never requires more steps than required to calculate $\varphi(B)$ above. It actually has the form shown in Figure 2. That this is true follows from the fact that the components of $\langle n \rangle \setminus \overline{\overline{\emptyset} \cup M}$ are components of $\langle n \rangle \setminus \overline{\emptyset}$. And this follows from an examination of cases, using the following.

Lemma. *Suppose $L \equiv \overline{\emptyset} \neq \langle n \rangle$, $K = \langle n \rangle \setminus L$. Then every exterior vertex $l \in L$ (i.e. $N(l) \cap K \neq \emptyset$) neighbors an isolated vertex $k$ of $K$ (i.e., $(k)_K = \{k\}$).*

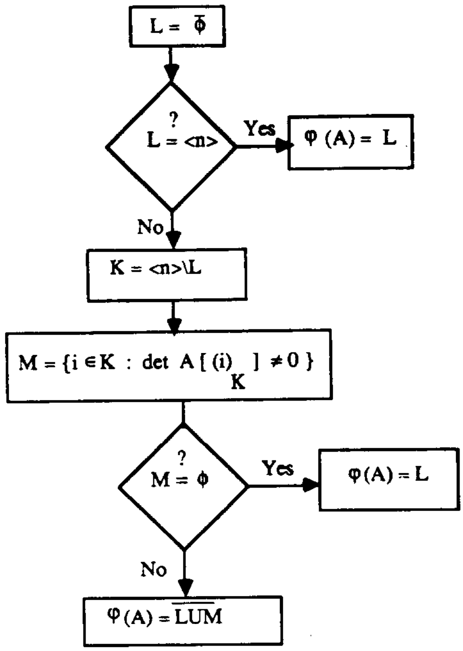



FIG. 2.

*Proof.* There exists an ordering $l_1, \dots, l_q$ of the elements of $L$ and distinct $v_1, \dots, v_q$ of $\langle n \rangle$ for which

$$\{l_1\} = N(v_1), \ \{l_2\} = N(v_2) \setminus \{l_2\}, \dots, \ \{l_q\} = N(v_q) \setminus \{l_1, \dots, l_{q-1}\}.$$

Suppose $L$ has $p$ exterior vertices, and $q - p$ vertices for which $N(l) \subseteq L$. Suppose $v_{i_1}, \dots, v_{i_s} \in K$ and $v_{i_1}, \dots, v_{i_{q-s}} \in L$. Then $l_{i_1}, \dots, l_{i_s}$ are distinct vertices of $L$ neighboring vertices of $K$, and $s \leqslant p$. On the other hand,

$$N(v_{j_t}) \subseteq \{l_i, \dots, l_{i_t}\} \subseteq L, \qquad t = 1, \dots, q - s,$$

so that $q - s \leqslant q - p$, i.e. $p \leqslant s$. Thus $p = s$. Finally,

$$N(v_{i_t}) \leqslant \{l_i, \dots, l_{i_t}\} \subseteq L, \qquad t = 1, \dots, p,$$

so that $(v_{j_t})_K = \{v_{j_t}\}$, and the exterior vertices $l_{i_1}, \dots, l_{i_s}$ all neighbor isolated vertices of $K$. ■

## 5. COMBINATORIALLY SYMMETRIC MATRICES WITH (POSSIBLY) NONZERO DIAGONAL ENTRIES

The nonsingularity criterion given in Theorem 1 is purely graph-theoretic [and does not apply to matrices over GF(2)]. In this section we give, in Theorem 2, a nonsingularity criterion involving $\varphi(A)$ which is *not* purely graph-theoretic and which holds over an arbitrary field.

PROPOSITION 4. *Let $A \in \mathbf{F}^{n,n}$ be combinatorially symmetric. If $Ax = 0$, then $\varphi(A) \subseteq \omega(x)$. Thus if $A$ is also singular, $\varphi(A) \neq \langle n \rangle$.*

*Proof.* Suppose $Ax = 0$; then $\omega = \omega(x) \in \Omega(G(A))$ by Proposition 1. Let $L \subseteq \omega$ and $K = \langle n \rangle \backslash L$. Then $A[K]x[K] = 0$, and for every $i \in K$, $A[(i)_K]x[(i)_K] = 0$. Whenever $A[(i)_K]$ is nonsingular, then $x[(i)_K] = 0$ and $(i)_K \subseteq \omega$. As $\varnothing \subseteq \omega$, $\overline{\varnothing} \subseteq \omega$, and it follows by the definitions of $\varphi(A)$ and of closure that in every step of the algorithm we have $L \subseteq \omega$, hence $\varphi(A) \subseteq \omega$.
The second statement is obvious. ■

COROLLARY 3. *Let $\mathbf{F} \neq$ GF(2), and let $A \in \mathbf{F}^{n,n}$ be irreducible and acyclic, with zero diagonal. Then $\varphi(A) = \overline{\varnothing}$.*

*Proof.* By Theorem 1, $\omega(x) = \overline{\varnothing}$ for some $x$, $Ax = 0$. By Proposition 4, $\varphi(A) \subseteq \overline{\varnothing}$. But also $\overline{\varnothing} \subseteq \varphi(A)$. ■

PROPOSITION 5. *Let $A \in \mathbf{F}^{n,n}$. If $\varphi(A) \neq \langle n \rangle$, then $A$ is singular.*

*Proof.* Our proof will be by induction on $n$. For $n = 1$, if $\varphi(A) \neq \langle n \rangle$ then $\varphi(A) = \varnothing$ and $A$ is singular. We assume the result true for matrices of order less than $n$, and consider a matrix $A$ of order $n$ for which $\varphi(A) \neq \langle n \rangle$. If $\overline{\varnothing} = \varnothing$, then $\varphi(A) \neq \langle n \rangle$ [and in fact $\varphi(A) = \varnothing$] iff $A$ is singular. If $\varnothing \neq \overline{\varnothing}$, then $G(A)$ has a vertex $i$ with only one neighbor $j$. If $j = i$, then $a_{ii} \neq 0$, $G(A[i])$ is a component of $G(A)$, $A$ is singular iff $A(i)$ is, and $\varphi(A) = \varphi(A(i)) \cup \{i\}$. As $\varphi(A) \neq \langle n \rangle$, then $\varphi(A(i)) \neq \langle n \rangle \backslash \{i\}$, and by induction $A(i)$ and $A$ are singular. We shall assume $j \neq i$. If $a_{ij} = 0$ or $a_{ji} = 0$, then $A$ would have a zero row or column and would be singular. Thus we may also assume $a_{ij} \neq 0$ and $a_{ji} \neq 0$, and

$$\det A = \pm a_{ij} a_{ji} \det A(i, j). \tag{3}$$

We next note that

$$\varphi(A) = \varphi(A(i, j)) \cup \sigma, \tag{4}$$

where

$$\sigma = \begin{cases} \{i, j\}, & N(j) \setminus \{i, j\} \subseteq \varphi(A(i, j)), \\ \{j\}, & N(j) \setminus \{i, j\} \not\subseteq \varphi(A(i, j)). \end{cases}$$

The proof of (4) requires the definition of $\varphi(A)$ and that of the closure of a subset $L$ of $\langle n \rangle$ as the smallest closed set containing $L$, and the application of these ideas to $A$ and $A(i, j)$.

Finally, if $\varphi(A) = \varphi(A(i, j)) \cup \{i, j\}$ and $\varphi(A) \neq \langle n \rangle$, then $\varphi(A(i, j)) \neq \langle n \rangle \setminus \{i, j\}$; and if not, as $N(j) \setminus \{i, j\} \subseteq \varphi(A(i, j))$, again $\varphi(A(i, j)) \neq \langle n \rangle \setminus \{i, j\}$. In either case, by induction, $\det A(i, j) = 0$, by (4) $\det A = 0$, and $A$ is singular. ■

Combining Propositions 4 and 5, we obtain immediately our second theorem.

THEOREM 2. *Let $A \in \mathbf{F}^{n,n}$ be combinatorially symmetric. Then $A$ is nonsingular iff $\varphi(A) = \langle n \rangle$.*

For surveys of other nonsingularity criteria involving combinatorial criteria, see [2] and [4].

## 6. AN EXTENSION FOR IRREDUCIBLE ACYCLIC MATRICES

Theorem 2 implies that, for any combinatorially symmetric $n \times n$ matrix $A$, if $L$ is a proper closed subset of $\langle n \rangle$ obtained by applications of the original algorithm to $\overline{\varnothing}$, and if all components of $A[\langle n \rangle \setminus L]$ are singular, then $A$ is singular. In this section we show that if $A$ is actually irreducible acyclic, this property holds for all proper closed subsets $L$ of $\langle n \rangle$.

THEOREM 3. *Let $A \in \mathbf{F}^{n,n}$ be irreducible and acyclic, and let $L$ be a proper closed subset of $\langle n \rangle$. If all components of $A[\langle n \rangle \backslash L]$ are singular, then $A$ is singular.*

*Proof.* We prove the theorem by induction on $n$. Whenever $L = \varnothing$, we have nothing to prove. If $n = 1$ or $n = 2$, the only possible $L$ is the empty set. Assume the assertion holds for $n < m$, and let $n = m$.

We can assume $L \neq \varnothing$. Let $C$ be a component of $G(A[L])$. Let $M = \langle n \rangle \backslash C$, and let $S_1$ be the set of all components of $G(A[M])$.

Since $A$ is irreducible and acyclic, it is combinatorially symmetric, and $G(A)$ is connected. Since also $C \neq \langle n \rangle$, there exists an $i \in C$ such that $\{i, j\} \in E$ for some $j \in M$. Given that $i \in C$, let

$$S_2 = \{\alpha \in S_1 : \{i, j\} \in E \text{ for some } j \in \alpha\} = \{\alpha_1, \dots, \alpha_h\}.$$

For $k = 1, \dots, h$, as $C$ and $\alpha_k$ are components of $G(A[L])$ and $G(A[M])$ respectively and as $A$ is acyclic, there exists a unique $j_k \in \alpha_k$ for which $\{i, j_k\} \in E$. Since $C$ is a component of the closed set $L$, we must have $h > 1$. Observe that for $k = 1, \dots, h$, $\alpha_k \cap L \neq \alpha_k$ as $j_k \in L$.

Observe also that for $k = 1, \dots, h$, every component of $A[\alpha_k \backslash L]$ is a component of $A[\langle n \rangle \backslash L]$, so that we may apply the inductive hypothesis to $A[\alpha_k]$, obtaining a nonzero vector $z^k$ for which $A[\alpha_k] z^k = 0$. If for some $k$, $k = 1, \dots, h$, we have $z_{j_k}^k = 0$, then we construct a nonzero vector $x$ for which $Ax = 0$ by

$$x_t = \begin{cases} z_t^k, & t \in \alpha_k, \\ 0, & t \notin \alpha_k. \end{cases}$$

If $z_{j_k}^k \neq 0$, $k = 1, \dots, h$, we define $h$ vectors

$$y^k = \left(a_{i j_k} z_{j_k}^k\right)^{-1} z^k, \qquad k = 1, \dots, h-1,$$

$$y^h = -\left[(h-1) a_{i j_h} z_{j_h}^h\right]^{-1} z^h.$$

(As $A$ is combinatorially symmetric, $a_{i j_k} \neq 0$, $k = 1, \dots, h$.) Observe that $y^k$ is a scalar multiple of $z^k$, $k = 1, \dots, h$, and that

$$\sum_{k=1}^{h} a_{i j_k} y_{j_k}^k = 0.$$

Thus, the nonzero vector $x$ defined by

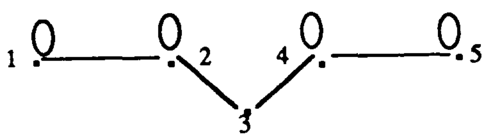
$$x_t = \begin{cases} y_t^k, & t \in \alpha_k, \quad k = 1, \ldots, h, \\ 0, & \text{otherwise.} \end{cases}$$

satisfies $Ax = 0$. ■

EXAMPLE. Theorem 3 is false if we replace the condition "all components of $A[\langle n \rangle \backslash L]$ are singular" by "$A[\langle n \rangle \backslash L]$ is singular," as is demonstrated by the following matrix. Over **F** of the characteristic not two,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

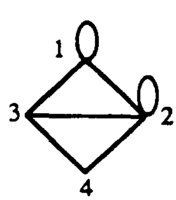
The matrix $A$ is nonsingular. Observe that

$G(A) =$ 


Let $L$ be the set $\{3\}$, which is closed. The matrix $A[\langle 5 \rangle \backslash \{3\}]$ is singular, but $A$ is nonsingular. Observe that $A[1,2]$ is a singular component of $A(3)$, while $A[4,5]$ is a nonsingular component of $A(3)$.

EXAMPLE. Theorem 3 does not hold in general for matrices whose graphs are not acyclic. For example consider the following nonsingular matrix over **F** of characteristic not two:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

whose graph is 

If we choose $L = \{4\}$, then $L$ is closed and the only component of $A(4)$ is the singular matrix $A[1,2,3]$.

## 7. CAN WE AVOID EVALUATING det $A[(i)_k]$?

As presented, our algorithm for $\varphi(A)$ is purely graph-theoretic in two cases:

if $\overline{\varnothing} = \langle n \rangle$, then $\varphi(A) = \langle n \rangle$ and $A$ is nonsingular;
if $\overline{\varnothing} \neq \langle n \rangle$, but all $|(i)_{\langle n \rangle \setminus \overline{\varnothing}}| = 1$, then $\varphi(A) = \overline{\varnothing} \neq \langle n \rangle$ and $A$ is singular.

On the other hand, as we have seen, we must calculate det $A$ in the case $\varnothing = \overline{\varnothing}$.

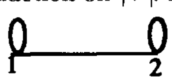
More generally, an interesting question is whether we can improve Theorem 2 by finding a graph-theoretical way to avoid the evaluations of the determinants det $A[(i)_K]$ in the algorithm to construct $\varphi(A)$. We can now show that such an improvement is impossible for acyclic combinatorially symmetric matrices over $\mathbf{F} \neq \mathrm{GF}(2)$.

Observe that in each step of the algorithm, for all $i \in K$, $\varnothing \in \Omega(G(A[(i)_K]))$. It will be enough to show that for any acyclic graph $G$ with $\varnothing = \overline{\varnothing}$ there exist both a singular $A$ and a nonsingular $B$ which are combinatorially symmetric and for which $G(A) = G(B) = G$. The existence of such a singular $A$ was established in Proposition 2 (actually, in the Note following Proposition 2).

Before showing the existence of a nonsingular $B$, we first define a *matching* $M$ on a graph $G$ to be set of pairwise disjoint edges which *covers* the vertices incident upon its edges. The matching $M$ is *complete* if $M$ covers $V$, the vertex set of $G$.

PROPOSITION 6. *Let $G$ be a connected acyclic graph with $|V| > 1$ and $\varnothing = \overline{\varnothing}$. Then there exists a complete matching of $V$.*

*Proof.* Our proof is by induction on $|V|$. If $|V| = 2$, then $G$ has the form



and $\{\{1,1\},\ \{2,2\}\}$ and $\{\{1,2\}\}$ are both complete matchings. Assume the assertion for $|V| < n$ and let $|V| = n$. As $G$ is acyclic and $|V| > 1$, there exists a vertex $i$ for which $N(i) \setminus \{i\} = \{j\}$ for some $j \neq 1$. As $\varnothing = \overline{\varnothing}$, we have $N(i) = \{i, j\}$.

*Case 1.* $j \in N(j)$. In this case $G \backslash \{i\}$ satisfies the hypotheses of the proposition, and by induction there is a complete matching of $V \backslash \{i\}$. Adding $\{i, i\}$ yields a complete matching of $V$.

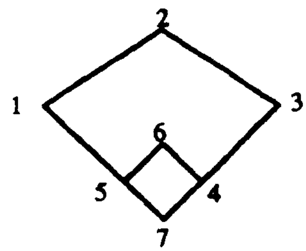
*Case 2.* $j \notin N(j)$. In this case we define graph $G'$ as the graph $G \backslash \{i\}$ with the edge $\{j, j\}$ added. Clearly $G'$ satisfies the hypotheses, and by induction there is a complete matching on $G'$. If $\{j, j\}$ is part of the complete matching on $G'$, replacing it by $\{i, j\}$ yields a complete matching on $G$. If $\{j, j\}$ is not part of the complete matching on $G'$, we add $\{i, i\}$ to obtain a complete matching on $G$. ∎

Returning to our discussion of the algorithm for $\varphi(A)$, let $C$ be a combinatorially symmetric matrix whose graph is $G$. If, for simplicity of notation, the complete matching obtained in Proposition 6 consisted of edges $\{1,2\},\dots,\{2l-1,2l\}$ and loops $\{2l+1,2l+1\},\dots,\{n,n\}$, then

$$a_{12}a_{21} \cdots a_{2l-1,2l}a_{2l,2l-1}a_{2l+1,2l+1} \cdots a_{n,n}$$

is a nonzero diagonal product, and by Proposition 3 of [2] there is a combinatorially symmetric nonsingular $B$ for which $G(B) = G(C) = G$.

Proposition 6 does not hold in general for non-acyclic graphs, as demonstrated by the graph




which has no complete matching. In view of this example, one might still hope to find a further graph-theoretic approach in determining the singularity of a non-acyclic matrix.

## 8. A RANK RESULT FOR COMBINATORIALLY SYMMETRIC MATRICES

Let $A$ be an $n \times n$ matrix over an arbitrary field $\mathbf{F}$, and let $S \subseteq \langle n \rangle$. We define the set

$$\gamma(S) = \{i \in S : N(i) \subseteq S\},$$

and we denote by $\rho(A)$ and $\nu(A)$ the rank and nullity respectively of $A$.

THEOREM 4. *Let $A \in \mathbf{F}^{n,n}$ be combinatorially symmetric. Then*

$$\nu(A) = \sum_{\alpha} \nu(A[\alpha]) - |\gamma(\varphi(A))|,$$

*where the summation is over the set of components of $G(A(\varphi(A)))$.*

*Proof.* Again, our proof will be by induction on $n$. As $A$ is combinatorially symmetric, it may be thought of as a direct sum of irreducible matrices. In fact, to prove the theorem it is sufficient to assume that $A$ itself is irreducible. If $n = 1$ the result is trivial. Assume that it holds for $n < m$, and let $n = m > 1$. If either $\varphi(A) = \varnothing$ or $\varphi(A) = \langle n \rangle$, there is nothing to prove, so we may assume that $\varnothing \subsetneqq \varphi(A) \subsetneqq \langle n \rangle$. This implies that $\varnothing \neq \overline{\varnothing}$. There exist $i, j \in \langle n \rangle$, $i \neq j$, for which $N(j) = \{i\}$. We distinguish two cases.

*Case 1.* $i \in \gamma(\varphi(A))$ *i.e.,* $i \in \varphi(A)$, *but* $N(i) \subseteq \varphi(A)$. In this case, $i$ has at least two neighbors not in $\varphi(A)$. As $i$ is the only neighbor of $j$, we have $a_{jj} = 0$ and $j \notin \overline{\varnothing}$. It follows that $j \notin \varphi(A)$. In this case,

$$\varphi(A(i)) = \varphi(A) \setminus \{i\}. \tag{5}$$

To see this, note first that

$$\overline{\varnothing}' = \overline{\varnothing} \setminus \{i\},$$

where $\overline{\varnothing}'$ is the closure of $\varnothing$ in $G(A(i))$, and that the components of $K = \langle n \rangle \setminus \overline{\varnothing}$ and $K' = (\langle n \rangle \setminus \{i\}) \setminus \overline{\varnothing}'$ are identical. From (5), the components of $G(B(\varphi(B)))$, where $B = A(i)$, are those of $G(A(\varphi(A)))$.

Applying the inductive assumption to $A(i)$, we get

$$\nu(A) = \sum_{\alpha} \nu(A[\alpha]) - |\gamma(\varphi(A))| + 1, \tag{6}$$

where $\alpha$ runs through the components of $G(A(\varphi(A)))$.

As is well known,

$$\rho(A) \leqslant \rho(A(i)) + 2,$$

which is equivalent to

$$\nu(A) \geqslant \nu(A(i)) - 1. \tag{7}$$

Let $N(A)$ denote the null space of $A$, and let $\tilde{N}(A)$ denote the set of vectors in $N(A)$ with $i$th coordinate deleted. Since $i \in \varphi(A)$, by Proposition 4 we have

$$\tilde{N}(A) \subseteq N(A(i)). \tag{8}$$

As $a_{kj} = \delta_{ik}$ (Kronecker delta), the vector $e_j$ with coordinates $\delta_{kj}$ (but $i$th coordinate deleted) belongs to $N(A(i))$ but not $\tilde{N}(A)$. Thus we have strict containment in (8), so that by (7) $\nu(A) \geqslant \nu(A(i)) - 1$. The result in this case now follows from (6).

*Case* 2. $i \notin \gamma(\varphi(A))$, *i.e.*, $i \in \varphi(A)$ (*as* $i \in \overline{\varnothing}$) *but* $N(i) \subseteq \varphi(A)$. In this case $j$ must be the only vertex whose only neighbor is $i$. (If $k$ were another such vertex, then $j, k \notin \overline{\varnothing}$, $a_{jj} = a_{kk} = 0$, so $j, k \notin \varphi(A)$, a contradiction.) Whether $j \in \overline{\varnothing}$ or $j \notin \overline{\varnothing}$,

$$\varphi(A(i)) = \varphi(A) \setminus \{i, j\},$$

by reasoning similar to that in Case 1 for (5). Also, the set of components of $G(B(\varphi(B))$ $[B = A(i)]$ is the set of components of $G(A(\varphi(A))$ together with $\{j\}$. Since $\gamma(\varphi(B)) = \gamma(\varphi(A))$, the induction hypothesis again yields (6), and the proof concludes as in Case 1. ∎

COROLLARY 4. *If* $\mathbf{F} \neq \mathrm{GF}(2)$ *and* $A$ *is combinatorially symmetric and acyclic with zero diagonal entries, then*

$$\nu(A) = n - |\overline{\varnothing}| - |\gamma(\overline{\varnothing})|.$$

*Proof.* By Corollary 3 $\varphi(A) = \overline{\varnothing}$. Also, every component of $A(\varphi(A))$ has order 1. ∎

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