The Association Schemes of Dual Polar Spaces of Type ${}^2A_{2d-1}(p^f)$ Are Characterized by Their Parameters If $d \geqslant 3$

A. A. Ivanov
Institute for System Studies
Academy of Sciences of the USSR
9, Prospect 60 Let Oktyabrya
117312, Moscow, USSR

and

S. V. Shpectorov
Institute of Problems of Cybernetics
Academy of Sciences of the USSR
37, Vavilov Street
117312, Moscow, USSR

Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

It is shown that the cliques in a distance-regular graph Γ whose parameters are that of the dual polar space graph of type ${}^2A_{2d-1}(p^f)$ have size p^f+1. This means that Γ is the point graph of a regular near $2d$-gon. If $d=2$, we obtain the well-known result due to P. J. Cameron, J. M. Goethals, and J. J. Seidel that a pseudogeometric graph with parameters of the point graph of a generalized quadrangle of type (q,q^2) is geometric. If $d \geqslant 3$, then some results due to P. J. Cameron, E. E. Shult, A. Yanushka, and J. Tits on near $2d$-gons imply that Γ coincides with the dual polar space graph of type ${}^2A_{2d-1}(p^f)$.

1. DEFINITIONS AND PRELIMINARIES

A symmetric association scheme with d classes (or simply scheme) $Y=(X,\{R_i\}_{0\leqslant i\leqslant d})$ consists of a finite set X and symmetric relations

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$R_0, R_1, \ldots, R_d$ on X possessing the following properties:

(i) $R_0 = \{(x, x) : x \in X\}$—the diagonal relation;

(ii) if $x, y \in X$, then $(x, y) \in R_i$ for exactly one i, $0 \leqslant i \leqslant d$;

(iii) if $0 \leqslant h, i, j \leqslant d$ and $(x, y) \in R_h$, then the number p_{ij}^h of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ depends only on h, i and j (the numbers p_{ij}^h are usually called the intersection numbers).

The adjacency matrices of the scheme Y are the matrices $A_0 = I, A_1, \ldots, A_d$ with the rows and columns indexed by the elements of X, where

$$(A_i)_{xy} = \begin{cases} 1 & \text{if} \quad (x, y) \in R_i, \\ 0 & \text{otherwise} \end{cases}$$

$(x, y \in X,\ 0 \leqslant i \leqslant d)$.

By (iii) the following equalities hold:

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \qquad (0 \leqslant i, j \leqslant d).$$

So the adjacency matrices span an algebra $\mathfrak{A}(Y)$ of dimension $d + 1$ over $\mathbb{R}$, which is known as the Bose-Mesner algebra.

Let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of $\mathfrak{A}(Y)$. We assume that $E_0 = (1/n)J$, where J is the all-unit matrix and $n = |X|$. The first and the second eigenmatrices P and Q of the scheme Y are the square matrices of size $d + 1$ with the (i, j) elements denoted by $p_j(i)$ and $q_j(i)$ such that

$$A_j = \sum_{i=0}^{d} p_j(i) E_i, \tag{1}$$

$$E_j = \frac{1}{n} \sum_{i=0}^{d} q_j(i) A_i. \tag{2}$$

The Krein parameters q_{ij}^h $(0 \leqslant h, i, j \leqslant d)$ are defined in the following way:

$$E_i \circ E_j = \frac{1}{n} \sum_{h=0}^{d} q_{ij}^h E_h,$$

where $\circ$ denotes the Hadamard (elementwise) product of matrices.

The $p_{ij}^h, q_{ij}^h, p_j(i), q_j(i)$ are the parameters of the association scheme Y. All of them are determined uniquely, for instance, by the intersection numbers.

Let $\mathfrak{A}(Y)$ act on the Hermitian space V with the inner product $\langle \ , \ \rangle$ and having the orthonormal basis indexed by the elements of the set X. Then V is the direct sum $V = V_0 \oplus V_1 \oplus \cdots \oplus V_d$ of the eigenspaces of the algebra $\mathfrak{A}(Y)$. Let π_j be the orthogonal projection of V onto V_j. We assume that the subspaces V_j are ordered in such a way that E_j is the matrix expressing π_j in the original basis. Since $E_j^2 = E_j$, ${}^tE_j = E_j$, $0 \leqslant j \leqslant d$, the matrix E_j coincides with the Gram matrix of the system of vectors $\{\pi_j(x) : x \in X\}$. So (2) implies that if $(x, y) \in R_i$ then

$$\left\langle \pi_j(x), \pi_j(y) \right\rangle = \frac{1}{n} q_j(i).$$

The scheme Y is called P-polynomial if

$$p_{1i}^j = 0 \quad \text{for } |i-j| \geqslant 2 \quad \text{and} \quad p_{1i}^j \neq 0 \quad \text{for } |i-j| = 1$$

for some ordering $R_0, R_1, \ldots, R_d$ of the relations. The following symbols are very useful:

$$b_i = p_{1,i+1}^i, \quad a_i = p_{1i}^i, \quad c_i = p_{1,i-1}^i, \quad k = b_0 = b_i + a_i + c_i, \qquad 0 \leqslant i \leqslant d.$$

P-polynomiality is equivalent to distance-regularity of the graph Γ with vertex set X and edge set R_1. The diameter of Γ is d, and the valency of Γ is k.

The association scheme Y is Q-polynomial if for some ordering of the primitive idempotents $E_0, E_1, \ldots, E_d$ the following condition holds:

$$q_{1i}^j = 0 \quad \text{for } |i-j| \geqslant 2 \quad \text{and} \quad q_{1i}^j \neq 0 \quad \text{for } |i-j| = 1.$$

In this case

$$b_i^* = q_{1,i+1}^i, \quad a_i^* = q_{1i}^i, \quad c_i^* = q_{1,i-1}^i, \quad m = b_0^* = b_i^* + a_i^* + c_i^*, \qquad 0 \leqslant i \leqslant d.$$

Here m is the rank of the idempotent E_1, that is, the dimension of the space V_1. Unlike P-polynomiality, Q-polynomiality does not possess a clear combinatorial interpretation. Nevertheless this notion is very important in the theory of t-designs.

It should be noted that a scheme can possess several P-polynomial and/or Q-polynomial structures corresponding to different orderings of the relations and/or of the primitive idempotents.

The most significant and interesting class of schemes is the class of (P and Q)-polynomial schemes.

The book [1] contains a list of known (P and Q)-polynomial schemes for sufficiently large values of diameter and a program for the classification of all (P and Q)-polynomial schemes. Leonard in [7] (cf. Theorem 5.1 in [1]) carried out the first important step in this program. He proved that all parameters of a (P and Q)-polynomial scheme are functions of a certain set of six parameters. Another important step in this program is the chacteriza-tion of all known (P and Q)-polynomial schemes in terms of their parameters. This means the determination of all schemes whose parameters coincide with that of some known (P and Q)-polynomial scheme.

Significant progress in this area was made recently by P. Terwilliger [10] and A. Neumaier [8] (see also the survey [2]). As a consequence all (P and Q)-polynomial schemes with $q = \pm 1$ (here q is one of the six parameters in the theorem of Leonard) were characterized. This class contains the well-known Hamming and Johnson schemes.

Another large class of known (P and Q)-polynomial schemes are the schemes of dual polar spaces, defined in the following way. Let W be a vector space over a finite field equipped with a nondegenerate form F (quadratic, symplectic, or Hermitian). Let X be the set of all maximal isotropic subspaces in W, and d be the dimension of such a subspace. The ith relation R_i on X is defined by the following:

$$(x, y) \in R_i \quad \Leftrightarrow \quad \dim(x \cap y) = d - i. \tag{3}$$

Then $Y = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ is a (P and Q)-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$ and

$$b_i = r^{i+e+1} \frac{r^{d-i} - 1}{r - 1}, \qquad c_i = \frac{r^i - 1}{r - 1}, \tag{4}$$

where r is the number of elements in the field and e equals $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$, depending on the type of the form F. If $e = -\frac{1}{2}$ we have the dual polar space of type ${}^2A_{2d-1}(p^f)$. This means that W is the $2d$-dimensional space over $\mathrm{GF}(p^{2f})$ and F is a nondegenerate Hermitian form. These schemes will be under consideration in the present article.

2. THE CHARACTERIZATION OF THE SCHEMES $A^2_{2d-1}(p^f)$

Let $Y = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a scheme with the parameters of the scheme of dual polar space ${}^2A_{2d-1}(p^f)$, and let $\Gamma = (X, R_1)$ be the corresponding distance-regular graph. Our aim is to prove that cliques in Γ have size $p^f + 1$.

The scheme ${}^2A_{2d-1}(p^f)$ has two Q-polynomial structures. In terms of the exceptional structure (which is absent in other schemes of dual polar spaces), the Leonard parameters are the following (cf. [1, p. 304]):

$$q = -p^f, \qquad s = p^{-2(d+1)f}, \qquad r_1 = -r_2 = p^{-(d+1)f}, \qquad s^* = r_3 = 0.$$

As mentioned above, these six values determine all parameters of the scheme Y. The explicit formulas are presented in Theorem 5.1 in [1].

Let $E_0, E_1, \ldots, E_d$ be the ordering corresponding to the Q-polynomial structure under consideration. Set $E = E_1$, $\pi = \pi_1$, and $\theta_i^* = q_1(i)$. Then the rank m of the matrix E is equal to

$$m = b_0^* = \frac{p^{2df} + p^f}{p^f + 1}.$$

Let $x \in X$, and $\{y_1, \ldots, y_k\} = \Gamma(x)$ be the set of all vertices of Γ adjacent to x. Then

$$k = b_0 = p^f \frac{p^{2df} - 1}{p^{2f} - 1}.$$

Let Δ be the subgraph of Γ induced by the set $\Gamma(x)$. Then Δ is a regular graph of valency $a_1 = p^f - 1$.

Let us consider the set $\{\pi(y) : y \in \Gamma(x)\}$ of vectors belonging to the subspace V_1. Let M be the Gram matrix of these vectors, i.e., the (i, j)th element of the matrix M is equal to $\langle \pi(y_i), \pi(y_j) \rangle$. Then M is a matrix of order k, and the rank of M does not exceed the dimension of V_1. Since $k > m$, we have some conditions on the subgraph Δ. It turns out that these conditions determine Δ uniquely up to isomorphism.

By the equality (3), the (i, j)th element of the matrix M depends only on the distance between the vertices y_i and y_j. Moreover, since $y_i, y_j \in \Gamma(x)$, then $d(y_i, y_j) \leqslant 2$. So we have

$$n\left\langle \pi(y_i), \pi(y_j) \right\rangle = \begin{cases} \theta_0^* & \text{if } \quad i = j, \\ \theta_1^* & \text{if } \quad y_i \text{ and } y_j \text{ are adjacent in } \Delta, \\ \theta_2^* & \text{otherwise,} \end{cases}$$

and

$$nM = \theta_2^* J + (\theta_0^* - \theta_2^*) I + (\theta_1^* - \theta_2^*) D, \tag{5}$$

where J is the all-unit matrix, I is the identity matrix, and D is the adjacency matrix of the subgraph Δ. It follows from the parameters of the graph Γ that $\theta_1^* = \theta_0^* - h^*(1 + p^f)/p^f$, $\theta_2^* = \theta_0^* + h^*(1 - p^{2f})/p^{2f}$ for some constant h^*.

Since Δ is a regular graph, its adjacency matrix D commutes with J. Hence J, D, and I have a basis of common eigenvectors. As the rank of M does not exceed m, this matrix has at least $k - m$ zero eigenvalues.

Let v be a common eigenvector of the matrices J, I, and D corresponding to zero eigenvalue of the matrix M. Then $Jv = \alpha v$, where $\alpha = k$ if all components of v are equal and $\alpha = 0$ otherwise; $Iv = v$, $Dv = \lambda v$, where λ is an eigenvalue of the matrix D. Let us assume at first that $\alpha = 0$. In this case

$$(\theta_0^* - \theta_2^*) + \lambda(\theta_1^* - \theta_2^*) = 0. \tag{6}$$

By substituting the expressions for θ_1^* and θ_2^* we obtain

$$\lambda h^* \frac{1 + p^f}{p^f} + (1 + \lambda) h^* \frac{1 - p^{2f}}{p^{2f}} = 0$$

and

$$p^f \lambda + (1 - p^f)\lambda = p^f - 1.$$

So $\lambda = p^f - 1$, which is equal to the valency of the graph Δ.

Now let $\alpha = k$. In this case v is a vector with equal components and $\lambda = p^f - 1$. But for this value of λ the equality (6) holds. Hence the vector v with equal components corresponds to a nonzero eigenvalue of M.

Thus the number of zero eigenvalues of the matrix M is one less than the multiplicity of eigenvalue $p^f - 1$ of the matrix D. Meanwhile it is well known (see for example Theorem 3.23 in [6]) that the multiplicity of the eigenvalue l of the adjacency matrix of a graph of valency l is equal to the number of connected components of the graph. This means that Γ has at least $k - m + 1$

connected components. Since $k - m + 1 = (p^{2df} - 1)/(p^{2f} - 1) = k/p^f$, each connected component has at most p^f vertices. But the valency of Δ is $p^f - 1$; hence Δ is the disjoint union of complete p^f-vertex subgraphs. So the cliques of Γ have size $p^f + 1$.

In the case $d = 2$ the result was proved by P. J. Cameron, J. M. Goethals, and J. J. Seidel in [5], where slightly different arguments were used. If $d = 2$, we obtain a generalized quadrangle of type (q, q^2). The complete description of these objects is known only for $q = 2$ and 3 (see the survey [11]).

If $d \geqslant 3$, then the existence of cliques of size $p^f + 1$ in Γ implies that Γ is the point graph of a regular near $2d$-gon. The results of P. J. Cameron [4], E. E. Shult and A. Yanushka [9] enable us to construct a dual polar space from such a graph. From the classification of dual polar spaces, obtained by J. Tits in [12], it follows that Γ is the dual polar space graph of type ${}^2A_{2d-1}(p^f)$ (see Section 8.4 in [3]).

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