A Problem on Integer Matrices

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

ABSTRACT

The existence of $n \times b$ matrices A of nonnegative integers satisfying $AJ = bJ$, $JA = nJ$, and $AA^T = bI + yJ$ is discussed.

1. INTRODUCTION

I will discuss the following problem: For which integers n and b is there an $n \times b$ matrix A with nonnegative integer entries, having row sums b and column sums n, and satisfying $AA^T = bI + [b(n-1)/n]J$, where J is the all-1 matrix?

I shall call such a matrix an *F-matrix*.

This problem was communicated to me by D. R. Hughes, who obtained it from R. Fletcher (Reading). What distinguishes it from similar problems is that the shape of the matrix is not restricted to be square. There are some trivial necessary conditions for the existence of an F-matrix. In this paper, I shall develop a small amount of "structure theory" and give some constructions.

The trivial necessary conditions are:

PROPOSITION 1.1. *If an $n \times b$ F-matrix exists, then*

(i) *n divides b;*
(ii) *if n is even then $2n$ divides b.*

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Proof. (i): $(n-1)b/n$ is an integer.

(ii): Set $b = tn$. The sum of the entries in a row is tn, and their sum of squares is $t(2n-1)$; these numbers are congruent mod 2.

2. DECOMPOSABILITY

We call two F-matrices *equivalent* if one can be obtained from the other by row and column permutations. Obviously such permutations do not affect the defining property.

It is clear that, if A_1 and A_2 are F-matrices with the same number of rows, then $(A_1 \; A_2)$ is an F-matrix. We call an F-matrix *indecomposable* if it is not equivalent to a matrix obtained by this construction. Clearly any F-matrix is equivalent to one obtained by composition of indecomposables; but the indecomposable "summands" are not unique, even up to equivalence. A structure theory should describe the indecomposables and specify the nonuniqueness in the construction.

THEOREM 2.1. *There are only finitely many indecomposable F-matrices with a given number of rows.*

The proof depends on the following "obvious" lemma:

LEMMA 2.2. *Let U be a subspace of $\mathbb{R}^n$ defined by a system of equations with integer coefficients, and let X be the set of vectors in U with nonnegative integer coordinates. Then there is a finite subset S of X such that every vector in X is a linear combination, with nonnegative integer coefficients, of vectors in S.*

[The intersection of S with the positive hyperoctant is a convex cone, and so there is a finite set S_0 of extreme vectors; without loss, their coordinates are integers. Now let P be the set of linear combinations of vectors in S_0 with coefficients between 0 and 1. Then P is bounded, so $P \cap X$ is finite; and we may take $S = S_0 \cup (P \cap X)$.]

Now let $c_1, \ldots, c_m$ be the column vectors of length n, having nonnegative integer entries with sum n. (*Exercise*: show that $m = \binom{2n-1}{n-1}$.) Set $b = tn$. Then an $n \times b$ matrix A, having x_i columns equal to c_i, $i = 1, \ldots, m$, is an F-matrix if and only if each row has sum tn and sum of squares $t(2n-1)$ and any two distinct rows have inner product $t(n-1)$; these are $2n + \binom{n}{2}$ linear equations in the variables $x_1, \ldots, x_m$ and t (not all independent). Thus the theorem follows from the lemma, on noting that the set S given by the lemma must contain every indecomposable.

The proof also suggests the way to describe the nonuniqueness of the decomposition. The linear dependencies among the indecomposables have a $\mathbb{Z}$-basis of integral vectors, since they form a free $\mathbb{Z}$-module. Any such dependence takes the form $E_1 = E_2$, where E_1 and E_2 are linear combinations of indecomposables with positive integer coefficients. That is, there is a "basic" set of nonunique decompositions from which all others follow.

For small n, the indecomposables can be found explicitly by solving the equations.

THEOREM 2.3.

(i) *For* $n = 2$, *there is a unique indecomposable F-matrix* (*up to column permutations*), *namely*

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

(ii) *For* $n = 3$, *there are just five indecomposable F-matrices up to equivalence, namely*

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 3 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

Proof. (i): The three columns are

$$\begin{pmatrix}2\\0\end{pmatrix}, \quad \begin{pmatrix}0\\2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1\\1\end{pmatrix};$$

among the equations are

$$x_1 + x_2 + x_3 = 2t,$$

$$2x_1 + x_3 = 2t,$$

$$x_3 = t,$$

so $x_1 = x_2 = \frac{1}{2}t$ and $x_3 = t$. Thus all solutions are equivalent to multiples of the indecomposable in the theorem.

(ii): Rather than solve 10 equations in 11 unknowns, we simplify the argument by observing that knowledge of an indecomposable can be used to restrict the variables; at least one column must occur fewer times in any other indecomposable. The ten columns are

$$\begin{matrix} 3 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 & 1 & 2 & 1 & 2 \end{matrix}.$$

In view of the first matrix in the theorem, we have $\min(x_4, x_8, x_9) = \min(x_5, x_6, x_{10}) = 0$. Thus, up to row permutations, there are three cases, viz. $x_4 = x_5 = 0$ (case 2), $x_6 = x_8 = 0$ (case 1), and $x_9 = x_{10} = 0$ (case 0). This choice distinguishes the first row, but retains symmetry between the second and third.

If a row has p threes, q twos, and r ones, then

$$9p + 4q + r = 5t,$$

$$3p + 2q + r = 3t,$$

so $3p + q = t$, $q + r = 2t$.

Case 2. $3x_1 = t,\ x_6 + x_7 + x_8 = 2t,$

$$3x_2 + x_6 + x_9 = t, \qquad x_6 + x_7 + x_9 + x_{10} = 2t,$$

$$3x_3 + x_8 + x_{10} = t, \qquad x_7 + x_8 + x_9 + x_{10} = 2t,$$

and $2x_6 + x_7 = 2t$. Then $x_6 = x_8 = x_9 + x_{10}$ and

$$x_6 + x_7 + x_8 = 2t = 3x_2 + 3x_3 + x_6 + x_8 + x_9 + x_{10},$$

so $x_7 = 3x_2 + 3x_3 + x_6$. Now clearly $x_1 > 0$. If $x_2 > 0$ and $x_3 > 0$, then $x_7 \geqslant 6$ and A contains a copy of A_2 (the second matrix in the theorem). So, by symmetry, $x_2 = 0$. Then everything is expressible in terms of x_3 and x_{10}, viz.

$$x_1 = 2x_3 + x_{10}, \qquad x_6 = x_8 = 3x_3 + 2x_{10},$$

$$x_7 = 6x_3 + 2x_{10}, \qquad x_9 = 2x_3 + x_{10},$$

and $(x_3, x_{10}) = (1,0)$ and $(0,1)$ are the only possible indecomposables, giving A_4 and A_3 respectively.

Case 1. Again, if x_1, x_2, x_3 are all positive, then $x_7 \geqslant 6$ and A contains A_2. So we can assume that $x_1 = 0$ or $x_2 = 0$. If $x_1 = 0$, we have $x_4 = x_2 + x_3$, $x_7 = 2x_4$, $x_9 = 2x_3 - x_2$, $x_{10} = 2x_2 - x_3$. Thus $2x_2 \geqslant x_3$ and $2x_3 \geqslant x_2$. Any solution is a positive integral combination of $(x_2, x_3) = (2,1)$, $(1,2)$, and $(1,1)$. The first and second are equivalent to A_4, the third to A_5. If $x_2 = 0$, we have $x_9 = t$, $x_4 = x_5 = x_9 + x_{10} \geqslant t$, so $3x_1 + x_4 + x_5 \geqslant 2t$, a contradiction.

Case 0. We find that $x_4 = x_5 = x_6 = x_8 = 0$, $x_7 = 2t$, and $3x_1 = 3x_2 = 3x_3 = t$, giving a multiple of A_2. ■

PROBLEM. Find a good bound on the size (or the number) of F-matrices with n rows.

3. CONSTRUCTIONS

PROPOSITION 3.1. *Every column vector of length n having nonnegative entries with sum n occurs in an F-matrix.*

The proof requires an auxiliary result. Call an $n \times b$ matrix B *hopeful* if its entries are nonnegative integers, it has row sums b and column sums n, and it satisfies $BB^T = xI + yJ$ for some x, y. Postmultiplying by J, we see that $bn = x + yn$; so a hopeful matrix is an F-matrix if and only if $x = b$. Accordingly, we call $x - b$ the *deficit* $d(B)$ of B .Note that if B_1 and B_2 are hopeful and have equally many rows, then $(B_1 \quad B_2)$ is hopeful, with

$$d((B_1 \quad B_2)) = d(B_1) + d(B_2).$$

We claim that any hopeful matrix can be embedded into an F-matrix. Then $n \times 1$ column of ones is hopeful with deficit -1; so any hopeful matrix of positive deficit can be converted into an F-matrix by adjoining columns of ones. Also, the matrix nI is hopeful with deficit $n(n-1)$. So, given a hopeful matrix with negative deficit, repeat it $n(n-1)$ times [so that the deficit is a multiple of $n(n-1)$], then adjoin an appropriate number of copies of nI.

For example, $(nI \ J)$, where J has $n(n-1)$ columns, is an $n \times n^2$ F-matrix, and is indecomposable.

Now to prove the proposition, let c be any such column, and B the matrix whose columns are all permutations of c. Then any row permutation of B has the same effect as some column permutation; so the row sums, row sums-of-squares, and row inner products are all constant, i.e., B is hopeful.

Let C be the incidence matrix of a 2-design, having the usual parameters v, k, λ, r, b, in which k divides v; say $v = ak$. Then aC is hopeful, and its deficit is $tv - b$, where $t = a(r - a\lambda)$. Straightforward calculation shows that the deficit is positive. Thus we have:

COROLLARY 3.2. *If a 2-design with parameters v, k, λ, r, b exists such that $v = ak$ (a integral), then a $v \times tv$ F-matrix exists where $t = a(r - a\lambda)$.*

The most interesting case is where t is as small as possible, viz. $t = 2$. This requires $a = 2$, $r = 2\lambda + 1$; that is, a $2-(2k, k, k-1)$ design. Such a design exists whenever there is a Hadamard matrix of order $4k$ (it is the residual of the corresponding Hadamard 2-design). Thus:

COROLLARY 3.3. *If a $4k \times 4k$ Hadamard matrix exists, then a $2k \times 4k$ F-matrix exists.*

This strongly supports the conjecture that $n \times 2n$ F-matrices exist for all even n. Note that not all such matrices come from the above construction.

For example, it can be shown that there are exactly five inequivalent 4×8 F-matrices, viz.

$$\begin{pmatrix} 3 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 1 & 1 & 0 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 & 2 & 2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 0 & 2 & 0 & 2 \\ 0 & 2 & 1 & 1 & 0 & 2 & 2 & 0 \end{pmatrix}.$$

4. SQUARE F-MATRICES

For which n does an $n \times n$ F-matrix exist? By Proposition 1.1, n must be odd. There is a further restriction:

PROPOSITION 4.1. *If an $n \times n$ F-matrix exists with $n \equiv 1 \pmod 4$, then n is the sum of two squares.*

The proof is completely standard. Following the Bruck-Ryser theorem (see [2, p. 87]), the equation

$$z^2 = nx^2 + (n-1)y^2$$

must have a solution in integers, not all zero. Writing the equation as $y^2 + z^2 = n(x^2 + y^2)$, we see that n is the sum of two squares.

[This technique gives no further condition if $n \equiv 3 \pmod 4$, for then the equation is $z^2 = nx^2 - (n-1)y^2$, and is satisfied by $x = y = z = 1$.]

Suppose now that an $n \times n$ F-matrix A has no entry greater than 2. Set $C = A - J$, a matrix with entries 1, 0, and -1 satisfying $CJ = JC = 0$, $CC^T = nI - J$. Such a matrix has a unique zero in each row or column; we may suppose it has zero on the diagonal and 1 or -1 elsewhere. It is shown in [1] that such a matrix is symmetric if $n \equiv 1 \pmod 4$, skew if $n \equiv -1 \pmod 4$. Moreover, for $n \equiv 1 \pmod 4$, such matrices coexist with strongly regular graphs having self-complementary parameters; for $n \equiv -1 \pmod 4$, with skew-type Hadamard matrices or strongly regular tournaments.

The simplest construction is that of Paley: n is an odd prime power, the rows and columns of C are indexed by the field of order n, and

$$c_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } j - i \text{ is a nonzero square,} \\ -1 & \text{otherwise.} \end{cases}$$

Note that there exist $n \times n$ F-matrices with entries greater than 2. The smallest such has $n = 9$; a particularly pretty example is

$$\begin{pmatrix} 3I + 2P & I + P^2 & I + P^2 \\ I + P^2 & 3I + 2P & I + P^2 \\ I + P^2 & I + P^2 & 3I + 2P \end{pmatrix}$$

where P is a cyclic permutation matrix of order 3.

PROBLEM. Clearly, in an F-matrix with n rows, no entry exceeds n. If n occurs, then there are at least n^2 columns, and this bound is exact. More generally, if the entry m occurs, then $m(m-1) \leqslant b(n-1)/n$. Is this bound anywhere near best possible? In particular, what is the largest possible entry in an F-matrix of size $n \times n$ (or $n \times 2n$)?

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