Quantum Double Models using SageMath

0.1 Simple Mathematics using SageMath

In [6]:
```
# Finding the gcd of two numbers
gcd?
```

In [7]:
```
gcd(25,30)
```

Out[7]:
```
5
```

In [8]:
```
factor(625)
```

Out[8]:
```
5^4
```

In [9]:
```
factor(2435)
```

Out[9]:
```
5 * 487
```

In [11]:
```
M = matrix([[1,2,4],[1,3,5],[5,4,2]]);M
```

Out[11]:
```
[1 2 4]
[1 3 5]
[5 4 2]
```

In [12]:
```
M.inverse()
```

Out[12]:
```
[   7/6     -1    1/6]
[-23/12    3/2   1/12]
[ 11/12   -1/2  -1/12]
```

In [13]:
```
M.eigenvalues()
```

Out[13]:
```
[-3.583940313126412?, 0.3631206519422502?, 9.22081966118417?]
```

0.2 Group Theory in SageMath

In [15]:
```
Z2 = SymmetricGroup(2); Z2
```

Out[15]:
```
Symmetric group of order 2! as a permutation group
```

In [16]:
```
S3 = SymmetricGroup(3); S3
```

Out[16]:
```
Symmetric group of order 3! as a permutation group
```

In [17]:
```
Z2.is_subgroup(S3)
```

Out[17]:
```
True
```

In [18]:
```
S3.is_cyclic()
```

Out[18]:
```
False
```

In [19]:
```
Z2.is_abelian()
```

Out[19]:
```
True
```

0.3 Excitations, Ribbon operators, Ground states in Quantum Double Models

Defining different models - Quantum Double of Z2, S3, D4

```
In [1]: QDM_Toric = SymmetricGroup(2)
        QDM_Toric

Out[1]: Symmetric group of order 2! as a permutation group

In [2]: QDM_S3 = SymmetricGroup(3)
        QDM_S3

Out[2]: Symmetric group of order 3! as a permutation group

In [3]: QDM_D4 = DihedralGroup(4)
        QDM_D4

Out[3]: Dihedral group of order 8 as a permutation group
```

Developing the machinery to compute the number of excitations.

1. **Computing the centralizers of the conjugacy class of the group.**

```
In [4]: def centralizer_conjugacy_class_QDM_generic(QDM_group):
            cent_QDM_group = []
            for conj_class in QDM_group.conjugacy_classes():
                centralizer = QDM_group.centralizer(conj_class.an_element())
                cent_QDM_group.append(centralizer)
            return cent_QDM_group

In [5]: cent_toric = centralizer_conjugacy_class_QDM_generic(QDM_Toric)
        cent_toric

Out[5]: [Subgroup of (Symmetric group of order 2! as a permutation group) generated by [(1,2)],
         Subgroup of (Symmetric group of order 2! as a permutation group) generated by [(1,2)]]

In [6]: cent_s3 = centralizer_conjugacy_class_QDM_generic(QDM_S3)
        cent_s3

Out[6]: [Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(2,3), (1,3)],
         Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(1,2)],
         Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(1,2,3)]]

In [7]: cent_d4 = centralizer_conjugacy_class_QDM_generic(QDM_D4)
        cent_d4

Out[7]: [Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(1,2,3,4), (1,4)(2
         Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(2,4), (1,3)(2,4)
         Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(1,2)(3,4), (1,3)
         Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(1,2,3,4), (1,3)(
         Subgroup of (Dihedral group of order 8 as a permutation group) generated by [(1,2,3,4), (1,4)(
```

2. The character table gives the trace of irreducible representations (but the trace is used at a later stage).

```
In [8]: def character_table_centralizers(centralizers_generic_group):
            char_table = []
            for subgroup in centralizers_generic_group:
                char_table.append(subgroup.character_table())
            return char_table
```

```
In [9]: cent_toric_centralizer_character = character_table_centralizers(cent_toric)
        cent_toric_centralizer_character
```

```
Out[9]: [
        [ 1 -1]  [ 1 -1]
        [ 1  1], [ 1  1]
        ]
```

```
In [10]: cent_s3_centralizer_character_table = character_table_centralizers(cent_s3)
         cent_s3_centralizer_character_table
```

```
Out[10]: [
         [ 1 -1  1]           [          1           1           1]
         [ 2  0 -1]  [ 1 -1]  [          1       zeta3 -zeta3 - 1]
         [ 1  1  1], [ 1  1], [          1 -zeta3 - 1       zeta3]
         ]
```

```
In [11]: cent_d4_centralizer_character_table = character_table_centralizers(cent_d4)
         cent_d4_centralizer_character_table
```

```
Out[11]: [
         [ 1  1  1  1  1]
         [ 1 -1 -1  1  1]  [ 1  1  1  1]  [ 1  1  1  1]
         [ 1 -1  1 -1  1]  [ 1 -1 -1  1]  [ 1 -1 -1  1]
         [ 1  1 -1 -1  1]  [ 1 -1  1 -1]  [ 1 -1  1 -1]
         [ 2  0  0  0 -2], [ 1  1 -1 -1], [ 1  1 -1 -1],

                                          [ 1  1  1  1  1]
         [      1      1      1      1]  [ 1 -1 -1  1  1]
         [      1     -1      1     -1]  [ 1 -1  1 -1  1]
         [      1 -zeta4     -1  zeta4]  [ 1  1 -1 -1  1]
         [      1  zeta4     -1 -zeta4], [ 2  0  0  0 -2]
         ]
```

3. Computing the number of excitations by counting the number of rows in the character table.

```
In [12]: def excitations_count(QDM_group):
             count = 0
             generic_centralizer_character_table = character_table_centralizers(centralizer_conjugacy_cl
             for char_table in generic_centralizer_character_table:
                 count += char_table.nrows()
             return count
```

```
In [13]: QDM_toric_excitations = excitations_count(QDM_Toric)
         QDM_toric_excitations
```

```
Out[13]: 4
```

```
In [14]: QDM_S3_excitations = excitations_count(QDM_S3)
         QDM_S3_excitations
```

```
Out[14]: 8
```

```
In [15]: QDM_D4_excitations = excitations_count(QDM_D4)
         QDM_D4_excitations
```

```
Out[15]: 22
```

Developing the machinery to compute the excitations that condense on a given boundary

1. **Computing the character related to the irreducible representation of the group.**

```
In [16]: def character_excitation(G, conjugacy_class, g, h):
             k_h = 0
             for g_1 in G:
                 if h*g_1 == g_1*conjugacy_class.an_element():
                     k_h = g_1
                     break
             if g*h == h*g and k_h != 0:
                 return k_h^-1*g*k_h
             else:
                 return 0
```

2. **Computing the character related to a particular boundary.**

```
In [17]: def character_subgroup(G, subgroup, g, h):
             sum = 0
             if h*g == g*h:
                 for g_1 in G:
                     if g_1*g*g_1^-1 in subgroup and g_1*h*g_1^-1 in subgroup:
                         sum = sum + 1
             return sum/len(subgroup)
```

3. **Computing the inner product terms of the above characters.**

```
In [18]: def inner_product_of_characters(QDM_group, subgroup, conjugacy_class):
             inner_product_terms = []
             for g in QDM_group:
                 for h in QDM_group:
                     if character_subgroup(QDM_group, subgroup, g, h) != 0 and character_excitation(QDM_
                         inner_product_terms.append([character_subgroup(QDM_group, subgroup, g, h), cha
             return inner_product_terms
```

```
In [19]: inner_product_of_characters(QDM_S3, QDM_S3.subgroups()[5], QDM_S3.conjugacy_classes()[0])
```

```
Out[19]: [[1, ()], [1, (1,2)], [1, (1,2,3)], [1, (1,3,2)], [1, (2,3)], [1, (1,3)]]
```

$1 * tr_{\pi_i}(e) + 1 * tr_{\pi_i}(1,2) + 1 * tr_{\pi_i}(1,2,3) + 1 * tr_{\pi_i}(1,3,2) + 1 * tr_{\pi_i}(2,3) + 1 * tr_{\pi_i}(1,3)$
From the character table for S_3, and labelling each excitation

$$\begin{array}{ccccccc} \{e\} & \{\tau\} & \{\sigma\} & & & & \\ 1 & -1 & 1 & -> & tr_{\pi_2} & -> & B \\ 2 & 0 & -1 & -> & tr_{\pi_3} & -> & C \\ 1 & 1 & 1 & -> & tr_{\pi_1} & -> & A \end{array}$$

Therefore A condenses on the boundary as the inner product is greater than zero, the others go to zero.

```
In [20]: inner_product_of_characters(QDM_S3, QDM_S3.subgroups()[5], QDM_S3.conjugacy_classes()[1])
```

```
Out[20]: [[1, ()], [1, ()], [1, ()], [1, (1,2)], [1, (1,2)], [1, (1,2)]]
```

$3 * tr_{\pi_i}(e) + 3 * tr_{\pi_i}(1,2)$
From the character table for Z_2, and labelling each excitation
$1 \quad -1 \quad tr_{\pi_2} \quad -> \quad E$
$1 \quad 1 \quad tr_{\pi_1} \quad -> \quad D$
Therefore D condenses on the boundary as the inner product is greater than zero, the others go to zero.

```
In [21]: inner_product_of_characters(QDM_S3, QDM_S3.subgroups()[5], QDM_S3.conjugacy_classes()[2])
```

```
Out[21]: [[1, ()], [1, ()], [1, (1,2,3)], [1, (1,3,2)], [1, (1,3,2)], [1, (1,2,3)]]
```

$2 * tr_{\pi_i}(e) + 2 * tr_{\pi_i}(1,2,3) + 2 * tr_{\pi_i}(1,3,2)$
From the character table for Z_3, and labelling each excitation
$1 \quad 1 \quad 1 \quad tr_{\pi_1} \quad -> \quad F$
$1 \quad zeta3 \quad -zeta3-1 \quad tr_{\pi_2} \quad -> \quad G$
$1 \quad -zeta3-1 \quad zeta3 \quad tr_{\pi_3} \quad -> \quad H$
Therefore F condenses on the boundary as the inner product is greater than zero, the others go to zero.
Hence, for the subgroup $K = G$, the excitations A, D, F condense on the boundary.

Similarly varying the boundaries (different subgroups) and using the inner product, the excitations which condense on the boundary can be determined.

```
In [22]: def boundary_condensates(QDM_group, QDM_subgroup):
             total_inner_product_terms = []
             for conj_class in QDM_group.conjugacy_classes():
                 total_inner_product_terms.append(inner_product_of_characters(QDM_group, QDM_subgroup, c
             return total_inner_product_terms
```

Boundary condensates for the boundary indexed by $\{e, \tau\}$

```
In [23]: boundary_condensates(QDM_S3, QDM_S3.subgroups()[1])
```

```
Out[23]: [[[3, ()], [1, (1,2)], [1, (2,3)], [1, (1,3)]],
          [[1, ()], [1, ()], [1, ()], [1, (1,2)], [1, (1,2)], [1, (1,2)]],
          []]
```

Observing the character table list, A, C, D condense given the boundary is indexed by $\{e, \tau\}$

Construction of the ribbon operators for lattice with boundary Given that the boundary is given by the boundary (subgroup K), the ribbon operator with an excitation in the bulk and the condensate on the boundary is given by
$T^{(k,g)} = \Sigma_{l \in K} F^{(lkl^{-1}, gl^{-1})}$ where $k \in K, g \in G$
Fixing the subgroup $K = \{e, \tau\}, (\{e, (2,3)\}$ for example)

```
In [24]: K = QDM_S3.subgroups()[1];K
```

```
Out[24]: Subgroup of (Symmetric group of order 3! as a permutation group) generated by [(2,3)]
```

```
In [25]: def ribbon_operator_constructs(QDM_group, subgroup):
             ribbon_operator_terms = []
             for k in subgroup:
                 for g in QDM_group:
                     for l in subgroup:
                         ribbon_operator_terms.append([k,g,l*k*l^-1, l*g^-1])
             return ribbon_operator_terms
         ribbon_operator_constructs(QDM_S3, K)
```

```
Out[25]: [[(), (), (), ()],
          [(), (), (), (2,3)],
          [(), (1,2), (), (1,2)],
          [(), (1,2), (), (1,2,3)],
          [(), (1,2,3), (), (1,3,2)],
          [(), (1,2,3), (), (1,3)],
          [(), (1,3,2), (), (1,2,3)],
          [(), (1,3,2), (), (1,2)],
          [(), (2,3), (), (2,3)],
          [(), (2,3), (), ()],
          [(), (1,3), (), (1,3)],
          [(), (1,3), (), (1,3,2)],
          [(2,3), (), (2,3), ()],
          [(2,3), (), (2,3), (2,3)],
          [(2,3), (1,2), (2,3), (1,2)],
          [(2,3), (1,2), (2,3), (1,2,3)],
          [(2,3), (1,2,3), (2,3), (1,3,2)],
          [(2,3), (1,2,3), (2,3), (1,3)],
          [(2,3), (1,3,2), (2,3), (1,2,3)],
          [(2,3), (1,3,2), (2,3), (1,2)],
          [(2,3), (2,3), (2,3), (2,3)],
          [(2,3), (2,3), (2,3), ()],
          [(2,3), (1,3), (2,3), (1,3)],
          [(2,3), (1,3), (2,3), (1,3,2)]]
```

$$T^{(e,e)} = F^{(e,e)} + F^{(e,(2,3))},$$
$$T^{(e,(1,2))} = F^{(e,(1,2))} + F^{(e,(1,2,3))},$$
$$T^{(e,(1,2,3))} = F^{(e,1,3,2)} + F^{(e,(1,3))},$$
$$T^{((2,3),e)} = F^{((2,3),e)} + F^{((2,3),(2,3))},$$
$$T^{((2,3),(1,2))} = F^{((2,3),(1,2))} + F^{((2,3),(1,2,3))},$$
$$T^{((2,3),(1,2,3)} = F^{((2,3),(1,3,2))} + F^{((2,3),(1,3))},$$

Similarly for various boundaries, various ribbon operators connecting the bulk to the boundary can be generated. It is observed that for every boundary (every subgroup) there are 6 unique ribbon operators connecting the bulk to boundary in the case of S_3

Ground states with respect to different `T` operators on a cylinder with a single lattice (implying boundary on both sides of the lattice) The lattice looks in the following way :

$$-g_1 \text{——} g_1-$$
$$|$$
$$|$$
$$g_2$$

|

|

$-g_3$——g_3-

Eigenstates of $\Pi\{\Sigma\ (vertex\ operators)\}\ (face\ operators) T$ are the ground states of the lattice with a ribbon operator. In the above lattice $g_1 and g_3$ are restricted to the subgroup (identified as boundary). There are three conditions to be satisfied, fixing the boundary to be $\{e, \tau\}$, due to the ribbon operators g_2 is restricted to $\{e, (2,3)\}$, due to the face operators the relationship between g_1, g_2, g_3 is as follows $g_3 g_2 g_1 g_2^{-1} = e$, and finally due to the vertex operators g_1, g_2, g_3 get mapped to $k_u g_1 k_u^{-1}, k_d g_2 k_u^{-1}, k_d g_3 k_d^{-1}$ respectively, where $k_u, k_d \in K$

In [26]:
```
def ground_state_terms(g1, g2, g3, ku,  kd):
    return ku*g1*ku^-1, kd*g2*ku^-1, kd*g3*kd^-1
```

In [27]:
```
def ground_state_sum(condition_set, subgroup):
    s = []
    for g2 in condition_set[1]:
        for g3 in subgroup:
            for g1 in subgroup:
                if condition_set[0]*g3*g2*condition_set[0]*g1 == g2:
                    s.append((condition_set[0]*g1,g2,condition_set[0]*g3))
                    for i in subgroup:
                        for j in subgroup:
                            s.append([ground_state_terms(condition_set[0]*g1, g2, condition_se
    return s
```

Observing that $T^{(e,e)} = F^{(e,e)} + F^{(e,(2,3))}$ the condition set is that $g_2 \in \{e, (2,3)\}$ similarly to determine the other ground states the condition set is required

In [28]:
```
ground_state_sum([QDM_S3[0],[QDM_S3[0], QDM_S3[4]]], QDM_S3.subgroups()[1])
```

Out[28]:
```
[((), (), ()),
 [((), (), ())],
 [((), (2,3), ())],
 [((), (2,3), ())],
 [((), (), ())],
 ((2,3), (), (2,3)),
 [((2,3), (), (2,3))],
 [((2,3), (2,3), (2,3))],
 [((2,3), (2,3), (2,3))],
 [((2,3), (), (2,3))],
 ((), (2,3), ()),
 [((), (2,3), ())],
 [((), (), ())],
 [((), (), ())],
 [((), (2,3), ())],
 ((2,3), (2,3), (2,3)),
 [((2,3), (2,3), (2,3))],
 [((2,3), (), (2,3))],
 [((2,3), (), (2,3))],
 [((2,3), (2,3), (2,3))]]
```

This implies for the operator $T^{(e,e)}$:

$Possible\ initial\ configuration$	$Ground\ state$
(e,e,e)	$2*(e,e,e) + 2*(e,(2,3),e)$
$((2,3),e,(2,3))$	$2*((2,3),e,(2,3)) + 2*((2,3),(2,3),(2,3))$
$(e,(2,3),e)$	$2*(e,e,e) + 2*(e,(2,3),e)$
$((2,3),(2,3),(2,3))$	$2*((2,3),e,(2,3)) + 2*((2,3),(2,3),(2,3))$

```
In [29]: ground_state_sum([QDM_S3[0],[QDM_S3[1], QDM_S3[2]]], QDM_S3.subgroups()[1])
```

```
Out[29]: [((), (1,2), ()),
          [((), (1,2), ())],
          [((), (1,2,3), ())],
          [((), (1,3,2), ())],
          [((), (1,3), ())],
          ((), (1,2,3), ()),
          [((), (1,2,3), ())],
          [((), (1,2), ())],
          [((), (1,3), ())],
          [((), (1,3,2), ())]]
```

This implies for the operator $T^{(e,(1,2))}$:

Possible initial configuration	*Ground state*
$(e,(1,2),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$
$(e,(1,2,3),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$

```
In [30]: ground_state_sum([QDM_S3[0],[QDM_S3[3], QDM_S3[5]]], QDM_S3.subgroups()[1])
```

```
Out[30]: [((), (1,3,2), ()),
          [((), (1,3,2), ())],
          [((), (1,3), ())],
          [((), (1,2), ())],
          [((), (1,2,3), ())],
          ((), (1,3), ()),
          [((), (1,3), ())],
          [((), (1,3,2), ())],
          [((), (1,2,3), ())],
          [((), (1,2), ())]]
```

This implies for the operator $T^{(e,(1,2,3))}$:

Possible initial configuration	*Ground state*
$(e,(1,3),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$
$(e,(1,3,2),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$

```
In [31]: ground_state_sum([QDM_S3[4],[QDM_S3[0], QDM_S3[4]]], QDM_S3.subgroups()[1])
```

```
Out[31]: [((2,3), (), (2,3)),
          [((2,3), (), (2,3))],
          [((2,3), (2,3), (2,3))],
          [((2,3), (2,3), (2,3))],
          [((2,3), (), (2,3))],
          ((), (), ()),
          [((), (), ())],
          [((), (2,3), ())],
          [((), (2,3), ())],
          [((), (), ())],
          ((2,3), (2,3), (2,3)),
          [((2,3), (2,3), (2,3))],
          [((2,3), (), (2,3))],
          [((2,3), (), (2,3))],
          [((2,3), (2,3), (2,3))],
          ((), (2,3), ()),
          [((), (2,3), ())],
          [((), (), ())],
          [((), (), ())],
          [((), (2,3), ())]]
```

This implies for the operator $T^{((2,3),e)}$:

Possible initial configuration	*Ground state*
(e,e,e)	$2*(e,e,e)+2*(e,(2,3),e)$
$((2,3),e,(2,3))$	$2*((2,3),e,(2,3))+2*((2,3),(2,3),(2,3))$
$(e,(2,3),e)$	$2*(e,e,e)+2*(e,(2,3),e)$
$((2,3),(2,3),(2,3))$	$2*((2,3),e,(2,3))+2*((2,3),(2,3),(2,3))$

```
In [32]: ground_state_sum([QDM_S3[4],[QDM_S3[1], QDM_S3[2]]], QDM_S3.subgroups()[1])
```

```
Out[32]: [((), (1,2), ()),
          [((), (1,2), ())],
          [((), (1,2,3), ())],
          [((), (1,3,2), ())],
          [((), (1,3), ())],
          ((), (1,2,3), ()),
          [((), (1,2,3), ())],
          [((), (1,2), ())],
          [((), (1,3), ())],
          [((), (1,3,2), ())]]
```

This implies for the operator $T^{((2,3),(1,2))}$:

Possible initial configuration	*Ground state*
$(e,(1,2),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$
$(e,(1,2,3),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$

```
In [33]: ground_state_sum([QDM_S3[4],[QDM_S3[3], QDM_S3[5]]], QDM_S3.subgroups()[1])
```

```
Out[33]: [((), (1,3,2), ()),
          [((), (1,3,2), ())],
          [((), (1,3), ())],
          [((), (1,2), ())],
          [((), (1,2,3), ())],
          ((), (1,3), ()),
          [((), (1,3), ())],
          [((), (1,3,2), ())],
          [((), (1,2,3), ())],
          [((), (1,2), ())]]
```

This implies for the operator $T^{((2,3),(1,2,3))}$:

Possible initial configuration	*Ground state*
$(e,(1,3,2),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$
$(e,(1,3),e)$	$(e,(1,2),e)+(e,(1,2,3),e)+(e,(1,3,2),e)+(e,(1,3),e)$

Therefore, there are 3 unique ground states for all possible configurations of ribbon operators with an excitation at one end and condensate at the other