1. Carothers 16.58 (Solution by Lander Ver Hoef)

Suppose that $m^*(E) < \infty$. Prove that E is measurable if and only if, for every $\varepsilon > 0$, there is a finite union of bounded intervals A such that $m^*(E \triangle A) < \varepsilon$ (where $E \triangle A$ is the symmetric difference of E and A).

Solution:
Suppose E is measurable, and let $\varepsilon > 0$. Then there exists an open cover I_n with $\sum_{n=1}^{\infty} l(I_n) < m^*(E) + \varepsilon$. In particular, the sum converges, so as n goes to infinity, $l(I_n)$ goes to 0. Thus, there exists an N such that $\sum_{n=N}^{\infty} l(I_n) < \varepsilon$. We will let $A = \bigcup_{n=1}^{N} I_n$.

Note that because E is measurable,

$$m^*\left(\bigcup_{n=1}^{\infty} I_n\right) = m^*\left(\bigcup_{n=1}^{\infty} I_n \cap E\right) + m^*\left(\bigcup_{n=1}^{\infty} I_n \cap E^c\right).$$

Note that because $E \subseteq \bigcup_{n=1}^{\infty} I_n$, we have that $E \cap \bigcup_{n=1}^{\infty} I_n = E$. Similarly, $\bigcup_{n=1}^{\infty} I_n \cap E^c = \bigcup_{n=1}^{\infty} I_n \setminus E$. This gives us that

$$m^*\left(\bigcup_{n=1}^{\infty} I_n\right) = m^*(E) + m^*\left(\bigcup_{n=1}^{\infty} I_n \setminus E\right).$$

Rearranging, then using countable subadditivity and out definition of the intervals I_n, we have that

$$\begin{aligned} m^*\left(\bigcup_{n=1}^{\infty} I_n \setminus E\right) &= m^*\left(\bigcup_{n=1}^{\infty} I_n\right) - m^*(E) \\ &\leq \sum_{n=1}^{\infty} l(I_n) - m^*(E) \\ &< m^*(E) + \varepsilon - m^*(E) \\ &= \varepsilon. \end{aligned}$$

Thus, $m^*(\bigcup_{n=1}^{\infty} I_n \setminus E) < \varepsilon$, and because taking the finite union gives a subset of the infinite union, we also have $m^*(A \setminus E) < \varepsilon$ by monotonicity.

Observe that because $\bigcup_{n=1}^{\infty} I_n \supseteq E$, everything in $E \setminus A$ must lie in the intervals we removed from the infinite sum. That is, $E \setminus A \subseteq \bigcup_{n=N}^{\infty} I_n$, so by monotonicity, $m^*(E \setminus A) < \varepsilon$.

$E \Delta A = (E \setminus A) \cup (A \setminus E)$, and because E and A are measurable sets and $E \setminus A$ and $A \setminus E$ are disjoint, we know that $m^*(E \triangle A) = m^*(E \setminus A) + m^*(A \setminus E) < 2\varepsilon$.

Now, suppose that for every $\varepsilon > 0$, there exists a finite union of bounded intervals $A = \bigcup_{n=1}^{N} I_n$ such that $m^*(E \triangle A) < \varepsilon$. Thus, for any n, there exists an A_n such that $m^*(A_n \triangle E) < 1/n$. By construction, as n goes to infinity, A_n comes to differ from E by at most a null set.

We define a new sequence of sets $B_k = \bigcap_{n=k}^{\infty} A_n$. As observed above, as n grows very large, the difference grows small, so when we intersect all the A_n past a certain point, we are guaranteed that we have discarded all points outside E but a point of measure 0. That is,

$m^*(B_k \setminus E) = 0$ for any k. On the other hand, when we intersected all the A_n, we may have thrown out a significant portion of the points in E, so $m^*(E \setminus B_k)$ is not necessarily 0. However, $E \setminus B_k \subseteq E$, so by monotonicity $m^*(E \setminus B_k) \leq m^*(E) < \infty$, so the outer measure is at least finite.

Furthermore, $B_{k+1} \supseteq B_k$, because we are discarding less. We define our final set C as

$$C = \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n.$$

Because the A_n are approaching E, as we discard less and less, the outer measure is approaching the outer measure of E. In fact, when we take the union, we have that $m^*(E \setminus C) = 0$, and we inherit from the B_k that $m^*(C \setminus E) = 0$.

Note that $E = (C \cup (E \setminus C)) \setminus (C \setminus E)$. However, $E \setminus C$ and $C \setminus E$ are both null sets, and thus are measurable, and C is an $F_{\delta\sigma}$ set, so it is also measurable. Because the set of measurable sets is an algebra, we know that E must be measurable.

2. Carothers 16.64 (Solution by Jody Gaines)

Suppose that $m^*(E) < \infty$. Then E is measurable if and only if, for every $\epsilon > 0$, there exists a compact set $F \subset E$ such that $m(F) > m^*(E) - \epsilon$.

Solution:
Suppose E is measurable. Then for each $n \in \mathbb{N}$, $E_n = E \cap [-n, n]$ is measurable. So for each $n \in \mathbb{N}$ pick a closed set K_n such that $K_n \subset E_n$ and $m(K_n) > m^*(E_n) - 1/n$. Observe $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, and thus $m^*(E) = m^*(E \cap \mathbb{R}) = \lim_{n\to\infty} m^*(E_n)$. So let $\epsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $m^*(E_n) > m^*(E) - \epsilon/2$. Now pick $N_2 \in \mathbb{N}$ so that $1/N_2 < \epsilon/2$ and let $N = \max\{N_1, N_2\}$. Then

$$m(K_N) > m^*(E_N) - \frac{1}{N} > \left(m^*(E) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} = m^*(E) - \epsilon.$$

Since $E \cap [-N, N]$ is bounded then so is each K_N, which implies K_N is compact.

Conversely, suppose for all $\epsilon > 0$ there exists a compact set $F \subset E$ such that $m(F) > m^*(E) - \epsilon$. Then for each $n \in \mathbb{N}$ pick $F_n \subset E$ such that F_n is compact, $F_n \subset F_{n+1}$, and $F_n > m^*(E) - 1/n$. Then $m(\cup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n) \geq m^*(E)$. Since $F_n \subset E$ for all $n \in \mathbb{N}$ then $m^*(E) \geq m(\cup_{n=1}^{\infty} F_n)$. Thus $m^*(E) = m(\cup_{n=1}^{\infty} F_n)$, and since $\cup_{n=1}^{\infty} F_n \subseteq E$ then $E = \cup_{n=1}^{\infty} F_n \cup N$ for some null set N. Hence E is the union of measurable sets $\cup_{n=1}^{\infty} F_n$ and N, which implies E is measurable.

3. [Sakti]

Suppose $E \subseteq \mathbb{R}$. Prove that E is measurable if and only if for any $\epsilon > 0$ there is an open set G and a closed set F such that $F \subseteq E \subseteq G$ and $m^*(G \setminus F) < \epsilon$. (This is your text's definition of measurability.)

4. (Solution by Mason Brewer)

Revisit 16.28 using the full power of the theorems we've developed for Lebesgue measure. That is, try to come up with a tidy short proof that $m(\Delta_\alpha) = \alpha$.

Solution:
Using continuity from below on the sets $C_n = [0,1] \backslash \Delta_{\alpha,n}$. $\Delta_{\alpha,n}$ are the nth step of making the cantor set. So C_1 is

Let $0 < \alpha < 1$ and define $\Delta_{\alpha,n}$ to be the n-th step of our construction. The amount that we have discarded at the n-th step is 2^{n-1} intervals each of length $(1-\alpha)3^{-n}$. Thus, at the n-th step, the amount that we have removed is

$$\frac{1-\alpha}{3} \sum_{k=1}^{n} \left(\frac{2}{3}\right)^{k-1},$$

and so

$$m^*(\Delta_{\alpha,n}) = 1 - \left(\frac{1-\alpha}{3} \sum_{k=0}^{n} \left(\frac{2}{3}\right)^{k}\right) = 1 - \frac{1-\alpha}{3} \cdot \frac{1-(2/3)^{n+1}}{1-(2/3)} = 1 - (1-\alpha) \cdot \left(1 - \left(\frac{2}{3}\right)^{n+1}\right),$$

and so note that $\lim_{n\to\infty} m^*(\Delta_{\alpha,n}) = \alpha$. Also, we know that $\Delta_{\alpha,1} \supseteq \Delta_{\alpha,2} \supseteq \Delta_{\alpha,3} \supseteq \ldots$ since at each iteration we are removing intervals. We also know that, by definition, $\Delta = \bigcap_{n=1}^{\infty} \Delta_{\alpha,n}$. Finally, note that $\Delta_{\alpha,n} = [0,1]$, and so has finite outer measure. Thus, by continuity from above, we get that

$$m^*(\Delta) = m^*\left(\bigcap_{n=1}^{\infty} \Delta_{\alpha,n}\right) = \lim_{n\to\infty} \Delta_{\alpha,n} = \alpha.$$

5. **[Carothers 16.73]** (Solution by Max Heldman)

If E is a measurable subset of the nonmeasurable set N, then $m(E) = 0$.

Solution:
Consider $E_r = E + r(\text{mod}1)$ for $r \in \mathbb{Q} \cap [0,1)$. Since $E \subseteq N$, E contains at most one member from each equivalence class of $\mathbb{Q}$ under addition mod1. Hence, as we saw in the proof that N is nonmeasurable, the sets E_r are pairwise disjoint.

Suppose that E is measurable. Then each E_r is measurable, and

$$\sum_{r\in\mathbb{Q}} m(E) = \sum_{r\in\mathbb{Q}} m(E_r) = m\left(\bigcup_{r\in\mathbb{Q}} E_r\right) \leq 1.$$

But this implies $m(E) = 0$.

6. Carothers 16.74 (Solution by Jody Gaines)

If $m^*(A) > 0$, show that A contains a nonmeasurable set, [Hint: we must have $m^*(A \cap [n, n+1)) > 0$ for some $n \in \mathbb{Z}$, and so we may suppose that $A \subset [0,1)$ (How?). It follows from Exercise 73 that one of the sets $E_r = A \cap N_r$ is nonmeasurable (Why?).]

Solution:
Since $m^*(A) > 0$ then there exists an interval $I \subset A$ such that $0 < m^*(I) \leq m^*(A)$. Thus $m^*(I \cap [n, n+1)) > 0$, which implies $m^*(A \cap [n, n+1)) > 0$ for some $n \in \mathbb{Z}$. Let $B = (A - n) \cap [0, 1)$. Since $m^*(A \cap [n, n+1)) > 0$ then $m^*((A - n) \cap [0, 1)) > 0$, or rather $m^*(B) > 0$. Since $B \subset [0, 1)$ and $m^*(B) > 0$ then $m^*(B \cap N_r) > 0$ for some $r \in \mathbb{Q} \cap [0, 1)$. By the contrapositive of Exercise 16.73 we have that $B \cap N_r$ is nonmeasurable. Hence $(B + n) \cap N_r$ is a nonmeasurable set in A.

7. Carothers 16.75 (Solution by Mason Brewer)

Solution:
Note that the Cantor function $f : \Delta \to [0, 1]$ is a bijective map. Since, $R \subseteq [0, 1]$ (the non-measurable set), then $f^{-1}(R) \subseteq \Delta$. But $m^*(\Delta) = 0$, and so $m^*(f^{-1}(R)) = 0$ by monotonicity, meaning that $f^{-1}(R)$ is null and therefore measurable. Thus, f maps the measurable set $f^{-1}(R)$ to the non-measurable set R.

8. Carothers 17.3 [Sakti]

9. Carothers 17.4 (Solution by Mason Brewer)

Solution:
First, by way of contrapositive, assume that E is not measurable. Then note that $\chi_E^{-1}((1/2, \infty)) = E$, which is not measurable, and so χ_E is not a measurable function. Then for the converse, let E be measurable. If $a \geq 1$, then $\chi_E^{-1}((a, \infty)) = \varnothing$; if $0 \leq a < 1$, then $\chi_E^{-1}((a, \infty)) = E$; and if $a < 0$, then $\chi_E^{-1}((a, \infty)) = \mathbb{R}$. In any case, $\chi_E^{-1}((a, \infty))$ is a measurable set, implying that χ_E is a measurable function.

10. Carothers 17.8 (Solution by Lander Ver Hoef)

Suppose that $D = A \cup B$, where A and B are measurable. Show that $f : D \to \mathbb{R}$ is measurable if and only if $f|_A$ and $f|_B$ are measurable (relative to their respective domains A and B of course).

Solution:
If f is measurable and U is any open set in $\mathbb{R}$, then $f^{-1}(U)$ is measurable. Hence, $f^{-1}(U) \cap A = f|_A^{-1}(U)$ is measurable, because A is measurable. Thus, $f|_A$ is measurable, as is $f|_B$ by an identical proof.

So, suppose $f|_A$ and $f|_B$ are measurable. Let U be an open set in $\mathbb{R}$. Then $f^{-1}(U) = (f^{-1}(U) \cap A) \cup (f^{-1}(U) \cap B)$, because $D = A \cup B$. But $f^{-1}(U) \cap A = f|_A^{-1}(U)$, which is measurable, as is $f^{-1}(U) \cap B = f|_B^{-1}(U)$. Therefore, $f^{-1}(U)$ is the union of two measurable sets, and is measurable. Hence, f is measurable.

11. **[Carothers 17.18]** (Solution by Max Heldman)

Let $f : [0, 1] \to [0, 1]$ be the Cantor function, and let $g(x) = f(x) + x$. Then:

a) g is a homeomorphism of $[0, 1]$ onto $[0, 2]$.

Solution:
Since f is increasing and surjective, f is continuous. Hence g is continuous. Moreover, if $x_1, x_2 \in [0,1]$ with $x_1 < x_2$. Then since f is increasing, $f(x_1) \leq f(x_2)$, so $g(x_1) = f(x_1) + x_1 < f(x_2) + x_2$. Hence g is strictly increasing, and therefore injective. Since g is increasing and continuous with $g(0) = 0$ and $g(1) = 2$, g is surjective. Finally, g maps the compact set $[0,1]$ continuously onto the Hausdorff space $[0,2]$, and so g is a closed map. Hence g^{-1} is continuous.

b) $g(\Delta)$ is measurable and $m(g(\Delta)) = 1$. In particular, $g(\Delta)$ contains a nonmeasurable set A.

Solution:
Observe that Δ^c has the form $\Delta^c = \bigcup_{k=1}^{\infty} I_k$, where the I_k are disjoint open intervals. Moreover, because g is bijective and continuous, $g(\Delta^c) = g\left(\bigcup_{k=1}^{\infty} I_k\right) = \bigcup_{k=1}^{\infty} g(I_k)$ is also a union of disjoint open intervals. To compute $m(g(\Delta^c))$, we need only compute the lengths of the $g(I_k)$.

Let $I_k = (a_k, b_k)$ for each k. Then $g(I_k) = (g(a_k), g(b_k))$, so $m(g(I_k)) = g(b_k) - g(a_k) = f(b_k) + b_k - (f(a_k) + a_k)$. But $f(a_k) = f(b_k)$, so $m(g(I_k)) = b_k - a_k = m(I_k)$. But this implies $m(g(\Delta^c)) = m(\Delta^c) = 1$. Thus $m(g(\Delta)) = m([0,2] \setminus \Delta^c) = m([0,2]) - m(\Delta^c) = 1$. It follows from **Exercise 16.74** that $g(\Delta)$ contains a nonmeasurable set A.

c) g maps some measurable set onto a nonmeasurable set.

Solution:
Observe that $g^{-1}(A)$ is measurable, since $g^{-1}(A)$ is a subset of the null set Δ. But $g(g^{-1}(A)) = A$ is nonmeasurable.

d) $B = g^{-1}(A)$ is Lebesgue measurable but *not* a Borel set.

Solution:
We showed in part **(c)** that B is Lebesgue measurable. But B is not a Borel set, since the homeomorphic image of a Borel set is itself a Borel set and hence Lebesgue measurable.

e) There is a Lebesgue measurable function F and a continuous function G such that $F \circ G$ is not Lebesgue measurable.

Solution:
Let $G = g^{-1}|_A : A \to B$. We note that since A is nonmeasurable, A is uncountable, and since g is a bijection this implies that $g^{-1}(A) = B$ is uncountable. Hence there exists a bijection $F : B \to (0,1)$.

Then $(F \circ G)^{-1}((0,1)) = G^{-1}(F^{-1}((0,1))) = G^{-1}(B) = g(B) = A$ is not Lebesgue measurable. Hence $F \circ G$ is not measurable, although G is continuous and F is Lebesgue measurable.