Linearization

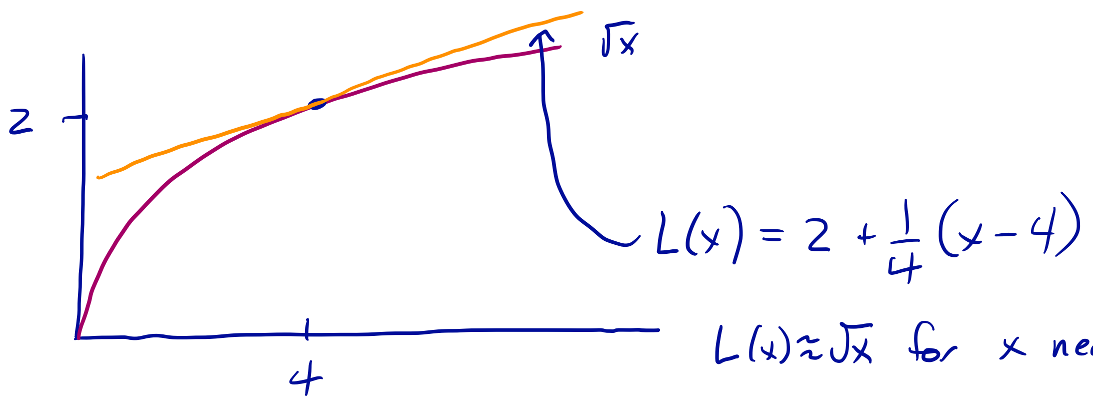
Given a function $f(x)$, its linearization at $x = a$ is the function

$$L(x) = f(a) + f'(a)(x - a).$$

For example, if $f(x) = \sqrt{x}$ and $a = 4$ then $f(4) = 2$ and $f'(4) = 1/(2\sqrt{4}) = 1/4$. So

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

The graph of the linearization is just the tangent line to the curve $y = \sqrt{x}$ at $x = 4$. So we expect that $L(x)$ is a good approximation for $\sqrt{x}$ for x near 4. The point is that computing square roots is hard work (even if your calculator makes it look easy) but computing the value of a linear function like L is easy. In fact your calculator is doing a more sophisticated generalization of the linear approximation: stay tuned in Calculus II!



$L(x) \approx \sqrt{x}$ for x near 4.

1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approxmiate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

$$L(4.1) = 2 + \frac{1}{4}(x-4)$$
$$= 2 + \frac{1}{4}\,\frac{1}{10} = 2 + \frac{1}{40} = 2.025$$

$\sqrt{4.1} = 2.0248\cdots$ error $\approx$ 0.0002

2. Use the linear approximation to approximate the cosine of $29° = \frac{29}{30}\frac{\pi}{6}$ radians.

Use linear approx of $\cos(x)$ at $x = \pi/6$

$\cos(\pi/6) = \sqrt{3}/2$

$\sin(\pi/6) = 1/2$

$$L(x) = \cos(\pi/6) + \cos'(\pi/6)(x - \pi/6)$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \pi/6)$$

$$L\left(\frac{29}{30}\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{1}{30}\cdot\frac{\pi}{6} = 0.87475\ldots$$

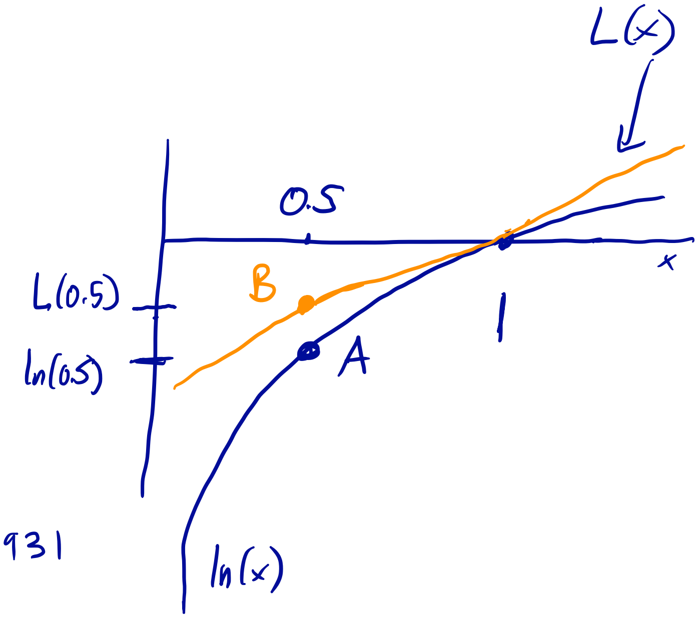
$$\cos\left(\frac{29}{30}\frac{\pi}{6}\right) = 0.87461$$

differ here.

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approxmate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5))$

$f(x) = \ln(x) \qquad f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = \frac{1}{1} = 1$



$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) \\ &= 0 + (x-1) \\ &= x - 1 \end{aligned}$$

$L(0.5) = -0.5$ vs. $\ln(0.5) = -0.6931$

$L(0.9) = -0.1$ vs $\ln(0.9) = -0.105\ldots$

4. Find the linear approximation of $f(x) = e^x$ at $a = 0$ and use it to approximate $e^{0.05}$ and e^1 Compare your approximations with your calculator's.

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= 1 + (x-0) \\ &= 1 + x \end{aligned}$$

$L(0.05) = 1.05$ vs. $e^{0.05} = 1.0512$

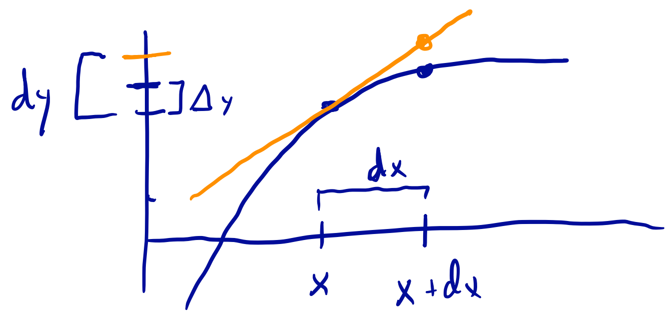
$L(1) = 2$ vs $e^1 = 2.718$

not so good, but 1 is "far" from 0, so the approximation will be worse.

Differentials Suppose we have a variable $y = f(x)$. We define its differential to be

$$dy = f'(x)dx$$

where x and dx are thought of as variables you can control. What's the point? The value of dy is an estimate of how much y changes if we change x into $x + dx$. See the graph:



$dy = f'(x)\,dx$ is the change in the linear approximation centred at x with step size dx

dy approximates $\Delta y = f(x+\Delta x) - f(x)$

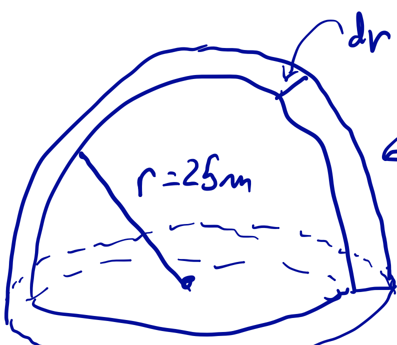
5. A tree is growing and the radius of its trunk in centemeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

$$dr = 2\,\frac{1}{2\sqrt{t}}\,dt = \frac{1}{\sqrt{t}}\,dt$$

$$t = 4,\quad dt = \frac{1}{12}$$

$$dr = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}\text{ cm}$$

6. A coat of paint of thinkness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]



paint is the extra volume

$$V = \frac{1}{2}\,\frac{4}{3}\pi r^3$$

hemisphere!

$$dV = 2\pi r^2\,dr$$

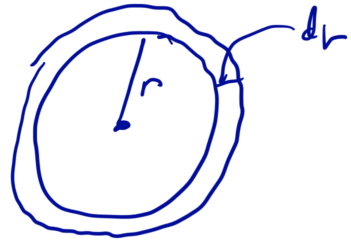
$r = 25$ m

$dr = 0.05\text{ cm} = \frac{0.05}{100}$ m.

$$dV = 2\pi\,(25)^2 \cdot \left(\frac{0.05}{100}\right)$$

$$= 1.96\text{ m}^3$$

7. The radius of a disc is 24cm with an error of ±0.5cm. Estimate the error in the area of the disc as an absolute and as a relative error.



$$A = \pi r^2$$

$$dA = 2\pi r \, dr \qquad r = 24\text{cm}$$

$$dr = 0.5\text{cm}$$

$$dA = 2\pi \cdot (24) \cdot \frac{1}{2}$$

$$= 75.4 \text{ cm}^2$$

$\uparrow$

This is absolute error.

Relative error compares the error with the measured value:

$$\frac{dA}{A} = \frac{75.4}{\pi (25)^2} = 0.0384 = 3.8\%$$