1. (Solution by David Maxwell)

In this problem, we will seek a solution to the initial value problem

$$f'(t) = F(t, f(t))$$
$$f(0) = a$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}$.

To obtain the existence result, we need to assume that F is sufficiently nice; we will assume that F is continuous, and moreover that there exists a constant K such that

$$|F(x, y_1) - F(x, y_2)| \le K|y_1 - y_2|$$

for all $x, y_1, y_2 \in \mathbb{R}$.

Define $G : C[-T, T] \to C[-T, T]$ by

$$G(f)(t) = a + \int_0^t F(s, f(s))\, ds.$$

a) Explain why $G(f) \in C[-T, T]$ if $f \in C[-T, T]$.

b) Show that if f solves the initial value problem for $t \in [-T, T]$, then $G(f) = f$.

c) Show that G is Lipschitz with Lipschitz constant TK.

d) Assuming $T < 1/K$, show that there exists a solution of $G(f) = f$ defined for $t \in [-T, T]$. You may use the fact that $C[-T, T]$ is complete; we'll show this later.

e) Assuming $T < 1/K$, show that there exists a unique solution of the initial value problem defined on $(-T, T)$.

f) Extra credit: Show that there exists a solution f of the initial value problem defined for all $t \in \mathbb{R}$.

Solution, part a:
Suppose $f \in C[-T, T]$. Let $g(s) = F(s, f(s))$. Then $g \in C[-T, T]$ as well. The Fundamental Theorem of Calculus implies that

$$H(t) = \int_t^x g(s)\, ds$$

is continuous on $[-T, T]$ and differentiable on $(-T, T)$ with derivative $H'(t) = g(t)$. But then $G(f)(t) = a + H(t)$ and hence G is continuous.

Solution, part b:
Suppose f solves the initial value problem. Then $f'(t) = F(t, f(t))$ for all $t \in (-T, T)$. By our previous remarks, for any $h \in C[-T, T]$, $G(h)'(t) = F(t, h(t))$. In particular,

$G(f)'(t) = F(t, f(t))$. That is, f and $G(f)$ are two functions that have the same derivative over an interval. They therefore differ by a constant. But $f(0) = G(f)(0) = a$. So they are the same functions. That is, $G(f) = f$.

Solution, part c:
Let $g, h \in C[-T, T]$. Then for $t \in [0, T]$

$$\begin{aligned}|G(g)(t) - G(h)(t)| &= \left|\int_0^t F(s, g(s)) - F(s, h(s))\, ds\right| \\ &\le \int_0^t |F(s, g(s)) - F(s, h(s))|\, ds \\ &\le \int_0^T K|g(s) - h(s)|\, ds \\ &\le KT\|g - h\|_\infty.\end{aligned}$$

A similar computation holds in the case $t \in [-T, 0]$, swapping the endpoints of the integrals and replacing T with $-T$ as and endpoint. Thus $|G(g)(t) - G(h)(t)| \le KT\|g - h\|_\infty$ for all $t \in [-T, T]$ and therefore $\|G(g) - G(h)\|_\infty \le KT\|g - h\|_\infty$. This proves that G is Lipschitz continuous with Lipschitz constant KT.

Solution, part d:
If $T < 1/K$ then G is a contraction and, since $C[-T, T]$ is complete, problem 1 implies that there is a unique fixed point f solving $G(f) = f$.

Solution, part e:
Suppose $T < 1/K$. Then there exists a fixed point f. The Fundamental Theorem of Calculus implies $G(f)'(t) = F(t, f(t))$ for all $t \in (-T, T)$. Since $G(f) = f$, $f'(t) = F(t, f(t))$ for all $t \in (-T, T)$. Since $f(0) = G(f)(0) = a$, f is a solution of the initial value problem on $(-T, T)$.

Now suppose $\hat{f}$ is some solution of the initial value problem on $(-T, T)$. Suppose $0 < T' < T$. Then $\hat{f}$ is a solution of the initial value problem on $(-T', T')$ as well and is continuous on $[-T', T']$. Let G_T and $G_{T'}$ be the functions G associated with the intervals $[-T, T]$ and $[-T', T']$. By subproblem b), $G_{T'}(\hat{f}) = \hat{f}$ on $[-T', T']$. But $G_{T'}(f) = G_T(f)|_{[-T',T']} = f$ on $[-T', T']$ as well. Since $KT' < KT < 1$, we know that $G_{T'}$ has exactly one fixed point. So $\hat{f} = f$ on $[-T', T']$. Since $T' < T$ is arbitrary, this also holds on $(-T, T)$. Hence there is one and only one solution of the IVP on this interval.

2. Carothers 8.66 [Max]

3. Carothers 8.76 [Sakti]

4. Carothers 8.77 [Jody]: Fix $k \ge 1$ and define $f : \ell_\infty \to \mathbb{R}$ by $f(x) = x_k$. Show that f is linear and has $\|f\| = 1$.

Solution:
Let $x, y \in \ell_\infty$ and $\alpha, \beta \in \mathbb{R}$. Then

$$f(\alpha x + \beta y) = (\alpha x + \beta y)_k = \alpha x_k + \beta y_k = \alpha f(x) + \beta f(y).$$

Hence f is linear. Let $x \in \ell_\infty - \{0\}$. Then $|T(x)| = |x_k|$ and $\|x\|_\infty \geq |x_k|$. Thus $\|f\| = |T(x)|/\|x\|_\infty \leq 1$. Note the sequence y defined by $y_n = 0$ for all $n \neq k$ and $y_k = 3$ gives the value $|T(y)|/\|y\|_\infty = 3/3 = 1$. Hence

$$\|T\| = \sup_{x \neq 0} \frac{|T(x)|}{\|x\|_\infty} = \frac{|T(y)|}{\|y\|_\infty} = 1.$$

5. Carothers 8.78 (Solution by Lander Ver Hoef)

Define a linear map $f : \ell_2 \to \ell_1$ by $f(x) = (x_n/n)_{n=1}^\infty$. Is f bounded? If so, what is $\|f\|$?

Solution:
f is not bounded. Choose M in $\mathbb{R}$. Then let x be the sequence in ℓ_2 defined by $x_n = M/n$. Then

$$\|f(x)\|_1 = \sum_{n=1}^\infty \left|\frac{M}{n^2}\right| = |M| \sum_{n=1}^\infty \left|\frac{1}{n^2}\right| = |M| \frac{\pi^2}{6} > |M| \geq M.$$

Hence, for all $M \in \mathbb{R}$, there exists an $x \in \ell_2$ such that $\|f(x)\| > M$. Thus, f is not bounded.

6. Carothers 8.80 [Mason]

7. Carothers 8.81 [Max]

8. Carothers 8.84 (Solution by Lander Ver Hoef)

Prove that $B(V, W)$ is complete whenever W is complete.

Solution:
Suppose W is complete, and let (T_n) be a Cauchy sequence in $B(V, W)$. We assert that for any fixed $v \in V$, the sequence $(T_n(v))$ is Cauchy in W. To show this, let $\varepsilon > 0$. Then there exists an N such that if $n, m \geq N$, then $\|T_n - T_m\| < \varepsilon$; that is,

$$\sup_{v \in V} \frac{\|T_n(v) - T_m(v)\|}{\|v\|} < \varepsilon.$$

Then, for fixed $v \in V$ and $n, m \geq N$, we have that

$$\|T_n(v) - T_m(v)\| \leq \|(T_n - T_m)(v)\| < \|v\|\,\varepsilon.$$

Thus, $(T_n(v))$ is Cauchy in W, and because W is complete, $(T_n(v))$ converges to some $T(v)$.

We must first show that this T is in fact a linear map, and then we will show that it is bounded. Observe that $\alpha, \beta \in \mathbb{R}$ and $u, v \in V$,

$$T(\alpha u + \beta v) = \lim_{n\to\infty} T_n(\alpha u + \beta v) = \alpha \lim_{n\to\infty} T_n(u) + \beta \lim_{n\to\infty} T_n(v) = \alpha T(u) + \beta T(v).$$

Thus, T is linear.

Next, because (T_n) is Cauchy, it is bounded. That is, there exists an M in $\mathbb{R}$ such that $\| T_n \| \leq M$ for all n. That is,

$$\sup_{v \in V} \frac{\| T_n(v) \|}{\| v \|} \leq M.$$

Then, for $v \in V$, observe that

$$\| T(v) \| = \lim_{n\to\infty} \| T_n(v) \| \leq \lim_{n\to\infty} M \| v \| = M \| v \| .$$

Therefore, T is bounded, and $T \in B(V, W)$.

Finally, let $\varepsilon > 0$. Then there exists an N such that if $n, m \geq N$, then $\| T_n - T_m \| < \varepsilon$. Fix $n \geq N$. Then

$$\| T_n - T \| = \left\| T_n - \lim_{m\to\infty} T_m \right\| = \lim_{m\to\infty} \| T_n - T_m \| ,$$

by the continuity of the norm. But for all $m \geq N$, $\| T_n - T_m \| < \varepsilon$, so $\lim_{m\to\infty} \| T_n - T_m \| < \varepsilon$, and $\| T_n - T \| < \varepsilon$.

9. Carothers 10.7 [Sakti]

10. Carothers 10.9 (No rigor, please!) [Mason]

11. Carothers 10.10 [Jody]: Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and define $f_n(x) = f(x + (1/n))$. Show that $f_n \xrightarrow{\text{uniformly}} f$ on $\mathbb{R}$.

Solution:
Let $\epsilon > 0$ and let $x \in \mathbb{R}$. Since f is uniformly continuous then there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$ for all $y \in \mathbb{R}$. Pick $N \in \mathbb{N}$ such that $1/N < \delta$. If $n \geq N$ then

$$\left| \left(x + \frac{1}{n} \right) - x \right| = \frac{1}{n} \leq \frac{1}{N} < \delta,$$

which implies

$$|f_n(x) - f(x)| = \left| f\left(x + \frac{1}{n} \right) - f(x) \right| < \epsilon.$$

Hence, noting that N is independent of x, $f_n \xrightarrow{\text{uniformly}} f$ on $\mathbb{R}$.