Back in the first week we discussed that exponential phenomena are doubling phenomena. If a population of bacteria is growing and it doubles every a hours then

$$P(t) = P_0 2^{t/a}.$$

where P_0 is the population at time $t = 0$. If a radioactive isotope loses half of its mass every a years (the half life) then the mass of the isotope is

$$m(t) = m_0 2^{-t/a}.$$

where m_0 is the mass at time $t = 0$. Base two isn't always convenient, however, and in calculus we find base e is best because e^x has such a nice derivative. We can always convert base 2 or base anything to base e since

$$2 = e^{\ln 2} \implies 2^{t/a} = e^{\frac{\ln 2}{a} t}.$$

So we work with expressions of the form e^{kt} where k is a constant ($k = \frac{\ln 2}{a}$ in the above example).

If a population is growing according to

$$P(t) = e^{kt}$$

then

$$P'(t) = ke^{kt} = kP(t).$$

This says that the rate of growth of the population ($P'(t)$) is proportional to the size of the population ($P(t)$). This sort of relation is fundamental in the sciences and it is the reason exponential growth shows up in numerous applications: in chemistry in discussing rates of reactions, in physics in radioactive decay and cooling, and in biology in population models among many other applications.

In this worksheet we look at Newton's Law of Cooling, which states that if an object has temperature $T(t)$ at time t, and is kept in an environment with constant temperature T_0, then

$$T'(t) = k(T(t) - T_0)$$

for some constant k. For example, you can think of $T(t)$ as the temperature of a cup of coffee and T_0 as the temperature of the room. Or you can think of $T(t)$ as the temperature inside a house, and T_0 the outside air temperature. It says the rate of change of temperature is proportional to the difference between the object's temperature and the temperature of its surroundings.

1. In Newton's law of cooling, if the temperature of the object is the same as the environment's temperature, what is the value of T'?

$$T' = k(T - T_0) = k(T_0 - T_0) = 0$$

2. In Newton's law of cooling, will the constant k be positive or negative?

Negative. If $T > T_0$ the object should cool, $T' < 0$.

But $T' = k\underbrace{(T - T_0)}_{>0}$ so $k < 0$.

3. Starting with Newton's Law of Cooling

$$T'(t) = k(T(t) - T_0)$$

introduce a new variable $u(t) = T(t) - T_0$. Show that $u'(t) = ku$. Conclude that $u(t) = u_0 e^{kt}$ for some constant u_0 is a solution of the law of cooling equation.

$$u' = T' - 0$$
$$= k(T - T_0)$$
$$= ku$$

So $u(t) = u_0 e^{kt}$ is a solution.

4. Compute $\lim_{t\to\infty} u(t)$ and $\lim_{t\to\infty} T(t)$.

Since $k<0$, $\lim_{t\to\infty} u(t) = u_0 \lim_{t\to\infty} e^{kt} = u_0 \cdot 0 = 0.$

Since $T(t) = u(t) + T_0$, $\lim_{t\to\infty} T(t) = 0 + T_0 = T_0$

5. A cup of coffee is cooling on a table. Air temperature is 20°C. The coffee has temperature 70°C at time $t = 0$ minutes. What is the value of u_0?

$u_0 = T(0) - T_0 = 70 - 20 = 50°C$

6. At time $t = 10$ minutes, the temperature of the coffee is measured to be 60°C. Determine the value of k.

$T(10) = 60$ so $u(10) = 40$

$u(t) = 50e^{kt}$ so $40 = 50e^{k10}$

$e^{k\cdot 10} = \frac{4}{5}$

$k = \frac{1}{10}\ln(4/5)$

≈ -0.0223

7. What is the temperature of the coffee at time $t = 20$ minutes?

$$T(20) = T_0 + u(20)$$
$$= T_0 + 50\, e^{\ln(4/5)\frac{20}{10}}$$
$$= 20 + 50 \cdot \left(\tfrac{4}{5}\right)^2 = 52°C$$