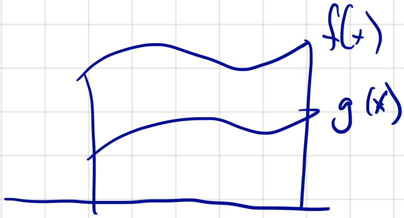
g) If $f(x) \geq 0$ $\quad \int_c^b f(x)\,dx \geq 0.$

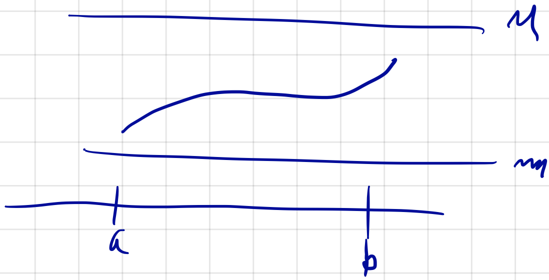
$a \leq b$

h) If $f(x) \geq g(x)$ $\quad \int_c^b f(x) \geq \int_c^b g(x)\,dx$

$a \leq x \leq b$



i)



$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

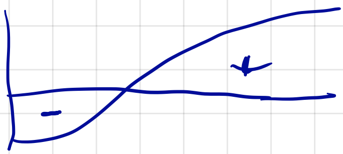
Last class:

Defined $\int_a^b f(x)\,dx$ (Riemann integral)

I should mention it's not defined for any function f,

but is if: f is cts, or is bounded and finitely many points of discont.

Geometric interpretation: net area



We also saw that it is a net change.

$v(t) \to$ velocity,

i.e. $\int_a^b v(t)\,dt$ is net distance traveled

from $t=a$ to $t=b$.

If $s(t)$ is position,

$$s'(t) = v(t)$$

And $\int_a^b v(t)\,dt = s(b) - s(a)$

net change in position.

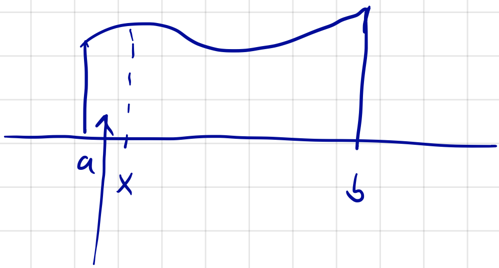
i.e. integrate a rate of change and get a net change.

This was just heuristic and today we make this rigorous.

Here's a funny looking function

$f(t)$ defined on $[a,b]$, continuous

$$G(t) = \int_a^x f(s)\,ds.$$



$G(x)$ is area in here.

$G(a) = 0$

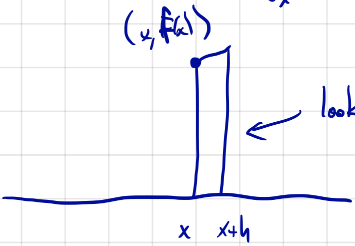
$$G(b) = \int_a^b F(x)\,dx$$

$$G'(\cancel{x}) = \lim_{h \to 0} \frac{F(\cancel{x}+h) - F(\cancel{x})}{h}$$

$$G(x+h) - G(x) = \int_a^{x+h} f(s)\,ds - \int_a^{x} f(s)\,dx$$

$$= \int_a^{x} f(s)\,ds + \int_x^{x+h} f(s)\,ds - \int_a^{x} f(s)\,dx$$

$$= \int_x^{x+h} f(s)\,ds$$



looks like a rectangle of height $F(x)$ and width h.

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h} \int_x^{x+h} f(s)\,ds \longrightarrow f(x).$$

$$G'(x) = f(x)$$

In other language, the area under the curve function $F(x)$ is an antiderivative

FTC (Part I)

If $f(x)$ is continuous on $[a,b]$ then

$$F(x) = \int_0^x f(s)\,ds$$

is diff on (a,b) and $F'(x) = f(x)$.

Moral: If $f(x)$ is continuous on and you really need an antiderivative, no sweat

$$G(x) = \int_a^x f(s)\,ds \text{ will do.}$$

Alt: $$\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$$

My guess is you are underwhelmed.

<u>But</u>

Suppose you want to integrate

$$\int_0^{\pi} \sin(x)\,dx$$

$$G(x) = \int_0^x \sin(s)\,ds$$

$$G'(x) = \sin(x)$$

Do you know any others?

$$\frac{d}{dx} -\cos(x) = \sin(x)$$

But then $F(x)$ and $-\sin(x)$ have same derivative on $[0,\pi]$. So

$$F(x) = -\cos(x) + C$$

I want $\int_0^{\pi} \sin(x)\,dx = F(\pi)$

If only I knew C.

But $F(0) = 0$ so $-\cos(0) + C = 0$

and $C = \cos(0)$

$$\int_0^{\pi} \sin(x)\,dx = \overset{F(\pi)}{-\cos(\pi)} + \cos(0)$$

$$= -(-1)+1 = 2$$

FTC (Part II)

Suppose $f(x)$ is continuous on $[a,b]$

and $F'(x) = f(x)$ on $[a,b]$

then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Want $\int_0^{\pi} \sin(x)\,dx$. Find $F(x)$, $F'(x) = \sin(x)$

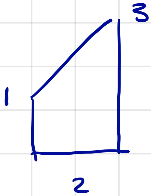
($F(x) = -\cos(x)$ will do!)

Then $\int_0^{\pi} \sin(x)\,dx = -\cos(\pi) + \cos(0)$

$$= -(-1)+1 = 2.$$

The proof is not much more than what we did for cos, so let's apply it instead.

$$\int_1^3 t\,dt$$



$$\frac{1}{2}(3+1)\cdot 2 = 4$$

$$F(t) = \frac{t^2}{2} \qquad F'(t) = t \checkmark$$

$$\int_1^3 (t)\,dt = F(3) - F(1)$$

$$= \frac{3^2}{2} - \frac{1}{2}$$

$$= \frac{9}{2} - \frac{1}{2} = 4 \checkmark$$

Shorthand: $$\int_1^3 t\,dt = \frac{t^2}{2}\Big|_1^3 \rightarrow \frac{3^2}{2} - \frac{1^2}{2}$$

$$F(x)\Big|_a^b = F(b) - F(a), \text{ by def.}$$

e.g. compute

$$\int_1^5 e^x \, dx = e^x \Big|_1^5 = e^5 - e^1$$