L'Hôpital's Rule

If f and g are differentiable and $g'(x) \neq 0$ on an interval containing a (except possibly at $x = a$). If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

so long as the right-hand limit exists, or is $\pm\infty$. Moreover, the same technique can be used

- if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$,
- for one-sided limits,
- for limits at infinity.

Compute the following limits.

1. $\lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)}$

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(3x)} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{5\cos(5x)}{3\cos(3x)} = \frac{5\cdot 1}{3\cdot 1} = \frac{5}{3}$$

2. $\lim_{x\to 0} \frac{\cos(x) - 1}{x}$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\sin(x)}{1} = \frac{-\sin(0)}{1} = 0$$

3. $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x^2}$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\sin(x)}{2x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\cos(x)}{2} = -\frac{1}{2}$$

4. $\lim\limits_{x\to -\infty} x e^x$.

$$\lim_{x\to -\infty} x e^x = \lim_{x\to -\infty} \frac{x}{e^{-x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to -\infty} \frac{1}{-e^{-x}} = \lim_{x\to -\infty} e^x = 0$$

5. $\lim\limits_{x\to 0} \dfrac{\arcsin(x)}{x}$

$$\lim_{x\to 0} \frac{\arcsin(x)}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = 1$$

6. $\lim_{x\to 0} \frac{e^x}{x+3}$. Careful!!

$$\lim_{x\to 0} \frac{e^x}{x+3} = \frac{e^0}{0+3} = \frac{1}{3}$$

Note: not a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, so l'Hôpital's rule does not apply.

7. $\lim_{x\to 0^+} \frac{e^{1/x}}{\ln x}$.

$$\lim_{x\to 0^+} \frac{e^{1/x}}{\ln(x)} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{e^{1/x}(-1/x^2)}{1/x} = \lim_{x\to 0^+} \frac{-e^{1/x}}{x}$$

$$= \lim_{x\to 0^+} -\frac{1}{x} \cdot e^{1/x}$$

$$= -\infty \cdot \infty = -\infty$$

8. $\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^{\cancel{5}x}$.

① Take ln first: $\lim_{x\to\infty} \ln\left(\left(1+\frac{5}{x}\right)^x\right) = \lim_{x\to\infty} x \ln\left(1+\frac{5}{x}\right)$

$$= \lim_{x\to\infty} \frac{\ln(1+5/x)}{1/x}$$

$$\overset{\frac{0}{0}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+5/x} \cdot \left(\frac{-5}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{5}{1+5/x} = 5$$

②: $\lim_{x\to\infty} \left(1+\frac{5}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{5}{x}\right)^x\right)}$

$$= \boxed{e^5}$$