1. Carothers 2.21 (Solution by Sakti Anupindi)

Show that any ternary decimal of the form $0.a_1a_2\ldots a_n11$, i.e., any finite-length decimal ending in two (or more) 1s, is not an element of Δ.

Solution:
There are only 2 ternary expansions for finite length decimal of the form $0.(a_1)(a_2)\ldots(a_n)11$,
(1) Ending in zeros i.e $0.(a_1)(a_2)\ldots(a_n)1100000\ldots$
(2) Ending in 2's i.e $0.(a_1)(a_2)\ldots(a_n)1022222\ldots$. Clearly both of these representations are not part of the Cantor set Δ.
Alternative Solution:
Let ternary representation of $x = 0.a_1a_2\ldots$ for any x $\in [0,1]$, where $a_i = 0, 1, 2$. Starting with $C_0 = [0,1]$, let's consider intervals below

$$C_0 = [0,1]$$
$$C_1 = C_0 - (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$
$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9} \cup [\frac{8}{9}, 1]$$
$$C_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{3}{27}] \cup [\frac{6}{27}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{9}{27}] \cup \frac{18}{27}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{21}{27}]$$

and continuing for $n = 0, 1, 2..$ define Cantor set C $= \cap_{n=0}^{\infty} C_n$.

Note that for finite ternary expansion for $a_1 = 0$, for 2 digits ending in '1' after a_0 give $0.011 = \frac{4}{27}$ while for $a_1 = 2$ and two digits ending in '1' give, $0.211 = \frac{22}{27}$ both which are excluded from cantor set as can be seen from above intervals. This shows for a start that any finite length decimal ending in two or more 1's is not an element of Cantor set. Also note that endpoints of the intervals in C_n are of form $\frac{m}{3^m}$ for integer m, for $0 \le m \le 3^m$, showing the end points are rational.

If x has a ternary representation not involving the digit 1, then from Carothers(Theorem 2.15) x has the ternary represenation of the form $x = 0.(2a_1)(2a_2)\ldots$ where $a_i = \{0, 2\}$. So a point x is in the cantor set if and only if at least one ternary representation of x doesnot have digit 1 in any place that means it has to be in the form above.

So, if $x = 0.a_1a_2\ldots$, with atleast one $a_n \neq 0, 2$, then the point x is in the open interval $(\frac{1}{3}, \frac{2}{3})$. It will be shown that this will be the pattern at the n^{th} stage of removing open intervals from the cantor set in [0,1]. It is clear in the cantor sets that the first interval that is removed from $[0,1]$ is the open interval $(\frac{1}{3} < x < \frac{2}{3})$

Assume that x lies in the open interval $(\frac{1}{3} < x < \frac{2}{3})$.It will be shown that there is a contradiction assuming each of the cases below. For a given ternary representation $x = 0.a_1a_2\ldots$ then
1. Consider $a_1 = 0$ which gives $x > \frac{1}{3}$ as below (a contradiction to above assumption)

$$x = \frac{0}{3} + \sum_{n=2}^{\infty} \frac{a_n}{3^n} \le \sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{2}{3^2} \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{2}{3^2} \frac{1}{1 - \frac{1}{3}} = \frac{1}{3}$$

2. Consider now, let $a_2 = 2$, it gives $x \ge \frac{2}{3}$ but $x < \frac{2}{3}$.

3. Finally consider $a_2 = a_3 = \ldots 0$ giving $x = \frac{1}{3}$, while $a_2 = a_3 \ldots 2$ giving $x = \frac{2}{3}$, So these cases suggest that $a_1 = 1$ and some of (at least one) $a_2, a_3 \ldots$ not equal to $\{0, 2\}$

Now,

$$\frac{1}{3} = 0.01, \frac{2}{3} = 0.2$$

which is the first open interval removed from the cantor set in ternary expansion in the 1st stage and

$$\frac{1}{3^2} = 0.01, \frac{2}{3^2} = 0.02, \frac{7}{3^2} = 0.21, \frac{8}{3^2} = 0.22$$

i.e these are intervals that are removed from cantor set in 2nd stage, so inductively the open interval removed at n^{th} stage is of the form $(0.(2a_1)(2a_2)\ldots(2a_{n-1})1, 0.(2a_1)(2a_2)\ldots(2a_{n-1})2$ and if $x = 0.(2a_1)(2a_2)\ldots(2a_{n-1})1a_{n+1}a_{n+2}\ldots$ with atleast one $a_{n+k} \neq \{0, 2\}$, then those open intervals are removed and not part of cantor set.

So a point $x \in [0, 1]$ is not removed if and only if it can be written in ternary form without using digit 1.

2. Carothers 2.22 (Solution by Mason Brewer)

Show that Δ contains no (nonempty) open intervals. In particular, show that if $x, y \in \Delta$ with $x < y$, then there is some $z \in [0, 1] \setminus \Delta$ with $x < z < y$. (It follows from this that Δ is *nowhere dense*, which is another way of saying that Δ is "small.")

Solution:
Let $x, y \in \Delta$ with $x < y$. Then let $x = 0.a_1a_2a_3...$ and $y = 0.b_1b_2b_3...$ be the base-3 expansions of x and y without any values of 1. Since they are not equal, there exists some i that is the first decimal place where x and y disagree. In other words, $a_n = b_n$ for $n < i$, and $a_i < b_i$, which must be the case because $x < y$. Thus it must be the case that $a_i = 0$ and $b_i = 2$ in order to satisfy $a_i < b_i$. Now define $z = 0.c_1c_2c_3...$ such that $c_n = a_n = b_n$ for $n < i$, and $c_n = 1$ for $n \geq i$. Note that since z has more than a single decimal equal to 1, it must be that $z \notin \Delta$, which implies that $z \neq x, y$. Since $a_i < c_i < b_i$, we know that $x < z < y$.

3. Carothers 2.25 (Solution by David Maxwell)

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 1$ if $x \in \Delta$, and $g(x) = 0$ otherwise. At which points of $\mathbb{R}$ is g continuous?

Solution:
Let $A = \mathbb{R} \setminus \Delta$. We claim that g is continuous exactly on A.

First, suppose $x \in A$, and let $\epsilon > 0$. Notice that Δ is a closed set, being an intersection of closed sets, and hence A, its complement, is open. Thus there is a $\delta > 0$ such that if $|x - y| < \delta$ then $y \in A$. But then $g(x) = g(y) = 0$ and $|g(x) - g(y)| = 0 < \epsilon$. Hence g is continuous at x

On the other hand, suppose $x \notin A$, so $x \in \Delta$. By the previous problem, for each $n \in \mathbb{N}$, we can find $x_n \in (x - 1/n, x + 1/n)$ such that $x_n \notin \Delta$. Now $|x - x_n| < 1/n$, so $x_n \to x$.

And $g(x_n) = 0$ for all n whereas $g(x) = 1$. Since $g(x_n) \not\to g(x)$, we conclude that g is not continuous at x.

4. Carothers 2.16 (Solution by Lander Ver Hoef)

The *algebraic numbers* are those real or complex numbers that are the roots of polynomials having *integer* coefficients. Prove that the set of algebraic numbers is countable. [Hint: First show that the set of polynomials having integer coefficients is countable.]

Solution:
First, observe that for a given n, there is a natural surjective mapping from the ordered n-tuple of integers with a non-zero final element (that is, an element of $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$) to the set of polynomials of degree $n-1$ with integer coefficients, given by

$$f(a_1, a_2, \ldots, a_n) = a_1 + a_2 x + \cdots + a_n x^{n-1},$$

where $a_n \neq 0$. Thus, the set P_n of all polynomials of degree n with integer coefficients is countable.

The set P of all polynomials with integer coefficients is the union of the P_n across n, and the countable union of countable sets is, itself, countable, so P is countable.

Each polynomial in P has countably many roots, so there is a surjective mapping from $P \times \mathbb{N}$ to the set of algebraic numbers, defined by (p, n) being mapped to the nth root of the polynomial p. Thus, there are only countably many algebraic numbers.

5. Carothers 3.7 (Solution by Max Heldman)

Let $f : [0, \infty) \to [0, \infty)$ be increasing and satisfy $f(0) = 0$, and $f(x) > 0$ for all $x > 0$. If $f(x)$ also satisfies $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$, then $f \circ d$ is a metric whenever d is a metric. Each of the following conditions is sufficient to ensure that $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$:

a) f has a second derivative satisfying $f'' \leq 0$;

b) f has a decreasing first derivative;

c) $f(x)/x$ is decreasing for $x > 0$;

Solution:
We first show that (a) $\implies$ (b) $\implies$ (c), and then prove that (c) is sufficient. To show (a) $\implies$ (b), we prove the contrapositive. Suppose $f'(x+h) > f'(x)$ for some $h > 0$. Then by the Mean Value Theorem there exists $c \in (x, x+h)$ such that $f''(c) = \frac{f'(x+h) - f'(x)}{h} > 0$.

For (b) $\implies$ (c), it is sufficient to show that $\frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{f'(x)x - f(x)}{x^2} \leq 0$ for $x > 0$. Observe that by Taylor's Theorem we have $f(x) = f(0) + f'(tx)x$, where $t \in [0, 1]$. Since f' is decreasing and $f(0) = 0$,

$$f(x) = f(0) + f'(tx)x = f'(tx)x \geq f'(x)x.$$

To complete the proof, suppose $f(x)/x$ is decreasing for $x > 0$. Let $x \geq y > 0$. Then $\frac{f(x+y)}{x+y} \leq \frac{f(x)}{x}$, and $\frac{f(x)}{x} \leq \frac{f(y)}{y}$, that is, $f(x)y \leq f(y)x$. Thus

$$f(x+y) \leq \frac{(x+y)f(x)}{x} = \frac{f(x)x + f(x)y}{x} \leq \frac{f(x)x + f(y)x}{x} = f(x) + f(y).$$

6. Carothers 3.15 (Solution by David Maxwell)

Show that a set A is bounded if and only if the diameter of the set is finite.

Solution:
Suppose A is bounded. So we can pick $x \in A$ and $R > 0$ such that $A \subseteq B_R(x)$. But then if $a, b \in A$, $d(a,b) \leq d(a,x) + d(x,b) < 2R$. Hence

$$\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\} \leq 2R.$$

So A has finite diameter.

Conversely, suppose A has finite diameter D. By hypothesis, A is not empty; let $x \in A$. For any $y \in A$, $d(y,x) \leq \operatorname{diam}(A) = D < 2D$. Thus $A \subseteq B_{2D}(x)$ and A is bounded.

7. Carothers 3.18 (Solution by Jody Gaines)

Show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for any $x \in \mathbb{R}^n$. Also check that $\|x\|_1 \leq n\|x\|_\infty$ and $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

Solution:
Let $y = \max_{1 \leq k \leq n} |x_k|$. Since $y \in \{x_1, \ldots, x_n\}$ then $y^2 \leq x_1^2 + \ldots + x_n^2$, which implies

$$\|x\|_\infty^2 = y^2 \leq \sum_{k=1}^{n} |x_k|^2 = \|x\|_2^2.$$

Moreover

$$\|x\|_2^2 = \sum_{k=1}^{n} |x_k|^2 = \sum_{k=1}^{n} |x_k||x_k| \leq \sum_{j=1}^{n} \sum_{i=1}^{n} |x_i||x_j| = \left(\sum_{k=1}^{n} |x_k| \right)^2 = \|x\|_1^2.$$

Since $\|x\|_1$, $\|x\|_2$, $\|x\|_\infty$ are nonnegative then $\|x\|_\infty^2 \leq \|x\|_2^2 \leq \|x\|_1^2$ implies $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$.

Now observe that

$$\|x\|_1 = \sum_{k=1}^{n} |x_k| \leq \sum_{k=1}^{n} y = ny = n\|x\|_\infty.$$

By the Cauchy-Schwartz inequality we have that

$$\sum_{k=1}^{n} |x_k| \leq \|x\|_2 \|(1, \ldots, 1)\|_2 = \sqrt{n}\|x\|_2.$$

Hence $\|x\|_1 \leq \sqrt{n}\|x\|_2$ and we are done.

8. Carothers 3.23 (Solution by Former 641 Student Yuanyuan Zhao)

The subset of ℓ_∞ consisting of all sequences that converge to 0 is denoted by c_0. Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$.

Solution:
Let $(x_n) \in c_0$. Then $(x_n) \to 0$, so (x_n) must be bounded. Therefore, $(x_n) \in \ell_\infty$ and $c_0 \subseteq \ell_\infty$. Since $x_n = 1$ does not converge to 0, the inclusion is proper.

Next, let $(x_n) \in \ell_2$. Then $\sum |x_n|^2 < \infty$, so $|x_n|^2 \to 0$. Hence, $x_n \to 0$ and $(x_n) \in c_0$. Thus, $\ell_2 \subseteq c_0$. Since $x_n = \frac{1}{\sqrt{n}} \to 0$, but $\sum |x_n|^2 = \sum 1/n$ diverges, the inclusion is proper.

Finally, let $(x_n) \in \ell_1$. Then $\sum |x_n| < \infty$, so $|x_n| \to 0$. In particular, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n| \leq 1$. Hence, $|x_n|^2 \leq |x_n|$ for $n \geq N$. By the Comparison Test, $\sum |x_n|^2$ must converge as well. Therefore, $(x_n) \in \ell_2$ and $\ell_1 \subseteq \ell_2$. If $x_n = 1/n$, then $\sum |x_n| = \sum 1/n$ diverges, but $\sum |x_n|^2 = \sum 1/n^2$ converges. Thus, $\ell_1 \subset \ell_2$ properly.