1. Let (a_n) and (b_n) be sequences of numbers such that $a_n \le b_n$ for all n.

a) Give a careful proof that

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n.$$

b) Show that it need not be true that

$$\limsup_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n.$$

2. Determine if the following functions are continuous.

a) $F : (C[0,1], L^2) \to \mathbb{R}$, $\quad F(f) = f(0)$

b) $G : (C[0,1], L^1) \to (C[0,1], L^2)$, $\quad G(f) = f$

c) $H : (C[0,1], L^2) \to (C[0,1], L^1)$, $\quad H(f) = f$

You are welcome to use the Cauchy-Schwartz inequality for integrals.

3. Carothers 8.53

4. Suppose $\{a_{ij} : i, j \in \mathbb{N}\}$ is a collection of non-negative real numbers. Suppose for each i, $\sum_{j=1}^{\infty} a_{ij}$ converges and moreover

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$$

converges to $L \in \mathbb{R}$. Show that for each j, $\sum_{i=1}^{\infty} a_{ij}$ converges and moreover

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = L.$$

Hint: Let $S_{nm} = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}$. Show that S_{nm} is bounded above by L for all n and m. Show moreover that given $\epsilon > 0$ that there is an N and M so that if $n \ge N$ and $m \ge M$ then $S_{nm} > L - \epsilon$.

5. Let X be a compact metric space, and let $\mathcal{X}$ be the set of non-empty closed subsets of X. If $a \in X$ and $B \subseteq X$, we define $d(a, B) = \inf_{b\in B} d(a, b)$. We define a metric, called the Hausdorff distance, on $\mathcal{X}$ by

$$H(A, B) = \sup_{a\in A} d(a, B) + \sup_{b\in B} d(b, A).$$

a) Suppose $X \subseteq \mathbb{R}^2$ is the closed ball of radius 100, A is the closed square with side length 1 centered at the origin, and B is the closed ball of radius 1/4 centered at the point (1/2, 1/2). Draw a picture of the arrangement and compute $H(A, B)$. (No rigor here please!)

b) Show that H is a metric on $\mathcal{X}$.

6. Let X be compact and let $\mathcal{X}$ be the set of non-empty closed subsets of X. Let H be the Hausdorff metric on $\mathcal{X}$ introduced previously. Suppose (F_n) is a sequence of nested closed sets, so

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots.$$

Show this is a convergent sequence, and compute its limit.

7. Carothers 11.65, and the following follow up problem:

Show that if $\int_a^b |K(x,t)|dt \leq 1$ for all $x \in [a,b]$, then the Arzela-Ascoli theorem implies that given any function f, the sequence $(T^{(n)}(f))$ has a subsequence that converges in $C[a,b]$.

8. Let $f \in C^{2\pi}$, and define

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

for $n = 0, 1, 2, \ldots$.

a) Suppose $h \in C^{2\pi}$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = 0, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx = 0$$

for $n = 0, 1, 2, \ldots$. Show that $h = 0$. *Hint:* See Application 11.6.

b) Suppose that $\sum_{n=1}^{\infty} |a_n| + |b_n|$ converges. Show that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

converges uniformly to a function $g \in C^{2\pi}$.

c) Let

$$\hat{a}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(kx)\,dx, \qquad \hat{b}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(kx)\,dx.$$

Show that $a_n = \hat{a}_n$ and $b_n = \hat{b}_n$ for all n.

d) Conclude that $f = g$.

Remark: In this problem you've shown that each element of $C^{2\pi}$ determines a different collection of numbers (a_n) and (b_n). Moreover, if these sequences converge to zero so fast that the series in part 2 converges, then the Fourier series of f converges uniformly to f. This begs the question: given an arbitrary element of $C^{2\pi}$, is it true that the series in part 2 converges? This is food for thought, not food for the exam.

9. Suppose $\sum |a_n|$ converges. By Carothers Exercise 10.26 we can define a function $f \in C(\mathbb{R})$ by $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$.

a) If there exist constants $M > 0$ and $\alpha > 2$ such that $|a_n| \leq \frac{M}{n^\alpha}$, prove that f is differentiable.

b) Visit `http://mathworld.wolfram.com/FourierSeriesTriangleWave.html`. Then remark on what that has to do with the current problem.

Remark: This problem illustrates a special case of the general principle that the faster the Fourier coefficients of a function converge to zero, the smoother that function is.