1. Consider the matrix

$$A = \begin{pmatrix} 23 & -8 & 4 \\ 21 & -8 & 5 \\ -126 & 42 & -19 \end{pmatrix}.$$

a) Show that $v_1 = [-1, -2, 3]$, $v_2 = [1, 3, 0]$ and $v_3 = [0, 1, 2]$ are eigenvectors of A, and determine their associated eigenvalues.

b) Compute the solution of

$$u' = Au$$

with initial condition $u(0) = v_3$. Show, by plugging your solution into the ODE, that your solution really is a solution.

c) Compute the solution of

$$u' = Au$$

with initial condition $u(0) = v_2 + v_3$. Show, by plugging your solution into the ODE, that your solution really is a solution.

d) Determine the exact solution of

$$u' = Au$$

with initial condition $u(0) = [1, 5, 5]$.

Solution, part a:
To show that x is an eigenvector, it is enough to show that $Ax = \lambda x$ for some number λ. A routine computation shows that $Av_1 = -5v_1$ $Av_2 = -v2$ and $Av_3 = 2v_3$ Thus $\lambda_1 = -5$, $\lambda_2 = -1$ and $\lambda_3 = 2$.

Solution, part b:
The solution is $u(t) = e^{2t}v_3$. Observe that $u(0) = e^0 v_3 = v_3$. Moreover, $u'(t) = 2e^{2t}v_3$ and $Au = e^{2t}Av_3 = e^{2t}2v_3 = 2u$. Comparing we find $u' = 2u = Au$.

Solution, part c:
The solution is $u(t) = e^{-t}v_2 + e^{2t}v_3$. Indeed,

$$u(0) = e^0 v_2 + e^0 v_3 = v_2 + v_3.$$

Moreover,

$$u'(t) = -e^{-t}v_2 + 2e^{2t}v_3$$

and

$$Au(t) = e^{-t}Av_2 + e^{2t}Av_3 = -e^{-t}Av_2 + 2e^{2t}Av_3.$$

So $u' = Au$.

Solution, part d:
We can write

$$[1, 5, 5] = v_1 + 2v_2 + v_3.$$

The solution of the IVP is therefore

$$u(t) = e^{-5t}v_1 + 2e^{-t}v_2 + e^{2t}v_3.$$

2. Suppose you wish to apply the RK4 method to solve the ODE of the previous problem. What is the largest time step you can use before issues concerning absolute stability arise in your solution?

Solution:
From the last homework assignment we know that the region of absolute stability for RK4 extends along the negative real axis up to about -2.9. For and ODE of the form $u' = \lambda u$ with $\lambda < 0$, we will encounter difficulties due to stability unless $z = \lambda h$ lies in the region of absolute stability. The ODE of the previous problem breaks into three decoupled ODEs of the form $u' = \lambda u$ along the eigenvectors. There are two negative eigenvalues, -1 and -5. Thus we need $-5h > -2.9$ and $-1h > -2.9$. Clearly the first condition is more restrictive and we need

$$h < \frac{2.9}{5} \approx 0.58.$$

3.

a) Use your Newton solver from last week's homework to implement the trapezoidal rule for solving systems of ODEs.

b) Determine the exact solution to the problem

$$\begin{aligned} u' &= 1 \\ v' &= v - u^2 \end{aligned} \tag{1}$$

with initial condition $u(0) = 0$ and $v(0) = 1$.

c) Test your solver against the previous exact solution and confirm that it has the predicted order of accuracy.

4. Consider this one-step (Runge-Kutta) method, the implicit midpoint method,

$$\begin{aligned} u_* &= u_n + \frac{h}{2} f(t_n + h/2, u_*) \\ u_{n+1} &= u_n + hf(t_n + h/2, u_*) \end{aligned} \tag{2}$$

The first equation (stage) is Backward Euler to determine an approximation to the value at the midpoint in time and the second stage is the midpoint method using this value.

a) Determine the order of accuracy of this method.

b) Determine the stability region.

c) Is this method A-stable?

Solution, part a:
Thinking of u_* as a function of h with u_n as a fixed a parameter we have $u_*(0) = u_n$ and

$$\frac{du_*}{dh} = \frac{1}{2}f(t_n + h/2, u_*) + \frac{h}{2} + \frac{d}{dh}f(t_n + h/2, u_*).$$

So $u_*'(0) = (1/2)f(t_n, u_n)$. It follows that

$$u_* = u_n + (1/2)f(t_n, u_n)h + O(h^2).$$

The local truncation error is

$$\tau = \frac{u(t+h) - u(t)}{h} - f(t + h/2, u_*(h; u(t))).$$

Now

$$u(t+h) = u(t) + u'(t)h + u''(t)h^2/2 + O(h^3) = u(t) + f(t,u)h + (f_t + f_u f)h^2/2 + O(h^3)$$

where we have used the fact that $u' = f(t,u)$ to rewrite u'' in terms of f. Thus

$$\frac{u(t+h) - u(t)}{h} = f(t,u) + (f_t + f_f f)\frac{h}{2} + O(h^2)$$

On the other hand, if

$$G(h) = f(t + h/2, u_*(h; u(t)))$$

we have

$$G(0) = f(t, u_*(0, u(t))) = f(t, u(t))$$

and

$$G'(0) = f_t(t, u(t))\frac{1}{2} + f_u u_*'(0; u(t)) = f_t(t, u(t))\frac{1}{2} + f_u\frac{1}{2}f(t, u(t))$$

by the computation above. Thus

$$f(t+h/2, u_*(h; u(t))) = G(h) = f(t, u(t)) + \frac{1}{2}\left[f_t(t, u(t)) + f_u(t, u(t))f(t, u(t))\right]h + O(h^2)$$

As a consequence,

$$\tau = f(t,u) + (f_t + f_f f)\frac{h}{2} + O(h^2) - (f(t, u(t)) + \frac{1}{2}\left[f_t(t, u(t)) + f_u(t, u(t))f(t, u(t))\right]h + O(h^2)) = O(h^2)$$

Thus the method is at least $O(h^2)$ accurate. A more detailed computation is needed to show that it is exactly $O(h^2)$ accurate. It uses the fact that

$$u''' = f_{tt} + 2f_{tu}f + f_{uu}f^2 + f_u(f_t + f_u t)$$

and

$$\frac{d^2}{dh^2}u_*|_{h=0} = \frac{1}{4}[f_t + f_u f].$$

The fact that no term of the kind f_{tt} appears here prevents the method from being $O(h^3)$

Solution, part b:
To determine the stability region we apply the method to $u' = \lambda u$. and find (using the notation $z = \lambda h$)

$$u_* = u_n + \frac{h}{2}\lambda u_* \implies u_* = \frac{1}{1-(z/2)}u_n$$

and therefore

$$u_{n+1} = u_n + z\frac{1}{1-z/2}u_n = \frac{2+z}{2-z}u_n.$$

The region of absolute stability is determined by the condition $|R(z)| \le 1$ where $R(z) = (2+z)/(2-z)$. Let us find the boundary of this region, which occurs when $|2+z|^2 = |2-z|^2$. That is,

$$4 + 4\mathfrak{R}(z) + |z|^2 = 4 - 4\mathfrak{R}(z) + |z|^2$$

These conditions combine to $8\mathfrak{R}(z) = 0$ and the boundary is therefore the imaginary axis. Noting that $R(-1) = 1/3$, we see that -1 lies in the region of absolute stability, which is therefore the entire left half plane.

Solution, part c:
The method is A-stable if its region of absolute stability includes the left half plane. In this case, like the trapezoidal rule, the region of absolute stability **is** the left half plane, and the method is A stable.