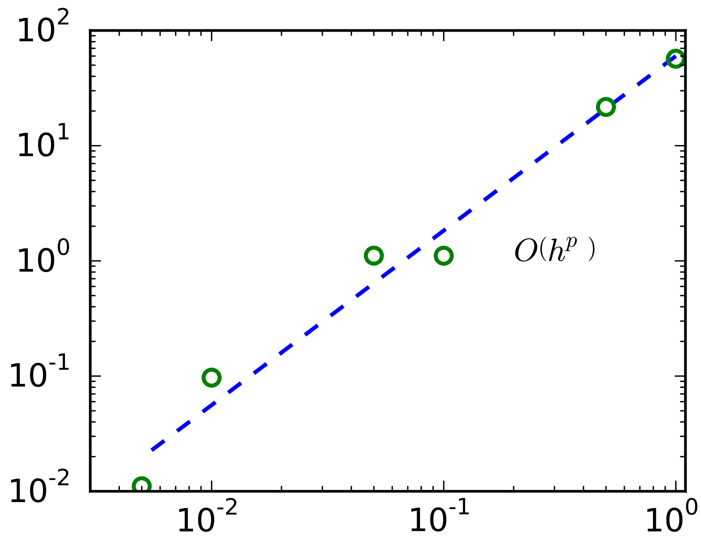
1. Suppose this table of "data" is samples of an $O(h^p)$ function:

h	1.0	0.5	0.1	0.05	0.01	0.005
Z	56.859	21.694	1.1081	1.1101	0.096909	0.011051

This data may be fitted (linear regression) by a function $f(h) = Mh^p$ for some values M and p, as in the following figure. Find p by fitting a straight line to the data, and reproduce the figure. Your version of the figure should have the value of p filled in.



Solution:

See Jupyter notebook.

2. Use Taylor's Theorem to verify the truncation term for the "Centered" row of Table 1.1 of your text. Hint: center all Taylor expansions at the same point.

Substantial partial credit will be awarded for showing the truncation term is $O(h^2)$, but try to get the exact expression with its constant. Hint: The average of two numbers lies in between the two numbers.

Solution:

Plugging a true solution of $u' = f$ into the centered difference scheme, the local truncation error is given by

$$-\tau_i = \frac{u(t_i + h) - u(t_i - h)}{2h} - u'(t_i).$$

Performing Taylor expansions at t_i we have

$$\begin{aligned} u(t_i + h) &= u(t_i) + u'(t_i)h + \frac{1}{2}u''(t_i)h^2 + \frac{1}{6}u'''(c_i)h^3 \\ u(t_i - h) &= u(t_i) - u'(t_i)h + \frac{1}{2}u''(t_i)h^2 - \frac{1}{6}u'''(\hat{c}_i)h^3 \end{aligned} \tag{1}$$

where c_i and $\hat{c}_i$ are unknown locations in $[t_i - h, t_i + h]$. Subsituting into the formula for τ_i we find

$$-\tau_i = \frac{h^2}{12}(u'''(c_i) + u'''(\hat{c}_i))$$

Assuming u''' is continuous, the average of $u'''(c_i)$ and $u'''(\hat{c}_i)$ is $u'''(d_i)$ for some other d_i in the interval. Thus

$$\tau_i = -\frac{h^2}{6}u'''(d_i).$$

3. Implement the following schemes for a scalar ODE:

1. Forward Euler
2. Backwards Euler
3. Trapezodial

Each method should be implemented with a function that takes the following arguments:

1. The right-hand side function $f(t,u)$.
2. The initial time t_0.
3. The initial value u_0.
4. The final time T.
5. The number M of time steps.

It should return a vector of sample times t_k, and a vector of solution values u_k.

Test your methods against $u' = -u$ and $u' = -\sin(t)$ with initial condition $u(0) = 1$ and confirm (using the technique of problem 1) that the order of convergence is the theoretically expected order for each method.

4. Consider the linear multistep method

$$u_{n+2} + 4u_{n+1} - 5u_n = h(4f_{n+1} + 2f_n)$$

where $f_k = f(t_k, u_k)$.

a) Show that this method is consistent.

b) In the case $f = 0$, the method reduces to a linear recurrence relation

$$u_{n+2} + 4u_{n+1} - 5u_n = 0.$$

The characteristic polynomial of this relation is $\sigma(\rho) = \rho^2 + 4\rho - 5$. Show that if ρ is a root of the characteristic polynomial, then $u_n = C\rho^n$ is a solution of the recurrence relation for any constant C. Moreover, if ρ_1 and ρ_2 are roots of the characteristic polynomial, then $u_n = C_1\rho_1^n + C_2\rho_2^n$ is a solution of the recurrence relation for any constants C_1 and C_2.

c) Compute the roots of the characteristic polynomial.

d) Implement this method (using Euler's method to compute u_1) and apply it to the IVP

$$\begin{aligned} u' &= -u \\ u(0) &= 1 \end{aligned}$$

on the t-interval $[0,1]$ with $M = 10$, 50 and 100.

e) Compute the global error in each of these three cases. Why is the error growing? Can you give an rough explanation for the rate of growth you observed?

Solution, part a:
From Taylor's Theorem,

$$\begin{aligned} u(t_i + 2h) &= u(t_i) + u'(t_i)(2h) + 2u''(t_i)h^2 + O(h^3) \\ u(t_i + h) &= u(t_i) + u'(t_i)h + \frac{1}{2}u''(t_i)h^2 + O(h^3) \\ u'(t_i + h) &= u'(t_i) + u''(t_i)h + O(h^2). \end{aligned} \tag{2}$$

Thus

$$\frac{u(t_i + 2h) + 4u(t_i + h) - 5u(t_i)}{h} = 6u'(t_i) + 4u''(t_i)h + O(h^2).$$

On the other hand,

$$4u'(t_i + h) + 2u'(t_i) = 6u'(t_i) + 4u''(t_i)h + O(h^2).$$

Comparing these last two equations we find that the truncation error for this method is $O(h^2)$ and the technique is consistent.

Solution, part b:
Let ρ be a solution of $\rho^2 + 4\rho - 5 = 0$. Setting $u_n = C\rho^2$ we find

$$u_{n+2} + 4u_{n+1} - 5u_n = C\rho^n(\rho^2 + r\rho - 5) = 0.$$

If ρ_1 and ρ_2 are two roots of the characteristic polynomial and $u_n = C_1\rho_1^n + C_2\rho_2^n$ we find

$$\begin{aligned} u_{n+2} + 4u_{n+1} - 5u_n &= (C_1\rho_1^{n+2} + C_2\rho_2^{n+2}) + 4(C_1\rho_1^{n+1} + C_2\rho_2^{n+1}) - 5(C_1\rho_1^n + C_2\rho_2^n) \\ &= C_1\rho_1^n(\rho_1^2 + 4\rho_1 - 5) + C_2\rho_2^n(\rho_2^2 + 4\rho_2 - 5) \\ &= 0. \end{aligned}$$

Solution, part c:
In this case, the roots of the characteristic polynomial are $\rho = 1$ and $\rho = -5$.

Solution, part d:

See Jupyter notebook.

Solution, part e:

See Jupyter notebook.

5. The two step Adams-Bashforth method is derived as follows. Suppose u_{i-1} and u_i have been computed already. There is a unique linear polynomial $p(t)$ that interpolates $(t_{i-1}, f(t_{i-1}, u_{i-1})$ and $(t_i, f(t_i, u_i))$. This linear polynomial provides an approximation for $f(t, u(t))$ on the interval $[t_i, t_{i+1}]$ and we replace the integral form of the ODE

$$u(t_{i+1}) = u(t_i) + \int_{t_i}^{t_{i+1}} f(t, u(t))\, dt$$

with

$$u_{i+1} = u_i + \int_{t_i}^{t_{i+1}} p(t)\, dt.$$

a) By explicitly integrating, show that this scheme can be written in the form

$$u_{i+1} = u_i + \frac{h}{2}(3f_i - f_{i-1}).$$

b) Compute the order of the local truncation error of this method.

c) This method is conditionally A-stable. Generate a plot of the boundary of the absolute stability region as follows.

(a) Write down the characteristic polynomial $p(\rho)$ for this method applied to the problem $u' = \lambda u$.

(b) The equation $p(\rho) = 0$ will involve the expresion λh. Solve for λh to write

$$\lambda h = f(\rho)$$

for some function f.

(c) Numerically determine values of $f(\rho)$ where ρ lives on the unit circle of complex numbers. These will generate values of λh where the associated root of the characteristic polynomial has size one, and is therefore potentially on the boundary of the stability region.

(d) Generate a plot of the values of $f(\rho)$ as ρ varies around the unit circle to see the boundary of the absolute stability region.

Solution, part a:
For a linear function $p(x)$ equal to $p(a)$ at a and $p(b)$ at b it is easy to see that

$$\int_a^b p(x)\, dx = \frac{p(a) + p(b)}{2}(b - a);$$

think about the area of area of a quaderalteral.

Consider a linear function $p(t)$ that equals that equals f_{i-1} at $t_i - h$ and that equals f_i at t_i. We can write it as

$$p(t) = f_{i-1}\frac{t_i - t}{h} + f_i\frac{t + h - t_i}{h}$$

and we observe

$$p(t_i + h) = f_{i-1}\frac{-h}{h} + f_i\frac{2h}{h} = -f_{i-1} + 2f_i.$$

Thus

$$\int_{t_i}^{t_i+h} p(t)\,dt = \frac{h}{2}\left[f_i + -f_{i-1} + 2f_i\right] = \frac{h}{2}(-f_{i-1} + 3f_i).$$

Solution, part b:
Let us write the scheme in the form

$$\frac{u_{i+2} - u_{i+1}}{h} - \frac{1}{2}(3f_{i+1} - f_i) = 0.$$

Now substitute the true solution of the differential equation into the left-hand side, using $u' = f$, to obtain

$$-\tau = \frac{u(t_i + 2h) - u(t_i + h)}{h} - \frac{1}{2}(3u'(t_i + h) - u'(t_i)).$$

Applying the Taylor expansions

$$\begin{aligned} u(t_i + 2h) &= u(t_i) + u'(t_i)2h + \frac{1}{2}u''(t_i)(2h)^2 + O(h^3) \\ u(t_i + h) &= u(t_i) + u'(t_i)h + \frac{1}{2}u''(t_i)(h)^2 + O(h^3) \\ u'(t_i + h) &= u'(t_i) + u''(t_i)h + O(h^2) \end{aligned} \tag{3}$$

we find

$$-\tau = u'(t_i) + \frac{3}{2}u''(t_i)h + O(h^2) - \frac{1}{2}(2u'(t_i) + 3u''(t_i)h + O(h^2)) = O(h^2).$$

So the the method is consistent and $O(h^2)$.

Solution, part c:
Delayed.