1. Carothers 16.4 [Sakti]

2. Carothers 16.12 [Mason]

3. Carothers 16.16 (Solution by Lander Ver Hoef)

   If $m^*(E) = 0$, show that $m^*(E \cup A) = m^*(A) = m^*(A \setminus E)$ for any A.

   Solution:
   Suppose we have some set E such that $m^*(E) = 0$, and let A be any subset of $\mathbb{R}$. Observe that $A \subseteq A \cup E$, so $m^*(A) \leq m^*(A \cup E)$. But $m^*(A \cup E) \leq m^*(A) + m^*(E) = m^*(A) + 0$, so $m^*(A) = m^*(A \cup E)$.

   Similarly, $A \setminus E \subseteq A$, so $m^*(A \setminus E) \leq m^*(A)$, and $A \subseteq (A \setminus E) \cup E$, so $m^*(A) \leq m^*(A \setminus E) + m^*(E) = m^*(A \setminus E)$. Thus, $m^*(A) = m^*(A \setminus E)$.

4. Carothers 16.22 [Jody]: Let $E = \cup_{n=1}^{\infty} E_n$. Show that $m^*(E) = 0$ if and only if $m^*(E_n) = 0$ for every n.

   Solution:
   Since m^* is countably subadditive, then if $m^*(E_n) = 0$ for every n we have that $m^*(E) \leq \sum_{n=1}^{\infty} \ell(E_n) = 0$. Since $m^*(A) \geq 0$ for all $A \subseteq \mathbb{R}$ then $m^*(E) \leq 0$ implies $m^*(E) = 0$. Now suppose $m^*(E) = 0$. Note for each E_n that $m^*(E_n) \leq m^*(E) = 0$, which implies $m^*(E_n) = 0$.

5. **[Carothers 16.24]** (Solution by Max Heldman)

   Given $E \subseteq \mathbb{R}$, there is a G_δ-set G so that $m^*(G) = m^*(E)$.

   Solution:
   By **Exercise 16.12**, $m^*(E) = \inf\{m^*(U) : U \text{ is open and } E \subseteq U\}$. Hence there is a sequence of open sets (U_n) so that $E \subseteq U_n$ for each n and $\lim_{n\to\infty} m^*(U_n) = m^*(E)$. Take $U = \bigcap_{n=1}^{\infty} U_n$, and observe that $E \subseteq U$, so $m^*(E) \leq m^*(U)$. Moreover, for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ so that $0 \leq m^*(U_n) - m^*(E) < \epsilon$. But since $U \subseteq U_n$, $m^*(U) \leq m^*(U_n)$, which implies
   $$0 \leq m^*(U) - m^*(E) \leq m^*(U_n) - m^*(E) < \epsilon.$$
   Hence $m^*(U) = m^*(E)$.

6. Carothers 16.25 (Solution by Lander Ver Hoef)

   Suppose that $m^*(E) > 0$. Given $0 < \alpha < 1$, show that there exists an open interval I such that $m^*(E \cap I) > \alpha m^*(I)$. [Hint: It is enough to consider the case $m^*(E) < \infty$. Now suppose that the conclusion fails.]

   Solution:
   Let $E_n = E \cap [-n, n]$. Observe that as n goes to infinity, $m^*(E_n)$ goes to $m^*(E)$, so there exists an n such that $m^*(E_n) > 0$, but $m^*(E_n) \leq 2n < \infty$, and $E_n \subseteq E$. Thus, if we can find an interval I that satisfies the condition for E_n, that same interval will satisfy the condition for E. Thus, we may consider only sets with finite outer measure.

So, suppose $0 < m^*(E) < \infty$. Choose α such that $0 < \alpha < 1$. Suppose to produce a contradiction that for all open intervals I,

$$m^*(E \cap I) \leq \alpha m^*(I).$$

Let $\varepsilon > 0$ such that $m^*(E) + \varepsilon < \frac{1}{\alpha} m^*(E)$. Then there exists a measuring cover $\{I_k\}$ such that

$$\sum_{k=1}^{\infty} m^*(I_k) < m^*(E) + \varepsilon < \frac{1}{\alpha} m^*(E).$$

That is, $m^*(E) > \alpha \sum_{k=1}^{\infty} m^*(I_k)$.

Next, note that because E is a subset of $\bigcup_{k=1}^{\infty} I_k$,

$$m^*(E) = m^*\left(E \cap \bigcup_{k=1}^{\infty} I_k\right) = m^*\left(\bigcup_{k=1}^{\infty} E \cap I_k\right) \leq \sum_{k=1}^{\infty} m^*(E \cap I_k),$$

by countable subadditivity. But then, by our hypothesis,

$$\sum_{k=1}^{\infty} m^*(E \cap I_k) \leq \sum_{k=1}^{\infty} \alpha m^*(I_k) = \alpha \sum_{k=1}^{\infty} m^*(I_k) < m^*(E).$$

Thus, $m^*(E) < m^*(E)$, which is a contradiction.

7. Carothers 16.28 [Mason]

8. Carothers 16.40 [Sakti]

9. **[Carothers 16.42]** (Solution by Max Heldman)

Suppose E is measurable with $m(E) = 1$. Then

a) There is a measurable set $F \subseteq E$ such that $m(F) = 1/2$.

Solution:
Define $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = m([-x, x] \cap E)$. Let $\epsilon > 0$, and suppose $x, y \in \mathbb{R}^+$ with $x > y$. Then

$$\begin{aligned}
|f(x) - f(y)| &= m([-x, x] \cap E) - m([-y, y] \cap E) \\
&= m(([-x, x] \setminus [-y, y]) \cap E) + m([-y, y] \cap E) - m([-y, y] \cap E) \\
&\leq m([-x, -y)) + m((y, x]) \\
&= 2|x - y|.
\end{aligned}$$

Hence f is Lipschitz continuous. Since $f(0) = 0$, by the Intermediate Value Theorem we need only show that there exists $b \in \mathbb{R}^+$ so that $f(b) \geq 1/2$.

For each $n \in \mathbb{N}$, let $E_n = [-n, n] \cap E$, and observe that $\bigcup_{n=1}^{\infty} E_n = E$. The E_n are nested, so $\lim_{n\to\infty} m(E_n) = m(E) = 1$. But $f(n) = m(E_n)$ for each $n \in \mathbb{N}$, so there exists $n \in \mathbb{N}$ so that $f(n) > 1/2$.

b) There is a closed set F, consisting entirely of irrationals, so that $F \subseteq E$ and $m(F) = 1/2$.

Solution:
Let $A = E \setminus \mathbb{Q}$. Since $m(\mathbb{Q}) = 0$, $m(A) = m(E) = 1$. Since A is measurable, there exists a sequence (V_n) of closed sets so that $V_n \subseteq A$ and $m(V_n) \to m(A) = 1$. In particular, there is a closed set $V \subseteq A$ so that $m(V) > 1/2$. By **(a)**, there exists $x \in \mathbb{R}$ so that $F = V \cap [-x, x]$ has measure $1/2$. Since F is an intersection of closed sets, F is closed, and F consists of only irrationals since $F \subseteq A \subseteq \mathbb{Q}^c$.

c) There is a compact set F with empty interior so that $F \subseteq E$ and $m(F) = 1/2$.

Solution:
The set $F \subseteq E$ produced in part **(b)** is closed, bounded (and therefore compact) and has an empty interior (since it contains only irrationals), and $m(F) = 1/2$.

10. Carothers 16.44 [Jody]: Let E be a measurable set with $m(E) > 0$. Prove that $E - E = \{x - y : x, y \in E\}$ contains an interval centered at 0.

Solution:
Since $m(E) > 0$ then by Exercise 16.25 there exists an open interval I such that $m(E \cap I) > 3m(I)/4$. Let $x \in \mathbb{R}$ so that $|x| < m(I)/2$. Then x shifts I by no more than $m(I)/2$, which implies $m((I + x)\backslash I) \leq m(I)/2$. Observe

$$m(I \cup (I + x)) = m(((I + x)\backslash I) \cup I) = m((I + x)\backslash I) + m(I) \leq m(I)/2 + m(I) = 3m(I)/2.$$

Suppose E and $E + x$ are disjoint. Since $E \cap I \subseteq E$ and $(E \cap I) + x \subseteq E + x$ then $E \cap I$ and $(E \cap I) + x$ are disjoint. Moreover $(E \cap I) \cup (E \cap I) + x \subseteq I \cup (I + x)$, thus monotonicity of m implies

$$m((E \cap I) \cup (E \cap I) + x) \leq m(I \cup (I + x)) \leq 3m(I)/2.$$

Since $E \cap I$ and $(E \cap I) + x$ are disjoint then

$$m((E \cap I) \cup (E \cap I) + x) = m(E \cap I) + m((E \cap I) + x) = 2m(E \cap I),$$

which implies $m(E \cap I) \leq 3m(I)/4$, a contradiction of $3m(I)/4 < m(E \cap I)$. So there exists $y \in (E + x) \cap E$. That is, $y \in E$ and $y = x + z$ for some $z \in E$, which implies $x = y - z \in E - E$. Hence for all $|R| < m(I)/2$ we have that $(-|R|, |R|) \subset E - E$.

11. Carothers 16.45 (Solution by former 641 student TJ Barry)
Let $f : X \to Y$ be any function.

a) If $\mathcal{B}$ is a σ-algebra of subsets of Y, prove that $\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra of subsets of X.

b) If $\mathcal{A}$ is a σ-algebra of subsets of X, prove that $\mathcal{B} = \{B : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra of subsets of Y.

Solution, part a:
Let $A \in \mathcal{A}$ be arbitrary. Note, $A = f^{-1}(B)$ for some $B \in \mathcal{B}$. Hence,

$$A^c = \left(f^{-1}(B)\right)^c = f^{-1}(B^c).$$

Since $\mathcal{B}$ is a σ-algebra, $B^c \in \mathcal{B}$, and hence $A^c \in \mathcal{A}$.

Similarly, let $\{A_i\}_{i\in I}$ be a countable collection of subsets of $\mathcal{A}$. Then for each i, $A_i = f^{-1}(B_i)$ for some $B_i \in \mathcal{B}$. Hence,

$$\bigcup_{i\in I} A_i = \bigcup_{i\in I} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i\in I} B_i\right).$$

Since $\mathcal{B}$ is a σ-algebra, $\bigcup_{i\in I} B_i \in \mathcal{B}$, and hence $\bigcup_{i\in I} A_i \in \mathcal{A}$.

Thus, $\mathcal{A}$ is closed under complements and countable unions. By De Morgan's laws we also find that $\mathcal{A}$ is closed under countable intersections. Therefore, we conclude that $\mathcal{A}$ is a σ-algebra.

Solution, part b:
Let $B \in \mathcal{B}$ be arbitrary. Note, $f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$. Since $f^{-1}(B) \in \mathcal{A}$ and $\mathcal{A}$ is a σ-algebra, $\left(f^{-1}(B)\right)^c \in \mathcal{A}$. Thus, $f^{-1}(B^c) \in \mathcal{A}$, and hence $B^c \in \mathcal{B}$.

Now, let $\{B_i\}_{i\in I}$ be a countable collection of subsets of $\mathcal{B}$. Then $f^{-1}(B_i) \in \mathcal{A}$ for each $i \in I$, and hence $\bigcup_{i\in I} f^{-1}(B_i) \in \mathcal{A}$. But $\bigcup_{i\in I} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i\in I} B_i\right)$, so we have $f^{-1}\left(\bigcup_{i\in I} B_i\right) \in \mathcal{A}$, and hence $\bigcup_{i\in I} B_i \in \mathcal{B}$.

Thus, $\mathcal{B}$ is closed under complements and countable unions. By De Morgan's laws we can easily show that $\mathcal{B}$ is closed under countable intersections. Therefore, we conclude that $\mathcal{B}$ is a σ-algebra.

12. Carothers 16.53 (Solution by David Maxwell)

Show that $\mathcal{B}$ is generated by each of the following:

(i) The open intervals $\mathcal{E}_1 = \{(a,b) : a < b\}$.

(ii) The closed intervals $\mathcal{E}_2 = \{[a,b] : a < b\}$.

(iii) The half-open intervals $\mathcal{E}_3 = \{(a,b], [a,b) : a < b\}$.

(iv) The open rays $\mathcal{E}_4 = \{(a,\infty), (-\infty,b)) : a,b \in \mathbb{R}\}$.

(iv) The closed rays $\mathcal{E}_5 = \{[a,\infty), (-\infty,b]) : a,b \in \mathbb{R}\}$.

Solution:
Note that each $\mathcal{E}_k$ is a collection of Borel sets. This is certainly true for $\mathcal{E}_1$, $\mathcal{E}_2$ $\mathcal{E}_4$, and $\mathcal{E}_5$ since these are collections of open or closed sets. Every half open interval $(a,b]$ or $[a,b)$ is a G_δ set (being the intersection $\cap_n(a-1/n,b)$ and $\cap_n(a,b+1/n)$) respectively). Hence

the elements of $\mathcal{E}_3$ are Borel sets as well. Hence for each k, $\mathcal{B}$ is a σ-algebra containing $\mathcal{E}_k$ and therefore

$$\sigma(\mathcal{E}_k) \subseteq \mathcal{B}.$$

Let $\mathcal{F} = \{(a, \infty) : a \in \mathbb{R}\}$. We will show that that $\sigma(\mathcal{F}) = \mathcal{B}$, and that $\mathcal{F} \subseteq \sigma(\mathcal{E}_k)$ for every k. Having done this, we can conclude that

$$\mathcal{B} = \sigma(\mathcal{F}) \subseteq \sigma(\mathcal{E}_k)$$

for each k and hence $\sigma(\mathcal{E}_k) = \mathcal{B}$.

Suppose U is an open set. Then U is a countable union of open intervals. To show that $U \in \sigma(\mathcal{F})$, it is enough to show that any open interval is in $\sigma(\mathcal{F})$. But $\sigma(\mathcal{F})$ contains, by taking complements, any closed ray $(-\infty, b])$ and by taking finite intersections any half open interval $(a, b]$. Now by taking countable unions of sets of the form $(a, b - 1/n]$ we have $\sigma(\mathcal{F})$ contains any bounded open intervals(a, b). But any unbounded open interval can be written as a countable union of open intervals. Hence $\sigma(\mathcal{F})$ contains the open intervals and hence the open sets. Since $\mathcal{B}$ is the smallest such σ-algebra, $\mathcal{B} \subseteq \sigma(\mathcal{F})$. But the elements of $\mathcal{F}$ are Borel sets, so $\sigma(\mathcal{F}) \subseteq \mathcal{B}$. Hence $\mathcal{B} = \sigma(\mathcal{F}))$.

Let $a \in \mathbb{R}$ and let $I = (a, \infty)$. We will show that $I \in \sigma(\mathcal{E}_k)$ for each k and hence $\mathcal{F} \subseteq \sigma(\mathcal{E}_k)$. Notice

$$\begin{aligned}
I &= \cup_{n\in\mathbb{N}}(a, a + n) \\
I &= \cup_{n\in\mathbb{N}}[a + 1/n, a + n] \\
I &= \cup_{n\in\mathbb{N}}(a, a + n] \\
I &= (a, \infty) \\
I &= \cup_{n\in\mathbb{N}}[a + 1/n, \infty).
\end{aligned}$$

In each case we have exhibited I as an element or countable union of elements of each $\mathcal{E}_k$. So $I \in \sigma(\mathcal{E}_k)$ for each k.