1. Recall the two step Adams-Bashforth method:

$$u_{i+1} = u_i + \frac{h}{2}(3f_i - f_{i-1}).$$

a) Write down the stability polynomial $p(\rho)$ for this method applied to the problem $u' = \lambda u$.

b) The equation $p(\rho) = 0$ will involve the expresion λh. Solve for λh to write

$$\lambda h = f(\rho)$$

for some function f.

c) Numerically determine values of $f(\rho)$ where ρ lives on the unit circle of complex numbers. These will generate values of λh where the associated root of the characteristic polynomial has size one, and is therefore potentially on the boundary of the stability region. Generate a plot of the values of $f(\rho)$ as ρ varies around the unit circle to see the boundary of the absolute stability region.

Solution, part a:
The stability polynomial is

$$\rho^2 - \rho - \frac{z}{2}(3\rho - 1)$$

Solution, part b:
Solving for z we have

$$z = 2\frac{\rho^2 - \rho}{3\rho - 1}$$

Solution, part c:
See Jupyter notebook.

2. Recall that a linear multistep method (LMM) has the form

$$\alpha_k u_{k+n} + \cdots + \alpha_1 u_{1+n} + \alpha_0 u_n = h(\beta_k f_{k+n} + \cdots + \beta_1 f_{1+n} + \beta_0 f_n)$$

a) Show that the method is consistent if and only if

$$\begin{aligned} \alpha_k + \cdots + \alpha_0 &= 0 \\ k\alpha_k + \cdots + 1\alpha_1 + 0\alpha_0 &= \beta_k + \cdots + \beta_0 \end{aligned} \tag{1}$$

b) Use the previous result to show that every consistent one-step LMM is zero stable.

Solution, part a:
For a solution of $u' = f(t, u)$ we have

$$\begin{aligned} u(t_n + kh) &= u(t_n) + ku'(t_n)h + O(h^2) \\ f(t_n + kh, u(t_n + kh)) = u'(t_n + kh) &= u'(t_n) + O(h) \end{aligned} \tag{2}$$

Subsituting into the LMM and dividing by h we find

$$\frac{1}{h}\sum_{j=1}^{k}\alpha_j[u(t_n)+ju'(t_n)h+O(h^2)]-\sum_{j=0}^{k}[\beta_j u'(t_n)+O(h)] = \frac{1}{h}u(t_n)\sum_{j=0}^{k}\alpha_j+u'(t_n)\sum_{j=1}^{n}[j\alpha_j-\beta_j]+O(h)$$

In order for the method to be consistent we need the coefficients of h^{-1} and h^0 to vanish, which occurs if and only if

$$\sum_{j=0}^{k}\alpha_j = 0$$

and

$$\sum_{j=0}^{k}[j\alpha_j - \beta_j]$$

Solution, part b:
For a single step method, the consistency conditions become $\alpha_1+\alpha_0 = 0$ and $\alpha_1 = (\beta_1+\beta_0)$. As a consequence, the characteristic polynomial is

$$\alpha_1\rho + \alpha_0 = \alpha_1\rho - \alpha_1 = \alpha_1(\rho - 1).$$

Thus only $\rho = 1$ is a root, and the LMM is zero stable.

3. Implement Euler's method and the Runge-Kutta RK4 method described in Table 1.3 for **vector** valued ODEs (i.e, systems). Test your work against the IVP $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with initial condition $\mathbf{x}_0 = (1, 0)$. Part of the exercise is to compute the true solution of this system. Hint: convert into a second-order scalar ODE. **Verify that your code has the theoretical order of convergence.**

Solution:
The exact solution of the system is $u(t) = [\cos(t), \sin(t)]$. For the remainder of the solution, see the Jupyter notebook.

4. You are going to solve the 2-body problem of gravitation for the Earth-moon system. Give two bodies with masses m_1 and m_2 at positions x_1 and x_2 in cartesian coordinates, the force of body 2 on body 1 is

$$F_{21} = Gm_1m_2\frac{x_2 - x_1}{|x_2 - x_1|^3}$$

where G is the gravitational constant. The force of body 1 on body 2 is the same, with the numbers 1 and 2 interchanged.

Newton's Laws then read

$$\begin{aligned} m_1\mathbf{x}_1'' &= F_{21} \\ m_2\mathbf{x}_2'' &= F_{12}. \end{aligned} \tag{3}$$

We have the following physical constants:

$$\begin{aligned} m_{\text{Earth}} &= 5.972 \times 10^{24}\text{kg} \\ m_{\text{moon}} &= 7.342 \times 10^{22}\text{kg} \\ G &= 6.67408 \times 10^{-11}\frac{\text{m}^3}{\text{kg s}^2} \end{aligned}$$

a) Convert system 4 into a first order system by introducing the variables $\mathbf{v}_1 = \mathbf{x}_1'$ and $\mathbf{v}_2 = \mathbf{x}_2'$.

b) Writing $\mathbf{x}_i = [x_i, y_i]$ and $\mathbf{v}_i = [v_i, w_i]$, we will keep track of all of the scalar system variables in vectors

$$[x_1, y_1, x_2, y_2, v_1, w_1, v_2, w_2]^T.$$

Using this convention, write a right-hand side function `z=earthmoon(t,u)` for the system you wrote down in part 1.

c) Suppose at time $t = 0$ the Earth is stationary and located at the origin, the moon has position $x = 3.565 \times 10^8$ and $y = 0$ meters with velocity $1.09 \times 10^3 m/s$ in the positive y direction.

Use these initial conditions, and each of the solvers from problem 2, to generate an approximate solutions over a 40 day time interval (convert to seconds!) using daily (M=40) and then hourly (M=40×24) time steps. That is, you are generating four different approximate solutions. Make basic plots of the computed trajectories. Describe in a few words what you see, and how these results relate to the local truncation errors of the schemes.

d) An important property of an isolated physical system is that its energy is conserved. For the 2-body problem, the energy is

$$E(t) = \frac{m_1}{2}|\mathbf{x}_1'|^2 + \frac{m_2}{2}|\mathbf{x}_2'|^2 + U(\mathbf{x}_1, \mathbf{x}_2)$$

where U is the potential energy

$$U(\mathbf{x}_1, \mathbf{x}_2) = -\frac{Gm_1m_2}{|\mathbf{x}_1 - \mathbf{x}_2|}.$$

If you have not seen this before, you should compute the derivative $E'(t)$ and show, using the differential equation, that it is zero (and hence $E(t)$ is constant).

Start by computing the value $E(0)$ exactly from the initial conditions. Then compute, for each of your four approximate solutions, an array of scalar energy values E(t) computed at the $M+1$ sample times. Plot the four energy curves from the four runs in one figure. Explain and comment. Describe an "energy error norm" which is small if the solution is of high accuracy, and report the values for the four runs.

Solution, part a:
Replacing $\mathbf{x}''$ with $\mathbf{v}'$ in the original ODE and adding the new equations relating $\mathbf{v}$ and $\mathbf{x}$ we obtain

$$\begin{aligned} \mathbf{x}_1' = \mathbf{v}_1 m_1\mathbf{v}_1' &= F_{21} \\ \mathbf{x}_2' = \mathbf{v}_2 m_2\mathbf{v}_2' &= F_{12}. \end{aligned} \tag{4}$$

Solution, part b:
See notebook.

Solution, part c:
See notebook. Here are brief observations about the solutions.

The daily Euler solution is an outlier, and is wildly different.

The $O(h)$ hourly Euler solution appears to have not reached the accuracy of even the daily RK4 solution.

The two RK4 solutions are visibily identical at the sample points.

The system is moving "up" as time progresses because the initial conditions have non-zero linear momentum.

Solution, part d:
See notebook.

5. Text: 1.10 a-d. Note that your text uses different notation from what we've used in class. My h is your text's k. My u_n is your text's y_n. You may find it helpful to show that if $u' = f(t,u)$ then

$$u''(t) = f_t(t,u) + f_u(t,u)f(t,u).$$

Solution:

Solution, part a:
Consider

$$g(k) = f(t+\alpha k, y+\beta k f).$$

Then $g(0) = f(t,y)$ and

$$\begin{aligned} g'(k) &= f_t(t+\alpha k, y+\beta k f)\alpha + f_y(t+\alpha k, y+\beta k f)\beta f \\ g''(k) &= f_{tt}(t+\alpha k, y+\beta k f)\alpha^2 + 2f_{ty}(t+\alpha k, y+\beta k f)\alpha\beta f + f_{yy}(t+\alpha k, y+\beta k f)(\beta f)^2. \end{aligned} \tag{5}$$

Thus

$$\begin{aligned} g'(0) &= \alpha f_t + \beta f_y f \\ g''(0) &= f_{tt}\alpha^2 + 2\alpha\beta f f_{ty} + (\beta f)^2 f_{yy} \end{aligned} \tag{6}$$

and

$$\begin{aligned} f(t+\alpha k, y+\beta k f) = g(k) &= g(0) + g'(0)k + \frac{1}{2}g''(0)k^2 + O(k^3) \\ &= f + (\alpha f_t + \beta f_y f)k + \frac{k^2}{2}D + O(k^3) \end{aligned} \tag{7}$$

where $D = (f_{tt}\alpha^2 + 2\alpha\beta f f_{ty} + (\beta f)^2 f_{yy})$.

Solution, part b:
For an exact solution of $y' = f(t,y)$ we have

$$\begin{aligned} y'' &= f_t + f_y y' = f_t + f_y f \\ y''' &= f_{tt} + 2f_{ty}f + f_{yy}f^2 + f_{yy}(f_t + f_y f). \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} y(t+k) &= y(t) + y'(t)k + y''(t)k^2/2 + y'''(t)k^3/6 + O(k^4) \\ &= y(t) + fk + \frac{k^2}{2}(f_t + f_y f) + \frac{k^3}{6}C + O(k^4) \end{aligned} \tag{9}$$

were $C = f_{tt} + 2f_{ty}f + f_{yy}f^2 + f_{yy}(f_t + f_y f)$. The local truncation error for the RK method is

$$\tau = \frac{y(t+k) - y(t)}{k} - k(af + bf(t+\alpha k, y+\beta k f))$$

Using our two expansions in k

$$\tau = f + \frac{k}{2}(f_t + f_y f) + \frac{k^2}{6}C - (a+b)f - b(\alpha f_t + \beta f_y f)k - \frac{bk^2}{2}D + O(k^3).$$

Simplifying this is

$$\tau = (1-(a+b))f + \frac{k}{2}(f_t + f_y f) - b(\alpha f_t + \beta f_y f)k + \frac{k^2}{6}C - \frac{bk^2}{2}D + O(k^3)$$

To get a consistent method we need $a + b = 1$. For an $O(k^2)$ method we need to make the coefficient on k vanish regardless of the values of f_t, f_y and f. This is possible only if $\alpha b = 1/2$ and $\beta b = 1/2$. These equations are equivalent to

$$\begin{aligned} a + b &= 1 \\ \alpha &= \beta \\ 2\alpha b &= 1 \end{aligned} \tag{10}$$

To get a higher order method we would need to make

$$C = 3bD$$

This requires $\alpha^2 b = 1$ and $2\alpha\beta b = 1$ among other restrictions. But $2\alpha b = 1$, so $\beta = 1$, as is α by our earlier restrictions. But then $b = 1/2$ and this violates $\alpha^2 b = 1$. So a higher order method is not possible. Whew.

Solution, part c:
Huen's method arises from the choice $\alpha = \beta = 1$ and $a = b = 1/2$.

Solution, part d:
Let us compute the stability polynomial for the RK method. We apply it to $u' = \lambda u$ to find

$$u_{k+1} = u_k + h(a\lambda u_k + b\lambda(u_k + \beta h \lambda u_k)) = u_k(1 + az + bz + b\beta z^2).$$

But $a + b = 1$ and $b\beta = 1/2$ for an $O(h^2)$ method. Thus the stability polynomial is

$$1 + z + \frac{1}{2}z^2$$

for all $O(h^2)$ methods and we cannot pick the parameter based on changing this region.

On the other hand, one could take $a = 0$, $b = 1$, $\alpha = \beta = 1/2$ for a moderately computationally simpler method.