1. Carothers 7.32 Max

2. Carothers 7.37 Mason (first read problem 7.36)

Solution:
Let $f : (a,b) \to (a,b)$ be differentiable at a fixed point $p \in (a,b)$. First, let us assume that $|f'(p)| < 1$, and we will show that p is an attracting point. So have that for some δ, we have that $|x-p| < \delta$ implies

$$\left|\frac{f(x)-f(p)}{x-p}\right| = \left|\frac{f(x)-f(p)}{x-p}\right| < 1,$$

and so $|f(x)-p| < |x-p|$. This means that there must be some $0 < \alpha_1 < 1$ such that $|f(x)-p| = \alpha_1|x-p|$. Also, note then that $|f(x)-p| < \delta$, and so $f(x) \in B_\delta(x)$. Now assume that $f^{(n)}(x) \in B_\delta(x)$, then $|f^{(n+1)}(x)-f(p)| = |f^{(n+1)}(x)-p| < |f^{(n)}(x)-p| < \delta$, and so this shows that $f^{(n)}(x) \in B_\delta(x)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we know that there exists some α_n such that $|f^{(n)}(x)-p| = \alpha_n|f^{(n-1)}(x)-p|$, and so let

$$\sigma_n = \prod_{i=1}^{n} \alpha_i,$$

and so we have that $|f^{(n)}(x)-p| = \sigma_n|x-p|$. Since σ_n converges to 0 (since each $\alpha < 1$), so does $\sigma_n|x-p|$, and so $|f^{(n)}(x)-p|$ also converges to 0. Thus $f^{(n)}(x)$ converges to p.

3. Carothers 7.42 (Solution by Lander Ver Hoef)

Define $T : C[0,1] \to C[0,1]$ by $(T(f))(x) = \int_0^x f(t)\,dt$. Show that T is not a strict contraction while T^2 is. What is the fixed point of T?

Solution:
Because the norm is not explicitly given, we assume the L_∞ norm on $C[0,1]$. To show that T is not a strict contraction, we must demonstrate an f and g in $C[0,1]$ such that $d(f,g) = d(T(f),T(g))$. Let $f(x) = 1$ and $g(x) = 2$. Clearly, these are both in $C[0,1]$, and $d(f,g) = 1$. But

$$d(T(f),T(g)) = \sup_{x\in[0,1]}\left|\int_0^x 1\,dt - \int_0^x 2\,dt\right| = \sup_{x\in[0,1]}\left|\int_0^x 1\,dt\right| = \sup_{x\in[0,1]}|x| = 1.$$

Thus, $d(T(f),T(g)) = d(f,g)$, and T cannot be a strict contraction.

Next, observe that

$$(T^2(f))(x) = \int_0^x \int_0^t f(y)\,dy\,dt,$$

so for $f,g \in C[0,1]$,

$$d(T^2(f),T^2(g)) = \sup_{x\in[0,1]}\left|\int_0^x \int_0^t f(y)-g(y)\,dy\,dt\right| \le \sup_{x\in[0,1]}\int_0^x \int_0^t |f(y)-g(y)|\,dy\,dt$$

by the linearity of the integral operator and the generalized triangle inequality. But for any $y \in [0,1]$, $|f(y) - g(y)| \le d(f,g)$, by the definition of the L_∞ norm, so

$$\sup_{x\in[0,1]} \int_0^x \int_0^t |f(y) - g(y)|\, dy\, dt \le \sup_{x\in[0,1]} \int_0^x \int_0^t d(f,g)\, dy\, dt.$$

Because the integrand is always positive, the supremum will occur at $x = 1$, so we may plug in $x = 1$ and carry out the integration. Observe that

$$\int_0^1 \int_0^t d(f,g)\, dy\, dt = \int_0^1 t\, d(f,g)\, dt = \frac{1}{2} t^2\, d(f,g)\Big|_0^1 = \frac{1}{2} d(f,g).$$

Therefore,

$$d(T^2(f), T^2(g)) \le \frac{1}{2}\, d(f,g),$$

and because $1/2 < 1$, T^2 is a strict contraction.

Observe that because $\int_0^x 0dt = 0$, the constant function $f = 0$ is the fixed point of T.

4. Carothers 8.4 Sakti

5. Carothers 8.13 (Solution by Lander Ver Hoef)

Given $c_n \ge 0$ for all n, prove that the set $\{x \in \ell_2 : |x_n| \le c_n, n \ge 1\}$ is compact in ℓ_2 if and only if $\sum_{n=1}^{\infty} c_n^2 < \infty$.

Solution:
We will first prove that $A = \{x \in \ell_2 : |x(k)| \le c(k), k \ge 1\}$ (where $x(k)$ indicates the kth entry in the sequence x) is closed in ℓ_2. Suppose there exists some sequence $(x_n) \in A$ that converges to some $x \in \ell_2$, and suppose to produce a contradiction that $x \notin A$. Then there exists some k_0 such that $|x(k_0)| > c(k_0)$. Let $\varepsilon = |x(k_0)| - c(k_0)$. Then, because $\varepsilon > 0$, there exists an N such that if $n \ge N$, then $\| x_n - x \|_2 < \varepsilon$. But for all n,

$$\| x_n - x \|_2^2 = \sum_{k=1}^{\infty} |x_n(k) - x(k)|^2 \ge |x_n(k_0) - x(k_0)|^2,$$

and by the reverse triangle inequality,

$$|x_n(k_0) - x(k_0)|^2 \ge [|x_n(k_0)| - |x(k_0)|]^2 = [|x_n(k_0)| - c(k_0) + (c(k_0) - |x(k_0)|)]^2 \ge (\varepsilon + 0)^2,$$

because $c(k_0) - |x(k_0)| \ge 0$. Therefore, $\| x_n - x \|_2 \ge \varepsilon$, which contradicts our supposition that x_n converged to x. Therefore, x must be in A, and A is closed. Because ℓ_2 is complete, and closed subsets of complete spaces are complete, A is complete.

Next, we will show that if $\sum_{k=1}^{\infty} c(k)^2 < \infty$, then A is within ε of the set $A_K = A \cap \{x \in \ell_2 : |x(k)| = 0, k \ge K\}$ for some K. Note that for every K, A_K is compact, because any sequence of elements (x_n) of A_K can differ on only finitely many coordinates $x_n(k)$, and each of those coordinates lies within the closed interval $|x_n(k)| \le c(k)$, which is compact in $\mathbb{R}$. Thus, each coordinate sequence has a convergent subsequence, and because there are only finitely many differing coordinates, the overall sequence has a convergent subsequence.

Assume $\sum_{k=1}^{\infty} c(k)^2 < \infty$ and let $\varepsilon > 0$. Then there exists a K such that if $k \geq K$, then $\sum_{k=K}^{\infty} c(k)^2 < \varepsilon^2$. Then, for $x \in A$, there exists a $y \in A_K$ with $y(k) = x(k)$ for $1 \leq k \leq K-1$. But then,

$$\|x-y\|_2^2 = \sum_{k=1}^{\infty} |x(k)-y(k)|^2 = \sum_{k=1}^{K-1} |x(k)-y(k)|^2 + \sum_{k=K}^{\infty} |x(k)-y(k)|^2 .$$

However, within the first sum, $x(k) = y(k)$, so $x(k) - y(k) = 0$. In the second sum, $y(k) = 0$, so this reduces to

$$\sum_{k=K}^{\infty} |x(k)|^2 \leq \sum_{k=K}^{\infty} c(k)^2 < \varepsilon^2.$$

Hence, $\|x-y\|_2 < \varepsilon$.

Let $\varepsilon > 0$. Then there exists a K such that for all $x \in A$, there exists a $y \in A_K$ with $\|x-y\|_2 < \varepsilon$. Because A_K is compact, it is totally bounded, and there exists an ε-net $\{a_1, a_2, \ldots, a_j\}$ for A_K. Choose $x \in A$. Then there exists a $y \in A_K$ such that $\|x-y\|_2 < \varepsilon$. There is also some a_i such that $\|y - a_i\| < \varepsilon$. So by the triangle inequality, $\|x - a_i\| < 2\varepsilon$, and the same points $\{a_1, a_2, \ldots, a_j\}$ form a 2ε-net for A, and A is totally bounded. Hence, A is compact.

Next, assume $\sum_{k=1}^{\infty} c(k)^2 = \infty$. Then for any M, there exists a K such that $\sum_{k=1}^{K} c(k)^2 > M$. Let x be the sequence $x(k) = c(k)$ for $1 \leq k \leq K$, and $x(k) = 0$ for $k > K$. Then clearly, $\|x\|_2^2 = \sum_{k=1}^{K} c(k)^2 = M$. However, $x \in A$, and $0 \in A$, so $\operatorname{diam}(A) \geq \sqrt{M}$ for all M, and A is unbounded. Therefore, it cannot be totally bounded, and is not compact.

Lemma 1: Let A and B be metric spaces and $f : A \to B$ be an isometry. Show that A is totally bounded if and only if $f(A)$ is totally bounded.

Solution:
Let $\epsilon > 0$ and let $a, x \in A$. Since f is an isometry then $d_A(a,x) < \epsilon$ if and only if $d_B(f(a), f(x)) < \epsilon$. So

$$a \in B_\epsilon(x) \iff d_A(a,x) < \epsilon \iff d_B(f(a), f(x)) < \epsilon \iff f(a) \in B_\epsilon(f(x)).$$

Hence

$$\begin{aligned} A \text{ is totally bounded} &\iff A \text{ has an } \epsilon-\text{net } \{x_1, \ldots, x_n\} \\ &\iff A \subseteq \cup_{k=1}^{n} B_\epsilon(x_k) \\ &\iff f(A) \subseteq \cup_{k=1}^{n} B_\epsilon(f(x_k)) \ (\text{ since } a \in B_\epsilon(x) \iff f(a) \in B_\epsilon(f(x))) \\ &\iff f(A) \text{ has an } \epsilon-\text{net } \{f(x_1), \ldots, f(x_n)\} \\ &\iff f(A) \text{ is totally bounded.} \end{aligned}$$

6. Carothers 8.16 Jody (read pages 102-103 on completions): Show that a metric space M is totally bounded if and only if its completion $\hat{M}$ is compact.

Solution:
Since $\hat{M}$ is a completion of M then there exists an isometry $i : M \to \hat{M}$ such that $\overline{i(M)} = \hat{M}$. Hence

$$\begin{aligned} M \text{ is totally bounded } &\iff i(M) \text{ is totally bounded (by Lemma 1)} \\ &\iff \overline{i(M)} \text{ is totally bounded (by homework 4 exercise)} \\ &\iff \hat{M} \text{ is totally bounded (since } \overline{i(M)} = \hat{M}) \\ &\iff \hat{M} \text{ is compact (since } \hat{M} \text{ is complete).} \end{aligned}$$

7. Carothers 8.17 Sakti

8. Carothers 8.29 Mason

Solution:
Let $f : M \to M$ satisfy $d(f(x), f(y)) < d(x, y)$. This means that f is Lipschitz continuous. The let $g : M \to \mathbb{R}$ be $g(x) = d(x, f(x))$, and note that g is continuous. Since M is compact by assumption, we know that $g(M)$ is compact, and since $g(M) \subset \mathbb{R}$, we know that $g(M)$ is closed, and therefore $\inf(g(M)) \in g(M)$. Call $\inf(g(M)) = m$, and we know that there exists an $x_0 \in M$ such that $g(x_0) = m$. Then note that $g(f(x_0)) \geq m$ since m is the infimum and $f(x_0) \in M$. But also notice that $g(f(x_0)) = d(f(x_0), f^2(x_0)) \leq d(x_0, f(x_0))$ (where the equality only attains equality if both equal 0), which means that $g(f(x_0)) = g(x_0)$, which is only true if $d(x_0, f(x_0)) = 0$. Thus it must be that $f(x_0) = x_0$, and so x_0 is a fixed point of f.

9. Carothers 8.38 Sakti

10. Carothers 8.40 (Solution by Lander Ver Hoef)

Let M be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Show that f is onto. [Hint: If $B_\varepsilon(x) \cap f(M) = \varnothing$, consider the sequence $(f^n(x))$.]

Solution:
Observe that because f is an isometry, f is continuous, and therefore $f(M)$ is a compact set. In particular, it is closed. Suppose to produce a contradiction that there exists some $x \in M$ not in $f(M)$, then there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap f(M) = \varnothing$. Observe that the sequence $(f^n(x))$ is in $f(M)$, a compact space, so it has a convergent subsequence $(f^{n_k}(x))$. However, for all $k \neq l$, with $k < l$ chosen arbitrarily, we have that $d(f^{n_k}(x), f^{n_l}(x)) = d(x, f^{n_l - n_k}(x)) \geq \varepsilon$. This implies that $(f^{n_k}(x))$ is not Cauchy, which is a contradiction for a convergent sequence. Therefore, no such x can exist, and f is onto.

11. Carothers 8.55 Max

12. Carothers 8.57 Max

13. Carothers 8.58 Mason

Solution:
Let $f : \mathbb{R} \to \mathbb{R}$ have a bounded derivative. This implies that f is continuous and differentiable

on all of $\mathbb{R}$. Let $M \in \mathbb{R}$ such that $f'(x) \leq M$ for all $x \in \mathbb{R}$, and let $x, y \in \mathbb{R}$, with $x < y$. By the mean value theorem, there exists some $c \in (x, y)$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \frac{|f(x) - f(y)|}{|x - y|} = f'(c) \leq M,$$

which means that $|f(x) - f(y)| \leq M|x - y|$, implying that f is Lipschitz of order 1.