1. Carothers 1.4 (Solution by David Maxwell)

Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Show there is a sequence (x_n) of elements of A converging to $\sup A$.

Solution:
Let $\alpha = \sup A$. Let $\epsilon > 0$. Then $\alpha - \epsilon$ is not an upper bound for A and there exists $a \in A$ such that $\alpha - \epsilon < a$. In particular, for each $n \in \mathbb{N}$ we can find $a_n \in A$ with $\alpha - (1/n) < a_n$. Note that $a_n \leq \alpha < \alpha + (1/n)$ since α is an upper bound for A. Hence

$$\alpha - (1/n) < a_n < \alpha + (1/n)$$

and therefore $|a_n - \alpha| < 1/n$. It follows that $\lim_{n\to\infty} a_n = \alpha$.

2. Carothers 1.11 (Solution by Jody Gaines)

Fix $a > 0$ and let $x_1 > \sqrt{a}$. For $n \geq 1$, define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = \frac{x_n^2 + a}{2x_n}.$$

Show that (x_n) converges and that $\lim_{n\to\infty} x_n = \sqrt{a}$.

Solution:
We wish to show that $x_n \geq \sqrt{a}$ for all $n \geq 1$. When $n = 1$ we have $x_1 \geq \sqrt{a}$. So suppose $x_n \geq \sqrt{a}$. Then $\sqrt{x_n} > 0$ and hence

$$x_{n+1} - \sqrt{a} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - \sqrt{a} = \frac{1}{2}\left(x_n - 2\sqrt{a} + \frac{a}{x_n}\right) = \frac{1}{2}\left(\sqrt{x_n} - \frac{\sqrt{a}}{\sqrt{x_n}}\right)^2 \geq 0,$$

which implies $x_{n+1} \geq \sqrt{a}$. Now observe

$$x_{n+1} - x_n = \frac{x_n^2 + a}{2x_n} - x_n = \frac{a - x_n^2}{2x_n} \leq \frac{a - \left(\sqrt{a}\right)^2}{2\sqrt{a}} = 0,$$

which implies $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. So (x_n) is a monotone decreasing sequence that is bounded below. Hence (x_n) converges to some $L \in \mathbb{R}$.

Since (x_n) is bounded below by $\sqrt{a}$ then $L \geq \sqrt{a}$. Note $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = L$, which implies

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^2 + a}{2x_n} = \frac{\lim_{n\to\infty}(x_n^2 + a)}{\lim_{n\to\infty} 2x_n} = \frac{L^2 + a}{2L}. \tag{1}$$

Simplifying (1) yields $L^2 = a$. Since $L \geq \sqrt{a}$ then $L^2 = a$ implies $L = \sqrt{a}$.

3. Carothers 1.15 (Solution by David Maxwell)

Show that a Cauchy sequence with a convergent subsequence actually converges.

Solution:
Let (x_n) be a Cauchy sequence with a convergent subsequence (x_{n_k}) converging to L. We wish to show the original sequence converges to L. Let $\epsilon > 0$. There is an $M \in \mathbb{N}$ such that if $n, m \geq N$ then $|x_n - x_m| < \epsilon/2$. And there is a $K \in \mathbb{N}$ such that if $k \geq K$, then $|x_{n_k} - L| < \epsilon/2$. Without loss of generality we can assume that $n_K \geq N$. Then if $n \geq N$,

$$|x_n - L| \leq |x_n - x_{n_K}| + |x_{n_K} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\lim x_n = L$.

Carothers 1.21 (Solution by David Maxwell)

Show that a number $x \in [0,1]$ has more than one base p expansion if and only if $x = \sum_{k=1}^{n} a_k/p^k$ for some n where $a_n \neq 0$. Show also that in this case x has exactly one other expansion,

$$x = \sum_{k=1}^{n-1} \frac{a_k}{p^k} + \frac{a_n - 1}{p^n} + \sum_{k=n+1}^{\infty} \frac{p-1}{p^n}. \tag{2}$$

Also, characterize the numbers in $[0,1]$ with repeating and eventually repeating p-ary expansions.

Solution:
Suppose x has two different expansions,

$$\begin{aligned} x &= \sum_{k=1}^{\infty} \frac{a_k}{p^k} \\ &= \sum_{k=1}^{\infty} \frac{b_k}{p^k}. \end{aligned}$$

Let N be the index in which they first differ, and without loss of generality assume that $a_N > b_N$. We will show that $a_N = b_N + 1$, $a_n = 0$ for $n > N$, and $b_n = p - 1$ for $n > N$. This will prove that if x has two different expansions, then one must be a terminating expansion, and that the only other expansion is the one of the form (2).

Let $y = \sum_{k=1}^{N-1} \frac{b_k}{p^k}$. Then

$$\begin{aligned} x &\leq \sum_{k=1}^{N-1} \frac{b_k}{p^k} + \frac{b_N}{p^N} + \sum_{k=N+1}^{\infty} \frac{p-1}{p^k} \\ &= y + \frac{b_N}{p^N} + \frac{p-1}{p^{N+1}} \frac{p}{p-1} \\ &= y + \frac{b_N}{p^N} + \frac{1}{p^N} \end{aligned}$$

with strict inequality unless $b_n = p - 1$ for all $n > N$. Similarly,

$$\begin{aligned} x &\geq \sum_{k=1}^{N-1} \frac{a_k}{p^k} + \frac{a_N}{p^N} + \sum_{k=N+1}^{\infty} \frac{0}{p^k} \\ &= y + \frac{a_N}{p^N} \end{aligned}$$

with strict inequality unless $a_n = 0$ for $n > N$. These inequalities together imply

$$\frac{a_N}{p^N} \leq \frac{b_N + 1}{p^N}$$

and hence $a_N \leq b_N + 1$ (with strict inequality unless $b_n = p - 1$ and $a_n = 0$ for $n > N$). But $a_N \geq b_N + 1$ since $a_N > b_N$ and since a_N and b_N are integers. Hence $a_N = b_N + 1$ and $b_n = p - 1$ and $a_n = 0$ for $n > N$.

If x has a terminating p-ary expansion, then clearly x is of the form

$$x = \frac{a}{p^N}$$

where $N \in \mathbb{N}$ and $0 \leq a < p^N$. The converse is also obvious.

Suppose x has a repeating p-ary expansion, so

$$x = 0.a_1 a_2 \ldots a_N a_1 a_2 \ldots a_N \ldots \quad \text{(base } p\text{)}.$$

Then

$$x = \sum_{k=1}^{N} \frac{a_k}{p^k} + \sum_{k=1}^{N} \frac{a_k}{p^{k+N}} + \cdots = \sum_{k=1}^{N} \frac{a_k}{p^k} \sum_{N=0}^{\infty} \frac{1}{p^N} = \left[\sum_{k=1}^{N} \frac{a_k}{p^k} \right] \frac{p^N}{p^N - 1}.$$

By our previous remarks for terminating expansions we then have

$$x = \frac{a}{p^N - 1}$$

where $0 \leq a < p^N$. Conversely, if $x = a/(P^N - 1)$, then x has a repeating expansion by reversing the arguments above.

Suppose x has an eventually repeating expansion, so

$$x = 0.a_1 \ldots a_M b_1 \ldots b_N b_1 \ldots b_N \ldots .$$

Then from our results for terminating and repeating expansions we have

$$x = \frac{a}{p^M} + \frac{1}{p^M} \frac{b}{p^N - 1}$$

where $a, b \in \mathbb{Z}$, $0 \leq a < p^M$, and $0 < b < p^N$. Reversing these arguments show that if x can be written in this form, then it has an eventually repeating expansion.

Suppose x has an eventually repeating expansion. Then clearly $x \in \mathbb{Q} \cap [0, 1]$. Conversely, suppose $x \in \mathbb{Q} \cap [0, 1]$, so

$$x = \frac{c}{d}$$

where $c, d \in \mathbb{Z}$ have no common devisors. Let $p = q_1^{n_1} \cdots q_m^{n_m}$ be the prime factorization of p. We can write $d = q_1^{a_1} \cdots q_m^{a_m} e$ where $\gcd(e, q_k) = 1$ for $1 \leq k \leq m$. Let $M = \max(a_1, \ldots, a_m)$. Then

$$p^M x = \frac{p^M c}{d}.$$

Removing common factors from the fraction we then have

$$p^M x = \frac{f}{e}$$

for some integer f. Since $\gcd(e, p) = 1$, by Euler's Theorem there is a natural number N such that

$$p^N \equiv 1 \pmod{e}$$

That is,

$$p^N - 1 = e \cdot g$$

for some $g \in \mathbb{Z}$. But then

$$p^M x = \frac{fg}{eg} = \frac{fg}{p^N - 1}.$$

Since $0 \leq p^M x \leq p^M$, we can write this fraction as

$$a + \frac{b}{p^N - 1}$$

where $0 \leq a < p^M$ and $0 < b < p^N$. Hence

$$x = \frac{a}{p^M} + \frac{1}{p^M}\frac{b}{p^N - 1}.$$

Hence x has an eventually repeating p-ary expansion.

4. Carothers 1.24 (Solution by Former student TJ Barry)

Show that $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

Lemma A: For a nonempty set $A \subset \mathbb{R}$, $\inf(-A) = -\sup(A)$, and $\sup(-A) = -\inf(A)$.

Proof. If $\sup(A) = \infty$, the result follows immediately.

Therefore, let A be bounded above, so $\sup(A)$ is finite. Consider $-A = \{-a \;:\; a \in A\}$. Note that this set is bounded below, and hence $\inf(-A)$ exists.

We claim that $\inf(-A) = -\sup(A)$.

Notice that since $\sup(A)$ is an upper bound for A, $-\sup(A)$ is a lower bound for $-A$. Hence, $\inf(-A) \geq -\sup(A)$.

Similarly, $\inf(-A)$ is a lower bound of $-A$, and hence $-\inf(-A)$ is an upper bound of A. Thus, $-\inf(-A) \geq \sup(A)$.

It follows that $\inf(-A) = -\sup(A)$, as desired.

The other claim $\sup(-A) = -\inf(A)$ can be shown similarly, or by defining $A' = -A$. $\square$

Solution:
Recall, $\limsup_{n\to\infty}(a_n) = \inf_{n\geq 1}(\sup\{a_n, a_{n+1}, \ldots\})$. Then, by the lemma,

$$\begin{aligned}\limsup_{n\to\infty}(-a_n) &= \inf_{n\geq 1}(\sup\{-a_n, -a_{n+1}, \ldots\}) = \\ &= \inf_{n\geq 1}(-\inf\{a_n, a_{n+1}, \ldots\}) = \\ &= -\sup_{n\geq 1}(\inf\{a_n, a_{n+1}, \ldots\}) = \\ &= -\liminf_{n\to\infty}(a_n)\end{aligned}$$

5. (Solution by Lander Ver Hoef)

Suppose $\limsup_{n\to\infty} x_n = -\infty$, as defined in terms of eventual upper bounds. Show that

$$\overline{\lim_{n\to\infty}} x_n = -\infty,$$

as defined in the text.

Solution:
Suppose $\limsup_{n\to\infty} x_n = -\infty$. Let $K \in \mathbb{R}$, and observe that K is an eventual upper bound for (x_n) – that is, there exists an $M \in \mathbb{N}$ such that K is an upper bound for $\{x_n\}_{n=M}^{\infty}$. But then

$$T_M = \sup_{n\geq M}\{x_n\} \leq K,$$

by the definition of a supremum. This gives us that

$$\inf_{N\geq 1}\{T_N\} \leq T_M \leq K.$$

Because K is an arbitrary real number, $\overline{\lim_{n\to\infty}} x_n = \inf_{N\geq 1}\{T_N\} = -\infty$.

6. (Solution by Mason Brewer)

Let (r_n) be an enumeration of $\mathbb{Q} \cap [0,1]$. Show that $\limsup n \to \infty = 1$.

Solution:
Let $N \in \mathbb{N}$. First note that every $r_n \leq 1$, and so 1 is an upper bound for $\{r_n\}_{n=N}^{\infty}$. Next, let $\epsilon > 0$. Observe that $1 - \epsilon < 1$, and it is true that there are infinitely many rational numbers between $1-\epsilon$ and 1, and thus there must be a rational number r such that $1-\epsilon < r < 1$ and $r \notin \{r_n\}_{n=1}^{N-1}$ (or else there would only be finitely many rational numbers between $1-\epsilon$ and 1). Thus 1 is the least upper bound for $\{r_n\}_{n=N}^{\infty}$. Thus $\sup_{n\geq N} r_n = 1$ for all natural numbers N. Therefore

$$\limsup_{n\to\infty} r_n = \inf_{N\geq 1}\sup_{n\geq N} r_n = \inf_{N\geq 1}\{1, 1, 1, \ldots\} = 1.$$

7. (Solution by Max Heldman)

If (x_n) and (y_n) are sequences, then

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

so long as neither of the right- or left-hand sides are of the form $\infty - \infty$.

Solution:
[In the case of finite values]

For the right-hand inequality, we see that if M_1 is an eventual upper bound for x_n and M_2 is an eventual upper bound for y_n, then there exists $n \in \mathbb{N}$ such that if $n \geq N$, $x_n \leq M_1$ and $y_n \leq M_2$. Then $x_n + y_n \leq M_1 + M_2$, so $M_1 + M_2$ is an eventual upper bound for $x_n + y_n$. Hence $\limsup(x_n + y_n) \leq M_1 + M_2$. Since this is true for all eventual upper bounds M_1 and M_2 for (x_n) and (y_n), $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$.

Now observe that, using the result from the previous paragraph and **Exercise 5**, we have:

$$\begin{aligned}
\limsup x_n + \liminf y_n &= \limsup(x_n + y_n - y_n) + \liminf y_n \\
&\leq \limsup(x_n + y_n) + \limsup(-y_n) + \liminf y_n \\
&= \limsup(x_n + y_n) - \liminf y_n + \liminf y_n \\
&= \limsup(x_n + y_n).
\end{aligned}$$

8. Carothers 1.36 (Solution by Former student Will Mitchell)

The root test: Let $a_n > 0$.

a) If $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$, show that $\sum_{n=1}^{\infty} < \infty$.

b) If $\liminf_{n\to\infty} \sqrt[n]{a_n} > 1$, show that $\sum_{n=1}^{\infty}$ diverges.

c) Find examples of both a convergent and a divergent series having $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

Solution, part a:
Our strategy is comparison to a geometric series. Suppose $\limsup \sqrt[n]{a_n} = s < 1$. Set $S = \frac{1+s}{2}$ so that $s < S < 1$. Then S is an eventual upper bound and we can choose an $N \in \mathbb{N}$ such that $n \geq N$ implies $\sqrt[n]{a_n} \leq S$. Then we have $a_n \leq S^n$ for all $n \geq N$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \textstyle\sum_{n=1}^{N-1} a_n + \displaystyle\sum_{n=N}^{\infty} a_n \\
&\leq \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} S^n \\
&= \sum_{n=1}^{N-1} a_n + \frac{S^N}{1-S}
\end{aligned}$$

which is finite.

Solution, part b:
Suppose $\liminf \sqrt[n]{a_n} = s > 1$. Then 1 is an eventual lower bound and we can choose $N \in \mathbb{N}$ such that $a_n \geq 1$ for all $n \geq N$. But then it is impossible to have $\lim_{n\to\infty} a_n = 0$ which is a necessary condition for convergence.

Solution, part c:
For a divergent series consider $a_n \equiv 1$.

For a convergent series consider $a_n = n^{-2}$. To see the convergence, write:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{2^n} + \cdots + a_{2^{n+1}-1} < \sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^{-n} = 2.$$

To show that $\lim_{n\to\infty} \sqrt[n]{n^{-2}} = 1$ we set $f(x) = \log \sqrt[x]{x^{-2}}$ and use l'Hôpital's Rule to calculate

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{-2\log x}{x} = \lim_{x\to\infty} \frac{-2/x}{1} = 0.$$

Thus $\lim_{n\to\infty} \log \sqrt[n]{n^{-2}} = 0$ and it follows that $\lim_{n\to\infty} \sqrt[n]{n^{-2}} = 1$.