**1.** In this problem, we will seek a solution to the initial value problem

$$\begin{aligned} f'(t) &= F(t, f(t)) \\ f(0) &= a \end{aligned}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}$.

To obtain the existence result, we need to assume that $F$ is sufficiently nice; we will assume that $F$ is continuous, and moreover that there exists a constant $K$ such that

$$|F(x, y_1) - F(x, y_2)| \le K|y_1 - y_2|$$

for all $x, y_1, y_2 \in \mathbb{R}$.

Define $G : C[-T, T] \to C[-T, T]$ by

$$G(f)(t) = a + \int_0^t F(s, f(s))\, ds.$$

a) Explain why $G(f) \in C[-T, T]$ if $f \in C[-T, T]$.

b) Show that if $f$ solves the initial value problem for $t \in [-T, T]$, then $G(f) = f$.

c) Show that $G$ is Lipschitz with Lipschitz constant $TK$.

d) Assuming $T < 1/K$, show that there exists a solution of $G(f) = f$ defined for $t \in [-T, T]$. You may use the fact that $C[-T, T]$ is complete; we'll show this later.

e) Assuming $T < 1/K$, show that there exists a unique solution of the initial value problem defined on $(-T, T)$.

f) Extra credit: Show that there exists a solution $f$ of the initial value problem defined for all $t \in \mathbb{R}$.

**2.** Carothers 8.76

**3.** Carothers 8.77

**4.** Carothers 8.78

**5.** Carothers 8.80

**6.** Carothers 8.81

**7.** Carothers 8.84

**8.** Carothers 8.66

**9.** Carothers 8.83

**10.** Carothers 10.7

**11.** Carothers 10.9 (No rigor, please!)

**12.** Carothers 10.10