**Rules and format:**

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You my not discuss the exam with anyone else until after the due date/time.
- You are permitted to reference any text you would like in solving these problems.
- Each problem is weighted equally.
- The due date/time is absolutely firm.

**1.** Text, 1.15

**2.** Text, 5.4. Then, taking $\ell = 1$, find the exact solution with $u(x,0) = \sin(\pi x)$. Finally, write a code that implements the method and verifies against this initial condition that the desired rate of convergence is acheived.

**3.** Consider the problem

$$u_{xx} + \gamma u^4 = f(x)$$

on the interval $0 \le x \le 1$ with $u(0) = u(1) = 0$.

a) If $u(x) = \sin(3\pi x)$, what is the value of $f(x)$?

b) Write a code to solve this problem based on the following approach:

(a) Use centered differences to as in Section 2.2 of your text to approximate the second derivative and derive an algebraic system to solve for a vector of unknowns $u_i$ that approximate $u(x_i)$.

(b) Implement a numerical method to solve the system (with user-supplied right-hand side $f$ and constant $\gamma$) by applying Newton's method to approximate the solution of the algebraic system. Newton's method will be applied to a system of the form $F(u) = 0$, and iterations should stop when the residual norm $||F(u)||$ has been reduced by $10^{-9}$ of its original value.

(c) Show that with the verification case from part a), and separately with $\gamma = 0$, that your code exhibits $O(h^2)$ convergence.

**4.** The TR-BDF2 method is an implicit second-order Runge-Kutta method of the following form.

$$\begin{aligned} u_* &= u_n + \frac{k}{4}\left[f(u_n) + f(u_*)\right] \\ u_{n+1} &= \frac{1}{3}\left[4u_* - u_n + kf(u_{n+1}\right] \end{aligned} \tag{1}$$

a) Show that this method is $L$-stable.

b) Write a numerical code using TR-BDF2 as the basis for solving the heat equation $u_t = u_{xx}$ in a Method of Lines approach. Verify, using the test case of Homework 6, problem 1c, that you observe $O(h^2)$ convergence.

c) Discuss the merits of this strategy versus Backwards Euler and Crank Nicolson.