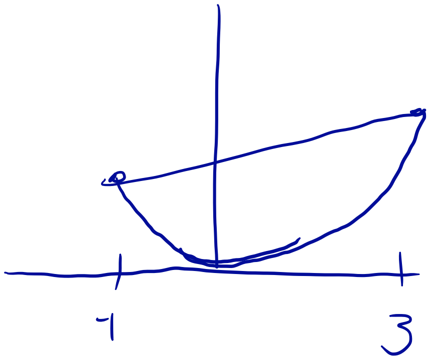
Consider the function $f(x) = x^2$ on the interval $[-1, 3]$

1. Find the slope of the secant line of the graph of $f(x)$ from $x = -1$ to $x = 3$.



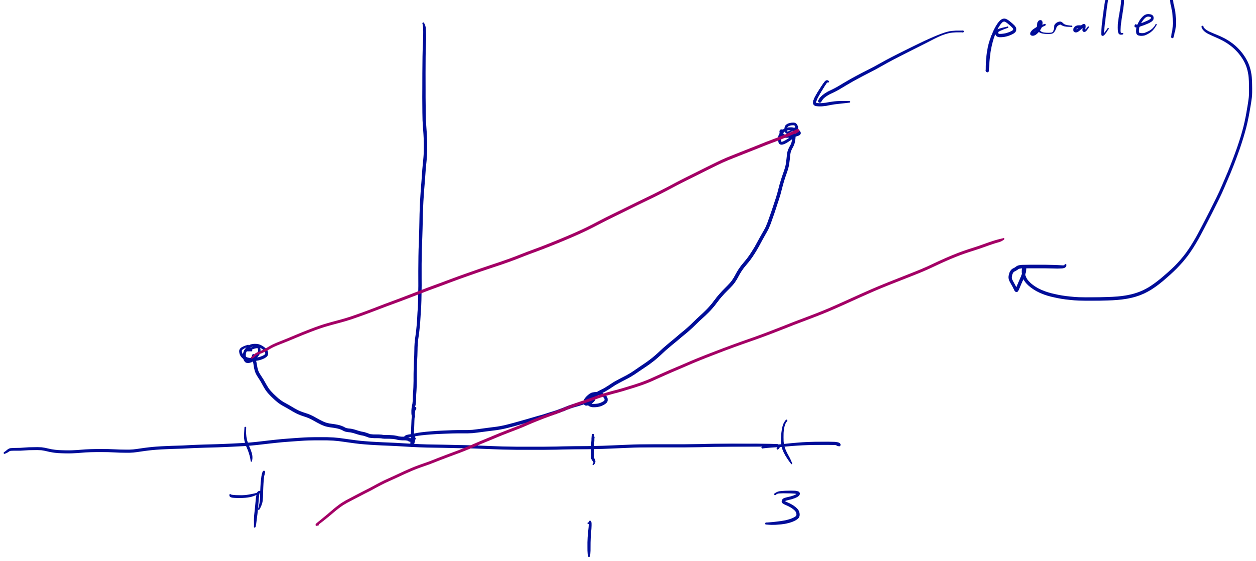
$$\frac{f(3)-f(-1)}{3-(-1)} = \frac{3^2-1}{3+1} = \frac{8}{4} = 2$$

2. Find a value of x in $[-1, 3]$ where $f'(x)$ equals the value you found in problem 1.

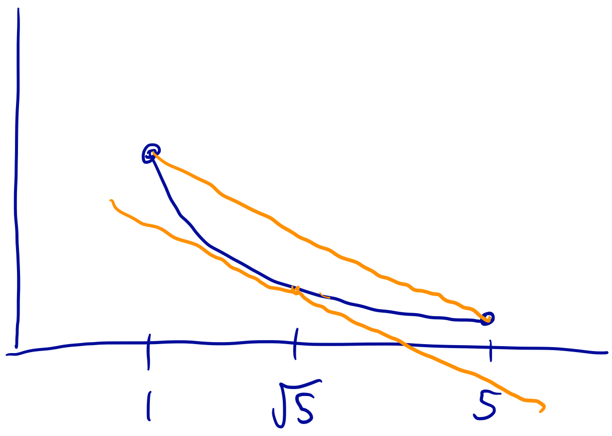
$f'(x) = 2x$

$2x = 2 \Rightarrow x = 1$

3. Make a sketch of the graph of $f(x)$ and add to it the secant line from problem 1 and the tangent line at the location found in problem 2. What property do the secant line and tangent line have?



4. Repeat the exercise of problems 1-3 with $g(x) = 1/x$ on [1,5].



slope of secant: $\dfrac{g(5)-g(1)}{5-1} = \dfrac{\frac{1}{5}-1}{4} = -\dfrac{1}{5}$

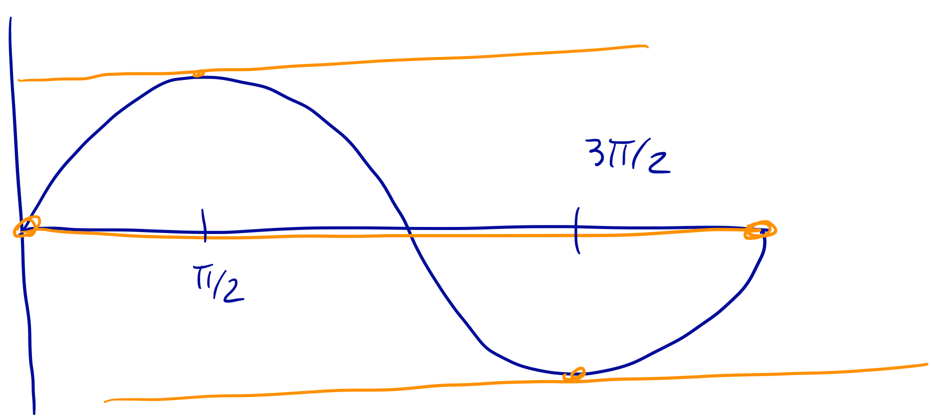
$f'(x) = -\dfrac{1}{x^2}$ $\quad -\dfrac{1}{x^2} = -\dfrac{1}{5} \Rightarrow x = \sqrt{5}$

$\sqrt{5} \approx 2.3$ so $1 \leq \sqrt{5} \leq 5$ as needed

5. Repeat the exercise of problems 1-3 with $\sin(x)$ on $[0, 2\pi]$.

slope of secant: $\dfrac{\sin(2\pi)-\sin(0)}{2\pi - 0} = \dfrac{0-0}{2\pi} = 0$

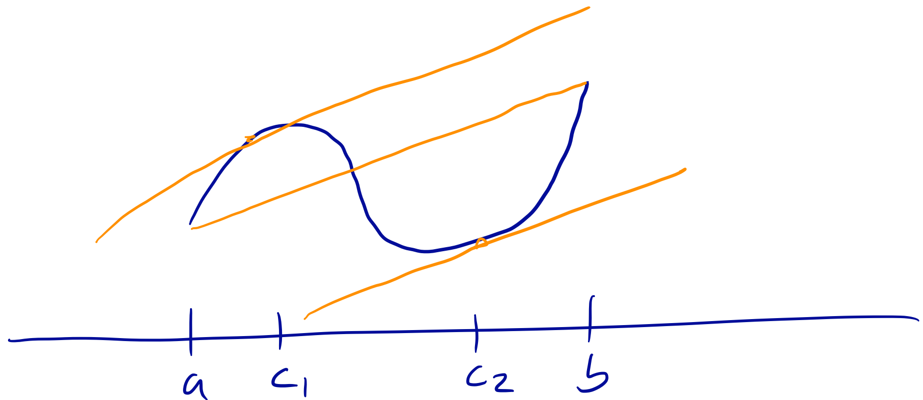
$\sin'(x) = \cos(x)$ $\quad \cos(x) = 0$ at $\pi/2$, $3\pi/2$



Mean Value Theorem. If f is a continuous function on an interval $[a, b]$ that has a derivative at every point in (a, b), then there is a point c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Picture from the board goes here:



6. What is the geometric meaning of the value $\frac{f(b)-f(a)}{b-a}$?

It's the slope of the secant line from $x=a$ to $x=b$

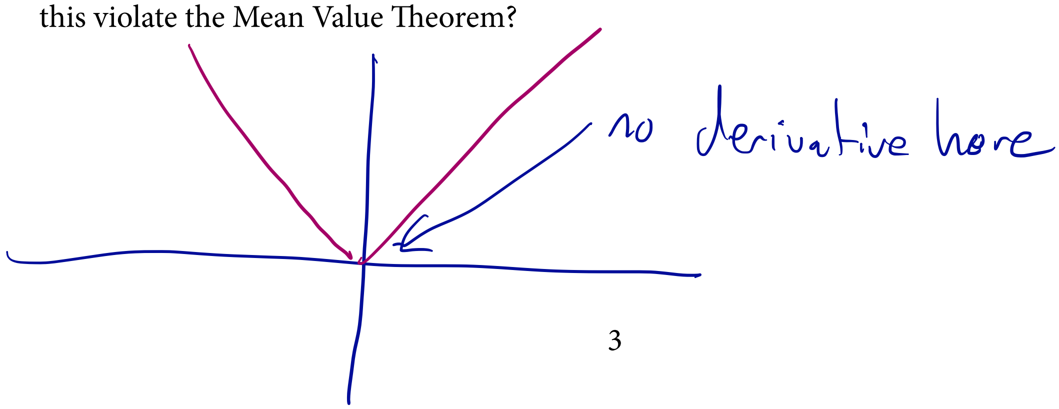
7. Suppose a car is traveling down the road and in 30 minutes it travels 32.7 miles. What does the Mean Value Theorem have to say about this?

Let $d(t)$ be distance traveled at time t.

$$\frac{d(30)-d(0)}{30-0} = d'(c)$$

$\frac{32.7}{30} = d'(c) \Rightarrow$ There is a time c where the speed of the car equals the average velocity

8. Draw the graph of $f(x) = |x|$ on the interval [-1,1]. Since $f(-1) = f(1)$, the Mean Value Theorem should say there is a c where $f'(c) = 0$. Is there such a choice of c? Why doesn't this violate the Mean Value Theorem?



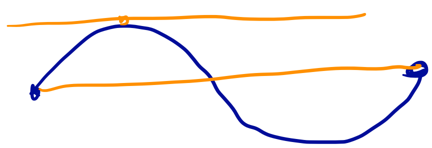
Rolle's Lemma (Baby Mean Value Theorem). If f is a continuous function on an interval $[a, b]$ that has a derivative at every point in (a, b), and if $f(a) = f(b)$, then there is a point c in (a, b) where

$$f'(c) = 0.$$

9. Why is this a special case of the Mean Value Theorem?

If $f(a) = f(b)$ $\quad \dfrac{f(b)-f(a)}{b-a} = 0$

10. Draw a picture that illustrates Rolle's Lemma.



Proof of Rolle's Lemma:

Since f is continuous on a closed, bounded interval, it attains a min and a max. If both occur at the endpoints, since $f(a)=f(b)$, f is constant, and $f'(x)=0$ on all (a,b). Otherwise, f admits a min or max at some $c \in (a,b)$ and Fermat's theorem implies $f'(c)=0$.

11. Suppose f is a continuous function on $[a, b]$ and $f'(x) \leq 0$ for every x in (a, b). How do $f(a)$ and $f(b)$ compare?

$$\frac{f(b)-f(a)}{b-a} = f'(c) \geq 0.$$

So $f(b) - f(a) \geq 0$

So $f(b) \geq f(a)$

12. Suppose f is a continuous function on $[a, b]$ and $f'(x) = 0$ for every x in (a, b). How do $f(a)$ and $f(b)$ compare?

$$\frac{f(b)-f(a)}{b-a} = f'(c) \leq 0.$$

So $f(b)-f(a) \leq 0.$

So $f(b) < f(a)$

Combine 10. and 11.

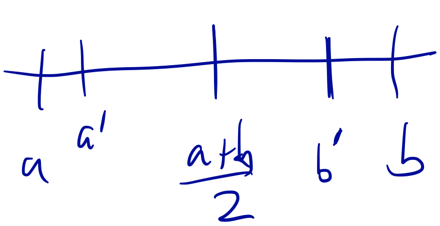
13. Suppose on some interval (a, b) that $f(x) = C$ for some constant C. What can you say about $f'(x)$ on (a, b)?

$f(a) \leq f(b)$ and $f(b) \leq f(a)$ so

$f(b) = f(a)$

$\frac{d}{dx} C = 0$ so $f'(x) = 0$ on (a,b)

14. Suppose $f'(x) = 0$ on an interval (a, b). Then there is a constant C such that $f(x) = C$ for all x in (a, b). Why?



By 12, $f\left(\frac{a+b}{2}\right) = f(b')$ for $\frac{a+b}{2} < b' \leq b.$

Also, $f(a') = f\left(\frac{a+b}{2}\right)$ for $a < a' < \frac{a+b}{2}.$ So $f(x) = f\left(\frac{a+b}{2}\right)$ for all $x \in (a,b).$

15. Suppose $f'(x) = g'(x)$ on an interval (a, b). Then there is a constant C where $g(x) = f(x) + C$. Why?

Let $h(x) = g(x) - f(x)$.

Then $h'(x) = g'(x) - f'(x) = 0$ on (a,b)

So $h(x) = C$ for some constant C.

So $g(x) = f(x) + C$.

Proof of Mean Value Theorem

Let $h(x) = f(x) - \ell(x)$ where $\ell(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$.

Then $h(a) = f(a) - \ell(a) = f(a) - f(a) = 0$.

And $h(b) = f(b) - \ell(b) = f(b) - f(b) = 0$.

Rolle's Lemma implies $h'(c) = 0$ for some $c \in (a,b)$.

So $f'(c) = \ell'(c)$. But $\ell'(x) = (f(b) - f(a))/(b-a)$.

So $f'(c) = \frac{f(b) - f(a)}{b-a}$