

In the first part of this worksheet we will get to know a method for computing an approximation of  $\sqrt{2}$  to many digits of accuracy using only addition, subtraction, multiplication and division, and indeed using only a few such operations.

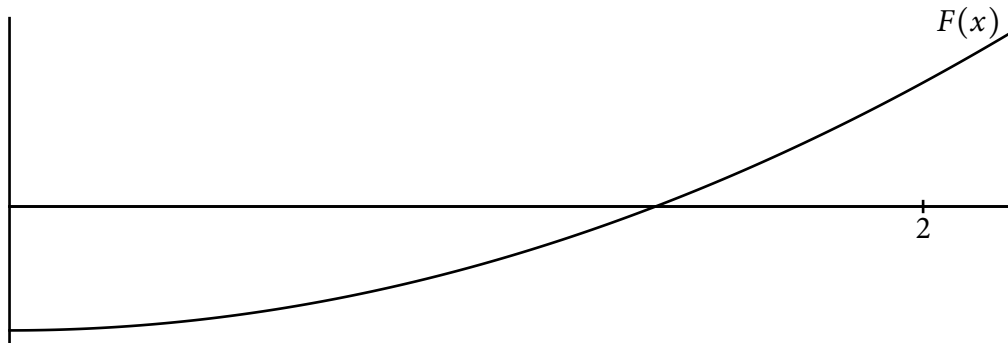
1. Consider the function

$$F(x) = x^2 - 2.$$

If we solve  $F(a) = 0$  for some  $a \geq 0$ , what is the value of  $a$ ?

2. Find the linearization  $L(x)$  of  $F(x)$  at  $x = 2$ . Leave your answer in point-slope form.

3. I've graphed  $F(x)$  for you below. Add to this diagram the graph of  $L(x)$ .



4. Find the number  $x_1$  such that  $L(x_1) = 0$ .
5. What good is the number  $x_1$ ? Keep in mind that you want to solve  $F(x) = 0$ . You solved  $L(x) = 0$  instead.
6. In the diagram above, label the point  $x_1$  on the  $x$ -axis.

7. Let's do it again! Find the linearization  $L(x)$  of  $F(x)$  at  $x = x_1$ .

8. Add the graph of this new linearization to your diagram on the first page.

9. Find the number  $x_2$  such that  $L(x_2) = 0$ . Then label the point  $x = x_2$  in the diagram.

10. To how many digits does  $x_2$  agree with  $\sqrt{2}$

11. Let's be a little more systematic. Suppose we have an estimate  $x_k$  for  $\sqrt{2}$ .

- Compute  $F(x_k)$ .
- Compute  $F'(x_k)$ .
- Compute the linearization of  $F(x)$  at  $x = x_k$ .

$$L(x) =$$

- Find the number  $x_{k+1}$  such that  $L(x_{k+1}) = 0$ . You should try to find as simple an expression as you can.

12. Starting with  $x_0 = 2$ , compute  $x_1$  and  $x_2$  with your shiny new formula. Verify that they agree with your earlier expressions for  $x_1$  and  $x_2$ .
13. Compute  $x_4$ . To how many digits does it agree with  $\sqrt{2}$ ?

### Newton's Method In General

We wish to solve  $F(x) = 0$  for a differentiable function  $F(x)$ . We have an initial estimate  $x_0$  for the solution.

14. Try to solve

$$e^{-x} - x = 0$$

by hand.

15. Explain why there is a solution between  $x = 0$  and  $x = 1$ .

16. Starting with  $x_0 = 1$ , find an approximation of the solution of  $e^{-x} - x = 0$  to 6 decimal places. During your computation, keep track of each  $x_k$  to at least 10 decimal places of accuracy.