1. Consider the heat equation $u_t = \kappa u_{xx}$ for $\kappa > 0$, $x \in [0,1]$, and Dirichlet boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$. Suppose we have initial condition $u(x,0) = sin(5\pi x)$.

a) Find an exact solution to this problem.

b) Implement the backward Euler (BE) method to solve this heat equation problem. Specifically, use diffusivity $\kappa = 1/20$ and final time $T = 0.1$. Note that you do not need to use Newton's method to solve the implicit equation, which is a linear system, but you should use sparse storage and an efficient linear solver (backslash in MATLAB will work).

c) Suppose the timestep k and the space step h are related by $k = 2h$. What do you expect for the convergence rate $O(h^p)$? Then measure it by using the exact solution from a), at the final time, and the infinity norm $||\cdot||_\infty$, and $h = 0.05, 0.02, 0.01, 0.005, 0.002, 0.001$. Make a log-log convergence plot of h versus the error.

d) Repeat parts b) and c) but with the trapezoidal rule instead of BE. (That is, implement and measure the convergence rate of Crank-Nicolson, with everything else the same.)

Solution, part a:
The exact solution is

$$u(x,t) = \exp(-\sqrt{5\kappa\pi}t)\sin(5\pi x)$$

Solution, part b:
See worksheet.

Solution, part c:
For Backwards Euler, the expected rate of convergence is $O(h)$. For Crank Nicolson the expected rate of convergence is $O(h^2)$. Log-log plots verifying these rates can be found in the worksheet.

2. Consider the PDE

$$u_t = \partial_x(p(x)u_x)$$

where $p(x)$ is a given function. We wish to solve the PDE on the region $0 \le x \le 1$, $0 \le t \le T$ with $u = 0$ at $x = 0, 1$ We will apply the following finite difference scheme to it:

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2}[(u_{i+1,j} - u_{i,j})p_{i+\frac{1}{2}} - (u_{i,j} - u_{i-1,j})p_{i-\frac{1}{2}}]$$

where $p_{i\pm\frac{1}{2}} = p(x_i \pm h/2)$.

a) Estimate the local truncation error in terms of powers of h and k and in terms of derivatives of u and derivatives of p. I'm looking for an answer akin to the estimate we derived for the heat equation of the form

$$|\tau| \le \max|u_{xxxx}|\left[\frac{k}{2} + \frac{h^2}{h}\right]$$

that we derived for the heat equation with no forcing term.

b) Show that the method is convergent, assuming $0 < p(x)k < h^2/2$. You will want to revist the proof from class that the explict method for the standard heat equation is convergent.

Solution:

Pending.

3.

a) Let

$$A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Compute $||A||_1$ and $||A||_\infty$.

b) Estimate $||A||_2$ as follows. Computer generate a figure containng the boundary of $A(B_1)$, where B_1 is the Euclidean ball of radius 1. Then use the figure to estimate the norm.

c) Suppose A is an $n \times n$ matrix, and choose $p \in [1, \infty]$. Show that $||A||_p = 0$ if and only if A is the 0 matrix.

d) For vectors in $\mathbb{R}^n$, it is known that $||x + y||_p \le ||x||_p + ||y||_p$ for any $p \in [1, \infty]$. This is the triangle inequality, and you need not prove it. But using this fact, show that the triangle inequality also holds for matrix norms $|| \cdot ||_p$ for p in the same range.

Solution, part a:

For a vector $x = (x_1, x_2)$, $Ax = (5x_1 + 6x_2, 7x_1 + 8x_2)$. Thus

$$||A_x||_1 = |5x_1 + 6x_2| + |7x_1 + 8x_2| \le 12|x_1| + 14|x_2| \le 14||x||_1.$$

Thus $||A||_1 \le 14$. But taking $x = (0, 1)$ we find

$$\frac{||Ax||_1}{||x||_1} = \frac{14}{1} = 14.$$

Thus$||A||_1 \ge 14$ as well and $||A||_1 = 14$.

Again, for a vector $x = (x_1, x_2)$, $Ax = (5x_1 + 6x_2, 7x_1 + 8x_2)$. Thus

$$\begin{aligned} ||A_x||_\infty &= \max |5x_1 + 6x_2|, |7x_1 + 8x_2| \\ &\le \max 5|x_1| + 6|x_2|, 7|x_1| + 8|x_2| \\ &\le \max(5||x||_\infty + 6||x_2||_\infty, 7||x||_\infty + 8||x||_\infty) \\ &\le 15||x||_\infty. \end{aligned} \tag{1}$$

Thus $||A||_\infty \le 15$. But taking $x = (1,1)$ we have $||x||_\infty = 1$ and $Ax = (11, 15)$ so

$$\frac{||Ax||_\infty}{||x||_\infty} = \frac{15}{1} = 15.$$

Thus $||A||_\infty \ge 15$ and hence $||A||_\infty = 15$.

Solution, part b:

See worksheet.

Solution, part c:
Suppose A is the zero matrix. Then $||Ax||_p = 0$ for any vector x and

$$||A||_p = \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p} = \sup_{x\neq 0} 0 = 0.$$

Now suppose A is not the zero matrix. Then one of its columns, column i, say, is not all zeros. Let e_i be the vector that is all zeros, except for a 1 in row i. Then $Ae_i = y$, where y is the i'th column of A. In particular $y \neq 0$ and $||y||_p > 0$. But then

$$||A||_p \ge \frac{||Ae_i||_p}{||e_i||_p} = \frac{||y||_p}{1} > 0.$$

Thus $A \neq 0$.

Thus $||A||_p = 0$ if and only if $A = 0$.

Solution, part d:
Suppose A and B are $n \times n$ matrices. Given a vector x,

$$||(A+B)x||_p = ||Ax + Bx||_p \le ||Ax||_p + ||Bx||_p \le ||A||_p||x||_p + ||B||_p||x||_p = (||A||_p + ||B||_p).$$

Note that we used the triangle inequality for vectors as well as the fundamental inequality for matrix norms:

$$||Ax||_p \le ||A||_p||x||_p.$$

Assuming that $x \neq 0$ we find

$$\frac{||(A+B)x||_p}{||x||p} \le ||A||_p + ||B||_p.$$

But the matrix norm is defined by

$$||A+B||_p = \sup_{x\neq 0} \frac{||(A+B)x||_p}{||x||p} \le ||A||_p + ||B||_p$$

by the above.

4. Text, problem 3.7

Solution, part a:
The space domain is $[0, \ell]$ and we assume that we make $N+1$ space steps of size $h = 1/(N+1)$. Letting $x_i = ih$ we know that $u(x_{N+1}, t) = 0$ always, but the value at x_0 is unknown; only the derivative there is specified. Thus we have extra unknowns $u_{0,j}$, compared with the Dirichlet problem.

We use centered differences for the second derivatives at x_i, $1 \le i \le N$ and we need another formula for the second derivative at x_0. We cannot use centered differences there since there are no values at x_{-1}. Instead, we use the formulas

$$\begin{aligned} u(h) &= u(0) + u'(0)h + \frac{1}{2}u''(0)h^2 + \frac{1}{6}u'''(0)h^3 + O(h^4) \\ &= u(0) + \alpha h + \frac{1}{2}u''(0)h^2 + \frac{1}{6}u'''(0)h^3 + O(h^4) \\ u(2h) &= u(0) + \alpha 2h + \frac{4}{2}u''(0)h^2 + \frac{4}{3}u'''(0)h^3 + O(h^4) \\ u(3h) &= u(0) + \alpha 3h + \frac{9}{2}u''(0)h^2 + \frac{9}{2}u'''(0)h^3 + O(h^4) \end{aligned} \tag{2}$$

Now suppose we have numbers c_1, c_2 and c_3 such that $c_1 + c_2 + c_3 = 0$. Then

$$c_1u(h)+c_2u(2h)+c_3u(3h) = (c_1+2c_2+3c_3)\alpha h+\frac{1}{2}(c_1+4c_2+9c_3)u''(0)h^2+\frac{1}{6}(c_1+8c_2+3^3c_3)u'''(0)h^3+O(h^4).$$

Now suppose in addition $c_1 + 8c_2 + 3^3c_3 = 0$. Then

$$\frac{2}{h^2}\frac{c_1u(h) + c_2u(2h) + c_3u(3h) - (c_1 + 2c_2 + 3c_3)\alpha h}{c_1 + 4c_2 + 9c_3} = u''(0) + O(h^2).$$

The general solution of

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + 8c_2 + 3^3c_3 &= 0 \end{aligned} \tag{3}$$

is given by (c_1, c_2, c_3) being a multiple of $(19, -26, 7)$. Thus

$$u''(0) = \frac{-19u(h) + 26u(2h) - 7u(3h)}{11h^2} - \frac{12}{11}\frac{\alpha}{h} + O(h^2)$$

This yields an $O(h^2)$ approximation

$$\frac{1}{h^2}\frac{-19u_{1,j} + 26u_{2,j} - 7u_{3,j}}{11} - \frac{12}{11}\frac{\alpha}{h}$$

for $u''(0, t_j)$.

To express our spatial discritization in matrix terms we introduce

$$A = \begin{pmatrix} -19/11 & 26/11 & -7/11 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -2 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \end{pmatrix}$$

and

$$N = \begin{pmatrix} -12/11 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Now let $\mathbf{u}(t)$ be the vector with components $u_i(t) \approx (u(x_i, t))$. Then our approximation of $u_{xx}(x_i, t)$ is

$$\frac{1}{h^2} A\mathbf{u}(t) + \frac{\alpha}{h} N\mathbf{e}_0$$

where $\mathbf{e}_0$ is the vector of all zeros, except its first entry is 1. Thus the Method of Lines yields the system of ODEs

$$\frac{d}{dt}\mathbf{u}(t) = \frac{1}{h^2} A\mathbf{u}(t) + \frac{\alpha}{h} N\mathbf{e}_0$$

To handle the time discretization we can apply Backwards Euler to obtain

$$\mathbf{u}_{j+1} = \mathbf{u}_j + k\left(\frac{1}{h^2} A\mathbf{u}_j + \frac{\alpha}{h} N\text{âČŮ}e_0\right)$$

This yields an $O(k) + O(h^2)$ order of accuracy.

On the other hand, for an $O(k^2) + O(h^2)$ order of accuracy we can apply the trapezoidal rule to the ODE to get

$$\mathbf{u}_{j+1} = \mathbf{u}_j + \frac{k}{2h^2}\left(A\mathbf{u}_j + A\mathbf{u}_{j+1}\right) + \frac{\alpha k}{h} N\mathbf{e}_0.$$

That is,

$$\left(I - \frac{k}{2h^2} A\right)\mathbf{u}_{j+1} = \left(I + \frac{k}{2h^2} A\right)\mathbf{u}_j + \frac{\alpha k}{h} N\mathbf{e}_0.$$