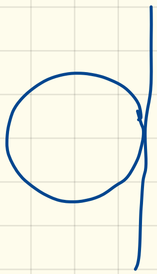
Last class:

Thought of explicit method as Euler's method applied to

$$u' = \frac{1}{h^2} D x$$

Expect eigenvalues of $\frac{1}{h^2} D \sim -\pi^2 n^2 \quad 1 \leq n \leq N$

Thus region of abs. stability is important



$$k\lambda \geq -2$$

$$-\pi^2 k N^2 \geq -2$$

$$k N^2 \leq \frac{2}{\pi^2}$$

$$\sim \frac{k}{h^2} \leq \frac{2}{\pi^2}$$

So $\underbrace{\frac{k}{h^2}}_{\lambda} < \frac{2}{\pi}$ ish.

This analysis is only heuristic: we don't yet know the eigenvalues of

$$\frac{1}{h^2} D.$$

To learn these, it's enough to study D.

We make a lucky guess

$$w_j = e^{Jrx_j} \qquad J^2 = -1 \qquad \text{(makes trig id's easy).}$$

$2 \le j \le N-1$

$$Dw_j = e^{Jrx_j}\left[e^{-Jrh} - 2 + e^{Jrh}\right] \qquad x_{j+1} = x_j + h$$

$$= -2w_j\left[1 + \cos(rh)\right] \qquad e^{J\theta} = \cos\theta + J\sin\theta$$

$$e^{-J\theta} = \cos\theta - J\sin\theta$$

$$= -4w_j\left[\sin^2\left(\frac{rh}{2}\right)\right]$$

So we almost have an eigenvalue: analysis doesn't apply at $j=1$, $j=N$.

But we let

$$v_j = \operatorname{Im}(w_j) \qquad r = n\pi$$

$$v_0 = 0 \qquad \left(h = \frac{1}{N+1}\right)$$
$$v_{N+1} = 0$$

$$\operatorname{Im}(Dw_j) = D\operatorname{Im}(w_j) = D v_j$$

$$\operatorname{Im}(Dw_j) = \operatorname{Im}\left(-\cancel{2}\sin^2\left(\tfrac{rh}{2}\right) w_j\right) = -\cancel{2}\sin^2\left(\tfrac{rh}{2}\right) \underbrace{\operatorname{Im}(w_j)}_{v_j}$$

So $\vec{v}_n$ is an eigenvector of D with e-value $-2\sin^2\left(\frac{r_n h}{2}\right)$

$$r_n = n\pi \qquad 1 \leq n \leq N.$$

Sweet.

Eigenvalues of $\frac{1}{h^2}D$ are $-\frac{4}{h^2}\sin^2\left(\frac{r_n h}{2}\right)$

If $\frac{r_n h}{2}$ is small, $\sim -\frac{4}{h^2}\left[\frac{n^2\pi^2}{4}h^2\right] = -n^2\pi^2$

(nh is small)

Analysis from Euler's method:

$$k\left(-\frac{4}{h^2}\right)\sin^2\left(\frac{r_n h}{2}\right) > -2$$

$$\frac{k}{h^2}\underbrace{\sin^2\left(\frac{r_n h}{2}\right)}_{\text{no control.}} < \frac{1}{2}$$

So $\quad \dfrac{k}{h^2} < \dfrac{1}{2}$

If $\quad k \geq \dfrac{h^2}{2}\quad$ then the time step is too long for the fastest transient modeled with this number of spatial steps.

Fourier Analysis: (a fast rule of thumb approach)

$$\vec{u}_{j+1} = (1 + \lambda D)\, u_j$$

$$v_k = e^{\sqrt{\ } r x_k}$$

For all but the boundary points

Suppose $u_j = v$

$$u_{i,j+1} = \lambda v_{i-1} + (1-2\lambda) v_i + \lambda v_{i+1}$$

$$= v_i \left[(1-2\lambda) + \lambda e^{-\sqrt{\ } r h} + \lambda e^{\sqrt{\ } r h} \right]$$

$$= v_i \left[1 + \lambda \left[-2 + \cos(rh) \right] \right]$$

$$= v_i \left[1 - 4\lambda \sin^2(rh) \right]$$

amplification factor

So $u_{j+1} = \left[1 - 4\lambda \sin^2(rh)\right] u_j$ except at boundary points.

To avoid instability want

$$-1 \leq 1 - 4\lambda \sin^2(rh) \leq 1$$

$$\frac{1}{2} \geq \lambda \sin^2(rh) \Rightarrow \frac{1}{2} \geq \lambda$$

same condition.

As time progresses, solution oscillates + grows if $\lambda > \frac{1}{2}$

Note: this condition is compatible with maximum principle.

Let's prove convergence assuming $\lambda \leq \frac{1}{2}$.

Specifically, $h_n \to 0$, $k_n \to 0$, $\frac{k_n}{h_n^2} \leq \frac{1}{2}$ $\forall n$.

I'd like to be a little more careful with LTE

$$\text{I} \quad \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} = u_t(x_i, t_j) + \frac{k}{2} u_{tt}(x_i, \eta_j)$$

$$t_j \leq \eta_j \leq t_j + h$$

$$\text{II} \quad \frac{u(x_i - h, t_j) - 2u(x_i, t_j) + u(x_i + h, t_j)}{h^2} = u_{xx}(x_i, t_j) + \frac{h^2}{6} u_{xxxx}(\hat{\eta}_i, t_j)$$

$$x_{i-1} \leq \hat{\eta}_i \leq x_{i+1}$$

$$\underbrace{\text{I} + \text{II} - \underbrace{f(x_i, t_j)}_{(u_t - u_{xx})(x_i, t_j)}}_{\tau} = \frac{k}{2} u_{tt}(x_i, \eta_j) + \frac{h^2}{6} u_{xxxx}(\hat{\eta}_i, t_j)$$

$$u_t = u_{xx} + f$$

$$u_{tt} = u_{txx} + f_t$$

$$= u_{xxt} + f_t$$

$$= u_{xxxx} + f_t$$

$$|\tau| \le \left[\frac{k}{2} + \frac{h^2}{6}\right] \max|u_{xxxx}| + \frac{k}{2} \max|f_t|$$

$u_{tt} = u_{xxxx} + f_t$ is, alas, a compatibility condition on the initial data:

$u_{tt} = 0$ on boundary.

If $f = 0$,

$u_{xxxx} = 0$ on boundary is needed.

$$C = \max|u_{xxxx}| + \max|f_t|$$

$$|\tau| \le \left[\frac{k}{2} + \frac{h^2}{6}\right] C$$

$U_{i,j}$ numerical solution

$$U_{i,j+1} = \lambda U_{i-1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i+1,j} + kf_{i,j}$$

$$u_{i,j} = u(x_i, t_j)$$

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j} + kf_{i,j} + k\tau_{i,j}$$

$$E_{i,j} = U_{i,j} - u_{i,j}$$

$$E_{i,j+1} = \lambda E_{i-1,j} + (1-2\lambda)E_{i,j} + \lambda E_{i+1,j} - k\tau_{i,j}$$

$$E_j = \max_i |E_{i,j}|$$

$$|E_{i,j+1}| \le |\lambda| |E_{i-1,j}| + |1-2\lambda| E_{i,j} + |\lambda| |E_{i+1}| + k|\tau_{i,j}|$$

use $\lambda \le \frac{1}{2}$

$$\le \lambda E_j + (1-2\lambda)E_j + \lambda E_j + k|\tau_{i,j}|$$

$$E_{j+1} \le E_j + k|\tau| \qquad |\tau| = \max_{i,j} |\tau_{i,j}|$$

$$\le C\left(\frac{k}{2} + \frac{h^2}{6}\right)$$

$$E_1 \le E_0 + k\tau$$

$$E_2 \le E_1 + k\tau$$

$$\le E_0 + 2k\tau$$

$$\vdots$$

$$E_M \le E_0 + \underbrace{Mk}_{T}\tau$$

$$E_j \le E_0 + TC\left[\frac{k}{2} + \frac{h^2}{6}\right]$$

$$0 \le j \le M.$$

$$\max_j E_j \le TC\left[\frac{k}{2} + \frac{h^2}{6}\right]$$

Thm: If $h_{(n)} \to 0$, $k_{(n)} \to 0$ $\frac{k}{h^2} \le \frac{1}{2}$, + compatability

$$(x_i^{(n)}, t_j^{(n)}) \to (x, t)$$

$$\Rightarrow \quad u_{i,j}^{(n)} \longrightarrow u(x_i, t_j)$$

Since we have issues with transients, it will be helpful to work with an A-stable scheme.

Backwards Euler:

$$\vec{u}_{j+1} = \vec{u}_j + \frac{k}{h^2} D \vec{u}_{j+1} + k\vec{f}_j$$

$$\left[I - \lambda D \right] \vec{u}_{j+1} = \vec{u}_j + k\vec{f}_j$$

This is a linear system to solve for $\vec{u}_{j+1}$.

But $A = (I - \lambda D)$ is tri-diagonal

$$-\lambda \quad 1+2\lambda \quad -\lambda$$
$$\ddots$$

So solving is $O(n)$.

Fourier Analysis:

$$\left[-\lambda e^{-jhr} + (1+2\lambda) - \lambda e^{jhr}\right] e^{jxr} = c e^{jxr}$$

$$c^{-1} = \left[1+2\lambda - \lambda 2\cos(hr)\right]$$

$$= \left[1 + 2\lambda\left(1-\cos(hr)\right)\right]$$

$$= \left[1 + 4\lambda \sin^2(hr/2)\right]$$

$$c = \frac{1}{1+4\lambda\sin^2\left(\frac{hr}{2}\right)} \leq 1, \text{ regardless of } \lambda.$$

We'll shortly see this method is

1) convergent, regardless of λ

2) $O(h) + O(k^2)$.

Generalization: Θ-method

$$\vec{u}_{j+1} = \vec{u}_j + \Theta \cdot \lambda D \vec{u}_j + (1-\Theta)\lambda D \vec{u}_{j+1} + \vec{f}_j$$

$$\left[1 - (1-\theta)\lambda D\right]\vec{u}_{j+1} = \left[1 + \theta\lambda\right]\vec{u}_j + \vec{f}_j$$

explicit when $\theta = 1$. Backwards Euler, $\theta = 0$.