Final Review – Chapter 3

1. Given $f(x) = 3x - x^2$, find $f'(x)$ *using the definition of the derivative.*

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)-(x+h)^2-3x+x^2}{h}$$

$$= \lim_{h\to 0} \frac{3h-2xh-h^2}{h}$$

$$= \lim_{h\to 0} 3-2x-h = 3-2x$$

2. Find dy/dx for each of the following.

   (a) $y^2e^x + 3 = x^2 + y$

$$2y\frac{dy}{dx}e^x + y^2e^x = 2x + \frac{dy}{dx}$$

$$(2ye^x-1)\frac{dy}{dx} = 2x - y^2e^x$$

$$\frac{dy}{dx} = \frac{2x-y^2e^x}{2ye^x-1}$$

   (b) $y = (\sin(x))^x$

$$\ln(y) = x\ln(\sin(x))$$

$$\frac{y'}{y} = \ln(\sin(x)) + \frac{x}{\sin(x)}\cdot\cos(x)$$

$$y' = y\left[\ln(\sin(x)) + x\cot(x)\right] = (\sin(x))^x\left[\ln(\sin(x)) + x\cot(x)\right]$$

3. Assume that the number of yeast cells in a laboratory culture is modeled by the function

$$n(t) = \frac{1000}{1+4e^{-t}},$$

where t is measured in hours.

(a) Find the average rate of change of the population in the first hour. You do not have to simplify your answer but you do have to give units.

$$\frac{n(1)-n(0)}{1-0} = \frac{1000}{1+4e^{-1}} - \frac{1000}{1+4}$$
$$= 1000\left(\frac{1}{1+4e^{-1}} - \frac{1}{5}\right) \approx 204.6$$
cells / hour

(b) Find $n'(t)$.

$$n'(t) = \frac{-1000}{(1+4e^{-t})^2} \cdot (-4e^{-t}) = \frac{4000e^{-t}}{(1+4e^{-t})^2}$$

(c) Find $n'(1)$ and interpret it in the context of the problem.

$$n'(1) = \frac{4000e^{-1}}{(1+4\cdot e^{-1})^2} = 240 \text{ cells / hour}$$

(d) Find and interpret $\lim_{t\to\infty} n(t)$.

long term population

$$\lim_{t\to\infty} \frac{1000}{1+4e^{-t}} = \frac{1000}{1+0} = 1000 \text{ cells}$$

(e) Find and interpret $\lim_{t\to\infty} n'(t)$.

$$\lim_{t\to\infty} \frac{4000e^{-t}}{(1+4e^{-t})^2} = \frac{4000\cdot 0}{(1+4\cdot 0)^2} = 0$$

Eventually, population size stops changing.