

## 1. Exercise 0.1 (Solution by John Gimbel)

If  $a$  and  $b$  are even integers, then so is  $a + b$ .

**Solution:**

Let  $a$  and  $b$  be even integers. Then there exist integers  $j$  and  $k$  such that  $a = 2j$  and  $b = 2k$ .

But then

$$a + b = 2j + 2k = 2(j + k). \quad (1)$$

Since  $j + k \in \mathbb{Z}$ ,  $a + b$  is even.

## 2. Exercise 0.2 (Solution by Jill Faudree)

Let  $X$  be a set.

a) An intersection of topologies on  $X$  is a topology on  $X$ .

b) A union of topologies on  $X$  need not be a topology.

**Solution, part a:**

Let  $\{\tau_\alpha\}$  be a family of topologies and let  $\tau = \cap_\alpha \tau_\alpha$ . Observe that  $\emptyset$  and  $X$  belong to  $\tau$  as they belong to each  $\tau_\alpha$ .

Suppose  $\{U_\beta\}$  is a family of sets in  $\tau$  and let  $U = \cup_\beta U_\beta$ . Fix  $\alpha$  and observe that each  $U_\beta \in \tau_\alpha$ . Since  $\tau_\alpha$  is a topology,  $U \in \tau_\alpha$ . Since  $\alpha$  is arbitrary,  $U \in \cap_\alpha \tau_\alpha = \tau$ .

The proof that a finite intersection of sets in  $\tau$  belongs to  $\tau$  is essentially similar.

**Solution, part b:**

Let  $X = \{1, 2, 3\}$ . Let  $\tau_1 = \{\emptyset, \{1\}, X\}$  and let  $\tau_2 = \{\emptyset, \{2\}, X\}$ . It is easy to see that these are topologies. Let  $T = \tau_1 \cup \tau_2 = \{\emptyset, \{1\}, \{2\}, X\}$ . Observe that  $T$  is not closed under taking unions as  $\{1\}$  and  $\{2\}$  are elements of  $T$  but  $\{1, 2\}$  is not.

## 3. Exercise 0.3 (Solution by Elizabeth Allman)

Let  $X$  be a metric space. Show that the collection of open balls in  $X$  forms the basis of a topology.

**Solution:**

We start with a technical lemma.

**Lemma A:** Suppose  $B_1 = B_{r_1}(x_1)$  and  $B_2 = B_{r_2}(x_2)$  are open balls in  $X$  and  $x_3 \in B_1 \cap B_2$ . Then there is an  $r > 0$  such that  $B_r(x_3) \subseteq B_1 \cap B_2$ .

*Proof.* Let  $r = \min(r_1 - d(x_3, x_1), r_2 - d(x_3, x_2))$  and observe that  $r > 0$ . Now suppose

$z \in B_r(x_3)$ . The triangle inequality implies

$$\begin{aligned} d(x_1, z) &\leq d(x_1, x_3) + d(x_3, z) \\ &< d(x_1, x_3) + r \\ &\leq d(x_1, x_3) + (r_1 - d(x_3, x_1)) \\ &= r_1 \end{aligned}$$

Hence  $z \in B_{r_1}(x_1) = x_1$ . Similarly  $z \in B_2$ , and hence  $B_r(z) \subseteq B_1 \cap B_2$ . □

Let  $\mathcal{B}$  be the collection of open balls in  $X$ . Fix  $x \in X$  and note that  $\cup_{r>0} B_r(x) = X$ . Hence  $\mathcal{B}$  covers  $X$ . Moreover, by Lemma A,  $\mathcal{B}$  satisfies the refinement property. Hence by the topology construction lemma,  $\mathcal{B}$  generates a topology on  $X$ , and the open sets are simply the unions of open balls.