1. Find two numbers whose difference is 100 and whose product is a minimum.

Numbers: x, y

$x - y = 100$

Minimize: $m = xy = (100+y)y$

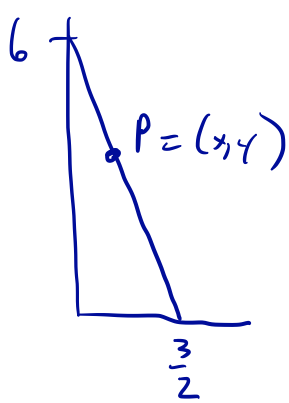
$m' = 100 + 2y$

$m' = 0 \Rightarrow y = -50$

$\Rightarrow x = 100 + y = 50$

$m'' = 2 > 0$ everywhere so we have an absolute minimum at $y = -50$

2. Find the point on the line $6x + y = 9$ that is closest to the origin. Hint: minimizing distance is equivalent to minimizing distance squared!



Minimize $L = x^2 + y^2$

$= x^2 + (9 - 6x)^2$

$L' = 2x + 2(9 - 6x)(-6)$

$= 2x + 2\cdot 36x - 2\cdot 54$

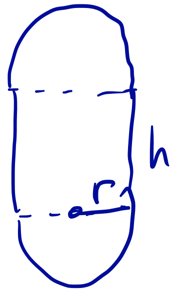
$= 2(37x - 54)$

$L' = 0 \Rightarrow x = 54/37$

$L'' = 66 > 0$ everywhere

$\Rightarrow$ abs min at $x = \frac{54}{37} \approx 1.46$ $\quad y = 9 - 6\left(\frac{54}{37}\right) \approx 0.23$

3. A stadium curve is the curve that bounds a rectangular region with half circles at opposite ends of the rectangle; think of a running track. Find the dimensions of a stadium curve that maximize the area of the enclosed rectangle if the perimeter of the stadium curve is 440 yards.



Area of square: $2rh$

Perimeter of curve: $h + h + 2\pi r = 2h + 2\pi r$

constraint: perimeter = 440 so

$$2h + 2\pi r = 440$$

$$h = 220 - \pi r$$

Area: $A = 2rh$

$$= 2r(220 - \pi r)$$

Maximize: $A' = 2\left[(220 - \pi r) + r(-\pi)\right]$

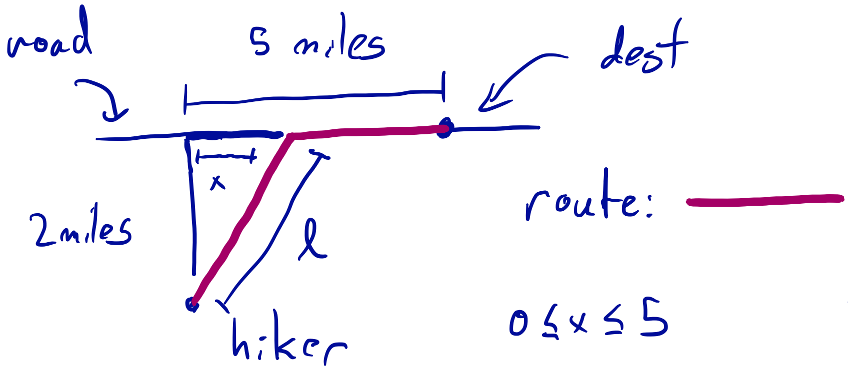
$$= 2\left[220 - 2\pi r\right]$$

$A' = 0 \Rightarrow r = 110/\pi \Rightarrow h = 220 - \pi\left(\frac{110}{\pi}\right) = 110.$

$A'' = -4\pi < 0$ everywhere, so it's a global max at $r = 110/\pi$

$$\boxed{r = \frac{110}{\pi},\ h = 110}$$

4. A hiker is on the tundra two miles south of a road. The road runs east-west the hiker wishes to reach a point on the road 5 miles to the east. The hiker can travel at 3 mph on the tundra and 4 mph on the road. What path should the hiker take to minimize their travel time to their destination?



$0 \le x \le 5$

Use closed interval method.

Compare

$T(0) = 1.91$

$T(5) = 1.795$

$T(6/\sqrt{7}) = 1.69$

absolute min

distance in tundra: ℓ miles

time in tundra: $\ell/3$ hours

distance on road: $5-x$ miles

time on road: $(5-x)/4$ hours

total time: $T = \dfrac{\ell}{3} + \dfrac{(5-x)}{4}$

But $\ell^2 = 2^2 + x^2$ $\quad \ell = \sqrt{4+x^2}$

$$T = \frac{\sqrt{4+x^2}}{3} + \frac{(5-x)}{4}$$

$$T' = \frac{1}{3}\frac{x}{\sqrt{4+x^2}} - \frac{1}{4}$$

$T' = 0 \quad 4x = 3\sqrt{4+x^2}$

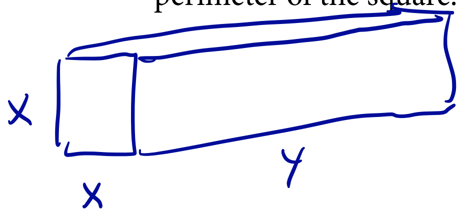
$16x^2 = 9\cdot(4+x^2)$

$7x^2 = 36$

$x = \dfrac{6}{\sqrt{7}}$

endpoints $x = 0, 5$

5. The USPS will accept a box for shipment if the sum of its length plus girth (total distance around) does not exceed 108 inches. What shape of box with a square end has maximum enclosed volume and is acceptable for shipping? You may assume that girth is measured as perimeter of the square.



Volume: $V = x^2 y$

constraint: $4x + y = 108, \quad 0 \leq x \leq 27$

$y = 108 - 4x$

$V = x^2[108 - 4x]$

$V' = 2 \cdot 108x - 12 \cdot x^2$

$= 2x[108 - 6x]$

Use closed interval method:

endpoints $x = 0, 27$

crit pts $x = 0, \frac{108}{6} = 18$

not interior; ignore.

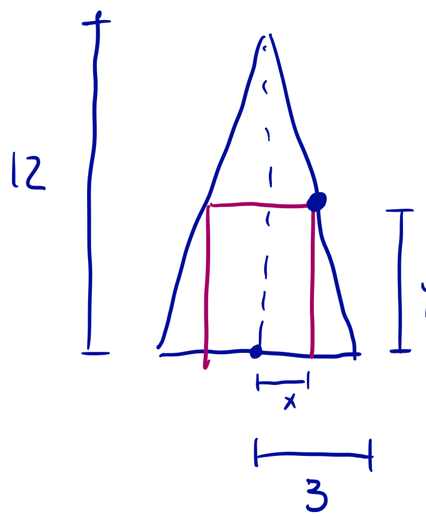
$V(0) = V(27) = 0$

$V(18) = 11664$

So maximum volume at

$x = 18, y = 36$

6. An isosceles triangle has base 6cm and height 12cm. Find the maximum possible area of a rectangle that can be placed inside the triangle with one side on the base of the triangle.



Area: $2xy$

similar triangles: $\frac{y}{3-x} = \frac{12}{3}$

$y = \frac{12}{3}(3-x)$

$A = \frac{24}{3}\left[3x - x^2\right]$

$A' = \frac{24}{3}\left[3 - 2x\right]; \quad A' = 0 \Rightarrow x = \frac{3}{2}$

$A'' = -\frac{48}{3} < 0$ everywhere, so an abs. max at $x = \frac{3}{2}$

$\boxed{\text{dimensions: } x = \frac{3}{2}\text{ cm}, \ y = 12\left(1 - \frac{1}{2}\right) = 6 \text{ cm}}$