In this worksheet we go through some guidelines (a recipe) for sketching the graph of a function using information that can be learned from calculus. In truth, in the modern era, computers and calculators are frequently used to graph functions. You should think of these exercises instead as pushups: nobody does pushups because they think that pushups are a good life skill. Rather, people do pushups to build upper-body strength to help them in other applications that they **do** think are useful. With that in mind, let's go through the steps for sketching our first curve.

You should also note that the guidelines presented here are just a little different from the ones presented in your text. Use either; I've presented mine here to emphasize what I think are helpful ways to approach this task and *think* about what you are doing.

1. Sketch the graph of $f(x) = \dfrac{x}{1+x^2}$.

a. Determine the domain of the function. You'll need to know this to decide on the set of x-values where you want to draw the sketch.

$f(x)$ is well defined for all x in $\mathbb{R}$. domain: $\mathbb{R}$

b. Decide if the function has any symmetry. Is it an even function ($f(-x) = f(x)$)? An odd function ($f(-x) = -f(x)$)? A periodic function?

f is odd: $f(-x) = \dfrac{-x}{1+(-x)^2} = \dfrac{-x}{1+x^2} = -\dfrac{x}{1+x^2} = -f(x)$

c. If you can, find a few choice values where you can compute the value of f. This will provide a bit of scaffolding for your sketch. Good candidates include $x = 0$ (i.e. computing the y-intercept) and finding roots (i.e. places where $f(x) = 0$). This step will vary function-by-function.

$f(0) = 0$

And $f(x) = 0$ only at $x = 0$.

d. Determine if the function has any interesting behavior at $\pm\infty$. Is $\lim_{x\to\infty} f(x) = L$? Then there will be a horizontal asymptote at $y = L$. Or maybe $\lim_{x\to\infty} f(x) = \infty$ and the function blows up at infinity.

$\lim_{x\to\infty} \dfrac{x}{1+x^2} = \lim_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{x^2}+1} = \dfrac{0}{0+1} = 0$

$\lim_{x\to-\infty} f(x) = 0$ since the function is odd.

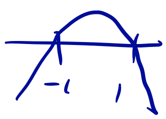
e. Determine if the function has any spots a where it blows up to $\pm\infty$ (i.e. $\lim_{x\to a} f(x) = \pm\infty$). There will be vertical asymptotes at these locations. You can often spot these by division by zero or by know behavior of functions like $\ln(x)$.

No singularities. No vertical asymptotes.

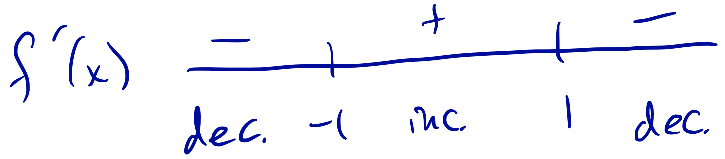
f. Determine where the function is increasing and where it is decreasing. I.e., decide where $f'(x) > 0$ and where $f'(x) < 0$. Doing so you will also determine critical points, where $f'(x) = 0$ and $f'(x)$ does not exist.

$$f'(x) = \frac{(1+x^2)-2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

← determines sign:



← positive

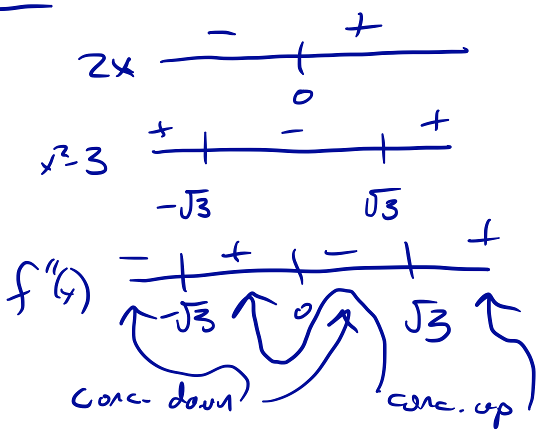


g. Determine where the function is concave up ($f''(x) > 0$) and concave down($f''(x) < 0$). Doing so you will also be able to spot points of inflection, where the concavity changes.

$$f''(x) = \frac{-2x(1+x^2)^2 - (1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3}$$

$$= \frac{2x(x^2-3)}{(1+x^2)^3}$$



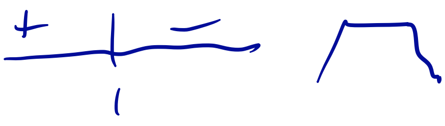
h. For each critical point, decide if it is a local min/local max or neither. The First Derivative Test and the Second Derivative Test are your friends.

points of infl. at $x = -\sqrt{3}, 0, \sqrt{3}$

At $x=-1$:

local min

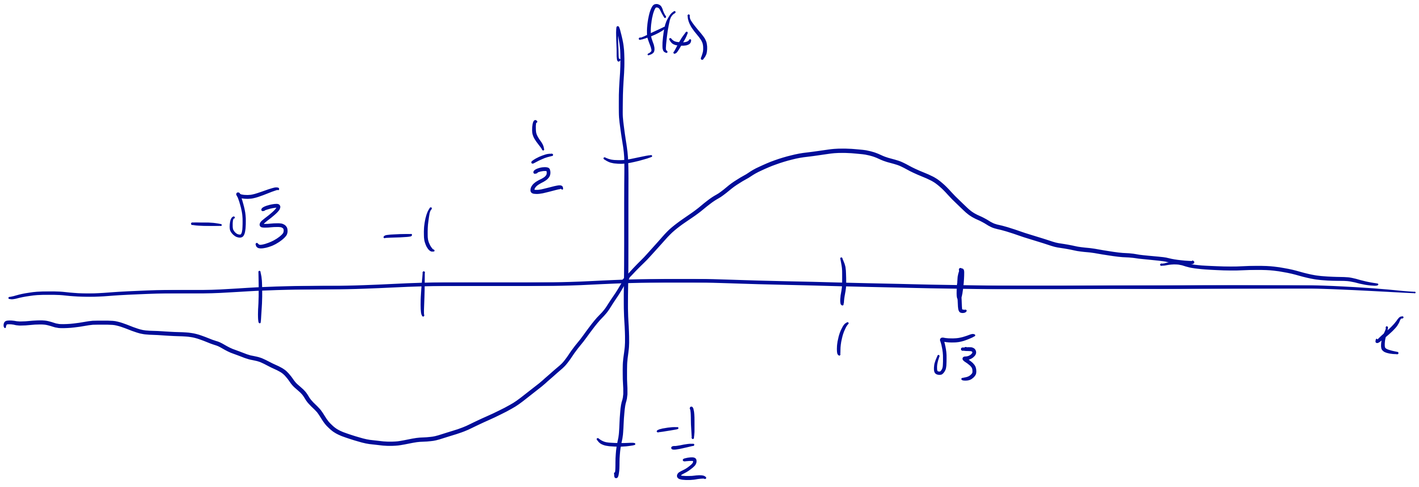
At $x=1$



local max

(First Derivative Test!)

i. Now you can put all this data together to sketch the curve.



2. Follow these guidelines to sketch the graph of

$$f(x) = \frac{2x^2}{x^2 - 4}.$$

a. What is the function's domain?

b. Does this function have any symmetry?

c. Find a few choice values of x to evaluate the function at. In this case, what's the y-intercept? Are there any roots?

d. What happens to this function at ±∞?

e. Does the function have any vertical asymptotes?

f. Find intervals where f is increasing/decreasing.

g. Find intervals where f is concave up/concave down.

h. What are the critical points? Classify each as a local min, local max, or neither.

i. Find any points of inflection.

j. Sketch the graph of $f(x)$.