**1.** Young's Inequality (Solution by Lander Ver Hoef)

Let $p \in (1, \infty)$ and define $q$ by $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $a, b \geq 0$. Show

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and that the inequality is strict unless either $a^p = ab$ or $b^q = ab$ (in which case both of these equalities hold!).

Hint: Fix $b \geq 0$ and consider $f(a) = a^p/p + b^q/q - ab$ on $[0, \infty)$. Look at the first and second derivatives of $f$.

Remark: Your proof should clearly note the place where $p > 1$ is used.

**Solution:**
Let $b \geq 0$ and let

$$f(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab,$$

which we will consider on the interval $[0, \infty)$. Notice that $f(a) \geq 0$ precisely when $ab \leq a^p/p + b^q/q$, so if we can show that $f(a) \geq 0$ for all $a$, we have established the inequality.

First, notice that

$$f'(a) = a^{p-1} - b,$$

and because $p > 1$, we know that

$$f''(a) = (p-1)a^{p-2}.$$

By setting $f' = 0$, we see that a stationary point of $f(a)$ occurs at $a^{p-1} = b$. But because $a \geq 0$ and $p > 1$, $f''(a)$ is greater than or equal to 0, and for all $a \neq 0$, $f''(a) > 0$. This means that $f(a)$ must be concave up, and the stationary point at $a^{p-1} = b$ is a global minimum.

Observe that if $a^{p-1} = b$, then $a^p = ab$. Furthermore, $a = b^{1/(p-1)}$, and by the definition of $q$,

$$\frac{1}{q} = 1 - \frac{1}{p} \Rightarrow q = \frac{p}{p-1}.$$

But if we subtract 1 from both sides, we get that

$$q - 1 = \frac{p}{p-1} - \frac{p-1}{p-1} = \frac{1}{p-1}.$$

Thus, $a = b^{q-1}$, and $ab = b^q$. Substituting into $f$, we see that at our global minimum, we have that

$$f(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab = \frac{ab}{p} + \frac{ab}{q} - ab = ab\left(\frac{1}{p} + \frac{1}{q} - 1\right) = 0.$$

Thus, $f(a) \geq 0$ for all $a$, and is equal to zero only when $a^p = ab$ or $b^q = ab$.

**2. [Carothers 3.34]**(Solution by Max Heldman)

If $x_n \to x$ and $y_n \to y$ in $(M, d)$, then $d(x_n, y_n) \to d(x, y)$ in $\mathbb{R}$.

**Solution:**
Observe that for any $n \in \mathbb{N}$,

$$\begin{aligned} -d(x_n, x) &= d(x_n, y) - d(x_n, y) - d(x_n, x) \\ &= d(x_n, y) - (d(x_n, y) + d(x_n, x)) \\ &\leq d(x_n, y) - d(x, y) \\ &\leq d(x_n, x) + d(x, y) - d(x, y) = d(x_n, x). \end{aligned}$$

Hence by the squeeze theorem, $d(x_n, y) - d(x, y) \to 0$, that is, $d(x_n, y) \to d(x, y)$. Thus

$$\begin{aligned} d(x, y_n) - d(x, x_n) &\leq d(x_n, y_n) + d(x_n, x) - d(x, x_n) \\ &= d(x_n, y_n) \leq d(x_n, x) + d(x, y_n). \end{aligned}$$

and by the squeeze theorem $d(x_n, y_n) \to d(x, y)$.

**3.** Carothers 3.36 (Solution by Sakti Anupindi)

A convergent sequence is Cauchy, and a Cauchy sequence is bounded (that is, the set $\{x_n : n \geq 1\}$ is bounded.

**Solution:**

**Convergent sequence is Cauchy**

Let $\frac{\epsilon_1}{2} > 0$. Given a convergent sequence $(x_n)$, choose $N \geq 1$, $N \in \mathbb{N}$ such that if $n \geq N$, $d(x_n, x) < \frac{\epsilon}{2}$. Now using triangle inequality in **M**, whenever $n, m \geq N$

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon < 2\epsilon.$$

Since $\epsilon$ is arbitrary, a convergent sequence is Cauchy.

**Cauchy sequence is bounded**

A subset A of **M** is said to be bounded if it is contained in some ball, that is if $A \subset B_r(x)$ for some x $\in$ **M** and r > 0. Since the sequence $(x_n)$ is Cauchy, given $\epsilon > 0$ there is an integer $N \geq 1$ such that $d(x_m, x_n) < \epsilon$ whenever $m, n \geq N$.
Take $\epsilon = 1$. Then there exists $N$ such that $d(x_m, x_n) < 1$ for all $m, n \geq N$. So, this means $x_n \in B_1(x_N)$ for all $n \geq N$. This helps in dealing with infinite terms. For finite terms put $\delta = 1 + max\{d(x_1, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N)\}$. Note $\delta > 0$. Then the set $\{x_n : n \geq 1\} \in B_\delta(x_N)$ for all n; so $(x_n)$ is bounded.

**4.** Carothers 3.39[Mason Brewer]

**Solution:**
Let us assume that $(x_n)$ does not converge to $x$ and show that this implies that $(x_n)$ has a subsequence with no further subsequence converging to $x$. Since $(x_n)$ does not converge $x$, then there exists some $\varepsilon$ such that for each $N \in \mathbb{N}$, there exists some $n \geq N$ such

that $|d(x_n, x)| \geq \varepsilon$. Now construct the subsequence $(x_{n_i})$ through the following iterative process. First, pick $n_0 \geq 1$ so that $|d(x_0, x)| \geq \varepsilon$, and then pick $n_i \geq n_{i-1} + 1 > n_{i-1} \in \mathbb{N}$ so that $|d(x_{n_i}, x)| \geq \varepsilon$. Each $n_i$ is guaranteed to exist by the fact that $(x_n)$ does not converge as stated earlier. Note $(x_{n_i})$ is in fact a subsequence of $(x_n)$, and $(x_{n_i})$ can't possibly have a further subsequence convergent to $x$ since for every $i \in \mathbb{N}$, we have that $|d(x_{n_i}, x)| \geq \varepsilon$. $\square$

**5.** Carothers 3.44

**6.** Carothers 3.46 (Solution by Jody Gaines)

: Given two metric spaces $(M, d)$ and $(N, \rho)$, we can define a metric on the product $M \times N$ in a variety of ways. Our only requirement is that a sequence of pairs $(a_n, x_n)$ in $M \times N$ should converge precisely when both coordinate sequences $(a_n)$ and $(x_n)$ converges (in $(M, d)$ and $(N, \rho)$, respectively). Show that each of the following define metrics on $M \times N$ that enjoy this property and that all three are equivalent:

$$\begin{aligned} d_1((a,x),(b,y)) &= d(a,b) + \rho(x,y), \\ d_2((a,x),(b,y)) &= (d(a,b)^2 + \rho(x,y)^2)^{1/2}, \\ d_\infty((a,x),(b,y)) &= \max\{d(a,b), \rho(x,y)\}. \end{aligned}$$

**Solution:**
Let $(a_n)$ and $(x_n)$ be sequences in $M$ and $N$ respectively such that $a_n \to a$ and $x_n \to x$ for some $a \in M$ and $x \in N$. Since $d(a_n, a) \to 0$ and $d(x_n, x) \to 0$ then

$$\begin{aligned} d_1((a_n,x_n),(a,x)) &= d(a_n,a) + \rho(x_n,x) \to 0 + 0 = 0, \\ d_2((a_n,x_n),(a,x)) &= (d(a_n,a)^2 + \rho(x_n,x)^2)^{1/2} \to (0^2 + 0^2)^{1/2} = 0, \\ d_\infty((a_n,x_n),(a,x)) &= \max\{d(a_n,a), \rho(x_n,x)\} \leq d(a_n,a) + \rho(x_n,x) \to 0 + 0 = 0. \end{aligned}$$

Suppose $(a_n, x_n) \to (a, x)$ in $d_1$. Then

$$\begin{aligned} d_2((a_n,x_n),(a,x)) &= (d(a_n,a)^2 + \rho(x_n,x)^2)^{1/2} \\ &\leq (d(a_n,a)^2 + 2d(a_n,a)\rho(x_n,x) + \rho(x_n,x)^2)^{1/2} \\ &= d(a_n,a) + \rho(x_n,x) \to 0. \end{aligned}$$

Likewise if $(a_n, x_n) \to (a, x)$ in $d_2$ then

$$d_\infty((a_n,x_n),(a,x)) = \max\{d(a_n,a), \rho(x_n,x)\} \leq (d(a_n,a)^2 + \rho(x_n,x)^2)^{1/2} \to 0.$$

Now suppose $(a_n, x_n) \to (a, x)$ in $d_\infty$. Then

$$d_1((a_n,x_n),(a,x)) = d(a_n,x_n) + \rho(a,x) \leq 2\max\{d(a_n,a), \rho(x_n,x)\} \to 0.$$

Hence $d_1, d_2, d_\infty$ are equivalent metrics.

**7.** Carother 4.3

**8.** Carothers 4.11 (Solution by Mason Brewer)

**Solution:**
Define $E = \{e^{(k)} \mid k \in \mathbb{N}\}$, and let $(x_n)$ be a sequence in $E$ that converges to some $x$, and so the sequence has the form $(x_n) = (e^{(k_n)})$. Since $(x_n)$ is convergent, then the sequence is Cauchy. Now by convergence, and then there exists some $N \in \mathbb{N}$ such that for $n, m \geq N$, we have that $d(x_n, x_m) < 1$, or $d(e^{(k_n)}, e^{(k_m)}) < 1$. But note that, since we are using the $l_1$ norm, our distance function is

$$d(e^{(k_n)}, e^{(k_m)}) = \|e^{(k_n)} - e^{(k_m)}\|_1 = \sum_{i=1}^{\infty} \left| e_i^{(k_n)} - e_i^{(k_m)} \right| = \sum_{i=1}^{\infty} |\delta_{i,k_n} - \delta_{i,k_m}|,$$

Thus, if $k_n \neq k_m$, we get that $d(e^{(k_n)}, e^{(k_m)}) = 2$, and if $k_n = k_m$, we get that $d(e^{(k_n)}, e^{(k_m)}) = 0$. For $n, m \geq N$, we have that $d(e^{(k_n)}, e^{(k_m)}) < 1$, and so it must be that $k_n = k_m$, and so call $e^{(k_n)} = e^{(k_m)} = e^k$. Then for all $\varepsilon > 0$, we have that $n \geq N$ implies $d(e^{(k_n)}, e^k) = 0 < \varepsilon$, and so our sequence converges to $e^k \in E$. Thus $E$ is closed.

**9.** Carothers 4.19 (Solution by Sakti Anupindi)

Show that $diam(A) = diam(\overline{A})$
**Solution:**
Let $p, q \in \overline{A}$. Pick $\epsilon > 0$. Then there exist $x, y \in A$ such that $x \in B_\epsilon(p)$ and $y \in B_\epsilon(q)$. But then

$$d(p,q) \leq d(p,x) + d(x,y) + d(y,q) \leq diam(A) + 2\epsilon.$$

Since $\epsilon$ is arbitrary, this implies that

$$diam(\overline{A}) \leq diam(A). \tag{1}$$

Also note that $A \subset \overline{A}$, so

$$diam(A) \leq diam(\overline{A}). \tag{2}$$

It follows from (3) and (4) that $diam(A) = diam(\overline{A})$.

**10.** Carothers 4.14 (Solution by Lander Ver Hoef)

Show that the set $A = \{x \in l_2 : |x_n| \leq 1/n, n = 1, 2, \dots\}$ is a closed set in $l_2$ but that $B = \{x \in l_2 : |x_n| < 1/n, n = 1, 2, \dots\}$ is not an open set. [Hint: Does $B \supset B_\varepsilon(0)$?]

**Solution:**
First, we will show that $A$ is closed. Let $(X_n)$ be a convergent sequence in $A$, and let $a$ be some element of $l_2$ that is not in $A$. Then there exists some $N$ such that $|a_N| > 1/N$, and hence $|a_N| - 1/N = \varepsilon$ for some $\varepsilon > 0$. But then for any $x$, a term in $(X_n)$,

$$\|x - a\|_2 = \left( \sum_{i=1}^{\infty} |x_n - a_n|^2 \right)^{1/2} \geq |a_N - x_N| \geq |a_N| - |x_N| \geq |a_N| - \frac{1}{N} = \varepsilon.$$

However, this implies that $(X_n)$ cannot converge to $a$, so $(X_n)$ must converge to an element of $A$, and $A$ is closed.

Next, we will show that $B$ is not open. Observe that $0 \in B$, so if $B$ were open, $B_\varepsilon(0) \subseteq B$ for sufficiently small $\varepsilon$. Let $\varepsilon > 0$. Then, for some $N \in \mathbb{N}$ such that $1/N < \varepsilon$, let $x = \{x_n : 1/n \text{ if } n = N, \text{ and } 0 \text{ otherwise}\}$. Clearly, $x \in B_\varepsilon(0)$, but $x \notin B$. Thus, $B$ is not open.

**11.** **[Carothers 5.17]**(Solution by Max Heldman)

Let $f, g : (M, d) \to (N, \rho)$ be continuous, and let $D$ be a dense subset of $M$. If $f(x) = g(x)$ for all $x \in D$, then $f(x) = g(x)$ for all $x \in M$. If $f$ is onto, then $f(D)$ is dense in $N$.

**Solution:**
Let $x \in M$. Then $x \in \overline{D}$, so there exists a sequence $(x_n) \in D$ converging to $x$. Then since $f$ and $g$ are continuous, $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. But each $x_n \in D$, so $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$. Since limits are unique, this implies $f(x) = g(x)$.

Now suppose $f$ is onto, and let $y \in N$. Then there exists $x \in M$ such that $f(x) = y$. Moreover, there exists a sequence $(x_n)$ in $D$ converging to $x$. Since $f$ is continuous, $f(x_n)$ is a sequence in $f(D)$ converging to $f(x) = y$. Thus $y \in \overline{f(D)}$, and $f(D)$ is dense.

**12.** Carothers 5.24 (Solution by former 641 student Dixon Jones)
Let $V$ be a normed vector space. If $y \in V$ is fixed, show that the maps $\alpha \mapsto \alpha\ y$ from $\mathbb{R}$ into $V$, and $x \mapsto x + y$, from $V$ into $V$, are continuous.

**Solution:**
Define $f : \mathbb{R} \to V$ by $f(\alpha) = \alpha y$, where $y \in V$ is fixed. Given any $\beta \in \mathbb{R}$ and $\epsilon > 0$, we must exhibit a $\delta > 0$ such that $\|f(\beta) - f(\alpha)\| < \epsilon$ whenever $|\beta - \alpha| < \delta$. We have

$$\begin{aligned} \|f(\beta) - f(\alpha)\| &= \|\beta y - \alpha y\| \\ &= \|(\beta - \alpha)y\| \\ &= |\beta - \alpha|\ \|y\|\ . \end{aligned}$$

The last step above follows from the scalar property of the norm [Carothers, p. 40, (iii)]. Taking $\delta = \epsilon/\|y\|$, if $|\beta - \alpha| < \delta$ we have

$$\|f(\beta) - f(\alpha)\| < \frac{\delta}{\|y\|}\|y\| = \epsilon\ ,$$

and $f$ is continuous. (Indeed, $f$ is uniformly continuous, because $\delta$ depends only on $\epsilon$ and the fixed $y$.)

Similarly, define $\phi : V \to V$ by $\phi(x) = x + y$, where $y \in V$ is fixed. Given any $z \in V$ and $\epsilon > 0$, we have

$$\|\phi(z) - \phi(x)\| = \|z + y - (x + y)\| = \|z - x\|\ .$$

Set $\delta = \epsilon$; then clearly $\|z - x\| < \delta$ implies $\|\phi(z) - \phi(x)\| < \epsilon$. We conclude that $\phi$ is (uniformly) continuous. □

**13.** Carothers 5.25 (Solution by Jody Gaines)

: A function $f : (M, d) \to (N, \rho)$ is called Lipschitz if there is a constant $K < \infty$ such that $\rho(f(x), f(y)) \leq Kd(x, y)$ for all $x, y \in M$. Prove that a Lipschitz mapping is continuous.

**Solution:**
Note that $K \geq 0$ since $\rho(f(x), f(y)) \geq 0$ for all $x, y \in M$. Moreover if $K = 0$ then $\rho(f(x), f(y)) \leq Kd(x, y) = 0 < r$ for all $r > 0$ and $x, y \in M$. So suppose $K > 0$. Let $\epsilon > 0$ and $\delta = \epsilon/K$. Suppose $(x_n) \to x$ for some $x \in M$. Then there exist $N \in \mathbb{N}$ such that if $n \geq N$ then $x_n \in B_\delta(x)$. Hence $n \geq N$ implies

$$\rho(f(x_n), f(x)) \leq Kd(x_n, x) < K\delta = \epsilon.$$

Since $f(x_n) \to f(x)$ then $f$ is continuous.