Mean Value Theorem. If f is a continuous function on an interval $[a,b]$ that has a derivative at every point in (a,b), then there is a point c in (a,b) where

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

1. Suppose f is a continuous function on $[a,b]$ that has a derivative at every point of (a,b). Suppose also that $f(b) \le f(a)$. What can you conclude from the Mean Value Theorem?

Since $\frac{f(b)-f(a)}{b-a} \le 0$ there is a point $c \in (a,b)$

where $f'(c) = \frac{f(b)-f(a)}{b-a} \le 0$.

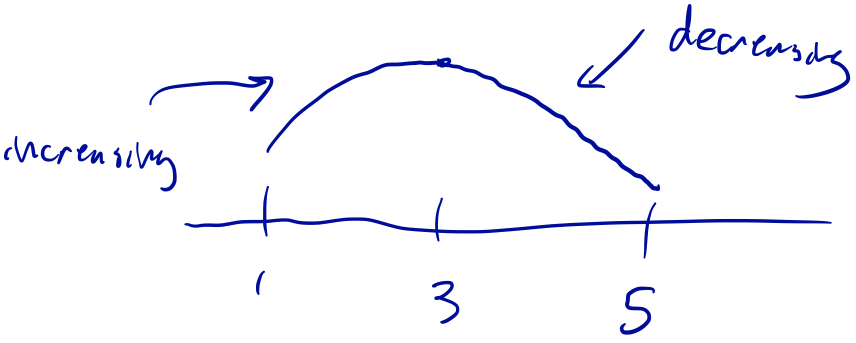
2. Suppose f is a continuous function on $[a,b]$ that has a derivative at every point of (a,b), and that $f'(x) > 0$ for each x in (a,b). Thinking about your answer to problem 1, what can you conclude about $f(a)$ and $f(b)$?

Since $f'(x) > 0$ always, it can't be that $f(b) \le f(a)$

for otherwise by problem 1 $f'(c) \le 0$ at some $c \in (a,b)$.

Thus $f(b) > f(a)$.

3. A function is said to be **increasing** on an interval (a,b) if whenever x and z are in the interval and $x < z$, then $f(x) < f(z)$. It is **decreasing** if whenever x and z are in the interval and $x < z$, then $f(x) > f(z)$ Sketch an example of a function that is increasing on $(1,3)$ and decreasing on $(3,5)$.



Increasing/Decreasing Test

Your answer to problem 2 implies the first item below; the second is justified by a similar argument.

- If $f'(x) > 0$ on an interval (a, b) then f is increasing on the interval.
- If $f'(x) < 0$ on an interval (a, b) then f is decreasing on the interval.

4. Use the increasing/decreasing test to find intervals where

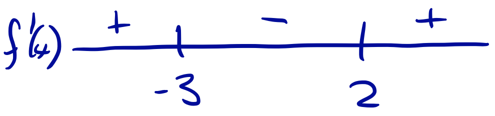
$$f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$$

is increasing and intervals where it is decreasing.

$f'(x) = 2x^2 + 2x - 12$

$= 2(x^2 + x - 6)$

$= 2(x+3)(x-2)$



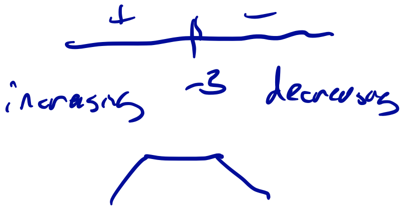
increasing: $(-\infty, -3) \cup (3, \infty)$

decreasing: $(-3, 2)$

5. Find the critical points of the function $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$ from the previous problem. There should be two, c_1 and c_2 with $c_1 < c_2$. Just pay attention to c_1.

1. Just to the left of c_1 is the function increasing or decreasing?
2. Just to the right of c_1 is the function increasing or decreasing?
3. Now decide intuitively, based on these two observations, if f has a local min, local max, or neither at c_1.

$c_1 = -3$

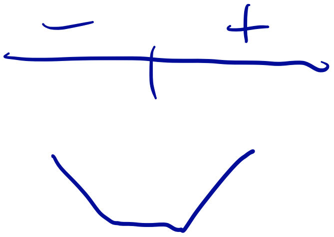


1) increasing

2) decreasing

3) local max

6. Repeat the previous exercise for the other critical point c_2.



1) decreasing

2) increasing

3) local min

You have just sketched the argument that justifies the following:

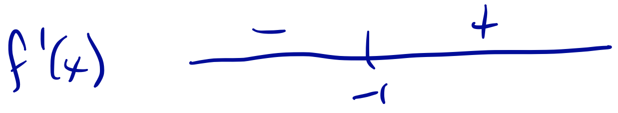
First Derivative Test
Suppose f is a function with a derivative on (a, b), and if c is a point in the interval with $f'(c) = 0$.

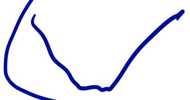
- If $f'(x) > 0$ for x just to the left of c and $f'(x) < 0$ for x just to the right of c, then f has a local max at c.
- If $f'(x) < 0$ for x just to the left of c and $f'(x) > 0$ for x just to the right of c, then f has a local min at c.

7. The function $f(x) = xe^x$ has exactly one critical point. Find it, and then use the First Derivative Test to determine if a local minimum or local maximum occurs there.

$$f'(x) = (1+x)\underbrace{e^x}_{>0 \text{ always!}}$$

c.p.: $x = -1$





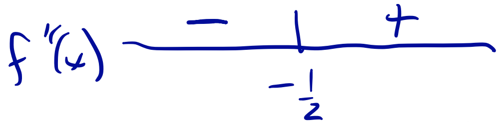
local min

8. Consider the function $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$. Find intervals such that the **derivative** of $f(x)$ is increasing or decreasing.

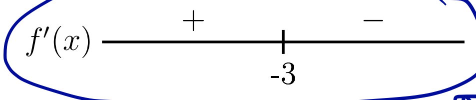
$$f'(x) = 2(x^2 + x - 6)$$

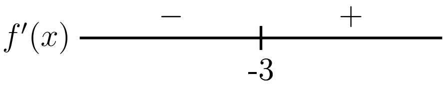
$$f''(x) = 2(2x+1)$$

f' is decreasing on $(-\infty, -\frac{1}{2})$ and increasing on $(-\frac{1}{2}, \infty)$



9. Earlier you computed that $f'(-3) = 0$. Is f' increasing near $x = -3$ or decreasing near $x = -3$? Which of the following two scenarios must we have?





Since -3 is in $(-\infty, -1/2)$, f' is decreasing at $x = -3$. So this must be the scenario.

You have just sketched out justification for the following.

Second Derivative Test
Suppose f is a function with a continuous second derivative on (a, b), and that c is a point in the interval with $f'(c) = 0$.

- If $f''(c) > 0$ then f has a local min at c.
- If $f''(c) < 0$ then f has a local max at c.

10. Use the Second Derivative Test to determine if $f(x) = xe^x$ has a local min/max at its only critical point.

$f'(x) = (1+x)e^x$ c.p.: $x = -1$

$f''(x) = (2+x)e^x$ $f''(-1) = e^{-1} > 0$

so a local min at $x = -1$.

11. Consider the function $f(x) = x^3$. Verify that $f'(0) = 0$. Then decide what the Second Derivative Test has to say, if anything, about whether a local min/max occurs at $x = 0$.

$f'(x) = 3x^2 \rightarrow f'(0) = 3 \cdot 0^2 = 0$

$f''(x) = 6x \rightarrow f''(0) = 0$, so 2nd derivative test is silent.

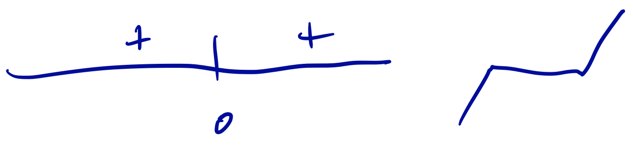
No conclusion.

First Derivative Test (Final Case)

- If $f'(c) = 0$ and $f'(x) < 0$ on both sides of c or $f'(x) > 0$ on both sides of c, then there is neither a local min nor a local max at c.

12. Decide what the First Derivative Test has to say, if anything, about whether a local min/max occurs at $x = 0$ for $f(x) = x^3$.

$f'(x)$



⟵ neither a local min nor a local max