1. Carothers 5.28 (Solution by former 641 student Lyman Gilispie)
Define $g : \ell_2 \to \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} x_n/n$. Is g continuous?

Solution:
We first show that g is well defined. Pick $x \in \ell_2$. By the Cauchy-Schrartz inequality

$$\sum_{k=1}^{\infty} \left| \frac{x_k}{k} \right| \leq \|x\|_2 \|(1/k)\|_2,$$

so the series $g(x) = \sum_{n=1}^{\infty} \frac{x_k}{k}$ is absolutely convergent, and therefore convergent.

Now, pick $x, y \in \ell_2$. Because the series $g(x)$ and $g(y)$ both converge and by the Cauchy-Schwartz inequality, we know that

$$\begin{aligned} \left| \sum_{k=1}^{\infty} \frac{x_k}{k} + \sum_{k=1}^{\infty} \frac{y_k}{k} \right| &= \left| \sum_{k=1}^{\infty} \frac{x_n - y_n}{k} \right| \\ &\leq \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{k} \\ &\leq \|x - y\|_2 \|(1/k)\|_2. \end{aligned}$$

This establishes that g is Lipschitz, and therefore continuous.

Lemma 1: Suppose A is totally bounded and $B \subseteq A$. Show that B is totally bounded. (Solution by Jody)

Solution:
Let $\epsilon > 0$. Since A is totally bounded then there exist an $\epsilon/2$-net for A, namely $\{x_1, \ldots, x_n\}$. Let C be the set consisting of all the elements x_k in $\{x_1, \ldots, x_n\}$, such that there exists some $b \in B$ in $B_{\epsilon/2}(x_k)$. Since $\{x_1, .., x_n\}$ is an $\epsilon/2$-net for A, and $B \subseteq A$, then $B \subseteq \cup_{u \in C} B_{\epsilon/2}(u)$. For each $u \in C$ pick $b_u \in B_{\epsilon/2}(x_k)$ such that $b_u \in B$. Let $b \in B$. Since $B \subseteq \cup_{u \in C} B_{\epsilon/2}(u)$ then $b \in B_{\epsilon/2}(u)$ for some $u \in C$. So

$$d(b, b_u) \leq d(b, u) + d(b_k, u) < \epsilon.$$

Hence $B \subseteq \cup_{u \in C} B_{\epsilon}(b_u)$. Since $\{x_1, .., x_n\}$ is finite then C is finite. Hence B is totally bounded.

2. Carothers 5.32 (Solution by Max)

Solution:
Suppose f is lower semicontinuous, and let $(x_n) \to x$ in M. If $\liminf_{n\to\infty} f(x_n) = \infty$, then it is always true that $f(x) \leq \liminf_{n\to\infty} f(x_n)$, so we assume $m = \liminf_{n\to\infty} f(x_n) < \infty$. Since f is lower semicontinuous, the set $A_\epsilon = \{t \in M : f(t) \leq m + \epsilon\}$ is closed for each $\epsilon > 0$. Moreover, this set always contains infinitely many terms of the sequence (x_n) – if, to the contrary, $A_\epsilon \cap \{x_n\}_{n=1}^{\infty}$ were finite, we could find $N = \max\{N \in \mathbb{N} : f(x_N) \leq m + \epsilon\}$, so that $f(x_n) > m + \epsilon$ for all $n \geq N$, and hence the contradiction

$$\liminf f(x_n) = \sup \inf_{n \geq K} f(x_n) \geq \inf_{n \geq N} f(x_n) \geq m + \epsilon > m.$$

Thus, for each $\epsilon > 0$, A_ϵ contains a tail of (x_n). Since each A_ϵ is closed, this implies that $\lim x_n = x \in \bigcap_{\epsilon>0} A_\epsilon = \{t \in M : f(t) \leq m\}$. Thus $f(x) \leq m$.

Now suppose $f(x) \leq \liminf_{n\to\infty} f(x_n)$ whenever $(x_n) \to x$ in M. Let $\alpha \in \mathbb{R}$, and let $A_\alpha = \{x \in M : f(x) \leq \alpha\}$. Since the empty set is closed, if A_α is empty we are done. Otherwise, let (x_n) be a sequence in A_α converging to some $x \in M$. Then $f(x_n) \leq \alpha$ for each $n \in \mathbb{N}$, which implies $f(x) \leq \liminf_{n\to\infty} f(x_n) \leq \alpha$. Thus $x \in A_\alpha$, and A_α is closed.

3. Carothers 7.5 (Solution by Jody)

Solution:
Note if $\overline{A}$ is totally bounded then by Lemma 1 A is totally bounded as well.

Suppose A is totally bounded. Let $\epsilon > 0$. Then there exist $x_1, \ldots, x_N \in A$ such that $A \subseteq \cup_{n=1}^{N} B_{\epsilon/2}(x_n)$. Let $a \in \overline{A}$. Then there exists (a_n) in A such that $a_n \to a$. So for all $n \in \mathbb{N}$ we have that $a_n \in \cup_{k=1}^{N} B_{\epsilon/2}(x_k)$. Moreover there exists $M \in \mathbb{N}$ such that $n \geq M$ implies $a_n \in B_{\epsilon/2}(a)$. Thus $a_M \in B_{\epsilon/2}(x_k) \cap B_{\epsilon/2}(a)$ for some $1 \leq k \leq N$. So

$$d(x_k, a) \leq d(a_M, a) + d(a_M, x_k) < \epsilon,$$

which implies $\overline{A} \subseteq \cup_{k=1}^{N} B_\epsilon(x_k)$. Since $x_1, \ldots, x_N \in A$ then $x_1, \ldots, x_N \in \overline{A}$. Hence $\overline{A} \subseteq \cup_{k=1}^{N} B_\epsilon(x_k)$ implies $\overline{A}$ is totally bounded.

4. Carothers 7.10 (Solution by Mason)

Solution:
For each $k \in \mathbb{N}$, let $\varepsilon_k = 1/k$, and then by total boundedness, there exists an ε_k-net, called D_k, for M. Note that each D_k is finite, and thus

$$D = \bigcup_{k=1}^{\infty} D_k,$$

is countable since it is a countable union of countable sets. Now I claim D is dense in M. Let $x \in M$. Construct the sequence (x_n) in the following way: for each n, pick $x_n \in D$ so that $x \in B_{\varepsilon_n}(x_n)$. Each x_n is guaranteed to exist by the fact that the ε_n-net D_n is a subset of D. This means that $d(x, x_n) < \varepsilon_n = 1/n$. Now let $\varepsilon > 0$. Note that there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $1/N < \varepsilon$. Thus for $n \geq N$, we have that $d(x, x_n) < 1/N < \varepsilon$. Thus (x_n) converges to x. □

5. Carothers 7.12 (Solution by Lander Ver Hoef)

Let A be a subset of an arbitrary metric space (M, d). If (A, d) is complete, show that A is closed in M.

Solution:
Suppose (x_n) is a sequence in A that converges to some x in M. Then (x_n) is Cauchy in M, and because A uses the inherited metric, (x_n) is also Cauchy in A. Because A is complete, (x_n) converges to some $a \in A$, and by the uniqueness of limits $x = a$, and hence $x \in A$ and A is closed.

6. Carothers 7.15 (Solution by Lander Ver Hoef)

If M is complete and $f : (M, d) \to (N, \rho)$ is continuous, then $f(M)$ need not be complete.

Solution:
Let $f(x) = e^{-(x^2)}$. f is continuous from $\mathbb{R}$ onto $(0, 1]$, and $\mathbb{R}$ is complete, but $(0, 1]$ is not complete, because (x_n) for $x_n = 1/n$ is Cauchy but not convergent in $(0, 1]$.

7. Carothers 7.18 (Solution by Mason)

Solution:

(i) ℓ_1 is complete.

Proof. Let (x_n) be a Cauchy sequence in ℓ_1. For each $k \in \mathbb{N}$, we know that the sequence $(x_n(k))_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathbb{R}$, we know that $(x_n(k))$ must converge to some $x(k)$. Then define x as the sequence $x = (x(1), x(2), \dots)$. We want to show that $x \in \ell_2$ and that (x_n) converges to x. First, let $K \in \mathbb{N}$, and note that

$$\lim_{m\to\infty} \sum_{k=1}^{K} |x_m(k)| = \sum_{k=1}^{K} |x(k)|.$$

Also, we know that since (x_n) is Cauchy, it must be bounded and thus there exists an $M \in \mathbb{R}$ such that $||x_m||_1 \leq M$ for all $m \in \mathbb{N}$. Since $|x_m(k)| \geq 0$ for all $m, k \in \mathbb{N}$, we know that

$$\sum_{k=1}^{K} |x_m(k)| \leq \sum_{k=1}^{\infty} |x_m(k)| = ||x_m||_1,$$

and thus $\sum_{k=1}^{K} |x_m(k)| \leq M$. Thus we have

$$\limsup_{m\to\infty} \sum_{k=1}^{K} |x_m(k)| \leq \limsup_{m\to\infty} \sum_{k=1}^{\infty} |x_m(k)| \leq M,$$

where the inequality involving M is true because M is an upper bound for $||x_m||_1$, and so is greater than the supremum and thus the limit supremum. But the limit exists, and so $\limsup_{m\to\infty} \sum_{k=1}^{K} |x_m(k)| = \sum_{k=1}^{K} |x(k)|$, meaning that $\sum_{k=1}^{K} |x(k)| \leq M$ for any $K \in \mathbb{N}$. Finally, note that $\sum_{k=1}^{K} |x(k)|$ is a bounded increasing real sequence in $\mathbb{R}$, and thus its limit exists. Since M is an upper bound for the partial sum, it must be that

$$\lim_{k\to\infty} \sum_{k=1}^{K} |x(k)| = \sum_{k=1}^{\infty} |x(k)| \leq M < \infty,$$

and thus $x \in \ell_1$.
Next, we will show that $x_n \to x$ in ℓ_1. Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ satisfy $n, m \geq N$

implies $\|x_n - x_m\|_1 < \varepsilon$. Now let $K \in \mathbb{N}$, then for $n \geq N$ we have that

$$\sum_{k=1}^{K} |x(k) - x_n(k)| = \lim_{m\to\infty} \sum_{k=1}^{K} |x_m(k) - x_n(k)| \leq \limsup_{m\to\infty} \sum_{k=1}^{\infty} |x_m(k) - x_n(k)|$$
$$= \limsup_{m\to\infty} \|x_m - x_n\|_1 \leq \varepsilon,$$

where the last inequality is true simply by the fact that ε is an eventual upper bound. Since this is true for all $K \in \mathbb{N}$, it must be true that

$$\sum_{k=1}^{\infty} |x(k) - x_n(k)| \leq \varepsilon,$$

and thus $x_n \to x$. And thus ℓ_1 is complete. □

(ii) ℓ_∞ is complete.

Proof. Let (x_n) be a Cauchy sequence in ℓ_∞. Thus for $k \in \mathbb{N}$, the real sequence $(x_n(k))_{n=1}^{\infty}$ is Cauchy and thus must converge to some $x(k) \in \mathbb{R}$. Now define $x = (x(k))$. We will first show that $x \in \ell_\infty$. To start, note that (x_n) is bounded by effect of it being Cauchy, and so there exists an $M \in \mathbb{R}$ such that $\|x_m\|_\infty \leq M$ for all $m \in \mathbb{N}$. Now let $k \in \mathbb{N}$, and then we have that

$$|x(k)| = |\lim_{m\to\infty} x_m(k)| = \lim_{m\to\infty} |x_m(k)| \leq M,$$

and so it must be that $(x(k))$ is a bounded sequence, and so $x \in \ell_\infty$.
Next, we show that $x_n \to x$ in ℓ_∞. Let $\varepsilon > 0$, and then we know that there exists an $N \in \mathbb{N}$ such that $n, m \geq N$ implies $\|x_m - x_n\| < \varepsilon$. Then if $k \in \mathbb{N}$, and $n \geq N$, we have that

$$|x(k) - x_n(k)| = \lim_{m\to\infty} |x_m(k) - x_n(k)| \leq \limsup_{m\to\infty} \|x_m - x_n\|_\infty < \varepsilon.$$

Since this is true for all $k \in \mathbb{N}$, it must be that $\|x - x_n\| \leq \varepsilon$ for $n \geq N$. □

8. Carothers 7.19 (Solution by Max)

Solution:
Let (f_n) be a Cauchy sequence in $c_0 \subseteq \ell_\infty$. Then since ℓ_∞ is complete, (f_n) converges to a limit $f \in \ell_\infty$.

To show that $f \in c_0$, let $\epsilon > 0$. Since $(f_n) \to f$, there exists $N_0 \in \mathbb{N}$ such that if $n \geq N_0$, $\|f_n - f\|_\infty < \epsilon/2$. Let $n \geq N_0$. Since $f_n \in c_0$, $\lim_{k\to\infty} f_n(k) = 0$, so there exists $N_1 \in \mathbb{N}$ such that if $k \geq N_1$, $|f_n(k)| < \epsilon/2$. Hence if $k \geq N_1$,

$$|f(k)| = |f(k) - f_n(k) + f_n(k)| \leq |f(k) - f_n(k)| + |f_n(k)| \leq \|f - f_n\|_\infty + |f_n(k)| < \epsilon.$$

9. Carothers 7.22 (Solution by former 641 student TJ Barry)
Let D be a dense subset of a metric space M and suppose that every Cauchy sequence from D converges to some point of M. Prove that M is complete.

Solution:
Let (x_n) be a Cauchy sequence in M. Since D is dense in M, for each point $x_n \in M$, and for any $\varepsilon > 0$, $B_\varepsilon(x_n) \cap D \neq \varnothing$.

Thus, we can pick $y_1 \in D$ such that $d(x_1, y_1) < 1$.

We can pick $y_2 \in D$ such that $d(x_2, y_2) < 1/2$.

Continuing inductively, for each $n \geq 1$, there exists $y_n \in D$ such that $d(x_n, y_n) < 1/n$. Thus, we have a sequence $(y_n) \subset D$, and we claim that this sequence is Cauchy.

Let $\varepsilon > 0$. Since (x_n) is Cauchy, let $N_1 \in \mathbb{N}$ be sufficiently large that if $n, m \geq N_1$, then $d(x_n, x_m) < \varepsilon$. Let $N_2 \in \mathbb{N}$ be sufficiently large such that $1/N_2 < \varepsilon$.

Then, for $n, m \geq \max\{N_1, N_2\}$, note that

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) \\ &< \frac{1}{n} + \varepsilon + \frac{1}{m} \\ &< 3\varepsilon \end{aligned}$$

and hence (y_n) is Cauchy, as claimed. By the hypothesis, this sequence converges to some point x of M.

We claim that (x_n) converges to x. Let $\varepsilon > 0$. Since $y_n \to x$, let $N_1 \in \mathbb{N}$ be sufficiently large that $d(y_n, x) < \varepsilon$. Let $N_2 \in \mathbb{N}$ be sufficiently large that $1/N_2 < \varepsilon$. Then, for $n \geq \max\{N_1, N_2\}$,

$$d(x_n, x) \leq d(x_n, y_n) + d(y_n, x) < \frac{1}{n} + \varepsilon < 2\varepsilon.$$

Hence, $d(x_n, x) \to 0$, and we conclude that $x_n \to x$.

Since Cauchy sequences in M converge to a point in M, we conclude that M is complete.

10. Carothers 7.25 (Solution by Jody)

Solution:
The statement is false. Consider the map $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \ln(x)$. Since $x > 0$ for all $x \in (0, 1]$ then f is continuous. Moreover, the sequence $x_n = 1/n$ in $(0, 1]$ is Cauchy since for each $\epsilon > 0$, if $N \in \mathbb{N}$ and $N > 1/\epsilon$, then $n, m \geq N$ implies $|1/n - 1/m| < \epsilon$. However

$$f(x_n) = \ln(1/n) \to -\infty.$$

Since $\mathbb{R}$ is complete, then $(f(x_n))$ not converging implies $(f(x_n))$ is not Cauchy.

If f is strictly increasing then the statement is still false. We just showed that for $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \ln(x)$, the sequence $(f(x_n))$ is not Cauchy when $x_n = 1/n$. Moreover $f(x) = \ln(x)$ is strictly increasing. Hence the statement is still false in the case that f is strictly increasing.

Suppose f is Lipschitz. Then there exists $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in A$. Note if $K = 0$ then for all $n, m \in \mathbb{N}$ we have that $|f(x_n) - f(x_m)| \leq 0|x - y| = 0 < \epsilon$.

So suppose $K > 0$. Since (x_n) is Cauchy then there exists $N \in \mathbb{N}$ such that if $n, m \geq N$ then $|x_n - x_n| < \epsilon/K$. Hence $n, m \geq N$ implies

$$|f(x_n) - f(x_m)| \leq K|x_n - x_m| < \epsilon.$$