1. [David] In this problem, we will seek a solution to the initial value problem

$$f'(t) = F(t, f(t))$$
$$f(0) = a$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}$.

To obtain the existence result, we need to assume that F is sufficiently nice; we will assume that F is continuous, and moreover that there exists a constant K such that

$$|F(x, y_1) - F(x, y_2)| \le K|y_1 - y_2|$$

for all $x, y_1, y_2 \in \mathbb{R}$.

Define $G : C[-T, T] \to C[-T, T]$ by

$$G(f)(t) = a + \int_0^t F(s, f(s))\, ds.$$

a) Explain why $G(f) \in C[-T, T]$ if $f \in C[-T, T]$.

b) Show that if f solves the initial value problem for $t \in [-T, T]$, then $G(f) = f$.

c) Show that G is Lipschitz with Lipschitz constant TK.

d) Assuming $T < 1/K$, show that there exists a solution of $G(f) = f$ defined for $t \in [-T, T]$. You may use the fact that $C[-T, T]$ is complete; we'll show this later.

e) Assuming $T < 1/K$, show that there exists a unique solution of the initial value problem defined on $(-T, T)$.

f) Extra credit: Show that there exists a solution f of the initial value problem defined for all $t \in \mathbb{R}$.

2. Carothers 8.66 [Max]

3. Carothers 8.76 [Sakti]

4. Carothers 8.77 [Jody]

5. Carothers 8.78 [Lander]

6. Carothers 8.80 [Mason]

7. Carothers 8.81 [Max]

8. Carothers 8.84 [Lander]

9. Carothers 10.7 [Sakti]

10. Carothers 10.9 (No rigor, please!) [Mason]

11. Carothers 10.10 [Jody]