1. Suppose $f \in L^1$ is uniformly continuous. Show that $\lim_{x\to\infty} f(x) = 0$.

2. Carothers 10.20

3. Carothers 8.79

4. Let

$$X_K = \left\{ f \in C([0,1]) : f \text{ is Lipschitz with constant } K \text{ and } \int_0^1 |f| \le 1 \right\}.$$

Show that X_K is compact in $C([0,1])$. Is X_K also compact in $L_1([0,1])$?

5. Let $\{f_n\}$ be a sequence of measurable real-valued functions. Let $E = \{x : (f_n(x)) \text{ converges}\}$. Show that E is measurable.

6. (Riemann integrable functions are continuous almost everywhere.)

a) Let (ψ_n) be an increasing sequence of step functions with $|\psi_n| \le M$ for some M. Show that $\lim \psi_n$ is continuous almost everywhere.

b) Show that Riemann integrable functions are continuous almost everywhere. Hint: Find functions g and G with $g \le f \le G$ where $G = g$ almost everywhere and where g and G are continuous almost everywhere.

7. (The approximate with wild abandon problem.)

Suppose $f \in L^1[a,b]$ and $\int_a^b fg = 0$ for every polynomial g. Show that $f = 0$ almost everywhere.

Hint: : First show that $\int_I f = 0$ for every interval in $[a,b]$. Then show that $\int_E f = 0$ for every measurable set in $[a,b]$. You might find Exercise 18.35 (the "even more is true" part) to be handy, as well.

8. Compute $\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \cos(x/n)\, dx$.

9. Assuming that Fatou's Lemma is true, prove the Monotone Convergence Theorem. (In some approaches to integration, Fatou's Lemma is proved first, in which case the MCT is a corollary.)

10. Consider the series $\sum_{k=1}^\infty a_k \sin(kx)$ on the domain $[0,2\pi]$. Suppose that $\sum_{k=1}^\infty (a_k)^2$ converges. Prove that the series converges in $L^2([0,2\pi])$. Compare this result with the first problem of the midterm.

11. A sequence (f_n) is Cauchy in measure if for every $\epsilon > 0$ there is an index N such that if $n, m \ge M$ then $m(\{|f_n - f_m| > \epsilon\}) < \epsilon$.

Show that if (f_n) is Cauchy in measure and has a subsequence that is convergent in measure, then the full sequence is Cauchy in measure.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- You may not discuss the exam with anyone else until after the due date/time.
- You are permitted to reference Carothers but no other text, nor may you consult the internet.
- Each problem is weighted equally.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- The due date/time is absolutely firm.
- We will hold a hint session during finals week, TBA.