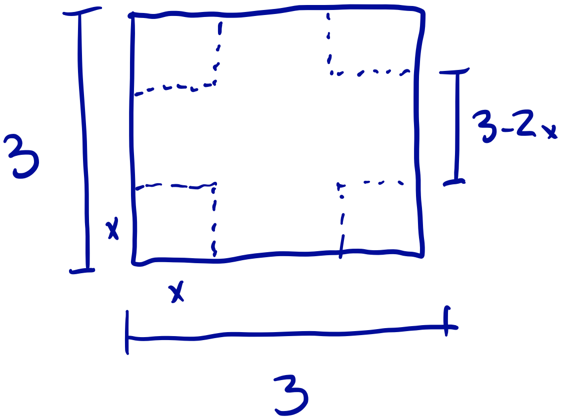
1. A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.



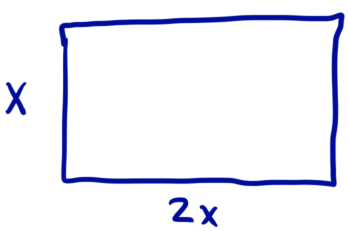
$$V = (3-2x)^2 x$$

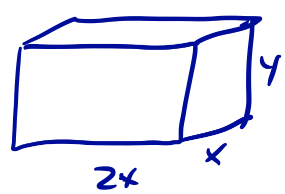
$$V' = -4(3-2x)x + (3-2x)^2$$
$$= (3-2x)(-4x+3-2x)$$
$$= (3-2x)(3-6x)$$
$$= 3(3-2x)(1-2x)$$

$V' = 0$ at $x = \frac{3}{2}, \frac{1}{2}$

Check: $x = 0, \frac{1}{2}, \frac{3}{2}$ $\quad V(0) = 0$ $\quad V(3/2) = 0$ $\quad V(1/2) = \frac{4}{2} = 2 \text{ ft}^2$

2. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of the base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$ 6 per square meter. Find the costs of materials for the cheapest such container.





$$V = 2x^2 y$$

$$y = V/2x^2 = \frac{5}{x^2}$$

Cost: $C = 10 \cdot 2x^2 + 6(2xy + 4xy)$

$$= 20x^2 + 36 \cdot \frac{5x}{x^2}, \quad x > 0$$
$$= 20x^2 + \frac{180}{x}$$

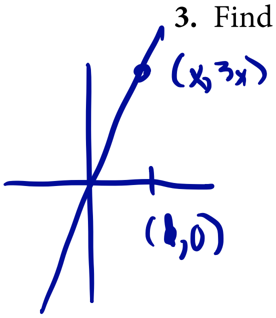
$$C' = 40x - \frac{180}{x^2}$$
$$= 20x\left[2 - \frac{9}{x^3}\right]$$

$C' = 0$ at $x = \left(\frac{9}{2}\right)^{1/3}$

$$C'' = 40 + \frac{360}{x^3} > 0$$

global min here

3. Find the point on the line $y = 3x$ that is closest to the point $(1, 0)$.



$$D = (x-1)^2 + 9x^2$$

$$D' = 2(x-1) + 18x$$

$$= 20x - 2$$

$$D'' = 20 \rightarrow \text{conc up everywhere}$$

crit. pt. is absolute min.

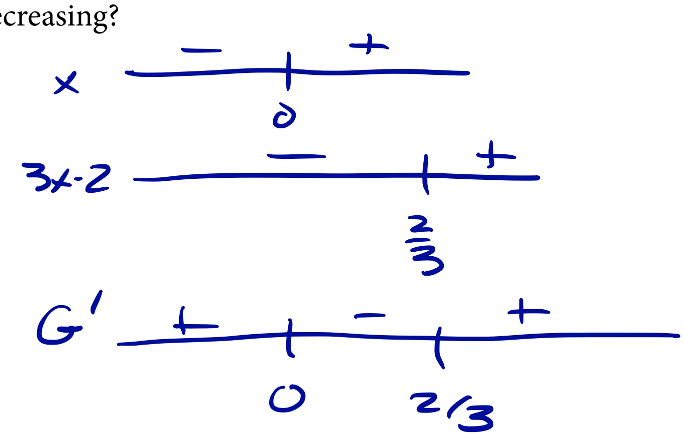
$$D' = 0 \text{ at } x = \frac{1}{10}, \; y = \frac{3}{10}$$

4. Consider the function $G(x) = x^3 - x^2$.

a. On what intervals is G increasing or decreasing?

$$G'(x) = 3x^2 - 2x$$

$$= x(3x-2)$$



increasing: $(-\infty, 0)$ and $(\frac{2}{3}, \infty)$

dec: $(0, 2/3)$

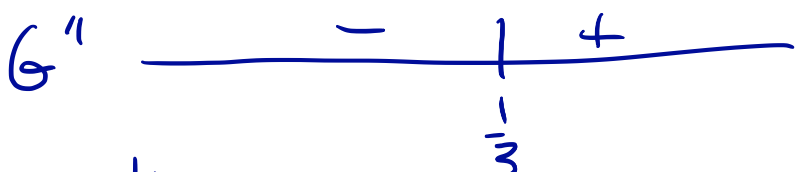
b. Find the locations of any local maximum and minimum values of G.

$x = 0$: local max here

$x = \frac{2}{3}$: local min here

c. Find the intervals of concavity and the inflection points.

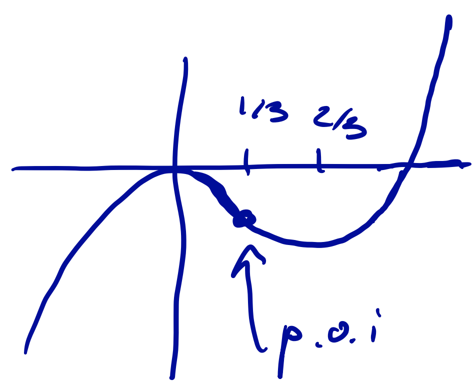
$$G''(x) = 6x - 2 = 3(3x-1)$$



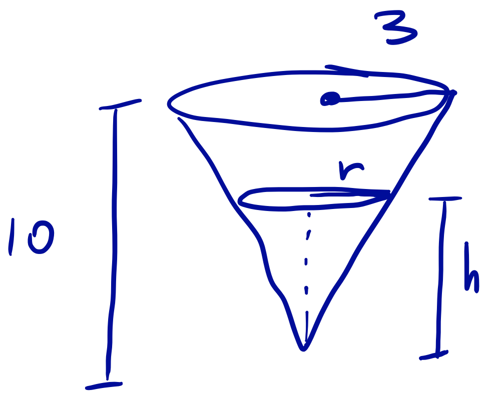
p.o.i: $x = 1/3$

conc up: $(1/3, \infty)$

conc down: $(-\infty, 1/3)$



5. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of $2\,\mathrm{cm}^3/\mathrm{sec}$, how fast is the water level rising when the water is 5 cm deep?



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{h}{r} = \frac{10}{3}$$

$$r = \frac{3h}{10}$$

$$V = \frac{1}{3}\pi \frac{9h^3}{100} = \frac{3\pi}{100}h^3$$

$$\frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt}$$

$$2 = \frac{9\pi}{100}\cdot 25\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{8}{9\pi}\ \mathrm{cm/sec}$$

6. Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$ and use it to estimate $\sqrt{4.1}$.

$$f(4) = 2$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x-4)$$

$$\sqrt{4.1} = f(4.1) \approx L(4.1) = 2 + \frac{1}{4}\cdot\frac{1}{10} = 2 + \frac{1}{40}$$

$$= 2.025$$

$$(\text{vs } 2.024845\ldots)$$

7. The position of a mass on the x axis is given by $x(t) = t(e^t - 2)$ for $t \geq 0$. Find an equation involving a derivative to solve to determine the time when $x(t)$ is at a minimum. You will not be able to solve the equation by hand, so don't sweat it.

$x'(t) = e^t - 2 + te^t$

$x(t)$ at a min requires $x'(t) = 0$

so $(1+t)e^t - 2 = 0$

8. We can use Newton's method in the previous problem to find an approximate solution.

a. Explain why you expect the minimum to occur somewhere between $t = 0$ and $t = \ln(2) \approx 0.7$.

$x(0) = 0$ and $x(t) < 0$ for $0 < t < \ln(2)$

$x(\ln(2)) = 0$ $x(t) > 0$ for $t > \ln(2)$

b. Apply one round of Newton's method to determine an approximate solution starting with $t = 1/2$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x) = (1+t)e^t - 2$

$f'(x) = e^t + (1+t)e^t$

$= (2+t)e^t$

$$x_1 = \frac{1}{2} - \frac{\frac{3}{2}e^{1/2} - 2}{\frac{5}{2}e^{1/2}}$$