1. Carothers 10.15 [Sakti]

2. **[Carothers 10.18]** (Solution by Max Heldman)

Dini's Theorem. Let X be a compact metric space and suppose that $(f_n) \in C(X)$ increases pointwise to a continuous function $f \in C(X)$. Then $f_n \rightrightarrows f$.

Solution:
Note that (f_n) increases to $f \in C(X)$ uniformly or pointwise if and only if $f - f_n$ decreases to 0 uniformly or pointwise. Since $f - f_n$ is continuous if and only if f is, we need only show that the proposition holds when $(f_n) \in C(X)$ decreases uniformly to the zero function.

Let (f_n) decrease pointwise to the zero function, and let $\epsilon > 0$. For each $n \in \mathbb{N}$, let $U_n = f_n^{-1}((-\epsilon, \epsilon))$. Observe that $U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$, and that each set U_n is open since each f_n is continuous.

Let $x \in X$. Then since $f_n(x) \to 0$, there exists $n \in \mathbb{N}$ so that $f_n(x) < \epsilon$. Hence $x \in U_n$. Thus $\{U_n\}_{n \in \mathbb{N}}$ forms an open cover of X, and since X is compact we can reduce to a finite subcover $\{U_{n_i}\}_{i=1}^k$ with $n_1 < n_2 < \ldots < n_k$. Let $N = n_k$. Observe that $U_N \supseteq U_{n_i}$ for all $i = 1, \ldots, k$, so $X = U_N$. Moreover, since $U_N \subseteq U_n$ for all $n \geq N$, $U_n = X$ for all $n \geq N$. Thus $f_n(x) < \epsilon$ for all $n \geq N$ and all $x \in X$. That is, $f_n \rightrightarrows 0$.

If f is not continuous, the proposition does not hold. Take $X = [0,1]$ and $f_n(x) = 1 - x^n$. The hypotheses of the theorem are satisfied – f_n increases pointwise to the function

$$f(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

for every x in the compact space X – but the convergence is not uniform.

3. Carothers 10.19 [Jody]: Suppose that (f_n) is a sequence of functions in $C[0,1]$ and that $f_n \rightrightarrows f$ on $[0,1]$. True or false? $\int_0^{1-(1/n)} f_n \to \int_0^1 f$.

Solution:
The statement is true. Since $f_n \rightrightarrows f$ and each f_n is in $C[0,1]$ then $f \in C[0,1]$. So f and each f_n is integrable on $[0,1]$. Thus

$$\begin{aligned}
\left| \int_0^{1-1/n} f_n(x)dx - \int_0^1 f(x)dx \right| &= \left| \int_0^{1-1/n} (f_n(x) - f(x))\, dx - \int_{1-1/n}^1 f(x)dx \right| \\
&\leq \int_0^{1-1/n} |f_n(x) - f(x)|dx + \int_{1-1/n}^1 |f(x)|dx \\
&\leq \left(\left[1 - \frac{1}{n}\right] - 0 \right) \sup_{x\in[0,1]} |f_n(x) - f(x)| + \left(1 - \left[1 - \frac{1}{n}\right]\right) \sup_{x\in[0,1]} |f(x)| \\
&= \left(1 - \frac{1}{n}\right) \|f_n - f\|_\infty + \frac{1}{n} \|f\|_\infty.
\end{aligned}$$

Since $f_n \rightrightarrows f$ then $\|f_n - f\|_\infty \to 0$. So $\|f_n - f\|_\infty \to 0$, $(1-1/n) \to 1$, and $1/n \to 0$ gives us that

$$\left(1 - \frac{1}{n}\right) \|f_n - f\|_\infty + \frac{1}{n}\|f\|_\infty \to 1 \cdot 0 + 0 \cdot \|f\|_\infty = 0,$$

which implies $\left|\int_0^{1-1/n} f_n - \int_0^1 f\right| \to 0$. Hence $\int_0^{1-1/n} f_n \to \int_0^1 f$.

4. Carothers 10.25 [Mason]

Solution:
Consider a set $D \subseteq B[0,1]$ that is dense in $B[0,1]$. Now for any $x \in \mathbb{R}$, define the function f_x as $f_x(y) = 0$ for $x \neq y$ and $f_x(y) = 1$ for $x = y$. Thus note that there is a unique f_x for each $x \in \mathbb{R}$ and so there are uncountably many of them. Also note that $||f_x - f_y||_\infty = 1$ whenever $x \neq y$, meaning that each f_x is a distance 1 from every other one. By the density of D, for every $x \in \mathbb{R}$, there must be some $g_x \in D$ such that $||g_x - f_x||_\infty < 1/2$. Then if $x \neq y$, we have that $||g_x - f_y||_\infty = ||g_x - f_x - (f_y - f_x)||_\infty$, and by the reverse triangle inequality, we have that $||g_x - f_x - (f_y - f_x)||_\infty \geq ||g_x - f_x||_\infty - ||f_y - f_x||_\infty > ||g_x - f_x||_\infty + 1 \geq 1 > 1/2$. Since $||g_x - f_y||_\infty > 1/2$ and $||g_y - f_y||_\infty < 1/2$, it must be that $g_y \neq g_x$. This means that there are uncountably many elements in D. Thus $B[0,1]$ is not separable.

5. (Carothers 10.26) (Solution by Lander)

If $\sum_{n=1}^\infty |a_n| < \infty$, prove that $\sum_{n=1}^\infty a_n \sin(nx)$ and $\sum_{n=1}^\infty a_n \cos(nx)$ are uniformly convergent on $\mathbb{R}$.

Solution:
Observe that each $|a_n|$ is an upper bound for $|a_n \sin(nx)|$ or $|a_n \cos(nx)|$, because $|\sin(nx)| \leq 1$ and $|\cos(nx)| \leq 1$ for all $x \in \mathbb{R}$. But then, because $\sum_{n=1}^\infty |a_n| < \infty$, by the Weierstrass M-test, $\sum_{n=1}^\infty a_n \sin(nx)$ and $\sum_{n=1}^\infty a_n \cos(nx)$ are uniformly convergent.

6. Carothers 10.27 [Sakti]

7. (Carothers 10.28) (Solution by Lander)

Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that (f_n) converges uniformly on $\mathbb{Q}$. Show that (f_n) actually converges uniformly on all of $\mathbb{R}$. [Hint: Show that (f_n) is uniformly Cauchy.]

Solution:
Let $\varepsilon > 0$. Then because (f_n) converges uniformly on $\mathbb{Q}$, it is uniformly Cauchy on $\mathbb{Q}$ as well. Thus, there exists an N such that if $n, m \geq N$, then $|f_n(y) - f_m(y)| < \varepsilon$ for all $y \in \mathbb{Q}$. Let $n, m \geq N$, and choose $x \in \mathbb{R}$. Then, by the continuity of f_n and f_m, there exists a $y \in \mathbb{Q}$ such that $|f_n(x) - f_n(y)| < \varepsilon$ and $|f_m(y) - f_m(x)| < \varepsilon$. But then

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f_n(y) + f_n(y) - f_m(y) + f_m(y) - f_m(x)| \\
&\leq |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(y) - f_m(x)| \\
&< 3\varepsilon.
\end{aligned}$$

This implies that (f_n) is uniformly Cauchy on $\mathbb{R}$. Therefore, it is pointwise Cauchy as well, and thus pointwise convergent to some function f on $\mathbb{R}$. Let $\varepsilon > 0$. Then, by the uniform Cauchiness of (f_n), there exists an N such that if $n, m \geq N$, then $d(f_n, f_m) < \varepsilon$. Choose $x \in \mathbb{R}$, and fix $n \geq N$. Then we have that

$$|f_n(x) - f(x)| = \left|f_n(x) - \lim_{m\to\infty} f_m(x)\right| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon,$$

because m is eventually greater than M. Thus, (f_n) converges uniformly to f.

8. Carothers 10.32 [Jody]: (a) If $\sum_{n=1}^{\infty} |a_n| < \infty$, show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[0, \infty)$.

Solution:
Define the map $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(t) = a_n e^{-nt}$. Then

$$\|f_n\|_\infty = \sup_{x\in[0,\infty)} |a_n e^{-nx}| = |a_n|e^{-n(0)} = |a_n| \cdot 1 = |a_n|$$

and hence $\sum_{n=1}^{\infty} \|f_n\|_\infty = \sum_{n=1}^{\infty} |a_n| < \infty$. By the Weierstrass M-test $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[0, \infty)$.

(b) If we assume that only (a_n) is bounded, show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[\delta, \infty)$ for every $\delta > 0$.

Solution:
Let $\delta > 0$. Note $e^{-nx} \leq e^{-n\delta}$ for all $n \in \mathbb{N}$ and $x \in [\delta, \infty)$. Define the map $f_n : [\delta, \infty) \to \mathbb{R}$ by $f_n(t) = a_n e^{-nt}$. Then

$$\|f_n\|_\infty = \sup_{x\in[\delta,\infty)} |a_n e^{-nx}| = |a_n|e^{-n\delta}.$$

Since (a_n) is bounded then there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, $e^{-n\delta} < 1$ since $-n\delta < 0$. Hence

$$\sum_{n=1}^{\infty} \|f_n\|_\infty = \sum_{n=1}^{\infty} |a_n|(e^{-\delta})^n \leq \sum_{n=1}^{\infty} M(e^{-\delta})^n = \frac{M}{1 - e^{-\delta}}.$$

By the Weierstrass M-test, $\sum_{n=1}^{\infty} a_n e^{-nx}$ is uniformly convergent on $[\delta, \infty)$.

9. Carothers 10.33 [Mason]

Solution:
Let $\varepsilon > 0$, (x_n) be a sequence, and (c_n) be a sequence with

$$\sum_{n=1}^{\infty} |c_n| = L < \infty.$$

Since $\mathbb{R}$ is complete, we know that (c_n) is summable since it is absolutely summable. Thus $\sum_{n=1}^{\infty} c_n \infty$. Also note that $I(x - x_n) \leq 1$, as it is only ever 0 or 1, and thus the series

$$\left|\sum_{n=1}^{\infty} c_n I(x - x_n)\right| \leq \left|\sum_{n=1}^{\infty} c_n\right| \leq \sum_{n=1}^{\infty} |c_n| = L,$$

for any $x \in \mathbb{R}$. Since this is precisely our function $f(x)$, the function is bounded. Now, let $x \in \mathbb{R}$ such that it is not equal to any x_n. One case is that there exists some $N \in \mathbb{N}$ such that $x_{N-1} < x < x_N$. Now define $\delta = \min(|x_{N-1} - x|, |x_N - x|)$. Then if $|x - y| < \delta$, then $I(x - x_n) = I(y - x_n)$ for all $n \in \mathbb{N}$, and thus

$$|f(x) - f(y)| = \left|\sum_{n=1}^{\infty} c_n I(x - x_n) - \sum_{n=1}^{\infty} c_n I(y - x_n)\right| = \left|\sum_{n=1}^{\infty} c_n (I(x - x_n) - I(y - x_n))\right| = 0,$$

and so definitely $|f(x) - f(y)| < \varepsilon$. The other two cases is if x is less than every x_n or greater than every x_n. In which case $I(x - x_n)$ is constantly 0 or 1, respectively, on an interval and so $f(x)$ must be continuous.

10. [Carothers 10.34] (Solution by Max Heldman)

Let $0 \le g_n \in C[a, b]$. If $\sum_{n=1}^{\infty} g_n$ converges pointwise to a continuous function on $[a, b]$, then $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[a, b]$.

Solution:
Let g be the pointwise limit of $\sum_{n=1}^{\infty} g_n$. For each $N \in \mathbb{N}$ and $x \in [a, b]$, let $G_N(x) = \sum_{n=1}^{N} g_n$. Then for each $x \in [a, b]$, $\lim_{N\to\infty} G_N(x) = \sum_{n=1}^{\infty} g_n(x) = g(x)$, so g is the pointwise limit of (G_N). Moreover, since $g_n \ge 0$ for each $n \in \mathbb{N}$, G_N increases to g. Since g is continuous, Dini's Theorem implies that the convergence of (G_N), and therefore of $\sum_{n=1}^{\infty} g_n$, is uniform.

11. 11.14 (Solution by former 641 student Will Mitchell)
Let $f \in C[a, b]$ be continuously differentiable, and let $\epsilon > 0$. Show that there is a polynomial p such that $\|f - p\|_\infty < \epsilon$ and $\|f' - p'\|_\infty < \epsilon$. Conclude that $C^{(1)}[a, b]$ is separable.

Solution:
If $a = b$ the result is trivial. Assume therefore that $a < b$.

Since $f' \in C[a, b]$ and the polynomials are dense in this space, we can find a polynomial p such that $\|p - f'\|_\infty < \epsilon/(b - a)$. Define $q(x) = f(a) + \int_a^x p(t)dt$. Then q is a polynomial and if $x \in [a, b]$ we have

$$\begin{aligned}
|q(x) - f(x)| &= \left|f(a) + \int_a^x p(t)\,dt - f(a) - \int_a^x f'(t)\,dt\right| \\
&= \left|\int_a^x \big(p(t) - f'(t)\big)dt\right| \\
&\le \int_a^x \big|p(t) - f'(t)\big|dt \\
&\le (x - a)\,\|p - f'\|_\infty \\
&< \epsilon.
\end{aligned}$$

Therefore the polynomial q has the desired property.

We now deal with separability. We claim that $\mathbb{Q}[x]$ is dense in $C^{(1)}[a, b]$. Indeed, let $f \in C^{(1)}[a, b]$ and let $\epsilon > 0$. By the result quoted above we can find a polynomial p with real coefficients such that $d(p, f) = \|p - f\|_\infty + \|p' - f'\|_\infty < \epsilon/2$. Note that p is of the form

$$p(x) = c_n x^n + \cdots + c_1 x + c_0$$

for some $n \in \mathbb{N}$ and $c_0, \cdots, c_n \in \mathbb{R}$. For all vectors $z \in \mathbb{R}^{n+1}$, let p_z be the polynomial defined by $p_z(x) = z_1 + z_2 x + \cdots + z_{n+1} x^n$. Now define the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$F(z) = \max_{x \in [a,b]} \{|p_z(x) - p(x)| + |p_z'(x) + p'(x)|\}.$$

Now F is continuous on $\mathbb{R}^{n+1}$ and $F(c) = 0$ where $c = (c_0, c_1, \cdots, c_n)$. Thus there exists some $\delta > 0$ such that $z \in B_\delta(c)$ implies $|F(z)| < \epsilon/2$. Because $\mathbb{Q}^{n+1}$ is dense in $\mathbb{R}^{n+1}$, we can find an element y in the intersection $\mathbb{Q}^{n+1} \cap B_\delta(c)$. That is, there is a polynomial $q = p_y$ with rational coefficients satisfying $\|p - q\|_{C^{(1)}} < \epsilon/2$. Now $\|f - q\|_{C^{(1)}} < \epsilon$ by the triangle inequality.

We have shown that every open ball in $C^{(1)}$ intersects $\mathbb{Q}[x]$, so $\mathbb{Q}[x]$ is dense in $C^{(1)}$. Note that the set of polynomials of degree n having rational coefficients has the cardinality of $\mathbb{Q}^n$, hence is countable. But then $\mathbb{Q}[x]$ is a countable union of countable sets, hence countable. This completes the proof.

12. Carothers 11.16 [David]