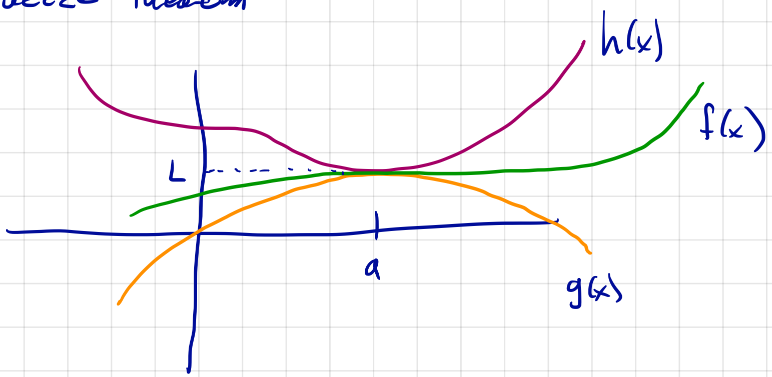
Discuss WS 2-3b 5, 6, 7

For 6, squeeze thm.

③ Squeeze theorem

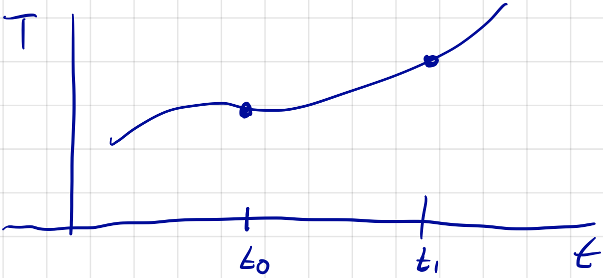


if $g(x) \le f(x) \le h(x)$

and $\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$ then

$\lim_{x \to a} f(x) = L$ also.

Why are we discussing limits?



Average rate of change $\dfrac{T(t_1) - T(t_0)}{t_1 - t_0}$

over interval $[t_0, t_1]$

Instantaneous rate of change: $t_1 = t_0$, but that's $\frac{0}{0}$.

Instantaneous rate of change: $\lim\limits_{t_1 \to t_0} \dfrac{T(t_1) - T(t_0)}{t_1 - t_0}$

We're doing limit computations so we can compute inst. rates of ch.

We've done limit computations with a common pattern

Algebra $\longrightarrow$ Limits don't care about one point

$\longrightarrow$ direct substitution principle

$\longrightarrow$ algebra

Lots of emphasis on limits don't care.

I want to focus on direct substitution now, and when it is justified.

$$\lim_{x \to 3} x^2 - 2x + 1 = 3^2 - 2 \cdot 3 + 1$$

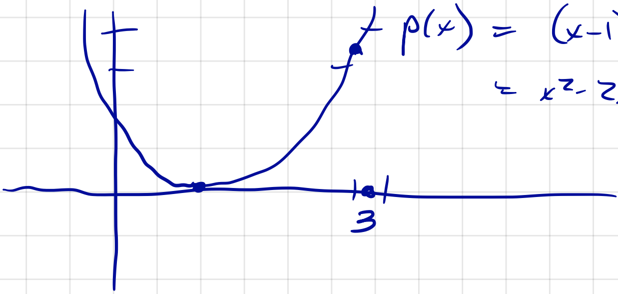
$$= 9 - 6 + 1 = 4$$

$$p(x) = x^2 - 2x + 1$$

$$\lim_{x \to 3} p(x) = p(3)$$

Think about what this says

As x gets close to 3, the values $p(x)$ get close to $p(3)$



$p(x) = (x-1)^2$

$= x^2 - 2x + 1$

Another way to think about this: $p(x)$ is forgiving. If x is close to 3 but not exactly 3, $p(x)$ will be close to $p(3)$. A little error in the input becomes a little error in the output.

Def: A function $f(x)$ is continuous at a point a in its domain if

$$\lim_{x \to a} f(x) = f(a)$$

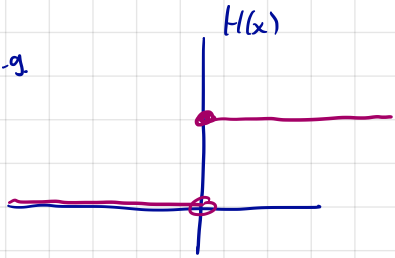
(i.e. if direct substitution applies)

Important for justifying limit calcs, but also in real world.

Def: A function $f(x)$ is <u>continuous</u>

if it is continuous at every point in its domain.

e.g.



continuous at $x=3$?

$H(x)=1$ near 3

"limits are nearsighted"

$$\lim_{x\to 3} H(x) = \lim_{x\to 3} 1$$

$$= 1$$

$H(3)=1$ ✓

continuous at $x=0$?

$$\lim_{x\to 0^+} H(x) = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} H(x) = \lim_{x\to 0^-} 0 = 0$$

$1 \neq 0$. So limit does not exist.

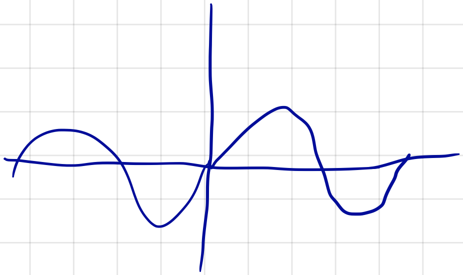
Not even a chance at being continuous.

A function that is not cts is said to be discont.

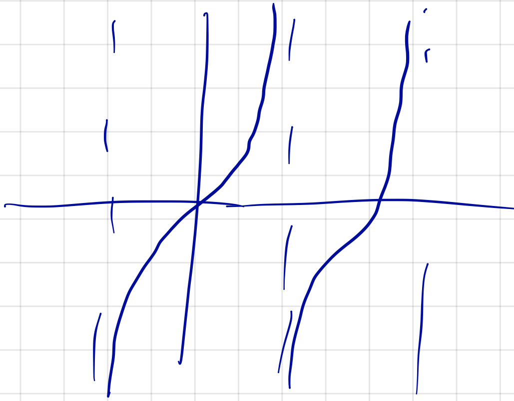
Is $H(x)$ a continuous function? Nope!

It takes just one bad point.

How about



$\sin(x)$



$\tan(x)$

yep!

yep!

If you believe $\sin(x)$ and $\cos(x)$ are cts, then so is $\tan(x)$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

a is in domain of $\tan(x)$ if and only if $\cos(a) \neq 0$.

$$\lim_{x\to a} \tan(x) = \lim_{x\to a} \frac{\sin(x)}{\cos(x)} = \frac{\lim_{x\to a} \sin(x)}{\lim_{x\to a} \cos(x)} = \frac{\sin(a)}{\cos(a)}$$

justified since $\cos(a) \neq 0$

Cts functions:

- polynomials
- rational
- root
- trig
- exp
- log
- abs

How about $\lim_{x\to 2} \sin(\sqrt{1+x^2}) \overset{?}{=} \sin(\sqrt{1+2^2})$

Yes: If $f(x)$ is continuous at a

and $g(x)$ is continuous at $f(a)$

$g(f(x))$ is cts at a.

I.e. $\lim_{x\to a} g(f(x)) = g(f(a))$

Simple version: A composition of continuous functions is continuous.

How about $\sin(3x) + 10^{-x}$?

$$\lim_{x\to 5} \sin(3x) + 10^{-x} = \lim_{x\to 5} \sin(3x) + \lim_{x\to 5} 10^{-x}$$

$$= \sin(3\cdot 5) + 10^{-5}$$

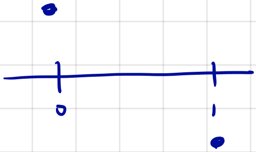
sums, products, differences, and division all ok (division by 0 would already be excluded)

Important theorem:

Consider $x^5 - 3x + 1 = p(x)$

$p(0) = 1$

$p(1) = -1$

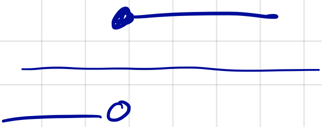


Somewhere in $[0,1]$ is a spot x where $p(x) = 0$.

This doesn't work for discontinuous functions

e.g.

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$



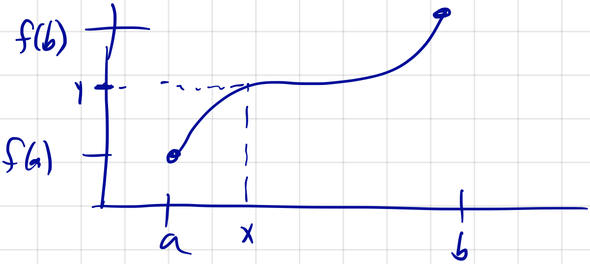
$f(x) \neq 0$ ever!

Intermediate Value Theorem

If $f(x)$ is a continuous function defined on an interval $[a,b]$, for any y between $f(a)$ and $f(b)$

there is $x \in [a,b]$ with $f(x) = y$.

[In particular, if $f(a) \geq 0$ and $f(b) \leq 0$ there is x in $[a,b]$ with $f(x) = 0$.]



(!)

In groups

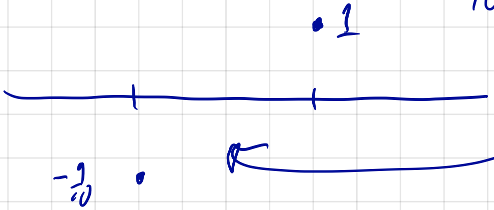
e.g. is there a number x with $10^x = x^2$

$f(x) = 10^x - x^2$ Want $f(x) = 0$.

$f(0) = 1$

$f(-1) = \frac{1}{10} - 1 = -\frac{9}{10}$

Aha!



there's a root somewhere in here!

In groups

a) show there is a number x with $10^x = x^2$

b) $$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$\mathrm{sgn}(-5) = -1$
$\mathrm{sgn}(3) = 1$

But $\mathrm{sgn}(x) \neq 1/2$ for all x.

Why doesn't this violate IVT?

[Carefully justify $\mathrm{sgn}(x)$ is discontinuous at $x = 0$]