1. (Solution by Mason Brewer)

Suppose $f \in \mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$. Show that $\alpha f \in \mathcal{R}[a,b]$ and

$$\int_a^b \alpha f = \alpha \int_a^b f$$

Solution:
Since $f \in \mathcal{R}[a,b]$, we know that

$$\overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f,$$

or

$$\sup_{G \in \mathcal{R}[a,b],\ G \geq f} \int_a^b G = \sup_{g \in \mathcal{R}[a,b],\ g \leq f} \int_a^b g.$$

Note that if $G \geq f$, then $\alpha G \geq \alpha f$, and similarly $g \leq f$ implies $\alpha g \leq \alpha f$. Thus

$$\overline{\int_a^b} \alpha f = \sup_{G \in \mathcal{R}[a,b],\ G \geq f} \int_a^b \alpha G = \alpha \sup_{G \in \mathcal{R}[a,b],\ G \geq f} \int_a^b G = \alpha \overline{\int_a^b} f,$$

and similarly

$$\underline{\int_a^b} \alpha f = \sup_{g \in \mathcal{R}[a,b],\ g \leq f} \int_a^b \alpha g = \alpha \sup_{g \in \mathcal{R}[a,b],\ g \leq f} \int_a^b g = \alpha \underline{\int_a^b} f.$$

Therefore we get that

$$\underline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f = \alpha \overline{\int_a^b} f = \overline{\int_a^b} \alpha f,$$

meaning that $\alpha f \in \mathcal{R}[a,b]$, and that

$$\int_a^b \alpha f = \overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f = \alpha \int_a^b f.$$

2. [Sakti] Show that the uniform limit of Riemann integrable functions is Riemann integrable. Conclude that $\mathcal{R}[a,b]$ is a closed subset of $B[a,b]$.

3. (Solution by Max Heldman)

Find (with proof) an element of $\mathcal{R}[a,b]$ that is not a uniform limit of step functions.

Solution:

4. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$$

We claim $f \in \mathcal{R}[0,1]$ with $\int_0^1 f = 0$. Indeed, if we define $g_n, G_n \in \text{step}[0,1]$ by

$$g_n(x) \equiv 0, \qquad G_n(x) = \begin{cases} 0 & x \in [0, 1-1/n) \\ 1 & x \in [1-1/n, 1] \end{cases}$$

for each $n \in \mathbb{N}$, then $g_n \leq f \leq G_n$ and moreover

$$\int_0^1 G_n - g_n = \int_0^1 G_n = 0 \cdot (1 - 1/n) + 1 \cdot (1 - (1 - 1/n)) = 1/n.$$

Hence $\lim_{n\to\infty} \int_0^1 G_n - g_n = 0$.

Now suppose that (f_n) is any sequence of step functions converging pointwise to f. Then there exists $N \in \mathbb{N}$ so that if $n \geq N$, $|f_n(1) - 1| = |f_n(1) - f(1)| < 1/2$. Hence $f_n(1) > 1/2$ for all $n \geq N$. But since each f_n is a step function, each f_n is constant on some interval $[1 - \epsilon_n, 1]$ for some $1 \geq \epsilon_n > 0$. But then

$$|f_n(1 - \epsilon_n) - f(1 - \epsilon_n)| = |f_n(1 - \epsilon) - 0| = |f_n(1 - \epsilon_n)| > 1/2.$$

Hence (f_n) does not converge to f uniformly.

5. (Solution by Jody Gaines)

Show that integration on $\mathcal{R}[a,b]$ (as a closed subset of $B[a,b]$) is continuous by showing that the map $T : \mathcal{R}[a,b] \to \mathbb{R}$ defined by

$$f \mapsto \int_a^b f$$

is a bounded linear map.

Solution:
Let $f \in \mathcal{R}[a,b]$. Note that $G(x) = \|f\|_\infty \in \text{Step}[a,b]$ and $G \geq f$. Hence

$$\left| \int_a^b f \right| \leq \left| \int_a^b G \right| = (b-a)\|f\|_\infty,$$

which implies that the map $f \mapsto \int_a^b f$ is bounded.

6. (Solution by Lander Ver Hoef)

Determine if $\chi_\Delta \in \mathcal{R}[0,1]$, where Δ is the Cantor set.

Solution:
First, let $g_n = 0$. Then $g_n \leq \chi_\Delta$, and $\int_0^1 g_n = 0$ for all n, so the sequence of integrals converges to 0.

Let P_n be the partition of $[0,1]$ defined by the endpoints of the nth step of the Cantor refinement. That is, $P_1 = \{0, 1/3, 2/3, 1\}$ and $P_2 = \{0, 1/9, 2/9, 1/3, 2/3, 7/9, 8/9, 1\}$. We then define the step functions G_n from these partitions by $G_n = 1$ on an interval from P_n

if there is an element from Δ in that interval. First note that $G_n \geq \chi_\Delta$. Next, observe that $\int_0^1 G_1 = 2/3$, while $\int_0^1 G_2 = 4/9$. In general, $\int_0^1 G_n = (2/3)^n$, which clearly converges to 0.

Thus, by our characterization of the Riemann integrable functions, χ_Δ is Riemann integrable with

$$\int_0^1 \chi_\Delta = 0.$$

7. (Solution by Jody Gaines)

Suppose $l : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ and that it is either additive or that it is countably additive. Show that either $l(\varnothing) = 0$ or $l(A) = \infty$ for all $A \in \mathcal{P}(\mathbb{R})$. Regardless, show that l is monotone.

Solution:
Suppose there exists $A \in \mathcal{P}(\mathbb{R})$ such that $l(A) < \infty$. Suppose l is additive. Then $l(A) = l(A \cup \varnothing) = l(A) + l(\varnothing)$, and since $l(A) < \infty$ then $l(A) = l(A) + l(\varnothing)$ implies $l(\varnothing) = 0$. Moreover for all $U, V \in \mathcal{P}(\mathbb{R})$ such that $U \subseteq V$,

$$l(V) = l(V \cup U) = l(V) + l(U) \geq l(U).$$

Now suppose l is countably additive. Then

$$l(A) = l(A \cup (\cup_{k=1}^{\infty} \varnothing)) = l(A) + \sum_{k=1}^{\infty} l(\varnothing),$$

and $l(A) < \infty$ implies $\sum_{k=1}^{\infty} l(\varnothing) = 0$ or rather $l(\varnothing) = l(\cup_{k=1}^{\infty} \varnothing) = \sum_{k=1}^{\infty} l(\varnothing) = 0$. If $U, V \in \mathcal{P}(\mathbb{R})$ such that $U \subseteq V$ then

$$l(V) = l((\cup_{k=1}^{\infty} V) \cup U) = l(U) + \sum_{k=1}^{\infty} l(V) \geq l(U).$$

Hence l is monotone and $l(\varnothing) = 0$.

8. (Solution by Mason Brewer)

Suppose $l : \mathcal{P}(\mathbb{R}) \to [0, \infty]$. Show that l is countably additive if and only if l is finitely additive and countably subadditive.

Solution:
First, assume that l is countable additive. It follows immediately that l is finitely additive. Now, let $\{A_k\}_{k=1}^{\infty}$ be and arbitrary collection of subsets of $\mathbb{R}$. For each $k \in \mathbb{N}$. Define

$$B_k = A_k \setminus \left(\bigcup_{i=1}^{k-1} A_i \right),$$

and note first that for each k, $B_k \subset A_k$, and also that each B_k is disjoint from one another. Finally, note that if $x \in \bigcup_{k=1}^{\infty}(A_k)$, then let k be the smallest integer such that $x \in A_k$. Then it must be that $x \in B_k$, or $x \in \bigcup_{k=1}^{\infty} B_k$, which implies that

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k.$$

Then, by countable additivity, we have that

$$l\left(\bigcup_{k=1}^{\infty} A_k\right) = l\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} l(B_k),$$

and by monotonicity, we get

$$l\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} l(A_k).$$

For the converse, assume that l is countably sub-additive and finitely additive and let $\{A_k\}_{k=1}^{\infty}$ be a disjoint collection of subsets of $\mathbb{R}$. By countable sub-additivity, we already know that $l(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} l(A_k)$. Now, for any $n \in \mathbb{N}$, it is true that $\bigcup_{k=1}^{n} A_k \subseteq \bigcup_{k=1}^{\infty} A_k$. Thus we have that

$$\sum_{k=1}^{n} l(A_k) = l\left(\bigcup_{k=1}^{n} A_k\right) \leq l\left(\bigcup_{k=1}^{\infty} A_k\right),$$

where the equality above is given by finite additivity, and the inequality above is given by monotonicity. Since this is true for every n, it must be that

$$\sum_{k=1}^{\infty} l(A_k) \leq l\left(\bigcup_{k=1}^{\infty} A_k\right).$$

Therefore l is countably additive.