Linear Algebra I – Fall 2017

Homework 3

Homework should be submitted through Moodle as a pdf file by 17:00 on Friday, November 17. Late submissions will not be accepted. Additional instructions are available on Moodle.

1. Determine whether $\mathbf{v_1} = \begin{pmatrix} 5 \\ 7 \\ 7 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ are linearly independent.

   Solution: The vectors are linearly dependent iff there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, not all zero, such that
$$\alpha_1 \begin{pmatrix} 5 \\ 7 \\ 7 \end{pmatrix} + \alpha_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
   This is aquivalent to the following homogenous system of linear equations, written in matrix form, having a nontrivial solution:
$$\begin{pmatrix} 5 & 4 & -2 \\ 7 & 5 & -4 \\ 7 & 6 & -6 \end{pmatrix}$$
   We apply row operations to bring it to an REF:
$$\begin{pmatrix} 5 & 4 & -2 \\ 7 & 5 & -4 \\ 7 & 6 & -6 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{5}R_1} \begin{pmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 7 & 5 & -4 \\ 7 & 6 & -6 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 7R_1} \begin{pmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & -\frac{3}{5} & -\frac{6}{5} \\ 0 & \frac{2}{5} & -\frac{16}{5} \end{pmatrix}$$
$$\xrightarrow{R_2 \to -\frac{5}{3}R_2} \begin{pmatrix} 1 & \frac{4}{5} & -\frac{2}{5} \\ 0 & 1 & 2 \\ 0 & \frac{2}{5} & -\frac{16}{5} \end{pmatrix} \xrightarrow[R_3 \to R_3 - \frac{2}{5}R_2]{R_1 \to R_1 - \frac{4}{5}R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{pmatrix} \xrightarrow{R_3 \to -\frac{1}{4}R_3} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$
   It is now evident that the system has only the trivial solution. Thus, the vectors are linearly independent.

2. In each of the following cases determine whether the set A spans V. If so, given an arbitrary vector $\mathbf{v} \in V$ ($\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ or $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$), express $\mathbf{v}$ as a linear combination of the vectors in A with the weights in terms of the entries of $\mathbf{v}$. If not, give an example of a vector which is not a linear combination of the vectors in A.

   (a) $A = \left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, V = \mathbb{R}^3.$

Solution: A vector $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ is in the span of A iff there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that the following system of linear equations, written in matrix form, has a solution:

$$\begin{pmatrix} -1 & 2 & 0 & a \\ -1 & 5 & 0 & b \\ -1 & 8 & 1 & c \end{pmatrix}$$

This system has the RREF:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{-5a+2b}{3} \\ 0 & 1 & 0 & \frac{-a+b}{3} \\ 0 & 0 & 1 & a-2b+c \end{pmatrix}$$

It follows that A spans V and

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{-5a+2b}{3} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \frac{-a+b}{3} \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + (a-2b+c) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) $A = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}, V = \mathbb{R}^2.$

Solution: A vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ is in the span of A iff there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that the following system of linear equations, written in matrix form, has a solution:

$$\begin{pmatrix} 1 & 2 & a \\ 1 & -1 & b \end{pmatrix}$$

This system has the RREF:

$$\begin{pmatrix} 1 & 0 & \frac{a+2b}{3} \\ 0 & 1 & \frac{a-b}{3} \end{pmatrix}$$

It follows that A spans V and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a+2b}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{a-b}{3} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

3. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Suppose $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent if and only if $\mathbf{w}$ is in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

   Solution: Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Thus, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, not all zero, such that

$$\alpha_1 \mathbf{u} + \alpha_2 \mathbf{v} + \alpha_3 \mathbf{w} = 0.$$

If $\alpha_3 = 0$, then at least one of α_1, α_2 is nonzero, in contradiction to the assumption that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Thus, $\alpha_3 \neq 0$. It follows that

$$\mathbf{w} = -\frac{\alpha_1}{\alpha_3}\mathbf{u} - \frac{\alpha_2}{\alpha_3}\mathbf{v} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}.$$

In the other direction, suppose $\mathbf{w}$ is in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$. Thus, there are $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\mathbf{w} = \alpha_1\mathbf{u} + \alpha_2\mathbf{v}$. Thus, $\alpha_1\mathbf{u} + \alpha_2\mathbf{v} - \mathbf{w} = 0$ is a nontrivial linear combination of zero and it follows that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

4. For each of the following solution sets find a set of vectors that spans it:

   (a) $\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + y + z = 0\}$.

   Solution: We have $\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + y + z = 0\} = \{\begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid y, z \in \mathbb{R}\} = \{y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \mid y, z \in \mathbb{R}\} = \mathrm{Span}\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$. Thus, an example of such a set is $\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$.

   (b) $\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 - x_2 + x_3 - x_4 = 0, 2x_1 + x_2 + 2x_3 + x_4 = 0\}$.

   Solution: One easily sees that the solution set of the system of linear equations is given by $\{\begin{pmatrix} -s \\ -t \\ s \\ t \end{pmatrix} \in \mathbb{R}^4 \mid s, t \in \mathbb{R}\} = \{s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4 \mid s, t \in \mathbb{R}\} = \mathrm{Span}\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\}$. Thus, an example of such a set is $\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\}$.

5. Let $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Determine whether the set

$$\{\mathbf{v_1}, \mathbf{v_1} + \mathbf{v_2}, \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}\}$$

   is linearly independent.

   Solution: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1\mathbf{v_1} + \alpha_2(\mathbf{v_1} + \mathbf{v_2}) + \alpha_3(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}) = 0.$$

Thus,

$$(\alpha_1 + \alpha_2 + \alpha_3)\mathbf{v_1} + (\alpha_2 + \alpha_3)\mathbf{v_2} + \alpha_3\mathbf{v_3} = 0.$$

As $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent, we have

$$\begin{aligned}
&\alpha_1 + \alpha_2 + \alpha_3 = 0 \\
&\alpha_2 + \alpha_3 = 0 \\
&\alpha_3 = 0
\end{aligned}$$

This system obviously has only the trivial solution, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = 0$. It follows that

$$\{\mathbf{v_1}, \mathbf{v_1} + \mathbf{v_2}, \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}\}$$

is linearly independent.

6. Prove: Let $A = \{\mathbf{v_1}, \dots, \mathbf{v_k}\}$ be a linearly independent set of vectors in $\mathbb{R}^n$ and let B be a nonempty subset of A. Then B is linearly independent.

   Solution: Without loss, $B = \{\mathbf{v_1}, \dots, \mathbf{v_m}\}$ where $1 \leq m \leq k$. Let $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, not all zero, such that

   $$\alpha_1\mathbf{v_1} + \cdots + \alpha_m\mathbf{v_m} = 0.$$

   Thus,

   $$\alpha_1\mathbf{v_1} + \cdots + \alpha_m\mathbf{v_m} + 0 \cdot \mathbf{v_{m+1}} + \cdots 0 \cdot \mathbf{v_k} = 0$$

   is a nontrival linear combination of zero. In contradiction to the assumption.

7. Let A be an $m \times n$ matrix in REF. Prove or provide a counterexample:

   (a) If A has a leading entry in every column then the columns of A are linearly independent.
   Solution: This is true. In this case A can be brought to an RREF of the form:

   $$\begin{pmatrix}
   1 & 0 & \cdots & 0 \\
   0 & 1 & & \vdots \\
   0 & 0 & \ddots & 0 \\
   \vdots & & & 1 \\
   0 & 0 & 0 & 0 \\
   \vdots & \vdots & \vdots & \vdots \\
   0 & 0 & 0 & 0
   \end{pmatrix}$$

   The columns of this matrix are obviously linearly independent and therefore also the columns of A are linearly independent.

   (b) If A has a leading entry in every column then the columns of A span $\mathbb{R}^m$.
   Solution: This is false. A counterexample is given by

   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Obviously, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ cannot be generated by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Thus, the columns of A do not span $\mathbb{R}^3$.

(c) If A has a leading entry in every row then the columns of A span $\mathbb{R}^m$.

Solution: In an REF each column can have at most one leading entry. Thus, the set of columns of A has a subset of the form $\left\{ \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ a_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} * \\ \\ \vdots \\ * \\ a_m \end{pmatrix} \right\}$ where the asteriks stand for real numbers and $a_1, \dots, a_m$ are all nonzero real numbers. It is clear that this subset spans $\mathbb{R}^m$ and therefore the set of columns of A spans $\mathbb{R}^m$.