Date

CS 512 - Homework 0

A. Given: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

Solutions

1. $2A - B =$

$$\begin{bmatrix} 2\times 1 & - & 4 \\ 2\times 2 & - & 5 \\ 2\times 3 & - & 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

2. $\|A\|$ and the angle of A relative to the positive X axis =

$$\|A\| = (x, y, z) = (1, 2, 3)$$

Length of point A from origin $= \sqrt{x^2+y^2+z^2}$

$$= \sqrt{1^2+2^2+3^2}$$
$$= \sqrt{14}$$

Angle from x axis $= \cos x = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$

$$= \frac{1}{\sqrt{14}} = 75^\circ$$

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A. 3. $\hat{A}$, a unit vector in direction of A.

$$\|A\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\therefore U_v = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

$$= \left(\frac{\sqrt{14}}{14}, \frac{2\sqrt{14}}{14}, \frac{3\sqrt{14}}{14}\right)$$

$$= \left(\frac{\sqrt{14}}{14}, \frac{\sqrt{14}}{7}, \frac{3\sqrt{14}}{14}\right)$$

4. The direction cosine of A.

A w.r.t $x = \boxed{75^\circ}$ A w.r.t $Y = \cos^{-1}\left(2/\sqrt{14}\right) = \boxed{58^\circ}$

$\cancel{= \frac{1}{\sqrt{1^2+2^2+3^2}}}$ $\cancel{= \frac{1}{\sqrt{14}}}$

A w.r.t $z = \cos^{-1}\left(3/\sqrt{14}\right) = \boxed{37^\circ}$

5. A . B and B . A

$$A \cdot B = 1\times 4 + 2\times 5 + 3\times 6 = 4+10+18$$
$$= 32$$

$$B \cdot A = A \cdot B = 32$$

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A. 6. The angle b/w A and B.

$A \cdot B = 32$

$\|A\| = \cancel{\sqrt{14}}\ \sqrt{14} \qquad \|B\| = \sqrt{4^2+5^2+6^2} = \sqrt{77}$

$\therefore \cos\theta = \dfrac{32}{\sqrt{14}\cdot\sqrt{77}} = \dfrac{32}{7\sqrt{22}} = 13^\circ$

7. A vector which is $\perp$ to A.

Let $\perp$ vector be $= V = (v_1, v_2, v_3)$

$\Rightarrow \quad A \cdot V = v_1 + 2v_2 + 3v_3$

since its $\perp$, $A \cdot V = 0$

$\therefore \quad v_1 + 2v_2 + 3v_3 = 0$

If v_1 & $v_2 = 1$

$v_3 = \dfrac{-3}{3} = -1$

$\therefore \perp$ vector is $(1, 1, -1)$ to A.

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A. 8. $A \times B$ & $B \times A$:

$$A = (1, 2, 3)^T$$
$$B = (4, 5, 6)^T$$

$$A \times B = (2 \times 6 - 3 \times 5,\ 3 \times 4 - 1 \times 6,\ 1 \times 5 - 4 \times 2)^T$$
$$= (12 - 15,\ 12 - 6,\ 5 - 8)^T$$
$$= (-3,\ 6,\ -3)^T$$

$$A \times B = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$

$$A \times B = -(B \times A)$$
$$\therefore B \times A = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}$$

9. A vector which is $\perp$ to both A & B.

Cross product of vector A & B is $\perp$ to each vector A & B.

$$\therefore \perp \text{ vectors is } \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$

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A. 10. The linear dependancy between A, B, C.

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 3 & 0 \end{array}\right]$$

$\Rightarrow$ 2^{nd} row - 2x 1st row & 3^{rd} row - 3x 1^{st} row

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right]$$

$\Rightarrow$ 3^{rd} row - 2 x 2^{nd} row

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ A, B, C are linearly dependant.

11. $A^T B$ & AB^T

$A \cdot B = A^T B = \cancel{AB} = (1\times4 + 2\times5 + 3\times6)$

$= 32$

$$A \cdot B^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

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B. Given: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$$

Solutions

1. $2A - B$:

$$= \begin{bmatrix} 2\times 1 - 1 & 2\times 2 - 2 & 2\times 3 - 1 \\ 2\times 4 - 2 & 2\times -2 - 1 & 2\times 3 + 4 \\ 2\times 0 - 3 & 2\times 5 + 2 & 2\times -1 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$$

2. AB and BA:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4+9 & 2+2-6 & 1-8+3 \\ 4-4+9 & 8-2-6 & 4+8+3 \\ 0+10-3 & 0+5+2 & 0-20-1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$$

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B. $= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1+8+0 & 2\cancel{+}^{-4}+5 & 3+6-1 \\ 2+4+0 & 4-2-20 & 6+3+4 \\ 3-8+0 & 6+4+5 & 9-6-1 \end{bmatrix}$

$$BA = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$$

3. $(AB)^T$ and $B^T A^T$

$$(AB)^T = B^T A^T = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}^T$$

$$= \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

4. $|A|$ and $|C|$

$$|A| = 1(2-15) - 2(-4-0) + 3(20+0)$$
$$= -13 + 8 + 60$$
$$= 55$$

$$|C| = 1(15-6) - 2(12+6) + 3(4+5)$$
$$= 9 - 36 + 27$$
$$= 0$$

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B. 5. the matrix (A, B or C) in which the row vectors form an orthogonal set

$$B \cdot B^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

6. A^{-1} & B^{-1}

$$A^{-1} = \frac{1}{|A|} \times \text{Adjugated}(A)$$

$$\text{Adj } A = \begin{bmatrix} 2-15 & -4 & 20 \\ (-) -2-15 & -1 & +5 \\ 6+6 & 3-12 & -2-8 \end{bmatrix}^T = \begin{bmatrix} -13 & -4 & 20 \\ -17 & -1 & 5 \\ 12 & -9 & -10 \end{bmatrix}^T$$

$$= \begin{bmatrix} -13 & 4 & 20 \\ 17 & -1 & -5 \\ 12 & 9 & -10 \end{bmatrix}^T = \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{55} \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$$

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6. $B^{-1} = \frac{1}{|B|} \cdot Adj(B)$

$$Adj(B) = (-)\begin{bmatrix} 1-8 & 2+12 & -2-3 \\ 2+2 & 1-3 & -2-6 \\ -4-1 & -4-2 & 1-4 \end{bmatrix}^T$$

$$= (-)\begin{bmatrix} -7 & 14 & -5 \\ 4 & -2 & -8 \\ -5 & -6 & -3 \end{bmatrix}^T = \begin{bmatrix} -7 & -14 & -5 \\ -4 & -2 & 8 \\ -5 & 6 & -3 \end{bmatrix}^T$$

$$Adj(B) = \begin{bmatrix} -7 & -4 & -5 \\ -14 & -2 & 6 \\ -5 & 8 & -3 \end{bmatrix}$$

$$|B| = 1(-7) - 2(14) + 1(-5)$$
$$= -7 - 28 - 5$$
$$= -40$$

$$B^{-1} = -\frac{1}{40}\begin{bmatrix} -7 & -4 & -5 \\ -14 & -2 & 6 \\ -5 & 8 & -3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 7 & 4 & 5 \\ 14 & 2 & -6 \\ 5 & -8 & 3 \end{bmatrix}$$

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C. Given: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

Solution.

1. the eigenvalues & corresponding eigenvectors of A.

∴ Let λ be eigenvalue of A

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{bmatrix} \right) = 0$$

$$(\lambda - 1)(\lambda - 2) - 6 = 0$$

$$\lambda^2 - 2\lambda - \lambda + 2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4 \text{ or } -1$$

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$A\vec{v} = \lambda \vec{v}$ where $\lambda = 4$ & -1

assuming non-zero eigenvector

$\vec{0} = \lambda \vec{v} - A\vec{v}$

$\vec{0} = \lambda I_n \vec{v} - A\vec{x}$

$\vec{0} = (\lambda I_n - A)\vec{v}$

$$\vec{0} = \left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right) \vec{v}$$

$$\vec{0} = \left(\begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \right) \vec{v}$$

$$\vec{0} = \left(\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \right) \cancel{4} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$3v_1 - 2v_2 = 0$$

$$3v_1 = 2v_2$$

$$v_1 = \frac{2}{3} v_2$$

$$E_4 = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

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$$E_{-1} = N\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}\right)$$

$$\cancel{E_{-1} = N\left(\left[\right.\right.}$$

$$-v_1 = v_2$$

$$E_{-1} = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix},\ t \in \mathbb{R} \right\}$$

C. 2. $\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

C. 3. $\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ -9 & 6 \end{bmatrix}$

C. 4. $\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\right) = 0$

$$\det\left(\begin{bmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 5 \end{bmatrix}\right) = 0$$

$$(\lambda - 2)(\lambda - 5) - 4 = 0$$

$$\lambda^2 - \lambda 5 - 2\lambda + 10 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 6 \text{ or } 1$$

$$E_6 = N\left(\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\right)$$

$$E_6 = N\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right)$$

$$E_6 = N\left(\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$2V_1 + V_2 = 0$$
$$V_1 = -\frac{1}{2}V_2$$

$$E_1 = N\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\right)$$

$$-V_1 + 2V_2 = 0$$
$$\frac{1}{2}V_1 = V_2$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = -2 + 2 = 0$$

$$\cancel{\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}}$$

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C. 5. The dot product of ~~eigh~~ eigenvectors of B is 0, therefore they are orthogonal vector.

D. given : $f(x) = x^2 + 3$; $g(x,y) = x^2 + y^2$,

solution

1. $f(x) = x^2 + 3$

$f'(x) = 2x$

$f''(x) = 2$

2. $g(x,y) = x^2 + y^2$

$\frac{\partial g}{\partial x} = 2x$ $\cancel{2y}$ $\qquad \frac{\partial g}{\partial y} = 2y$

$= 2 \qquad\qquad = 2$

3. $\nabla g(x,y) = x^2 + y^2$

$g'(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

$g''(x,y) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

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4. $p(x \mid \mu, \sigma^2) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{Z} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

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What makes you happy?

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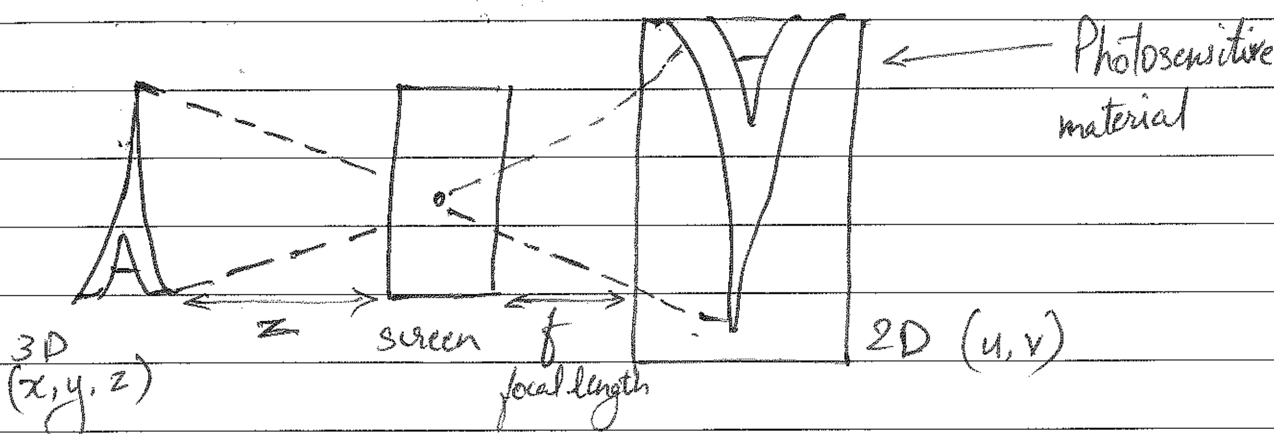
Geometric Image Formation

* Pinhole camera model



→ Disadvantage of Pinhole camera
- Focus is not achieved, blurred. Solvable by making the hole small.
- If the hole is small, not much light will pass
- Interference

Note: The bigger the (f) focal length is, the bigger the image will be.
- Z is the z axis on camera coordinate.

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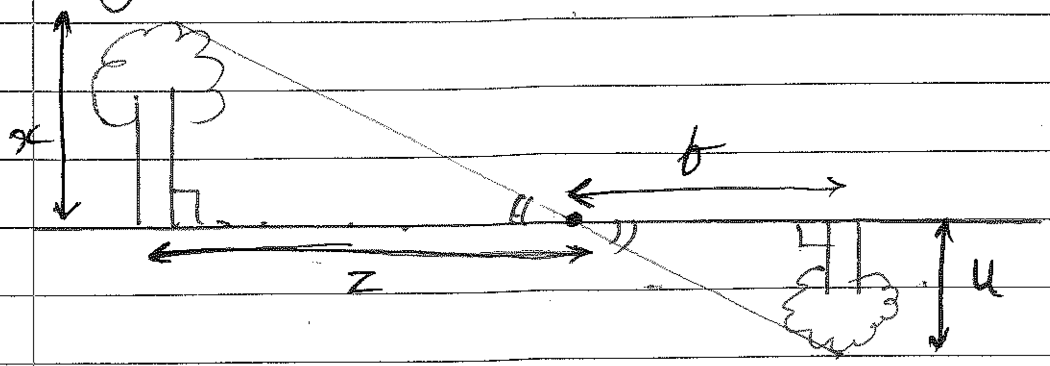
* Projection Equations



Since two similar triangle are formed we can equate the ratio of sides.

$$\frac{x}{z} = \frac{-u}{f} \Rightarrow u = -f\frac{x}{z}$$

3D $\longrightarrow$ 2D
world Image

$$\frac{-v}{f} = \frac{y}{z} \Rightarrow v = -f\frac{y}{z}$$

In matrix form,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{z}\begin{bmatrix} -f & 0 \\ 0 & -f \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- non-linear (cos of z div)
- fore shortening (div by z)
- all in camera coods

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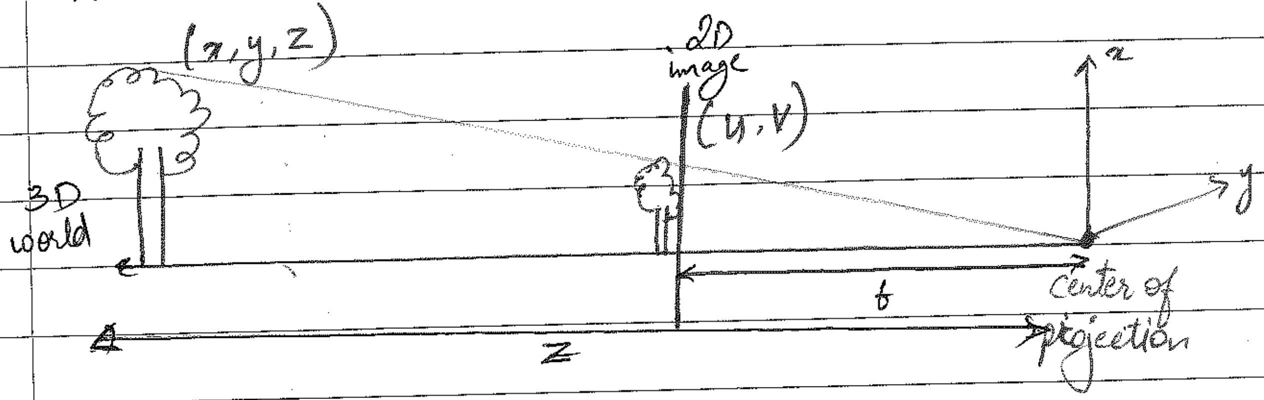
Alternative model



$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{z} \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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* Homogeneous Coordinates

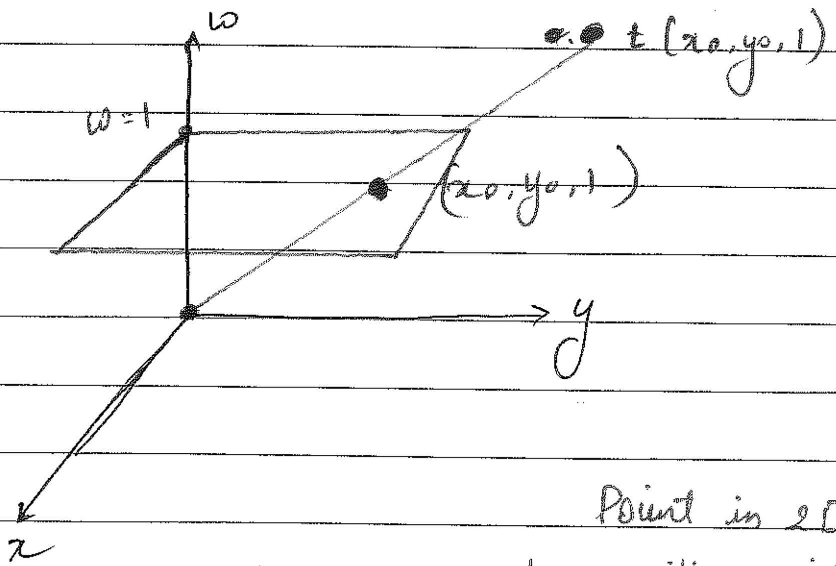
2D to 2DH

$$\underset{(x, y)}{2D} \longrightarrow \underset{(x, y, 1)}{2DH}$$

$$(x, y, 1) \longleftrightarrow (tx, ty, t) \quad t \neq 0$$

2DH to 2D

$$\left(\frac{x}{w}, \frac{y}{w}\right) \xleftarrow{w \neq 0} (x, y, w)$$



Note: always look at image in 2D. Not 2DH.

Point in 2D = line in 2DH

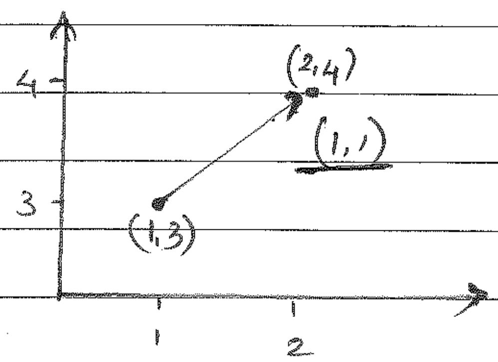
homogeniting = intersecting with plane ($w=1$).

* Points at infinity

$(x, y, 0) \longrightarrow \left(\frac{x}{0}, \frac{y}{0}\right)$ is called a point at infinity cos its huge. = direction

2DH   2D

It represents a direction:



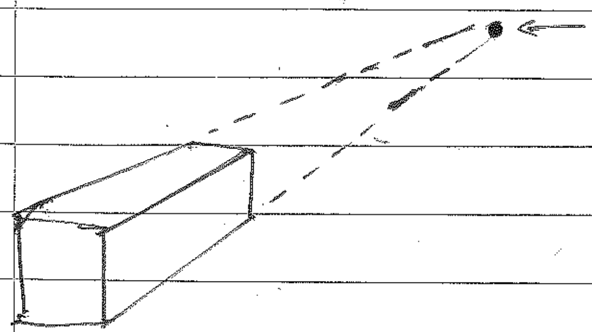
$(2, 4, 1) - (1, 3, 1) = (1, 1, 0)$

direction

* Just as there is 2D & 2DH, there is 3D & 3DH.

3D $\longrightarrow$ 3DH

3 components   4 components



Vanishing point = projection of direction

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* Matrix Representation that is linear.

Represent projection using ~~some~~ linear equation in the homogeneous coordinates.

$$\underset{\text{2DH}}{\begin{bmatrix} U \\ V \\ W \end{bmatrix}} = \underset{\text{3×3 Proj matrix}}{\begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix}} \underset{\text{3D}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \Rightarrow \left.\begin{aligned} U &= fx \\ V &= fy \\ W &= z \end{aligned}\right\} \text{Projected points in 2DH}$$

$$\left.\begin{aligned} u &= \frac{U}{W} = \frac{fx}{z} \\ v &= \frac{V}{W} = \frac{fy}{z} \end{aligned}\right\} \text{same projection equations as 2D}$$

Thus we use 2DH coordinates matrix equations as they are linear and gives same equation ~~of~~ on conversion to 2D. But it is not completely balanced.

$$\underset{\text{2DH}}{\begin{bmatrix} U \\ V \\ W \end{bmatrix}} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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* Balanced Matrix Representation.

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

2DH 3X4 proj matrix 3DH

$$u = \frac{U}{W} = \frac{fx}{z}$$

$$v = \frac{V}{W} = \frac{fy}{z}$$

Its balance because LHS & RHS have homogeneous coordinates.

Alternative Representation

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1/f & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

divided proj materix by f.

$$u = \frac{U}{W} = \frac{x}{z/f} = \frac{fx}{z} \quad \Big| \quad v = \frac{V}{W} = \frac{y}{z/f} = \frac{fy}{z}$$

Same output. Same proj equation.

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* Block Notation

$$\underset{p}{\overset{2DH}{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}} = \overset{3\times4 \text{ Proj M}}{\left[\begin{array}{c|c} \underbrace{\begin{matrix} f & & \\ & f & \\ & & 1 \end{matrix}}_{K} & \underbrace{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}}_{0} \end{array}\right]} \underset{\underline{P}}{\overset{3DH}{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}}}$$

$$\underset{3\times1}{p} = [\underset{3\times3}{K} | \underset{3\times1}{0}] \underset{4\times1}{\underline{P}}$$

$$\Rightarrow \underset{2DH}{p} = \underbrace{K[I|0]}_{\text{Proj Matrix}} \underset{3DH}{\underline{P}}$$

$[I|0]$ = external parameter

K = Internal parameter

External Parameter :

I will represent rotation

0 will represent translation

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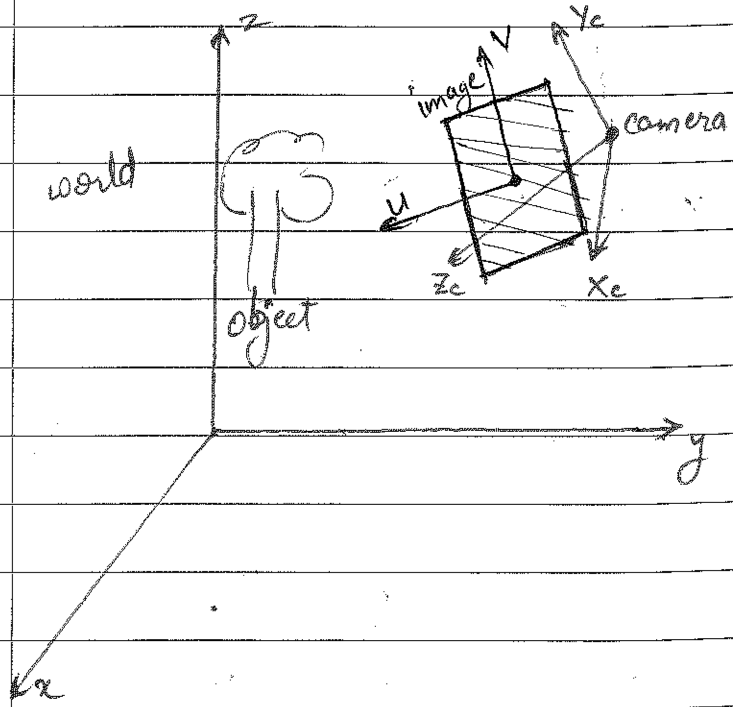
Lecture - 4

Geometric Image Formation (contd.)

The problem was, everything was in camera coordinate system which is not realistic. (thing from lec-3)

We want to have a world coordinate system.



Reason: If everything done is in camera coordinates; if we move the camera, everything changes. (transformation)

Types of transformation: - Rotation
- Translation
- Scaling
- Shear

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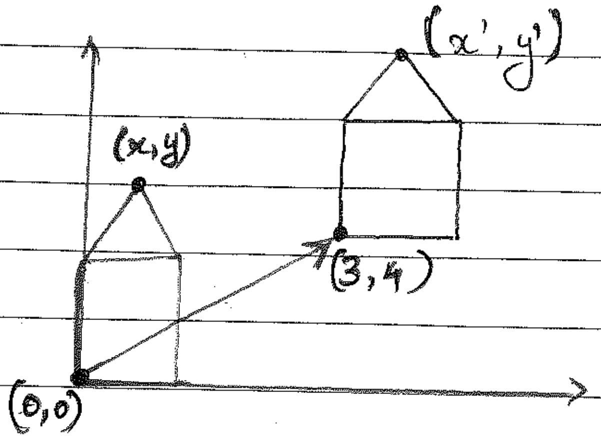
* Translation



$x' = x+3$
$y' = y+4$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$ } Matrix equation for translation

Translation in homogeneous coordinates

$$\underset{\text{3DH}}{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}} = \begin{bmatrix} 1 & & & t_x \\ & 1 & & t_y \\ & & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\text{3DH}}{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}$$

$x' = x + t_x$
$y' = y + t_y$
$z' = z + t_z$

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Block Notation

$$\underline{p'} \quad \begin{bmatrix} \overbrace{\begin{matrix}1&&\\&1&\\&&1\end{matrix}}^{I} & \overbrace{\begin{matrix}t_x\\t_y\\t_z\end{matrix}}^{t} \\ \underbrace{\begin{matrix}0&0&0\end{matrix}}_{0} & \underbrace{1}_{1}\end{bmatrix} \quad \underline{P}$$

$$\underset{3DH}{\underline{p'}} = \left[\begin{array}{c|c} \overset{3\times 3}{I} & \overset{3\times 1}{t} \\ \hline \underset{1\times 3}{0} & \underset{1\times 1}{1} \end{array}\right] \underset{3DH}{\underline{P}} \iff \underset{4\times 1}{\underline{p'}} = \underset{4\times 4}{T(t)}\, \underset{4\times 1}{\underline{P}}$$

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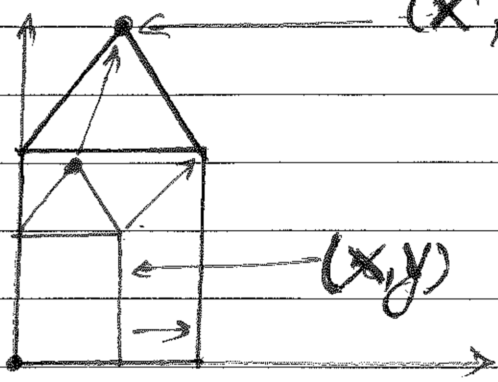
* Scale



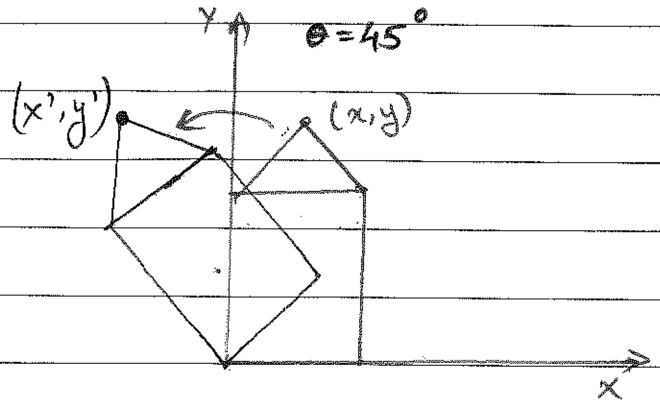
$$x' = S_x\,x$$
$$y' = S_y\,y$$
$$z' = S_z\,z$$

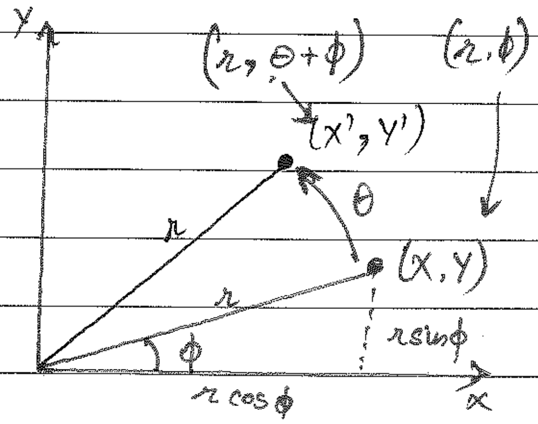
$$\underset{3DH}{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}} = \underset{4\times 4}{\begin{bmatrix} S_x & & & 0 \\ & S_y & & 0 \\ & & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{3DH}{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}$$

Block Notation:

$$\underline{P'} = \left[\begin{array}{c|c} \overset{3\times 3}{S} & \overset{3\times 1}{0} \\ \hline \underset{1\times 3}{0^T} & \underset{1\times 1}{1} \end{array}\right] \underline{P}$$

* Rotation 2D





- we represent using polar coordinates, (r, ϕ) instead of (x, y).

Relation b/w (x, y) & (r, ϕ):

$$X = r\cos\phi$$
$$Y = r\sin\phi$$

$$X' = r\cos(\phi+\theta)$$
$$Y' = r\sin(\phi+\theta)$$

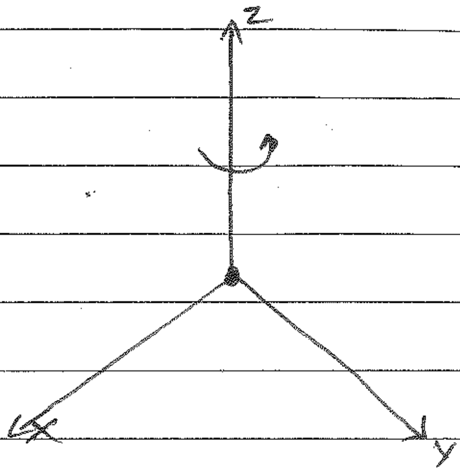
$$X' = \overbrace{r\cos\phi}^{x}\cos\theta - \overbrace{r\sin\phi}^{y}\sin\theta$$
$$Y' = \underbrace{r\cos\phi}_{x}\sin\theta + \underbrace{r\sin\phi}_{y}\cos\theta$$

$$X' = X\cos\theta - Y\sin\theta$$
$$Y' = X\sin\theta - Y\cos\theta$$

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$ } 2D rotation about origin.

* 3D Rotation

- 3D Rotation about z



$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}}_{3DH} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4\times4} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{3DH}$$

Block Notation :

$$\underline{p}' = \begin{array}{cc} 3\times3 & 3\times1 \\ \left[\begin{array}{c|c} r & 0 \\ \hline 0^T & 1 \end{array}\right] \\ 1\times3 & 1\times1 \end{array} \underline{P} \iff \underbrace{\underline{p}'}_{4\times1} = \underbrace{R_z(\theta)}_{4\times4} \underbrace{\underline{P}}_{4\times1}$$

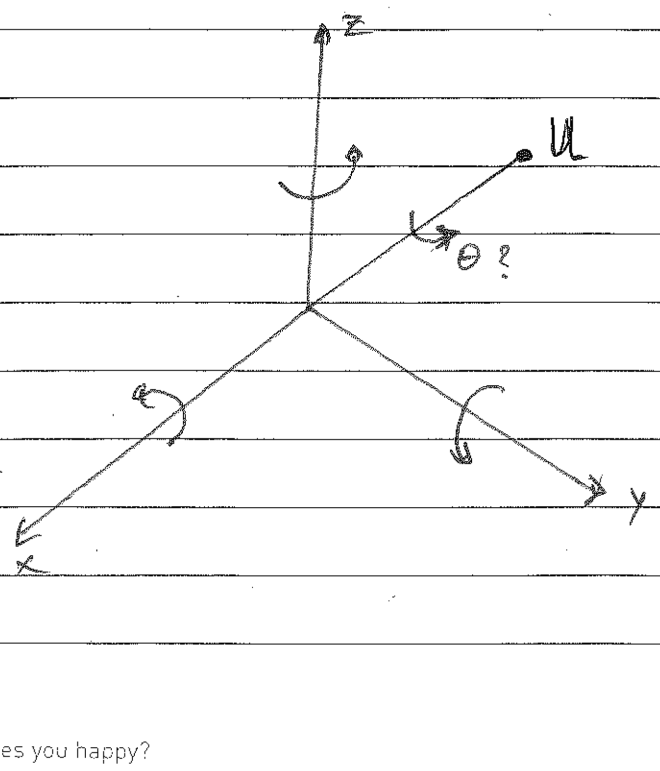
Date

- 3D Rotation about x –

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- 3D Rotation about y –

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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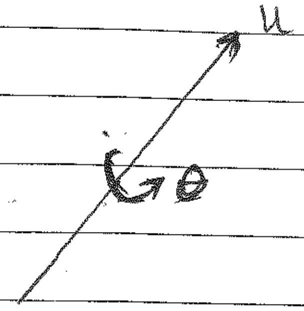
* Aibitary 3D Rotation:



Rodrigues Formula

$$\underset{3\times3}{r} = \underset{3\times3}{I} + \sin\theta \, \underset{3\times3}{Q} + (1-\cos\theta)\, \underset{3\times3}{Q^2}$$

gives the Block for rotation matrix

$$p' = \left[\begin{array}{c|c} r & 0 \\ \hline 0^T & 1 \end{array}\right] p$$

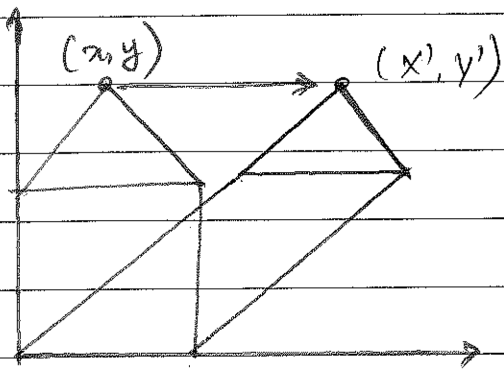
where,

$$Q = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

Skew symmetric matrix

Date

* 2D Skew (Shear)



$y' = y$

$x' = x + S_x y$

Shear factor

$$\underbrace{\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}}_{2DH} = \begin{bmatrix} 1 & S_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{2DH} \Bigg\} \text{ Shear in X}$$

Date

* Apply transformations

combined transformation Matrix.

$\underline{p}' = M\,\underline{P}$

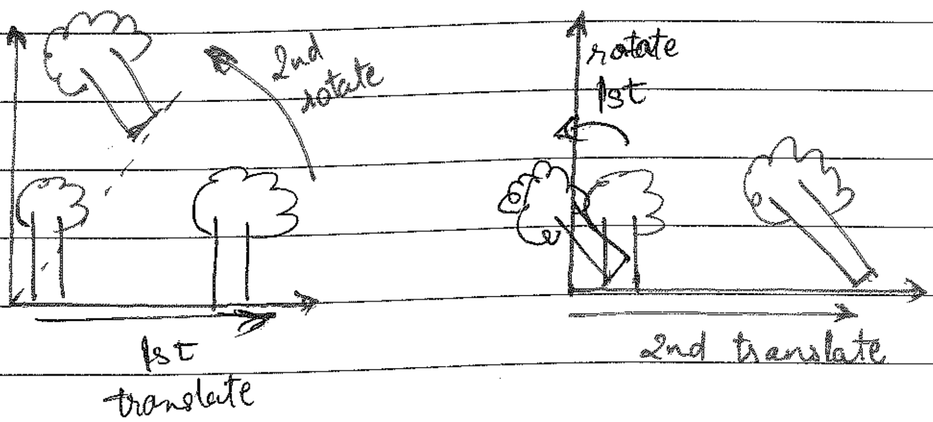
The order of matrix multiplication matter since $RT \neq TR$

$\underline{p}' = RT\underline{P}$

first

Second

why the order matters?



$\underline{p}' = RT\underline{P}$ $\qquad$ $\underline{p}' = TR\underline{P}$

What makes you happy?
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Date

* Combining transformations

- How can we rotate about an arbitory point?

Sol^n Trick is to translate the point to origin, then rotate, then canckl the translation.

$$R_{p,u}(\theta) = T(P)\ R_u(\theta)\ T(-P)$$

Rotate about point p using direction u & angle θ.

- How can we ~~trans~~ scale an image from a point that's not at origin.

Sol^n Trick is to move / translate point to origin, then scale, then cancel the translation.

$$S_p(S_x, S_y, S_z) = T(P)\ S(S_{x_1}, S_{x_2}, S_{x_3})\ T(-P)$$

Date

* Inverse Transformation

What if we want to undo the transformations.

- Invert matrices.

$$\underline{p}' = M\underline{p} \Rightarrow \underline{p} = M^{-1}\underline{p}'$$

transformation   Inverse transformation

- Invert a sequence of transformations.

$$(M_1 M_2 M_3 M_4)^{-1} = M_4^{-1} M_3^{-1} M_2^{-1} M_1^{-1}$$

Example:

$$\underline{p}' = TR\underline{p} \Rightarrow \underline{p} = (TR)^{-1}\underline{p}'$$

$$\Rightarrow \underline{p} = R^{-1}T^{-1}\underline{p}'$$

* Special cases

$$T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z)$$

$$S^{-1}(s_x, s_y, s_z) = S(1/s_x, 1/s_y, 1/s_z)$$

$$R_u^{-1}(\theta) = R_u(-\theta) = R_u^T(\theta)$$

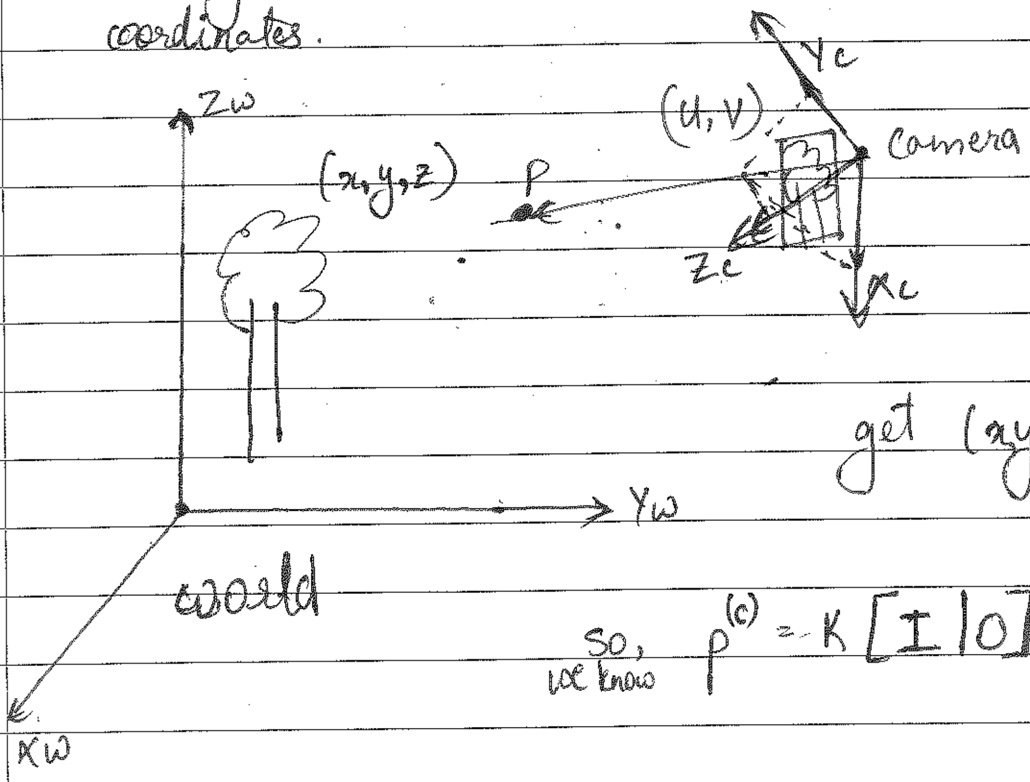
Date

Lecture 5

☆ TRANSFORMATION B/W COORDINATE SYS.

Q. Transformation $w \rightarrow C$

A. Align the camera b/w camera & world coordinates.



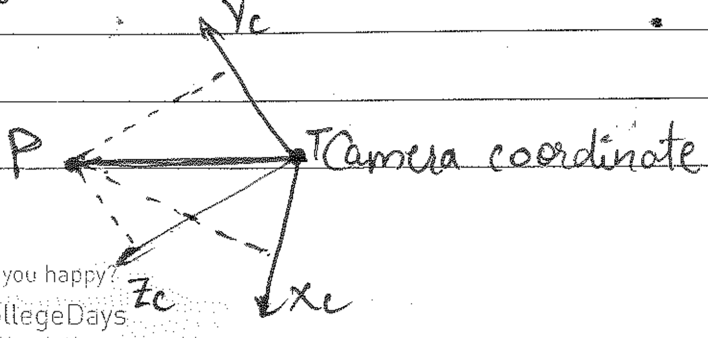
get $(x,y,z) \rightarrow (u,v)$

So, we know $p^{(c)} = K[I|0]\,\underline{P}^{(c)}$

→ Want: $p^{(i)} = M\,\underline{P}^{(w)}$

∗ Assume the camera is rotated by R and translated by T wrt world.

Find coordinates of point P in camera system
⇒ project $(P-T)$ onto $\hat{X}_c, \hat{Y}_c, \hat{Z}_c$.



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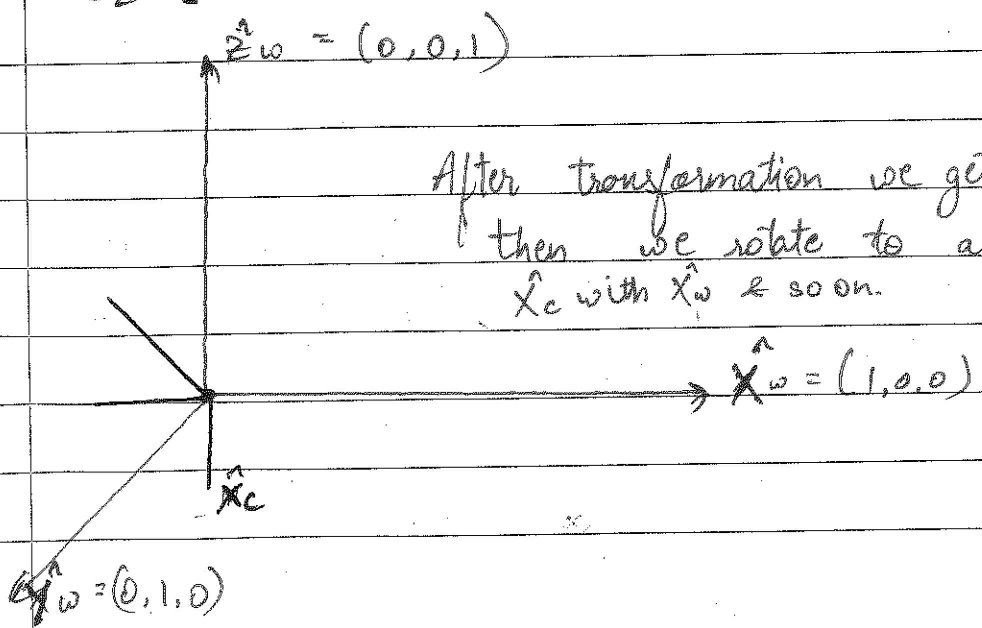
camera coordinates

$$x' = (P-T) \cdot \hat{X}_c = \underbrace{\hat{X}_c^T}_{\text{world}} (P-T)$$
$$y' = (P-T) \cdot \hat{Y}_c = \hat{Y}_c^T (P-T)$$
$$z' = (P-T) \cdot \hat{Z}_c = \hat{Z}_c^T (P-T)$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{X}_c^T \\ \hat{Y}_c^T \\ \hat{Z}_c^T \end{bmatrix}}_{\text{Rotate}} \underbrace{(P-T)}_{\text{translate to origin}}$$

Align the camera w/ the world. (cancel translate, rotate)

$\begin{bmatrix} \hat{X}_c^T \\ \hat{Y}_c^T \\ \hat{Z}_c^T \end{bmatrix}$ is rotation matrix because its ~~rotation~~ a matrix orthogonal.



After transformation we get ~~to~~ this. then we rotate to align camera $\hat{X}_c$ with $\hat{X}_w$ & so on.

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Thus,

$$R^T = \begin{bmatrix} \hat{X_c}^T \\ \hat{Y_c}^T \\ \hat{Z_c}^T \end{bmatrix}$$

$$p' = R^T(P - T)$$

First invest translation

Then invest rotate.

R^T align camera with world,

$$R^T \hat{X_c} = \begin{bmatrix} \hat{X_c}^T \cdot \hat{X_c} \\ \hat{Y_c}^T \hat{X_c} \\ \hat{Z_c}^T \hat{X_c} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = X_w$$

* **Conclusion:** The transformation between world and camera is obtained by aligning ~~the~~ the camera with world.

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* General camera model (for perspective projection)

So far: projection equations relates 3D points to 2D projections, all in the same coord. system.

Want: relate 3D point in world coord system to 2D points in image coordinates (pixels).

we have: $p^{(c)} = K[I|0] \underline{P}^{(c)}$

we want: $\underset{2DH}{p^{(i)}} = \underset{3\times4}{\underbrace{M_{i\leftarrow w}}} \underset{3DH}{\underline{P}^{(w)}}$

Projection

we can get this by,

$\overset{2DH}{p^{(i)}} = \underset{3\times3}{\underbrace{M_{i\leftarrow c}}} \overset{2DH}{p^{(c)}}$

So, $p^{(i)} = M_{i\leftarrow c} \overbrace{K[I|0]}^{3\times4} \overset{3DH}{\underline{P}^{(c)}}$

then, $p^{(i)} = M_{i\leftarrow c}\ K[I|0]\ M_{c\leftarrow w}\ \underline{P}^{(w)}$ 3DH

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Date

? So what is $M_{c \leftarrow w}$,

it is to invert translation and rotation on ~~camera~~ world cordinates.

$$M_{c \leftarrow w} = \tilde{R}^{-1} \tilde{T}^{-1}$$

$$= \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{c|c} I & T \\ \hline 0 & 1 \end{array}\right]^{-1}$$

$$= \left[\begin{array}{c|c} R^T & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & -T \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R^T & -R^T T \\ \hline 0 & 1 \end{array}\right] \equiv \left[\begin{array}{c|c} R^* & T^* \\ \hline 0 & 1 \end{array}\right]$$

$$\underline{\underline{M_{c \leftarrow w}}} = \left[\begin{array}{c|c} R^* & T^* \\ 0 & 1 \end{array}\right]$$

Note: R^*, T^*: rotation of world w.r.t camera.
R, T : R & T ~~wrt~~ of camera w.r.t world.

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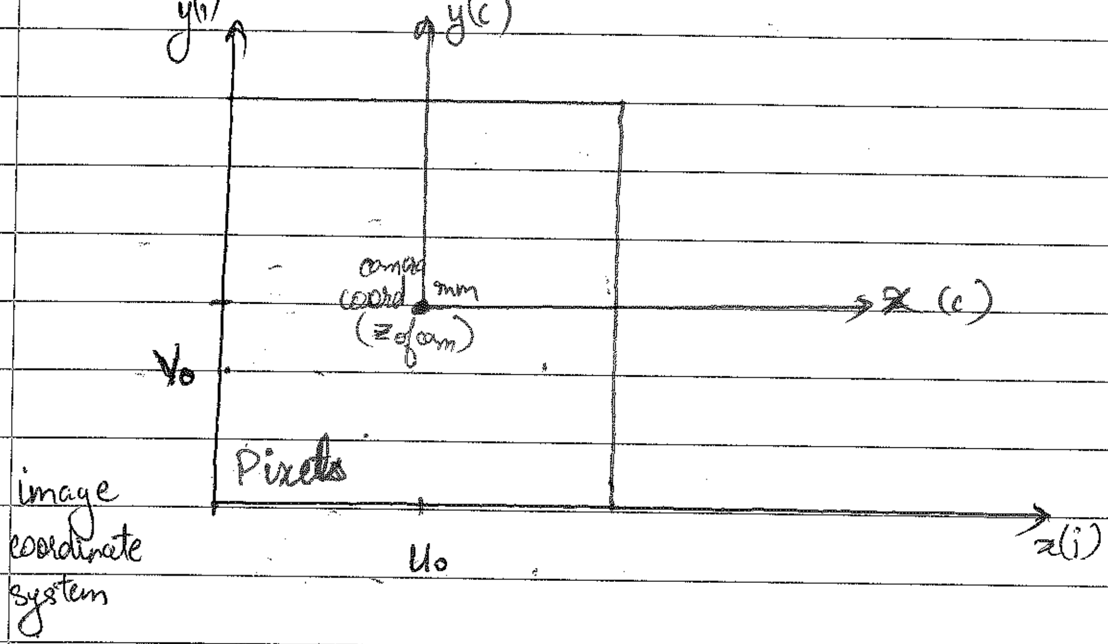
? So what is $M_{i \leftarrow c}$

first we see what $M_{c \leftarrow i}$ is:



we need to move camera origin /middle to image origin. [u_0, v_0 → translate to optical center (pixels)]

k_u, k_v = scale [pixels / mm]

Align camera with image:

$$\underline{M_{c \leftarrow i}} = \begin{bmatrix} 1/k_u & & \\ & 1/k_v & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -u_0 \\ & 1 & -v_0 \\ & & 1 \end{bmatrix}$$

(2) inverse scale (1) inverse translation

Date

$$M_{i \leftarrow c} = M_{c \leftarrow i}^{-1} \qquad (AB)^{-1} = B^{-1}A^{-1}$$

$$= \begin{bmatrix} 1 & & -u_0 \\ & 1 & -v_0 \\ & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1/k_u & & \\ & 1/k_v & \\ & & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & u_0 \\ & 1 & v_0 \\ & & 1 \end{bmatrix} \begin{bmatrix} k_u & & \\ & k_v & \\ & & 1 \end{bmatrix}$$

$$M_{i \leftarrow c} = \begin{bmatrix} k_u & & u_0 \\ & k_v & v_0 \\ & & 1 \end{bmatrix}$$

So the materix becomes,

$$p^{(i)} = M_{i \leftarrow c} \; K [I|0] \; M_{c \leftarrow w} \; \underline{P}^{(w)}$$

$$= \begin{bmatrix} k_u & & u_0 \\ & k_v & v_0 \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} [I|0] \left[\begin{array}{c|c} R^* & T^* \\ \hline 0 & 1 \end{array}\right] \underline{P}^{(w)}$$

$$= \begin{bmatrix} f k_u & & u_0 \\ & f k_v & v_0 \\ & & 1 \end{bmatrix} [R^* | T^*] \; \underline{P}^{(w)}$$

$$= \underbrace{K^*}_{\text{Internal parameters}} \underbrace{[R^* | T^*]}_{\text{external parameter}} \underline{P}^{(w)}$$

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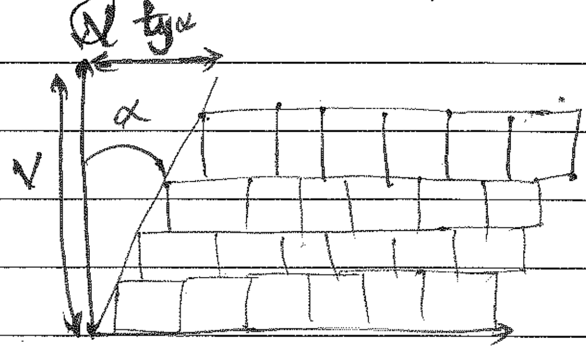
$$K^* = \begin{bmatrix} f k_u & & u_0 \\ & f k_v & v_0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & & u_0 \\ & \alpha_B & v_0 \\ & & 1 \end{bmatrix}$$

focal len in pixels

☆ Adding Skew Parameters



$$M_{skew\,2D} = \begin{bmatrix} 1 & tg\alpha & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$M_{i \leftarrow e} = \begin{bmatrix} \alpha_u & & u_0 \\ & \alpha_v & v_0 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & tg\alpha & \\ & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & \alpha_u tg\alpha & u_0 \\ & \alpha_v & v_0 \\ & & 1 \end{bmatrix}$$

Date

* Parameter Summary:

notation	meaning	units
K_u K_v	scale in x & y pixels to mm	pixels/mm
u_0 v_0	translation of principal axis to O	pixels
f	focal length	mm
α_u, α_v	focal length (fK_u & fK_v)	pixels

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* Radial lens distortion:

$$p^{(i)} = \begin{bmatrix} 1/\lambda & & \\ & 1/\lambda & \\ & & 1 \end{bmatrix} K^* [R^* | I^*] \underline{p^{(w)}}$$

linear distortion coeff

$$\lambda = 1 + K_1 d + K_2 d^2$$

quadratic distortion coefficient

d = distance from center

* Weak perspective camera:

where foreshotting doesn't happen.

$$M_\infty = \begin{bmatrix} & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

depth perspective camera is correct when depth variation in the scene is small compared with depth from camera.

depth variation

$$e = \left| M_\infty \overset{2DH}{\underline{P}} - M \overset{2DH}{\underline{P}} \right| = \frac{\Delta}{d_0} \left(\underline{MP - P_0} \right)$$

distance from center

distance from camera

Photometric image

* Affine camera :

$$\text{Matrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

computational model

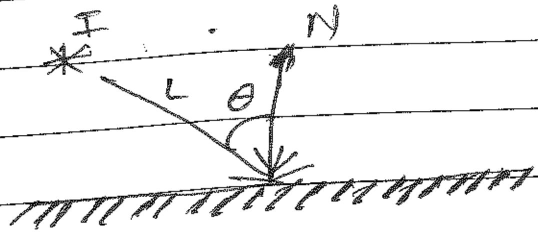
Date

PHOTOMETRIC Image Formation (not core-review)

* Relate light source intensity to reflected light intensity
* Lambertian surface → diffuse reflection

$\cos\theta = N \cdot L$

$N \cdot L < 0 \Rightarrow$ surface not visible



surface albedo $\in [0,1]$ (ref. coef)

$I_{ref} = I \cdot \rho \cdot \cos\theta = I \cdot \rho \cdot (N \cdot L)$

Intensity of reflection

Intensity of source

* Radiosity Model

Relate light in the scene (surface radiance) to light in the image (image irradiance)

$L(P)$ = power of light per unit area reflected from surface (surface radiance)

$E(P)$ = power of light per unit area reflected at the image. (image radiance).

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$$E(p) = L(p)\,\frac{\pi}{4}\left(\frac{d}{f}\right)^2 (\cos\alpha)^4$$

light at img

light at surface

diameter of lens

focal length

angle b/w principle axis & surface normal.

$d\uparrow \Rightarrow E(p)\uparrow$

$f\uparrow \Rightarrow E(P)\downarrow$

$\alpha\uparrow \Rightarrow E(P)\downarrow$

* Color (info topic)

visible light : 380 − 720 nm

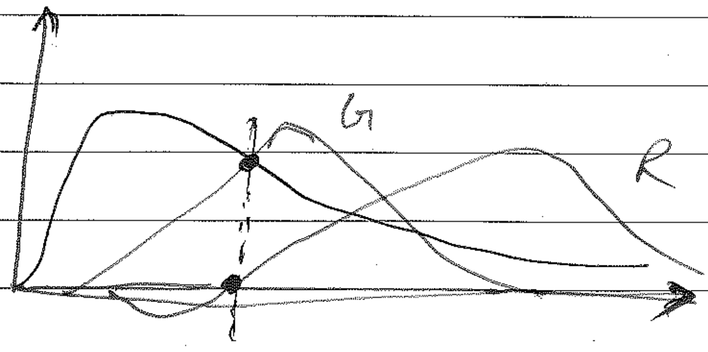
higher freq

lower freq

Human vision : RGB receptors

* YIQ model :

Y is grayscale & important



Lecture 6

* NTSC transmission system :

Y = luminance $\qquad Y = 0.299 R_N + 0.587 G_N + 0.114 B_N$

I, Q = chrominance $\qquad I =$

$Q =$

Euclidean distance in RGB space does not correspond to human perception.

Whereas euclidean distance in LAB space does correspond to perception.

RGB → $L^* a^* b^*$ → compare colors

Ex : $|C_1 - R| < |C_2 - R|$

$\Downarrow$

C_1 is more similar to R.

? Aliasing ?

The weird pattern seen in densely packed images.

What makes you happy?
#HappyCollegeDays

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☆ Noise &

Noise and filtering:

- sampling noise (aliasing)
- color quantification
- Noise model:

$$I \rightarrow \boxed{H} \rightarrow \overset{n\ (\text{gaussian})}{\overset{\downarrow}{\Sigma}} \rightarrow \hat{I}$$

Signal to Noise ratio

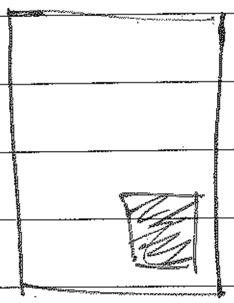
$$SNR = \frac{E_s}{E_n} = \frac{\sigma_s^2}{\sigma_n^2} = \frac{\frac{1}{n}\sum_{i,j}\left(I(i,j) - \bar{I}\right)^2}{\sigma_n^2}$$

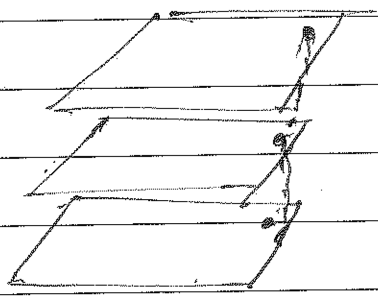
variance

σ_n^2 = variance for multiple frames of a static scene

OR...

variance in a uniform image region





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$$SNR[db] = 10\log_{10}\frac{E_s}{E_n}$$

10 db ⇒ E_s is 10 times bigger than E_n

Noise filtering

- Remove noise with smoothing
- smooth using convolution

convolution

$$I_A(i,j) = I(i,j) * A(i,j)$$

$$= \sum_{h=-\frac{m}{2}}^{m/2} \sum_{K=-\frac{n}{2}}^{n/2} A(h,K)\, I(i-h, j-K)$$

Convolution properties:

$$f*g = g*f$$

$$f*(g*h) = (f*g)*h$$

$$f*(g+h) = (f*g)+(f*h)$$

$$\frac{d}{dx}(f*g) = \frac{d}{dx}f*g = f*\frac{d}{dx}g$$

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Date

for boundries,

(who does this) — mirror/

① zero padding or ② Replicate or ③ ignore

0	0	0	0
0			
0			

5	5	6	7
5	5	6	7
5	5	8	9
8	8	5	7

IP = 1000 x 1000

OP = 998 x 998

Remember:- Store ~~[illegible]~~ result in new image.
- Store result in float ~~[illegible]~~ array.

Smoothing using convolution

- convolution is a linear filter
- simple smoothing filter

$$\frac{1}{9} * \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Low pass filter interpretation

smoothing = remove high frequencies in image

Date

~~CS 550 - Lec 2~~

Convolution applications:

Blurring $\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sharpening $\frac{1}{9}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 18 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

vertical edge detection $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

Horizontal edge detection $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}$