HW 12

Q1

As an exercise, let

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Answer the following questions.

(a) Using $\mathbb{R}^3_+$, is it true that $x \succeq_{\mathbb{R}^3_+} y$? Explain why. Plot $x - y$ and $\mathbb{R}^3_+$.

(b) Using $\mathbb{L}^3$, is it true that $x \succeq_{\mathbb{L}^3} y$? Explain why. Plot $x - y$ and $\mathbb{L}^3$.

(c) Now define three matrices

$$A = \begin{bmatrix} -6 & 7 & 8 \\ 7 & -8 & 9 \\ 8 & 9 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} -10 & 9 & 8 \\ 9 & -8 & 7 \\ 8 & 7 & -12 \end{bmatrix}.$$

Is it true that $A \preceq_{\mathbb{S}^3_+} B$? Is it true that $A \succeq_{\mathbb{S}^3_+} C$? Explain why. Remember A real symmetric matrix is positive semidefinite if and only if all of its eigenvalues are nonnegative.

Solution Q1.2

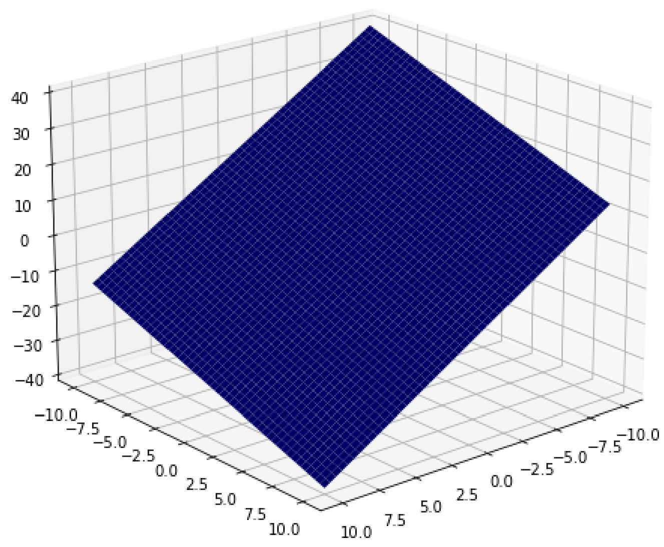
False, since the first component is less than zero:

$$x - y = \begin{bmatrix} 1-3 \\ 2-2 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \tag{1}$$

In [2]:

```
import matplotlib.pyplot as plt
import numpy as np

fig_1, ax_1 = plt.subplots(subplot_kw={"projection": "3d"}, figsize=(8, 8))
fig_1.patch.set_facecolor("xkcd:white")

d = np.linspace(-10, 10, 1000)
x, y = np.meshgrid(d, d)

z1 = (x + 2 * y) / 3
z2 = (3 * x + 2 * y)


ax_1.plot_surface(x, y, z1-z2, color="blue", vmax=20)
ax_1.view_init(20, 50)
ax_1.set_title("Q1.a")

plt.show()
```

Q1.a



Solution Q1.b

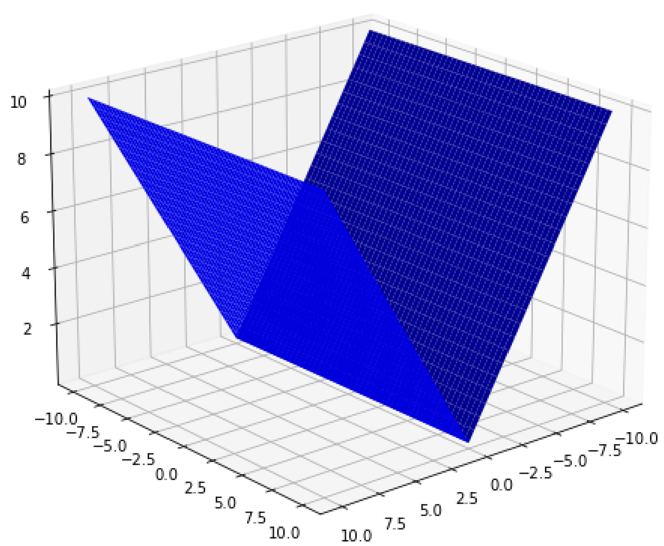
True. Using the Second Order Cone:

$$\begin{gathered}\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \leq x_3-y_3 \\ \sqrt{(-2)^2+(0)^2} \leq 2 \\ 2 \leq 2\end{gathered} \tag{2}$$

In [3]:

```
fig_2, ax_2 = plt.subplots(subplot_kw={"projection": "3d"}, figsize=(8, 8))
fig_2.patch.set_facecolor("xkcd:white")

z = np.sqrt((-2 * x) ** 2) / 2

ax_2.plot_surface(x, y, z, color="blue")
ax_2.view_init(20, 50)
ax_2.set_title("Q1.b")

plt.show()
```

Q1.b



Solution Q1.c

- $A \preceq_{\mathbb{S}^3_+} B$?

$$B - A \succeq_{\mathbb{S}^3_+} 0$$
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} -6 & 7 & 8 \\ 7 & -8 & 9 \\ 8 & 9 & -10 \end{bmatrix} \succeq_{\mathbb{S}^3_+} 0$$
$$\begin{bmatrix} 6 & -6 & -6 \\ -6 & 10 & -6 \\ -6 & -6 & 14 \end{bmatrix} \succeq_{\mathbb{S}^3_+} 0$$

Next, let's calculate the eigen values from the matrix above

In [4]:
```
import array_to_latex as a2l
from IPython.display import display, Markdown

A_1 = np.array([
    [6, -6, -6],
    [-6, 10, -6],
    [-6, -6, 14]
])

eig_vals_1 = np.linalg.eigvals(A_1)
display(Markdown(a2l.to_ltx(eig_vals_1, print_out=False)))
```

$$\begin{bmatrix} -2.58 & 14.00 & 18.58 \end{bmatrix}$$

Since not all the eigenvalues are positive, the matrix is not positive-semidefinite. Hence, $A \preceq_{\mathbb{S}^3_+} B$ is **False**

- $A \succeq_{\mathbb{S}^3_+} C$?

$$A - C \succeq_{\mathbb{S}^3_+} 0$$
$$\begin{bmatrix} -6 & 7 & 8 \\ 7 & -8 & 9 \\ 8 & 9 & -10 \end{bmatrix} - \begin{bmatrix} -10 & 9 & 8 \\ 9 & -8 & 7 \\ 8 & 7 & -12 \end{bmatrix} \succeq_{\mathbb{S}^3_+} 0$$
$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} \succeq_{\mathbb{S}^3_+} 0$$

In [5]:
```
A_2 = np.array([
    [4, -2, 0],
    [-2, 0, 2],
    [0, 2, 2]
])

eig_vals_2 = np.linalg.eigvals(A_2)
display(Markdown(a2l.to_ltx(eig_vals_2, print_out=False)))
```

$$\begin{bmatrix} -1.76 & 5.06 & 2.69 \end{bmatrix}$$

Since not all the eigenvalues are positive, the matrix is not positive-semidefinite. Hence, $A \succeq_{\mathbb{S}^3_+} C$ is **False**

Solution Q2

$\min \max \|u - w\|$

Using the epigraph reformulation we have:

$\min t$ s.t $\max \|u - w\| \leq t$

Since the maximum of the SOCs is the same as testing the inequality for all i, we have:

$\min t$ s.t $\|u_i - w\| \leq t, \quad \forall_i = 1, m$

u is the matrix of the location of the villages, while w is the location of the fire station. The objective is to minimize t with the defined constraints.

In [43]:
```
import cvxpy as cp

n = 20  # Number of villages
```

```
u = np.random.rand(n, 2) * 10  # Creation of villages matrix
w = cp.Variable(2)  # The fire station variable
t = cp.Variable()  # The maximum distance to minimize from the epigraph reformulation

soc_constraints = [
      cp.SOC(t, u[i] - w) for i in range(n)
]

prob = cp.Problem(cp.Minimize(t),
                  soc_constraints + [t >= 0])
prob.solve()

max_dist = round(prob.value, 2)

print("The max distance to be minimized from the villages to the fire station is:", max_dist)
print("The fire station is located at point:", w.value)
```

```
The max distance to be minimized from the villages to the fire station is: 5.88
The fire station is located at point: [5.31286705 4.70636214]
```

In [44]:

```
plt.scatter(u[:, 0], u[:, 1], label="villages")
plt.scatter(w.value[0], w.value[1], color="r", label="fire station", marker="^")
plt.title("Villages and Fire Station Location")
plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.show()
```