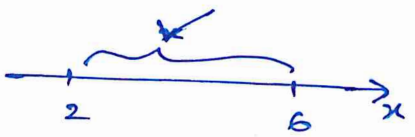
MATHEMATICAL LOGIC

⊛ Why study logic?

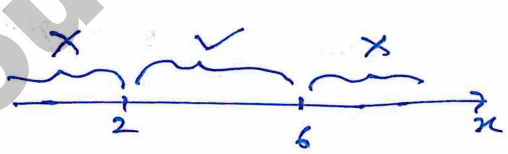
We will learn about Propositional logic and Predicate logic.

Suppose person P_1 and P_2 write the following code

P_1
```
if ((x > 2) && (x < 6))
{
    ≡ do something
}
```



P_2
```
if (! ((x <= 2) || (x >= 6)))
{
    ≡ do something
}
```



As a software engineer, people do code reviews and software testers would test if the code is behaving in the expected manner. So verifying and validating the code is a part of software engineering and in general a part of programming.

Logic gives us a bunch of tools which consists of symbols, notations, properties and laws to verify or solve the problem if given two programs are equivalent or not. When we have large s/w like Windows O.S or Linux O.S, these tools can be utilized to verify and validate the code.

Suppose $x > 2$ and x is an integer. Given 2 statements:

S_1: if x is prime then x is odd

S_2: if x is not odd then x is not prime

With the help of logic we can also verify, if two given statements are equivalent or not.

For $n > 2$

S_1: for all $x \in \{2, 3, \dots n-1\}$ if $\underline{x \text{ doesnot } n}$, $x \nmid n$

then n is prime.

S_2: If there exists an element $x \in \{2, 3, \dots, n-1\}$ such that $\underbrace{x \mid n}_{x \text{ divides } n}$, then n is not prime.

x divides n
$2 \mid 6$
$2 \nmid 5$
$3 \mid 6$
$3 \nmid 8$

There is an area in computer science and engineering called formal verification. Formal verification is an extension of Formal logic (that we will learn). Formal verification is used by electronic engineers to verify circuit designs. It is also used in software code verification. It is also used in computer networks design to verify the behaviour of the network.

⊛ Propositional Logic – I :

Propositional Logic / calculus / 0^{th} order logic :

Proposition : It is a declarative (declares a fact) statement that is either true or false but not both.

Propositions are often represented as $p, q, r, s, \dots$

E.g.

p : New Delhi is the capital of India — True

q : $1+2=3$ — True

r : $4+6=9$ — False

s : what's the time? — Not a proposition.

P_1 : $x+1=2 \longrightarrow \left.\begin{array}{l} T \text{ if } x=1 \\ F \text{ if } x \neq 1 \end{array}\right\}$ Not a proposition

P_2 : there exists $x \in \mathbb{R}$, such that $x+1=2$

↳ True

P_3 : He is tall — Not a proposition
(He: pronoun)

P_4 : $\underbrace{\text{This sentence is false}}_{\text{This statement is called a Liar Paradox in Logic}}$ — a proposition

Note:

Most of these concepts that we learn in propositional logic have been first developed formally by Aristotle around 2300 years ago.

Logical operators on propositions:

1. Negation (unary operation): Negation is denoted by $\{\neg p, \tilde{p}, p', \bar{p}\}$
   (operation on 1 proposition)

p: Today is friday

$\neg p$: Today is not friday

It's not friday today

It's not the case that today is friday

Truth Table

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

$\boxed{\neg(\neg p) = p}$

2. Conjunction: $\rightarrow p \wedge q$, $p \cdot q$, pq

p: Today is friday

q: It is raining today

$p \wedge q$: Today is friday and it's raining today

$\wedge$ is a logical AND. In C programming, logical AND is represented as &&. In Digital circuit design, we have learnt about AND GATE, which operates the same way.

Conjunction is a binary operator as it operates on 2 propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

③ Disjunction : $p \vee q$, $p + q$, logical OR.

In C programming → ||

In DLC → OR GATE

$p \vee q$: $\underbrace{\text{Today is friday}}_{p}$ or $\underbrace{\text{it's raining today}}_{q}$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

④ Exclusive OR : $\oplus$

In DLC → XOR Gate

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q)$ ← using $\neg, \vee, \wedge$

$= (p \vee q) \wedge (\neg p \vee \neg q)$

Proof:

p	q	$p \oplus q$	$(p \wedge \neg q)$	$(\neg p \wedge q)$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
T	T	F	$T \wedge F = F$	$F \wedge T = F$	F
T	F	T	$T \wedge T = T$	$F \wedge F = F$	T
F	T	T	$F \wedge F = F$	$T \wedge T = T$	T
F	F	F	$F \wedge T = F$	$T \wedge F = F$	F

$\cong$

⑤ Conditional operation / Implication:

$p \rightarrow q$

- p: hypothesis, premise, antecedent
- q: conclusion, consequence

- if p then q
- if p is true, then q is true
- p implies q
- p only if q
- q if p
- q when p
- q follows from p
- q whenever p
- q unless $\neg p$
- A sufficient condition for q is p.
- A necessary condition for p is q
- q is necessary for p.
- if p, q

Eg: if $\underbrace{\text{you get 100 marks}}_{p}$, then $\underbrace{\text{you will get A grade}}_{q}$

$\longrightarrow$

you get 100 marks $\longrightarrow$ you get A grade

p	q	$p \to q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$$p \to q = \neg p \vee q$$

E.g. p: X learns Discrete Maths

q: X gets a good job.

$p \to q$: if X learns Discrete maths, then X gets a good job.

$(p \to q)$ and $(q \to p)$ are not equivalent.

$p \to q$: X gets a good job unless X doesnot learn Discrete Maths.

Converse: $q \to p$

Contrapositive: $\neg q \to \neg p$

Inverse: $\neg p \to \neg q$

E.g. $\underbrace{\text{I will stay}}_{p}$ $\xrightarrow{\text{only if}}$ $\underbrace{\text{you go}}_{q}$

I will stay $\to$ you go.

Converse: $q \to p$

If you go then I will stay.

6. Biconditional / iff :

$$p \leftrightarrow q = (p \to q) \land (q \to p)$$

p	q	$p \leftrightarrow q$	p	q	$\neg(p \oplus q)$
T	T	T	F		T
T	F	F	T		F
F	T	F	T		F
F	F	T	F		T

$p \leftrightarrow q = \neg(p \oplus q)$

Eg: $\underbrace{\text{you can fly}}_{p}$ iff $\underbrace{\text{you have a ticket}}_{q}$.

$p \leftrightarrow q$

$p \leftrightarrow q$:
- p if and if q.
- p is necessary & sufficient for q
- if p then q and and conversely
- p iff q
- if p then q and if q then p
- q unless $\neg p$ and p unless $\neg q$

Note :

Logical Equivalence : $\Leftrightarrow$, $\equiv$

$p \Leftrightarrow q$ means ~~that~~ $p \leftrightarrow q$ is a tautology.

Logical implication : $\Rightarrow$

$p \Rightarrow q$ means that $p \to q$ is a tautology.

Note! We will see the definition of tautology soon.

(7) NAND & NOR $\rightarrow (p \downarrow q) = \neg(p \vee q)$

$(p \uparrow q) = \neg(p \wedge q)$

p	q	$(p \uparrow q)$	$(p \downarrow q)$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

(*) The Operator precedence in Logic :

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

Eg., $\neg p \vee q$

$= (\neg p) \vee q$

Eg. $\neg p \wedge q \vee p$

$= ((\neg p) \wedge q) \vee p$

Eg. $\neg p \vee q \rightarrow s$

$((\neg p) \vee q) \rightarrow s$

⊛ Propositional Logic - II

⊛ Logic and Algebraic structures :

We have seen Lattice in algebraic structures, which is a 6-tuple $(A, \vee, \wedge, \sim, 1, 0)$

A : set
$\vee$: LUB
$\wedge$: GLB
$\sim$: complement
1 : greatest element
0 : least element

A lattice where every pair of element has an LUB, GLB, every element has a complement and if the greatest and the least elements exists, such a lattice is called a Boolean Algebra.

From set theory perspective,

$(S, \cup, \cap, A^c, U, \phi)$

S : set of all sets
$\cup$: union, LUB
$\cap$: intersection, GLB
A^c : complement of element A
U : greatest element (Universal set)
ϕ : least element (Null set)

From logic perspective,
$\vee$ is OR
$\vee$ is AND
1 is True
0 is False

The smallest Boolean Algebra, $A = \{0, 1\}$ (0 : False, 1 : True),

$(A, \vee, \wedge, \sim, 1, 0)$, it contains only 2 elements the greatest and the least element.

A propositional logic is nothing but the boolean algebra that we have studied in the algebraic structures.

$\therefore$ Propositional logic is distributed, complemented and bounded lattice.

Now we will study the properties of propositional logic using the properties of Boolean Algebra.

Properties:

① Closure: $\forall x, y \in A$, $(x \vee y) \in A$

$x = T$
$y = F$

$x \vee y = T$

Note: The smallest Boolean Algebra,
$A = \{F, T\}$.

Again, $x \wedge y \in A$

$\bar{x} = \neg x \in A$

These are propositional logic perspective.

Now, from set theory perspective,

$\forall A, B \in S$,

$A \cup B \in S$
$A \cap B \in S$
$A^c = \bar{A} \in S$

Note: Algebraic structures connect the concepts in sets and propositional logic via boolean algebra (which is a special type of lattice.

② Commutative:

$x \vee y = y \vee x$
$x \wedge y = y \wedge x$
} propositional logic perspective

$A \cap B = B \cap A$
$A \cup B = B \cup A$
} set theoretic view.

③ Distributive property:

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

④ Identity:

$x \vee 0 = x = x \vee F$

$x \wedge 1 = x = x \wedge T$

$A \cup \phi = A$

$A \cap \phi = A$

⑤ Complement

$(\exists \ \neg x \ \forall x)$

There exists complement of x for all x.

$x \vee \neg x = T$

$x \wedge \neg x = F$

$A \cup \bar{A} = U$

$A \cap \bar{A} = \phi$

⑥ Idempotent law :

$x \wedge x = x$

$x \vee x = x$

$A \cup A = A$

$A \cap A = A$

Note: In Algebraic structures, $(x * x = x)$ where $*$ is any operator.

⑦ Absorption Law :

$x \wedge (x \vee y) = x$

$x \vee (x \wedge y) = x$

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

⑧ De Morgan's Law :

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

$\neg(p \vee q) = \neg p \wedge \neg q$

$\neg(p \wedge q) = \neg p \vee \neg q$

⑨ Law of double complement:

$$\neg(\neg p) = p$$

$$(A')' = A$$

⑩ Domination Law:

$$p \vee 1 = 1$$

$$p \wedge 0 = 0$$

$A \cup U = U$, where U is the universal set.

$$A \cap \phi = \phi$$

(*) Tautology, Contradiction, Contingency & Commutative and Associative Operators

Tautology: Any proposition (simple or complex) is a tautology if it always True.

In case of complex proposition, if its Truth value is True for all possible values, then the given complex proposition is a tautology.

E.g.,

p	T	$p \vee T$
T	T	T
F	T	T

E.g., $s: ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

We need to prove if the given complex proposition(s) is a tautology or not.

p	q	r	s
			T
			⋮

But, to find if all values in s are True for all possibilities of p, q and r, is a very cumbersome process.

Instead let us simplify the given complex proposition (s).

$$\therefore \quad s: ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

$$\equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r)$$

Apply Demorgan's Law:

$$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg p \vee r)$$

$$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$$

$$\equiv \neg p \vee (p \wedge \neg q) \vee r \vee (q \wedge \neg r)$$

$$\equiv (\neg p \vee p) \wedge (\neg p \vee \neg q) \vee (r \vee q) \wedge (r \vee \neg r)$$

$$\equiv T \wedge (\neg p \vee \underline{q \vee \neg q} \vee r) \wedge T$$

$$\equiv T \wedge (\neg p \vee T \vee r) \wedge T$$

$$\equiv T \wedge T \wedge T$$

$$\equiv T$$

$\therefore$ complex proposition S is always True.
Therefore S is a tautology.

Note:

If we have n-variables, $P_1, P_2, \ldots P_n$.
Each of these variables can be True/False, ie., two possibilities.
Therefore the total number of rows that we can have is $2 * 2 * 2 * 2 * \ldots * n$ times

$= 2^n$.

If we have 2 variables, number of rows $= 2^2 = 4$

4 variables — 16 rows
3 variables — 8 rows.

	P_1	P_2	. . . . . .	P_n	f
1.					True/False
2.					T/F
3.					
⋮			⋮		⋮
2^n					True/ False

The number of proposition function f possible is $2 * 2 * 2 * 2 * \ldots * 2^n$ times

$= 2^{2^n}$

2^{2^n} is superexponential.

For $n=3$, $2^n = 8$, $2^{2^n} = 2^8 = 256$

Therefore, the truth table approach maynot be always feasible, as when n increases, number of rows increases exponentially.

It is usually better to use the properties and laws of propositional logic to prove the propositions.

Contradiction / fallacy / invalid :

If a proposition p is always false, then we call the proposition as contradiction / fallacy / invalid. In other words, if the truth table results in False for all combinations of propositional variables, then it is called a contradiction.

Contingency :

If a proposition is sometimes true and sometimes false.

Example : $(p \vee q)$ is a contingency.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

E.g. $s: ((p \to q) \land (q \to r)) \to (r \to p)$

$\equiv \neg((\neg p \lor q) \land (\neg q \lor r)) \lor (\neg r \lor p)$

$\equiv \neg(\neg p \lor q) \lor \neg(\neg q \lor r) \lor (\neg r \lor p)$

$\equiv (p \land \neg q) \lor (q \land \neg r) \lor (\neg r \lor p)$

$\equiv p \lor (p \lor \neg q) \lor \neg r \lor (q \lor \neg r)$

$\equiv p \lor \neg r$ (using absorption rule)

Therefore s is sometimes true and sometimes false.

Note: If an expression (propositional statement) is a tautology or a contingency, it is said to be satisfiable.

If an expression is a contradiction, it is said to be non-satisfiable as no combination of Truth values (T/F) assigments to the propositional variables, will result in a True for the whole expression.

Example:

$p \Rightarrow (q \land r) \equiv (p \Rightarrow q) \land (p \Rightarrow r)$

RHS. $(p \Rightarrow q) \land (p \Rightarrow r)$

$\equiv (\neg p \lor q) \land (\neg p \lor r)$

$\equiv \neg p \lor (q \land r) =$ LHS

$\therefore$ The given propositional statement is a tautology.

(Q) $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$

LHS.

$\neg p \vee (q \vee r)$

$(\neg p \vee q \vee r)$

RHS.

$(\neg p \vee q) \vee (\neg p \vee r)$

$(\neg p \vee q \vee r)$

$\therefore$ LHS = RHS

$\therefore$ The given propositional statement is also a tautology.

(Q) $(p \Leftrightarrow q) \wedge (p' \Leftrightarrow q)$

$\equiv (\neg p \vee q) \wedge (\neg q \vee p) \wedge (p \vee q) \wedge (\neg q \wedge \neg p)$

$\equiv (\neg p \vee (q \wedge \neg q)) \wedge (p \vee (\neg q \wedge q))$

$\equiv (\neg p) \wedge (p)$

$\equiv F$

which means the propositional statement is a contradiction.

Interesting properties of operators :

① $\wedge$, $\vee$, XOR , $\Leftrightarrow$ —— All these operators are commutative and associative.

$a \vee b \equiv b \vee a$ —— commutative

$a \vee (b \vee c) \equiv (a \vee b) \vee c$ —— associative.

② $\Rightarrow$ —— not commutative, not associative

$(p \Rightarrow q) \not\equiv (q \Rightarrow p)$

$\because (\neg p \vee q) \not\equiv (\neg q \vee p)$

$\therefore$ Not commutative.

③ NAND $(\uparrow)$, NOR $(\downarrow)$ } Commutative but not associative.

⊛ Functionally complete set, Normal Forms (PDNF, PCNF)

Functional completeness (FC):

A set F of boolean functions is functionally complete if we can express any boolean function using functions in set F.

E.g: $F: \{\wedge, \vee, \neg\}$: FC set

Any complex boolean function using XOR, NAND, NOR, ...
can be represented using an FC set.

Q. Why is the concept of functional completeness important?

If we are designing digital logic circuits, we will have logic gates, like AND gate, OR gate, NOT gate, which are actual physical electronic circuits.
What functional completeness tells us is that using AND, OR and NOT gate we can represent any boolean function we need.

E.g: $\{\wedge, \vee, \neg, \rightarrow, \Leftrightarrow, \uparrow, \downarrow\}$: FC set.

$\{\wedge, \vee\}$: Not FC set because we can't represent $\neg p$ given proposition p.

$\left.\begin{array}{l}\{\wedge, \neg\}\\ \{\vee, \neg\}\\ \{\rightarrow, \neg\}\\ \{\uparrow\}\\ \{\downarrow\}\end{array}\right\}$: FC and also minimally FC.

Note: Minimally FC set: These are the FC set for which there is no subset of F is FC.

E.g: F: $\{\wedge, \vee, \neg\}$ is not minimally FC because the subset $\{\wedge, \neg\}$ or $\{\vee, \neg\}$ are FC.

Note: $\{\leftrightarrow, \neg\}$ is not a FC set.

In Digital Logic circuit perspective, using NAND ($\uparrow$) /NOR ($\downarrow$) gates, we can construct any boolean function.

E.g. Let's show that $\{\uparrow\}$ is FC. Given proposition p,

$$p \uparrow p = \neg(p \wedge p) = \neg p$$

Given propositions p and q,

$$(p \uparrow p) \uparrow (q \uparrow q)$$

$$\equiv \neg(p \wedge p) \uparrow \neg(q \wedge q)$$

$$\equiv \neg(p) \uparrow \neg(q)$$

$$\equiv \neg(\neg p \wedge \neg q)$$

$\equiv p \vee q$ (using DeMorgan's law)

Similarly,

$$(p \uparrow q) \uparrow (p \uparrow q)$$

$$\equiv p \wedge q$$

Therefore using $\{\uparrow\}$ we are able to simulate $\{\neg, \wedge, \vee\}$. Therefore $\{\uparrow\}$ is also F.C.

E.g. Let's see $\{\downarrow\}$ or NOR is FC or not.

Using the similar approach that we used for $\{\uparrow\}$, we can show that,

$$p \downarrow p = \neg(p \vee p) = \neg p$$

$$p \vee q = (p \downarrow q) \downarrow (p \downarrow q)$$

$$p \wedge q = (p \downarrow p) \downarrow (q \downarrow q)$$

Therefore, FC sets are a extremely interesting concept in Digital Logic Circuits that help us understand how many types of circuits that we need to be able to simulate/generate any boolean function.

Normal Forms:

To explain the Normal Forms, let's take an example to understand it.

E.g. $p \leftrightarrow (q \rightarrow r')$ where r' or $\bar{r}$ or $\neg r$ are same.

Let 0 represent False and 1 represent True.

p	q	r	$q \rightarrow r'$	$p \leftrightarrow (q \rightarrow r')$	min term ($\wedge$ 0: p', 1: p)	max term ($\vee$ 0: p, 1: p')
0	0	0	1	0	$m_0 =$	$M_0 = p \vee q \vee r$
0	0	1	1	0	$m_1 =$	$M_1 = p \vee q \vee r'$
0	1	0	1	0	$m_2 =$	$M_2 = p \vee q' \vee r$
0	1	1	0	1	$m_3 = p' \wedge q \wedge r$	$M_3 =$
1	0	0	1	1	$m_4 = p \wedge q' \wedge r'$	$M_4 =$
1	0	1	1	1	$m_5 = p \wedge q' \wedge r$	$M_5 =$
1	1	0	1	1	$m_6 = p \wedge q \wedge r'$	$M_6 =$
1	1	1	0	0	$m_7 =$	$M_7 = p' \vee q' \vee r'$

$p \leftrightarrow (q \to r')$

$q \to r'$

$= \neg q \vee \neg r$

$= q' \vee r'$

we know,

$p \leftrightarrow q$

$= (p \to q) \wedge (q \to p)$

Therefore, $p \leftrightarrow (q \to r') \equiv (m_3 \vee m_4 \vee m_5 \vee m_6)$

Each m is built using $\{p, q, r, \text{negation}, \wedge\}$

And $(m_3 \vee m_4 \vee m_5 \vee m_6)$ is in the standard form.

This standard form is called principal disjunctive Normal Form (PDNF)

Also,

$p \leftrightarrow (q \to r') \equiv M_0 \wedge M_1 \wedge M_2 \wedge M_7$

Each M is built using $\{p, q, r, \text{negation}, \vee\}$

This standard form is called Principal conjunctive Normal Form (PCNF).

This is extremely useful in Digital logic circuit design. We can represent any functions using $\{\neg, \vee, \wedge\}$, ie, OR, NOT and AND gates.

Note: 1. Given any expression, PCNF & PDNF are unique.

2. Given $x = y$,
we can check for:

$$PDNF(x) = PDNF(y)$$

$$PCNF(x) = PCNF(y)$$

3. (# terms in PCNF) + (# terms in PCNF) = 2^n

Arguments and Rules of Inference:

Arguments

(e.g.) S1: If $\underbrace{\text{you work hard}}_{p}$, $\underbrace{\text{you will get an A grade}}_{q}$: $p \to q$

$$p \to q$$

S2: $\underbrace{\text{You are working hard}}_{p}$: p

premises

$$\begin{array}{l} p \\ \hline \therefore q \end{array}$$

Conclusion

Arguments are basically a set of premises followed by a logically valid conclusion.

Let,

$$\left.\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{array}\right\} \text{Premises}$$

$$\overline{\therefore q} \;\} \text{ conclusion}$$

In a world where all of $P_1, P_2, P_3, \ldots P_n$ are True, q should logically follow (always).

$$((P_1 \wedge P_2 \wedge P_3 \wedge P_4 \ldots \wedge P_n) \longrightarrow q) \text{ is tautology}$$

$$\left.\begin{aligned} &\equiv T \\ &\equiv 1 \end{aligned}\right\} \text{always}$$

$\therefore$ To conclude q from $(P_1 \wedge P_2)$ we need to prove that $((P_1 \wedge P_2) \rightarrow q)$ is True always or is a tautology.

$P_1 : p \longrightarrow q$

$P_2 : p$

$c : q$

$$(P_1 \wedge P_2 \longrightarrow q)$$

$$\equiv ((p \rightarrow q) \wedge p) \rightarrow q$$

$$\equiv ((\neg p \vee q) \wedge p) \rightarrow q$$

$$\equiv \neg((\neg p \vee q) \wedge p) \vee q$$

$$\equiv (\neg p \vee p) \wedge (\neg p \vee \neg q) \vee q$$

$$\equiv T \wedge (\neg p \vee \neg q \vee q)$$

$$\equiv T \wedge (\neg p \vee T)$$

$$\equiv T \wedge T$$

$$\equiv T \equiv 1 \text{ (RHS)}$$

Because $((p_1 \wedge p_2) \to q)$ is True, we can conclude q given premises p_1 and p_2.

If we have to make conclusions from premises, this method is too time taking. Therefore we will learn to use some simple rules called the the rules of inference and see how these simple rules can actually simplify the whole process for us. We will also see how to use this rules to make conclusions with the given premises.

Rules of Inference:

① Addition : $\dfrac{p}{\therefore p \vee q}$

In a world where p is true, would p and q be also true?

Yes, because if p is true, we don't have to think about q.

② Simplification: $\dfrac{p \wedge q}{\therefore p}$ $\qquad$ $\dfrac{p \wedge q}{\therefore q}$

In a world where both p and q are True, is p True?

Yes, ofcourse.

③ Conjunction :

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

In a world where p and q are both True, $(p \wedge q)$ is also True.

④ Modus Ponens :

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

It is also called Rule of Detachment.

⑤ Modus Tollens :

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

We can re-write it :

$$\Big(\underbrace{((p \rightarrow q) \wedge \neg q) \longrightarrow \neg p}_{\text{Is this a Tautology.}} \Big) \equiv 1 .$$

It is also called proof by contradiction.

⑥ Hypothetical syllogism :

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

This is transitive property of implication.

⑦ Constructive dilemma : $p \to q$

$r \to s$

$p \vee r$

$\therefore (q \vee s)$

⑧ Destructive dilemma : $p \to q$

$r \to s$

$\neg q \vee \neg s$

$\therefore \neg p \vee \neg r$

⑨ Disjunctive syllogism : $p \vee q$

$\neg p$

$\therefore q$

This is also called proof by elimination.

In a world where $(p \vee q)$ is true and $\neg p$ is true, i.e., p is not true, therefore q is true.

E.g.,

$p \to (s \oplus r)$

p

r

$\therefore ?$

using modus ponens :

$\to s \oplus r \equiv (s \wedge \neg r) \vee (r \wedge \neg s)$

r

$\therefore ?$

If r is True, $\neg r$ is false. $\therefore (s \wedge \neg r)$ is False.

Now, $F \vee (r \wedge \neg s) \equiv (r \wedge \neg s)$

since r is True, $T \wedge \neg s \equiv \neg s$

$\therefore r \wedge \neg s \equiv T \wedge \neg s \equiv \neg s$

E.g.,

$$\left.\begin{array}{l} p \to (s \vee r) \\ p \end{array}\right\} \xrightarrow[\text{ponens}]{\text{modus}} \left.\begin{array}{l} s \vee \overset{F}{r} \\ \underbrace{\neg r}_{T} \\ \hline \therefore s \end{array}\right\} \text{Proof by elimination}$$

$$\begin{array}{l} \neg r \\ \hline \therefore ? \end{array}$$

E.g.,

$$\left.\begin{array}{l} p \to (s \oplus r) \\ p \end{array}\right\} \xrightarrow[\text{ponens}]{\text{modus}} s \oplus r \equiv (s \wedge \overset{T}{\neg r}) \vee \overset{F}{(\neg s \wedge \overset{F}{r})}$$

$$\begin{array}{l} \neg r \\ \hline \therefore ? \end{array} \qquad T\,\{\neg r \quad \begin{array}{l} \hline \therefore s \\ \hline \end{array}$$

Given $\neg r$ is True,

means r = False.

$\neg s \wedge F \equiv F$

and $(s \wedge \neg r) \equiv (s \wedge T) \equiv s$

$\therefore$ s is the conclusion.

(*) Some more Translations from English to logical statements:

More English Translations (This is extremely important for non-native english speakers).

① $\underbrace{\text{It's raining}}_{p}$ but $\underbrace{\text{it's sunny}}_{q}$.

p but q

$p \wedge q$

② $\overbrace{\text{You will not get an A grade}}^{\neg p}$ unless $\underbrace{\text{you work hard}}_{q}$.

$$\neg p \vee q \equiv p \to q$$

<u>Note</u>: p unless $q \equiv p \vee q$

③ $\underbrace{\text{I worked hard}}_{p}$ also $\underbrace{\text{I got a good grade}}_{q}$

$$p \text{ also } q \equiv p \wedge q$$

④ $\underbrace{\text{I worked hard}}_{p}$ as well $\underbrace{\text{I got a good grade}}_{q}$

$$p \text{ as well } q \equiv p \text{ as well as } q \equiv p \wedge q$$

⑤ p or q is true but not both
Exactly one of p or q (is True) $\Bigg\} \to$ (XOR) $p \oplus q$

⑥ Atleast one of p or q
Either p or q $\Bigg\} \to p \vee q$

⑦ Neither p nor q $\} \longrightarrow$ (NOR) $p \downarrow q \equiv \neg p \wedge \neg q \equiv \neg(p \vee q)$

⑧ Atleast one of p or q is false
Either p is false or q is false $\Bigg\}$ (NAND) $p \uparrow q \equiv \neg p \vee \neg q$
$\equiv \neg(p \wedge q)$

(9) p is necessary for q $\equiv q \to p$

p is required for q

p is a must for q

(10) p is sufficient for q $\equiv p \to q$

p is enough for q

E.g., $\underbrace{\text{A degree}}_{p}$ is required $\underbrace{\text{to apply for this job}}_{q}$.

$\equiv q \to p$ where q: you can apply for this job

p: you have a degree.

$\underbrace{\text{A degree}}_{p}$ is enough to $\underbrace{\text{apply for this job}}_{q}$

$\equiv p \to q$

(11) You cannot ride a roller coaster if you are under 3 feet tall unless you are older than 10 years.

p: you can ride a roller coaster

q: you are under 3 feet tall

r: you are older than 10 years.

p unless q $\equiv p \vee q$

q if p $\equiv p \to q$

p if q $\equiv q \to p$

$\underbrace{\neg p \text{ if } q}$ $\underline{\text{unless}}$ r

$q \to \neg p \vee r$

$\equiv (q \to \neg p) \vee r$

⊛ Predicate Logic : An Introduction

Predicate logic / Predicate Calculus / 1st order logic :

While studying propositional logic, we have seen statements like :

$x + 2 = 5$ and we said that it is not a proposition, because for some values of x, the statement holds true, while in other cases the statement is false, ie, the same statement is Both True and False. ∴ It is not a proposition.

Let $x \in \mathbb{R}$

$x + 2 = 5$ is True if $x = 3$
False if $x \neq 3$

We can say,

there exists $x \in \mathbb{R}$ such that $x + 2 = 5$.

This is a proposition.

$\underbrace{\exists x}_{\text{There exists } x \text{ such that}}$ $x + 2 = 5$

We have to define the domain of x clearly, ie., whether x is an integer, or real number or complex number, or x is an integer greater than 5, etc.

Domain of x means all the possible values of x.

If Domain of $x = \mathbb{R} - \{3\}$,

then $\exists x\,(x+2=5)$ is False.

If Domain of $x = \{3\}$,

then $\exists x\,(x+2=5)$ is True.

$\exists$ is called an existential quantifier as it is quantifying what x would be.

It is often written as:

$\exists x \; P(x)$ where $P(x) : x+2=5$

<u>In</u> <u>English</u>:

There exists x such that $P(x)$

There is an x such that $P(x)$

There is atleast one x such that $P(x)$.

E.g. $(x+1) > x$: $P(x)$ $x \in \mathbb{R}$

For all $x \in \mathbb{R}$, $(x+1) > x$

$\underbrace{\forall x}_{\text{Universal quantifier}} P(x)$; Domain of $x = \mathbb{R}$

English: for all x, $P(x)$
for every x, $P(x)$
for each x, $P(x)$
for any x, $P(x)$
given any x, $P(x)$
All of x, $P(x)$
for arbitrary x, $P(x)$

$\forall x\, P(x)$ $\quad x \in \{x_1, x_2, x_3, \ldots\}$ = Domain of x

① $(P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge P(x_4) \wedge \ldots) \Rightarrow \forall x\, P(x)$

$((x_1+1 > x_1) \wedge (x_2+1 > x_2) \wedge (x_3+1 > x_3) \wedge \ldots) \Rightarrow \forall x\, P(x)$

② $\exists x\, P(x)$ $\quad x \in \{x_1, x_2, x_3, \ldots\}$

$(P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots) \Rightarrow \exists x\, P(x)$

E.g. $x \in Z^+$

$P(x)$: $\sqrt{x}$ is real

$\neg P(x)$: $\sqrt{x}$ is not real

Note: $\sqrt{Z^+}$ = real but $\sqrt{Z^-}$ = complex number.

$\forall x\, P(x) \equiv T$: For all values of x, $\sqrt{x}$ is real. It is true.

$\exists x\, P(x) \equiv T$

$\downarrow$

There exists a value of x for which $\sqrt{x}$ is real. It is true.

Similarly,

$\forall x \; \neg P(x) \equiv F$

$\exists x \; \neg P(x) \equiv F$

<u>E.g.</u> $x \in \mathbb{R}$

$P(x) : x+2=5$

$\forall x\, P(x) \equiv$ False

$\exists x P(x) \equiv$ True

<u>Uniqueness</u> <u>Quantifier</u>: (Not widely used)

$\exists!\; x P(x)$ or $\exists_1 x\, P(x)$

There exists a unique x s.t (such that) $P(x)$

There is one and only x such that $P(x)$

$\exists_1 x\; (x+2=5)$ s.t $x \in \mathbb{R}$

Statements and Quantifiers on 2 variables:

(e.g) $x, y \in \mathbb{R}$

$P(x,y) : x+y = 5$

① $\forall x \forall y \; P(x,y) \equiv \underset{6}{\forall x} \underset{4}{\forall y} \; (x+y=5) \equiv$ False

$\equiv$ For all x and for all y

$(x+y) = 5$

② $\exists x \exists y \; P(x,y) \equiv \underset{2}{\exists x} \underset{3}{\exists y} \; (x+y=5) \equiv$ True

③ $\forall x \exists y \; P(x,y) \equiv \forall x \exists y \; (x+y=5)$

$x+y=5$

$\Rightarrow y = 5-x$

$\therefore \underset{x}{\forall x} \underset{5-x}{\exists y} \; (x+y=5) \equiv$ True

④ $\exists x \forall y \; P(x,y) \equiv \exists x \forall y \; (x+y)=5 \equiv$ False

For a fixed value of x, y also becomes fixed since $y = 5-x$. Therefore y doesnot have all possible values in the real numbers range.

⑤ $\forall y \exists x \; (x+y)=5 \equiv$ True

For any value y, we can find a value of x such that $(x+y) = 5$.

(e.g) $x, y \in Z^+$

$P(x,y) \equiv x+y=5$ | $y=5-x$, $x=5-y$

$\underbrace{\forall x}_{6} \underbrace{\forall y}_{3} \; P(x,y) \equiv F$

$\underbrace{\exists x}_{1} \underbrace{\exists y}_{4} \; P(x,y) \equiv T$

$\exists x \; \forall y \; P(x,y) \equiv F$

$\underbrace{\forall y}_{6} \; \exists x \; P(x,y) \equiv F$ $\qquad (6+x)=5$, $x \in Z^+$

(e.g) $P(x,y)$: $x+y^2=10$ $\qquad x, y \in \mathbb{R}$

ⓐ $\underbrace{\forall x}_{1} \underbrace{\forall y}_{9} \quad x+y^2=10 \equiv$ False

ⓑ $\underbrace{\exists x}_{1} \underbrace{\exists y}_{3} \quad x+y^2=10 \equiv$ True

ⓒ $\underbrace{\exists x}_{x_1} \; \forall y \; (x+y^2=10) \equiv F \quad ; \therefore y=\sqrt{10-x_1}$

$\because$ x and y are dependent here, selecting x_1 makes $y=\sqrt{10-x_1}$. Therefore we can't take any values of y. Therefore the statement is false.

ⓓ $\underbrace{\exists y}_{y_1} \; \forall x \; ((x+y^2=10) \equiv F \quad ; \therefore 10-y_1^2=x$

(e) $\forall x \; \exists y \; (x+y^2=10) \equiv F$

$x = 11$

$y = \sqrt{10-x}$

$= \sqrt{10-11}$

$= \sqrt{-1} \notin \mathbb{R}$

(f) $\forall y \; \exists x \; (x+y^2=10) \equiv T$

$x = 10-y^2$

If y is a real number, y^2 is also a real number, $(10-y^2)$ is also a real number.

$\therefore$ The statement is True.

Statements and quantifiers on 3 variables:

Let $x, y, z \in \mathbb{R}$

$P(x,y,z): \; x+y+z=10$

(a) $\forall x \; \forall y \; \forall z \; P(x,y,z) \equiv$ False

For all x, for all y, for all z, the statement is not True.

(b) $\forall x \; \forall y \; \exists z \; P(x,y,z) \equiv$ True

Take any values (real numbers) of x and y, $z = (10-x-y)$ is also a real number. $\therefore$ True.

(c)

$\forall x \; \exists z \; \forall y \; P(x, y, z) \equiv$ False

$y = 10 - z - x$.

$= 10 - z_1 - 11$ (say $z = z_1$)

$= -1 - z_1$

The moment we set $z = z_1$, y is also fixed as y is dependent on z_1.

$\therefore$ y cannot take any value that it wants.

$\therefore$ It fails. $\therefore$ False.

<u>Note</u>: The sequence of assignment is always left to right, i.e., first assign x, then assign z and then y.

(d) $\underbrace{\exists z}_{z_1} \; \forall x \; \forall y \; (x + y + z = 10) \equiv$ False.

$x + y = 10 - z_1$

If we set $z = z_1$, for all values of x and for all values of y, the statement need not be True.

(e) $\exists z \; \exists y \; \forall x \; (x+y+z=10) \equiv$ False.

$\underset{z_1}{\exists z} \; \underset{y_1}{\exists y}$

If we fix z and y as z_1 and y_1, this statement will not hold for all values of x.

(f) $\underbrace{\exists x}_{1} \; \underbrace{\exists y}_{2} \; \underbrace{\exists z}_{7} \; (x+y+z=10) \equiv$ True

(g) $\forall x \; \exists y \; \exists z \; (x+y+z=10) \equiv$ True

$y+z = \underbrace{10-x}_{\mathbb{R}}$

We can allocate y and z, values which are real numbers and satisfy the statement.

Properties of Predicate Logic

1. $\forall x \, P(x) \Rightarrow \exists x \, P(x)$

$P \rightarrow Q$

P	Q	$P \rightarrow Q \equiv \neg P \vee Q$
T	T	T ✓
T	F	F
F	F	T ✓
F	T	T

$P \rightarrow Q$ is 'if P then Q.'

It means, if P is True, then Q has to be True.
if P is False, we don't care what Q is, $P \rightarrow Q$ will always be True.

$$\forall x P(x) \equiv (P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \dots)$$

$$x \in \{x_1, x_2, x_3, \dots\}$$

To make notations simpler,

$P(x_i) = P_i$

$Q(x_j) = Q_j$

Let's try to prove

$$\forall x P(x) \Rightarrow \exists x P(x)$$

LHS.

$$\forall x P(x) \equiv (P(x_1) \wedge P(x_2) \wedge \dots)$$

$$\equiv P1 \wedge P2 \wedge P3 \wedge \dots$$

RHS.

$$\exists x P(x) \equiv P_1 \vee P_2 \vee P_3 \vee \dots$$

From modus ponens :-

$$\begin{array}{l} P \rightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

Given P is True, then for $P \rightarrow Q$ to be True, Q has to be True.

$$\therefore \underset{T}{(P1} \wedge \underset{T}{P2} \wedge \underset{T}{P3} \wedge \dots) \Rightarrow (\underset{T}{P_1} \vee \underset{T}{P_2} \vee \underset{T}{P_3} \dots)$$

$\therefore$ LHS implies RHS.

Let's now try to see:

$$\exists x\, P(x) \overset{?}{\Rightarrow} \forall x\, P(x)$$

$$\underbrace{P1 \vee P2 \vee P3 \ldots}_{\text{For this to be true if only } P_1 \text{ is True and other propositions are False, will hold. } \therefore \text{ LHS is True.}} \Rightarrow \underbrace{P_1 \wedge P_2 \wedge P_3 \wedge \ldots}_{\text{But for the same Truth value (True/False) assignment of the propositions in RHS, RHS will result in False.}}$$

For this to be true if only P_1 is True and other propositions are False, will hold.
$\therefore$ LHS is True.

But for the same Truth value (True/False) assignment of the propositions in RHS, RHS will result in False.

$$\therefore \underbrace{T \Rightarrow F}_{F}$$

It is not a tautology.

$$\therefore \exists x\, P(x) \not\Rightarrow \forall x\, P(x)$$

② De'Morgans Laws:

ⓐ $\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$

ⓑ $\neg(\exists x\, P(x)) \equiv \forall x(\neg P(x))$

Let's try to prove ⓐ :-

$$\neg(\forall x\, P(x)) \Leftrightarrow \exists x\, \neg P(x)$$

First, we try to prove:

$$\neg(\underbrace{\forall x\, P(x)}) \Rightarrow \exists x(\neg P(x))$$

LHS: $\neg(P_1 \wedge P_2 \wedge P_3 \ldots)$ RHS: $(\neg P_1 \vee \neg P_2 \vee \neg P_3 \ldots)$

Using De Morgan's, we can prove LHS → RHS.
Now, let's try to prove RHS → LHS.

$\exists x \neg P(x) \Rightarrow \neg(\forall x P(x))$

$(\neg P_1 \vee \neg P_2 \vee \neg P_3 \ldots) \Rightarrow \neg(P_1 \wedge P_2 \wedge P_3 \ldots)$

Again, we use De morgan's and prove that RHS → LHS.

$\therefore \neg(\forall x P(x)) \Longleftrightarrow (\exists x \neg P(x))$

③ Nested Quantifiers:

$\exists x \forall y P(x,y) \Rightarrow \forall y \exists x P(x,y)$

LHS:

$\exists x \forall y P(x,y)$

$\equiv \exists x (\forall y P(x,y))$

$\equiv \exists x (P(x,y_1) \wedge P(x,y_2) \wedge P(x,y_3) \wedge \ldots)$

$\equiv (P(x_1,y_1) \wedge P(x_1,y_2) \wedge P(x_1,y_3) \wedge \ldots)$

$\equiv (P(x_1,y_1) \wedge P(x_1,y_2) \wedge P(x_2,y_3) \wedge \ldots)$
$\vee (P(x_2,y_1) \wedge P(x_2,y_2) \wedge P(x_3,y_3) \wedge \ldots)$
$\vee (P(x_3,y_1) \wedge P(x_3,y_2) \wedge P(x_3,y_3) \wedge \ldots)$

$\equiv (P_{11} \wedge P_{12} \wedge P_{13} \wedge \ldots) \vee$
$(P_{21} \wedge P_{22} \wedge P_{23} \wedge \ldots) \vee$
$(P_{31} \wedge P_{32} \wedge P_{33} \wedge \ldots)$

Note: While expanding $\forall$ gives $\wedge$ and $\exists$ gives $\vee$.

But, $\forall x \exists y\, P(x,y) \not\Rightarrow \exists y \forall x\, P(x,y)$

Shortcut to remember :

$\exists x \forall y\, P(x,y) \Rightarrow \forall y \exists x\, P(x,y)$

We can pullout $\forall y$ ahead of $\exists x$

$\forall x \exists y\, P(x,y) \not\Rightarrow \exists y \forall x\, P(x,y)$

X

We cannot pullout $\exists y$ ahead of $\forall x$.

E.g., $\forall x \exists y\ (x+y=0) \not\Rightarrow \exists y \forall x\ (x+y=0)$

$x, y \in \mathbb{R}$

The RHS is False as if we fix 'y', $\forall x$ doesnot hold. $\therefore \exists y \forall x\ (x+y=0)$ is False.

But LHS is True.

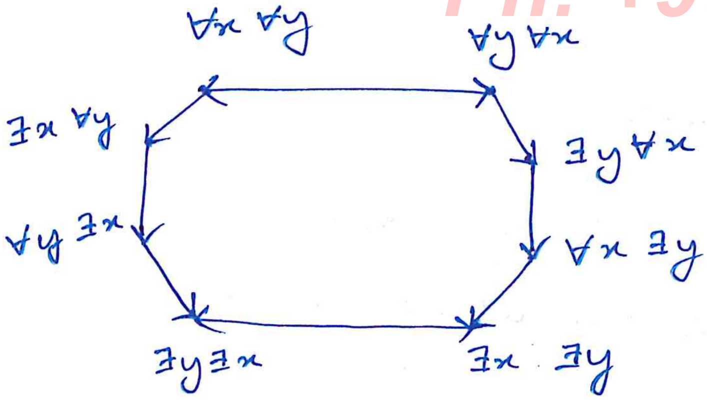
$\therefore \underbrace{T \rightarrow F}_{F}$

$\therefore$ LHS $\not\Rightarrow$ RHS.

This Theorem is called One-way Theorem, As LHS $\Rightarrow$ RHS but RHS $\not\Rightarrow$ LHS.

(4) $\forall x \forall y\, P(x,y) \Leftrightarrow \forall y \forall x\, P(x,y)$

$\exists x \exists y\, P(x,y) \Leftrightarrow \exists y \exists x\, P(x,y)$



$\forall x \forall y\ P(x,y) \longrightarrow \exists x \forall y\ P(x,y)$: True

$\forall x \forall y\ P(x,y) \longrightarrow \forall x \exists y\ P(x,y)$: True

$\forall x \exists y\ P(x,y) \longrightarrow \exists y \forall x\ P(x,y)$: False

(5.) Distributive property:

(A.) $\forall x\ (P(x) \wedge Q(x)) \equiv (\forall x\ P(x)) \wedge (\forall x\ Q(x))$

$\equiv (P_1 \wedge Q_1) \wedge (P_2 \wedge Q_2) \wedge (P_3 \wedge Q_3) \ldots \equiv (P_1 \wedge P_2 \wedge P_3 \ldots) \wedge (Q_1 \wedge Q_2 \wedge Q_3 \wedge \ldots)$

Hence, LHS is equivalent to RHS.

(B.) $\exists x\ (P(x) \vee Q(x)) \equiv (\exists x\ P(x)) \vee (\exists x\ Q(x))$

$\equiv (P_1 \vee Q_1) \vee (P_2 \vee Q_2) \vee \ldots \equiv (P_1 \vee P_2 \vee P_3 \vee \ldots) \vee (Q_1 \vee Q_3 \vee Q_4 \vee \ldots)$

Hence, LHS is equivalent to RHS.

(c) $\forall x\,(P(x) \vee Q(x)) \overset{?}{\Rightarrow} (\forall x\,(P(x)) \vee (\forall x\, Q(x))$

$\equiv (P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \wedge (P_3 \vee Q_3) \wedge \ldots$ | $\equiv (P_1 \wedge P_2 \wedge P_3 \ldots) \vee (Q_1 \wedge Q_2 \wedge Q_3 \ldots)$

We try to find out if an assignment of the Truth values result in

$T \to F$

Then it would mean that $(T \to F)$ is False.

LHS.

$(\overset{T}{P_1} \vee \overset{F}{Q_1}) \wedge (\overset{F}{P_2} \vee \overset{T}{Q_2}) \wedge (\underset{T}{P_3} \vee \underset{F}{Q_3}) \wedge \ldots$

$\equiv T$

RHS.

$(\overset{T}{P_1} \wedge \overset{F}{P_2} \wedge \overset{T}{P_3} \ldots) \vee (\underset{F}{Q_1} \wedge \underset{T}{Q_2} \wedge \underset{F}{Q_3} \ldots)$

$\equiv F \vee F$

$\equiv F$

$\therefore$ LHS $\longrightarrow$ RHS

$\equiv T \longrightarrow F$

$\equiv F$.

$\therefore \forall x\,(P(x) \vee Q(x)) \not\Rightarrow (\forall x\, P(x)) \vee (\forall x\, Q(x))$.

(D) $(\forall x\, P(x)) \vee (\forall x\, Q(x)) \overset{?}{\Rightarrow} \forall x\,(P(x) \vee Q(x))$

$\equiv (P_1 \wedge P_2 \wedge P_3 \ldots) \vee (Q_1 \wedge Q_2 \wedge Q_3 \ldots)$ | $\equiv (P_1 \vee Q_1) \wedge (P_2 \vee Q_2) \wedge (P_3 \vee Q_3) \wedge \ldots$

For LHS to be True, either each of the propositions $P_1, P_2, P_3, \ldots$ has to be True or, $Q_1, Q_2, Q_3 \ldots$ has to be True or both has to be True.

Given the Truth values, RHS will have:

(i) $\underset{T\quad F}{(P_1 \vee Q_1)} \wedge \underset{T\quad F}{(P_2 \vee Q_2)} \wedge \ldots$

(ii) $\underset{F\quad T}{(P_1 \vee Q_1)} \wedge \underset{F\quad T}{(P_2 \vee Q_2)} \wedge \ldots$

(iii) $\underset{T\quad T}{(P_1 \vee Q_1)} \wedge \underset{T\quad T}{(P_2 \vee Q_2)} \wedge \ldots$

In all the above possibilities RHS will be True.

$\therefore \underset{(LHS)}{T} \longrightarrow \underset{(RHS)}{T}$

$\therefore (\forall x\, P(x)) \vee (\forall x\, Q(x)) \Rightarrow \forall x\, (P(x) \vee Q(x))$

(E) $\exists x\, (P(x) \wedge Q(x)) \Rightarrow (\exists x\, P(x)) \wedge (\exists x\, Q(x))$

(F) $(\exists x\, P(x)) \wedge (\exists x\, Q(x)) \not\Rightarrow \exists x (P(x) \wedge Q(x))$

(G) $\forall x (P(x) \wedge Q) \equiv (\forall x\, P(x)) \wedge Q$.

$\equiv (P_1 \wedge Q) \wedge (P_2 \wedge Q) \wedge (P_3 \wedge Q) \wedge \ldots$

$\equiv (P_1 \wedge P_2 \wedge P_3 \ldots) \wedge Q$.

$\equiv (\forall x\, P(x)) \wedge Q \equiv RHS$.

Note: Q is not a statement which contains x.

(H) $\exists x\,(P(x) \land Q) \equiv (\exists x\, P(x)) \land Q$

(I) $\forall x\,(P(x) \lor Q) \equiv (\forall x\, P(x)) \lor Q$

(J) $\exists x\,(P(x) \lor Q) \equiv (\exists x\, P(x)) \lor Q$

E.g. $\forall x\,(P(x) \lor Q) \Longleftrightarrow (\exists x\, P(x)) \lor Q$

Let's check for LHS $\Rightarrow$ RHS first.

$\equiv (\forall x\, P(x)) \lor Q \Rightarrow (\exists x\, P(x)) \lor Q$

$\therefore$ LHS $\Rightarrow$ RHS

Let's check for RHS $\Rightarrow$ LHS now.

$\exists x\, P(x) \not\Rightarrow \forall x\, P(x)$

$\therefore$ RHS $\not\Rightarrow$ LHS.

E.g. $\forall x\,(P(x) \rightarrow Q(x)) \Rightarrow (\forall x\, P(x) \rightarrow (\forall x\, Q(x))$

$\equiv (\neg P_1 \lor Q_1) \land (\neg P_2 \lor Q_2)$
$\land (\neg P_3 \lor Q_3) \land \dots$

$\equiv \neg(P_1 \land P_2 \land P_3 \dots)$
$\lor (Q_1 \land Q_2 \land Q_3 \land \dots)$
$\equiv (\neg P_1 \lor \neg P_2 \lor \neg P_3 \dots)$
$\lor (Q_1 \land Q_2 \land Q_3 \land \dots)$

Whenever LHS is True, RHS is also True.

$T \longrightarrow T$ holds

$\therefore$ LHS $\Rightarrow$ RHS.

Note: To expand we use the property:
$P \rightarrow Q \equiv \neg P \lor Q$.

E.g. $(\forall x\, P(x)) \rightarrow (\forall x\, Q(x)) \not\Rightarrow \forall x\,(P(x) \rightarrow Q(x))$

$$\overset{T}{\neg P_1} \lor \overset{F}{\neg P_2} \lor \overset{F}{\neg P_3} \ldots$$
$$\lor\,(\underset{F}{Q_1} \land \underset{F}{Q_2} \land \underset{F}{Q_3} \land \ldots)$$
$$\equiv T$$

$$(\overset{T}{\neg P_1} \lor \overset{F}{Q_1}) \land (\overset{F}{\neg P_2} \lor \overset{F}{Q_2})$$
$$\land\,(\underset{F}{\neg P_3} \lor \underset{F}{Q_3}) \land \ldots$$
$$\equiv (T \lor F) \land (F \lor F)$$
$$\land\,(F \lor F) \land \ldots$$
$$\equiv T \land F \land F \land \ldots$$
$$\equiv F$$

$\therefore$ $T \longrightarrow F$
(LHS) (RHS)

$\therefore$ we show that,

$$(\forall x\, P(x)) \rightarrow (\forall x\, Q(x)) \not\Rightarrow \forall x\,(P(x) \rightarrow Q(x))$$

E.g. $\exists x\,(P(x) \rightarrow Q(x)) \not\Rightarrow (\exists x\, P(x)) \rightarrow (\exists x\, Q(x))$

It can be proved by finding

$T \rightarrow F$.

E.g. $(\exists x\, P(x)) \longrightarrow (\exists x\, Q(x)) \Rightarrow \exists x\,(P(x) \rightarrow Q(x))$

Here, $T \longrightarrow T$ holds.

(*) Translation from English to Predicate Logic and vice-versa.

Precedence : $\forall, \exists$ have the highest priority

$$\forall x\, P(x) \rightarrow Q(x)$$

$$\equiv (\forall x\, P(x)) \rightarrow Q(x)$$

$$\forall x\, P(x) \rightarrow \exists x\, Q(x)$$

$$(\forall x\, P(x)) \rightarrow (\exists x\, Q(x))$$

Translations :

Predicate statements $\longleftrightarrow$ English.

Basic Rules :

(1) $\exists$: some
atleast one

$\forall$: all
any
every

Eg. Every student in the class has studied DM.

$x \in \underbrace{\text{students in the class}}_{\text{Domain / Universe}}$

x is a variable that represents any element as long as the element belongs to the set (students in the class).

$S(x)$: x has studied DM.

$\therefore\ \forall x\, S(x) \equiv$ All the students in the class have studied DM.

Note:
Incase where domain is not clearly defined, we need to define domain as broadly as possible.

$x \in \underbrace{\text{set of persons}}_{\text{Domain}}$

$c(x)$: x is a student in the class.

$\forall x\, (c(x) \longrightarrow S(x))$

For all x, if x is a student in the class, then x have studied D.M.
And if x is not a student in the class, then we don't care about the person, as we don't have that information.

The following statement is incorrect:

$\forall x\, (c(x) \wedge S(x))$ ——— common mistake

It means for every person in the universe, they should belong to my class and they should have studied DM.
And there may be some person x, who

doesnot belong to the class
This is not what has been said in the given statement. Therefore is an error/mistake while translation that we should be aware of.

<u>Note</u>: $\forall$ with $\rightarrow$ is correct.
$\forall$ with $\wedge$ could lead to incorrect translation.

<u>E.g.</u>

Some students in the class have studied DM.

$x \in$ set of students in class.

<u>Case 1</u>: Domain is clearly stated.

$\exists x \; S(x)$

<u>Case 2</u>: Domain is not clearly stated.

$\left.x \in \text{set of people}\right\}$ we define a broad domain.

$\exists x (C(x) \wedge S(x))$

The following is a common mistake:

$\exists x (C(x) \longrightarrow S(x))$

What if there are no students in the class, then $S(x)$ may have any Truth value (T/F) and $(C(x) \rightarrow S(x))$ will always be True.

$\therefore$ x will exist.

$\therefore$ It is a wrong translation.

E.g. who is / which is } AND

There is someone $\underbrace{\text{who is}}$ taller than 7 feet

P(x) : x is a person

x ∈ all objects } broad domain

T(x) : x is taller than 7 feet.

$\therefore \exists x (P(x) \wedge T(x))$
(AND)

E.g. Humming Bird is small.

x ∈ all objects } broad domain

H(x) : x is a Humming bird

S(x) : x is small

$\therefore \exists x (H(x) \wedge S(x))$

'All humming birds are small', will be

$\forall x (H(x) \rightarrow S(x)) \equiv$ Humming birds are small

E.g. Every tall boy in the class plays BasketBall.

x ∈ students in the class

T(x) : x is Tall

B(x) : x is a boy.

$BB(x)$: x plays basketball.

$$\forall x (\{B(x) \wedge T(x)\} \longrightarrow BB(x))$$

E.g: No & Not.

No hardworking person is poor.

$x \in$ set of persons (in the universe)

$P(x)$: x is poor.

$H(x)$: x is hardworking

Let, $\neg P(x) \equiv R(x)$: x is rich.

No hardworking person is poor

$\equiv \neg$ (a hardworking person is poor)

$\equiv \neg (\exists x \; H(x) \wedge P(x))$

$\equiv \forall x (\neg H(x) \vee \neg P(x))$

$\equiv \forall x (H(x) \longrightarrow \neg P(x))$

$\equiv \forall x (H(x) \longrightarrow R(x))$

$\equiv$ Every hardworking person is rich.

E.g. Not all hardworking people are poor.

$\equiv \neg$ (all hardworking people are poor)

$\equiv \neg (\forall x (H(x) \longrightarrow P(x)))$ De morgan's law

$\equiv \exists x (H(x) \wedge \neg P(x))$

$\equiv \exists x (H(x) \wedge R(x))$

≡ There are some people who are both rich and hardworking.

Eg. Pearls and Rubies are precious.

$x \in$ all objects

$P(x)$: x is a pearl

$R(x)$: x is a Ruby

$Pr(x)$: x is precious.

$$\forall x((P(x) \vee R(x)) \rightarrow Pr(x))$$

(*) More Translations and examples

Eg Lions and Tigers attack when hungry or threatened.

x : any object

$$S1: \forall x\,(L(x) \vee T(x)) \Rightarrow [(H(x) \vee Th(x)) \Rightarrow A(x)]$$

Given,

$L(x)$: x is a Lion

$T(x)$: x is a Tiger

$H(x)$: x is Hungry

$Th(x)$: x is Threatened.

$A(x)$: x attacks.

Let's check if it is a correct translation.

$\forall x \, (L(x) \vee T(x)) \Rightarrow [(H(x) \vee Th(x)) \Rightarrow A(x)]$

$\forall x$ (if ✓ ✓ then { if ✓ ✓ then ✓ })

```
if ( ____ )
{
    if ( ____ )
    {
        ✓
    }
}
```

Nested if

It can be very complicated sometimes to read using nested if.

Therefore let's simplify it.

Let $P = L(x) \vee T(x)$

$Q = (H(x) \vee Th(x))$

$R = A(x)$

$\therefore$ We have:- (we are ignoring $\forall x$ for now).

$$P \Rightarrow (Q \Rightarrow R)$$

$$\equiv \neg p \vee (\neg q \vee r)$$

$$\equiv \neg p \vee \neg q \vee r$$

$$\equiv \neg (p \wedge q) \vee r$$

$$\equiv (p \wedge q) \rightarrow r$$

$\therefore$ We have,

S2: $\forall x \, [\{L(x) \vee T(x)\} \wedge \{H(x) \vee Th(x)\}] \Rightarrow A(x)$

≡ For all x, if x is a lion or a tiger, and, if x is hungry or threatened, then it will attack.

∴ It is a correct translation.

Note: $\forall x(L(x) \wedge T(x))$ is incorrect because an animal cannot be both lion and a tiger.

And,

$$S_3: \underbrace{\underbrace{\forall x(L(x) \wedge T(x))}_{F} \Rightarrow \underbrace{[(H(x) \vee Th(x)) \Rightarrow A(x)]}_{T/F}}_{\text{It will always be True}}$$

∴ It is a wrong translation.

$$S_4: \forall x\left[\{\underbrace{H(x) \vee Th(x)}_{p}\} \rightarrow \{(\underbrace{L(x) \vee T(x)}_{q}) \rightarrow \underbrace{A(x)}_{r}\}\right]$$

$$\begin{aligned} p \rightarrow (q \rightarrow r) &\equiv \neg p \vee \neg q \vee r \\ &\equiv \neg(p \vee q) \vee r \\ &\equiv (p \wedge q) \rightarrow r \\ &\equiv S_2 \end{aligned}$$

∴ S_4 is also a correct translation.

e.g. All purple mushrooms are poisonous.

S1: $\forall x\,[(P(x) \wedge M(x)) \Rightarrow Po(x)]$

It says:

For all x, if x is a mushroom and x is purple, then it is poisonous.

It is a correct translation as the statement is only talking about purple mushrooms.

S2: $\forall x\,[P(x) \rightarrow \{M(x) \rightarrow P_o(x)\}]$

$$P \rightarrow \{M \rightarrow P_o\} \equiv \neg(P \wedge M) \vee P_o \equiv (P \wedge M) \rightarrow P_o$$
$$\equiv S1.$$

S3: $\forall x\,[M(x) \rightarrow \{P(x) \rightarrow P_o(x)\}] \equiv S1.$

S4: $\forall x\,[\{P(x) \rightarrow M(x)\} \rightarrow P_o(x)\}] \equiv \neg(\neg P \vee M) \vee P_o$

$$\equiv (P \wedge \neg M) \vee P_o$$
$$\equiv \neg(\neg P \vee M) \vee P_o$$
$$\equiv (\neg P \vee M) \rightarrow P_o$$
$$\not\equiv S1$$

$\therefore$ S_4 is an incorrect translation.

Note: Try to always write a simple statement with one or minimum possible implications. Then compare other statements with the simple statement.

(e.g.) No one can fool everyone all the time

$x, y \in$ set of people.

$F(x,y) \equiv x$ can fool y all the time.

$\neg(\underbrace{\exists x \forall y F(x,y)})$

There exists x such that x can fool all the y all the time.

$\equiv \forall x \exists y \neg F(x,y)$

For all x, there exists a y such that x cannot fool y all the time.

(e.g.) $x \in$ set of students in a college

$F(x,y) : x$ & y are friends

$\forall x \forall y F(x,y)$: All students in the college are friendly with each other

$\forall x \exists y F(x,y)$: Every student is friendly with someone in a college.

$\exists y \forall x F(x,y)$: Some students are friendly with everyone in a college.

$\exists y \exists x F(x,y)$: There is atleast one pair of friends in a college.

(e.g.) Some boys are taller than all the girls in the class.

Let, $x \in$ students in the class.

$\exists x [B(x) \wedge \{\forall y\ G(y) \longrightarrow T(x,y)\}]$

Mistake: $\exists x [B(x) \wedge \{\forall y\ G(y) \underline{\wedge} T(x,y)\}]$

(e.g.)

$x \in$ all painters

$y \in$ all paintings.

$p(x,y)$: x has painted y

$\exists x \forall y\ p(x,y)$: some painters have painted all paintings.

$\forall y \forall x\ p(x,y)$: Some paintings are painted by every painters.

$\exists x \forall y\ p(x,y)$: For every painting, there exists someone who has painted it.

(e.g.) $\forall x \{(x \neq 0) \longrightarrow \exists y (xy = 1)\}$

$y, x \in \mathbb{R}$

$\equiv$ Every non-zero real number has an inverse (multiplicative inverse).

$\equiv$ Every non zero real number x, there exists another real number y such that $x \cdot y = 1$.

(e.g.) $\neg(x=1) \wedge \forall y\,[\exists z\,(x=y*z) \Rightarrow \{(y=x) \vee (y=1)\}]$

$x, y, z \in \mathbb{R}$

If x is not equal to 1 and (if y is a divisor of x, then y is either x or y is 1).

The statement says :

$x \neq 1$

y and z are divisors of x.

For every y there exists a z.

If the above is true then $y=x$ or $y=1$

Eg.: $x=6$

$x=6=\underset{y}{2} * \underset{z}{3}$

Eg. $x=5=\underset{y}{5} * \underset{z}{1}$

$x=5=\underset{y}{1} * \underset{z}{5}$

$x \neq 1$ but divisors of x are x and 1.

Such a number is called a prime number.

Let's try to simplify the statement:-

$\forall y\,[\exists z\,(x=y*z) \Rightarrow \{(y=z) \vee (y=1)\}]$

$\equiv \forall y\,[\neg(\exists z\; x=y*z) \vee (y=x) \vee (y=1)]$

$\equiv \forall y\,[\forall z\,(x \mathrel{!=} y*z) \vee (y=x) \vee (y=1)]$

For the statement to be true one of the above has to be true. But in this case, the simplification doesnot seem to be intuitive.

E.g. There is exactly one apple on the table.

$y, x \in$ all objects on the table.

S_1: $\exists x \, (A(x) \wedge \forall y \, \{ y \neq x \longrightarrow \neg A(y) \})$

$A(x)$: x is an Apple.

The statement S_1 means:

There exists an Apple x and for all y, if object y is not same as object x (Apple), then y is not an apple.

$\equiv \exists x \, (A(x) \wedge \forall y \, \{ A(y) \rightarrow (y = x) \})$

$\equiv \exists_1 x \; A(x)$

$$\equiv \underbrace{\underbrace{\exists x \, A(x)}_{\geq 1} \wedge \underbrace{\forall x \forall y \, (A(x) \wedge A(y) \rightarrow (x = y))}_{\leq 1}}_{=1}$$

exactly one apple.

Atleast 2 apples :

$$\exists x \left[A(x) \wedge \exists y \left(A(y) \wedge (x \neq y) \right) \right]$$

We have one item (Apple x) and another item (Apple y) and they both are not the same. That is, there are atleast 2 apples.

Exactly 2 apples :

$$\overbrace{\exists x \left[A(x) \wedge \exists y \left(A(y) \wedge (x \neq y) \right) \right]}^{\text{atleast}} \wedge$$

$$\underbrace{\forall z \left\{ A(z) \Rightarrow \{ (z = x) \vee (z = y) \} \right\}}_{\text{atmost}}$$

$$\equiv \exists x \exists y \left[(x \neq y) \wedge A(x) \wedge A(y) \wedge \forall z \left(A(z) \Leftrightarrow (z = x \vee z = y) \right) \right]$$

Atmost 2 :

$$\forall x \forall y \forall z \left(\{ A(x) \wedge A(y) \wedge A(z) \} \Rightarrow \{ z = x \text{ or } z = y \text{ or } x = y \} \right)$$

If we have three apples, then two of them are the same.