Measure Theory

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§0. Notation

$\mathbb{N}$ represents the set $\{1, 2, \ldots\}$.
$\mathbb{N}_0$ represents the set $\{0, 1, 2, \ldots\}$.
For $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!}$$

is the generalised binomial coefficient.

For $n \in \mathbb{N}$, we denote $\{1, 2, \ldots, n\}$ as $[n]$ and $\{0, 1, 2, \ldots, n\}$ as $[n]_0$

For $a \in \mathbb{R}^n$, we denote the ith coordinate of a by a_i for each $i = 1, 2, \ldots, n$.
For $a, b \in \mathbb{R}^n$, we write $a < b$ if $a_i < b_i$ for each $i = 1, 2, \ldots, n$.

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of reals. Then

$$\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \left(\sup_{m \geq n} a_n \right) = \inf_{n \geq 0} \left(\sup_{m \geq n} a_n \right)$$
$$\liminf_{n \to \infty} a_n = \lim_{n \to \infty} \left(\inf_{m \geq n} a_n \right) = \sup_{n \geq 0} \left(\inf_{m \geq n} a_n \right)$$

Let A and B be two sets. We denote by

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

the *symmetric difference* of A and B.

If sets A and B are disjoint, we represent their union as $A \uplus B$ (similar to the $\oplus$ notation in linear algebra).

If $f, g : \Omega \to \overline{\mathbb{R}}$ such that $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$, we write $f \leq g$. We similarly write $f < g$, $f \leq 0$ and other expressions.

If Ω is a set and $f : \Omega \to \overline{\mathbb{R}}$, then we define the functions $f^+, f^- : \Omega \to \overline{\mathbb{R}}$ by

$$f^+ = \max\{0, f\} \text{ and } f^- = \max\{0, -f\}.$$

§1. Measure Theory

Before beginning a rigorous study of probability theory, it is necessary to understand some parts of basic measure theory.

1.1. Classes of Sets

Let Ω be a non-empty set and $\mathcal{A} \subseteq 2^\Omega$, where 2^Ω is the power set of Ω. Then

Definition 1.1. $\mathcal{A}$ is called

- $\cap$-closed (closed under intersections) or a π-system if $A \cap B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.
- σ-$\cap$-closed (closed under countable intersections) if $\bigcap_{i=1}^\infty A_i \in \mathcal{A}$ for any choice of countably many sets $A_1, A_2, \ldots \in \mathcal{A}$.
- $\cup$-closed (closed under unions) if $A \cup B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.
- σ-$\cup$-closed (closed under countable unions) if $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$ for any choice of countably many sets $A_1, A_2, \ldots \in \mathcal{A}$.
- $\setminus$-closed (closed under differences) if $A \setminus B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.
- closed under complements if $A^c = \Omega \setminus A \in \mathcal{A}$ for all $A \in \mathcal{A}$.

Theorem 1.1. Let $\mathcal{A}$ be closed under complements. Then $\mathcal{A}$ is $\cup$-closed (σ-$\cup$-closed) if and only if $\mathcal{A}$ is closed $\cap$-closed (σ-$\cap$-closed).

The above is relatively straightforward to prove using De Morgan's Laws.

Theorem 1.2. Let $\mathcal{A}$ be $\setminus$-closed. Then

(a) $\mathcal{A}$ is $\cap$-closed,

(b) if $\mathcal{A}$ is σ-$\cup$-closed, then $\mathcal{A}$ is σ-$\cap$-closed.

(c) Any countable union of sets in $\mathcal{A}$ can be expressed as a countable union of pairwise disjoint sets in $\mathcal{A}$.

Proof.

(a) For $A, B \in \mathcal{A}$, $A \cap B = A \setminus (A \setminus B) \in \mathcal{A}$.

(b) Let $A_1, A_2, \ldots \in \mathcal{A}$. Then

$$\begin{aligned}\bigcap_{i=1}^\infty A_i &= \bigcap_{i=1}^\infty (A_1 \cap A_i) \\ &= \bigcap_{i=1}^\infty A_1 \setminus (A_1 \setminus A_i) \\ &= A_1 \setminus \bigcup_{i=1}^\infty (A_1 \setminus A_i).\end{aligned}$$

(c) Let $A_1, A_2, \ldots \in \mathcal{A}$. We then have

$$\bigcup_{i=1}^\infty A_i = A_1 \uplus (A_2 \setminus A_1) \uplus ((A_3 \setminus A_2) \setminus A_1) \uplus \cdots$$

The result follows.

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This equivalence between $\cap$ and $\cup$ if the class is $\setminus$-closed is apparent from De Morgan's laws.

Definition 1.2 (Algebra)**.** A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called an *algebra* if

(i) $\Omega \in \mathcal{A}$,

(ii) $\mathcal{A}$ is $\setminus$-closed, and

(iii) $\mathcal{A}$ is $\cup$-closed.

Definition 1.3 (σ-algebra)**.** A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a *σ-algebra* if

(i) $\Omega \in \mathcal{A}$,

(ii) $\mathcal{A}$ is closed under complements, and

(iii) $\mathcal{A}$ is σ-$\cup$-closed.

σ-algebras are also known as *σ-fields*.
Note that any σ-algebra is an algebra (but the converse is not true).

Theorem 1.3. A class of sets $\mathcal{A} \subseteq 2^\Omega$ is an algebra if and only if

(a) $\Omega \in \mathcal{A}$,

(b) $\mathcal{A}$ is closed under complements, and

(c) $\mathcal{A}$ is $\cap$-closed.

The proof of the above is left as an exercise to the reader.

Definition 1.4 (Ring)**.** A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a *ring* if

(i) $\varnothing \in \mathcal{A}$,

(ii) $\mathcal{A}$ is $\setminus$-closed, and

(iii) $\mathcal{A}$ is $\cup$-closed.

Further, a ring is a *σ-ring* if it is σ-$\cup$-closed.

Definition 1.5 (Semiring)**.** A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a *semiring* if

(i) $\varnothing \in \mathcal{A}$,

(ii) for any $A, B \in \mathcal{A}$, $A \setminus B$ is a finite union of mutually disjoint sets in $\mathcal{A}$, and

(iii) $\mathcal{A}$ is $\cap$-closed.

Definition 1.6 (λ-system)**.** A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a *λ-system* (or *Dynkin's λ-system*) if

(i) $\Omega \in \mathcal{A}$,

(ii) for any $A, B \in \mathcal{A}$ with $B \subseteq A$, $A \setminus B \in \mathcal{A}$, and

(iii) $\biguplus_{i=1}^{\infty} A_i \in \mathcal{A}$ for any choice of countably many pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{A}$.

Among the above classes of sets, σ-algebras in particular are extremely important as we shall use them when defining probability.

Theorem 1.4.

(a) Every σ-algebra is also a λ-system, an algebra and a σ-ring.

(b) Every σ-ring is a ring, and every ring is a semiring.

(c) Every algebra is a ring. An algebra on a finite set Ω is a σ-algebra.

Proof.

(a) Let $\mathcal{A}$ be a σ-algebra. Then for any $A, B \in \mathcal{A}$, $A \setminus B = (A^c \cup B)^c \in \mathcal{A}$ and $A \cap B = (A^c \cup B^c)^c \in \mathcal{A}$, that is, $\mathcal{A}$ is $\setminus$-closed and $\cup$-closed. The result follows.

(b) Let $\mathcal{A}$ be a ring. Then theorem 1.1 implies that $\mathcal{A}$ is $\cap$-closed. The result follows.

(c) Let $\mathcal{A}$ be an algebra. With proof similar to the first part of this theorem, it is seen that $\mathcal{A}$ is $\setminus$-closed. We have $\varnothing = \Omega \setminus \Omega \in \mathcal{A}$ and thus, it is a ring. If Ω is finite, then $\mathcal{A}$ is finite. Thus any countable union of sets is a finite union of sets and the result follows.

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Definition 1.7. Let $A_1, A_2, \ldots$ be subsets of Ω. Then

$$\liminf_{n\to\infty} A_n := \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j \text{ and } \limsup_{n\to\infty} A_n := \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j$$

are respectively called the *limit inferior* and *limit superior*, of the sequence $(A_n)_{n\in\mathbb{N}}$.

The above may be rewritten as

$$A_* := \liminf_{n\to\infty} A_n = \{\omega \in \Omega : |n \in \mathbb{N} : \omega \notin A_n| < \infty\}$$
$$A^* := \limsup_{n\to\infty} A_n = \{\omega \in \Omega : |n \in \mathbb{N} : \omega \in A_n| = \infty\}$$

That is, A_* represents the set of elements that do not appear in a finite number of sets and A^* represents the set of elements that appear in an infinite number of sets. This implies that $A_* \subseteq A^*$. (Why is the opposite not necessarily true?)

Definition 1.8 (Indicator function)**.** Let A be a subset of Ω. The *indicator function on* A is defined by

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

With the above notation, it may be shown that

$$\mathbb{1}_{A_*} = \liminf_{n\to\infty} \mathbb{1}_{A_n} \text{ and } \mathbb{1}_{A^*} = \limsup_{n\to\infty} \mathbb{1}_{A_n}.$$

If $\mathcal{A} \subseteq 2^\Omega$ is a σ-algebra and if $A_n \in \mathcal{A}$ for every $n \in \mathbb{N}$, then $A_* \in \mathcal{A}$ and $A^* \in \mathcal{A}$. This is clear from the fact that σ-algebras are closed under countable unions and intersections.
Proving the above statements is left as an exercise to the reader.

Theorem 1.5. Let I be some index set and $\mathcal{A}_i$ be a σ-algebra for each $i \in I$. Then the intersection $\mathcal{A}_I = \bigcap_{i\in I} \mathcal{A}_i$ is also a σ-algebra.

Proof. We can prove this by using the three conditions in the definition of a σ-algebra.

(i) Since $\Omega \in \mathcal{A}_i$ for every $i \in I$, $\Omega \in \mathcal{A}_I$.

(ii) Let $A \in \mathcal{A}_I$. Then $A \in \mathcal{A}_i$ for each $i \in I$ and thus $A^c \in \mathcal{A}_i$ for each $i \in I$. Therefore, $A^c \in \mathcal{A}_I$.

(iii) Let $A_1, A_2, \ldots \in \mathcal{A}_I$. Then $A_n \in \mathcal{A}_i$ for each $n \in \mathbb{N}$ and $i \in I$. Thus $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_i$ for each i as well. The result follows.

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A similar statement holds for λ-systems.

Theorem 1.6. Let $\mathcal{E} \subseteq 2^\Omega$. Then there exists a smallest σ-algebra $\sigma(\mathcal{E})$ with $\mathcal{E} \subseteq \sigma(\mathcal{E})$:

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{A} \subseteq 2^\Omega \text{ is a } \sigma\text{-algebra} \\ \mathcal{E} \subseteq \mathcal{A}}} \mathcal{A}.$$

$\sigma(\mathcal{E})$ is called the *σ-algebra generated by* $\mathcal{E}$ and $\mathcal{E}$ is called a *generator of* $\sigma(\mathcal{E})$.

Proof. 2^Ω is a σ-algebra that contains $\mathcal{E}$ so the intersection is non-empty. By theorem 1.5, $\sigma(\mathcal{E})$ is a σ-algebra. ∎

Similar to the above, $\delta(\mathcal{E})$ is defined as the λ-system generated by $\mathcal{E}$.

We always have the following:

1. $\mathcal{E} \subseteq \sigma(\mathcal{E})$.
2. If $\mathcal{E}_1 \subseteq \mathcal{E}_2$, then $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.
3. $\mathcal{A}$ is a σ-algebra if and only if $\sigma(\mathcal{A}) = \mathcal{A}$.

Similar statements hold for λ-systems. Further, $\delta(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. This is to be expected as σ-algebras have more "structure" than λ-systems.

Theorem 1.7 ($\cap$-closed λ-system)**.** Let $\mathcal{D} \subseteq 2^\Omega$ be a λ-system. Then $\mathcal{D}$ is a π-system if and only if $\mathcal{D}$ is a σ-algebra.

Proof. If $\mathcal{D}$ is a σ-algebra, then it is obviously a π-system. Let $\mathcal{D}$ be a π-system. Then

(a) As $\mathcal{D}$ is a λ-system, $\Omega \in \mathcal{D}$.

(b) Let $A \in \mathbb{D}$. Since $\Omega \in \mathcal{D}$ and $\mathcal{D}$ is a λ-system, $A^c = \Omega \setminus A \in \mathcal{D}$.

(c) Let $A, B \in \mathcal{D}$. We have $A \cap B \in \mathcal{D}$. We now have $A \setminus B = A \setminus (A \cap B) \in \mathcal{D}$, that is, $\mathcal{D}$ is $\setminus$-closed.
Let $A_1, A_2, \ldots \in \mathcal{D}$. Then by theorem 1.2, there exist $B_1, B_2, \ldots \in \mathcal{D}$ such that

$$\bigcup_{i=1}^\infty A_i = \biguplus_{i=1}^\infty B_i \in \mathcal{D}.$$

This completes the proof. ∎

Theorem 1.8 (Dynkin's π-λ theorem)**.** If $\mathcal{E} \subseteq 2^\Omega$ is a π-system, then $\delta(\mathcal{E}) = \sigma(\mathcal{E})$.

Proof. We already have $\delta(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. We must now prove the reverse inclusion. We shall show that $\delta(\mathcal{E})$ is a π-system.
For each $E \in \delta(\mathcal{E})$, let

$$\mathcal{D}_E = \{A \in \delta(\mathcal{E}) : A \cap E \in \delta(\mathcal{E})\}.$$

To show that $\delta(\mathcal{E})$ is a π-system, it suffices to show that $\delta(\mathcal{E}) \subseteq \mathcal{D}_E$ for all $E \in \delta(\mathcal{E})$. We shall first show that $\mathcal{D}_E$ is a λ-system for each $E \in \mathcal{E}$ by checking each of the conditions in definition 1.6.

(a) We clearly have $\Omega \in \mathcal{D}_E$ as $\Omega \cap E = E$.

(b) For any $A, B \in \mathcal{D}_E$ with $A \subseteq B$,

$$(B \setminus A) \cap E = (B \cap E) \setminus (A \cap E) \in \delta(\mathcal{E}).$$

(c) Let $A_1, A_2, \ldots \in \mathcal{D}_E$ be mutually disjoint sets. Then

$$\left(\biguplus_{i=1}^\infty A_i\right) \cap E = \biguplus_{i=1}^\infty (A_i \cap E) \in \delta(\mathcal{E}).$$

Now since $\mathcal{D}_E$ is a λ-system and $\mathcal{E} \subseteq \mathcal{D}_E$ (Why?), $\delta(\mathcal{E}) \subseteq \mathcal{D}_E$.
Now that we have shown that $\delta(\mathcal{E})$ is a π-system, the result follows by theorem 1.7. ■

Definition 1.9 (Topology)**.** Let $\Omega \neq \varnothing$ be an arbitrary set. A class of sets $\tau \subseteq 2^\Omega$ is called a *topology* on 2^Ω if

(i) $\varnothing, \Omega \in \tau$,

(ii) τ is $\cap$-closed, and

(iii) for any $\mathcal{F} \subseteq \tau$, $\bigcup_{A \in \mathcal{F}} A \in \tau$.

In the above case, the pair (Ω, τ) is called a *topological space*. The sets $A \in \tau$ are called *open* and the sets $A \subseteq \Omega$ with $A^c \in \tau$ are called *closed*.

Note that in contrast with σ-algebras, topologies are closed under only finite intersections but are also closed under arbitrary unions.

For example, consider the natural topology on $\mathbb{R}$ which consists of all open intervals in $\mathbb{R}$ and any arbitrary union of them.

Definition 1.10 (Borel σ-algebra)**.** Let (Ω, τ) be a topological space. The σ-algebra

$$\mathcal{B}(\Omega) = \mathcal{B}(\Omega, \tau) = \sigma(\tau)$$

that is generated by the open sets is called the *Borel σ-algebra on Ω*. The elements $A \in \mathcal{B}(\Omega, \tau)$ are called *Borel sets* or *Borel measurable sets*.

A Borel σ-algebra that we shall often encounter is $\mathcal{B}(\mathbb{R}^n)$ for $n \in \mathbb{N}$. Consider the following classes of sets:

$$\begin{aligned}
\mathcal{A}_1 &= \{A \subseteq \mathbb{R}^n : A \text{ is open}\} \\
\mathcal{A}_2 &= \{A \subseteq \mathbb{R}^n : A \text{ is closed}\} \\
\mathcal{A}_3 &= \{A \subseteq \mathbb{R}^n : A \text{ is compact}\} \\
\mathcal{A}_4 &= \{(a, b) : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\
\mathcal{A}_5 &= \{(a, b] : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\
\mathcal{A}_6 &= \{[a, b) : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\
\mathcal{A}_7 &= \{[a, b] : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\
\mathcal{A}_8 &= \{(-\infty, b) : b \in \mathbb{Q}^n\} \\
\mathcal{A}_9 &= \{(-\infty, b] : b \in \mathbb{Q}^n\} \\
\mathcal{A}_{10} &= \{(a, \infty) : a \in \mathbb{Q}^n\} \\
\mathcal{A}_{11} &= \{[a, \infty) : a \in \mathbb{Q}^n\}
\end{aligned}$$

It may be proved that $\mathcal{B}(\mathbb{R}^n)$ is generated by any of the classes of sets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{11}$.

For $A \in \mathcal{B}(\mathbb{R})$, we represent by $\mathcal{B}(\mathbb{R})|_A$ the restriction of $\mathcal{B}(\mathbb{R})$ to A. It may be proved that this is equal to $\mathcal{B}(A)$, the σ-algebra generated by the open subsets of A.

1.2. Measure

Definition 1.11. Let $\mathcal{A} \subseteq 2^\Omega$ and let $\mu : \mathcal{A} \to [0, \infty]$ be a set function. We say that μ is

(i) *monotone* if for any $A, B \in \mathcal{A}$, $A \subseteq B$ implies that $\mu(A) \leq \mu(B)$,

(ii) *additive* if for any choice of finitely many mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ with $\biguplus_{i=1}^n A_i \in \mathcal{A}$,

$$\mu\left(\biguplus_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i),$$

(iii) σ-additive if for any choice of countably many mutually disjoint sets $A_1, A_2, \ldots \in \mathcal{A}$ with $\biguplus_{i=1}^{\infty} A_i \in \mathcal{A}$,

$$\mu\left(\biguplus_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i),$$

(iv) subadditive if for any choice of finitely many sets $A, A_1, A_2, \ldots, A_n \in \mathcal{A}$ with $A \subseteq \bigcup_{i=1}^{n} A_i$, we have

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i), \text{ and}$$

(v) σ-subadditive if for any choice of countably many sets $A, A_1, A_2, \ldots \in \mathcal{A}$ with $A \subseteq \bigcup_{i=1}^{\infty} A_i$, we have

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 1.12. Let $\mathcal{A}$ be a semiring and $\mu : \mathcal{A} \to [0, \infty]$ be a set function with $\mu(\varnothing) = 0$. μ is called a

(i) *content* if μ is additive,

(ii) *premeasure* if μ is σ-additive, and

(iii) *measure* if μ is σ-additive and $\mathcal{A}$ is a σ-algebra.

Theorem 1.9 (Properties of contents)**.** Let $\mathcal{A}$ be a semiring and μ be a content on $\mathcal{A}$. Then

(a) If $\mathcal{A}$ is a ring, then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$.

(b) μ is monotone. If $\mathcal{A}$ is a ring, then $\mu(B) = \mu(A) + \mu(B \setminus A)$ for any $A, B \in \mathcal{A}$ with $A \subseteq B$.

(c) μ is subadditive. If μ is σ-additive, then it is also σ-subadditive.

(d) If $\mathcal{A}$ is a ring, then

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

for any choice of countably many mutually disjoint sets $A_1, A_2, \ldots \in \mathcal{A}$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Proof.

(a) Note that $A \cup B = A \uplus (B \setminus A)$ and $B = (A \cap B) \uplus (B \setminus A)$. As μ is additive,

$$\mu(A \cup B) = \mu(A) + \mu(B \setminus A) \text{ and } \mu(B) = \mu(A \cap B) + \mu(B \setminus A).$$

The result follows.

(b) Let $A \subseteq B$. If $B \setminus A \in \mathcal{A}$ (which is true in the case of a ring), we have $B = A \uplus (B \setminus A)$ and thus

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

If $\mathcal{A}$ is just a semiring, then there exist $n \in \mathbb{N}$ and mutually disjoint sets $C_1, C_2, \ldots, C_n \in \mathcal{A}$ such that

$$B \setminus A = \biguplus_{i=1}^{n} C_i.$$

In either case, we have $\mu(A) \leq \mu(B)$.

(c) Let $A, A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that $A \subseteq \bigcup_{i=1}^n A_i$. Let $B_1 = A_1$ and for each $k = 2, 3, \ldots, n$, let

$$B_k = A_k \setminus \left(\bigcup_{i=1}^{k-1} A_i \right).$$

Note that any two B_is are disjoint. As μ is additive and monotone, we have

$$\begin{aligned} \mu(A) &\leq \mu\left(\bigcup_{i=1}^n A_i \right) \\ &= \mu\left(\biguplus_{i=1}^n B_i \right) \\ &= \sum_{i=1}^n \mu(B_i) \leq \sum_{i=1}^n \mu(A_i). \end{aligned}$$

We can similarly prove that if μ is σ-additive, then it is σ-subadditive.

(d) Let $A = \bigcup_{i=1}^\infty A_i \in \mathcal{A}$. Since μ is additive and monotone,

$$\sum_{i=1}^m \mu(A_i) = \mu\left(\biguplus_{i=1}^m A_i \right) \leq \mu(A) \text{ for any } m \in \mathbb{N}.$$

The result follows.

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Note that if equality holds in the fourth part of the above theorem, μ is a premeasure.

Definition 1.13 (Finite content)**.** Let $\mathcal{A}$ be a semiring. A content μ on A is called

(i) *finite* if $\mu(A) < \infty$ for all $A \in \mathcal{A}$ and

(ii) *σ-finite* if there exists a sequence of sets $\Omega_1, \Omega_2, \ldots \in \mathcal{A}$ such that $\Omega = \bigcup_{i=1}^\infty \Omega_i$ and $\mu(\Omega_i) < \infty$ for every $i \in \mathbb{N}$.

Definition 1.14. Let $A, A_1, A_2, \ldots$ be sets. We write

(i) $A_n \uparrow A$ if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and $\bigcup_{i=1}^\infty A_i = A$. In this case, we say that A_n increases to A.

(ii) $A_n \downarrow A$ if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ and $\bigcap_{i=1}^\infty A_i = A$. In this case, we say that A_n decreases to A.

For example, if $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$, then $A_n \downarrow \{0\}$.

Definition 1.15 (Continuity of contents)**.** Let μ be a content on the ring $\mathcal{A}$. μ is called

(i) *lower semicontinuous* if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ for any $A \in \mathcal{A}$ and sequence $(A_n)_{n\in\mathbb{N}}$ such that $A_n \uparrow A$,

(ii) *upper semicontinuous* if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ for any $A \in \mathcal{A}$ and sequence $(A_n)_{n\in\mathbb{N}}$ such that $\mu(A_n) < \infty$ for some n (this implies that it holds for all $n \in \mathbb{N}$) and $A_n \downarrow A$,

(iii) *$\varnothing$-continuous* if (ii) holds for $A = \varnothing$.

Theorem 1.10. Let μ be a content on the ring $\mathcal{A}$. The following properties are equivalent:

(a) μ is σ-additive (and hence a premeasure).

(b) μ is σ-subadditive.

(c) μ is lower semicontinuous.

(d) μ is $\varnothing$-continuous.

(e) μ is upper semicontinuous.

Then (a) $\iff$ (b) $\iff$ (c) $\implies$ (d) $\iff$ (e). If μ is finite, then all five statements are equivalent.

Proof.

- (a) $\implies$ (b) (σ-additivity implies σ-subadditivity).
  This follows from theorem 1.9(c).
- (b) $\implies$ (a) (σ-subadditivity implies σ-additivity).
  This follows from theorem 1.9(d).
- (a) $\implies$ (c) (σ-additivity implies lower semicontinuity).
  Let μ be a premeasure and $A \in \mathcal{A}$. Let $A_1, A_2, \ldots \in \mathcal{A}$ such that $A_n \uparrow A$ and let $A_0 = \varnothing$. Then
  $$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i \setminus A_{i-1}) = \lim_{n\to\infty} \mu(A_n).$$
- (c) $\implies$ (a) (lower semicontinuity implies σ-additivity).
  Let $B_1, B_2, \ldots \in \mathcal{A}$ be mutually disjoint and let $B = \biguplus_{n=1}^{\infty} B_n \in \mathcal{A}$. Let $A_n = \biguplus_{i=1}^{n} B_i$ for each $n \in \mathbb{N}$. Then
  $$\mu(B) = \lim_{n\to\infty} \mu(A_n) = \sum_{i=1}^{\infty} \mu(B_i).$$
  Thus μ is σ-additive.
- (d) $\implies$ (e) ($\varnothing$-continuity implies upper semicontinuity).
  Let $A, A_1, A_2, \ldots \in \mathcal{A}$ with $A_n \downarrow A$ and $\mu(A_1) < \infty$. Define $B_n = A_n \setminus A \in \mathcal{A}$ for valid n. Then $B_n \downarrow \varnothing$. Thus
  $$\lim_{n\to\infty} \mu(A_n) - \mu(A) = \lim_{n\to\infty} \mu(B_n) = 0$$
  and the result is proved.
- (e) $\implies$ (d) (upper semicontinuity implies $\varnothing$-continuity).
  This is obvious.
- (c) $\implies$ (d) (lower semicontinuity implies $\varnothing$-continuity).
  Let $A_1, A_2, \ldots \in \mathcal{A}$ with $A_n \downarrow \varnothing$ and $\mu(A_1) < \infty$. Then $A_1 \setminus A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ and $A_1 \setminus A_n \uparrow A_1$. Thus
  $$\mu(A_1) = \lim_{n\to\infty} \mu(A_1) - \mu(A_n).$$
  Since $\mu(A_1) < \infty$, $\lim_{n\to\infty} A_n = 0$ and the result is proved.
- (d) $\implies$ (c) ($\varnothing$-continuity implies lower semicontinuity) if μ is finite.
  Let $A, A_1, A_2, \ldots \in \mathcal{A}$ with $A_n \uparrow A$. Then $A \setminus A_n \downarrow \varnothing$ and
  $$\lim_{n\to\infty} \mu(A) - \mu(A_n) = \lim_{n\to\infty} \mu(A \setminus A_n) = 0.$$
  The result follows.

■

Definition 1.16 (Measurable spaces)**.**

(i) A pair $(\Omega, \mathcal{A})$ consisting of a nonempty set Ω and a σ-algebra $\mathcal{A} \subseteq 2^{\Omega}$ is called a *measurable space*. The sets $A \in \mathcal{A}$ are called *measurable sets*. If Ω is countable and $\mathcal{A} = 2^{\Omega}$, then the space $(\Omega, 2^{\Omega})$ is called *discrete*.

(ii) A triple $(\Omega, \mathcal{A}, \mu)$ is called a *measure space* if $(\Omega, \mathcal{A})$ is a measurable space and μ is a measure on $\mathcal{A}$.

1.3. Measurable Maps

In measure theory, the homomorphisms (structure-preserving maps between objects) are studied as measurable maps.

Definition 1.17 (Measurable map)**.** Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces. A map $X : \Omega \to \Omega'$ is called $\mathcal{A} - \mathcal{A}'$*-measurable* (or just measurable) if

$$X^{-1}(A') \in \mathcal{A} \text{ for any } A' \in \mathcal{A}'.$$

In this case, we write $X : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$.

Theorem 1.11 (Generated σ-algebra)**.** Let $(\Omega', \mathcal{A}')$ be a measurable space and Ω be a nonempty set. Let $X : \Omega \to \Omega'$ be a map. Then

$$X^{-1}(\mathcal{A}') = \{X^{-1}(A') : A' \in \mathcal{A}'\}$$

is the smallest σ-algebra with respect to which X is measurable. We call $X^{-1}(\mathcal{A}')$ the *σ-algebra generated by X* and denote it as $\sigma(X)$.

Proof. Let X be measurable with respect to some σ-algebra $\mathcal{A}$. Then $X^{-1}(A') \in \mathcal{A}$ for any $A' \in \mathcal{A}'$, that is, $\sigma(X) \subseteq \mathcal{A}$. Let us now prove that $\sigma(X)$ is a σ-algebra by checking each of the axioms in definition 1.3.

1. As $\Omega' \in \mathcal{A}'$ and $X^{-1}(\Omega') = \Omega$, $\Omega \in \sigma(X)$.
2. Let $A \in \sigma(X)$ and $A' \in \mathcal{A}'$ such that $X^{-1}(A') = A$. Then as $\mathcal{A}'$ is closed under complements,
$$\Omega \setminus A = X^{-1}(\Omega') \setminus X^{-1}(A') = X^{-1}(\Omega' \setminus A') \in \sigma(X).$$
Therefore, $\sigma(X)$ is closed under complements.
3. Let $A_1, A_2 \ldots \in \sigma(X)$ and $A'_1, A'_2, \ldots \in \mathcal{A}'$ such that $A_i = X^{-1}(A'_i)$ for each $i \in \mathbb{N}$. Then as $\mathcal{A}'$ is σ-$\cup$-closed,
$$\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} X^{-1}(A'_i) = X^{-1}\left(\bigcup_{i \in \mathbb{N}} A'_i\right) \in \sigma(X)$$

Therefore, $\sigma(X)$ is a σ-algebra. ■

Theorem 1.12. Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces and $X : \Omega \to \Omega'$ be a map. Let $\mathcal{E}' \subseteq \mathcal{A}'$ be a class of sets. Then $\sigma(X^{-1}(\mathcal{E}')) = X^{-1}(\sigma(\mathcal{E}'))$.

Proof. We have that $X^{-1}(\mathcal{E}) \subseteq X^{-1}(\sigma(\mathcal{E})) = \sigma(X^{-1}(\sigma(\mathcal{E})))$. This implies that

$$\sigma(X^{-1}(\mathcal{E})) \subseteq X^{-1}(\sigma(\mathcal{E})).$$

To establish the reverse inclusion, consider

$$\mathcal{A}'_0 = \{A' \in \sigma(\mathcal{E}') : X^{-1}(A') \in \sigma(X^{-1}(\mathcal{E}'))\}$$

We shall show that $\mathcal{A}'_0$ is a σ-algebra.

(a) Clearly, $\Omega' \in \mathcal{A}'_0$ as $\Omega \in \sigma(X^{-1}(\mathcal{E}'))$ and $\Omega' \in \sigma(\mathcal{E}')$.

(b) Let $A'_0 \in \mathcal{A}'_0$. Then
$$X^{-1}((A'_0)^c) = (X^{-1}(A'_0))^c \in \sigma(X^{-1}(\mathcal{E}'))$$
and thus $\mathcal{A}_{\prime}{}'$ is closed under complements.

(c) Let $A'_1, A'_2, \ldots \in \mathcal{A}'_0$. Then
$$X^{-1}\left(\bigcup_{i=1}^{\infty} A'_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(A'_i) \in \sigma(X^{-1}(\mathcal{E}')).$$
Thus, $\mathcal{A}'_0$ is σ-$\cup$-closed.

Now, note that $\mathcal{E}' \subseteq \mathcal{A}'_0$ and $\mathcal{A}'_0 \subseteq \sigma(\mathcal{E}')$. This implies that $\mathcal{A}'_0 = \sigma(\mathcal{E}')$, and thus $X^{-1}(\sigma(\mathcal{E}')) \subseteq \sigma(X^{-1}(\mathcal{E}'))$. This proves the result. ∎

Corollary 1.13. Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces and $X : \Omega \to \Omega'$ be a map. Let $\mathcal{E}' \subseteq \mathcal{A}'$ be a class of sets. Then X is $\mathcal{A}$-$\sigma(\mathcal{E}')$ measurable if and only if $X^{-1}(\mathcal{E}') \in \mathcal{A}$. If in particular $\sigma(\mathcal{E}') = \mathcal{A}'$, then X is $\mathcal{A} - \mathcal{A}'$-measurable if and only if $X^{-1}(\mathcal{E}') \subseteq \mathcal{A}'$.

Theorem 1.14. Let $(\Omega, \mathcal{A})$, $(\Omega', \mathcal{A}')$, and $(\Omega'', \mathcal{A}'')$ be measurable spaces and let $X : \Omega \to \Omega'$ and $X' : \Omega' \to \Omega''$ be measurable. Then $Y = X' \circ X : \Omega \to \Omega''$ is $\mathcal{A} - \mathcal{A}''$-measurable.

Proof. This is due to the fact that

$$Y^{-1}(\mathcal{A}'') \subseteq X^{-1}((X^{-1})(\mathcal{A}'')) \subseteq X^{-1}(\mathcal{A}') \subseteq \mathcal{A}.$$

∎

The above theorem just states that the composition of two measurable maps is measurable.

Theorem 1.15 (Measurability of Continuous Maps)**.** Let (Ω, τ) and (Ω', τ') be topological spaces and let $f : \Omega \to \Omega'$ be a continuous map. Then f is $\mathcal{B}(\Omega) - \mathcal{B}(\Omega')$-measurable.

Proof. As $\mathcal{B}(\Omega') = \sigma(\tau')$, by corollary 1.13 it is enough to show that $f^{-1}(A') \in \sigma(\tau)$ for all $A' \in \tau'$. However, since f is continuous, $f^{-1}(A') \in \tau$ for all $A' \in \tau'$ so the result follows. ∎

Theorem 1.16. Let $X_1, X_2, \ldots$ be measurable maps $(\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$. Then $\inf_{n\in\mathbb{N}} X_n$, $\sup_{n\in\mathbb{N}} X_n$, $\liminf_{n\in\mathbb{N}} X_n$, and $\limsup_{n\in\mathbb{N}} X_n$ are also measurable.

Proof. For any $x \in \overline{\mathbb{R}}$,

$$\left(\inf_{n\in\mathbb{N}} X_n\right)^{-1}([-\infty, x)) = \bigcup_{n=1}^{\infty} (X_n)^{-1}([-\infty, x)) \in \mathcal{A}.$$

The first part of the result follows by corollary 1.13. The proof for $\sup_{n\in\mathbb{N}} X_n$ is similar.
For $\limsup_{n\in\mathbb{N}}$, consider the sequence $(Y_n)_{n\in\mathbb{N}}$ where $Y_n = \sup_{m\geq n} X_m$. Each Y_m is measurable. Then since $\inf_{n\in\mathbb{N}} Y_m$ is measurable, the result follows. ∎

Definition 1.18 (Simple Function)**.** Let $(\Omega, \mathcal{A})$ be a measurable space. A map $f : \Omega \to \mathbb{R}$ is called *simple* if there exists some $n \in \mathbb{N}$, mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}.$$

Definition 1.19. Let $f, f_1, f_2, \ldots$ be maps $\Omega \to \overline{\mathbb{R}}$ such that

$$f_1(\omega) \leq f_2(\omega) \leq \cdots \text{ and } \lim_{n\to\infty} f_n(\omega) = \omega \text{ for all } \omega \in \Omega.$$

We then write $f_n \uparrow f$. Similarly, we write $f_n \downarrow f$ if $(-f_n) \uparrow (-f)$.

Theorem 1.17. Let $(\Omega, \mathcal{A})$ be a measurable space and let $f : \Omega \to [0, \infty]$ be measurable. Then

(a) There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of non-negative simple functions such that $f_n \uparrow f$.

(b) There are measurable sets $A_1, A_2, \ldots \in \mathcal{A}$ and $\alpha_1, \alpha_2, \ldots \in [0, \infty)$ such that $f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$.

Proof.

(a) For $n \in \mathbb{N}_0$, define

$$f_n = \min\{n, 2^{-n}\lfloor 2^n f\rfloor\}.$$

Each f_n is measurable. (Why?) Since it can take at most $n2^n + 1$ distinct values, each f_n is simple. Clearly, $f_n \uparrow f$.

(b) Let f_n be the same as above. For $n \in \mathbb{N}$ and $i \in [2^n]$, define

$$B_{n,i} = \{\omega : f_n(\omega) - f_{n-1}(\omega) = i2^{-n}\} \text{ and } \beta_{n,i} = i2^{-n}.$$

Then $f_n - f_{n-1} = \sum_{i=1}^{2^n} \beta_{n,i} \mathbb{1}_{B_{n,i}}$. Changing the enumeration from (n,i) to m, we get some $(\alpha_m)_{m\in\mathbb{N}}$ and $(A_m)_{m\in\mathbb{N}}$ such that

$$f = f_0 + \sum_{n=1}^{\infty}(f_n - f_{n-1}) = \sum_{m=1}^{\infty} \alpha_m \mathbb{1}_{A_m}.$$

■

§2. Some Results

2.1. Outer Measure

Lemma 2.1. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and $\mathcal{E} \subseteq \mathcal{A}$ be a π-system that generates $\mathcal{A}$. Assume there exists sequence $\Omega_1, \Omega_2 \ldots \in \mathcal{E}$ such that $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and $\mu(\Omega_i) < \infty$ for all $i \in \mathbb{N}$. Then μ is uniquely determined by the values $\mu(E)$, $E \in \mathcal{E}$.
If $\Omega \in \mathcal{A}$ and $\mu(\Omega) = 1$, then the existence of the sequence $(\Omega_n)_{n\in\mathbb{N}}$ is not required.

Proof. Let ν be a σ-finite measure on $(\Omega, \mathcal{A})$ such that $\mu(E) = \nu(E)$ for all $E \in \mathcal{E}$.
Let $E \in \mathcal{E}$ with $\mu(E) < \infty$. Consider

$$\mathcal{D}_E = \{A \in \mathcal{A} : \mu(A \cap E) = \nu(A \cap E)\}.$$

We claim that $\mathcal{D}_E$ is a λ-system. We shall prove this by checking each of the conditions of definition 1.6.

(a) Clearly, $\Omega \in \mathcal{D}_E$.

(b) Let $A, B \in \mathcal{D}_E$ with $B \subseteq A$. Then

$$\begin{aligned} \mu((A \setminus B) \cap E) &= \mu(A \cap E) - \mu(B \cap E) \quad \text{(using theorem 1.9)} \\ &= \nu(A \cap E) - \nu(B \cap E) \\ &= \nu((A \setminus B) \cap E). \end{aligned}$$

That is, $(A \setminus B) \in \mathcal{D}_E$.

(c) Let $A_1, A_2, \ldots \in \mathcal{D}_E$ be mutually disjoint sets. Then

$$\begin{aligned} \mu\left(\left(\biguplus_{i=1}^{\infty} A_i\right) \cap E\right) &= \sum_{i=1}^{\infty} \mu(A_i \cap E) \\ &= \sum_{i=1}^{\infty} \nu(A_i \cap E) \\ &= \nu\left(\left(\biguplus_{i=1}^{\infty} A_i\right) \cap E\right). \end{aligned}$$

Therefore, $\biguplus_{i=1}^{\infty} A_i \in \mathcal{D}_E$ and $\mathcal{D}_E$ is a λ-system.

As $\mathcal{E} \subseteq \mathcal{D}_E$ (Why?), $\delta(\mathcal{E}) \subseteq \mathcal{D}_E$. Since $\mathcal{E}$ is a π-system, theorem 1.8 implies that

$$\mathcal{A} \supseteq \mathcal{D}_E \supseteq \delta(\mathcal{E}) = \sigma(\mathcal{E}) = \mathcal{A}.$$

Hence $\mathcal{D}_E = \mathcal{A}$.
Therefore, $\mu(A \cap E) = \nu(A \cap E)$ for any $A \in \mathcal{A}$ and $E \in \mathcal{E}$ with $\mu(E) < \infty$.
Now, let $\Omega_1, \Omega_2, \ldots \in \mathcal{E}$ be a sequence such that $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and $\mu(\Omega_i) < \infty$ for all $i \in \mathbb{N}$. Let $E_0 = \varnothing$ and $E_n = \bigcup_{i=1}^{n} \Omega_i$ for each $n \in \mathbb{N}$. Note that

$$E_n = \biguplus_{i=1}^{n} (E_{i-1}^c \cap \Omega_i).$$

Therefore for any $A \in \mathcal{A}$ and $n \in \mathbb{N}$,

$$\begin{aligned} \mu(A \cap E_n) &= \sum_{i=1}^{n} \mu((A \cap E_{i-1}^c) \cap \Omega_i) \\ &= \sum_{i=1}^{n} \nu((A \cap E_{i-1}^c) \cap \Omega_i) = \nu(A \cap E_n). \end{aligned}$$

Now, since $E_n \uparrow \Omega$ and μ, ν are lower semicontinuous (by theorem 1.10),

$$\begin{aligned}\mu(A) &= \lim_{n\to\infty} \mu(A \cap E_n) \\ &= \lim_{n\to\infty} \nu(A \cap E_n) = \nu(A)\end{aligned}$$

This proves the result.

The second part of the theorem is trivial as $\mathcal{E} \cup \{\Omega\}$ is a π-system that generates $\mathcal{A}$. Hence one can choose the constant sequence $E_n = \Omega, n \in \mathbb{N}$. ∎

Definition 2.1 (Outer Measure)**.** A function $\mu^* : 2^\Omega \to [0, \infty]$ is called an *outer measure* if

(i) $\mu^*(\varnothing) = 0$,

(ii) μ^* is monotone, and

(iii) μ^* is σ-subadditive.

Lemma 2.2. Let $\mathcal{A} \subseteq 2^\Omega$ be an arbitrary class of sets with $\varnothing \in \mathcal{A}$ and let μ be a nonnegative set function on $\mathcal{A}$ with $\mu(\varnothing) = 0$. For $A \subseteq \Omega$, define the set of countable coverings $\mathcal{F}$ with sets $F \in \mathcal{A}$

$$\mathcal{U}(A) = \left\{ \mathcal{F} \subseteq \mathcal{A} : \mathcal{F} \text{ is countable and } A \subseteq \bigcup_{F \in \mathcal{F}} F \right\}.$$

Define

$$\mu^*(A) = \inf \left\{ \sum_{F \in \mathcal{F}} \mu(F) : \mathcal{F} \in \mathcal{U}(A) \right\}$$

where $\inf \varnothing = \infty$. Then μ^* is an outer measure. If μ is σ-subadditive then $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof. Let us check each of the three conditions in the definition of an outer measure.

(a) Since $\varnothing \in \mathcal{A}$, we have $\{\varnothing\} \in \mathcal{U}(\varnothing)$ and hence $\mu(\varnothing) = 0$.

(b) If $A \subseteq B$, then $\mathcal{U}(A) \subseteq \mathcal{U}(B)$, and hence $\mu^*(A) \leq \mu^*(B)$.

(c) Let $A, A_1, A_2, \ldots \subseteq \Omega$ such that $A \subseteq \bigcup_{i=1}^\infty A_i$. We claim that $\mu^*(A) \leq \sum_{i=1}^\infty \mu^*(A_i)$.
Without loss of generality, assume that $\mu^*(A_i) < \infty$ and hence $\mathcal{U}(A_i) \neq \varnothing$ for all $i \in \mathbb{N}$. Fix some $\varepsilon > 0$. Now, for every $n \in \mathbb{N}$, we may choose a covering $\mathcal{F}_n \in \mathcal{U}(A_n)$ such that

$$\sum_{F \in \mathcal{F}_n} \mu(F) \leq \mu^*(A_n) + \varepsilon 2^{-n}.$$

Then let $\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{F}_n \in \mathcal{U}(A)$.

$$\mu^*(A) \leq \sum_{F \in \mathcal{F}} \mu(F) \leq \sum_{n=1}^\infty \sum_{F \in \mathcal{F}_n} \mu(F) \leq \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

This proves the first part of the result.

To prove the next part of the result, first note that since $\{A\} \in \mathcal{U}(A)$, we have $\mu^*(A) \leq \mu(A)$. If μ is σ-subadditive, then for any $\mathcal{F} \in \mathcal{U}(A)$,

$$\sum_{F \in \mathcal{F}} \mu(F) \geq \mu(A).$$

It follows that $\mu^*(A) \geq \mu(A)$. ∎

Definition 2.2 (μ^*-measurable sets)**.** Let μ^* be an outer measure. A set $A \in 2^\Omega$ is called μ^**-measurable* if

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E) \text{ for any } E \in 2^\Omega.$$

We write $\mathcal{M}(\mu^*) = \{A \subseteq \Omega : A \text{ is } \mu^*\text{-measurable}\}$.

Lemma 2.3. $A \in \mathcal{M}(\mu^*)$ if and only if

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E) \text{ for any } E \in 2^\Omega.$$

Proof. As μ^* is subadditive, we trivially have

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \geq \mu^*(E) \text{ for any } E \in 2^\Omega.$$

The result follows. ■

Lemma 2.4. $\mathcal{M}(\mu^*)$ is an algebra.

Proof. We shall check the conditions given in the definition of an algebra definition 1.2.

(a) We clearly have $\Omega \in \mathcal{M}(\mu^*)$.

(b) By definition, $\mathcal{M}(\mu^*)$ is closed under complements.

(c) We must check that $\mathcal{M}(\mu^*)$ is closed under intersections. Let $A, B \in \mathcal{M}(\mu^*)$ and $E \subseteq \Omega$. Then

$$\begin{aligned}
\mu^*((A \cap B) \cap E) + \mu^*((A \cap B)^c \cap E) &= \mu^*((A \cap B) \cap E) \\
&\quad + \mu^*((A \cap B^c \cap E) \cup (A^c \cap B \cap E) \cup (A^c \cap B^c \cap E)) \\
&\leq \mu^*(A \cap (B \cap E)) + \mu^*(A \cap (B^c \cap E)) \\
&\quad + \mu^*(A^c \cap (B \cap E)) + \mu^*(A^c \cap (B^c \cap E)) \\
&= \mu^*(B \cap E) + \mu^*(B^c \cap E) \quad (\text{since } A \in \mathcal{M}(\mu^*)) \\
&= \mu^*(E). \quad (\text{since } B \in \mathcal{M}(\mu^*))
\end{aligned}$$

This proves the result. ■

Lemma 2.5. An outer measure μ^* is σ-additive on $\mathcal{M}(\mu^*)$.

Proof. Let $A, B \in \mathcal{M}(\mu^*)$ with $A \cap B = \varnothing$. Then

$$\begin{aligned}
\mu^*(A \cup B) &= \mu^*(A \cap (A \cup B)) + \mu^*(A^c \cap (A \cup B)) \\
&= \mu^*(A) + \mu^*(B).
\end{aligned}$$

That is, μ^* is additive (and thus a content) on $\mathcal{M}(\mu^*)$. Since μ^* is σ-subadditive, theorem 1.10 gives the required result. ■

Lemma 2.6. If μ^* is an outer measure, $\mathcal{M}(\mu^*)$ is a σ-algebra.

Proof. We have already shown that $\mathcal{M}(\mu^*)$ is an algebra (and thus a π-system). Using theorem 1.7, it is sufficient to show that $\mathcal{M}(\mu^*)$ is a λ-system.
Let $A_1, A_2, \ldots \in \mathcal{M}(\mu^*)$ be mutually disjoint sets and let $A = \biguplus_{i=1}^\infty A_i$. Further, for each $n \in \mathbb{N}$, let $B_n = \biguplus_{i=1}^n A_i$. We must show that $M \in \mathcal{M}(\mu^*)$.
For any E and valid $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mu^*(E \cap B_{n+1}) &= \mu^*((E \cap B_{n+1}) \cap B_n) + \mu^*((E \cap B_{n+1}) \cap B_n^c) \\
&= \mu^*(E \cap B_n) + \mu^*(E \cap A_{n+1}).
\end{aligned}$$

By a simple induction, it follows that

$$\mu(E \cap B_n) = \sum_{i=1}^n \mu^*(E \cap A_i).$$

Since μ^* is monotonic, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &\geq \mu^*(E \cap B_n) + \mu^*(E \cap A^c) \\ &= \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap A^c).\end{aligned}$$

Letting $n \to \infty$ and using the fact that μ^* is σ-subadditive, we have

$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap A^c) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)\end{aligned}$$

Therefore, $A \in \mathcal{M}(\mu^*)$ and this completes the proof. ■

2.2. The Approximation and Extension Theorems

Theorem 2.7 (Approximation Theorem for Measures)**.** Let $\mathcal{A} \subseteq 2^\Omega$ be a semiring and let μ be a measure on $\sigma(\mathcal{A})$ that is σ-finite on $\mathcal{A}$. For any $A \in \sigma(\mathcal{A})$ with $\mu(\mathcal{A}) < \infty$ and any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and mutually disjoint sets $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that $\mu\left(A \triangle \bigcup_{i=1}^n A_n\right) < \varepsilon$.

Proof. Consider the outer measure μ^* as defined in lemma 2.2. Note that by lemma 2.2 and lemma 2.1, μ and μ^* are equal on $\sigma(\mathcal{A})$. By the definition of μ^*, for any $A \in \mathcal{A}$, there exists a covering $B_1, B_2, \ldots \in \mathcal{A}$ of A such that

$$\mu(A) \geq \sum_{i=1}^{\infty} \mu(B_i) - \varepsilon/2.$$

Since $\mu(A) < \infty$, there exists some $n \in \mathbb{N}$ such that $\sum_{i=n+1}^{\infty} \mu(B_i) < \varepsilon/2$. Now, let $D = \bigcup_{i=1}^n B_i$ and $E = \bigcup_{i=n+1}^{\infty} B_i$. We have

$$\begin{aligned}A \triangle D &= (D \setminus A) \cup (A \setminus D) \\ &\subseteq (D \setminus A) \cup (A \setminus (D \cup E)) \cup E \\ &\subseteq (A \triangle (D \cup E)) \cup E.\end{aligned}$$

This together with the fact that $A \subseteq \bigcup_{i=1}^{\infty} B_i$ implies that

$$\begin{aligned}\mu(A \triangle D) &\leq \mu(A \triangle (D \cup E)) + \mu(E) \\ &\leq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) - \mu(A) + \frac{\varepsilon}{2} \\ &\leq \varepsilon.\end{aligned}$$

Now define $A_1 = B_1$ and for each $i \geq 2$, $A_i = B_i \setminus \bigcup_{j=1}^{i=1} B_j$. By definition, $A_1, A_2 \ldots$ are mutually disjoint. This proves the result. ■

The following theorem allows us to "extend" measures from a semiring to the σ-algebra generated by it. This allows us to define measures over an entire σ-algebra by defining its values over just a semiring that generates it.

Theorem 2.8 (Measure Extension Theorem)**.** Let $\mathcal{A}$ be a semiring and let $\mu : \mathcal{A} \to [0, \infty]$ be an additive, σ-subadditive and σ-finite set function with $\mu(\varnothing) = 0$. Then there is a unique σ-finite measure $\tilde{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$ such that $\tilde{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof. Since $\mathcal{A}$ is a π-system, if such a $\tilde{\mu}$ exists, it is uniquely defined due to lemma 2.1.
We shall explicitly construct a function that satisfies the given conditions. In order to do so, define as in lemma 2.2

$$\mu^*(A) = \inf\left\{\sum_{F\in\mathcal{F}} \mu(F) : \mathcal{F} \in \mathcal{U}(A)\right\} \text{ for any } A \subseteq \Omega.$$

By lemma 2.2, μ^* is an outer measure and $\mu^*(A) = \mu(A)$ for any $A \in \mathcal{A}$.

We first claim that $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$.
To prove this, let $A \in \mathcal{A}$ and $E \subseteq \Omega$ with $\mu^*(E) < \infty$. Fix some $\varepsilon > 0$. Then by the definition of μ^*, there exists a sequence $E_1, E_2, \ldots \in \mathcal{A}$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and } \sum_{i=1}^{\infty} \mu(E_i) \leq \mu^*(E) + \varepsilon.$$

For each n, define $B_n = E_n \cap A$. Since $\mathcal{A}$ is a semiring, there exists for each n some $m_n \in \mathbb{N}$ and mutually disjoint sets $C_{n,1}, C_{n,2}, \ldots, C_{n,m_n}$ such that

$$E_n \setminus A = E_n \setminus B_n = \biguplus_{i=1}^{m_n} C_{n,i}$$

Then we have that

$$\begin{aligned}
E \cap A &\subseteq \bigcup_{n=1}^{\infty} B_n, \\
E \cap A^c &\subseteq \bigcup_{n=1}^{\infty} \biguplus_{i=1}^{m_n} C_{n,i}, \text{ and} \\
E_n &= B_n \uplus \biguplus_{i=1}^{m_n} C_{n,i}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \sum_{n=1}^{\infty} \mu(B_n) + \sum_{n=1}^{\infty}\sum_{i=1}^{m_n} \mu(C_{n,i}) \quad \text{(since } \mu \text{ is } \sigma\text{-subadditive)} \\
&= \sum_{n=1}^{\infty}\left(\mu(B_n) + \sum_{i=1}^{m_n} \mu(C_{n,i})\right) \\
&= \sum_{n=1}^{\infty} \mu(E_n) \quad \text{(since } \mu \text{ is additive)} \\
&\leq \mu^*(E) + \varepsilon.
\end{aligned}$$

lemma 2.3 implies that $A \in \mathcal{M}(\mu^*)$, that is, $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$. This in turn in implies that $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mu^*)$. Define the required function by $\tilde{\mu} : \sigma(\mathcal{A}) \to [0, \infty]$, $A \mapsto \mu^*(A)$. By lemma 2.5, $\tilde{\mu}$ is σ-additive. Since μ is σ-finite, $\tilde{\mu}$ is σ-finite as well. This proves the result. ∎

2.3. Important Examples of Measures

Now that we have the Measure Extension Theorem, we may introduce the Lebesgue-Stieltjes measure, a very useful measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which is given as follows.

Definition 2.3 (Lebesgue-Stieltjes Measure)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be monotone increasing and right continuous. The measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\mu_F((a, b]) = F(b) - F(a) \text{ for all } a, b \in \mathbb{R} \text{ such that } a < b$$

is called the *Lebesgue-Stieltjes measure* with distribution function F.

The Lebesgue-Stieltjes measure is well-defined due to the Measure Extension Theorem theorem 2.8.

To see this more clearly, let $\mathcal{A} = \{(a,b] : a,b \in \mathbb{R} \text{ and } a \leq b\}$. It may be checked that $\mathcal{A}$ is a semiring. Further, $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Now, define the function $\tilde{\mu}_F : \mathcal{A} \to [0,\infty)$ by $(a,b] \mapsto F(b) - F(a)$. Clearly $\tilde{\mu}_F(\varnothing) = 0$ and the function is additive. It remains to check that $\tilde{\mu}_F$ is σ-subadditive.

Let $(a,b], (a_1,b_1], (a_2,b_2], \ldots \in \mathcal{A}$ such that $(a,b] \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i]$. Fix some $\varepsilon > 0$ and choose $a_\varepsilon \in (a,b)$ such that

$$F(a_\varepsilon) - F(a) < \varepsilon/2 \implies \tilde{\mu}_F((a,b]) - \tilde{\mu}_F((a_\varepsilon,b]) < \varepsilon/2.$$

It is possible to choose such an ε due to the right continuity of F. Also, for any $k \in \mathbb{N}$, choose $b_{k,\varepsilon}$ such that

$$F(b_{k,\varepsilon}) - F(b_k) < \varepsilon 2^{-k-1} \implies \tilde{\mu}_F((a_k,b_{k,\varepsilon}]) - \tilde{\mu}_F((a_k,b_k]) < \varepsilon 2^{-k-1}.$$

We now have

$$[a_\varepsilon, b] \subseteq (a,b] \subseteq \bigcup_{i=1}^{\infty}(a_k,b_k] \subseteq \bigcup_{k=1}^{\infty}(a_k,b_{k,\varepsilon}]$$

Due to the compactness of $[a_\varepsilon,b]$, there then exists some $k_0 \in \mathbb{N}$ such that

$$(a_\varepsilon, b] \subseteq \bigcup_{k=1}^{k_0}(a_k,b_{k,\varepsilon}].$$

This implies that

$$\begin{aligned}
\tilde{\mu}_F((a,b]) &\leq \frac{\varepsilon}{2} + \tilde{\mu}_F((a,b]) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=1}^{k_0} \tilde{\mu}_F((a_k,b_{k,\varepsilon}]) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=1}^{k_0} \left(\tilde{\mu}_F((a_k,b_k]) + \varepsilon 2^{-k-1}\right) \\
&\leq \varepsilon + \sum_{k=1}^{\infty} \tilde{\mu}_F((a_k,b_k])
\end{aligned}$$

As this is true for any choice of ε, $\tilde{\mu}_F$ is σ-subadditive.

Then the extension of $\tilde{\mu}_F$ uniquely to a σ-finite measure is guaranteed by theorem 2.8. This measure is known as the Lebesgue-Stieltjes measure.

The measure that results when the function F is equal to the identity function is referred to the *Lebesgue measure* on $\mathbb{R}^1$. Similar to this, we can define the Lebesgue measure in general as follows.

Definition 2.4 (Lebesgue Measure)**.** There exists a unique measure λ^n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that for all $a,b \in \mathbb{R}^n$ with $a < b$,

$$\lambda^n((a,b]) = \prod_{i=1}^{n}(b_i - a_i).$$

λ^n is called the *Lebesgue measure* on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ or the *Lebesgue-Borel measure.*

Let E be a finite nonempty set and $\Omega = E^{\mathbb{N}}$. If $\omega_1, \omega_2, \ldots, \omega_n \in E$, we define the following.

$$[\omega_1, \omega_2, \ldots, \omega_n] = \{\omega' \in \Omega : \omega'_i = \omega_i \text{ for } i \in [n]\}.$$

This represents the set of all sequences whose first n elements are $\omega_1, \omega_2, \ldots, \omega_n$.

Theorem 2.9 (Finite Products of Measures)**.** Let $n \in \mathbb{N}$ and $\mu_1, \mu_2, \ldots, \mu_n$ be Lebesgue-Stieltjes measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then there exists a unique σ-finite measure μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that for all $a, b \in \mathbb{R}^n$ with $a < b$,

$$\mu((a,b]) = \prod_{i=1}^{n} \mu_i((a_i, b_i])$$

We call μ the *product measure of* $\mu_1, \mu_2, \ldots, \mu_n$ and denote it by $\bigotimes_{i=1}^n \mu_i$.

The proof of the above is similar to that of theorem 2.8. We choose intervals $(a, b_\varepsilon]$ and so on such that $\mu((a, b_\varepsilon]) < \mu((a,b]) + \varepsilon$. Such b_ε exists due to the right continuity of each of the F_is corresponding to each of the μ_is.

§3. The Integral

In the following, we assume $(\Omega, \mathcal{A}, \mu)$ to be a measure space. We denote by $\mathcal{E}$ the vector space of simple functions on $(\Omega, \mathcal{A})$ and by $\mathbb{E}^+ = \{f \in E : f \geq 0\}$ the cone of nonnegative simple functions. If

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$$

for some $n \in \mathbb{N}$, mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ and $\alpha_1, \ldots, \alpha_n \in (0, \infty)$ then this representation is said to be a *normal representation of* f.

3.1. Set Up to Define the Integral

Lemma 3.1. Let $f = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{A_i}$ and $f = \sum_{j=1}^{n} \beta_j \mathbb{1}_{B_j}$ be two normal representations of $f \in \mathbb{E}^+$. Then

$$\sum_{i=1}^{m} \alpha_i \mu(A_i) = \sum_{j=1}^{n} \beta_j \mu(B_j)$$

Proof. Clearly, if $\alpha_i \neq 0$ for some i, then $A_i \subseteq \bigcup_{j=1}^{n} B_j$. A similar result holds for B_j. Thus,

$$\sum_{i=1}^{m} \alpha_i \mu(A_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \mu(A_i \cap B_j).$$

If $\mu(A_i \cap B_j) \neq 0$, then $f(\omega) = \alpha_i = \beta_j$ for any $\omega \in A_i \cap B_j$. Therefore,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \mu(A_i \cap B_j) = \sum_{j=1}^{n} \sum_{i=1}^{m} \beta_j \mu(A_i \cap B_j) = \sum_{j=1}^{n} \beta_j \mu(B_j).$$

■

Definition 3.1. Define $I : \mathbb{E}^+ \to [0, \infty]$ by

$$I(f) = \sum_{i=1}^{m} \alpha_i \mu(A_i)$$

if f has the normal representation $f = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{A_i}$.

The above definition makes sense due to the previous lemma.

Lemma 3.2. Let $f, g \in \mathbb{E}^+$ and $\alpha \geq 0$. Then

(a) $I(\alpha f) = \alpha I(f)$,

(b) $I(f + g) = I(f) + I(g)$, and

(c) If $f \leq g$, then $I(f) \leq I(g)$.

We leave the proof of this theorem as an exercise to the reader.

Definition 3.2 (Integral)**.** If $f : \Omega \to [0, \infty]$ is measurable, then we define the *integral* of f with respect to μ by

$$\int f \, \mathrm{d}\mu = \sup \left\{ I(g) : g \in \mathbb{E}^+, g \leq f \right\}.$$

Note that by lemma 3.2(iii), $I(f) = \int f \, \mathrm{d}\mu$ for any $f \in \mathbb{E}^+$. That is, the integral is an extension of I from $\mathbb{E}^+$ to the set of non-negative measurable functions. We expand this to measurable functions in general in the next subsection.

Let $f, g : \Omega \to \overline{\mathbb{R}}$. Similar to how we write $f \leq g$ if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$, we write $f \leq g$ almost everywhere if there exists some set $N \in \mathcal{A}$ such that $\mu(N) = 0$ and $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega \setminus N$.

Theorem 3.3. Let $f, g, f_1, f_2, \ldots$ be measurable maps $\Omega \to [0, \infty]$. Then

(a) If $f \leq g$, then $\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$.

(b) If $f_n \uparrow f$, then $\int f_n \,\mathrm{d}\mu \uparrow \int f \,\mathrm{d}\mu$.

(c) If $\alpha, \beta \in [0, \infty]$, then

$$\int (\alpha f + \beta g) \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu + \beta \int g \,\mathrm{d}\mu$$

where we take $\infty \cdot 0 = 0$.

Proof.

(a) This is obvious from the definition of the integral.

(b) By the definition of the integral,

$$\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\mu$$

We must now show that $\int f \,\mathrm{d}\mu \leq \sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu$. Let $g \in \mathbb{E}^+$ with $g \leq f$. It is enough to show that $\sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu \geq \int g \,\mathrm{d}\mu$. Fix some $t \in (0, 1)$. For each $n \in \mathbb{N}$, define

$$A_n = \{\omega \in \Omega : f_n(x) \geq tg(x)\}.$$

Note that each A_n is measurable (Why?) and that $A_i \subset A_{i+1}$ for each $i \in \mathbb{N}$.

We first claim that $\bigcup_{i=1}^\infty A_i = \Omega$. We prove this as follows. For any $\omega \in \Omega$,

- If $f(\omega) \leq tg(\omega)$, then $f(\omega) = 0$ and $\omega \in A_n$ for every $n \in \mathbb{N}$.
- If $f(\omega) > tg(\omega)$, then there exists some $n \in \mathbb{N}$ such that $f_n(\omega) > tg(\omega)$. It follows that $\omega \in A_n$.

Therefore, $\Omega \subseteq \bigcup_{i=1}^\infty A_i$. Since the reverse inclusion is obviously true, we have $\bigcup_{i=1}^n A_i = \Omega$.

Now,

$$\int f_n \,\mathrm{d}\mu \geq \int tg\mathbb{1}_{A_n} \,\mathrm{d}\mu$$

Taking the limit as $n \to \infty$ and $t \to 1$, we have

$$\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu \geq \int g \,\mathrm{d}\mu.$$

This completes the proof.

(c) By theorem 1.17, there exist sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{E}^+$ such that $f_n \uparrow f$ and $g_n \uparrow g$. Then by lemma 3.2 and (ii),

$$\begin{aligned}\int (\alpha f + \beta g) \,\mathrm{d}\mu &= \lim_{n\to\infty} \int (\alpha f + \beta g) \,\mathrm{d}\mu \\ &= \alpha \lim_{n\to\infty} \int f \,\mathrm{d}\mu + \beta \lim_{n\to\infty} \int g \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu + \beta \int g \,\mathrm{d}\mu.\end{aligned}$$

∎

3.2. The Integral and some Properties

We have now introduced enough to define the integral for measurable functions in general.

Definition 3.3 (Integrals of Measurable Functions)**.** Let $f : \Omega \to \overline{\mathbb{R}}$ be measurable. We call f *μ-integrable* if $\int |f| \,\mathrm{d}\mu < \infty$ and write

$$\mathcal{L}^1(\mu) = \mathcal{L}^1(\Omega, \mathcal{A}, \mu) = \left\{f : \Omega \to \overline{\mathbb{R}} : f \text{ is } \mu\text{-integrable}\right\}.$$

For $f \in \mathcal{L}^1(\mu)$, we define the integral of f with respect to μ by

$$\int f(\omega)\mu(\mathrm{d}\omega) = \int f \,\mathrm{d}\mu = \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu.$$

For $A \in \mathcal{A}$, we define

$$\int_A f \,\mathrm{d}\mu = \int (f \mathbb{1}_A) \,\mathrm{d}\mu.$$

Theorem 3.4. Let $f : \Omega \to [0, \infty]$ be measurable.

(a) $f = 0$ almost everywhere if and only if $\int f \,\mathrm{d}\mu = 0$.

(b) If $\int f \,\mathrm{d}\mu < \infty$, then $f < \infty$ almost everywhere.

Proof.

(a) Let us first prove the forward implication. Let $P = \{\omega \in \Omega : f(\omega) > 0\}$. Then $f \leq \infty \cdot \mathbb{1}_P$. As $n\mathbb{1}_N \uparrow \infty\mathbb{1}_N$, by theorem 3.3,

$$0 \leq \int f \,\mathrm{d}\mu \leq \lim_{n\to\infty} \int n\mathbb{1}_N \,\mathrm{d}\mu = 0.$$

For the backward implication, let $A_n = \{\omega \in \Omega : f(\omega) \geq 1/n\}$. Then $A_n \uparrow P$ and for any $n \in \mathbb{N}$,

$$0 = \int f \,\mathrm{d}\mu \geq \int \frac{1}{n}\mathbb{1}_{A_n} = \frac{\mu(A_n)}{n}.$$

This implies that $\mu(A_n) = 0$ for any $n \in \mathbb{N}$ and therefore, $\mu(P) = 0$.

(b) Let $A = \{\omega \in \Omega : f(\omega) = \infty\}$. Then for any $n \in \mathbb{N}$,

$$\mu(A) = \int \mathbb{1}_A \,\mathrm{d}\mu \leq \frac{1}{n} \int f\mathbb{1}_{f\geq n} \leq \frac{1}{n} \int f \,\mathrm{d}\mu.$$

Taking the limit as $n \to \infty$, we get $\mu(A) = 0$.

■

We now expand some of the properties that we proved earlier for non-negative measurable functions to measurable functions in general.

Theorem 3.5. Let $f, g \in \mathcal{L}^1(\mu)$.

(a) (Monotonicity). If $f \leq g$ almost everywhere, then $\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$. In particular, if $f = g$ almost everywhere, then $\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$.

(b) (Triangle Inequality). $\left|\int f \,\mathrm{d}\mu\right| \leq \int |f| \,\mathrm{d}\mu$.

(c) (Linearity). If $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathcal{L}^1(\mu)$ and

$$\int \alpha f + \beta g \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu + \beta \int g \,\mathrm{d}\mu.$$

Proof.

(a) Since $f \leq g$ almost everywhere, $f^+ \leq g^+$ and $f^- \geq g^-$ almost everywhere. It is enough to show that $\int f^+ \, \mathrm{d}\mu \leq \int g^+ \, \mathrm{d}\mu$ and $\int f^- \, \mathrm{d}\mu \geq \int g^- \, \mathrm{d}\mu$. Let us prove the former. The latter can similarly be shown.

Let $h = g^+ - f^+$. As $h^- = 0$ almost everywhere, by theorem 3.4, $\int h \, \mathrm{d}\mu = \int h^+ \, \mathrm{d}\mu \geq 0$. The result follows.

(b) We have

$$\begin{aligned}\left|\int f \, \mathrm{d}\mu\right| &= \left|\int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu\right| \\ &\leq \int (f^+ + f^-) \, \mathrm{d}\mu \\ &= \int |f| \, \mathrm{d}\mu.\end{aligned}$$

(c) To show linearity, it suffices to show that for any $\alpha \in [0, \infty)$,

- $\int (f+g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$,
- $\int \alpha f \, \mathrm{d}\mu = \alpha \int f \, \mathrm{d}\mu$, and
- $\int (-f) \, \mathrm{d}\mu = -\int f \, \mathrm{d}\mu$.

These are easily shown by splitting each function f into f^+ and f^- and using theorem 3.3 wherever necessary.

■

Definition 3.4. Let $f : \Omega \to [0, \infty)$ be measurable. Define the measure ν by

$$\nu(A) = \int (\mathbb{1}_A f) \, \mathrm{d}\mu \text{ for } A \in \mathcal{A}.$$

Then ν is said to have *density* f with respect to μ. We also denote ν by $f\mu$.

In showing that ν is a measure using theorem 1.10, finite additivity follows from finite additivity of the integral and lower semicontinuity follows from theorem 3.9.

Theorem 3.6. Let $f : \Omega \to [0, \infty)$ and $g : \Omega \to \overline{\mathbb{R}}$ be measurable. Then $g \in \mathcal{L}^1(f\mu)$ if and only if $(gf) \in \mathcal{L}^1(\mu)$. In this case,

$$\int g \, \mathrm{d}(f\mu) = \int (gf) \, \mathrm{d}\mu.$$

We omit the proof of the above. It may be shown by first assuming g to be an indicator function, then extending this to simple functions, non-negative measurable functions and measurable functions.

Definition 3.5. Let $f : \Omega \to \overline{\mathbb{R}}$ be measurable and $p \in [1, \infty)$. Define

$$\|f\|_p = \left(\int |f|^p \, \mathrm{d}\mu\right)^{1/p} \text{ and}$$

$$\|f\|_\infty = \inf\{k \geq 0 : \mu(\{|f| \geq k\}) = 0\}.$$

Further, for any $p \in [0, \infty]$, we define the vector space

$$\mathcal{L}^p(\mu) = \{f : \Omega \to \overline{\mathbb{R}} : f \text{ is measurable and } \|f\|_p < \infty\}.$$

Theorem 3.7. The map $\|\cdot\|_1$ is a seminorm on $\mathcal{L}^1(\mu)$, that is, for any $f, g \in \mathcal{L}^1(\mu)$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}\|\alpha f\|_1 &= |\alpha| \cdot \|f\|_1 \\ \|f+g\|_1 &\leq \|f\|_1 + \|g\|_1 \\ \|f\|_1 &\geq 0 \text{ with equality if and only if } f = 0 \text{ almost everywhere.}\end{aligned}$$

Proof. The first and third (in)equalities follow from theorem 3.5(c) and theorem 3.4(a) respectively. The second inequality follows from the fact that $|f+g| \leq |f| + |g|$. We leave the details of the proof to the reader. ■

In fact, $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(\mu)$ for any $p \in [1, \infty]$. The proofs of the first and third (in)equalities are similarly straightforward. The proof of the second however requires Minkowski's inequality.

Theorem 3.8. Let $\mu(\Omega) < \infty$ and $1 \leq p' \leq p \leq \infty$. Then $\mathcal{L}^p(\mu) \subseteq \mathcal{L}^{p'}(\mu)$ and further, the canonical inclusion $\mathcal{L}^p(\mu) \hookrightarrow \mathcal{L}^{p'}(\mu)$ given by $f \mapsto f$ is continuous.

Proof. Let us first take the case where $p = \infty$. For any $f \in \mathcal{L}^\infty(\mu)$, since $|f| \leq \|f\|_\infty$ almost everywhere,

$$\int |f|^{p'} \, \mathrm{d}\mu \leq \int \|f\|_\infty^{p'} \, \mathrm{d}\mu = \mu(\Omega) \|f\|_\infty^{p'} < \infty.$$

It follows that for any $f, g \in \mathcal{L}^\infty(\mu)$,

$$\|f - g\|_{p'} \leq (\mu(\Omega))^{1/p'} \|f - g\|_\infty$$

and so the inclusion map is continuous.
Let us next take the case where p is finite. Then for any $f \in \mathcal{L}^p(\mu)$ since $|f|^{p'} \leq 1 + |f|^p$,

$$\int |f|^{p'} \, \mathrm{d}\mu \leq \int 1 + |f|^p \, \mathrm{d}\mu = \mu(\Omega) + \int |f|^p \, \mathrm{d}\mu < \infty.$$

Now, for any $f, g \in \mathcal{L}^p(\mu)$, let $c = \|f - g\|_p$. Then

$$\begin{aligned} |f-g|^{p'} &= |f-g|^{p'} \mathbb{1}_{|f-g| \leq c} + |f-g|^{p'} \mathbb{1}_{|f-g| > c} \\ &\leq c^{p'} + c^{p'-p} |f-g|^p \end{aligned}$$

This implies that

$$\begin{aligned} \|f-g\|_{p'} &\leq c \left(\int 1 + c^{-p} |f-g|^p \, \mathrm{d}\mu \right)^{1/p'} \\ &= \|f-g\|_p \, (1 + \mu(\Omega))^{1/p'}. \end{aligned}$$

This completes the proof. ■

3.3. Monotone Convergence and Fatou's Lemma

Under what conditions can we exchange the limit and the integral? We answered this question in part in theorem 3.3(b). Over the course of this subsection, we attempt to answer this.

Theorem 3.9 (Monotone Convergence, Beppo-Levi Theorem)**.** Let $f_1, f_2, \ldots \in \mathcal{L}^1(\mu)$ and $f : \Omega \to \overline{\mathbb{R}}$ be measurable. Assume that $f_n \uparrow f$ almost everywhere. Then

$$\lim_{n \to \infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$$

where both sides can equal ∞.

Proof. Let N be a set such that $\mu(N) = 0$ and $f_n(\omega) \uparrow f(\omega)$ for all $\omega \in N^c$. For each $n \in \mathbb{N}$, define

$$f_n' = (f_n - f_1) \mathbb{1}_{N^c} \text{ and } f' = (f - f_1) \mathbb{1}_{N^c}.$$

Note that $f_n' \uparrow f'$ and each of these functions are non-negative. By theorem 3.3(b), $\int f_n' \, \mathrm{d}\mu \uparrow \int f' \, \mathrm{d}\mu$. Then by theorem 3.5(a),

$$\begin{aligned} \lim_{n \to \infty} \int f_n \, \mathrm{d}\mu &= \lim_{n \to \infty} \left(\int f_n' \, \mathrm{d}\mu + \int f_1 \, \mathrm{d}\mu \right) \\ &= \int f' \, \mathrm{d}\mu + \int f_1 \, \mathrm{d}\mu \\ &= \int f \, \mathrm{d}\mu. \end{aligned}$$

■

Theorem 3.10 (Fatou's Lemma)**.** Let $f \in \mathcal{L}^1(\mu)$ and let $f_1, f_2, \ldots$ be measurable with $f_n \geq f$ almost everywhere for all $n \in \mathbb{N}$. Then
$$\int \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n \,\mathrm{d}\mu$$

Proof. Considering $f_n - f$ for each $n \in \mathbb{N}$, we may assume each f_n to be non-negative almost everywhere. For each $m \in \mathbb{N}$, consider $g_m = \inf_{n\geq m} f_n$. Note that $g_m \uparrow \liminf_{n\to\infty} f_n$. Thus, using theorem 3.5(a) and theorem 3.9,

$$\int \liminf_{n\to\infty} f_n = \int \lim_{m\to\infty} g_m \,\mathrm{d}\mu = \lim_{m\to\infty} \int g_m \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n \,\mathrm{d}\mu.$$

■

In the above theorem, we require an integrable f for the statement to hold. This f is called a "minorant".